Homework1

Problem 1. (5pt) A valuation distribution meets the *monotone hazard rate (MHR)* condition if its *hazard rate* $\frac{f_i(v_i)}{1-F_i(v_i)}$ is non-decreasing in v_i. Prove that every distribution meeting the MHR condition is regular.

proof:

We have known that $\frac{f_i(v_i)}{1-F_i(v_i)}$ is non-decreasing, set $g_i(v_i) = \frac{f_i(v_i)}{1-F_i(v_i)}$,

$$\therefore g_i'(v_i) = \frac{f_i'(v_i)[1-F_i(v_i)] + f_i^2(v_i)}{(1-F_i(v_i))^2} \geq 0$$
$$\Rightarrow f_i'(v_i)[1-F_i(v_i)] + f_i^2(v_i) \geq 0$$

set $h_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$,

$$\begin{aligned}\therefore h_i'(v_i) &= 1 - [\frac{-f_i^2(v_i) - f_i'(v_i)[1-F_i(v_i)]}{f_i^2(v_i)}] \\ &= 1 + [\frac{f_i^2(v_i) + f_i'(v_i)[1-F_i(v_i)]}{f_i^2(v_i)}] \\ &\geq 0\end{aligned}$$

Thus, every distribution meeting the MHR condition is regular.

Problem 2. (5pt) Prove that for every single-parameter environment and regular valuation distributions $F_1, \ldots, F_n$, the virtual-welfare-maximizing allocation rule is monotone. Assuming that ties are broken lexicographically with respect to some fixed total ordering over the feasible outcomes.

> We can compare it to the previous chapters to prove that DSIC must be monotone.

proof:

Assume that (x, p) is virtual welfare maximized (here x is allocation rule and p is payment rule).

Because $F_1, \ldots, F_n$ is regular, its virtual price $\varphi(x)$ is monotone.

Thus for any $0 \leq y < z$, we have:

$$0 \leq \varphi(y) < \varphi(z)$$

For any i and b_{-i}, we have:

$$z * x(\varphi(z)) - p(z) \geq z * x(\varphi(y)) - p(\varphi(y))$$
$$y * x(\varphi(y)) - p(\varphi(y)) \geq y * x(\varphi(z)) - p(\varphi(z))$$

Here, $x(z) = x_i(z, b_{-i}), p(z) = p_i(z, b_{-i})$

$$\therefore z * [x(\varphi(y)) - x(\varphi(z))] \le p(\varphi(y)) - p(\varphi(z)) \le y * [x(\varphi(y)) - x(\varphi(z))]$$
$$\therefore (y - z) * [x(\varphi(y)) - x(\varphi(z))] \ge 0$$
$$\Rightarrow non - decreasing!$$

Namely, allocation rule is monotone.

Problem 3. This problem is to prove the Sperner's Lemma, a combinatorial version of Brouwer's Fixed Point Theorem. Given a grid as Figure 1, we first color the boundary using three colors in a legal way as the figure says, and then color the internal nodes arbitrarily. Prove that there exists one tri-chromatic triangle, i.e., a small unit triangle whose nodes are colored by all the three colors. You should prove this lemma using two methods as follows.

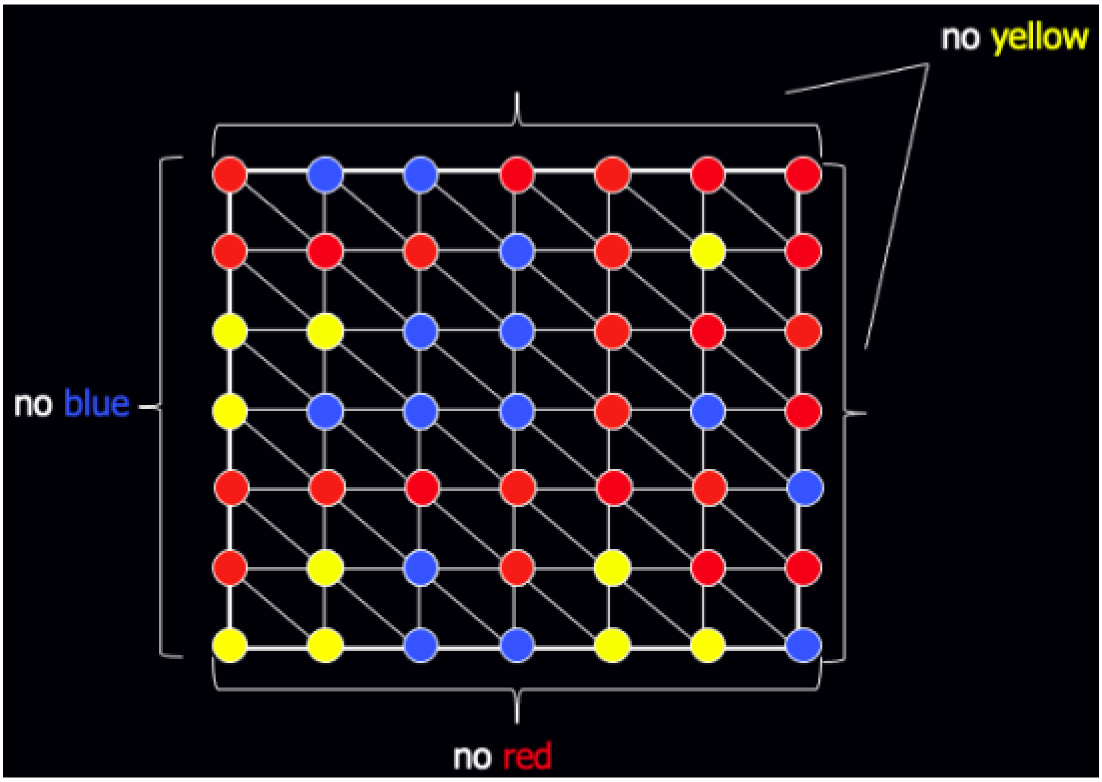
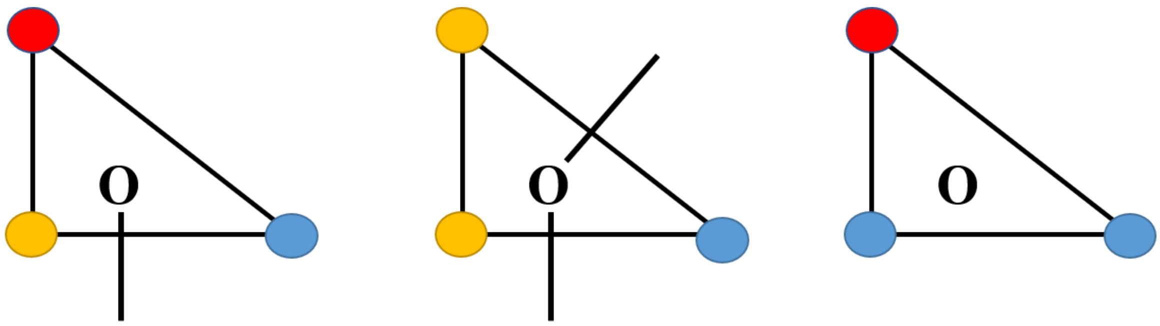


Figure 1: An example of Sperner's Lemma

1. (3pt) The first method is using *double-counting*, that is, we count the number of some object from two different views. In this problem, we can prove the lemma by counting the number of yellow-blue edges of all the unit triangles.

proof:

Assume that there is a point A outside the quadrilateral, and each small triangle has a vertex O at the center. If the small triangle contains yellow and blue vertices, the edge formed by the yellow and blue points from O will form an edge (that is, the out degree), For a more vivid illustration, I drew a picture as follows:



Out-degree is 1 **Out-degree is 2** **Out-degree is 0**

According to the handshake theorem, the sum of the degrees must be an even number. At the bottom "no red" side of the grid, the observation shows that the degree must be an odd number, that is, there must be an odd number of line segments on the yellow-blue side and blue-yellow side. There must be a triangle of odd degree in the triangle, so that the sum of the out-degree of the "no red" side must be an even number, namely, a triangle of 1 degree must exist. This is proved.

2. (7pt) The second method is using *path-following*. Actually, PPAD is inspired by this lemma! (Recall the problem End-of-A-Line) One can define each triangle as a node in the graph. How to define directed edges is the crucial part. Another issue is the initial source node (0^n in the problem EoAL).

Reference here: [Sperner's Lemma (maths.org)]() in the triangle situation, this question is a quadrilateral.

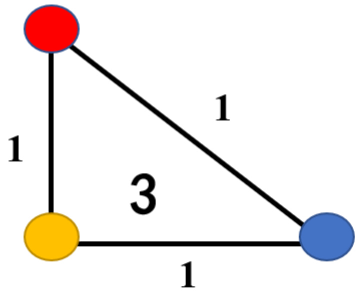
proof:

Here, red, blue and yellow are recorded as clockwise (hereinafter will be abbreviated as RBY). If an edge (directed) is of the same color, the edge is recorded as 0, which represents the cost from one point to another. If the vertex color of this edge is clockwise, it is marked as 1, otherwise it is marked as -1. Add up the three sides of a triangle as the sum of the "area" of the triangle.

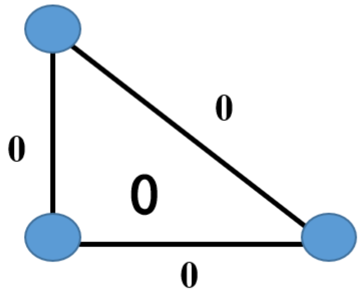
Then there will be and only the following four situations will occur:

1. The vertex color appears clockwise, the three sides are marked as 1, and the "area sum" is 3.
2. The vertex color appears counterclockwise, the three sides are marked as -1, and the "area sum" is -3.
3. The vertex colors are the same, the three sides are marked as 0, and the "area sum" is 0.
4. The two vertices have the same color, and the other is different, so one edge is marked as 0, the two outside two are respectively 1, -1, and the "area sum" is 0.

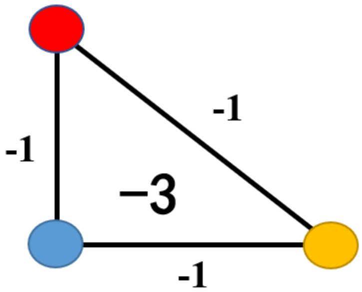
Also, I drew a picture here to illustrate, as follows:



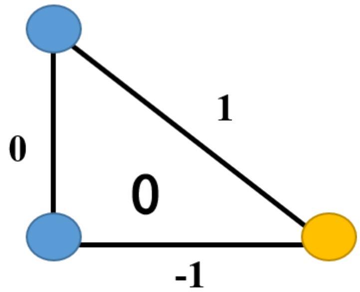
(1) R-B-Y



(2) X-X-X



(1) R-Y-B



(2) X-X-Y

Easy to prove: For the starting red point A and ending blue point B (*there are no yellow point between them*), the mark synthesis of the passing edges is only related to the color of vertices A and B, and has nothing to do with the number of edges passed and the color of the vertices.

Therefore, for the original grid, the red dot from the upper left corner moves in a clockwise direction, and the sum from the upper left corner to the lower right corner is 1. In the same way, the sum from the lower right corner to the lower left corner is 1, and the sum from the lower left corner to the upper left corner is 1, so the sum of the marked edges of the outermost edge of the quadrilateral is 3.

And because the sum of the sides of the inner sides of the quadrilateral is 0 (because it is not 0+0, or -1+1). Thus, the sum of the marks on the sides of the entire quad is 3.

Now, the sum marked on each side must be the same size as the total "area sum", so there must be a triangle with a "area sum" of 3 or -3 (the clockwise case is always one more than the counterclockwise case), that is, there are three The color is the triangle with the vertex, the proof is complete.

Problem 4. This problem derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density f on the interval $[0, v_{\max}]$ (with $v_{\max} < +\infty$).

For $q \in [0,1]$, define $V(q) = F^{-1}(1-q)$ as the posted price resulting in a probability q of a sale (for a single bidder with valuation drawn from F). Define $R(q) = q \cdot V(q)$ as the expected revenue obtained when (for a single bidder) the probability of a sale is q. The function $R(q)$, for $q \in [0,1]$, is often called the "revenue curve" of a distribution F. Note that $R(0) = R(1) = 0$.

1. (3pt) What is the revenue curve for the uniform distribution on $[0,1]$?

solve:

$$\because X \sim U[0,\ 1]$$

$$\therefore F(x) = \begin{cases} 0,\ x < 0 \\ x,\ 0 \leq x \leq 1 \\ 1,\ x > 1 \end{cases}$$

$$\therefore F^{-1}(x) = \begin{cases} 0,\ x < 0 \\ x,\ 0 \leq x \leq 1 \\ 1,\ x > 1 \end{cases}$$

$$又 \because R(0) = R(1) = 0$$

$$V(q) = F^{-1}(1-q) = 1-q \quad, q \in [0,1]$$

$$R(q) = q * (1-q) \quad, q \in [0,1]$$

2. (2pt) Prove that the slope of the revenue curve at q (i.e., $R'(q)$) is precisely $\varphi(V(q))$, where φ is the virtual valuation function of F.

proof:

$$\because R(q) = q * V(q) = q * F^{-1}(1-q)$$

$$\therefore R'(q) = F^{-1}(1-q) - q * (F^{-1}(1-q))'$$

$$\begin{aligned} \because \varphi(V(q)) &= V(q) - \frac{1 - F(V(q))}{f(V(q))} \\ &= F^{-1}(1-q) - \frac{1 - F(F^{-1}(1-q))}{f(F^{-1}(1-q))} \\ &= F^{-1}(1-q) - \frac{q}{f(F^{-1}(1-q))} \end{aligned}$$

If we want to prove that $R'(q) = \varphi(V(q))$, that is equal to prove:

$$\frac{1}{f(F^{-1}(1-q))} = (F^{-1}(1-q))'$$

According to the property of the inverse function, we have $F(F^{-1}(x)) = x$. Thus, we can get

$$\begin{aligned} 1 &= F'(F^{-1}(x)) \\ &= f(F^{-1}(x)) * (F^{-1}(x))' \end{aligned}$$

Namely,

$$\frac{1}{f(F^{-1}(x))} = (F^{-1}(x))'$$

Replace x with 1-q,

$$\frac{1}{f(F^{-1}(1-q))} = (F^{-1}(1-q))'$$

Thus,

$$\frac{q}{f(F^{-1}(1-q))} = q * (F^{-1}(1-q))'$$
$$F^{-1}(1-q) - \frac{q}{f(F^{-1}(1-q))} = F^{-1}(1-q) - q * (F^{-1}(1-q))'$$
$$R'(q) = \varphi(V(q))$$

The prove is complete.

3. (2pt) Prove that a distribution is regular if and only if its revenue curve is concave.

proof:

$$\because R'(q) = \varphi(V(q))$$
$$\therefore R''(q) = \varphi'(V(q))V'(q)$$
$$\because \frac{d}{dq}F^{-1}(q) = \frac{1}{F'(F^{-1}(q))}$$
$$\therefore F'(q) > 0 \Rightarrow (F^{-1}(q))' > 0 \Rightarrow (F^{-1}(1-q))' < 0 \Leftrightarrow V'(q) < 0$$
$$\therefore R''(q) \leq 0 \Leftrightarrow \varphi'(V(q)) \geq 0$$

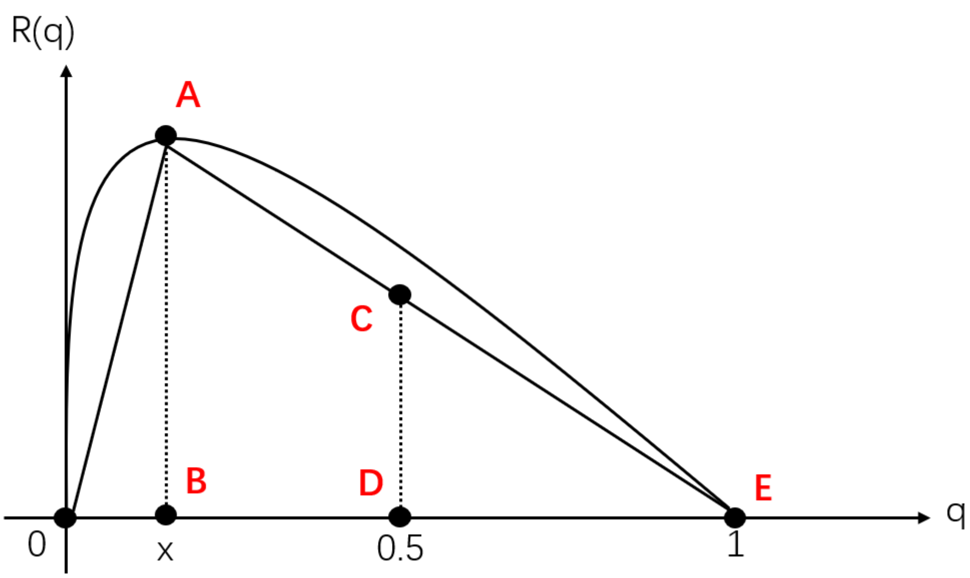
In summary, $R(q)$ is concave $\Leftrightarrow$ the distribution F is regular.

4. (3pt) Prove that, for a regular distribution, the median price $V(1/2)$ is a $(1/2)$-approximation of the optimal posted price. That is, prove that $R(1/2) \geq 1/2 \cdot \max_{q\in[0,1]} R(q)$.

proof:

$$\because R(q)\ is\ a\ concave\ fuction$$
$$R(0) = R(1) = 0$$

1. If R(x) is maximum When $x = \frac{1}{2}$, the conclusion is obvious.
2. if R(x) is maximum When $x < \frac{1}{2}$.



$$\because \triangle AEB \sim \triangle CDE$$

$$\therefore \frac{1}{2}AB < CD$$

$$\frac{1}{2}R(x) < R(\frac{1}{2})$$

$$\therefore R(\frac{1}{2}) > \frac{1}{2}max_{q\in[0,1]}R(q)$$

3. the conclusion is same when $x > \frac{1}{2}$.