1 [12 p] avrg: 9.9 (83%), sd: 3.2

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$. For any non-empty set S of numbers, $\max S$ and $\min S$ are the largest and smallest numbers S, respectively.

1. [3p] $\#\{(a,b) \in A \times A : a \leq b\} =$ 55

2. [4p] $\max\left\{\frac{a}{b} : a, b \in A\right\} - \min\left\{\frac{a}{b} : a, b \in A\right\} =$ 99/10

3. [5p] $\min\left\{\frac{a}{b} : a, b \in A, a > b\right\} - \max\left\{\frac{a}{b} : a, b \in A, a \leq b\right\} =$ 1/9

2 [20 p] avrg: 13.0 (65%), sd: 3.5

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ as before and a family of relations
$R_i = \{(a,b) \in A \times A : \mod (b,a) = i\}$ for any $i \in \mathbb{N}$, with $\mod (b,a)$ the remainder when dividing positive integer b by positive integer a. So, for example,
$R_3 = \{(a,b) \in A \times A : \mod (b,a) = 3\}$.

1. [2p] $\#R_4 =$ 8

2. [2p] $\#R_5 =$ 5

3. [2p] $\#R_0 =$ 27

4. [2p] $\#R_{10} =$ 0

5. [3p] $R_3(7) =$ {3, 10}

6. [3p] $R_1(2) =$ {1, 3, 5, 7, 9}

7. [3p] $R_1(A) =$ {1, 3, 4, 5, 6, 7, 8, 9, 10}

8. [3p] $R_3(A) =$ {3, 7, 8, 9, 10}

   Hint: Please review the definition of "image", and be sure of what the kind of result is expected here.

The most frequent question I get about these kinds of tasks is whether there is a "shortcut". In this case, one way to go about answering this question is to construct a little table representing the modulo operation for numbers between 1 and 10. Note that the table contains a number of regularities, and it can be constructed very easily. Just follow the first few rows to see how this is built.

		b									
	b mod a	1	2	3	4	5	6	7	8	9	10
a	1	0	0	0	0	0	0	0	0	0	0
	2	1	0	1	0	1	0	1	0	1	0
	3	1	2	0	1	2	0	1	2	0	1
	4	1	2	3	0	1	2	3	0	1	2
	5	1	2	3	4	0	1	2	3	4	0
	6	1	2	3	4	5	0	1	2	3	4
	7	1	2	3	4	5	6	0	1	2	3
	8	1	2	3	4	5	6	7	0	1	2
	9	1	2	3	4	5	6	7	8	0	1
	10	1	2	3	4	5	6	7	8	9	0

Once you have this table, answering 2.1 – 2.8 becomes pretty straightforward.

3 [16 p] avrg: 7.6 (47%), sd: 4.8

Suppose, as previously, $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a relation
$R_3 = \{(a, b) \in A \times A : \mod(b, a) = 3\}$.

Let $T = R_3 \circ R_3^{-1}$.

Here, too, things become a lot simpler once you have explicitly constructed the extensions of the relations involved. So for reference:

$R_3 = \{(4,3),(4,7),(5,3),(5,8),(6,3),(6,9),(7,3),(7,10),(8,3),(9,3),(10,3)\}$

$R_3^{-1} = \{(3,4),(7,4),(3,5),(8,5),(3,6),(9,6),(3,7),(10,7),(3,8),(3,9),(3,10)\}$

$T = \{(3,3),(7,7),(8,8),(9,9),(10,10),(3,7),(3,8),(3,9),(3,10),(10,3),(9,3),(8,3),(7,3)\}$

1. [3 p] $\#T =$ 13
2. [3 p] $T(1) =$ {}
3. [3 p] $T(7) =$ {3, 7}
4. [3 p] $T(A) =$ {3, 7, 8, 9, 10}
5. [4 p] T is ... (circle those that apply)

… reflexive	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

4 [18 p] avrg: 10.0 (57%), sd: 6.0

Suppose you have **injections** $f : A \hookrightarrow B$ and $g : A \hookrightarrow B$, as well as a **non-empty** set $S \subset A$ (note that S is a **proper** subset of A). Now let's define a function $h : A \longrightarrow B$ as follows:

$$h : x \mapsto \begin{cases} f(x) & \text{for } x \in S \\ g(x) & \text{for } x \notin S \end{cases}$$

This function is not, in general, injective.
Whether it is injective depends on the definitions of A, B, f, g, and S.

1. [5p] Give definitions for A, B, f, g, and S such that the h above is injective.

   A = {a, b}

   B = {x, y}

   f = {(a, x), (b,y)}

   g = {(a, x), (b,y)}

   S = {a}

2. [5p] Give definitions for A, B, f, g, and S such that the h above is **not** injective.

   A = {a, b}

   B = {x, y}

   f = {(a, x), (b,y)}

   g = {(a, y), (b,x)}

   S = {a}

3. [8p] Give a general formal criterion, depending only on A, B, f, g, and S (not necessarily all of them), that defines the condition under which h is injective. (Hint: Remember, f and g are already injective.)

   h is injective iff $f(S) \cap g(A \setminus S) = \emptyset$

Note: You are **not** supposed to reiterate the definition of injectivity for h, but rather give an expression involving at most A, B, f, g, and S (but **not** h) that is true if an only if they lead to an injective h.

Common mistakes on the first two subquestions included answers that defined f and g in such a way that they were not functions from A to B. Often, f was defined only on S and g on A\S. Occasionally, the image of A under f or g was not in B.

On the third subquestion, some answers missed that the criterion was supposed to be true *if and only if* (*iff*) h was injective. So, for example, if f(A) and g(A) are disjoint, then h will be injective, but that's not required: h will also be injective in cases where f(A) and g(A) aren't disjoint (for example, it could be that f=g, and in fact then h will always be injective, for any S), so that criterion is too strict.

5 **[18 p]** avrg: 5.6 (31%), sd: 6.3

Suppose we have a rooted tree (T, R) with nodes T, links $R \subseteq T \times T$, and root a as well as a labeling function $\lambda : T \longrightarrow \mathbb{N}$ assigning each node in the tree a natural number.

We want to define a function $L : T \longrightarrow \mathbb{N}$ that computes for each node $n \in T$ the lowest number a node in the subtree rooted at n is labeled with (that subtree includes n itself). If the subtree consists only of n, its label $\lambda(n)$ is the lowest number.

As before, for any non-empty set S of numbers, $\min S$ is the lowest number in that set.

1. [10p] Define L using well-founded recursion. (Hint: You may use cases if you like, but it is possible to define this function without an explicit "base case.")

$$L : n \mapsto \min\{\lambda(n)\} \cup \{L(n') : nRn'\}$$

Note that the nRn' takes care of the "termination": if there is no child, is means there is no n', and the second set in the union above will simply be empty.

2. [8p] Define a strict partial order $\prec$ on T such that the poset $(T, \prec)$ is well-founded and your definition of L performs well-founded recursion on that poset. For all $n, n' \in T$...

$$n' \prec n \iff nR^*n'$$

R^* is the transitive closure of R. Specifying *only* the link relation itself would not result in a poset, since it is not a partial order.

There are other ways of answering that question, for example using the closure of {n} under R, i.e. R[{n}]: $n' \prec n \iff n' \in R[\{n\}]$

Many answers tried to use L to define the order. However, that does not work since, for instance, the labeling could give the same number to every node (there is nothing that would require it not to), and so there would be no way to distinguish between nodes, which a strict order would have to.

Also, it would not make much sense to do so, since the point of this order is to demonstrate the well-definedness of L, so using L to define it would be oddly circular. However, this in itself would not make the answer wrong (only useless for its intended purpose).

No proof is required. It is sufficient that the strict partial order is well-founded and your definition of L conforms to it.

Hint: Make sure the partial order you define actually is one, i.e. that it has all the properties required from a strict partial order, including, for example, transitivity.

6 [10 p] avrg: 10.0 (100%), sd: 0.0

Suppose we have a set of four five characters $C = \{"a", "b", "(", ")"\}$ $C = \{"a", "b", "(", ")", ","\}$, the set $S = \{"a", "b"\}$ consisting only of the letters a and b, and a relation $R = \{(s_1, s_2, "("s_1","s_2")") : s_1, s_2 \in C^*\}$.

As you can see, R is a 3-place relation, and we compute the image of some set of strings $X \subseteq C^*$ under R by applying the relation to all pairs of strings in X, i.e. $R(X) = \{y : x_1, x_2 \in X, (x_1, x_2, y) \in R\}$.

Now let $R^n(X)$ be the set that results from computing the image of some set $X \subseteq C^*$ under R n times in a row, with $R^0(X) = X$, $R^1(X) = R(X)$, $R^2 = R(R(X))$ and so forth, so that $R^{n+1}(X) = R(R^n(X))$.

1. [1p] Give an element in $R^0(C)$: $R^0(S)$: a

2. [2p] Give an element in $R^1(C)$: $R^1(S)$: (a,b)

3. [2p] Give an element in $R^2(C)$: $R^2(S)$: ((a,b),(b,a))

4. [5p] Suppose $U = \bigcup_{n \in \mathbb{N}} R^n(C)$. $U = \bigcup_{n \in \mathbb{N}} R^n(S)$

   Give an element in $R[C] \setminus U$, $R[S] \setminus U$ or write “none” if no such element

   exists: (a,(a,b))

   Note:
   $R[C]$ $R[S]$ here is the closure of C S under the set of relations that only consists of the relation R.

Unfortunately, the version that was printed in the exam papers contained several errors (see the corrections above in red – the blue answers relate to the corrected questions). Some answers seem to have guessed the intention and corrected for it, others answered the question as asked, some did something in between. Either way, I saw no fair way to grade the answers, so I decided to give everybody full marks on this question, thereby effectively removing the question from consideration (and slightly raising the average score).

7 **[10 p]** avrg: 9.3 (93%), sd: 1.9

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.
Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$((\forall z(P\dot{y} \to Q\underset{\bullet}{z}\dot{x})) \leftrightarrow P\dot{z}) \to (Q\dot{x}\dot{z} \wedge ((\exists x(Q\underset{\bullet}{x}\dot{z})) \leftrightarrow \exists z(P\underset{\bullet}{z} \to P\dot{x})))$$

bound

8 **[15 p]** avrg: 7.9 (53%), sd: 5.7

Find a DNF for each of the following formulae. Write "none" if a formula has no DNF.

1. [5 p] $(r \vee \neg q) \leftrightarrow (p \wedge q)$

   $(\neg p \wedge q \wedge \neg r) \vee (p \wedge q \wedge r)$

```
(p q r)
(0 0 0)  -->  0
(0 0 1)  -->  0
(0 1 0)  -->  1
(0 1 1)  -->  0
(1 0 0)  -->  0
(1 0 1)  -->  0
(1 1 0)  -->  0
(1 1 1)  -->  1
```

2. [5 p] $(p \barwedge (q \barwedge (r \barwedge s)))$

$\neg p \vee (q \wedge \neg s) \vee (q \wedge \neg r)$

```
(p q r s)
(0 0 0 0)  -->  1
(0 0 0 1)  -->  1
(0 0 1 0)  -->  1
(0 0 1 1)  -->  1
(0 1 0 0)  -->  1
(0 1 0 1)  -->  1
(0 1 1 0)  -->  1
(0 1 1 1)  -->  1
(1 0 0 0)  -->  0
(1 0 0 1)  -->  0
(1 0 1 0)  -->  0
(1 0 1 1)  -->  0
(1 1 0 0)  -->  1
(1 1 0 1)  -->  1
(1 1 1 0)  -->  1
(1 1 1 1)  -->  0
```

3. [5 p] $(p \rightarrow q) \barwedge ((q \rightarrow r) \barwedge (r \rightarrow p))$

$(p \wedge r) \vee (p \wedge \neg q) \vee (\neg q \wedge \neg r)$

```
(p q r)
(0 0 0)  -->  1
(0 0 1)  -->  0
(0 1 0)  -->  0
(0 1 1)  -->  0
(1 0 0)  -->  1
(1 0 1)  -->  1
(1 1 0)  -->  0
(1 1 1)  -->  1
```

As always, the minimal requirement was that the answer be a DNF. In some cases, the answer was “close” (e.g., a negation too many/too few, or one basic conjunction missing), and I gave partial points for those.

Also, of course, there is more than one correct DNF for each of these. The answers typically depend on whether they were obtained by transforming the formula or through a truth table. Either way is fine, neither is better than the other.