specifying sets

enumeration of its elements

$$\{red, green, blue\}$$

set builder notation / set comprehensions

flavor 1

$$\{x \in \mathbb{N}^+ : x \text{ is prime}\}$$

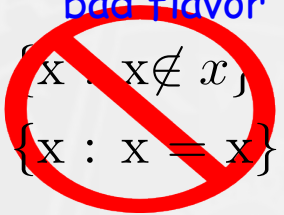
$$\{x \in \mathbb{N}^+ \mid x \text{ is prime}\}$$

flavor 2

$$\{x : x \in \mathbb{N}^+, x \text{ is prime}\}$$

$$\{2x : x \in \mathbb{N}^+\}$$

bad flavor



$$\{x : x \notin x\}$$

$$\{x : x = x\}$$

recursive definition

(we will discuss this later)

enumeration w/ suspension points/ellipsis

$$\{1, 2, 3, 4, 5, ...\}$$

(informal stand-in for a recursive definition)

$$12 \in \{2, 3, 5, 8, ...\} \quad ?$$

set algebra

some properties of intersection, union, and set difference:

idempotence	$A \cup A = A \cap A = A$
commutativity	$A \cup B = B \cup A$
commutativity	$A \cap B = B \cap A$
associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
associativity	$(A \cap B) \cap C = A \cap (B \cap C)$
	$A \cup \emptyset = A$
	$A \cap \emptyset = \emptyset$
	$A \supseteq A \cap B \subseteq B$
	$A \subseteq A \cup B \supseteq B$
distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cap B \subseteq A \cup B$
	$A \setminus A = \emptyset$
	$A \setminus \emptyset = A$
	$-(-B) = B$

(more in the exercises of 1.4.1, 1.4.2, and 1.4.3 in SLAM)

terminology: source, target, domain, range

For binary relations $R \subseteq A \times B$:

A is a *source*. B is a *target*.

Note that for any R, source and target are *not uniquely determined*:

$$R \subseteq A \times B$$

For any $A' \supseteq A$ and $B' \supseteq B$, we have $A \times B \subseteq A' \times B'$.

$$R \subseteq A \times B \subseteq A' \times B'$$

By contrast, these *are* uniquely determined:

the *domain* of R: $\mathrm{dom}(R) = \{a : (a, b) \in R \text{ for some } b\}$

the *range* of R: $\mathrm{range}(R) = \{b : (a, b) \in R \text{ for some } a\}$

For any relation $R \subseteq A \times B$ it is always the case that

$\mathrm{dom}(R) \subseteq A$ and $\mathrm{range}(R) \subseteq B$

converse vs complement

$$R = \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,c)\}$$

R	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

$$R^{-1} = \{(a,a),(a,c)(b,a),(b,b),(c,b),(c,c)\}$$

R	a	b	c
a	1	0	1
b	1	1	0
c	0	1	1

converse: mirror at the diagonal

$$\overline{R} = \{(a,c),(b,a),(c,b)\}$$

R	a	b	c
a	0	0	1
b	1	0	0
c	0	1	0

complement: flip zeros and ones

properties: a(nti)symmetry

Consider $\leq$ and $<$ on the natural numbers. Neither is symmetric, but in slightly different ways.

For $<$, it is **never** the case that $a < b$ and $b < a$.
This is called **asymmetry**.

For $\leq$, it sometimes is, but only when $a = b$.
This is called **antisymmetry**.

Both relations are antisymmetric. Only $<$ is asymmetric.

A binary relation $R \subseteq A \times A$ is *asymmetric* iff for all $a, b \in A$

$$\text{if } aRb \text{ then not } bRa$$

A binary relation $R \subseteq A \times A$ is *antisymmetric* iff for all $a, b \in A$

$$\text{if } aRb \text{ and } bRa \text{ then } a = b$$

equivalence relations

A binary relation $\approx \subseteq A \times A$ is an *equivalence relation* iff it is

1. reflexive
2. symmetric
3. transitive

What about these:

- equality
- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- relatively prime: $m \perp n$ iff there is no $k \geq 2 : k \mid m$ and $k \mid n$

equivalence relations

A binary relation $\approx \subseteq A \times A$

1. reflexive
2. symmetric
3. transitive

equality: yes, as it is reflexive, symmetric, and transitive.
divides: no, because it is not symmetric;
2 divides 4, but 4 does not divide 2.
relatively prime: no, it is neither reflexive (3 is not relatively prime to 3), nor transitive (2 is relatively prime to 5, 5 to 6, but not 2 to 6.

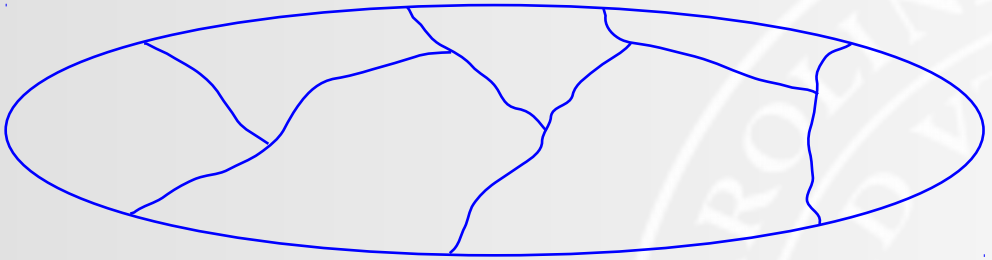
What about these:

- equality
- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- relatively prime: $m \perp n$ iff there is no $k \geq 2 : k \mid m$ and $k \mid n$

partitions

Given a set A, a *partition of A* is a set of pairwise disjoint sets $\{B_i : i \in I\}$, such that

$$A = \bigcup_{i \in I} B_i$$



A: EU citizens, I: EU member states, B_i: citizens of country i

A: atoms, I: elements, B_i: atoms of element i

A: natural numbers, I: primes, B_i: multiples of i (excluding i)

partitions

Given a set A, a *partition* $\{B_i : i \in I\}$, such that

EU citizens: no, since a EU citizen may be a citizen of more than one country
atoms: yes, all atoms belong to exactly one element
natural numbers: no, 1. the multiple of different primes are not disjoint, and 2. without the primes themselves their union does not include all natural numbers.

A: EU citizens, I: EU member states, B_i: citizens of country i

A: atoms, I: elements, B_i: atoms of element i

A: natural numbers, I: primes, B_i: multiples of i (excluding i)

equivalence class, quotient set

Equivalence relations can be used to generate partitions.

Given a set A and an equivalence relation $\approx$ on A, for any $a \in A$ we define the *equivalence class of a* $[a]_\approx$ as

$$[a]_\approx = \{b \in A : a \approx b\}$$

Alternative syntax:

SLAM $\left\{ \begin{array}{l} [a] \\ |a| \\ |a|_\approx \end{array} \right\}$ when the relation is understood

Given a set A and an equivalence relation $\approx$ on A, the *quotient (set)* $A/\approx$ is defined as

$$A/\approx \; = \{|a|_\approx : a \in A\}$$

SLAM 2.5.4:

1. Every partition is the quotient of an equivalence relation.
2. Every quotient set is a partition.

Review the proof in the book. Connect it to these definitions.

order relation, poset

A binary relation $\preceq \subseteq A \times A$ is an (*inclusive* or *non-strict*) (*partial*) *order* iff it is

1. reflexive
2. antisymmetric
3. transitive

What about these:

- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- set inclusion: $\subseteq$
- on numbers: $\leq$ and $<$
- proper set inclusion: $\subset$

divides: yes
set inclusion: yes
less than or equal to: yes
less than: no
proper set inclusion: no

order relation, poset

A binary relation $\preceq \subseteq A \times A$ is an (*inclusive* or *non-strict*) (*partial*) *order* iff it is

1. reflexive
2. antisymmetric
3. transitive

What about these:

- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- set inclusion: $\subseteq$
- on numbers: $\leq$ and $<$
- proper set inclusion: $\subset$

A pair $(A, \preceq)$ where A is a set and $\preceq \subseteq A \times A$ a partial order on A is called a *partially ordered set* or *poset*.

Examples: $(\mathbb{N}^+, |)$

$(\mathcal{P}(A), \subseteq)$

strict (partial) order

A binary relation $\prec \subseteq A \times A$ is a *strict (partial) order* iff it is

1. irreflexive
2. transitive

Note: Irreflexivity and transitivity imply asymmetry.

How?

irreflexivity: $a \not\prec a$

transitivity: if $a \prec b$ and $b \prec c$ then $a \prec c$

asymmetry: if $a \prec b$ then $b \not\prec a$

total (or linear) order

A binary relation $\preceq \subseteq A \times A$ is a *(non-strict) total (or linear) order* iff it is

1. reflexive
2. antisymmetric
3. transitive
4. total (complete): $a \preceq b$ or $b \preceq a$

divides: no, not total
set inclusion: no, not total
less then or equal: yes
less than: no, not reflexive

What about these:

- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- set inclusion: $\subseteq$
- on numbers: $\leq$ and $<$

transitive closure

The *transitive closure* R^+ of a binary relation $R \subseteq A \times A$ is defined as follows:

$$R^+ = \bigcup_{i \in \mathbb{N}} R_i \text{ with}$$

$$R_0 = R$$

$$R_{n+1} = R_n \cup \{(a, c) : \text{if } aR_nb \text{ and } bR_nc \text{ for some } b \in A\}$$

R^*

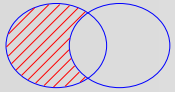
alternative syntax (SLAM)

$$\bowtie = \{(F,E),(E,F),(F,B),(B,F),(F,D),(D,F),(F,CH),(CH,F),(F,I),$$
$$(I,F),(B,NL),(NL,B),(B,D),(D,B),(D,NL),(NL,D),(D,CH),$$
$$(CH,D),(CH,I),(I,CH),(GB,IRL),(IRL,GB)\}$$

What is the meaning of $\bowtie^+$?
What are its properties?

$R = \{x : x \notin x\}$

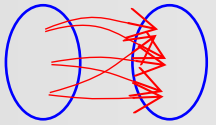
sets



$\supset$

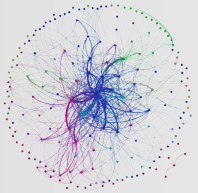
$\heartsuit \subseteq P \times Q$

relations



$\cup$

graphs



$\supset$

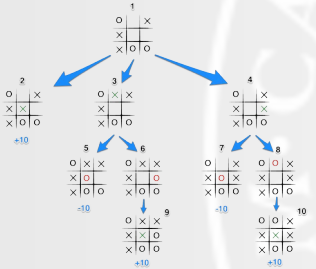
$f : A \longrightarrow B$

functions

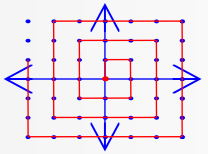
$A \hookrightarrow B$

$\supset$

trees



investigate



infinity

working with infinite
(or arbitrarily large) stuff

definition, construction,
recursion, induction
(also: proofs, logic)

domain, range, codomain

When talking about functions, some of the terminology is the same as for relations in general, some is not:

	$R \subseteq A \times B$	$f : A \longrightarrow B$
actual values, left	domain	domain
actual values, right	range	range
A	source	domain
B	target	codomain

$\mathrm{dom}(f) = A$

about B^A ...

Why B^A ?

f	1	2	3	4
a	1	0	0	0
b	0	0	1	0
c	1	0	0	0

each time, we choose from these #(B) options

we make a choice for each of these, #(A) times

$$\underbrace{\#(B) \cdot \ldots \cdot \#(B)}_{\#(A) \text{ times}} = \#(B)^{\#(A)}$$

So for the number of functions from A to B, we have $\#(B^A) = \#(B)^{\#(A)}$.

How many *relations* $R \subseteq A \times B$?

about B^A ...

Why B^A ?

f	1	2	3	4
a	1	0	0	0
b	0	0	1	0
c	1	0	0	0

← each time, we chose from these #(B) options

$$\underbrace{\#(B) \cdot \ldots \cdot \#(B)}_{\#(A) \text{ times}} = \#(B)^{\#(A)}$$

we make a choice
for each of these #(A) times

So for the number o

This question asks for the number of subsets of $A \times B$, i.e.

$$\#\mathcal{P}(A \times B) = 2^{\#(A \times B)} = 2^{\#A \#B}$$

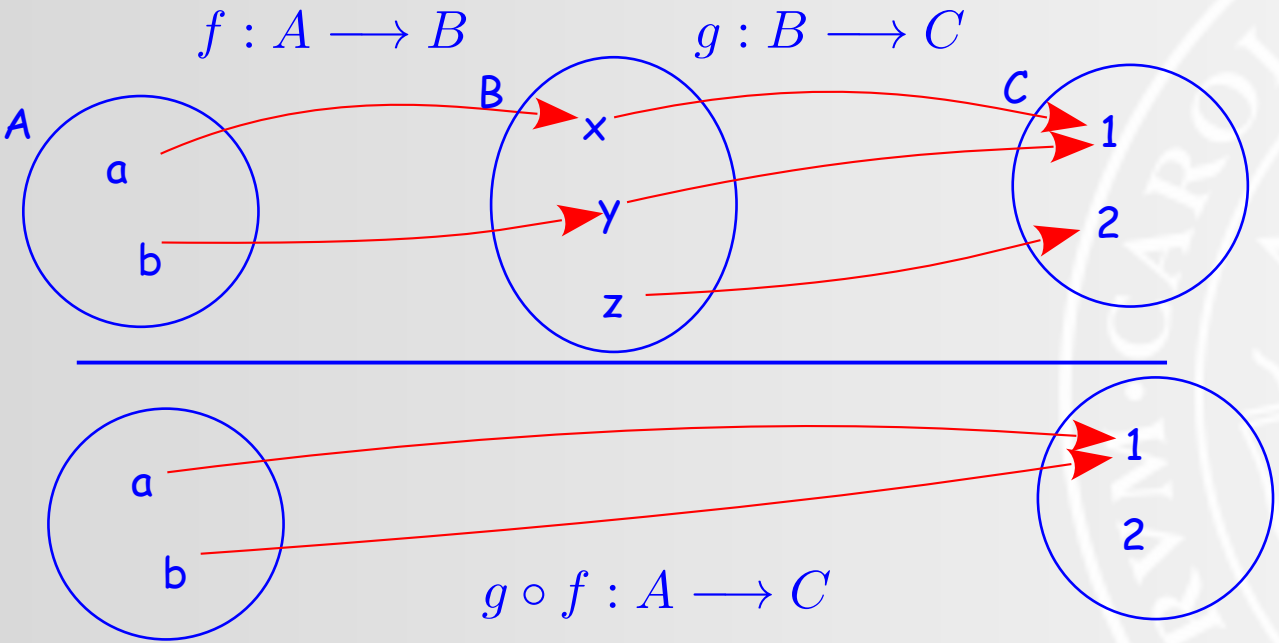
How many *relations* $R \subseteq A \times B$?

composition

Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ their *composition* $g \circ f : A \longrightarrow C$ defined as:

$$g \circ f(a) = g(f(a))$$

Function composition is just a special case of composition of relations:



It is associative:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

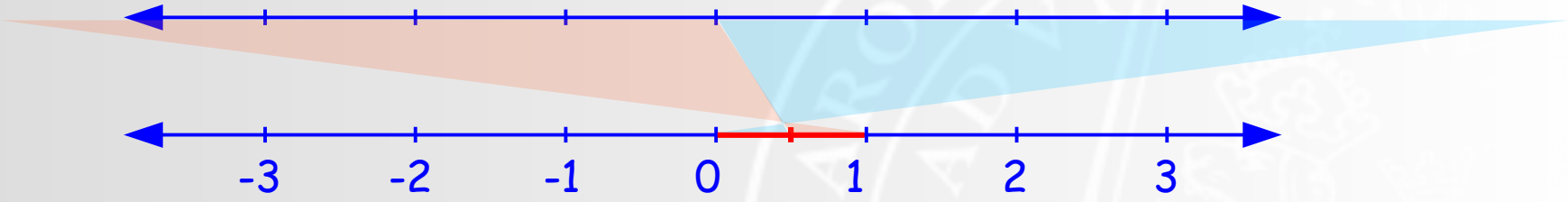
So we can omit the parentheses and write

$$h \circ g \circ f$$

$\mathbb{Q}$: the rational numbers

Some properties:

1. $\mathbb{Q}$ is *dense*: between any two distinct $r, s \in \mathbb{Q}$ there is $\frac{r+s}{2} \in \mathbb{Q}$
2. Any non-empty open interval $]r, s[\subset \mathbb{Q}$ is equinumerous to $\mathbb{Q}$



For example:

$p_{\mathbb{Q}} : \;]0,1[_{\mathbb{Q}} \longleftrightarrow \mathbb{Q}$

$$x \longmapsto \begin{cases} \frac{1}{x} - 2 \text{ for } x < \frac{1}{2} \\ \frac{1}{x-1} + 2 \text{ for } x \geq \frac{1}{2} \end{cases}$$

function definitions

What, exactly, is the relationship between the $\mathrm{f}\colon\ \mathrm{x}\mapsto$ ... style of defining a function, and concept of a function as a set of pairs?

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto 2x$$

$$f = \{(x, 2x) : x \in \mathbb{R}\}$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \mapsto \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

$\{(x, x) : x \in \mathbb{R}, x \geq 0\}$

$\{(x, -x) : x \in \mathbb{R}, x < 0\}$

$$g = \{(x, x) : x \in \mathbb{R}, x \geq 0\} \cup \{(x, -x) : x \in \mathbb{R}, x < 0\}$$

recursive function definitions (v 1.0)

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

$$n \mapsto \begin{cases} 1 & \text{for } n = 0 \\ n \cdot f(n-1) & \text{for } n > 0 \end{cases}$$

$$S_0 = X$$

$$S_{k+1} = S_k \cup F(S_k)$$

$$S = \bigcup_{k \in \mathbb{N}} S_k$$

$$f = \bigcup_{i \in \mathbb{N}} f_i$$

$$f_0 = \emptyset$$

$$f_{i+1} = f_i \cup \{(0,1)\} \cup \{(n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0\}$$

Note that f_i is a relation, so $f_i(n-1)$ computes the image of $n-1$ under f_i, which is a set of values. If it is empty, nothing bad happens, there just is no value for v.

recursive function definitions

$$f = \bigcup_{i \in \mathbb{N}} f_i$$

$$f_0 = \emptyset$$

$$f_{i+1} = f_i \cup \{(0,1)\} \cup \{(n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0\}$$

i	f_i	$\{(n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0\}$
0	$\emptyset \longrightarrow \cup\{(0,1)\}\cup \longleftarrow$	$\emptyset$
1	$\{(0,1)\}$	$\{(1,1)\}$
2	$\{(0,1),(1,1)\}$	$\{(1,1),(2,2)\}$
3	$\{(0,1),(1,1),(2,2)\}$	$\{(1,1),(2,2),(3,6)\}$
4	$\{(0,1),(1,1),(2,2),(3,6)\}$	$\{(1,1),(2,2),(3,6),(4,24)\}$

closure under relation

Given a relation $R \subseteq A \times A$ and a set $X \subseteq A$, the *closure of X under R* $R[X]$ is defined as the smallest $Y \subseteq A$ such that

$$X \subseteq Y \text{ and } R(Y) \subseteq Y$$

Construction:

$$Y_0 = X$$
$$Y_{n+1} = Y_n \cup R(Y_n)$$
$$R[X] = \bigcup_{i \in \mathbb{N}} Y_i$$

$R[C]$?

$R[\{\varepsilon\}]$?

$R[C]$: the set of all odd-length palindromes (in C)

$R[\{\varepsilon\}]$: the set of all even-length palindromes

Example:

$C = \{"a", ..., "z"\}$

$R \subseteq C^* \times C^*$

$R = \{(s, asa) : s \in C^*, a \in C\}$

How to express the set of all palindromes?

closure under relations, rules, generators

Given a set A, a family of relations on A $R = \{R_i \subseteq A^{n_i} : i \in I\}$ and a set $X \subseteq A$, the *closure of X under R* $R[X]$ is defined as the smallest $Y \subseteq A$ such that

$$X \subseteq Y \text{ and } R_i(Y^{n_i - 1}) \subseteq Y \text{ for all } i \in I$$

Construction:

$$Y_0 = X$$

$$Y_{n+1} = Y_n \cup \bigcup_{i \in I} R_i(Y_n^{n_i - 1})$$

$$R[X] = \bigcup_{i \in \mathbb{N}} Y_i$$

The elements of R are also called *rules, constructors, generators.*

$R[V]$ is the set of all expressions with unary - and binary + and *, and the variable symbols in V

Example:

$$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$$

$$R_1 = \{(s, \text{"-"}\ s) : s \in C^*\}$$

$$R_2 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$

$$R_3 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$

$$V = \{\text{"a"}, ..., \text{"z"}\}^* \setminus \{\varepsilon\}$$

$R[V]$?

structural recursion

$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$

$V = \{\text{"a"}, ..., \text{"z"}\}^* \setminus \{\varepsilon\}$

$$R_1 = \{(s, \text{"-"}\ s) : s \in C^*\}$$
$$R_2 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$
$$R_3 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$

Let's write a function that evaluates an expression in $R[V]$.

Assume a function $E : V \longrightarrow \mathbb{R}$ that assigns every variable a value.

$$\text{eval}_E : R[V] \longrightarrow \mathbb{R}$$

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ -\text{eval}_E(s') & \text{for } s' \in R[V], s = \text{"-"}s' \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"} \\ \text{eval}_E(s_1) \cdot \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"} \end{cases}$$

Note how the structure is *decomposed* (or *deconstructed*) in these clauses!

Write a function that returns for every expression s the set of variables that occur in it.

unique decomposability

$R = \{R_1, R_2\}, C = \text{UTF-16}$

$V = \{\text{"a"}, ..., \text{"z"}\}^* \setminus \{\varepsilon\}$

$R_1 = \{(s_1, s_2, s_1 \text{ "+" } s_2) : s_1, s_2 \in C^*\}$

$R_2 = \{(s_1, s_2, s_1 \text{ "-" } s_2) : s_1, s_2 \in C^*\}$

Let's look at a variant of the previous example:

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = s_1 \text{ "+" } s_2 \\ \text{eval}_E(s_1) - \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = s_1 \text{ "-" } s_2 \end{cases}$$

What is the problem with this function definition?

It does **not** uniquely decompose a string such as "a – b + c – d".

unique decomposability

Suppose we have generators $R = \{R_1, ..., R_k\}$ with $R_k \subseteq A^{n_k+1}$, and basis $X \subseteq A$.

The general form of a recursive function $f : R[X] \longrightarrow V$ is

$$\mathrm{f} : x \mapsto \begin{cases} h_X(x) & \text{for } x \in X \\ h_1(f(x_1), ..., f(x_{n_1})) & \text{for } x_i, x \in R[X], (x_1, ..., x_{n_1}, x) \in R_1 \\ & ... \\ h_k(f(x_1), ..., f(x_{n_k})) & \text{for } x_i, x \in R[X], (x_1, ..., x_{n_k}, x) \in R_k \end{cases}$$

It is only well-defined if all $x \in R[X]$ are *uniquely decomposable.*

cf. section 4.6.3 in SLAM.

the idea

$$A(m,n) = \begin{cases} n+1 & \text{for } m = 0 \\ A(m-1,1) & \text{for } m > 0, n = 0 \\ A(m-1, A(m,n-1)) & \text{otherwise} \end{cases}$$

We need to "order" the elements in domains such as $R[V]$ and $\mathbb{N} \times \mathbb{N}$ in such a way that
(a) recursive calls are always made with "smaller" argument values than the call received that they are made from, and
(b) from any possible argument value, you cannot "descend" forever --- eventually, the function will call itself with a value for which there is no "smaller" value to recursively call itself with.

$$\text{eval}_E : R[V] \longrightarrow \mathbb{R}$$

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ -\text{eval}_E(s') & \text{for } s' \in R[V], s = \text{"-"} s' \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"} \\ \text{eval}_E(s_1) \cdot \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"} \end{cases}$$

well-founded sets

A poset $(A, <)$ is *well-founded* iff all non-empty subsets $X \subseteq A$ have a minimal element, i.e.

$$\text{for some } m \in X \text{ and all } a \in X, a \not< m$$

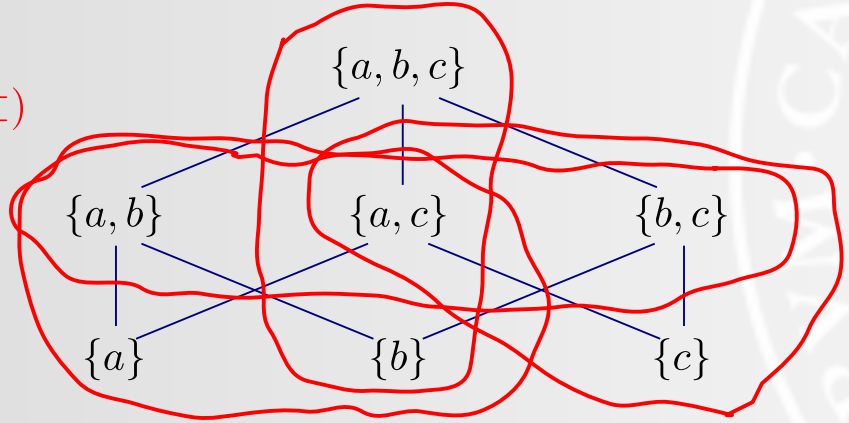
Intuitively, this means there are no infinite descending chains:

$$\ldots < a_n < a_{n-1} < \ldots < a_2 < a_1 < a_0$$

minimal element != minimum element:

- minimal means that there is no smaller element
- minimum means all other elements are greater

$(\mathcal{P}(\{a,b,c\}) \setminus \{\emptyset\}, \subset)$



well-founded sets

A poset $(A, <)$ is *well-founded* iff all non-empty subsets $X \subseteq A$ have a minimal element, i.e.

$$\text{for some } m \in X \text{ and all } a \in X, a \not< m$$

$(\mathbb{N}, <)$ ✓

$(\mathbb{Z}, <)$ ✗

$(\mathbb{Q}_0^+, <)$ ✗

$(\mathcal{P}(\mathbb{N}), \subset)$ ✗

$(\mathbb{N}^2, <_{\text{lex}})$ ✓

$(\mathbb{N}^2, <_{\text{prod}})$ ✓

$(a_1, a_2) <_{\text{prod}} (b_1, b_2)$ iff $a_1 < b_1$ and $a_2 < b_2$

$(a_1, a_2) <_{\text{lex}} (b_1, b_2)$ iff $a_1 < b_1$ or $(a_1 = b_1$ and $a_2 < b_2)$

well-founded recursion

Given a well-founded set $(W, <)$ and a recursive function definition for a function $f : W \longrightarrow X$,
f is *well-defined* if it computes the value for every $w \in W$ only depending on values of f for $v < w$.

So how do we use this in practice? How can we tell that the Ackermann function is well-defined?

$$A : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$$

$$A(m,n) = \begin{cases} n+1 & \text{for } m = 0 \\ A(m-1,1) & \text{for } m > 0, n = 0 \\ A(m-1, A(m, n-1)) & \text{otherwise} \end{cases}$$

$(m-1,1) \prec (m,0)$

$(m-1, A(m,n-1)) \quad (m-1,x) \prec (m,n) \quad (m,n-1) \prec (m,n)$

$(m-1,1) \prec (m,0)$

$(m-1,x) \quad (m,n-1) \prec (m,n)$

well-founded recursion

Given a well-founded set $(W, <)$ and a recursive function definition for a function $f : W \longrightarrow X$,
f is *well-defined* if it computes the value for every $w \in W$ only depending on values of f for $v < w$.

$$(a_1, a_2) <_{\text{lex}} (b_1, b_2) \text{ iff } a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 < b_2)$$

Which order would make $\mathbb{N} \times \mathbb{N}$ well-founded, such that the inequalities on the right would be true?

$$(m-1, 1) \prec (m, 0)$$

$$\begin{matrix}(m-1, x) \\ (m, n-1)\end{matrix} \prec (m, n)$$

Now take a look at the "buggy" version of the Ackermann function,
and track down, how it would fail to to meet the requirements of well-founded recursion.

a simple inductive proof

$$\text{sum}(n) = \sum_{i=1\ldots n} i = 1 + \ldots + n$$

Hypothesis: $\text{sum}(n) = \frac{n(n+1)}{2}$

Basis: $\text{sum}(1) = \frac{1(1+1)}{2} = 1$

Induction step:

induction hypothesis

Assuming that $\text{sum}(k) = \frac{k(k+1)}{2}$

show that $\text{sum}(k+1) = \frac{(k+1)(k+2)}{2}$

induction goal

$$\begin{aligned}
\text{sum(k+1)} &= \text{sum(k)} + \text{(k+1)} && \text{def sum()} \\
&= \frac{k(k+1)}{2} + (k+1) && \text{IH} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

a simple induction principle

Hypothesis: $P(n)$ for all n

Basis: Show that $P(1)$ (or $P(0)$ or some other, depending on circumstances)

Induction step: Assuming that $\underbrace{P(k)}$ show that $\underbrace{P(k+1)}$

induction hypothesis — induction goal

Things that often go wrong:

- mixing basis and induction step: do not try to do everything at once
- confusing induction hypothesis and induction goal
- not using the induction hypothesis: it ain't cheating!
- getting lost: proving the goal can be messy, keep eyes on prize

cumulative (complete) induction

Cumulative, also *complete* or *strong*, induction uses an induction hypothesis that assumed the truth of the hypothesis for <u>all</u> smaller values, instead of just the previous one.

Hypothesis: $g(n) \leq 2^n$

Basis: $g(0) \leq 2^0 = 1$

$g(1) \leq 2^1 = 2$

Induction step:

induction hypothesis

Assuming that $g(m) \leq 2^m$ for all $m < k$

show that $g(k) \leq 2^k$

induction goal

$$g(n) = \begin{cases} 1 & \text{for } n \leq 1 \\ g(n-1) + g(n-2) & \text{otherwise} \end{cases}$$

$$\begin{aligned} g(k) &= g(k-1) + g(k-2) && \text{def g()} \\ &\leq 2^{k-1} + 2^{k-2} && \text{IH} \\ &= \frac{1}{2}2^k + \frac{1}{4}2^k \\ &\leq 2^k \end{aligned}$$

cumulative induction principle

Hypothesis: $P(n)$

Basis: $P(0), \ldots$

Induction step:

Assuming that $\underbrace{P(m) \text{ for all } m < k}_{\text{induction hypothesis}}$ show that $\underbrace{P(k)}_{\text{induction goal}}$

Note that the basis is the vacuous form of the induction step, for k=0.
As a result, the basis is subsumed by the induction step.
In practice, it is often treated separately.

structural induction

$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$

$V = \{\text{"a"}, ..., \text{"z"}\}^* \setminus \{\varepsilon\}$

$$R_1 = \{(s, \text{"-"}\ s) : s \in C^*\}$$
$$R_2 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$
$$R_3 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$

Hypothesis: Every $s \in R[V]$ contains equal numbers of opening and closing parentheses.

Basis: Every $s \in V$ contains equal numbers of opening and closing parentheses.

Induction step:

induction hypothesis

All rules $R_i \in R$ preserve the property: if their "input" objects have it, then so does their "output" object.

induction goal

$$R_1 = \{(s, \text{"-"}\ s) : s \in C^*\}$$
$$R_2 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"+"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$
$$R_3 = \{(s_1, s_2, \text{"("}\ s_1\ \text{"*"}\ s_2\ \text{")"}) : s_1, s_2 \in C^*\}$$

structural induction principle

Structural induction is a variant of cumulative induction, but instead of natural numbers we induce over the structure of rule-generated objects in some structurally-recursive set $R[X]$.

Hypothesis: $P(x)$ for all $x \in R[X]$

Basis: $P(x)$ for all $x \in X$

Induction step:

induction hypothesis

All rules $R_i \in R$ preserve the property: if their "input" objects have it, then so does their "output" object.

induction goal

well-founded induction

Emmy Noether
1882-1935

Cumulative (complete, strong) induction assumed that a property needed to be shown over the natural numbers, structural induction that the set was generated inductively through generators.
Well-founded induction generalizes the idea to all well-founded sets.

Given a well-founded set $(A, <)$ and a property $P(a), a \in A$, if

$$\text{for all } a \in A : \underbrace{P(w) \text{ for all } w < a}_{\text{induction hypothesis}} \text{ implies } \underbrace{P(a)}_{\text{induction goal}}$$

then $P(a)$ for all $a \in A$

As with cumulative induction, the base case is subsumed by the induction step.
In practice, it is still sometimes handled separately.

well-founded induction

Hypothesis: Every $n \geq 2$ can be factored into primes.

well-founded order: $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$ (divides-relation, strict version)

Descending chains?
Minimal elements?

Descending chains look like this: $7|_{\neq}21|_{\neq}42|_{\neq}126|_{\neq}252|_{\neq}\ldots$

The minimal elements are the primes.

well-founded induction

Hypothesis: Every $n \geq 2$ can be factored into primes.

well-founded order: $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$ (divides-relation, strict version)

(Base:) Trivially true for every minimal element in $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$

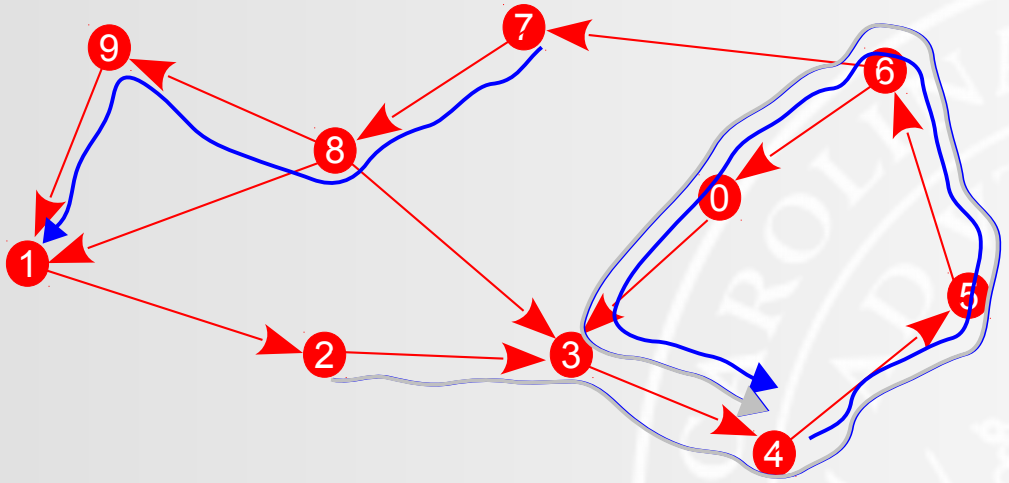
Induction step: If n is not minimal, then there is a k such that $k|_{\neq}n$, and thus there is some m such that $n = km$. k and m can be prime-factored, therefore so can n.

How do we know that m can be factored into primes?

Because $m|_{\neq}n$, too.

paths

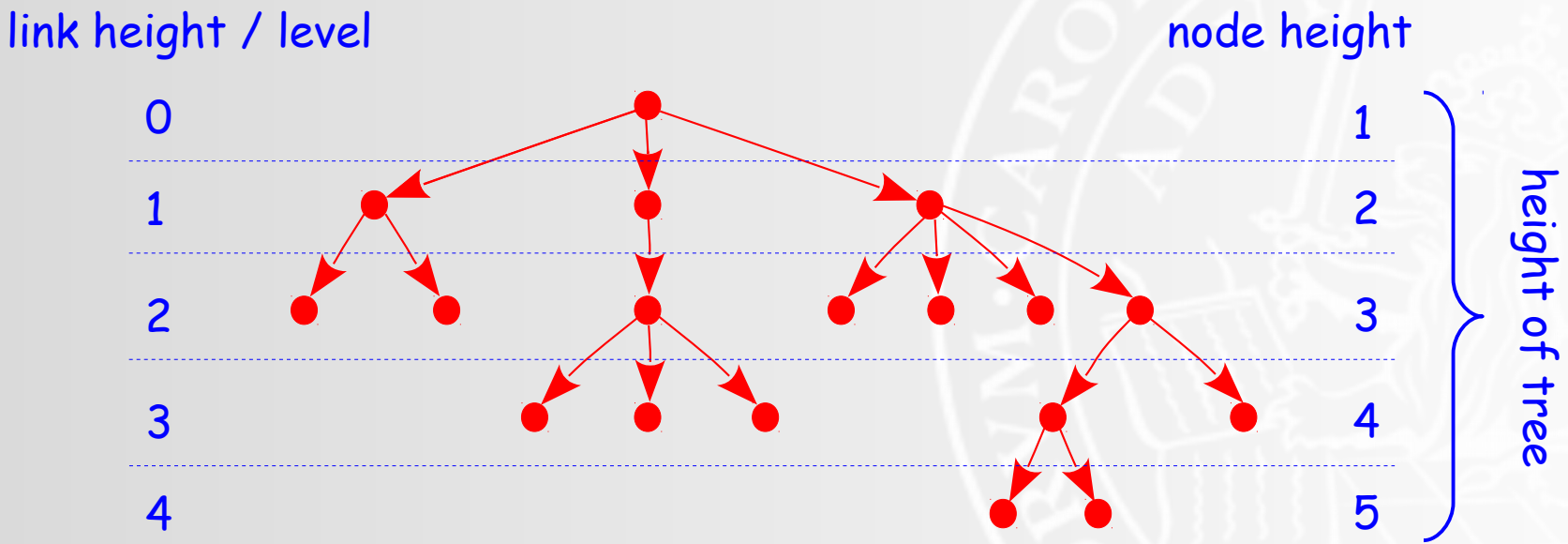
Given a graph (V, E), a *path* is a finite sequence $a_0, \ldots, a_n$ in V with $n \geq 1$ such that $(a_{k-1}, a_k) \in E$ for $1 \leq k \leq n$ .The *length* of the path is n.



A *cycle* is a path $a_0, \ldots, a_n$ where $a_0 = a_n$.
A graph that does not contain cycles is called *acyclic*.

properties

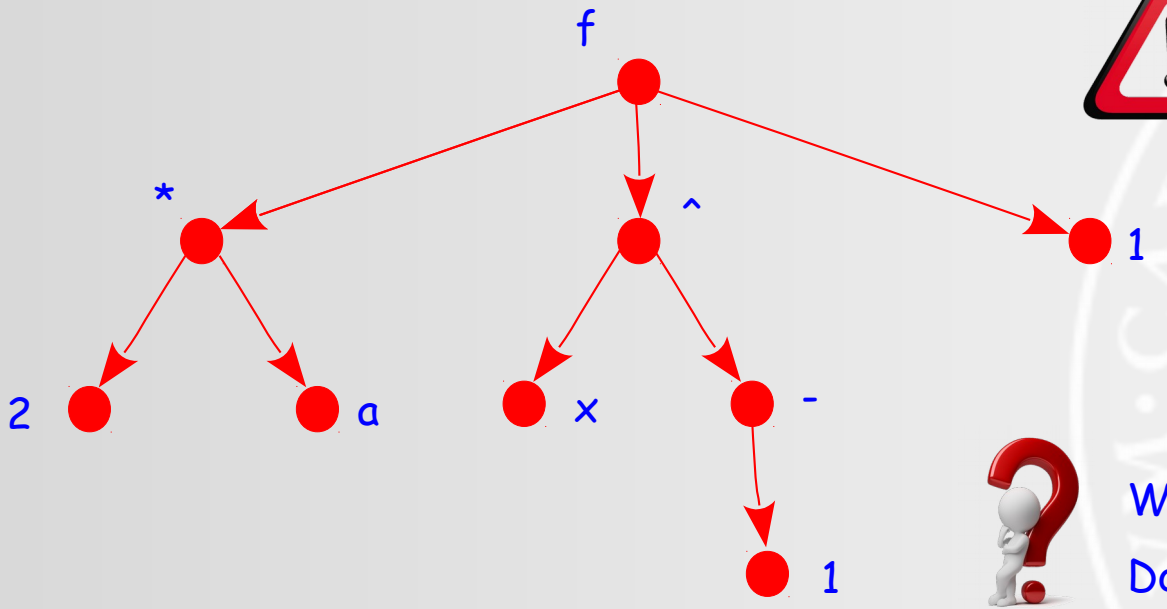
(1) Any non-empty tree has a unique root.
(2) A root has no parent.
(3) Every non-root has exactly one parent.
(4) A tree with n nodes has n-1 links.
(5) A tree contains no cycles.



labeled trees

Given a tree (T, R), and a set of *labels* L, a *labeling* is a function $\lambda : T \longrightarrow L$

A tree with a labeling is called a *labeled tree*.



In practice, the labeling function is often realized by adding data to the nodes of a tree.

Not uniquely – the tree definition contains no ordering information. It represents just as well f(1, 2*a, (-1)^x)

What expression might the tree represent?

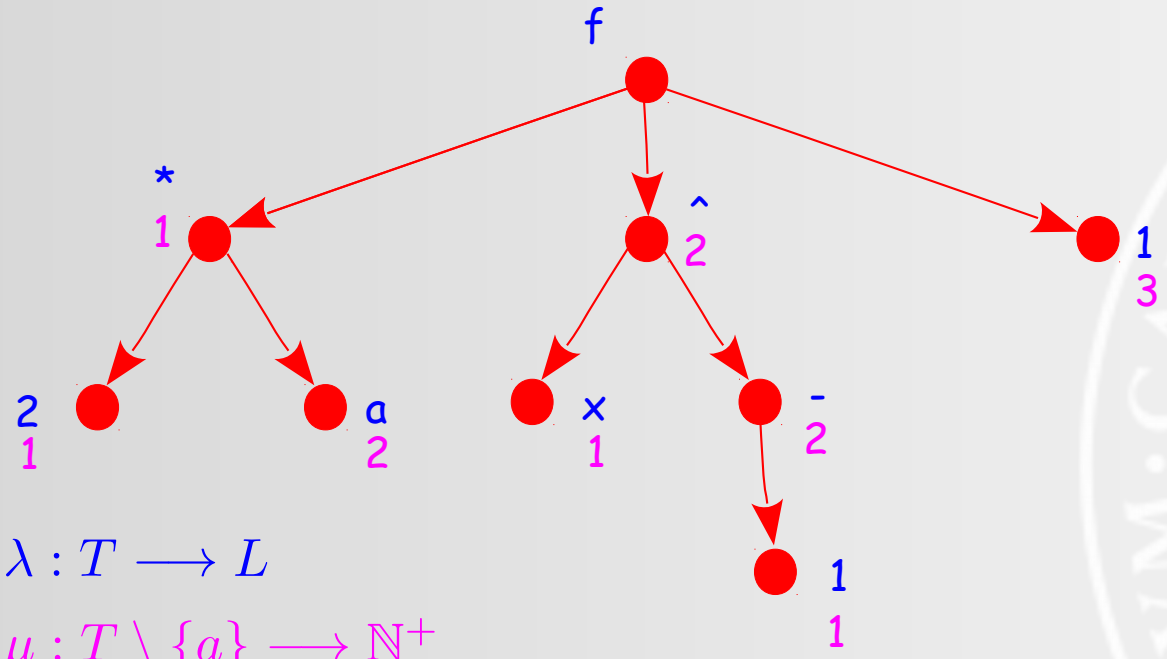
Does it?

ordered trees

Given a tree (T, R) with root a, we say it is *ordered* if there is a function

$$\mu : T \setminus \{a\} \longrightarrow \mathbb{N}^+$$

such that for every node its n children are labeled $1..n$.



$\lambda : T \longrightarrow L$

$\mu : T \setminus \{a\} \longrightarrow \mathbb{N}^+$

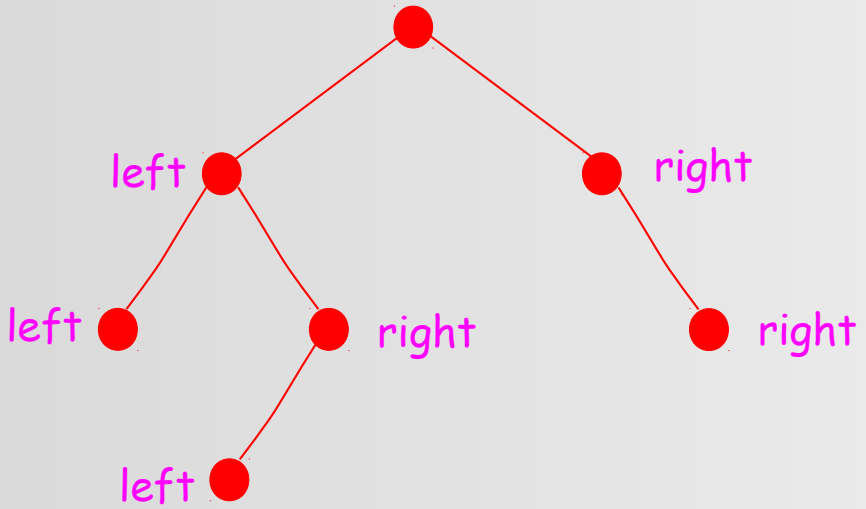
Ordering is usually represented implicitly by the left-to-right order of child nodes in the tree data structure.

binary trees

Given a tree (T, R) with root a, we say it is *binary* if every node has at most two children and there is a labeling function

$$\beta : T \setminus \{a\} \longrightarrow \{\text{left}, \text{right}\}$$

such that no two children of the same node are labeled identically.



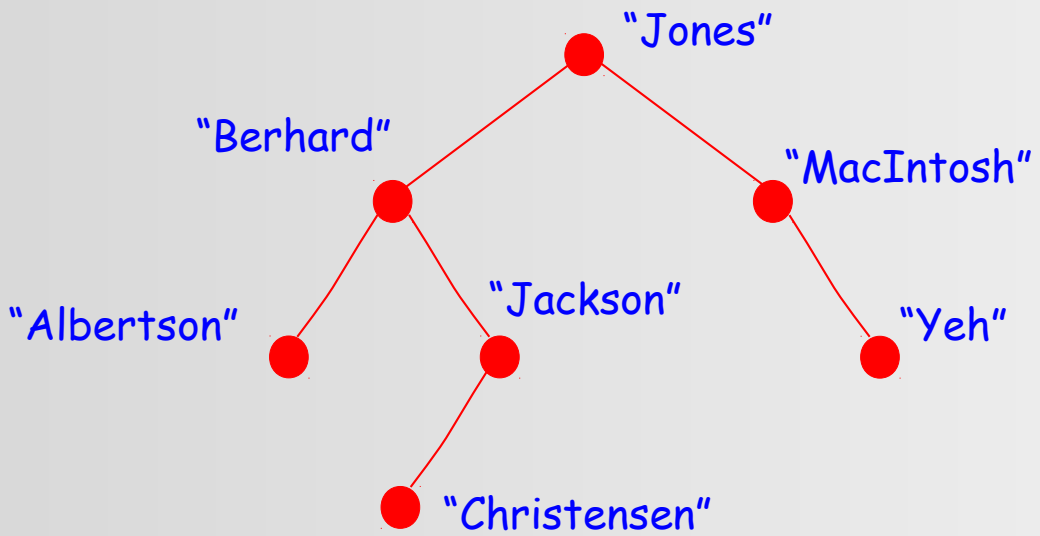
$$\beta : T \setminus \{a\} \longrightarrow \{\text{left}, \text{right}\}$$

Labels are usually represented by the left-to-right order of child nodes and angled links.

binary search trees

Given a binary tree (T, R), with root a, binary labels $\beta : T \longrightarrow \{\text{left}, \text{right}\}$, and labeling function $\lambda : T \longrightarrow L$ and a totally ordered set $(L, \leq)$.
It is a *binary search tree* iff for all nodes their label is greater than any label in their left subtree, and less than any label in their right subtree.

It could't.
The labels of an ordered tree establish a total order *among the children* of each node.
The labels in a binary tree also relate the children with respect to the *parent*.

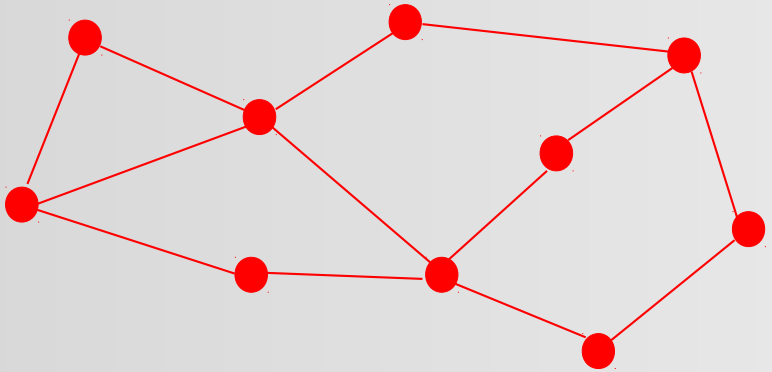


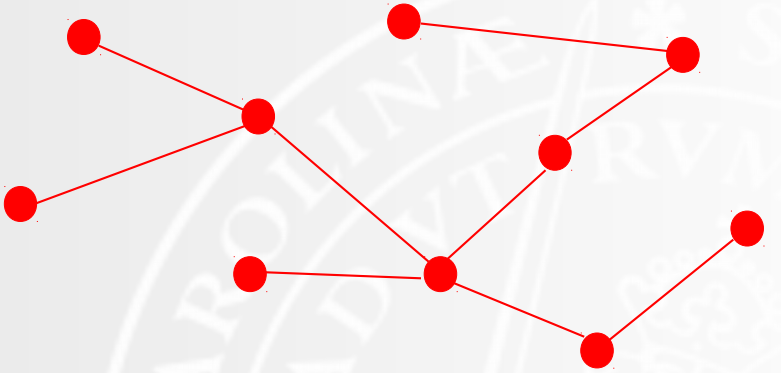
Could the same be achieved with an ordered tree, instead of a binary tree?

IOW, is a binary tree a special case of an ordered tree, or something else?

spanning trees

Given an undirected graph (T, S), an unrooted tree (T, R) is a *spanning tree* for it iff $R \subseteq S$





There may be many spanning trees for any given graph.

from truth table to formula

p	q	r	f(p, q, r)
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

$p \wedge q \wedge r$

$p \wedge \neg q \wedge r$

$\neg p \wedge \neg q \wedge r$

$f : (p, q, r) \mapsto (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg p \wedge r)$

Are there truth tables for which this method does not produce a formula?

What does this mean for the connectives not, and, or?

Only the one where always $f(...) = 0$.
This can be expressed as $f(p, q, r) = p \wedge \neg p$.

It means these connectives are universal.
All propositional formulae can be expressed using not, and, or.

tautological implication

Given a set of formulae A and a formula β we say that A *tautologically implies* β if there is no valuation v such that

$$v(\alpha) = 1 \text{ for all } \alpha \in A \text{ and } v(\beta) = 0$$

$A \vdash \beta$

$A \models \beta$

Example:
(modus ponens) $p, p \to q \vdash q$

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Name	LHS	RHS
Simplification, $\wedge^-$	$\alpha \wedge \beta$	α
	$\alpha \wedge \beta$	β
Conjunction, $\wedge^+$	α, β	$\alpha \wedge \beta$
Disjunction, $\vee^+$	α	$\alpha \vee \beta$
	β	$\alpha \vee \beta$
Modus ponens, MP, $\to^-$	$\alpha, \alpha \to \beta$	β
Modus tollens, MT	$\neg\beta, \alpha \to \beta$	$\neg\alpha$
Disjunctive syllogism, DS	$\alpha \vee \beta, \neg\alpha$	β
Transitivity	$\alpha \to \beta, \beta \to \gamma$	$\alpha \to \gamma$
Material implication	β	$\alpha \to \beta$
	$\neg\alpha$	$\alpha \to \beta$
Limiting cases	γ	$\beta \vee \neg\beta$
	$\alpha \wedge \neg\alpha$	γ

SLAM, p. 259

tautological equivalence

Given two formulae α and β, we say that they are *tautologically equivalent* if they tautologically imply each other.

$$\alpha \dashv\vdash \beta$$

Name	LHS	RHS
Double negation	α	$\neg\neg\alpha$
Commutation for $\wedge$	$\alpha\wedge\beta$	$\beta\wedge\alpha$
Association for $\wedge$	$\alpha\wedge(\beta\wedge\gamma)$	$(\alpha\wedge\beta)\wedge\gamma$
Commutation for $\vee$	$\alpha\vee\beta$	$\beta\vee\alpha$
Association for $\vee$	$\alpha\vee(\beta\vee\gamma)$	$(\alpha\vee\beta)\vee\gamma$
Distribution of $\wedge$ over $\vee$	$\alpha\wedge(\beta\vee\gamma)$	$(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)$
Distribution of $\vee$ over $\wedge$	$\alpha\vee(\beta\wedge\gamma)$	$(\alpha\vee\beta)\wedge(\alpha\vee\gamma)$
Absorption	α	$\alpha\wedge(\alpha\vee\beta)$
	α	$\alpha\vee(\alpha\wedge\beta)$
Expansion	α	$(\alpha\wedge\beta)\vee(\alpha\wedge\neg\beta)$
	α	$(\alpha\vee\beta)\wedge(\alpha\vee\neg\beta)$
de Morgan	$\neg(\alpha\wedge\beta)$	$\neg\alpha\vee\neg\beta$
	$\neg(\alpha\vee\beta)$	$\neg\alpha\wedge\neg\beta$
	$\alpha\wedge\beta$	$\neg(\neg\alpha\vee\neg\beta)$
	$\alpha\vee\beta$	$\neg(\neg\alpha\wedge\neg\beta)$
Limiting cases	$\alpha\wedge\neg\alpha$	$\beta\wedge\neg\beta$
	$\alpha\vee\neg\alpha$	$\beta\vee\neg\beta$

SLAM, p. 262

tautological equivalence

Name	LHS	RHS
Contraposition	$\alpha \to \beta$	$\neg\beta \to \neg\alpha$
	$\alpha \to \neg\beta$	$\beta \to \neg\alpha$
	$\neg\alpha \to \beta$	$\neg\beta \to \alpha$
Import/export	$\alpha \to (\beta \to \gamma)$	$(\alpha \wedge \beta) \to \gamma$
	$\alpha \to (\beta \to \gamma)$	$\beta \to (\alpha \to \gamma)$
Consequential mirabilis (miraculous consequence)	$\alpha \to \neg\alpha$	$\neg\alpha$
	$\neg\alpha \to \alpha$	α
Commutation for $\leftrightarrow$	$\alpha \leftrightarrow \beta$	$\beta \leftrightarrow \alpha$
Association for $\leftrightarrow$	$\alpha \leftrightarrow (\beta \leftrightarrow \gamma)$	$(\alpha \leftrightarrow \beta) \leftrightarrow \gamma$
$\neg$ through $\leftrightarrow$	$\neg(\alpha \leftrightarrow \beta)$	$\alpha \leftrightarrow \neg\beta$
Translations between connectives	$\alpha \leftrightarrow \beta$	$(\alpha \to \beta) \wedge (\beta \to \alpha)$
	$\alpha \leftrightarrow \beta$	$(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$
	$\alpha \to \beta$	$\neg(\alpha \wedge \neg\beta)$
	$\alpha \to \beta$	$\neg\alpha \vee \beta$
	$\alpha \vee \beta$	$\neg\alpha \to \beta$
Translations of negations of connectives	$\neg(\alpha \to \beta)$	$\alpha \wedge \neg\beta$
	$\neg(\alpha \wedge \beta)$	$\alpha \to \neg\beta$
	$\neg(\alpha \leftrightarrow \beta)$	$(\alpha \wedge \neg\beta) \vee (\beta \wedge \neg\alpha)$

SLAM, p. 263

constructing the DNF

Given a formula, there are two common ways of constructing its DNF:

1. via its truth table
2. through successive syntactic transformations

Basic algorithm for building a DNF (SLAM p. 272):

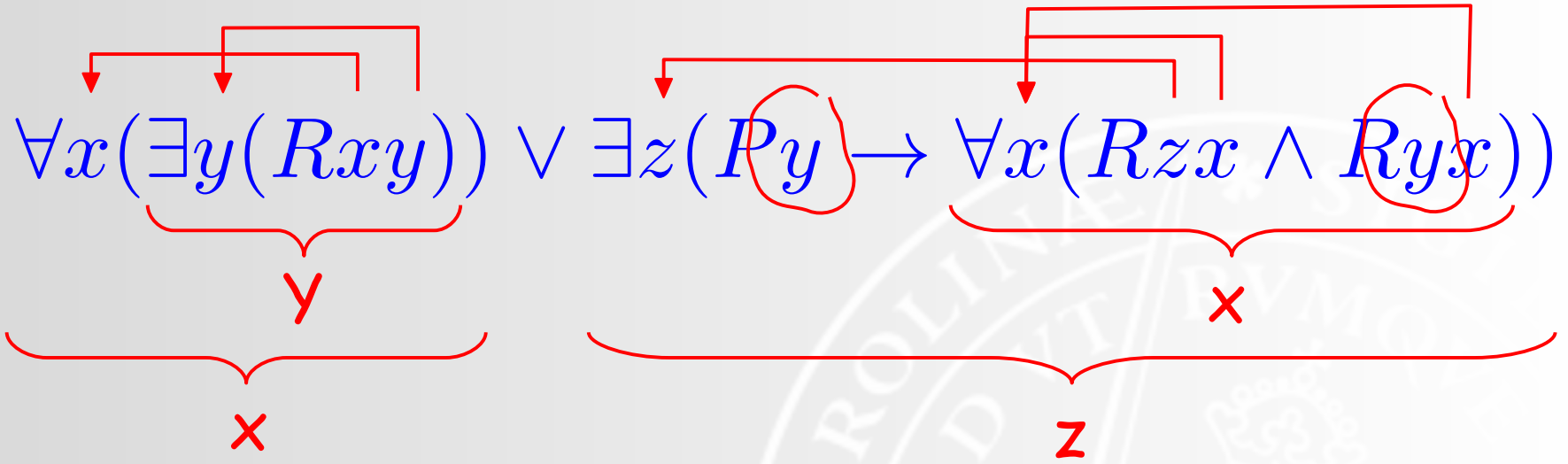
1. express all connectives through negation, conjunction, disjunction
2. move negations inward (de Morgan), eliminate double negation
3. move conjunctions inward (distribution)
4. remove repetitions in basic conjunctions
5. remove basic conjunctions with a letter and its negation

Example: $\neg((p \vee \neg q) \rightarrow (p \wedge \neg q))$

1. $\neg(\neg(p \vee \neg q) \vee (p \wedge \neg q))$
2.1 $\neg\neg(p \vee \neg q) \wedge \neg(p \wedge \neg q)$
2.2 $\neg\neg(p \vee \neg q) \wedge (\neg p \vee q)$
2.3 $(p \vee \neg q) \wedge (\neg p \vee q)$
3. $(p \wedge \neg p) \vee (p \wedge q) \vee (\neg q \wedge \neg p) \vee (\neg q \wedge q)$
5. $(p \wedge q) \vee (\neg p \wedge \neg q)$

quantifier scopes

variable *uses*, and the quantifier they are bound by



quantifier scopes, and the variables bound in/by them

free and bound variable occurrences

A variable occurrence is *bound* iff it occurs inside the scope of a quantifier that binds that variable.

It is *free* otherwise.

A formula with no free variable occurrences is called *closed*.
A closed formula is a *sentence*.

free occurrences

bound occurrences

$$\forall z[Rxz \rightarrow \exists y(Rzy)]$$

definitions, theorems, proofs

A definition is a statement that gives a precise meaning to a term or a symbol.

$$A \subseteq B \text{ iff } \forall x \ (x \in A \rightarrow x \in B)$$

$$n \in \mathbb{Z} \text{ is } \textit{even} \text{ iff } \exists k \in \mathbb{Z} \ (n = 2k)$$

$$n \in \mathbb{Z} \text{ is } \textit{odd} \text{ iff } \exists k \in \mathbb{Z} \ (n = 2k + 1)$$

A theorem is a statement that needs to be proven based on definitions (and axioms).

$$A \times (B \cap C) = A \times B \cap A \times C$$

$$\#(\mathbb{N}) < \#(2^{\mathbb{N}})$$

There are infinitely many prime numbers.

Other words for theorem: proposition, lemma, corollary.

A proof is a is a chain of logical reasoning showing the truth of a theorem.

kinds of proofs

Proofs come in different flavors, which depend on the form of the theorem, and the chain of reasoning best suited to prove it.

Many theorems are conditional statements, i.e. they have the form "premise implies conclusion, or

$$P \to C$$

$\forall x \in \mathbb{Z}\ (x \text{ is odd } \to x^2 \text{ is odd})$

$\forall a, b, c \in \mathbb{Z}\ ((a|b \wedge b|c) \to a|c)$

P	C	$P \to C$
T	T	T
T	F	F
F	T	T
F	F	T

direct proof

Theorem: If P, then C.

Proof: Suppose P.

…

Therefore C.

Theorem:

$$x \text{ is odd } \rightarrow x^2 \text{ is odd}$$

Proof:

Suppose x is odd.

Therefore, there is an integer k such that $x = 2k + 1$.

Thus $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Note that $2k^2 + 2k$ is an integer.

Thus there is an integer n such that $x^2 = 2n + 1$.

Therefore x^2 is odd.

direct proof with cases

Sometimes, the premise consists of several *cases*, and it becomes easier to study each case by itself.

n	$1+(-1)^n(2n-1)$
1	0
2	4
3	-4
4	8
5	-8
6	12

Theorem: If $n \in \mathbb{N}$ then $1+(-1)^n(2n-1)$ is a multiple of 4.

Proof: Suppose $n \in \mathbb{N}$. Then n is either even or odd.

Case 1: Suppose n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$.

Thus $1+(-1)^{2k}(2(2k)-1) = 1+1^k(4k-1) = 4k$.

That is a multiple of 4.

Case 2: Suppose n is odd. Then $n = 2k+1$ for some $k \in \mathbb{Z}$.

Thus $1+(-1)^{2k+1}(2(2k+1)-1) = 1-(4k+2-1) = -4k$.

That is also a multiple of 4.

The result in both cases is a multiple of 4.

contrapositive proof

In some cases, it is easier to reason about a theorem in *contrapositive* form.

Theorem:

If $x^2 - 6x + 5$ is even, then x is odd.

Proof:

Suppose $x^2 - 6x + 5$ is even, i.e. there exists an integer a such that $x^2 - 6x + 5 = 2a$.

...

Thus there is an integer b such that $x = 2b + 1$.

Therefore b is odd.

direct proof:

Theorem: If P, then C.

Proof: Suppose P.

…

Therefore C.

contrapositive proof

Contrapositive form: $\neg C \to \neg P$

P	C	$P \to C$	$\neg C$	$\neg P$	$\neg C \to \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Theorem: If P, then C.

Proof: Suppose not C.

…

Therefore not P.

Theorem:

If $x^2 - 6x + 5$ is even, then x is odd.

Proof:

Suppose x is even.

There is an integer a such that $x = 2a$.

$x^2 - 6x + 5 = 4a^2 - 12a + 4 + 1 = 2(2a^2 - 6a + 2) + 1$

So there is an integer b s.t. $x^2 - 6x + 5 = 2b + 1$.

Therefore $x^2 - 6x + 5$ is not even.

proof by contradiction

Suppose we want to prove a proposition P, not necessarily in conditional form.
Proof by contradiction uses the fact that if we can show that not P results in a logical contradiction, e.g. it implies some conclusion C as well as its opposite, not C, then not P cannot be true, and so P must be true.

P	C	$\neg P$	$C \wedge \neg C$	$\neg P \rightarrow C \wedge \neg C$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Theorem:

If $a, b \in \mathbb{Z}$ then $a^2 - 4b \neq 2$.

Proof:

Suppose there are $a, b \in \mathbb{Z}$ s.t. $a^2 - 4b = 2$.
Since this implies $a^2 = 4b + 2 = 2(2b + 1)$, a^2 is even.
Hence a is even, so $a = 2c$ for some integer c.
Thus $4c^2 - 4b = 2$, i.e. $2c^2 - 2b = 1$.
Therefore $2(c^2 - b) = 1$ with $c^2 - b \in \mathbb{Z}$.
So 1 is even.

Theorem: P.

Proof: Suppose not P.

…

Or any other false proposition!

Therefore C and not C.