1 [6 p]

Given $A = \{4, 5, 6, 7, 8, 9\}$, suppose we define the following sets
(see the last page for the $\perp$ operator):

$$B = \left\{ \frac{a-b}{a+b} : a, b \in A \right\}$$

$$C = \left\{ \frac{a}{b} : a, b \in A \wedge a \perp b \right\}$$

Give the number of elements in these sets as follows:

1. [3 p] $\#(B) =$ 29

2. [3 p] $\#(C) =$ 22

Note that the question is about the *cardinality* of sets, so the answers are numbers.

2 [8 p]

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a \perp b\}$ compute the following images of R:

1. [2 p] $R(6) =$ {1, 5, 7, 11, 13, 17, 19}

2. [2 p] $R(7) =$ {1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20}

3. [2 p] $R(2) =$ {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

4. [2 p] $R(\{2, 5\}) =$ {1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19}

3 [23 p]

With $A = \{2,3,4,5,6,7\}$, $R = \{(a,b) \in A^2 : a \perp b\}$, and $S = \{(a,b) \in A^2 : a | b\}$. We are looking at the composition $S \circ R$ in this task.

1. [3 p] $\#(S \circ R) =$ 25
2. [3 p] $S \circ R(2) =$ {3, 5, 6, 7}
3. [3 p] $S \circ R(3) =$ {2, 4, 5, 6, 7}
4. [3 p] $S \circ R(6) =$ {5, 7}
5. [3 p] $S \circ R(7) =$ {2, 3, 4, 5, 6}
6. [4 p] $S \circ R$ is ... (circle those that apply)

… reflexive	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

7. [4 p] R is ... (circle those that apply)

… reflexive	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

4 [12 p]

Assume you have a surjection $s : A \twoheadrightarrow B$ and an injection $j : B \hookrightarrow C$.

1. [1 p] Is their composition $j \circ s$ always injective?

   YES NO

2. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A, B, and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not injective.)

   $A = C = \{a, c\}, B = \{b\}$
   $s = \{(a, b), (c, b)\}, j = \{(b, c)\}$
   $j \circ s = \{(a, c), (c, c)\}$

   The composition is not injective, since $j \circ s(a) = j \circ s(c) = c$.

3. [1 p] Is their composition $j \circ s$ always surjective?

   YES NO

4. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A, B, and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not surjective.)

   Same definitions as before. The composition is not surjective because $j \circ s(A) = \{c\} \neq C$.

One mistake that occurred a few times was giving definitions for j and s that were not, in fact, functions at all (either because they weren't unique mappings, or because they omitted values from the domain).

5 [8 p]

As we saw in the lecture on quantificational logic, $\exists x \forall y Rxy$ always implies $\forall x \exists y Rxy$ for any relation *R*. The converse, however, is not necessarily the case: $\forall x \exists y Rxy$ does not always mean that $\exists x \forall y Rxy$ is true.

1. [6 p] Define a binary relation R over a non-empty universe D (that you also need to define) such that $\forall x \exists y Rxy$ is true, and $\exists x \forall y Rxy$ is false.
   Hint: Keep in mind that the $\forall$ and $\exists$ operators are quantified over D.

   D = {a, b}

   R = {(a, a), (b, b)}

2. [2 p] Suppose $D = R = \emptyset$. What are the values of the formulae then?

$\forall x \exists y Rxy$	TRUE	FALSE
$\exists x \forall y Rxy$	TRUE	FALSE

Of course, there are many possible ways of constructing D and R. If they are empty, the first formula in the second part is true, because that is what happens when one universally quantifies over an empty set. Conversely, existentially quantifying over an empty set always yields false.

6 [17 p]

1. [2 p] Suppose we have a set A partially ordered by a strict order $<$. What would you need to show in order to demonstrate that $(A, <)$ is **not** well-founded? (in English/Swedish)

   An infinite descending chain in A. [or]
   A non-empty subset of A without a minimal element.

2. [8 p] The set $\mathcal{P}(\mathbb{Q})$ under strict set inclusion $\subset$ is not well-founded. Show this.

   $$S = \left\{ \left[0, \frac{1}{n}\right] : n \in \mathbb{N}^{+} \right\}$$
   S has no minimal element.

   There are, of course, many ways of constructing an infinite descending chain. Including, incidentally, the one in the fourth part of this problem.

3. [1 p] Is the set $\mathcal{P}(\mathbb{N})$ under strict set inclusion $\subset$ well-founded? (circle answer)

   YES NO

4. [6 p] Prove it or provide a counterexample. (Hint: You can use the bijection between $\mathbb{N}$ and $\mathbb{Q}$ we discussed in the course without having to define it here.)

   $$S = \{\{i : i \in \mathbb{N}^{+}, i > n\} : n \in \mathbb{N}^{+}\}$$
   S has no minimal element.

   The bijection hint was meant for those who wouldn't find a simpler solution like the one above. If you found an answer to the second part of this problem, you already had an infinite descending chain in the rational numbers. With the bijection between those and the natural numbers, you could turn that into an infinite descending chain of sets of natural numbers "for free".

7 [10 p]

Suppose we have a directed graph (V, E). We want to define a function $r : V \longrightarrow \mathcal{P}(V)$ that computes for each vertex the set of vertices that one can reach from it by following zero or more directed edges.

We do this using a helper function $r' : V \times \mathcal{P}(V) \longrightarrow V$ that keeps track of the vertices we have visited already, so that we do not get stuck in cycles. Then r itself simply becomes

$$r(a) = r'(a, \emptyset)$$

The second argument of r' is the set of vertices we have visited already, initially empty.

Your task is to define r' using recursion:

$$r' : (a, S) \mapsto \begin{cases} S & \text{for } a \in S \\ S \cup \{a\} \cup \bigcup_{v \in E(a)} r'(v, S \cup \{a\}) & \text{otherwise} \end{cases}$$

Hint 1: Note that the edge relation E can be used to compute all the nodes that can be reached from a given vertex a in one hop: that set is simply the image of a under E, i.e. $E(a)$.

Hint 2: You might want to recall the notion of a "generalized union", along with the associated notation.

Judging from the responses, this seemed a very difficult problem.

The first case handles the case where we run into a node we have visited already, i.e. at the end of a cycle. In that case, we return all the nodes we have seen so far.

In the second case we deal with a node we haven't encountered before. We return the union of two things: $S \cup \{a\}$, the set of all nodes we have seen so far (plus the current one), and $\bigcup_{v \in E(a)} r'(v, S \cup \{a\})$, the union of all those nodes we can reach from here along the edges going out from the current node (not forgetting to add the current node to the set of "visited nodes" we pass into r').

One might think that this second part would be sufficient, and that we would not need to explicitly add $S \cup \{a\}$ at the beginning, because all the calls to r' eventually run into the first case and will return the set S then. Except they don't: if a node has no outgoing edges, E(a) is the empty set, and no calls to r' will be made. That is why we need to include $S \cup \{a\}$. Alternatively, one could write r' to explicitly distinguish the case where the current node has outgoing arcs from when it doesn't, which would add another case to the two given in the problem.

8 [9 p]

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.
Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$(\exists y(P\underset{\bullet}{y} \lor Q\overset{\bullet}{x}\overset{\bullet}{z}\underset{\bullet}{y})) \to \forall x(R\underset{\bullet}{x}\overset{\bullet}{y} \leftrightarrow \exists y(P\overset{\bullet}{z} \to \forall x(R\overset{\bullet}{z}\underset{\bullet}{x})))$$

bound

9 [15 p]

Find a DNF for each of the following formulae. Write “none” if a formula has no DNF.

1. [5 p] $(r \lor q) \to (q \lor p))$

   $p \lor q \lor (\neg q \land \neg r)$

2. [5 p] $(p \to q) \to ((q \to r) \lor (r \to p))$

   $p \lor \neg q \lor r \lor \neg r \lor (p \land \neg q) = p \lor \neg q \lor r \lor \neg r = \text{true}$

3. [5 p] $(p \to (q \land r)) \lor (q \to (p \land r)) \lor (r \to (p \land q))$

   $\neg p \lor \neg q \lor \neg r \lor (p \land q) \lor (p \land r) \lor (q \land r)$

The last two DNFs evaluate to TRUE. That does not mean they don't exist, only that they aren't very interesting. For example, $p \lor \neg q \lor r \lor \neg r$ is a perfectly respectable DNF, even if it is obviously always TRUE.

Either way, responses that said that there was no DNF were scored incorrect. One exception was a response that said there was no DNF because the formulae were TRUE. I gave points for that one.

Also, any response that wasn't a DNF was marked incorrect.