DNF solving

Intervals

Injective and surjectiv

Properties of paths, graphs and trees

Other important defintions

Node height/depth

The link-height (alias level) of a node in a tree is defined recursively: that of the root is 0, and that of each of the children of a node is one greater than that of the node. The node-height of a node (alias just its height in many texts) is defined by the same recursion, except that the node-height of the root is set at 1. Thus, for every node x, node-height(x) = link-height(x) + 1. As trees are usually drawn upside-down, the term 'depth' is often used instead of 'height'.

p	q	r	s	$\overset{a}{p \to q}$	$\overset{b}{q \to r}$	$\overset{c}{r \to s}$	$\overset{d}{s \to p}$	$\overset{e}{a \barwedge b}$	$\overset{f}{c \barwedge d}$	$e \land f$
1	1	1	1	1	1	1	1	0	0	0
1	1	1	0	1	1	0	1	0	1	0
1	1	0	1	1	0	1	1	1	0	0
1	1	0	0	1	0	1	1	1	0	0
1	0	1	1	0	1	1	1	1	0	0
1	0	1	0	0	1	0	1	1	1	1 ←
1	0	0	1	0	1	1	1	1	0	0
1	0	0	0	0	1	1	1	1	0	0
0	1	1	1	1	1	1	0	0	1	0
0	1	1	0	1	1	0	1	0	1	0
0	1	0	1	1	0	1	0	1	1	1 ←
0	1	0	0	1	0	1	1	1	0	0
0	0	1	1	1	1	1	0	0	1	0
0	0	1	0	1	1	0	1	0	1	0
0	0	0	1	1	1	1	0	0	1	0
0	0	0	0	1	1	1	1	0	0	0

DNF for when $((p \to q) \barwedge (q \to r)) \land ((r \to s) \barwedge (s \to p))$ is true:

$$(p \land \neg q \land r \land \neg s) \lor (\neg p \land q \land \neg r \land s)$$

För att lösa DNF uppgifter kan man skapa ett truth table med alla literaler som förekommer i uttrycket. Sedan bygger man stegvis upp hela uttrycket med fler kolumner. Se s.12 i Definitions för *basic logic connectives.*

1 **[20 p]**

For the following sets of numbers, specify the smallest and the largest numbers, write NONE if there is no smallest or largest number, or EMPTY (in one of the two columns) if the set is empty.

All intervals are supposed to be intervals in the real numbers, $\mathbb{R}$. Similarly, all relations and operators are on the real numbers, unless explicitly stated otherwise.

set	smallest element	largest element
$\{x \in \mathbb{Z} : x > 4 \wedge x < 2\}$	EMPTY	
$\{1, 2, 3, 4\}$	1	4
$]1, 4]$	NONE	4
$\bigcap_{i\in\mathbb{N}+} \left[0, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i\in\mathbb{N}+} \left]0, \frac{1}{i}\right] = \emptyset$	EMPTY	
$\bigcap_{i\in\mathbb{N}+} \left[\frac{-1}{i}, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i\in\mathbb{N}+} \left]\frac{-1}{i}, 1\right] = [0, 1]$	0	1
$\mathrm{sqrt}([0, 0.1]) = [0, \sqrt{0.1}]$	0	$\sqrt{0.1}$
$\mathrm{sqrt}[[0, 0.1]] = [0, 1[$	0	NONE
$\mathrm{sqrt}[]0, 0.1]] =]0, 1[$	NONE	NONE
$\bigcup_{i\in\mathbb{N}+} \left]\frac{1}{i+1}, \frac{1}{i}\right[=]0, 1[$	NONE	NONE
$\mathrm{inv}(]0, 0.1]) = [10, +\infty[$	10	NONE
$\mathrm{inv}[]0, 0.1]] =]0, 0.1] \cup [10, +\infty[$	NONE	NONE

$\mathrm{sqrt} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the positive square root function, i.e. for every non-negative real number a, $\mathrm{sqrt}(a)$ is the non-negative real number such that $a = \mathrm{sqrt}(a) \cdot \mathrm{sqrt}(a)$.

$\mathrm{inv} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ is the inversion function, defined by $\mathrm{inv} : r \mapsto \frac{1}{r}$.

Here is a question about understanding how intervals change when sending through functions and meet. TODO: Understand answer and [[]] syntax

When looking at intersections of an infinite number of sets, it is important to keep in mind that any value that is an element of the intersection must be an element of each and every one of those sets. Take for example $\bigcap_{i\in\mathbb{N}+}\left[0,\frac{1}{i}\right]$. If that intersection contained any positive real number in addition to 0, it would mean that there is a real number $r>0$, such that $r\in\left[0,\frac{1}{i}\right]$ for every $i\in\mathbb{N}^+$. It's easy to see that this cannot be the case: for any $r>0$, there is some natural number $k>\frac{1}{r}$, which means that $\frac{1}{k}<r$, and so $r\notin\left[0,\frac{1}{k}\right]$, and so r cannot be in the intersection.

This is a explanation of how to think with intervals in more detail.

2 [6p]

Define two sets A and B, as well as a function $f : A \longrightarrow B$, such that f is **surjective** and **not injective**.

$A = \{a, b\}$

$B = \{x\}$

$f : a \mapsto x$

Of course, if you already found the answer to the next question, you could "reuse" it here by simply choosing to make A and B the same set.

The idea was to start with an easier question to get you to think about surjectivity and injectivity in a simpler setting first, before tackling the harder problem.

This is an easier example for injective and surjective functions. The reason why this question is easier is because we can define the domain and codomain seperatly.

3 [10p]

Define **one** set A, as well as a function $f : A \longrightarrow A$, such that f is **surjective** and **not injective**.

$A = \mathbb{N}$

$$f : a \mapsto \begin{cases} 0 & \text{if } a = 0 \\ a - 1 & \text{otherwise} \end{cases}$$

The crux here is that A has to be infinite.

4 [4p]

Suppose there is a function $f : A \longrightarrow A$ which is surjective and **not** injective, like the one you were asked to define in the previous task. This question is about a property of A (the domain and codomain of f) that implies that such a function exists, and which is also implied by the existence of such a function. (You do not need to prove this here.)

A surjective and **not** injective function $f : A \longrightarrow A$ exists if and only if

A is infinite.

(this must be a statement about the set A, and cannot involve f)

If you want to get a deeper understanding of this point, try to prove it. You can do this in two steps: (1) You show that if A is infinite, a function exists on it that is surjective but not injective. This you can show by taking our definition of an infinite set (one that is equivalent to a proper subset of itself), and use that definition to construct such a function.

(2) Now you need to show that if such a function exists, then A is infinite. Being infinite means that A must be equinumerous to a proper subset of itself. So given a function that is surjective but not injective, you need to find a proper subset of A that is the same size as A.

If you find this confusing, have a look at the solution above, and try to figure out what a suitable proper subset of A would be, and how it is related to f.

With all of these weird example where something should be surjective and not injective and the codomain = domain we need to utilize infinte. What is happening here is that any item in the codomain could be reached from the domain by takin the item plus one. Meaning it is surjective but we also point to the same element twice.

5 [10p]

Recall that a *directed graph* (V, E) is defined as a finite set V of vertices and a relation $E \subseteq V \times V$ between them.

This question is about the properties of that relation. In the table below, make one mark in each row for the property in the left column, depending on whether all, some, or no relations defining a graph have that property. Put the mark in the corresponding ALL box, if **all relations** defining a graph have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over V		X	
transitive		X	
symmetric		X	
antisymmetric		X	
asymmetric		X	

Graphs in our definition make no special assumptions about the relations that define them, so any (finite) relation could be a graph.

Here is an example of properties of an directed graph. The interesting thing to remember is that no property is guaranteed. We don't put any demands on our directed graph meaning it could be any kind of relation, every node could be symmetric but then we would need to edges per node.

6 [10p]

Recall that a *rooted tree* is a graph (T, R) such that, if the set T of nodes is not empty, then there is a node $a \in T$ (the root) such that for every $x \in T$ with $x \neq a$ there is exactly one path from a to x. Like V in the previous question, $R \in T \times T$ is a relation on the set of nodes. To make things simpler, for this question we only consider non-empty trees, that is $T \neq \emptyset$.

This question is about the properties of the relations defining trees. In the table below, make one mark in each row for the corresponding property in the left column, depending on whether all, some, or no relations defining a tree have that property. Put the mark in the corresponding ALL box, if **all relations** defining a tree have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over T			X
transitive		X	
symmetric		X	
antisymmetric	X		
asymmetric	X		

The situation is different for trees, which are much more specialized and constrained structures than graphs. Since we only consider non-empty trees, none of them can be reflexive. (If we allowed the empty tree, then its link-relation R would also be empty, which is reflexive over the empty set.)

But there are trees whose link relation is transitive, viz. all those of link height 0 or 1. (Make sure you understand why that is.) And there is a tree whose link relation is symmetric, namely the tree consisting of only a root, whose link relation is therefore empty, which is symmetric.

This is an example of properties for a rooted tree. This differs a bit from the example above, here some demands are put on the directed graph. The reason why it is antisymmetric and asymetric is because if there exist (a,b) and (b,a) then there is a two ways to reach a, you can go directly to be or go to a and then go to b and then a. The reason why we don't allow antisymmetric is the same reason it is not reflexive.