MAT 300 1-29 HW

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Problem 1. Direct Proof

Use a direct proof to prove that "If x + y is even and y + z is even, then x + z is even".

Solution

Part (a)

Theorem 0.0.1. *If $x + y$ is even and $y + z$ is even, then $x + z$ is even.*

Proof The definition of an even number is that the number is divisible by two such that $n = 2a$. By this definition, $x + y = 2a$ and $y + z = 2b$. So $x + y + y + z = 2a + 2b$, which simplifies to $x + z = 2(a + b - y)$. Since $a + b - y$ is just an integer (call it c), then $x + z = 2c$ and therefore $x + z$ is even. ■

Problem 2. Contrapositives

Find the contrapositives of the following statements. Write in positive terms where possible.

(a) If $x < 0$, then $x^2 > 0$.

(b) If $x \neq 0$, then there exists y for which $xy = 1$.

(c) If x is an even integer, then x^2 is an even integer.

(d) If $x + y$ is odd and $y + z$ is odd, then $x + z$ is odd.

(e) If f is a polynomial of odd degree, then f has at least one real root.

Solution

Part (a)

(a) If $x^2 \leq 0$, then $x \geq 0$.

(b) If for all y, $xy \neq 1$, then $x = 0$.

(c) If x^2 is an odd integer, then x is an odd integer.

(d) If $x + z$ is even, then $x + y$ is even and $y + z$ is even.

(e) If f does not have at least one real root, then f is a polynomial of even degree.

Problem 3. Line creation proof

Suppose that (a, b) and (c, d) are two distinct points in R^2. Prove that there exists a unique line passing through the two points.

Solution

Part (a)

Theorem 0.0.2. *Given two distinct points (a, b) and (c, d) in R^2, there exists a unique line passing through the two points.*

Proof A line is defined as either $y = mx + n$ (n is used to distinguish from b, which is already defined), or $x = l$, where m, n, and l are real numbers.

If $a = c$, then the line is vertical and takes the form $x = l$, where $l = a = c$. No other value of l will work, because if another value for l is chosen, then $l \neq a \neq c$ and therefore neither point is on the line. Therefore in this case the line is unique.

If $a \neq c$, then the line takes the form $y = mx + n$. In this case, m is defined as $\frac{(b-d)}{(a-c)}$, and n is then solved for as $b = \frac{(b-d)}{(a-c)} * a + n$. This means that $n = b - \frac{(b-d)}{(a-c)} * a$. Since both m and n, the defining variables of the line, are in terms of a, b, c, and d, the line is unique with respect to the points. ■

Problem 4. Subset Transitivity

Prove that if $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.

Solution

Part (a)

Theorem 0.0.3. *If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$.*

Proof If $A \subseteq B$, for every element $x \in A$, $x \in B$. Similarly, if $B \subseteq C$, for every element $x \in B$, $x \in C$. Therefore, for every element $x \in A$, $x \in C$. ■

Problem 5. Pondering Counting

What would a theory of counting look like in terms of sets? If I say "These two sets have the same number of elements," what does that mean mathematically? If I say, "This set has twice as many elements as that set," what does *that* mean? How might we define this rigorously?

Solution

Part (a)

Two sets A and B would have the same number of elements if every element in set A corresponded to an element in set B. Similarly, if a set B has twice as many elements as a set A, then there are two elements in B for each element in set A.

Problem 6. Tautology Proof

This question contained a typo. I believe that the right hand side of the statement should be $((A \Rightarrow C) \wedge (B \Rightarrow C))$, but was $((A \Rightarrow C) \vee (B \Rightarrow C))$.

Prove that the following statement is a tautology:

$((A \vee B) \Rightarrow C) \Leftrightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C))$

Solution

Part (a)

Theorem 0.0.4. $((A \vee B) \Rightarrow C) \Leftrightarrow ((A \Rightarrow C) \vee (B \Rightarrow C))$ *is a tautology.*

Proof The left hand side $((A \vee B) \Rightarrow C)$ can be converted into $C \vee \neg(A \vee C)$ by rule nine (Absorption). Then, this can be transformed using DeMorgan's law: $C \vee (\neg A \wedge \neg B)$. Further, by distributivity, the statement is broken apart: $(C \vee \neg A) \wedge (C \vee \neg B)$. Each side of the statement can again be turned into an implication, resulting in $(A \Rightarrow C) \wedge (B \Rightarrow C)$. $\blacksquare$