

## specifying sets

enumeration of its elements

$\{red, green, blue\}$

set builder notation / set comprehensions

flavor 1

$\{x \in \mathbb{N}^+ : x \text{ is prime}\}$

$\{x \in \mathbb{N}^+ | x \text{ is prime}\}$

flavor 2

$\{x : x \in \mathbb{N}^+, x \text{ is prime}\}$

$\{2x : x \in \mathbb{N}^+\}$

bad flavor

$\{(x : x \notin x)\}$

$\{(x : x = x)\}$

recursive definition

(we will discuss this later)

enumeration w/ suspension points/ellipsis

$\{1, 2, 3, 4, 5, \dots\}$

(informal stand-in for a recursive definition)



$12 \in \{2, 3, 5, 8, \dots\}$  ?



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## set algebra

some properties of intersection, union, and set difference:

idempotence  $A \cup A = A \cap A = A$

commutativity  $A \cup B = B \cup A$

commutativity  $A \cap B = B \cap A$

associativity  $(A \cup B) \cup C = A \cup (B \cup C)$

associativity  $(A \cap B) \cap C = A \cap (B \cap C)$

$A \cup \emptyset = A$

$A \cap \emptyset = \emptyset$

$A \supseteq A \cap B \subseteq B$

$A \subseteq A \cup B \supseteq B$

distributivity  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

distributivity  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cap B \subseteq A \cup B$

$A \setminus A = \emptyset$

$A \setminus \emptyset = A$

$-(-B) = B$

(more in the exercises of  
1.4.1, 1.4.2, and 1.4.3 in SLAM)

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## terminology: source, target, domain, range

For binary relations  $R \subseteq A \times B$  :

$A$  is a source.

$B$  is a target.

Note that for any  $R$ , source and target are *not uniquely determined*:

$R \subseteq A \times B$

For any  $A' \supseteq A$  and  $B' \supseteq B$ , we have  $A \times B \subseteq A' \times B'$ .

$R \subseteq A \times B \subseteq A' \times B'$

By contrast, these are uniquely determined:

the domain of  $R$ :  $\text{dom}(R) = \{a : (a, b) \in R \text{ for some } b\}$

the range of  $R$ :  $\text{range}(R) = \{b : (a, b) \in R \text{ for some } a\}$

For any relation  $R \subseteq A \times B$  it is always the case that

$\text{dom}(R) \subseteq A$  and  $\text{range}(R) \subseteq B$

## converse vs complement

$R = \{(a, a), (a, b), (b, b), (b, c), (c, a), (c, c)\}$

R	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

$R^{-1} = \{(a, a), (a, c)(b, a), (b, b), (c, b), (c, c)\}$

R	a	b	c
a	1	0	1
b	1	1	0
c	0	1	1

converse: mirror at the diagonal

$\bar{R} = \{(a, c), (b, a), (c, b)\}$

R	a	b	c
a	0	0	1
b	1	0	0
c	0	1	0

complement: flip zeros and ones

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## properties: a(anti)symmetry

Consider  $\leq$  and  $<$  on the natural numbers. Neither is symmetric, but in slightly different ways.

For  $<$ , it is **never** the case that  $a < b$  and  $b < a$ .

This is called **asymmetry**.

For  $\leq$ , it sometimes is, but only when  $a = b$ .

This is called **antisymmetry**.

Both relations are antisymmetric. Only  $<$  is asymmetric.

A binary relation  $R \subseteq A \times A$  is **asymmetric** iff for all  $a, b \in A$   
if  $aRb$  then not  $bRa$

A binary relation  $R \subseteq A \times A$  is **antisymmetric** iff for all  $a, b \in A$   
if  $aRb$  and  $bRa$  then  $a = b$

## equivalence relations

A binary relation  $\approx \subseteq A \times A$  is an **equivalence relation** iff it is

1. reflexive
2. symmetric
3. transitive

What about these:

- equality
- divides:  $m | n$  iff there is  $k \geq 1 : km = n$
- relatively prime:  $m \perp n$  iff there is no  $k \geq 2 : k | m$  and  $k | n$



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## equivalence relations

A binary relation  $\approx \subseteq A \times A$

1. reflexive
2. symmetric
3. transitive

equality: yes, as it is reflexive, symmetric, and transitive.  
divides: no, because it is not symmetric;  
    2 divides 4, but 4 does not divide 2.  
relatively prime: no, it is neither reflexive (3 is not  
    relatively prime to 3), nor transitive (2 is relatively  
    prime to 5, 5 to 6, but not 2 to 6).

What about these:

- equality
- divides:  $m | n$  iff there is  $k \geq 1 : km = n$
- relatively prime:  $m \perp n$  iff there is no  $k \geq 2 : k | m$  and  $k | n$



## partitions

Given a set  $A$ , a **partition** of  $A$  is a set of pairwise disjoint sets

$\{B_i : i \in I\}$ , such that

$$A = \bigcup_{i \in I} B_i$$



A: EU citizens, I: EU member states,  $B_i$ : citizens of country  $i$

A: atoms, I: elements,  $B_i$ : atoms of element  $i$

A: natural numbers, I: primes,  $B_i$ : multiples of  $i$  (excluding  $i$ )



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## partitions

Given a set  $A$ , a partition  
 $\{B_i : i \in I\}$ , such that

- EU citizens: no, since a EU citizen may be a citizen of more than one country
- atoms: yes, all atoms belong to exactly one element
- natural numbers: no, 1. the multiple of different primes are not disjoint, and 2. without the primes themselves their union does not include all natural numbers.



- $A$ : EU citizens,  $I$ : EU member states,  $B_i$ : citizens of country  $i$
- $A$ : atoms,  $I$ : elements,  $B_i$ : atoms of element  $i$
- $A$ : natural numbers,  $I$ : primes,  $B_i$ : multiples of  $i$  (excluding  $i$ )

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## equivalence class, quotient set

Equivalence relations can be used to generate partitions.

Given a set  $A$  and an equivalence relation  $\approx$  on  $A$ , for any  $a \in A$  we define the equivalence class of  $a$   $[a]_{\approx}$  as  $[a]_{\approx} = \{b \in A : a \approx b\}$

Alternative syntax:

SLAM  $\left\{ \begin{array}{l} [a] \\ |a| \\ |a|_{\approx} \end{array} \right\}$  when the relation is understood

Given a set  $A$  and an equivalence relation  $\approx$  on  $A$ , the quotient (set)  $A/\approx$  is defined as  $A/\approx = \{|a|_{\approx} : a \in A\}$

SLAM 2.5.4: 1. Every partition is the quotient of an equivalence relation.

2. Every quotient set is a partition.



Review the proof in the book. Connect it to these definitions.

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## order relation, poset

A binary relation  $\preceq \subseteq A \times A$  is an (inclusive or non-strict) (partial) order iff it is

1. reflexive
2. antisymmetric
3. transitive



What about these:

- divides:  $m \mid n$  iff there is  $k \geq 1 : km = n$
- set inclusion:  $\subseteq$
- on numbers:  $\leq$  and  $<$
- proper set inclusion:  $\subset$

- divides: yes
- set inclusion: yes
- less than or equal to: yes
- less than: no
- proper set inclusion: no

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## order relation, poset

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- on numbers:  $\leq$  and  $<$
- proper set inclusion:  $\subset$

A pair  $(A, \preceq)$  where  $A$  is a set and  $\preceq \subseteq A \times A$  a partial order on  $A$  is called a partially ordered set or poset.

Examples:  $(\mathbb{N}^+, |)$   
 $(\mathcal{P}(A), \subseteq)$

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## strict (partial) order

A binary relation  $\prec \subseteq A \times A$  is a *strict (partial) order* iff it is

1. irreflexive
2. transitive

Note: Irreflexivity and transitivity imply asymmetry.



How?

- irreflexivity:  $a \not\prec a$
- transitivity: if  $a \prec b$  and  $b \prec c$  then  $a \prec c$
- asymmetry: if  $a \prec b$  then  $b \not\prec a$

## total (or linear) order

A binary relation  $\preceq \subseteq A \times A$  is a *(non-strict) total (or linear) order* iff it is

1. reflexive
2. antisymmetric
3. transitive
4. total (complete):  $a \preceq b$  or  $b \preceq a$

divides: no, not total  
set inclusion: no, not total  
less then or equal: yes  
less than: no, not reflexive

What about these:



- divides:  $m | n$  iff there is  $k \geq 1 : km = n$
- set inclusion:  $\subseteq$
- on numbers:  $\leq$  and  $<$

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## transitive closure

The *transitive closure*  $R^+$  of a binary relation  $R \subseteq A \times A$  is defined as follows:

$$R^+ = \bigcup_{i \in \mathbb{N}} R_i \text{ with } R^* \text{ alternative syntax (SLAM)}$$

$$R_0 = R$$

$$R_{n+1} = R_n \cup \{(a, c) : \text{if } aR_nb \text{ and } bR_nc \text{ for some } b \in A\}$$

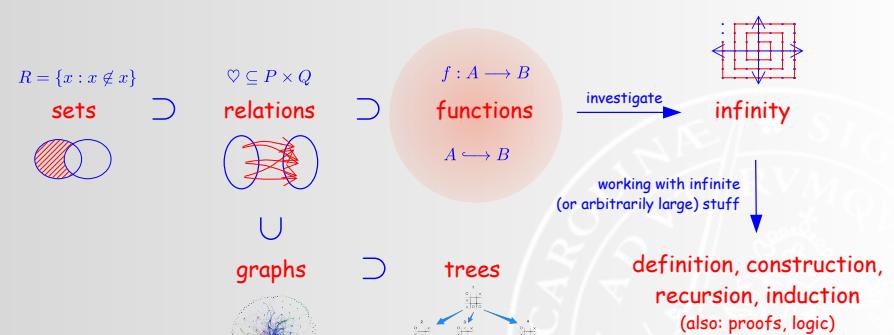
$$\bowtie = \{(F, E), (E, F), (F, B), (B, F), (F, D), (D, F), (F, CH), (CH, F), (F, I), (I, F), (B, NL), (NL, B), (B, D), (D, B), (D, NL), (NL, D), (D, CH), (CH, D), (CH, I), (I, CH), (GB, IRL), (IRL, GB)\}$$



What is the meaning of  $\bowtie^+$ ?  
What are its properties?



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## domain, range, codomain

When talking about functions, some of the terminology is the same as for relations in general, some is not:

	$R \subseteq A \times B$	$f : A \rightarrow B$
actual values, left	domain	domain
actual values, right	range	range
A	source	domain
B	target	codomain

$$\text{dom}(f) = A$$

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## about $B^A$ ...

Why  $B^A$  ?

f	1	2	3	4
a	1	0	0	0
b	0	0	1	0
c	1	0	0	0

we make a choice  
for each of these  $\#(A)$  times

← each time, we choose from these  $\#(B)$  options

$$\underbrace{\#(B) \cdot \dots \cdot \#(B)}_{\#(A) \text{ times}} = \#(B)^{\#(A)}$$

So for the number of functions from A to B, we have  $\#(B^A) = \#(B)^{\#(A)}$ .



How many relations  $R \subseteq A \times B$  ?



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## about $B^A$ ...

Why  $B^A$  ?

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← each time, we chose from these  $\#(B)$  options

$$\underbrace{\#(B) \cdot \dots \cdot \#(B)}_{\#(A) \text{ times}} = \#(B)^{\#(A)}$$

we make a choice  
for each of these  $\#(A)$  times

So for the number of

This question asks for the number of subsets of  $A \times B$ ,  
i.e.  
 $\#P(A \times B) = 2^{\#(A \times B)} = 2^{\#A \#B}$



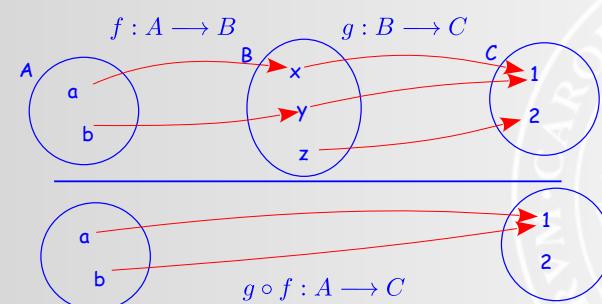
How many relations  $R \subseteq A \times B$  ?

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## composition

Given functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$  their composition  
 $g \circ f : A \rightarrow C$  defined as:  
 $g \circ f(a) = g(f(a))$

Function composition is just a special case of composition of relations:



It is associative:  
 $(h \circ g) \circ f = h \circ (g \circ f)$

So we can omit the parentheses and write  
 $h \circ g \circ f$

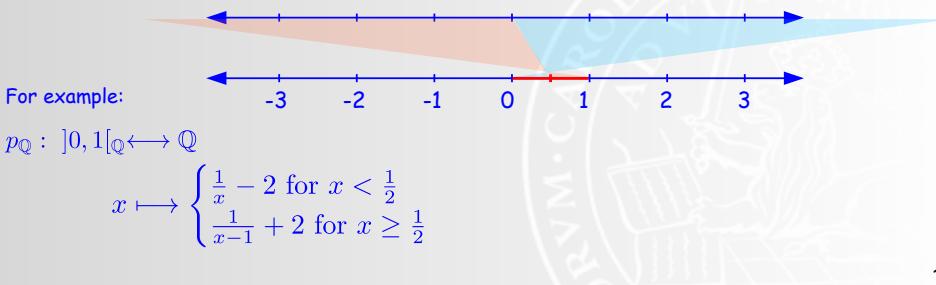
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## $\mathbb{Q}$ : the rational numbers

Some properties:

1.  $\mathbb{Q}$  is *dense*: between any two distinct  $r, s \in \mathbb{Q}$  there is  $\frac{r+s}{2} \in \mathbb{Q}$

2. Any non-empty open interval  $]r, s[ \subset \mathbb{Q}$  is equinumerous to  $\mathbb{Q}$



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## function definitions

What, exactly, is the relationship between the  $f : x \mapsto \dots$  style of defining a function, and concept of a function as a set of pairs?

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto 2x$$

$$f = \{(x, 2x) : x \in \mathbb{R}\}$$

$$g : \mathbb{R} \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

$$\{ (x, x) : x \in \mathbb{R}, x \geq 0 \}$$

$$\{ (x, -x) : x \in \mathbb{R}, x < 0 \}$$

$$g = \{ (x, x) : x \in \mathbb{R}, x \geq 0 \} \cup \{ (x, -x) : x \in \mathbb{R}, x < 0 \}$$

## recursive function definitions (v 1.0)

$$f : \mathbb{N} \rightarrow \mathbb{N}$$

$$n \mapsto \begin{cases} 1 & \text{for } n = 0 \\ n \cdot f(n-1) & \text{for } n > 0 \end{cases}$$

$$f = \bigcup_{i \in \mathbb{N}} f_i$$

$$f_0 = \emptyset$$

$$f_{i+1} = f_i \cup \{ (0, 1) \} \cup \{ (n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0 \}$$

$$S_0 = X$$

$$S_{k+1} = S_k \cup F(S_k)$$

$$S = \bigcup_{k \in \mathbb{N}} S_k$$



Note that  $f_i$  is a relation, so  $f_i(n-1)$  computes the image of  $n-1$  under  $f_i$ , which is a set of values. If it is empty, nothing bad happens, there just is no value for  $v$ .

## recursive function definitions

$$f = \bigcup_{i \in \mathbb{N}} f_i$$

$$f_0 = \emptyset$$

$$f_{i+1} = f_i \cup \{ (0, 1) \} \cup \{ (n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0 \}$$

$i$	$f_i$	$\{(n, n \cdot v) : n \in \mathbb{N}, v \in f_i(n-1), n > 0\}$
0	$\emptyset$	$\emptyset$
1	$\{ (0, 1) \}$	$\{ (1, 1) \}$
2	$\{ (0, 1), (1, 1) \}$	$\{ (1, 1), (2, 2) \}$
3	$\{ (0, 1), (1, 1), (2, 2) \}$	$\{ (1, 1), (2, 2), (3, 6) \}$
4	$\{ (0, 1), (1, 1), (2, 2), (3, 6) \}$	$\{ (1, 1), (2, 2), (3, 6), (4, 24) \}$

## closure under relation

Given a relation  $R \subseteq A \times A$  and a set  $X \subseteq A$ , the closure of  $X$  under  $R$   $R[X]$  is defined as the smallest  $Y \subseteq A$  such that  $X \subseteq Y$  and  $R(Y) \subseteq Y$

**Construction:**

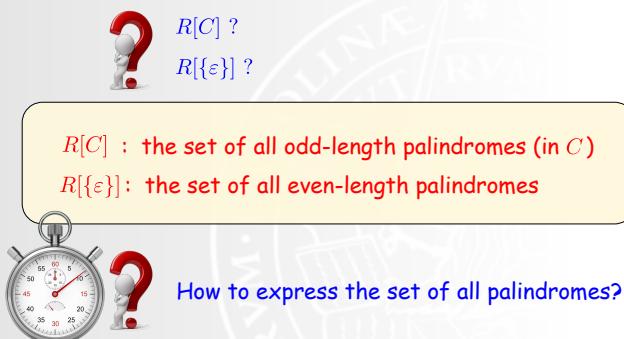
$$\begin{aligned} Y_0 &= X \\ Y_{n+1} &= Y_n \cup R(Y_n) \\ R[X] &= \bigcup_{i \in \mathbb{N}} Y_i \end{aligned}$$

**Example:**

$$C = \{"a", \dots, "z"\}$$

$$R \subseteq C^* \times C^*$$

$$R = \{(s, asa) : s \in C^*, a \in C\}$$



## structural recursion

$$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$$

$$V = \{"a", \dots, "z"\}^* \setminus \{\epsilon\}$$

$$\begin{aligned} R_1 &= \{(s, "-" s) : s \in C^*\} \\ R_2 &= \{(s_1, s_2, "(" s_1 "+" s_2 ")") : s_1, s_2 \in C^*\} \\ R_3 &= \{(s_1, s_2, "(" s_1 "*" s_2 ")") : s_1, s_2 \in C^*\} \end{aligned}$$

Let's write a function that evaluates an expression in  $R[V]$ .

Assume a function  $E : V \rightarrow \mathbb{R}$  that assigns every variable a value.

$$\text{eval}_E : R[V] \rightarrow \mathbb{R}$$

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ -\text{eval}_E(s') & \text{for } s' \in R[V], s = "-" s' \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = "(" s_1 "+" s_2 ")" \\ \text{eval}_E(s_1) \cdot \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = "(" s_1 "*" s_2 ")" \end{cases}$$

Note how the structure is decomposed (or deconstructed) in these clauses!



Write a function that returns for every expression  $s$  the set of variables that occur in it.

## closure under relations, rules, generators

Given a set  $A$ , a family of relations on  $A$   $R = \{R_i \subseteq A^{n_i} : i \in I\}$  and a set  $X \subseteq A$ , the closure of  $X$  under  $R$   $R[X]$  is defined as the smallest  $Y \subseteq A$  such that

$$X \subseteq Y \text{ and } R_i(Y^{n_i-1}) \subseteq Y \text{ for all } i \in I$$

**Construction:**

$$\begin{aligned} Y_0 &= X \\ Y_{n+1} &= Y_n \cup \bigcup_{i \in I} R_i(Y_n^{n_i-1}) \\ R[X] &= \bigcup_{i \in I} Y_i \end{aligned}$$

The elements of  $R$  are also called *rules, constructors, generators*.

$R[V]$  is the set of all expressions with unary - and binary + and \*, and the variable symbols in  $V$

**Example:**

$$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$$

$$R_1 = \{(s, "-" s) : s \in C^*\}$$

$$R_2 = \{(s_1, s_2, "(" s_1 "+" s_2 ")"") : s_1, s_2 \in C^*\}$$

$$R_3 = \{(s_1, s_2, "(" s_1 "*" s_2 ")"") : s_1, s_2 \in C^*\}$$

$$V = \{"a", \dots, "z"\}^* \setminus \{\epsilon\}$$



## unique decomposability

$$R = \{R_1, R_2\}, C = \text{UTF-16}$$

$$V = \{"a", \dots, "z"\}^* \setminus \{\epsilon\}$$

$$R_1 = \{(s_1, s_2, s_1 "+" s_2) : s_1, s_2 \in C^*\}$$

$$R_2 = \{(s_1, s_2, s_1 "-" s_2) : s_1, s_2 \in C^*\}$$

Let's look at a variant of the previous example:

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = s_1 "+" s_2 \\ \text{eval}_E(s_1) - \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = s_1 "-" s_2 \end{cases}$$



What is the problem with this function definition?

It does not uniquely decompose a string such as "a - b + c - d".

## unique decomposability

Suppose we have generators  $R = \{R_1, \dots, R_k\}$  with  $R_k \subseteq A^{n_k+1}$ , and basis  $X \subseteq A$ ,

The general form of a recursive function  $f : R[X] \rightarrow V$  is

$$f : x \mapsto \begin{cases} h_X(x) & \text{for } x \in X \\ h_1(f(x_1), \dots, f(x_{n_1})) & \text{for } x_i, x \in R[X], (x_1, \dots, x_{n_1}, x) \in R_1 \\ \dots & \dots \\ h_k(f(x_1), \dots, f(x_{n_k})) & \text{for } x_i, x \in R[X], (x_1, \dots, x_{n_k}, x) \in R_k \end{cases}$$

It is only well-defined if all  $x \in R[X]$  are uniquely decomposable.



cf. section 4.6.3 in SLAM.

## the idea

$$A(m, n) = \begin{cases} n + 1 & \text{for } m = 0 \\ A(m - 1, 1) & \text{for } m > 0, n = 0 \\ A(m - 1, A(m, n - 1)) & \text{otherwise} \end{cases}$$

We need to "order" the elements in domains such as  $R[V]$  and  $\mathbb{N} \times \mathbb{N}$  in such a way that

- (a) recursive calls are always made with "smaller" argument values than the call received that they are made from, and
- (b) from any possible argument value, you cannot "descend" forever --- eventually, the function will call itself with a value for which there is no "smaller" value to recursively call itself with.

$$\text{eval}_E : R[V] \rightarrow \mathbb{R}$$

$$\text{eval}_E : s \mapsto \begin{cases} E(s) & \text{for } s \in V \\ -\text{eval}_E(s') & \text{for } s' \in R[V], s = "\neg" s' \\ \text{eval}_E(s_1) + \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = "(" s_1 "+" s_2 ")" \\ \text{eval}_E(s_1) \cdot \text{eval}_E(s_2) & \text{for } s_1, s_2 \in R[V], s = "(" s_1 "*" s_2 ")" \end{cases}$$

## well-founded sets

A poset  $(A, \leq)$  is well-founded iff all non-empty subsets  $X \subseteq A$  have a minimal element, i.e.  
for some  $m \in X$  and all  $a \in X, a \not\leq m$

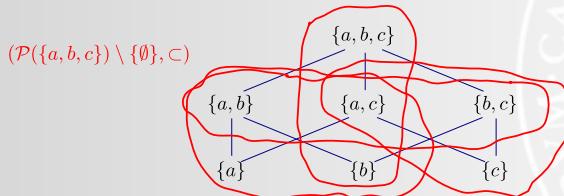
Intuitively, this means there are no infinite descending chains:

$$\dots < a_n < a_{n-1} < \dots < a_2 < a_1 < a_0$$



minimal element != minimum element:

- minimal means that there is no smaller element
- minimum means all other elements are greater



## well-founded sets

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for some  $m \in X$  and all  $a \in X, a \not\leq m$

$(\mathbb{N}, \leq)$	✓
$(\mathbb{Z}, \leq)$	✗
$(\mathbb{Q}^+, \leq)$	✗
$(P(\mathbb{N}), \subseteq)$	✗
$(\mathbb{N}^2, \leq_{\text{lex}})$	✓
$(\mathbb{N}^2, \leq_{\text{prod}})$	✓

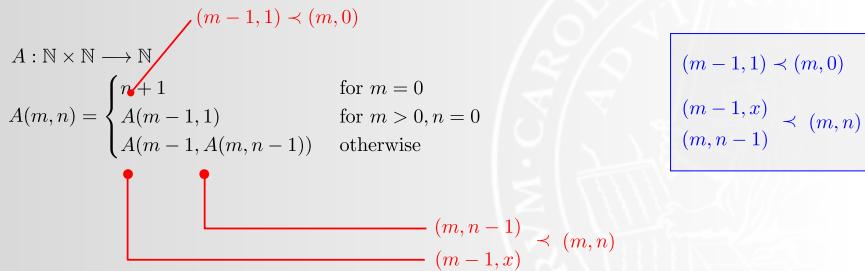
$$(a_1, a_2) <_{\text{prod}} (b_1, b_2) \text{ iff } a_1 < b_1 \text{ and } a_2 < b_2$$

$$(a_1, a_2) <_{\text{lex}} (b_1, b_2) \text{ iff } a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 < b_2)$$

## well-founded recursion

Given a well-founded set  $(W, <)$  and a recursive function definition for a function  $f : W \rightarrow X$ ,  $f$  is **well-defined** if it computes the value for every  $w \in W$  only depending on values of  $f$  for  $v < w$ .

So how do we use this in practice? How can we tell that the Ackermann function is well-defined?



## well-founded recursion

Given a well-founded set  $(W, <)$  and a recursive function definition for a function  $f : W \rightarrow X$ ,  $f$  is **well-defined** if it computes the value for every  $w \in W$  only depending on values of  $f$  for  $v < w$ .

$$(a_1, a_2) <_{\text{lex}} (b_1, b_2) \text{ iff } a_1 < b_1 \text{ or } (a_1 = b_1 \text{ and } a_2 < b_2)$$



Which order would make  $\mathbb{N} \times \mathbb{N}$  well-founded, such that the inequalities on the right would be true?

$$\begin{aligned} (m-1, 1) &\prec (m, 0) \\ (m-1, x) &\prec (m, n) \\ (m, n-1) &\prec (m, n) \end{aligned}$$



Now take a look at the "buggy" version of the Ackermann function, and track down, how it would fail to meet the requirements of well-founded recursion.

## a simple inductive proof

$$\text{sum}(n) = \sum_{i=1 \dots n} i = 1 + \dots + n$$

Hypothesis:  $\text{sum}(n) = \frac{n(n+1)}{2}$



Basis:  $\text{sum}(1) = \frac{1(1+1)}{2} = 1$



Induction step:

induction hypothesis

Assuming that  $\text{sum}(k) = \frac{k(k+1)}{2}$   
 show that  $\text{sum}(k+1) = \frac{(k+1)(k+2)}{2}$

induction goal

$$\begin{aligned} \text{sum}(k+1) &= \text{sum}(k) + (k+1) && \text{def sum()} \\ &= \frac{k(k+1)}{2} + (k+1) && \text{IH} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

## a simple induction principle

Hypothesis:  $P(n)$  for all  $n$

Basis: Show that  $P(1)$  (or  $P(0)$  or some other, depending on circumstances)

Induction step: Assuming that  $\underbrace{P(k)}$  show that  $\underbrace{P(k+1)}$

induction hypothesis induction goal

Things that often go wrong:

- mixing basis and induction step: do not try to do everything at once
- confusing induction hypothesis and induction goal
- not using the induction hypothesis: it ain't cheating!
- getting lost: proving the goal can be messy, keep eyes on prize



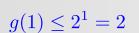
## cumulative (complete) induction

Cumulative, also *complete* or *strong*, induction uses an induction hypothesis that assumed the truth of the hypothesis for all smaller values, instead of just the previous one.

Hypothesis:  $g(n) \leq 2^n$



Basis:  $g(0) \leq 2^0 = 1$



$g(1) \leq 2^1 = 2$

Induction step: induction hypothesis

Assuming that  $g(m) \leq 2^m$  for all  $m < k$

show that  $g(k) \leq 2^k$

induction goal

$$g(n) = \begin{cases} 1 & \text{for } n \leq 1 \\ g(n-1) + g(n-2) & \text{otherwise} \end{cases}$$

$$\begin{aligned} g(k) &= g(k-1) + g(k-2) && \text{def } g() \\ &\leq 2^{k-1} + 2^{k-2} && \text{IH} \\ &= \frac{1}{2}2^k + \frac{1}{4}2^k \\ &\leq 2^k \end{aligned}$$

## cumulative induction principle

Hypothesis:  $P(n)$

Basis:  $P(0), \dots$

Induction step:

Assuming that  $\underbrace{P(m) \text{ for all } m < k}_{\text{induction hypothesis}}$  show that  $\underbrace{P(k)}_{\text{induction goal}}$

Note that the basis is the vacuous form of the induction step, for  $k=0$ . As a result, the basis is subsumed by the induction step. In practice, it is often treated separately.

## structural induction

$$R = \{R_1, R_2, R_3\}, C = \text{UTF-16}$$

$$V = \{"a", \dots, "z"\}^* \setminus \{\varepsilon\}$$

$$R_1 = \{(s, "-" s) : s \in C^*\}$$

$$R_2 = \{(s_1, s_2, "(" s_1 "+" s_2 ")") : s_1, s_2 \in C^*\}$$

$$R_3 = \{(s_1, s_2, "(" s_1 "*" s_2 ")") : s_1, s_2 \in C^*\}$$

Hypothesis: Every  $s \in R[V]$  contains equal numbers of opening and closing parentheses.



Basis: Every  $s \in V$  contains equal numbers of opening and closing parentheses.



Induction step: induction hypothesis

All rules  $R_i \in R$  preserve the property: if their "input" objects have it, then so does their "output" object.

induction goal

$$R_1 = \{(s, "-" s) : s \in C^*\}$$

$$R_2 = \{(s_1, s_2, "(" s_1 "+" s_2 ")") : s_1, s_2 \in C^*\}$$

$$R_3 = \{(s_1, s_2, "(" s_1 "*" s_2 ")") : s_1, s_2 \in C^*\}$$



## structural induction principle

Structural induction is a variant of cumulative induction, but instead of natural numbers we induce over the structure of rule-generated objects in some structurally-recursive set  $R[X]$ .

Hypothesis:  $P(x)$  for all  $x \in R[X]$

Basis:  $P(x)$  for all  $x \in X$

Induction step:

All rules  $R_i \in R$  preserve the property: if their "input" objects have it, then so does their "output" object.

induction goal

induction hypothesis

## well-founded induction

Cumulative (complete, strong) induction assumed that a property needed to be shown over the natural numbers, structural induction that the set was generated inductively through generators.

Well-founded induction generalizes the idea to all well-founded sets.

Given a well-founded set  $(A, <)$  and a property  $P(a), a \in A$ , if

for all  $a \in A : P(w)$  for all  $w < a$  implies  $P(a)$   
then  $P(a)$  for all  $a \in A$

induction hypothesis      induction goal



Emmy Noether  
1882-1935

As with cumulative induction, the base case is subsumed by the induction step.  
In practice, it is still sometimes handled separately.

## well-founded induction

Hypothesis: Every  $n \geq 2$  can be factored into primes.

well-founded order:  $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$  (divides-relation, strict version)



Descending chains?  
Minimal elements?

Descending chains look like this:  $7|_{\neq} 21|_{\neq} 42|_{\neq} 126|_{\neq} 252|_{\neq} \dots$

The minimal elements are the primes.

## well-founded induction

Hypothesis: Every  $n \geq 2$  can be factored into primes.

well-founded order:  $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$  (divides-relation, strict version)

(Base:) Trivially true for every minimal element in  $(\{n \in \mathbb{N} : n \geq 2\}, |_{\neq})$

Induction step: If  $n$  is not minimal, then there is a  $k$  such that  $k|_{\neq} n$ , and thus there is some  $m$  such that  $n = km$ .  $k$  and  $m$  can be prime-factored, therefore so can  $n$ .

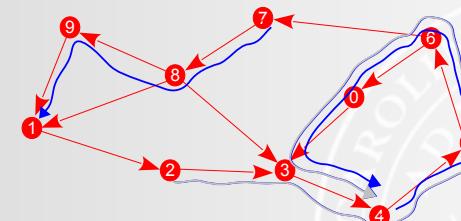


How do we know that  $m$  can be factored into primes?

Because  $m|_{\neq} n$ , too.

## paths

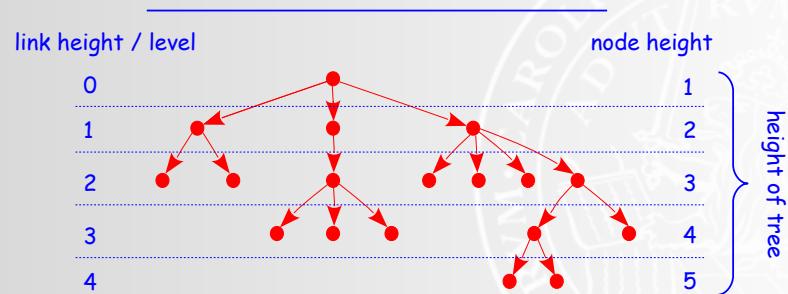
Given a graph  $(V, E)$ , a path is a finite sequence  $a_0, \dots, a_n$  in  $V$  with  $n \geq 1$  such that  $(a_{k-1}, a_k) \in E$  for  $1 \leq k \leq n$ . The length of the path is  $n$ .



A cycle is a path  $a_0, \dots, a_n$  where  $a_0 = a_n$ .  
A graph that does not contain cycles is called acyclic.

## properties

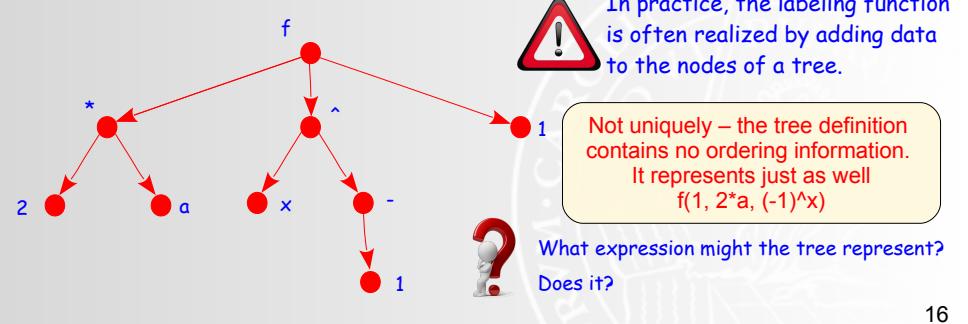
- (1) Any non-empty tree has a unique root.
- (2) A root has no parent.
- (3) Every non-root has exactly one parent.
- (4) A tree with  $n$  nodes has  $n-1$  links.
- (5) A tree contains no cycles.



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## labeled trees

Given a tree  $(T, R)$ , and a set of *labels*  $L$ , a *labeling* is a function  $\lambda : T \rightarrow L$   
A tree with a labeling is called a *labeled tree*.

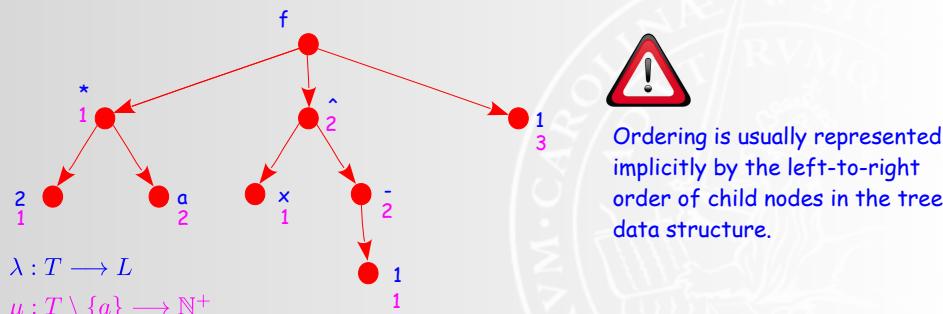


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## ordered trees

Given a tree  $(T, R)$  with root  $a$ , we say it is *ordered* if there is a function  
 $\mu : T \setminus \{a\} \rightarrow \mathbb{N}^+$

such that for every node its  $n$  children are labeled  $1..n$ .



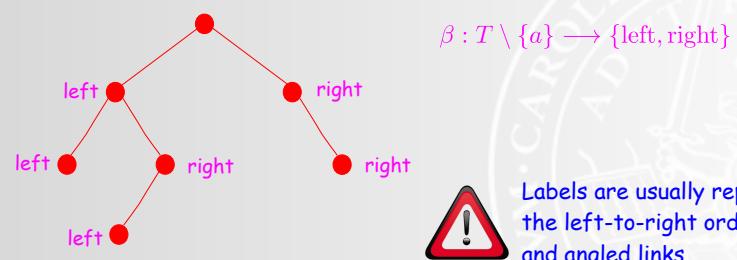
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## binary trees

Given a tree  $(T, R)$  with root  $a$ , we say it is *binary* if every node has at most two children and there is a labeling function

$\beta : T \setminus \{a\} \rightarrow \{\text{left, right}\}$

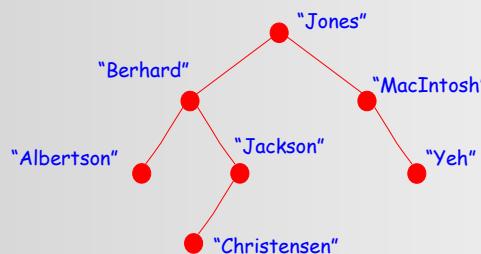
such that no two children of the same node are labeled identically.



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## binary search trees

Given a binary tree  $(T, R)$ , with root  $\alpha$ , binary labels  $L \subseteq \{left, right\}$ . It could't. →  $\{\text{left}, \text{right}\}$   
and labeling function  $\lambda : T \rightarrow L$  and a total order  $\leq$  on  $L$ .  
The labels of an ordered tree establish a total order among the children of each node.  
It is a *binary search tree* iff for all nodes  $\alpha$ :  $\lambda(\alpha) = \text{left} \Leftrightarrow \forall \beta \in \text{left subtree}(\alpha) \text{ s.t. } \lambda(\beta) = \text{right} \Rightarrow \beta < \alpha$   
The labels in a binary tree also relate any label in their left subtree, and less than or equal to the label in their right subtree.



Could the same be achieved with an ordered tree, instead of a binary tree?

IOW, is a binary tree a special case of an ordered tree, or something else?

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## from truth table to formula

p	q	r	$f(p, q, r)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

←  $p \wedge q \wedge r$

←  $p \wedge \neg q \wedge r$



Are there truth tables for which this method does not produce a formula?

What does this mean for the connectives not, and, or?

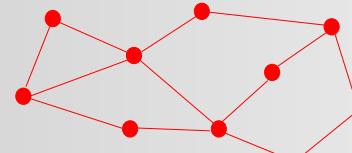
Only the one where always  $f(\dots) = 0$ . This can be expressed as  $f(p, q, r) = p \wedge \neg p$ .

$$f : (p, q, r) \mapsto (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$$

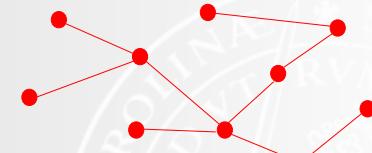
It means these connectives are universal. All propositional formulae can be expressed using not, and, or.

## spanning trees

Given an undirected graph  $(T, S)$ , an unrooted tree  $(T, R)$  is a *spanning tree* for it iff  $R \subseteq S$



There may be many spanning trees for any given graph.



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## tautological implication

Given a set of formulae  $A$  and a formula  $\beta$  we say that  $A$  tautologically implies  $\beta$  if there is no valuation  $v$  such that

$$v(\alpha) = 1 \text{ for all } \alpha \in A \text{ and } v(\beta) = 0$$

$$\begin{aligned} A \vdash \beta \\ A \models \beta \end{aligned}$$

Example:  
(modus ponens)  $p, p \rightarrow q \vdash q$

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Name	LHS	RHS
Simplification, $\wedge^-$	$\alpha \wedge \beta$	$\alpha$
	$\alpha \wedge \beta$	$\beta$
Conjunction, $\wedge^+$	$\alpha, \beta$	$\alpha \wedge \beta$
Disjunction, $\vee^+$	$\alpha$	$\alpha \vee \beta$
	$\beta$	$\alpha \vee \beta$
Modus ponens, MP, $\rightarrow^-$	$\alpha, \alpha \rightarrow \beta$	$\beta$
Modus tollens, MT	$\neg \beta, \alpha \rightarrow \beta$	$\neg \alpha$
Disjunctive syllogism, DS	$\alpha \vee \beta, \neg \alpha$	$\beta$
Transitivity	$\alpha \rightarrow \beta, \beta \rightarrow \gamma$	$\alpha \rightarrow \gamma$
Material implication	$\beta$	$\alpha \rightarrow \beta$
	$\neg \alpha$	$\alpha \rightarrow \beta$
Limiting cases	$\gamma$	$\beta \vee \neg \beta$
	$\alpha \wedge \neg \alpha$	$\gamma$

SLAM, p. 259

## tautological equivalence

Given two formulae  $\alpha$  and  $\beta$ , we say that they are *tautologically equivalent* if they tautologically imply each other.  $\alpha \dashv\vdash \beta$

Name	LHS	RHS
Double negation	$\alpha$	$\neg\neg\alpha$
Commutation for $\wedge$	$\alpha \wedge \beta$	$\beta \wedge \alpha$
Association for $\wedge$	$\alpha \wedge (\beta \wedge \gamma)$	$(\alpha \wedge \beta) \wedge \gamma$
Commutation for $\vee$	$\alpha \vee \beta$	$\beta \vee \alpha$
Association for $\vee$	$\alpha \vee (\beta \vee \gamma)$	$(\alpha \vee \beta) \vee \gamma$
Distribution of $\wedge$ over $\vee$	$\alpha \wedge (\beta \vee \gamma)$	$(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
Distribution of $\vee$ over $\wedge$	$\alpha \vee (\beta \wedge \gamma)$	$(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
Absorption	$\alpha$	$\alpha \wedge (\alpha \vee \beta)$
	$\alpha$	$\alpha \vee (\alpha \wedge \beta)$
Expansion	$\alpha$	$(\alpha \wedge \beta) \vee (\alpha \wedge \neg\beta)$
	$\alpha$	$(\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$
de Morgan	$\neg(\alpha \wedge \beta)$	$\neg\alpha \vee \neg\beta$
	$\neg(\alpha \vee \beta)$	$\neg\alpha \wedge \neg\beta$
	$\alpha \wedge \beta$	$\neg(\neg\alpha \vee \neg\beta)$
	$\alpha \vee \beta$	$\neg(\neg\alpha \wedge \neg\beta)$
Limiting cases	$\alpha \wedge \neg\alpha$	$\beta \wedge \neg\beta$
	$\alpha \vee \neg\alpha$	$\beta \vee \neg\beta$

SLAM, p. 262

## tautological equivalence

Name	LHS	RHS
Contraposition	$\alpha \rightarrow \beta$	$\neg\beta \rightarrow \neg\alpha$
	$\alpha \rightarrow \neg\beta$	$\beta \rightarrow \neg\alpha$
	$\neg\alpha \rightarrow \beta$	$\neg\beta \rightarrow \alpha$
Import/export	$\alpha \rightarrow (\beta \rightarrow \gamma)$	$(\alpha \wedge \beta) \rightarrow \gamma$
	$\alpha \rightarrow (\beta \rightarrow \gamma)$	$\beta \rightarrow (\alpha \rightarrow \gamma)$
Consequential mirabilis (miraculous consequence)	$\alpha \rightarrow \neg\alpha$	$\neg\alpha$
	$\neg\alpha \rightarrow \alpha$	$\alpha$
Commutation for $\leftrightarrow$	$\alpha \leftrightarrow \beta$	$\beta \leftrightarrow \alpha$
Association for $\leftrightarrow$	$\alpha \leftrightarrow (\beta \leftrightarrow \gamma)$	$(\alpha \leftrightarrow \beta) \leftrightarrow \gamma$
$\neg$ through $\leftrightarrow$	$\neg(\alpha \leftrightarrow \beta)$	$\alpha \leftrightarrow \neg\beta$
Translations between connectives	$\alpha \leftrightarrow \beta$	$(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$
	$\alpha \leftrightarrow \beta$	$(\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$
	$\alpha \rightarrow \beta$	$\neg(\alpha \wedge \neg\beta)$
	$\alpha \rightarrow \beta$	$\neg\alpha \vee \beta$
	$\alpha \vee \beta$	$\neg\alpha \rightarrow \beta$
Translations of negations of connectives	$\neg(\alpha \rightarrow \beta)$	$\alpha \wedge \neg\beta$
	$\neg(\alpha \wedge \beta)$	$\alpha \rightarrow \neg\beta$
	$\neg(\alpha \leftrightarrow \beta)$	$(\alpha \wedge \neg\beta) \vee (\beta \wedge \neg\alpha)$

SLAM, p. 263

## constructing the DNF

Given a formula, there are two common ways of constructing its DNF:

1. via its truth table
2. through successive syntactic transformations

### Basic algorithm for building a DNF (SLAM p. 272):

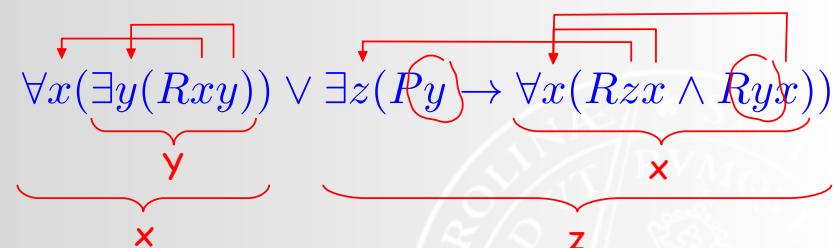
1. express all connectives through negation, conjunction, disjunction
2. move negations inward (de Morgan), eliminate double negation
3. move conjunctions inward (distribution)
4. remove repetitions in basic conjunctions
5. remove basic conjunctions with a letter and its negation

Example:  $\neg((p \vee \neg q) \rightarrow (p \wedge \neg q))$

1.  $\neg(\neg(p \vee \neg q) \vee (p \wedge \neg q))$  2.3.  $(p \vee \neg q) \wedge (\neg p \vee q)$
- 2.1.  $\neg\neg(p \vee \neg q) \wedge \neg(p \wedge \neg q)$  3.  $(p \wedge \neg p) \vee (p \wedge q) \vee (\neg q \wedge \neg p) \vee (\neg q \wedge q)$
- 2.2.  $\neg\neg(p \vee \neg q) \wedge (\neg p \vee q)$  5.  $(p \wedge q) \vee (\neg p \wedge \neg q)$

## quantifier scopes

variable uses, and the quantifier they are bound by



quantifier scopes, and the variables bound in/by them

## free and bound variable occurrences

A variable occurrence is *bound* iff it occurs inside the scope of a quantifier that binds that variable.

It is *free* otherwise.

A formula with no free variable occurrences is called *closed*.

A closed formula is a *sentence*.

free occurrences  
 $\forall z[Rxz \rightarrow \exists y(Rzy)]$   
bound occurrences

## definitions, theorems, proofs

A **definition** is a statement that gives a precise meaning to a term or a symbol.

$A \subseteq B$  iff  $\forall x (x \in A \rightarrow x \in B)$

$n \in \mathbb{Z}$  is *even* iff  $\exists k \in \mathbb{Z} (n = 2k)$

$n \in \mathbb{Z}$  is *odd* iff  $\exists k \in \mathbb{Z} (n = 2k + 1)$

A **theorem** is a statement that needs to be proven based on definitions (and axioms).

$A \times (B \cap C) = A \times B \cap A \times C$

$\#(\mathbb{N}) < \#(2^{\mathbb{N}})$

There are infinitely many prime numbers.

A **proof** is a chain of logical reasoning showing the truth of a theorem.

Other words for theorem:  
proposition, lemma, corollary.

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## kinds of proofs

Proofs come in different flavors, which depend on the **form of the theorem**, and the chain of reasoning best suited to prove it.

Many theorems are conditional statements, i.e. they have the form "premise implies conclusion, or"

$$P \rightarrow C$$

$$\forall x \in \mathbb{Z} (x \text{ is odd} \rightarrow x^2 \text{ is odd})$$

$$\forall a, b, c \in \mathbb{Z} ((a|b \wedge b|c) \rightarrow a|c)$$

P	C	$P \rightarrow C$
T	T	T
T	F	F
F	T	T
F	F	T

## direct proof

**Theorem:** If P, then C.

**Proof:** Suppose P.

...

Therefore C.

**Theorem:**

$$x \text{ is odd} \rightarrow x^2 \text{ is odd}$$

**Proof:**

Suppose  $x$  is odd.

Therefore, there is an integer  $k$  such that  $x = 2k + 1$ .

$$\text{Thus } x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Note that  $2k^2 + 2k$  is an integer.

$$\text{Thus there is an integer } n \text{ such that } x^2 = 2n + 1.$$

Therefore  $x^2$  is odd.

## direct proof with cases

Sometimes, the premise consists of several cases, and it becomes easier to study each case by itself.

$n$	$1 + (-1)^n(2n - 1)$
1	0
2	4
3	-4
4	8
5	-8
6	12

**Theorem:** If  $n \in \mathbb{N}$  then  $1 + (-1)^n(2n - 1)$  is a multiple of 4.

**Proof:** Suppose  $n \in \mathbb{N}$ . Then  $n$  is either even or odd.

**Case 1:** Suppose  $n$  is even. Then  $n = 2k$  for some  $k \in \mathbb{Z}$ .

$$\text{Thus } 1 + (-1)^{2k}(2(2k) - 1) = 1 + 1^k(4k - 1) = 4k.$$

That is a multiple of 4.

**Case 2:** Suppose  $n$  is odd. Then  $n = 2k + 1$  for some  $k \in \mathbb{Z}$ .

$$\text{Thus } 1 + (-1)^{2k+1}(2(2k + 1) - 1) = 1 - (4k + 2 - 1) = -4k.$$

That is also a multiple of 4.

The result in both cases is a multiple of 4.

## contrapositive proof

In some cases, it is easier to reason about a theorem in contrapositive form.

**Theorem:**

If  $x^2 - 6x + 5$  is even, then  $x$  is odd.

**Proof:**

Suppose  $x^2 - 6x + 5$  is even, i.e. there exists an integer  $a$  such that  $x^2 - 6x + 5 = 2a$ .

...

Thus there is an integer  $b$  such that  $x = 2b + 1$ .

Therefore  $b$  is odd.

**direct proof:**

**Theorem:** If  $P$ , then  $C$ .

**Proof:** Suppose  $P$ .

...

Therefore  $C$ .

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## contrapositive proof

Contrapositive form:  $\neg C \rightarrow \neg P$

$P$	$C$	$P \rightarrow C$	$\neg C$	$\neg P$	$\neg C \rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

**Theorem:** If  $P$ , then  $C$ .

**Proof:** Suppose not  $C$ .

...

Therefore not  $P$ .

**Theorem:**

If  $x^2 - 6x + 5$  is even, then  $x$  is odd.

**Proof:**

Suppose  $x$  is even.

There is an integer  $a$  such that  $x = 2a$ .

$$x^2 - 6x + 5 = 4a^2 - 12a + 4 + 1 = 2(2a^2 - 6a + 2) + 1$$

So there is an integer  $b$  s.t.  $x^2 - 6x + 5 = 2b + 1$ .

Therefore  $x^2 - 6x + 5$  is not even.

## proof by contradiction

Suppose we want to prove a proposition  $P$ , not necessarily in conditional form.

Proof by contradiction uses the fact that if we can show that not  $P$  results in a logical contradiction, e.g. it implies some conclusion  $C$  as well as its opposite, not  $C$ , then not  $P$  cannot be true, and so  $P$  must be true.

**Theorem:**

If  $a, b \in \mathbb{Z}$  then  $a^2 - 4b \neq 2$ .

**Proof:**

Suppose there are  $a, b \in \mathbb{Z}$  s.t.  $a^2 - 4b = 2$ .

Since this implies  $a^2 = 4b + 2 = 2(2b + 1)$ ,  $a^2$  is even.

Hence  $a$  is even, so  $a = 2c$  for some integer  $c$ .

Thus  $4c^2 - 4b = 2$ , i.e.  $2c^2 - 2b = 1$ .

Therefore  $2(c^2 - b) = 1$  with  $c^2 - b \in \mathbb{Z}$ .

So 1 is even.

$P$	$C$	$\neg P$	$C \wedge \neg C$	$\neg P \rightarrow C \wedge \neg C$
T	T	F	F	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

**Theorem:**  $P$ .

**Proof:** Suppose not  $P$ .

...  
Or any other  
false proposition!

Therefore  $C$  and not  $C$ .

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