46770 Integrated energy grids

Lecture 5 – Optimal Power Flow

Types of optimization problems and course structure

	One node	Network			
		Power			
One time step	Economic dispatch or One-node dispatch optimization (Lecture 2)	Linearized AC power flow (Lecture 4)	AC power flow (Lecture 5)	Gas flow (Lecture 6)	Heat flow (Lecture 7)
Multiple time steps	Multi-period optimization Join capacity and dispatch optimization in one node (Lecture 8)	Join capacity and dispatch optimization in a network (Lecture 10)			

Recap from previous lectures

Learning goals

- Write the system cost minimization problem including AC optimal power flow
- Define and calculate the line admittance matrix and the bus admittance matrix
- Formulate the optimal power flow problem on a computer.
- Describe approaches to convexify the AC optimal power flow

For a comprehensive discussion on the setting the equations for AC power flows, calculating the Jacobian matrix, and Newton-Raphson technique for solving AC power flows, check Lecture 7 in DTU course 46700 Introduction to Electric Power Systems,

Dispatch optimization in a network

Assume we have a network of nodes i. In every node, we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity $G_{s,i}$ and a linear variable cost o_S. Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d_i in a certain hour and the optimal AC power flow while minimizing the total system cost.

Economic dispatch in one node

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0$$

$$0 \leq g_s \leq G_s$$

How can we include power flows between nodes in our optimization problem? Or more specifically, what are our nodal energy balance equations and links equations?

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = \boxed{?}$$

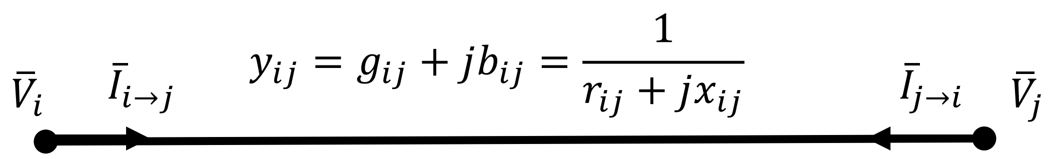
$$0 \leq g_s \leq G_s$$

$$\boxed{?}$$

Modeling AC transmission lines

previous lecture

Simple series admittance model



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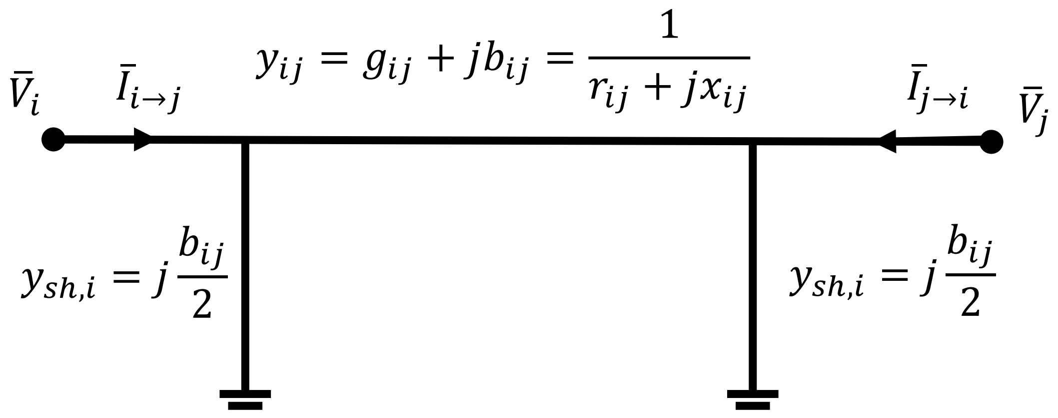
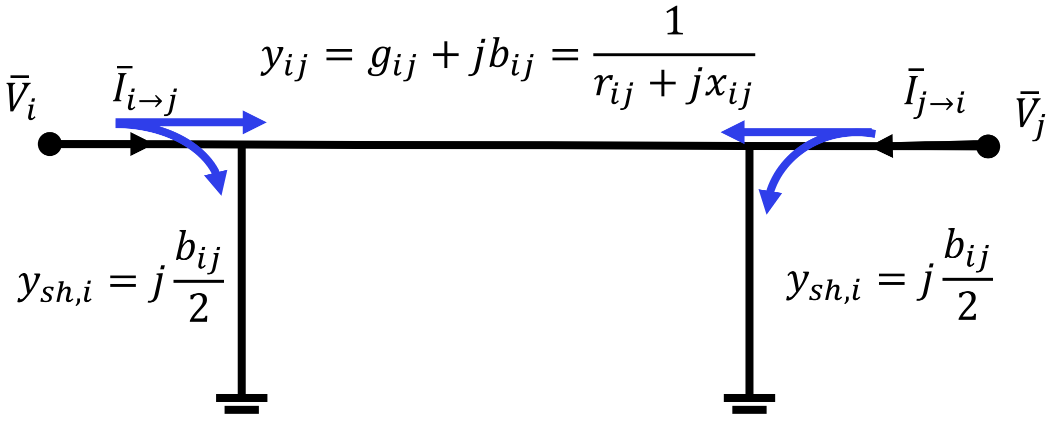
π model



Line model – equivalent π model



The current entering node i flows along the series admittance y_{ij} but some flows along the shunt susceptance $y_{sh,i}$

$$\bar{I}_{i\to j} = y_{sh,i}\bar{V}_i + y_{ij}(\bar{V}_i - \bar{V}_j)$$

Similarly, for the current entering node j

$$\bar{I}_{j\to i} = y_{sh,j}\bar{V}_j + y_{ij}(\bar{V}_j - \bar{V}_i)$$

We can express the equations using a matrix

$$\begin{pmatrix} \bar{I}_{i\to j} \\ \bar{I}_{j\to i} \end{pmatrix} = \begin{pmatrix} y_{sh,i} + y_{ij} & -y_{ij} \\ -y_{ij} & y_{sh,j} + y_{ij} \end{pmatrix} \begin{pmatrix} \bar{V}_i \\ \bar{V}_J \end{pmatrix}$$

Incidence Matrix or Line Admittance Matrix

The **Line Admittance Matrix $\bar{Y}_{lines,ij}$** relates the bus voltajes $\bar{V}_i$ to the current flows $\bar{I}_{i\to j}$

$$\begin{pmatrix} \bar{I}_{i\to j} \\ \bar{I}_{j\to i} \end{pmatrix} = \begin{pmatrix} y_{sh,i} + y_{ij} & -y_{ij} \\ -y_{ij} & y_{sh,j} + y_{ij} \end{pmatrix} \begin{pmatrix} \bar{V}_i \\ \bar{V}_j \end{pmatrix} \qquad \bar{I}_{i\to j} = \sum_j \bar{Y}_{line,i\to j} \bar{V}_j$$

The Line Admittance Matrix is equivalent to a weighted incidence matrix K_{ij}.

In AC systems, the net complex power flowing through a line is equal to the product of the voltage and the conjugate of the current (both complex numbers) $\bar{S}_{i\to j} = \bar{V}_i \bar{I}^*_{i\to j}$

We can use the Line Admittance Matrix to impose that the maximum power flow through every line is lower that the line capacity

$$\left| \bar{V}_i \sum_j \bar{Y}^*_{line,i\to j} \bar{V}^*_j \right| \leq S_{i\to j,max}$$

$$\left| \bar{V}_i \sum_j \bar{Y}^*_{line,j\to i} \bar{V}^*_j \right| \leq S_{j\to i\;max}$$

Weighted Laplacian matrix or Bus Admittance Matrix

Following Kirchoff's current law, the net current injection at a bus is equal to the sum of the currents leaving the bus

$$I_i = I_{i\to j} + I_{i\to k} = (y_{sh,ij}+y_{ij})V_i - y_{ij}V_j + (y_{sh,ik}+y_{ik})V_i - y_{ik}V_k$$

$$\begin{pmatrix} I_i \\ I_j \\ I_k \end{pmatrix} = \begin{pmatrix} y_{sh,ij} + y_{ij} + y_{sh,ik} + y_{ik} & -y_{ij} & -y_{ik} \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} V_i \\ V_j \\ V_k \end{pmatrix}$$

The **Bus Admittance Matrix $\bar{Y}_{bus,ij}$** relates the current injected and the voltage in the different buses

$\bar{I}_i = \sum_j \bar{Y}_{bus,ij} \bar{V}_j$

Economic dispatch with AC optimal power flow

Assume we have a network of nodes i. In every node, we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity $G_{s,i}$ and a linear variable cost o_S. Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d_i in a certain hour and the optimal AC power flow while minimizing the total system cost.

Economic dispatch in one node

$$\min_{g_s} \sum_s o_s g_s$$

subject to:

$$\sum_s g_s - d = 0$$

$$0 \leq g_s \leq G_s$$

Economic dispatch with AC power flow

$$\min_{g_{s,i}} \sum_{s,i} o_s g_{s,i}$$

subject to:

$$\sum_{s,i} g_{s,i} - d_i = \bar{V}_i \Big(\sum_j \overline{Y_{bus,ij}}\, \bar{V}_j\Big)^*$$

$$0 \leq g_s \leq G_s$$

$$\left| \bar{V}_i \sum_j \bar{Y}^*_{line,i\to j} \bar{V}^*_j \right| \leq S_{i\to j,max}$$

$$0 \leq p_s \leq P_s$$

$$-Q_s \leq q_s \leq Q_s$$

These equations include complex numbers which entail constrains for both the real and imaginary parts

Nodal balance equations

Capacities limits

The bar above a variable denotes complex number

AC power flow analysis

$$\sum_{s,i} g_{s,i} - d_i = \bar{V}_i (\sum_j \overline{Y_{bus,ij}}\, \bar{V}_j)^*$$

$$\bar{S}_{i\to j} = p_{i\to j} + \mathrm{j} q_{i\to j}$$

$$\bar{Y}^*_{ij} = g_{ij} - \mathrm{j} b_{ij}$$

$$\bar{V}_j^* \bar{V}_i = V_i V_j \angle \theta_i - \theta_j = |V_i||V_j| \cos(\theta_i - \theta_j) + j|V_i||V_j| \sin(\theta_i - \theta_j)]$$

Split the complex power into real (active power) and imaginary part (reactive power) and using rectangular form

$$p_i = V_i \sum_j V_j \, [g_{ij}\cos(\theta_i - \theta_j) + b_{ij}\sin(\theta_i - \theta_j)]$$

$$q_i = V_i \sum_j V_j \, [g_{ij}\sin(\theta_i - \theta_j) - b_{ij}\cos(\theta_i - \theta_j)]$$

AC power flow analysis (IV)

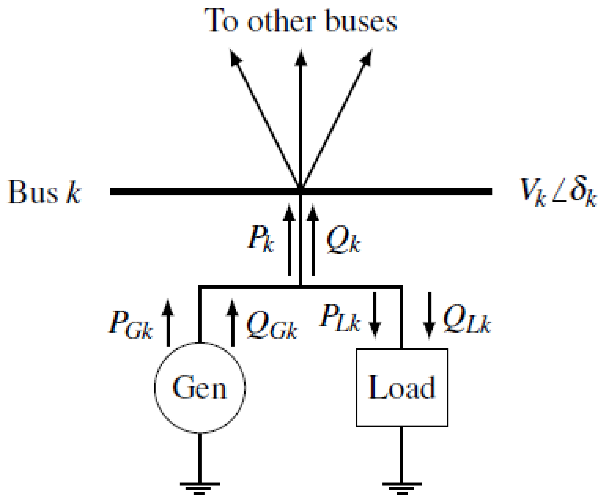
Objectives of AC power flow analysis:

- Find the power flows in the lines of a network given the injection pattern for the nodes
- Determines voltage magnitudes and angles at each bus

We have a set of nonlinear equations that relate the power flow and the injection in the nodes

$$P_{Gi} - P_{Li} = V_i \sum_j Y_{bus,ij} V_j \cos(\theta_i - \theta_j - \theta_{ij})$$

$$Q_{Gi} - Q_{Li} = V_i \sum_j Y_{bus,ij} V_j \sin(\theta_i - \theta_j - \theta_{ij})$$



Here, we follow the convention that departing flows are positive and arriving flows are negative

Every bus has for variables: net real power $P_{Gi} - P_{Li}$, net reactive power $Q_{Gi} - Q_{Li}$, voltage magnitude V_i, and phase angle θ_i

- **Load (PQ) bus**: $P_{Gi} - P_{Li}$ and $Q_{Gi} - Q_{Li}$ given
- **Voltage controlled (PV) bus**: : $P_{Gi} - P_{Li}$ and V_i
- **Slack bus**: V_i and θ_i given

Economic dispatch with AC optimal power flow

Assume we have a network of nodes i. In every node, we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity $G_{s,i}$ and a linear variable cost o_S. Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d_n in a certain hour and the optimal AC power flow while minimizing the total system cost.

Economic dispatch in one node

$$\left\{ \begin{array}{l} \min\limits_{g_s} \sum\limits_s o_s g_s \\ \\ \text{subject to:} \\ \sum\limits_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ \\ 0 \leq g_s \leq G_s \end{array} \right.$$

Economic dispatch with AC power flow

$$\left\{ \begin{array}{l} \min\limits_{g_{s,i}} \sum\limits_{s,i} o_s g_{s,i} \\ \\ \text{subject to:} \\ \sum\limits_{s,i} g_{s,i} - d_i = \bar{V}_i (\sum\limits_J \overline{Y_{bus,ij}}\, \bar{V}_j)^* \quad \leftrightarrow \quad \lambda_i \\ 0 \leq g_s \leq G_s \\ \left| \bar{V}_i \bar{Y}^*_{line,i\rightarrow j} \bar{V}^*_j \right| \leq S_{i\rightarrow j,max} \\ \left| \bar{V}_i \bar{Y}^*_{line,j\rightarrow i} \bar{V}^*_j \right| \leq S_{j\rightarrow i,max} \\ 0 \leq p_s \leq P_s \\ -Q_s \leq q_s \leq Q_s \end{array} \right.$$

Power flow equations

Apparent power flow Capacities

The bar above a variable denotes complex number

Modelling approaches for power flow in AC networks

Net Transfer Capacities	Linearized AC power flow (DC Power flow)	AC Power flow
$\sum_{s,i} g_{s,i} - d_i = \sum_{l} K_{il} p_l$ $\lvert p_l \rvert \leq P_l$	$\sum_{s,i} g_{s,i} - d_i = \sum_{l} K_{il} p_l$ $\lvert p_l \rvert \leq P_l$ $\sum_{l} C_{lc}\, x_l\, p_l = 0$	$\sum_{s,i} g_{s,i} - d_i = \bar{V}_i \left(\sum_{J} \overline{Y_{bus,ij}}\, \bar{V}_j \right)^*$ $\lvert \bar{V}_i \bar{Y}^*_{line,i\to j} \bar{V}^*_j \rvert \leq S_{i\to j,max}$ $\lvert \bar{V}_i \bar{Y}^*_{line,j\to i} \bar{V}^*_j \rvert \leq S_{j\to i,max}$
	✓ Linear ✓ Optimality guaranteed ✓ Computational tractable ✓ Good enough for long-term planning	✓ Feasible AC power flows
	✗ Not guarantee feasible power flows ✗ No representation of power losses ✗ Not good enough for heavily-loaded systems, fast changes, restart from blackout, network splitting, distribution networks, etc.	✗ Non-linear ✗ No optimality guaranteed ✗ High computational complexity

Power flow vs optimal power flow

Power flow vs Optimal Power Flow

	Power flow analysis	Optimal power flow
Objective	• Find the flows in the links of a network given the injection pattern for the nodes. • Determines voltage magnitudes and angles at each bus • Computes real and reactive power flows for lines connecting buses, as well as lines losses	• Find the optimal flows in the links of a network that minimizes the total system costs (while supplying demand in every node)
Applications	• Feasibility analysis: for dispatch (obtained in market clearing, identify grid overlading) • Security analysis: simulate outages to identify possible grid overloading • Transmission adequacy analysis: identify possible bottlenecks in long-term planning	• Find optimal dispatch that minimizes system cost and ensures power flow feasibility • Find optimal capacities to avoid bottlenecks (long-term planning) • Find optimal redispatch after an outage to restore feasibility

Newton-Raphson method

Newton-Raphson method (I)

Assume we have a set of nonlinear equations, each of them $f(x) = y$

We do a Taylor series expansion $y = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x=x_0} (x - x_0) + (higher\ order\ terms)$

0

We neglect higher order terms and solve for x:

$$x = x_0 + \left[\left.\frac{\partial f}{\partial x}\right|_{x=x_0}\right]^{-1} [y - f(x_0)]$$

The Newton-Raphson method replaces x by old values $x(i)$ and x by new values $x(i+1)$:

$$x(i+1) = x(i) + \left[\left.\frac{\partial f}{\partial x}\right|_{x=x(i)}\right]^{-1} [y - f(x(i))]$$

These slides are reproduced from Lecture 7 in DTU course 46700 Introduction to Power Systems, prepared by Tilman Weckesser

Newton-Raphson method (II)

The Jacobian matrix collects all the partial derivatives

$$J(i) = \left.\frac{\partial f}{\partial x}\right|_{x=x(i)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots \\ \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{x=x(i)}$$

$$x(i+1) = x(i) + \underbrace{\left[\left.\frac{\partial f}{\partial x}\right|_{x=x(i)}\right]^{-1}}_{J^{-1}(i)} [y - f(x(i))]$$

We avoid the inversion of $\mathbf{J}$ which is computationally expensive by re-writing as

$$J(i)[x(i+1) - x(i)] = [y - f(x(i))]$$

Newton-Raphson method (III)

Steps to compute solution

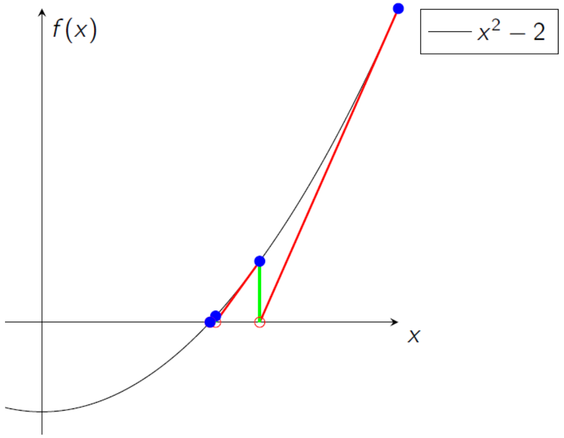
1. Select an initial guess for $x(i)$
2. Compute $y - f(x(i))$ and $J(i)$
3. Solve $J(i)[x(i+1) - x(i)] = [y - f(x(i))]$
4. Compute $x(i+1)$
5. Check for convergence $\left|\frac{x(i+1)-x(i)}{x(i)}\right| < \varepsilon$

Example: $y = f(x) = x^2 - 2 = 0$

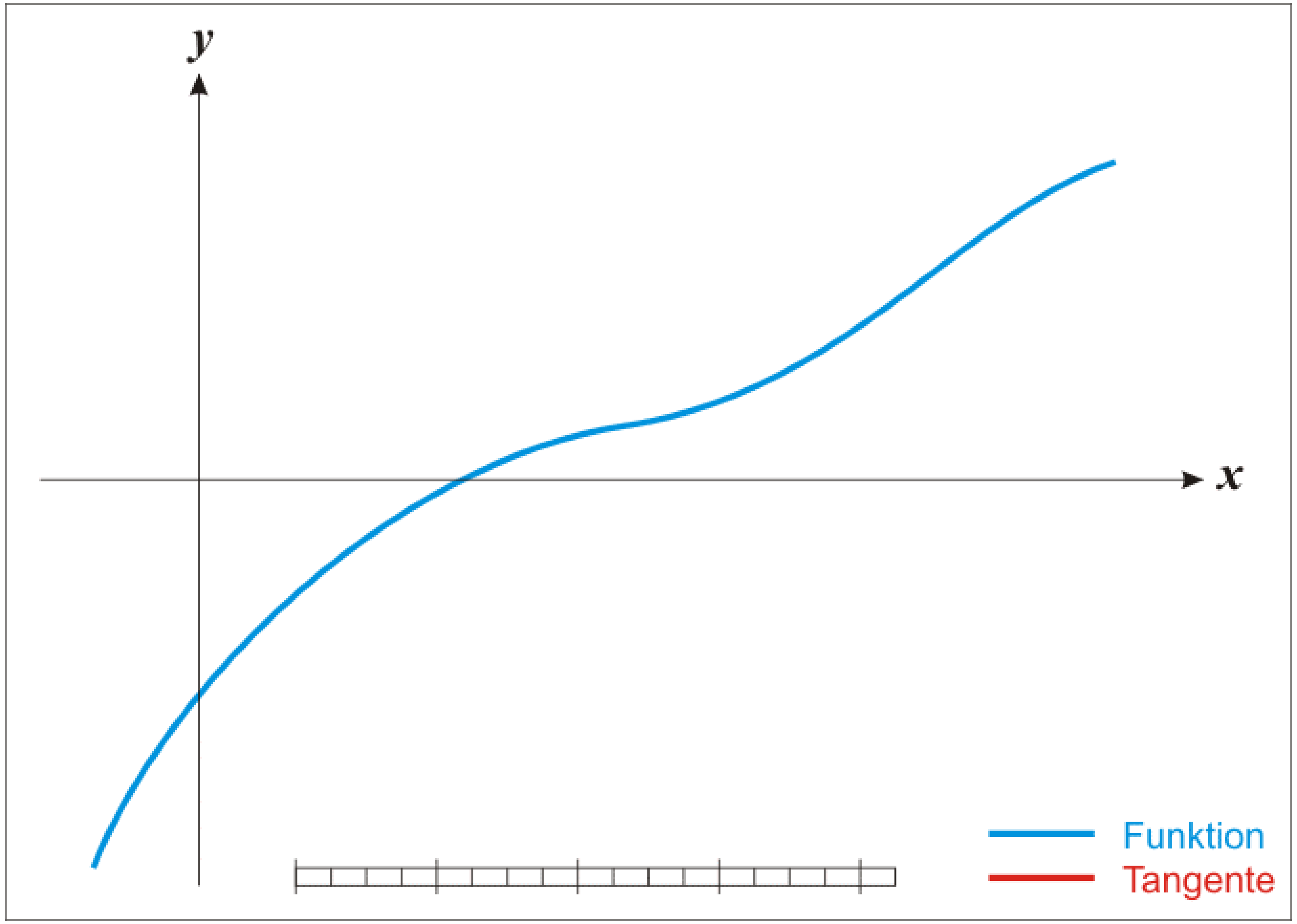
$$x(i+1) = x(i) + \left[\frac{\partial f}{\partial x}\bigg|_{x=x(i)}\right]^{-1} [y - f(x(i))]$$

$$x(i+1) = x(i) + \frac{1}{2x(i)} [0 - (x(i)^2 - 2)]$$

i	$x(i)$	$J(i)$	$y - f(x(i))$	$\Delta x(i)$
0	3	0.167	-7	-1.167
1	1.833	0.272	-1.361	-0.371
2	1.462	0.342	-0.138	-0.047
3	1.415	0.353	-0.002	-0.001



Newton-Raphson method (III)



Newton-Raphson method (IV)

When we apply the method to the power flow equations, the y values are the active and reactive power, the x values are the angles and voltage magnitudes, and the functions $f(x)$ are the AC power flow equations

$$y = \begin{bmatrix} P \\ Q \end{bmatrix} \qquad x = \begin{bmatrix} \theta \\ V \end{bmatrix} \qquad f(x) = \begin{bmatrix} P(\theta, V) \\ Q(\theta, V) \end{bmatrix}$$

$$\downarrow$$

$$P_{Gi} - P_{Li} = V_i \sum_j Y_{bus,ij} V_j \cos\left(\theta_i - \theta_j - \theta_{ij}\right)$$

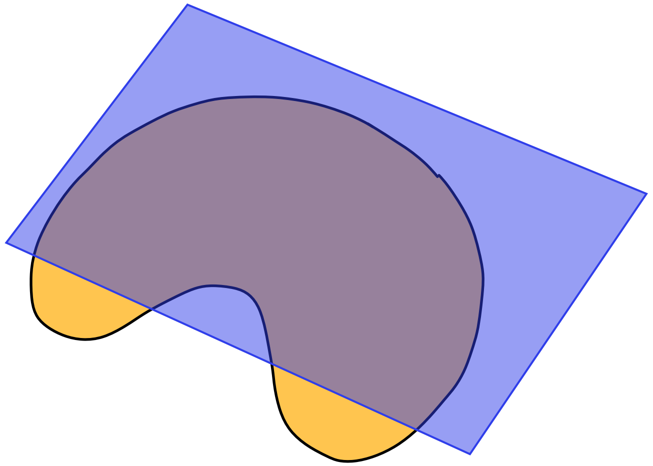
$$Q_{Gi} - Q_{Li} = V_i \sum_j Y_{bus,ij} V_j \sin\left(\theta_i - \theta_j - \theta_{ij}\right)$$

Convex relaxation to include losses in linearized AC power flows

Limitations of the Linearized AC Optimal Power flow

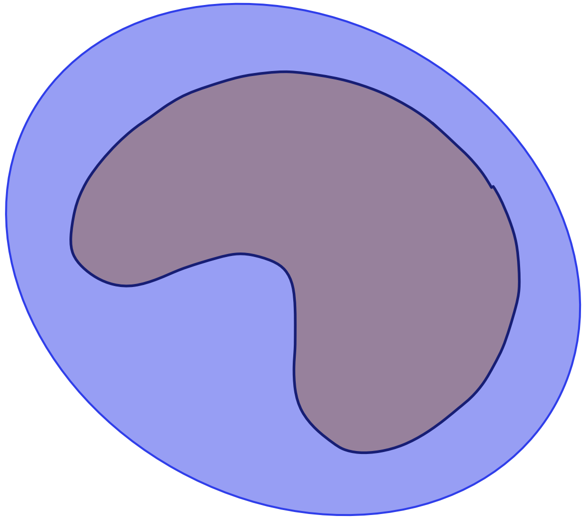
Linearization of AC power flow is an **approximation** of the feasible space which might:

- may contain points that are not feasible
- may not contain some feasible points
- may not include original optimal solution



Convexification of AC power flow is an **extension** of the feasible space which might:

- may contain points that are not feasible
- Contains all feasible points and
- Contains the true optimal solution



Losses in AC power flow (I)

The **Line Admittance Matrix $\bar{Y}_{lines,ij}$** relates the bus voltajes V_i to the current flows $I_{i\to j}$

$$\bar{I}_{i\to j} = \sum_j \bar{Y}_{line,j}\bar{V}_j$$

In AC systems, the net complex power flowing through a line is equal to the product of the voltage and the conjugate of the current (both complex numbers)

$$\bar{S}_i = P_i + \mathrm{j}Q_i = \bar{V}_i\bar{I}_i^* = \bar{V}_i\bar{Y}^*_{line,i\to j}\bar{V}_j^*$$

Split the complex power into real (active power) and imaginary part (reactive power) and using rectangular form

$$p_{i\to j} = g_{ij}|V_i|^2 - |V_i||V_j|\,[g_{ij}\cos(\theta_i - \theta_j) + b_{ij}\sin(\theta_i - \theta_j)]$$

$$q_{i\to j} = b_{ij}|V_i|^2 - |V_i||V_j|\,[g_{ij}\sin(\theta_i - \theta_j) - b_{ij}\cos(\theta_i - \theta_j)]$$

Losses in AC power flow (II)

$$p_{i\to j} = g_{ij}|V_i|^2 - |V_i||V_j|\,[g_{ij}\cos(\theta_i-\theta_j) + b_{ij}\sin(\theta_i-\theta_j)]$$

Following the convention that departing flows are positive and arriving flows are negative, the losses are the difference between power sent from i to j and power received from i to j.

$$\Psi_l = p_{i\to j} + p_{j\to i} = g_{ij}|V_i|^2 - |V_i||V_j|\,[g_{ij}\cos(\theta_i-\theta_j) + b_{ij}\sin(\theta_i-\theta_j)]$$

$$+g_{ij}|V_j|^2 - |V_j||V_i|\,[g_{ij}\cos(\theta_j-\theta_i) + b_{ij}\sin(\theta_j-\theta_i)]$$

$$\cos(-a) = \cos(a)$$
$$\sin(-a) = -\sin(a)$$

$$\Psi_l = p_{i\to j} + p_{j\to i} = g_{ij}(|V_i|^2 + |V_j|^2) - 2|V_i||V_j|\,g_{ij}\cos(\theta_i-\theta_j)$$

Losses in AC power flow (II)

$$\Psi_l = p_{i\to j} + p_{j\to i} = g_{ij}(|V_i|^2 + |V_j|^2) - 2|V_i||V_j|\, g_{ij}\cos(\theta_i - \theta_j)$$

Losses reduce the transmission capacity at the received end of a line $|p_l| \leq P_l - \Psi_l$

Losses must be accounted for in the nodal balance equation $p_l = \sum_l K_{il} p_l + \frac{|K_{il}|}{2}\Psi_l$

Assuming close to nominal per-unit voltage magnitudes $|V_i| \sim 1$

$$\Psi_l = 2g_{ij}(1 - 2\cos(\theta_i - \theta_j))$$

$$p_l = \frac{\theta_i - \theta_j}{x_l} \qquad \Psi_l = 2g_{ij}(1 - \cos(p_l x_l))$$

Losses in AC power flow (III)

Approximating $\cos(a) \approx 1 - \frac{a^2}{2}$

$$\Psi_l = 2g_{ij}(1 - \cos(p_l x_l)) = 2g_{ij}\left(1 - 1 + \frac{p_l^2 x_l^2}{2}\right) = g_{ij} p_l^2 x_l^2$$

By inserting $g_l \approx \frac{r_l}{x_l^2}$

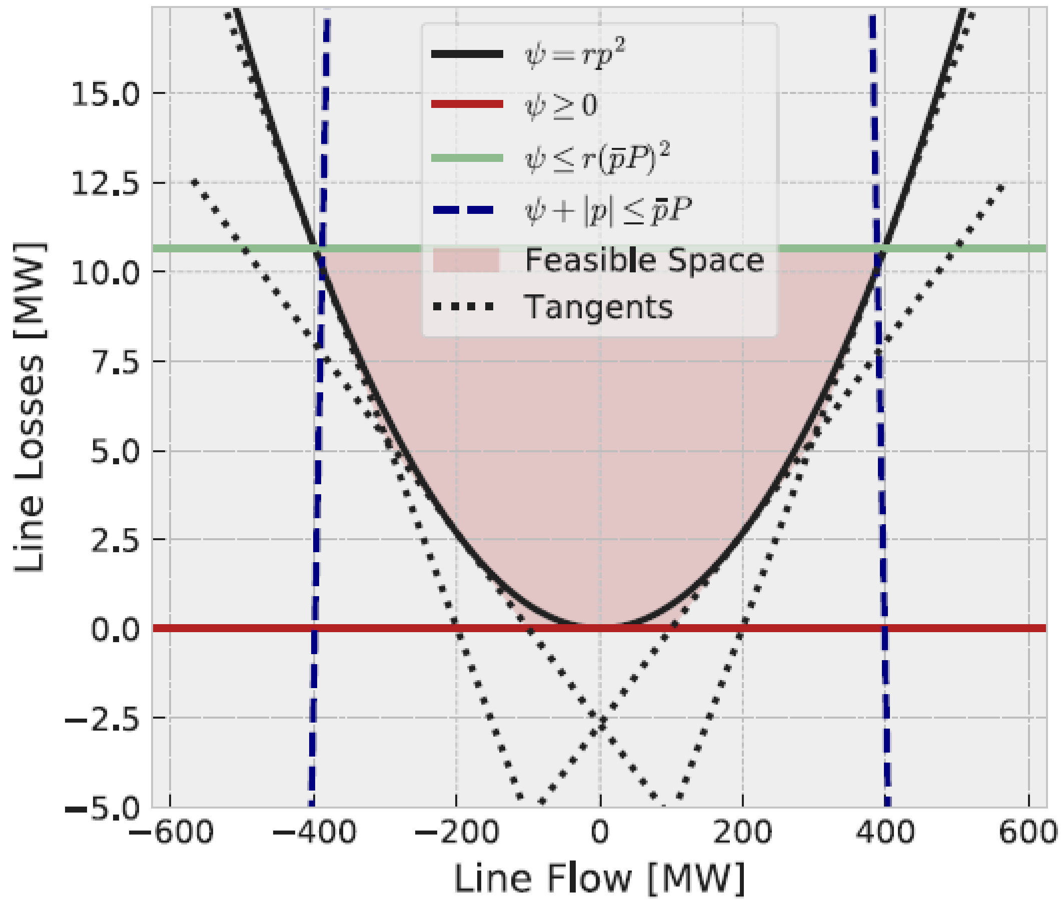
$$\Psi_l = r_l p_l^2$$

This is still a quadratic equality constraint that leads to a quadratically constrained quadratic program (QCQP)

We can make it convex by building a convex envelope around the constraint.

This way we keep the problem linear while including transmission losses.

Losses in AC power flow (IV)



We can make it convex by building a convex envelope around the constraint.

$\boxed{\Psi_l = r_l p_l^2}$ substituted by

$$\Psi_l \geq 0$$

$$p_l \leq P_l \quad \rightarrow \quad \Psi_l \leq r_l P_l^2$$

$$\begin{array}{lll} \Psi_l \geq 0 & m_k \cdot p_l + a_k & \forall k = 1 \ldots . n \\ \Psi_l \leq 0 & -m_k \cdot p_l + a_k & \forall k = 1 \ldots . n \end{array}$$

Set of linear equations that represent a convex envelope of the nonlinear constraint

Economic dispatch with DC optimal power flow and losses

Assume we have a network of nodes i. In every node, we have a set of generators s (e.g., onshore wind, solar PV, gas power plant…) each of them has an installed capacity $G_{s,i}$ and a linear variable cost o_S. Determine the optimal economic dispatch (how much energy is being produced by each of them) to supply the demand d_n in a certain hour and the optimal AC power flow while minimizing the total system cost.

Economic dispatch in one node

$$\begin{cases} \min\limits_{g_s} \sum\limits_s o_s g_s \\ \text{subject to:} \\ \sum\limits_s g_s - d = 0 \quad \leftrightarrow \quad \lambda \\ 0 \leq g_s \leq G_s \end{cases}$$

Economic dispatch with AC power flow

$$\begin{cases} \min\limits_{g_{s,i}} \sum\limits_{s,i} o_s g_{s,i} \\ \text{subject to:} \\ \sum\limits_{s,i} g_{s,i} - d_i = \sum\limits_l K_{il} p_l + \frac{|K_{il}|}{2} \Psi_l \leftrightarrow \lambda_i \\ 0 \leq g_s \leq G_s \\ |p_l| \leq P_l - \Psi_l \\ \sum\limits_l C_{lc}\, x_l\, p_l = 0 \end{cases}$$

Nodal power balance

Power flow Capacities

Kirchoff’s Voltage Law

n-1 security criterion

Contingency and N-1 criterion

Contingency: an incident that deviate from the planned operation and can affect the security of the power system. (line outages, load outages, deviations of the power generation due to solar or wind forecast errors)

N-1 security criterion:

Power system must remain secure in the event of a component outage that leads to a system operation with N-1 components.

An approximation for the n-1 security criterion when using DC power flow consists in limiting the power that can flow for every line to 70% of its thermal rating. This also allows room for reactive power

Problems for this lecture

Problems 5.1, 5.2 (**Group 10**)

Problems 5.3, 5.4 (**Group 11**)