DNF solving

You can use a truth table to solve DNF problems. Start by adding a column for each literal with all combinations of 1s and 0s. Then you add columns to your truth table for each subexpression until you've expressed the entire expression. The rows containing a 1 in the final column are your answer. See p.12 in Definitions för *basic logic connectives*.

p	q	r	s	a $p \to q$	b $q \to r$	c $r \to s$	d $s \to p$	e $a \bar{\wedge} b$	f $c \bar{\wedge} d$	$e \wedge f$
1	1	1	1	1	1	1	1	0	0	0
1	1	1	0	1	1	0	1	0	1	0
1	1	0	1	1	0	1	1	1	0	0
1	1	0	0	1	0	1	1	1	0	0
1	0	1	1	0	1	1	1	1	0	0
1	0	1	0	0	1	0	1	1	1	1 ←
1	0	0	1	0	1	1	1	1	0	0
1	0	0	0	0	1	1	1	1	0	0
0	1	1	1	1	1	1	0	0	1	0
0	1	1	0	1	1	0	1	0	1	0
0	1	0	1	1	0	1	0	1	1	1 ←
0	1	0	0	1	0	1	1	1	0	0
0	0	1	1	1	1	1	0	0	1	0
0	0	1	0	1	1	0	1	0	1	0
0	0	0	1	1	1	1	0	0	1	0
0	0	0	0	1	1	1	1	0	0	0

DNF for when $\underbrace{((\underbrace{p \to q}_{a}) \bar{\wedge} (\underbrace{q \to r}_{b}))}_{e} \wedge \underbrace{((\underbrace{r \to s}_{c}) \bar{\wedge} (\underbrace{s \to p}_{d}))}_{f}$ is true:

$(p \wedge \neg q \wedge r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge s)$

Intervals

Here is a question about understanding how intervals change when sending through functions and meet.

For the following sets of numbers, specify the smallest and the largest numbers, write NONE if there is no smallest or largest number, or EMPTY (in one of the two columns) if the set is empty.

All intervals are supposed to be intervals in the real numbers, $\mathbb{R}$. Similarly, all relations and operators are on the real numbers, unless explicitly stated otherwise.

set	smallest element	largest element
$\{x \in \mathbb{Z} : x > 4 \wedge x < 2\}$	EMPTY	
$\{1, 2, 3, 4\}$	1	4
$]1, 4]$	NONE	4
$\bigcap_{i \in \mathbb{N}^+} \left[0, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i \in \mathbb{N}^+} \left]0, \frac{1}{i}\right] = \emptyset$	EMPTY	
$\bigcap_{i \in \mathbb{N}^+} \left[\frac{-1}{i}, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i \in \mathbb{N}^+} \left]\frac{-1}{i}, 1\right] = [0, 1]$	0	1
$\mathrm{sqrt}([0, 0.1]) = [0, \sqrt{0.1}]$	0	$\sqrt{0.1}$
$\mathrm{sqrt}[[0, 0.1[] = [0, 1[$	0	NONE
$\mathrm{sqrt}[]0, 0.1[] =]0, 1[$	NONE	NONE
$\bigcup_{i \in \mathbb{N}^+} \left]\frac{1}{i+1}, \frac{1}{i}\right[=]0, 1[$	NONE	NONE
$\mathrm{inv}(]0, 0.1]) = [10, +\infty[$	10	NONE
$\mathrm{inv}[]0, 0.1]] =]0, 0.1] \cup [10, +\infty[$	NONE	NONE

$\mathrm{sqrt} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the positive square root function, i.e. for every non-negative real number a, $\mathrm{sqrt}(a)$ is the non-negative real number such that $a = \mathrm{sqrt}(a) \cdot \mathrm{sqrt}(a)$.

$\mathrm{inv} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ is the inversion function, defined by $\mathrm{inv} : r \mapsto \frac{1}{r}$.

Below is a explanation of how to think with intervals in more detail.

When looking at intersections of an infinite number of sets, it is important to keep in mind that any value that is an element of the intersection must be an element of each and every one of those sets.

Take for example $\bigcap_{i \in \mathbb{N}^+} \left[0, \frac{1}{i}\right]$. If that intersection contained any positive real number in addition to 0, it would mean that there is a real number $r > 0$, such that $r \in \left[0, \frac{1}{i}\right]$ for every $i \in \mathbb{N}^+$. It's easy to see that this cannot be the case: for any $r > 0$, there is some natural number $k > \frac{1}{r}$, which means that $\frac{1}{k} < r$, and so $r \notin \left[0, \frac{1}{k}\right]$, and so r cannot be in the intersection.

Injective and surjectiv

This is a simpler example of injective and surjective functions. The reason why this question is simpler is because we can define the domain and codomain separatly.

Define two sets A and B, as well as a function $f : A \longrightarrow B$, such that f is **surjective** and **not injective**.

$A = \{a, b\}$ $B = \{x\}$ $f : a \mapsto x$

Of course, if you already found the answer to the next question, you could "reuse" it here by simply choosing to make A and B the same set.

The idea was to start with an easier question to get you to think about surjectivity and injectivity in a simpler setting first, before tackling the harder problem.

When something should be surjective but **not** injective and the *codomain* = *domain* we need to utilize infinity. Below, any element in the codomain could be reached from the domain by adding one to it. Meaning it is surjective but we also point to the same element twice, 0 in this case.

Define **one** set A, as well as a function $f : A \longrightarrow A$, such that f is **surjective** and **not injective**.

$A = \mathbb{N}$ $\quad f : a \mapsto \begin{cases} 0 & \text{if } a = 0 \\ a - 1 & \text{otherwise} \end{cases}$

The crux here is that A has to be infinite.

Suppose there is a function $f : A \longrightarrow A$ which is surjective and **not** injective, like the one you were asked to define in the previous task. This question is about a property of A (the domain and codomain of f) that implies that such a function exists, and which is also implied by the existence of such a function. (You do not need to prove this here.)

A surjective and **not** injective function $f : A \longrightarrow A$ exists if and only if

A is infinite.

(this must be a statement about the set A, and cannot involve f)

If you want to get a deeper understanding of this point, try to prove it. You can do this in two steps: (1) You show that if A is infinite, a function exists on it that is surjective but not injective. This you can show by taking our definition of an infinite set (one that is equivalent to a proper subset of itself), and use that definition to construct such a function.

(2) Now you need to show that if such a function exists, then A is infinite. Being infinite means that A must be equinumerous to a proper subset of itself. So given a function that is surjective but not injective, you need to find a proper subset of A that is the same size as A.

If you find this confusing, have a look at the solution above, and try to figure out what a suitable proper subset of A would be, and how it is related to f.

Properties of paths, graphs and trees

Properties of a **directed graph**, worth remembering is that no property is guaranteed. We don't put any constraints on our graph meaning it could be any kind of relation, for example every edge could be symmetric.

Recall that a *directed graph* (V, E) is defined as a finite set V of vertices and a relation $E \subseteq V \times V$ between them.

This question is about the properties of that relation. In the table below, make one mark in each row for the property in the left column, depending on whether all, some, or no relations defining a graph have that property. Put the mark in the corresponding ALL box, if **all relations** defining a graph have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over V		X	
transitive		X	
symmetric		X	
antisymmetric		X	
asymmetric		X	

Graphs in our definition make no special assumptions about the relations that define them, so any (finite) relation could be a graph.

Below are the properties of a **rooted tree**. The reason it's antisymmetric and asymmetric is because if there exist (a,b) and (b,a) then there are multiple ways to reach b (ie. $a \to b$, $a \to b \to a \to b$). The reason why it's antisymmetric is the same reason it is not reflexive.

Recall that a *rooted tree* is a graph (T, R) such that, if the set T of nodes is not empty, then there is a node $a \in T$ (the root) such that for every $x \in T$ with $x \neq a$ there is exactly one path from a to x. Like V in the previous question, $R \in T \times T$ is a relation on the set of nodes. To make things simpler, for this question we only consider non-empty trees, that is $T \neq \emptyset$.

This question is about the properties of the relations defining trees. In the table below, make one mark in each row for the corresponding property in the left column, depending on whether all, some, or no relations defining a tree have that property. Put the mark in the corresponding ALL box, if **all relations** defining a tree have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over T			X
transitive		X	
symmetric		X	
antisymmetric	X		
asymmetric	X		

The situation is different for trees, which are much more specialized and constrained structures than graphs. Since we only consider non-empty trees, none of them can be reflexive. (If we allowed the empty tree, then its link-relation R would also be empty, which is reflexive over the empty set.)

But there are trees whose link relation is transitive, viz. all those of link height 0 or 1. (Make sure you understand why that is.) And there is a tree whose link relation is symmetric, namely the tree consisting of only a root, whose link relation is therefore empty, which is symmetric.

Other important definitions

Node height/depth

The **link-height** (*alias:* level) in a tree is defined recursively: that of the root is 0, and that of each of the children of a node is one greater than that of the node.

The **node-height** (*alias:* height) is defined by the same recursion, except that the node-height of the root is set to 1. Thus, for every node x, $node-height(x) = link-height(x) + 1$. As trees are usually drawn upside-down, the term 'depth' is often used instead of 'height'.

CNF

Conjunctive normal form is like disjunctive normal form but 'upside-down': the roles of disjunction and conjunction are reversed. A basic disjunction is defined to be any disjunction of (one or more) literals in which no letter occurs more than once. A formula is said to be in conjunctive normal form (CNF) iff it is a conjunction of (one or more) basic disjunctions

Edge cases for always/sometimes/never problems

If you have a graph (V, E), then you should check the edge cases where:

1. $V = \emptyset$ or $E = \emptyset$.
2. V only contains 1 node or 2 nodes.
3. E contains a connection in one direction but not back. And when it contains a connection in both directions.
4. All nodes are connected to all nodes.

Quantifiers

$\forall x \in \emptyset(\ldots) = true.$
$\exists x \in \emptyset(\ldots) = false.$

Logic operators

1. $\alpha \barwedge \beta = \neg(\alpha \wedge \beta)$ and $\neg\alpha = (\alpha \barwedge \alpha)$

Image of n-ary

When computing the image, we treat the last element of a tuple as the 'output'.

$$R(a_1, \ldots, a_{n-1}) = \{a \in A : (a_1, \ldots, a_{n-1}, a) \in R\} \tag{1}$$