MAT 300 2-12 HW

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Problem 1. Powersets have cardinality 2^n

Let S be any finite set, and suppose $x \notin S$. Let $K = S \cup x$.

(a) Prove that $\mathcal{P}(K)$ is the disjoint union of $\mathcal{P}(S)$ and $X = T \subseteq K : x \in T$

(b) Prove that every element of X is the union of a subset of S with x, and if you take different subsets of S you get different elements of X. Argue that, therefore, X has the same number of elements as $\mathcal{P}(S)$

Solution

Part (a)

The second theorem below is proven while proving the first, so I've combined both here.
Theorem 0.0.1. *$\mathcal{P}(K)$ is the disjoint union of $\mathcal{P}(S)$ and $X = \{T \subseteq K : x \in T\}$ (i.e. $\mathcal{P}(K) = \mathcal{P}(S) \cup X$ and $\mathcal{P}(S) \cap X = \varnothing$)*
Theorem 0.0.2. *Every element of X is the union of a subset of S with $\{x\}$, and that if you take different subsets of S you get different elements of X. Argue that, therefore, X has the same number of elements as $\mathcal{P}(S)$*

Proof The powerset, $\mathcal{P}(U)$ is defined as the set of all subsets of of the set U. Each subset (of U, say) is defined as the set V where each element $v \in V$ is also in U. So, the powerset definition, in total, is $\mathcal{P}(U) = \{B | b \in U \text{ for all } b \in B\}$. As a new element, x is added to S to create K, the new powerset must now include subsets containing x (since subsets are any set which contains elements in the parent set), which are based on the original elements of $\mathcal{P}(S)$, such that $X = \{B \cup \{x\} | B \in \mathcal{P}(S)\}$. The powerset is then the union of these sets ($\mathcal{P}(S)$and$X$), by definition. It is very simple to see that these sets ($\mathcal{P}(S)$and$X$) have the same cardinality, from the definition of X. (This shows that the cardinality of $\mathcal{P}(K)$ is twice that of $\mathcal{P}(S)$). It is also very simple to see that the intersection of $\mathcal{P}(S)$and$X$ is the null set, because every set in X contains x, and every set in $\mathcal{P}(S)$ does not. ■

Part (b)

Provided above as part of the same proof.

Problem 2. Induction 1

Let $n \in \mathcal{N}$. Conjecture a formula for:

$$a_n = \frac{1}{(1)(2)} + \frac{1}{(2)(3)} + ... + \frac{1}{(n)(n+1)}$$

Solution

Part (a)

Theorem 0.0.3. *The formula* $a_n = \frac{n}{n+1}$ *describes the summation.*

Proof Proceeding by induction, it is first established that the formula works for a_1, because $a_1 = \frac{1}{(1)*(2)} = \frac{1}{2}$. Then, suppose the formula is true for an arbitrary $n \geq 1$, i.e. that $a_n = \frac{n}{n+1}$. From this, $a_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)(n+1)}{(n+1)(n+2)} = \frac{n+1}{n+2}$. By induction, this formula describes the summation a_n.

■

0.0.1 Scratch work

$a_1 = (1/2)$
$a_2 = (1/2) + (1/6) = 4/6 = 2/3$
$a_3 = (1/2) + (1/6) + (1/12) = 9/12 = 3/4$

A likely formula is $a_n = (n/n + 1)$

Problem 3. Induction 2

Let m and $n \in \mathcal{N}$. Define what it means to say that m divides n. Now prove that for all $n \in \mathcal{N}$, 6 divides $n^3 - n$.

Solution

Part (a)

It is said that m divides n if there exists a natural number o such that $om = n$.
Theorem 0.0.4. *For all $n \in \mathcal{N}$, 6 divides $n^3 - n$.*

Proof Proceeding by induction, it is first shown that 6 divides $n = 2$ (as $n = 1$ is a trivial, but true, case, where $o = 0$). $n^3 - n = 6$, which is obviously divided by 6 with $o = 1$. Assume the induction hypothesis, namely that $n^3 - n = 6 * o$, where o is a natural number. For $n + 1$, the expression becomes $(n+1)^3 - (n+1) = 6 * p$, where p is another natural number. This expands to $n^3 + 3n^2 + 2n = 6 * p$. Using the induction hypothesis, $n^3 - n = 6 * o$, $(6 * o) + 3n^2 + 3n = 6 * p$. Then, $3(n)(n+1) = 6(p - o)$. Then, $(n)(n+1) = n^2 + n = 2(p - o)$, which simply means that $n^2 + n$ must be even for any n in the natural numbers. If n is even ($n = 2a$), then $n^2 + n = 4a^2 + 2a = 2(2a^2 + a)$, and the expression is even. If n is odd ($n = 2b + 1$), then $n^2 + n = 4b^2 + 4b + 1 + 2b + 1 = 2(2b^2 + 3b + 1)$, and the expression is even. Therefore, by induction, for all $n \in \mathcal{N}$, 6 divides $n^3 - n$. ■

Problem 4. Induction 3

Prove the following? Let $x \neq 1$ be a real number. For all $n \in \mathcal{N}$,

$\frac{(x^n-1)}{x-1} = (x^(n-1) + x^(n-2) + .. + x + 1$

Solution

Part (a)

Theorem 0.0.5. *The formula* $\frac{(x^n-1)}{x-1}$ *describes the summation* $(x^(n-1)+x^(n-2)+..+x+1$ *for all* $n \in \mathcal{N}$*.*

Proof Proceeding by induction, it is first shown that the formula describes the summation for $n = 1$, i.e. $\frac{x^n-1}{x-1} = \frac{x-1}{x-1} = 1$. Then, the induction hypothesis is assumed to be true, such that $\frac{x^n-1}{x-1} = (x^(n-1)+x^(n-2)+..+x+1)$. So, $\frac{x^{n+1}-1}{x-1} = (x^n+x^(n-1)+x^(n-2)+..+x+1)$. Which simplifies to $\frac{x^{n+1}-1}{x-1} = x^n + \frac{x^n-1}{x-1} = \frac{x^n-1+(x^n)(x-1)}{x-1} = \frac{x^{n+1}-1}{x-1}$, which is what was to be shown. By induction, the formula $\frac{(x^n-1)}{x-1}$ describes the summation $(x^(n-1)+x^(n-2)+..+x+1$ for all $n \in \mathcal{N}$. ■

Problem 5. Induction 4

Prove that every reducible polynomial can be written as a product of irreducible polynomials. (*Hint*: Proceed by complete induction on the degree of the polynomial)

Solution

Part (a)

Theorem 0.0.6. *Every reducible polynomial can be written as a product of irreducible polynomials.*

Proof Proceeding by complete induction, it is first shown that a single reducible polynomial can be written as a product of irreducible polynomials. For instance, consider $x^2 + 2x + 1$, which reduces to $(x + 1) * (x + 1)$. Assume the induction hypothesis: that for some n, any reducible polynomial of degree $k < n$ can be factored into a product of irreducible polynomials (of degree $n' < k$). Then assume p is reducible and has degree $n+1$. So, $p = qr$, and the degrees of q and r sum to $n + 1$. Each q and r will be called sub-polynomials and are either be irreducible or reducible. If a sub-polynomial of degree m is reducible, then by definition it can be broken into two separate polynomials whose degrees sum to m, as in the induction hypothesis. Lastly, all polynomials of degree zero are irreducible, and some polynomials of higher degree are also irreducible (such as the "polynomial" x). Since the induction hypothesis requires a reduction in degree, it is shown recursively, by induction, that any reducible polynomial will be expressed as a product of irreducible polynomials. ■