MAT 300 4-09 HW

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Problem 1. 5.5.21

Let A be a set, and (s_i) be a sequence in A.

(a) If (s_i) is constant, every subsequence of (s_i) is constant.

(b) If $(s_i$ has distinct terms, then every subsequence of (s_i) has distinct terms.

Solution

Part (a)

Proof Suppose there existed a non-constant subsequence of s_i, call it s_{a_k}. Then, by definition, $s_{a_i} \neq s_{a_j}$ for distinct natural numbers i and j. Evaluate a_i, and a_j, which will be natural numbers. Call them x and y respectively. This means that $s_x s_y$ for distinct x and y (x and y must be distinct because, if they were not, then s_{a_i} would equal s_{a_j}). This leads to a contradiction, namely that $s_x \neq s_y$ for distinct x and y, or the original subsequence would be constant. By contradiction, every subsequence of a constant sequence must also be constant. ■

Proof Suppose that there existed a non-distinct subsequence of s_i. Call this subsequence s_{a_k}. By negating the definition of distinct, $s_{a_i} = s_{a_j}$ for some distinct i and j. Evaluate a_i and a_j to natural numbers x and y, so that $s_x = s_y$ for distinct x and y. This is a contradiction with the fact that s_i is distinct, so it is not possible to have a constant subsequence of a distinct sequence. ■

Problem 2. 6.1.4

Use the well-ordering of $\mathcal{N}$ (Every non-empty set of natural numbers contains a least element) to prove the following:

(a) Every nonempty subset of $\mathcal{Z}$ that has a lower bound has a least element.

(b) Every nonempty subset of $\mathcal{Z}$ that has a upper bound has a greatest element.

Solution

In each of these solutions, the subset on $\mathcal{Z}$ must be converted to a subset of $\mathcal{N}$.

Part (a)

Proof Call this lower bounded set A, with lower bound b. Create a new set $B = \{x-b+1 | x \in A\}$, which is a subset of the natural numbers (because it has lower bound $c = b-b+1 = 1$), and thus now has a least element, by the well ordering principle. Let $y \in B$, so that $y = x - b + 1$ for $x \in A$. Since b is a lower bound for A, $x \geq b$, $x - b + 1 \geq 1$, so $y \geq 1$, and thus $y \in \mathcal{N}$. Now observe that B is not empty. Since $A \neq \varnothing$, it contains at least one element, x_0. Then $y_0 = x_0 - b + 1 \in B$, so $B \neq \varnothing$. By the well ordering principle, B has least element $y_1 = x_1 - b + 1$. Consider an arbitrary $x \in A$. Then, $x - b + 1 \in B$, y_1 is the least element of B, $x - b + 1 \geq y_1 = x_1 - b + 1$. Add $b - 1$ to both sides, so that $x \geq x_1$. Since x is arbitrary, x_1 is the least element of A ■

Part (b)

Proof Call this upper bounded set S, with upper bound s_u. Create a new set, $T = \{-s + s_u + 1\}$, which is a subset of the natural numbers. Consider that $s \in S$ is less than or equal to s_u, so $-s_u \leq -s$. Then $-s_u + s_u + 1 \leq -s + s_u + 1$, so $t = -s + s_u + 1 \geq 1$, and T is a subset of the natural numbers. Observe that T is not empty, because S is not empty (it contains some s_0), and then $t_0 = -s_0 + s_u + 1$ and t_0 is in T. Since t_1 is a least element in T (by the well ordering principle), and $t_1 = -s_1 + s_u + 1$ for some s_1. So $-s_1$ is a least element of S, and s_1 is a greatest element: Let $s \in S$. Then $-s + s_u + 1 \in T$. t_1 is the least element of T, so $-s + s_u + 1 \geq t_1$. Also, $t_1 = -s_1 + s_u + 1$, so $-s + s_u + 1 \geq -s_1 + s_u + 1$ This implies that $s \leq s_1$, and thus s_1 is a greatest element of S, since s is arbitrary. ■

Problem 3. 6.2.10

Prove the following about the partial ordering on $\mathcal{N}$.

(a) Prove that $\mathcal{N}$ is partially ordered under the relation $|$.

(b) Is $|$ a *total* order on $\mathcal{N}$? Explain.

(c) Draw a lattice diagram the depicts the order $|$ on the set $\{1, 2..15\}$.

(d) Does $\{2, 3, 4, 5..\}$ have any minimal or maximal elements with respect to the order $|$?

Solution

Part (a)

A partial ordering must be reflexive, anti-symmetric, and transitive. $|$ is reflexive, because any number (say, $x \in \mathcal{N}$) divides itself into one and itself: $x = x * 1$. $|$ is anti-symmetric, because for natural numbers a, b, x and y if $x|y$ and $y|x$, x and y must be equal. This can be seen by supposing to the contrary that $x|y$ and $y|x$ and $x \neq y$. In this case, $y = xa$ and $x = yb$, so $y = y * a * b$. Both a and b must be 1, or this leads to a contradition, but for a and b to be 1, x must equal y. $|$ is transitive, i.e. for natural numbers a, b, c, x, y if $a|b$ and $b|c$, then $a|c$. This can be seen when written explicitly: $b = ax$ and $c = by$, so $c = axy$. Let $z = xy$, and now $c = az$, so a divides c. Thus, $|$ is a partial ordering on N.

Part (b)

No, because there are some numbers which do not divide others, and a total ordering requires the relation to exist between arbitrary elements. For instance, 3 does not divide 5, because 5 is prime.

Part (c)

```
       8 4-12
       |/ /  \
 15 10 | 6 9 | 14
 | \| \|/|/  |/|
(3) 5  | 3---| 7 11 13
       |     |
       |     |
       2------
(all divided by)
1
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Part (d)

1 is minimal (and least) because it divides everything, but there is no maximal element because the set is infinite.

Problem 4. 6.2.3

Let a, b be natural numbers. Suppose $a = qb + r$ with $0 \leq r < b$. What does the division algorithm yield when $-a$ is divided by b? Justify your answer. Those natural numbers n that, when divided into m using the division algorithm, yield a remainder of zero have a special status with respect to m. We say that n divides m evenly or simply that n divides m.

Solution

Part (a)

Clearly, $-a$ is not a natural number. However, since q divides positive a, $-q$ divides $-a$, such that $-a = -qb - r$, by algebraic manipulation.

Problem 5. 6.2.8

Let a, b, c be integers.

(a) If $a|b$ and $b|a$, then $a = +/ - b$.

(b) If $a|b$ and $b|c$, then $a|c$.

Solution

Part (a)

If $a|b$ and $b|a$, then $b = ax$ and $a = by$, so $a = axy$, which is only true when xy is one. So either $x = -1$ and $y = -1$ or $x = 1$ and $y = 1$. Substitute these into the original statements, and either $a = -b$ or $a = b$.

Part (b)

If $a|b$ and $b|c$, then $b = ax$ and $c = by$. So $c = axy$, and thus $c = az$ for $z = xy$, so a divides c.