MAT 300 2-12 HW

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Problem 1. Powersets have cardinality 2^n

Let S be any finite set, and suppose $x \notin S$. Let $K = S \cup x$.

(a) Prove that $\mathcal{P}(K)$ is the disjoint union of $\mathcal{P}(S)$ and $X = T \subseteq K : x \in T$

(b) Prove that every element of X is the union of a subset of S with x, and if you take different subsets of S you get different elements of X. Argue that, therefore, X has the same number of elements as $\mathcal{P}(S)$

Solution

Part (a)

The second theorem below is proven while proving the first, so I've combined both here.
Theorem 0.0.1. *$\mathcal{P}(K)$ is the disjoint union of $\mathcal{P}(S)$ and $X = \{T \subseteq K : x \in T\}$ (i.e. $\mathcal{P}(K) = \mathcal{P}(S) \cup X$ and $\mathcal{P}(S) \cap X = \varnothing$)*
Theorem 0.0.2. *Every element of X is the union of a subset of S with $\{x\}$, and that if you take different subsets of S you get different elements of X. Argue that, therefore, X has the same number of elements as $\mathcal{P}(S)$*

Proof The powerset, $\mathcal{P}(U)$ is defined as the set of all subsets of of the set U. Each subset (of U, say) is defined as the set V where each element $v \in V$ is also in U. So, the powerset definition, in total, is $\mathcal{P}(U) = \{B | b \in U \text{ for all } b \in B\}$. As a new element, x is added to S to create K, the new powerset must now include subsets containing x (since subsets are any set which contains elements in the parent set), which are based on the original elements of $\mathcal{P}(S)$, such that $X = \{B \cup \{x\} | B \in \mathcal{P}(S)\}$. The powerset is then the union of these sets ($\mathcal{P}(S)$and$X$), by definition. It is very simple to see that these sets ($\mathcal{P}(S)$and$X$) have the same cardinality, from the definition of X. (This shows that the cardinality of $\mathcal{P}(K)$ is twice that of $\mathcal{P}(S)$). It is also very simple to see that the intersection of $\mathcal{P}(S)$and$X$ is the null set, because every set in X contains x, and every set in $\mathcal{P}(S)$ does not. ■

Part (b)

Provided above as part of the same proof.

Problem 2. Induction 1

Let $n \in \mathcal{N}$. Conjecture a formula for:

$$a_n = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)}$$

Solution

Part (a)

Theorem 0.0.3. *The formula $a_n = \frac{n}{n+1}$ describes the summation.*

Proof Proceeding by induction, it is first established that the formula works for a_1, because $a_1 = \frac{1}{(1)*(2)} = \frac{1}{2}$. Then, suppose the formula is true for an arbitrary $n \geq 1$, i.e. that $a_n = \frac{n}{n+1}$. From this, $a_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)(n+1)}{(n+1)(n+2)} = \frac{n+1}{n+2}$. By induction, this formula describes the summation a_n.

■

0.0.1 Scratch work

$a_1 = (1/2)$
$a_2 = (1/2) + (1/6) = 4/6 = 2/3$
$a_3 = (1/2) + (1/6) + (1/12) = 9/12 = 3/4$

A likely formula is $a_n = (n/n+1)$

Problem 3. Induction 2

Let m and $n \in \mathcal{N}$. Define what it means to say that m divides n. Now prove that for all $n \in \mathcal{N}$, 6 divides $n^3 - n$.

Solution

Part (a)

It is said that m divides n if there exists a natural number o such that $om = n$.
Theorem 0.0.4. *For all $n \in \mathcal{N}$, 6 divides $n^3 - n$.*

Proof Proceeding by induction, it is first shown that 6 divides $n = 2$ (as $n = 1$ is a trivial, but true, case, where $o = 0$). $n^3 - n = 6$, which is obviously divided by 6 with $o = 1$. Assume the induction hypothesis, namely that $n^3 - n = 6 * o$, where o is a natural number. For $n + 1$, the expression becomes $(n+1)^3 - (n+1) = 6 * p$, where p is another natural number. This expands to $n^3 + 3n^2 + 2n = 6 * p$. Using the induction hypothesis, $n^3 - n = 6 * o$, $(6 * o) + 3n^2 + 3n = 6 * p$ Then, $3(n)(n+1) = 6(p - o)$ Then, $(n)(n+1) = n^2 + n = 2(p - o)$, which simply means that $n^2 + n$ must be even for any n in the natural numbers. If n is even ($n = 2a$), then $n^2 + n = 4a^2 + 2a = 2(2a^2 + a)$, and the expression is even. If n is odd ($n = 2b + 1$), then $n^2 + n = 4b^2 + 4b + 1 + 2b + 1 = 2(2b^2 + 3b + 1)$, and the expression is even. Therefore, by induction, for all $n \in \mathcal{N}$, 6 divides $n^3 - n$. ■

Problem 4. Induction 3

Prove the following? Let $x \neq 1$ be a real number. For all $n \in \mathcal{N}$,
$\frac{(x^n-1)}{x-1} = (x^(n-1) + x^(n-2) + .. + x + 1$

Solution

Part (a)

Solution

Problem 5. Induction 4

Prove that every reducible polynomial can be written as a product of irreducible polynomials. (*Hint*: Proceed by complete induction on the degree of the polynomial)

Solution

Part (a)

Solution