MAT 300 2-26 HW

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Problem 1. 4.1.10

Indicate whether the following relations on the given sets are reflexive, symmetric, anti-symmetric, or transitive.

Solution

Part (a)

(a) $A = \{p : p \text{ is a person in alaska}\}$. $x\ y$ if x is at least as tall as y.
Reflexive, not symmetric, anti-symmetric, and transitive.

(b) $A = \mathcal{N}$. $x\ y$ if $x + y$ is even.
Reflexive, symmetric, not anti-symmetric, and transitive (though somewhat vacuously).

(c) $A = \mathcal{N}$. $x\ y$ if $x + y$ is odd.
Reflexive, symmetric, not anti-symmetric, and transitive, as above.

(d) $A = \mathcal{P}(\mathcal{N})$. $x\ y$ if $x \subseteq y$.
Reflexive, not symmetric, anti-symmetric, and transitive.

(e) $A = \mathcal{R}$. $x\ y$ if $x = 2y$.
Not reflexive, not symmetric, not anti-symmetric, and not transitive.

(f) $A = \mathcal{R}$. $x\ y$ if $x - y$ is irrational.
Not reflexive, symmetric, not anti-symmetric, and not transitive.

(g) $A = \{l : l \text{is a line in the Cartesian plane}\}$. $x\ y$ if x and y are parallel lines or if x and y are the same line.
Reflexive, symmetric, not anti-symmetric, and transitive.

Problem 2. 4.2.4

Let A be a set. Show that $\mathcal{P}(A)$ need not be totally ordered under the relation $\subseteq$.

Solution

Part (a)

Theorem 0.0.1. *$\mathcal{P}(A)$ is not totally ordered under the relation $\subseteq$.*

Proof Recall that a set A is said to be totally ordered if it has a relation which is anti-symmetric, transitive, and satisfies the "connex property": a b or b a for any a, b in the set A. While $\subseteq$ satisfies anti-symmetry and transitivity for $\mathcal{P}(A)$, it does not satisfy the connex property. For instance, the powerset of $\{0, 1\}$ contains the elements $\{0\}$ and $\{1\}$. Let these be the variables a and b in the connex property, and it is clear that it is not satisfied (because $\{0\} \not\subseteq \{1\}$ and $\{1\} \not\subseteq \{0\}$). ■

Problem 3. 4.2.15

Let A be a partially ordered set. Prove that if A has a greatest element, then the greatest element is unique. (Assume two greatest elements and show they are the same).

Solution

Part (a)

Theorem 0.0.2. *If A has a greatest element, then the greatest element is unique.*

Proof Suppose A has a greatest element, a. For all y in a, $y \sim x$. Suppose that there were another greatest element, b, which satisfied the same property. This implies that $a \sim b$ and $b \sim a$. Then, because any partial ordering satisfies anti-symmetry by definition, $a = b$, and thus the greatest element of A is unique. ■

Problem 4. A2

Let F_i be the fibonacci numbers. Use complete induction to prove that $F_n \geq \alpha^{n-2}$ for all $n \geq 1$ where $\alpha = (1+\sqrt{5})/2$.

Solution

Part (a)

Theorem 0.0.3. *The formula $F_n \geq \alpha^{n2}$ is true for all $n \geq 1$ where $\alpha = (1+\sqrt{5})/2$.*

Proof First consider the base cases $n = 1$ and $n = 2$. First, $F_1 = 1$ and $F_2 = 2$. For $n = 1$, $\alpha^{n-2} = \frac{2}{\sqrt{5}+1} = 0.618..$, which is less than 1. For $n = 2$, $\alpha^{n-2} = 1$, which is less than 2. Now, suppose that for all k in $n \geq k \geq 1$, $F_k \geq \alpha^{n-2}$. Using the induction hypothesis and the formula for the fibonacci numbers, $F_{n+1} = F_n + F_{n-1} \geq \alpha^{n-2} + \alpha^{n-3}$. Consider $\alpha = (1+\sqrt{5}/2)$. $\alpha^2 = \frac{6+2*\sqrt{5}}{4} = \frac{2+1+\sqrt{5}/2}{=} 1 + \alpha$. From above we can now say $F_{n+1} \geq \alpha^{n-3} * (\alpha + 1) = \alpha n - 3 * \alpha^2 = \alpha n - 1$. Thus, since $F_{n+1} \geq \alpha n - 1$, by the principle of complete mathematical induction, the hypothesis holds. ■