# EDAA40 Exam

26 August 2017

## Instructions

**Things you CAN use during the exam.**
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

**Things you CANNOT use during the exam.**
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

**WRITE CLEARLY.** If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

**A sheet with common symbols and notations is attached at the end.**

Good luck!

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | total |
|---|---|---|---|---|---|---|---|---|
| 20 | 6 | 10 | 4 | 10 | 10 | 16 | 20 | 96 |
| | | | | | | | | |

**points required for 3: 50**

**points required for 4: 65**

**points required for 5: 84**

# 1 [20 p]

For the following sets of numbers, specify the smallest and the largest numbers, write NONE if there is no smallest or largest number, or EMPTY (in one of the two columns) if the set is empty.

All intervals are supposed to be intervals in the real numbers, $\mathbb{R}$. Similarly, all relations and operators are on the real numbers, unless explicitly stated otherwise.

| set | smallest element | largest element |
|---|---|---|
| $\{x \in \mathbb{Z} : x > 4 \wedge x < 2\}$ | EMPTY | |
| $\{1, 2, 3, 4\}$ | 1 | 4 |
| $]1, 4]$ | NONE | 4 |
| $\bigcap_{i \in \mathbb{N}^+} \left[0, \frac{1}{i}\right]$ | 0 | NONE |
| $\bigcap_{i \in \mathbb{N}^+} \left]0, \frac{1}{i}\right]$ | EMPTY | |
| $\bigcap_{i \in \mathbb{N}^+} \left[\frac{-1}{i}, \frac{1}{i}\right]$ | EMPTY | |
| $\bigcap_{i \in \mathbb{N}^+} \left]\frac{-1}{i}, 1\right]$ | NONE | 1 |
| $\mathrm{sqrt}([0, 0.1])$ | 0 | sqrt(0.1) |
| $\mathrm{sqrt}[[0, 0.1]]$ | 0 | sqrt(0.1) |
| $\mathrm{sqrt}[]0, 0.1]]$ | none | sqrt(0.1) |
| $\bigcup_{i \in \mathbb{N}^+} \left]\frac{1}{i+1}, \frac{1}{i}\right[$ | EMPTY | EMPTY |
| $\mathrm{inv}(]0, 0.1])$ | 10 | NONE |
| $\mathrm{inv}[]0, 0.1]]$ | | |

$\mathrm{sqrt} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the positive square root function, i.e. for every non-negative real number $a$, $\mathrm{sqrt}(a)$ is the non-negative real number such that $a = \mathrm{sqrt}(a) \cdot \mathrm{sqrt}(a)$.

$\mathrm{inv} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ is the inversion function, defined by $\mathrm{inv} : r \mapsto \frac{1}{r}$.

## 2 [6p]

Define two sets $A$ and $B$, as well as a function $f : A \longrightarrow B$, such that $f$ is **surjective** and **not injective**.

$A =$

$B =$

$f : a \mapsto$

## 3 [10p]

Define **one** set $A$, as well as a function $f : A \longrightarrow A$, such that $f$ is **surjective** and **not injective**.

$A =$

$f : a \mapsto$

## 4 [4p]

Suppose there is a function $f : A \longrightarrow A$ which is surjective and **not** injective, like the one you were asked to define in the previous task. This question is about a property of $A$ (the domain and codomain of $f$) that implies that such a function exists, and which is also implied by the existence of such a function. (You do not need to prove this here.)

A surjective and **not** injective function $f : A \longrightarrow A$ exists <u>if and only if</u>

____________________________________________.

(this must be a statement about the set $A$, and cannot involve $f$)

## 5 [10p]

Recall that a *directed graph* $(V, E)$ is defined as a finite set $V$ of vertices and a relation $E \subseteq V \times V$ between them.

This question is about the properties of that relation. In the table below, make one mark in each row for the property in the left column, depending on whether all, some, or no relations defining a graph have that property. Put the mark in the corresponding ALL box, if **all relations** defining a graph have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

| | ALL | SOME | NONE |
|---|---|---|---|
| reflexive over $V$ | | | X |
| transitive | | X | |
| symmetric | | X | |
| antisymmetric | | X | |
| asymmetric | | X | |

## 6 [10p]

Recall that a *rooted tree* is a graph $(T, R)$ such that, if the set $T$ of nodes is not empty, then there is a node $a \in T$ (the root) such that for every $x \in T$ with $x \neq a$ there is exactly one path from $a$ to $x$. Like $V$ in the previous question, $R \in T \times T$ is a relation on the set of nodes. To make things simpler, for this question we only consider non-empty trees, that is $T \neq \emptyset$.

This question is about the properties of the relations defining trees. In the table below, make one mark in each row for the corresponding property in the left column, depending on whether all, some, or no relations defining a tree have that property. Put the mark in the corresponding ALL box, if **all relations** defining a tree have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

| | ALL | SOME | NONE |
|---|---|---|---|
| reflexive over $T$ | | | X |
| transitive | | | X |
| symmetric | | | X |
| antisymmetric | X | | |
| asymmetric | X | | |

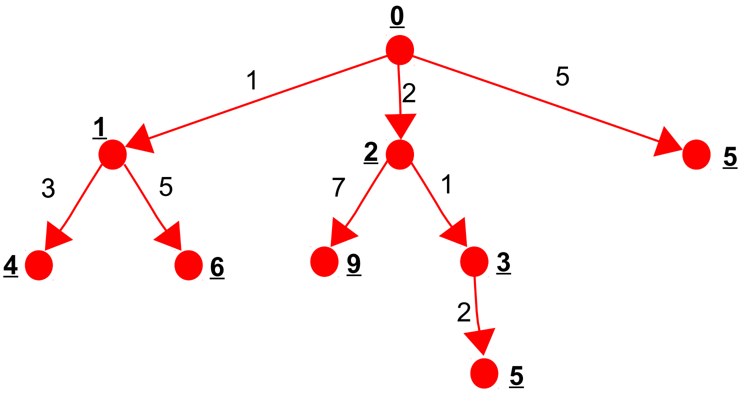
# 7 **[16p]**

Suppose we have a rooted tree $(T, R)$ with nodes $T$, links $R \subseteq T \times T$.

We are also given a function $w : R \longrightarrow \mathbb{N}$ that assigns each *link* a natural number, let's call it the "weight" of that link.

1. [8p] Define a function $W : T \longrightarrow \mathbb{N}$ that maps each node in the tree to its "path weight", that is the sum of the weights of the links on the path from the root to that node.

   <ins>Example:</ins>




   The numbers next to the links are those assigned to them by the (given) function $w : R \longrightarrow \mathbb{N}$. The underlined numbers next to the nodes are those that your function $W : T \longrightarrow \mathbb{N}$ is supposed to compute in the case of this example.

   $$W : n \mapsto$$

2. [4p] Formally (using math, not natural language) define the set $L \subseteq T$ of *leaf nodes* of a tree, i.e. nodes that do not have any children:

   $$L =$$

3. [4p] Formally (again, using math, not natural language) define the set $M \subseteq L$ of leaf nodes in $T$ with the smallest path weight (you may, indeed should, use $L$ from the previous subtask):

   $$M =$$

# 8 [20 p]

Find a DNF for each of the following formulae. Write “none” if a formula has no DNF.

1. [5 p] $\neg((r \barwedge \neg q) \leftrightarrow (p \vee q))$

2. [5 p] $\neg((p \barwedge q) \barwedge (r \barwedge s))$

3. [5 p] $\neg((p \barwedge q) \rightarrow (q \barwedge r))$

4. [5 p] $((p \rightarrow q) \barwedge (q \rightarrow r)) \barwedge (r \rightarrow p)$

# Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k (k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- $R^{-1}$ of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a,b], ]a,b[, ]a,b], [a,b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- $A^*$ for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, $\varepsilon$
- $\sum S$ sum of all elements of $S$
- $\prod S$ product of all elements of $S$
- $\bigcup S$ union of all elements in $S$
- $\bigcap S$ intersection of all elements in $S$
- $\bigcup\limits_{a \in S} E(a)$ , $\bigcap\limits_{a \in S} E(a)$ generalized union / intersection of the sets computed for every a in S