MAT 300 2-26 HW

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Problem 1. 4.11

Consider $\mathcal{P}(\mathcal{N})$ under the partial ordering $\subseteq$.

(a) Give an example of a nonempty subset of $\mathcal{P}(\mathcal{N})$ with no greatest element.

(b) Let $K = \{\{2, 3, 4, 12\}, \{3, 6, 9, 12\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\}$. Find three upper bounds for K and three lower bounds for K in $\mathcal{P}(\mathcal{N})$. Does K have a least upper bound? Does K have a greatest upper bound?

(c) Let X be any set. Suppose that K is a nonempty subset of $\mathcal{P}(\mathcal{X})$, ordered under $\subseteq$. How would you construct the least upper bound of K? How about the greatest lower bound of K?

Solution

Part (a)

(a) The set $\{\{1\}, \{2\}\}$, which is a subset of the powerset of the natural numbers, has no greatest element because the two inner sets are not comparable (They only share the null set as an element, so neither is a subset of the other).

(b) Three upper bounds are $\{1, 2..12\}, \{1, 2..13\}, and \mathcal{Z}$. Three lower bounds are $\varnothing, \{3\}$, and $\{12\}$. There exists a least upper bound, namely $\{1, 2..12\}$, but no greatest lower bound, because the set of lower bounds has no greatest element because its items are not comparable (as in question 1).

(c) The least upper bound would be constructed by creating a set of all upper bounds, and then attempting to find the least element in that set. Similarly, the greatest lower bound would be constructed by creating a set of all lower bounds, and attempting to find the greatest element in that set. However, because the relation is $\subseteq$, X in general will have no greatest lower bound, unless X is the null set, or a set of one element.

Problem 2. 4.2.17

Show by giving an example that immediate successors and immediate predecessors are not necessarily unique.

Solution

Part (a)

Non-unique immediate successors and immediate predecessors are possible if a set has a partial ordering, but not a total one. For instance, consider an ordering over $\{a, b, c\}$ where $a < b$ and $a < c$, but no other orderings are defined. In this case, the successors to a, which are b and c, are clearly not unique. If the relation is inverted, then the same is true for predecessors.

Problem 3. 4.2.26

Prove the following. Let A be a partially ordered set that has the least upper bound property. Then every nonempty subset of A that is bounded below has a greatest lower bound. (Or: Every partially ordered set with the least upper bound property also has the greatest lower bound property). (Hint: Use Lemma 4.2.25).

Solution

Part (a)

Theorem 0.0.1. *Every non-empty subset of A that is bounded below (has a lower bound) has a greatest lower bound.*

Proof By the least upper bound property it is known that every non-empty subset K of a poset A with an upper bound will have a least upper bound in A, because there is a least element in the set of upper bounds, which are known to exist. By lemma 4.2.25, the least upper bound of the set of lower bounds of K, is the greatest lower bound of K. Suppose K is bounded below, i.e. the set of lower bounds, L_k, is non-empty. Just like K, L_k is a non-empty subset of A, and will itself be bounded, and is guaranteed to have a least upper bound, because the poset A has the least upper bound property. Then, by lemma 4.2.25, this least upper bound is the greatest lower bound of K. ■

Problem 4. 4.3.10

Let the set A be $\{1..6\}$. Suppose a relation r is defined as:
$\{(1,1),(2,2),(2,3),(2,5),(3,5),(4,2),(4,3),(4,5),(5,2),(5,3),(5,5)\}$
For each $\alpha \in A$, find T_α, and then find Ω_r (Short answer only).

Solution

Part (a)

Consider each element in A:

- $T_1 = \{1\}$
- $T_2 = \{2,3,5\}$
- $T_3 = \{5\}$
- $T_4 = \{2,3,5\}$
- $T_5 = \{2,3,5\}$
- $T_6 = \varnothing$

Thus, Ω_r is the set $\{\{1\},\{5\},\{2,3,5\},\varnothing\}$.

Problem 5. 4.3.23

Show that the following relations r on the specified set S are equivalence relations. In each case do this in two ways:

- By identifying the equivalence classes and noting that they partition S.
- By showing directly that r is reflexive, symmetric, and transitive.

Solution

Part (a)

(a) $S = \{p : p \text{ is a person in Ohio}\}$. A rel B represents that person A and person B were born in the same year.
Let Ω_r be the set of equivalence classes, a collection of subsets of S associated with relation r, where each T_α in Ω_r is the set of relatives of α. Years are natural numbers with a lower bound. Thus, there is a finite number of them. In each year, if person A is born in the same year as person B, then the reverse is also true (Thus r is symmetric). Thus, each year has a set of people born in that year, and this set is an equivalence class T_α. Each T_α is disjoint from other T_βs, and covers all people because a person must be born in a particular year, and only one particular year. This shows that the equivalence classes partition S. Now, r is also reflexive and transitive because each person is born in the same year as themself, and if person a is born in the same year as person b and person b is born in the same year as person c, then the people a, b and c (of interest: a and c) are all born in the same year. So, r is symmetric, reflexive, and transitive.

(b) $S = \mathcal{Z}$. a rel b represents that $a = b$
It is only true that $a = b$ when either $a = b$ or $a = -b$ (or $-a = b$, though a and b are general). So, to start with, this is actually a combination of two equivalence relations. Consider the equivalence classes T_α. For any element a in S, it is known that T_α includes two elements, and no others: a and $-a$. r is symmetric because the underlying relation $=$ is symmetric. Because r is symmetric, T_α is disjoint from other equivalence classes, but in total Ω_r includes all elements in S. This shows that the equivalence classes partition S. Now, r is also reflexive and transitive because each number is equal to itself, regardless of sign, and the transitive property holds on the underlying equality relationship.

Problem 6. 4.7

Is it possible for a partially ordered set to have both a least element and a minimal element that is *not* a least element?

Solution

Part (a)

Theorem 0.0.2. *It is not possible for a partially ordered set to have both a least element and a minimal element that is* not *a least element.*

Proof Proceeding by contradiction, consider a poset $A, \leq$ which has a least element, a. Least simply means that for all elements $x \in A$, $a \leq x$. So, suppose there there existed a minimal element, call it b, which was *not* a least element. The definition of minimal is weaker, and simply states that b is not greater than (i.e. $\leq$) any element in A. This scenario is not possible, by cases:

(a) Either the minimal element b is not related to the least element, a, and thus a is not a least element.

(b) Or, b is equal to a, in which case b is a least element.

Since both cases lead to a contradiction, the theorem holds. ■