MAT 300 3-26 HW

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Problem 1. 5.2.10

Theorem 0.0.1. *Suppose that $h: A \to B$ and $k: B \to C$ are functions. If h and k are both one-to-one and onto, $(k \circ h)^{-1}$ is a function and $(k \circ h)^{-1} = h^{-1} \circ k^{-1}$.*

Solution

Part (a)

Proof First, since h and k are both one-to-one and onto, then $k \circ h$ is also one-to-one and onto, by a previously proven theorem. Similarly, if any function f is one-to-one and onto, then f^{-1} is also a function, which is one-to-one and onto. Thus, $(k \circ h)^{-1}$ is a function, which is one-to-one and onto. Now, $k \circ h$ will map A to C. h^{-1} maps $B \to A$ and k^{-1} maps $C \to B$. Thus, $k^{-1} \circ h^{-1}$ maps $C \to A$, as does $(k \circ h)^{-1}$. Also, both $(k \circ h)^{-1}$ and $h^{-1} \circ k^{-1}$ are equal to $h^{-1}(k^{-1}(x))$ (The first instance is such because it is the inverse of $k(h(x))$, the second case is just a re-writing). Thus, they are equal. ■

Problem 2. 5.3.12

Let $f\colon A \to B$ be a function.

(a) Give an example of a function $f\colon A \to B$ and two subsets X and Y of A such that $f(X \cap Y) \neq f(X) \cap f(Y)$

(b) Show that $f(\cap_{\alpha\in\Lambda} T_\alpha) = \cap_{\alpha\in\Lambda} f(T_\alpha)$ for all choices of f if and only if f is one-to-one.

Solution

Part (a)

Consider the sets $X = \{1\}, Y = \{2\}$, and $f = (1, 1,), (2, 1)$. Then, $f(X \cap Y) = \varnothing$, but $f(X) \cap f(Y) = \{1\}$.

Part (b)

Theorem 0.0.2. *Given a function f, the intersection of the images of all subsets of A is equal to the image of the intersection of all subsets of A if and only if f is one-to-one.*

Proof Suppose the condition of the contrapositive, namely that f is not one-to-one, i.e. that for some b in the codomain, there are two distinct values x and y that map to another distinct value, b. Now consider any subset Y, containing y and any other subset X, containing x, where y is not in X and x is not in Y. By definition, the intersection of X and Y will contain neither x nor y. However, since x and y both map to b, b is an element of the intersection of the images on X and Y. Now recall that $b \neq x$ and $b \neq y$, so that the intersection of the images of X and Y is not equal to the image of the intersection of X and Y. Thus, by contrapositive, if f is one-to-one, then the intersection of the images of all subsets of f is equal to the image of the intersection of all subsets.

Now suppose the opposite direction, that the intersection of the images of X and Y is not equal to the image of the intersection of X and Y. Call the intersection of images C and the image of intersections D. This means, by definition, that, without loss of generality between C and D, there is some $c \in C$ and $c \notin D$, and this c is reachable by $f(x)|x \in A$. However, the only way for c to be in C and not in D is if f is not one-to-one. This is because either c is in the intersection of the images or the image of the intersections, but not both. This is possible in the case where c is in the interesection of the images, but not the image of the intersections, when x and y map onto the same value, b, and thus f is not one-to-one. Thus, by contrapositive, if the intersection of the images of all subsets of f is equal to the image of the intersection of all subsets, then f is one-to-one. ∎

Problem 3. 5.3.5

Let $f\colon : \mathcal{R} \to \mathcal{R}$ be given by $f(x) = x^2$. Find the following:

(a) $f^{-1}(\{4\})$

(b) $f^{-1}([-2, 9])$

(c) $f^{-1}((1, 4])$

Solution

Part (a)

$\{2\}$

Part (b)

$[0, 3]$

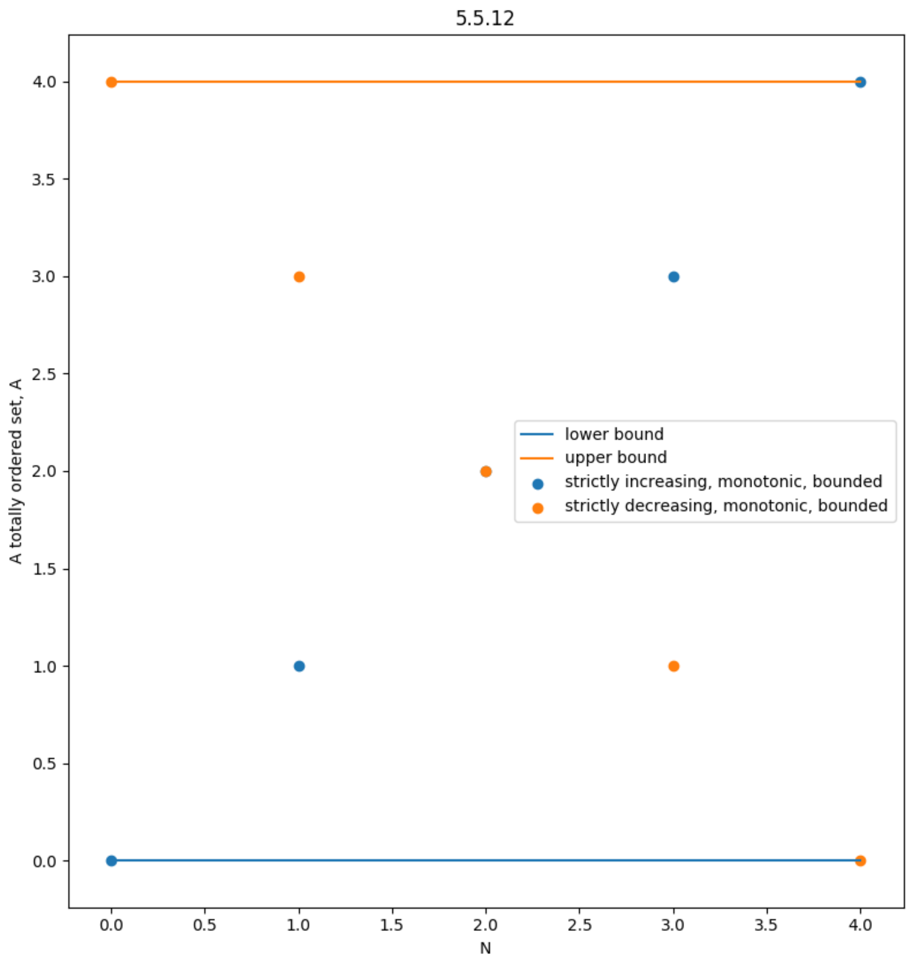
Part (c)

$[1, 2]$

Problem 4. 5.5.13

Solution

Part (a)



Problem 5. 5.11

Let $f\colon A \to B$ be a function. Let X and Y be subsets of A, and U and V be subsets of B.

(a) Prove that $f^{-1}(U)\ f^{-1}(V) = f^{-1}(U\ V)$.

(b) Prove that $f(X)\ f(Y) \subseteq f(X\ Y)$.

Solution

Part (a)

Theorem 0.0.3. *The difference of the preimages of U and V is equal to the preimage of the difference of U and V.*

Proof The difference of the preimages contains all elements x that reach U but do not reach V. Similarly, the preimage of the difference of U and V contains all elements x which reach U, but not V. Thus, these two sets are the same. ∎

Part (b)

Theorem 0.0.4. *The difference of images on X and Y is a subset of the image on the difference of X and Y.*

Proof Consider the difference of images on X and Y (Call this A). This set consists of all elements reached by elements in X, but not reached by elements in Y. The image on the difference of X and Y (B) also consists of all elements which X maps to, and so every element of A is in B, i.e. $A \subseteq B$. ∎

Problem 6. 5.12

Let $f\colon A \to B$ be a function. Consider sets of the form $f(f^{-1}(S))$.

(a) Show that for all subsets S of B, $f(f^{-1}(S)) \subseteq S$.

(b) Give an example to show that $f(f^{-1}(S))$ need not be equal to S.

(c) Complete and prove the following statement: $f(f^{-1}(S)) = S$ for all subsets S of B if and only if..

Solution

Part (a)

f maps $A \to B$, and f^{-1} maps $B \to A$, then for some subset S of B, f^{-1} maps S to $T \subseteq A$, and similarly f maps $T \to V \subseteq B$. However, V is a subset of S, because V consists of all elements which follow the form $f(f^{-1}(x))$, where x is originally from S. $f(f^{-1}(x))$ cannot map onto an element not in S, because the "middle" set, i.e. $f^{-1}(S)$ is mapped to from S, and so the image of this set must be a subset of S.

Part (b)

Consider $f(x) = x^2$ defined on the integers. Both 2 and -2 map to 4, but the inverse does not map back to -2.

Part (c)

...If and only if f is one-to-one and onto.