ii) $y = \cos(\cos x)$

$$\frac{dy}{dx} = -\sin(\cos x) \cdot \frac{d}{dx}(\cos x)$$

$$= -\sin(\cos x) \cdot (-\sin x)$$

$$= \sin(\cos x) \cdot \sin x$$

iii)

$$\frac{dy}{dx} = \frac{1}{2}(x^3 + \operatorname{cosec} x)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^3 + \operatorname{cosec} x)$$

$$= \frac{3x^2 - \operatorname{cosec} \cot x}{2\sqrt{x^3 + \operatorname{cosec} x}}$$

(x) If $y = \cot^3(\pi - \theta)$. find $\frac{dy}{d\theta}$

$$\Rightarrow \frac{dy}{dx} = 3\cot^2(\pi-\theta) \cdot \frac{d}{d\theta}(\cot(\pi-\theta))$$

$$= 3\cot^2(\pi-\theta) \cdot -\operatorname{cosec}^2(\pi-\theta) \cdot \frac{d}{d\theta}(\pi-\theta)$$

$$= 3\cot^2(\pi-\theta) \cdot -\operatorname{cosec}^2(\pi-\theta) - 1$$

$$= 3\cot^2(\pi-\theta) \cdot \operatorname{cosec}^2(\pi-\theta)$$

※ $y = (1+\cos 2x)^2$

$$\frac{dy}{dx} = \frac{d}{dx}(1+\cos 2x)^2$$

$$= 2(1+\cos 2x)\cdot(-\sin 2x\cdot 2)$$

Chapter - 3

3.1
Implicit Differentiation

Implicit Function	Explicit Function
	$y = f(x)$
$x^2 + 2xy + y^2 = 0$	$y = x^2 + 2x + 1$
$xy = 1$	$\longrightarrow y = \frac{1}{x}$
$xy + x = 2$	$\longrightarrow y = \frac{2-x}{x}$
$f(x,y) = g(x,y)$	$\longrightarrow y = \pm\sqrt{1-x^2}$
$x^2 + y^2 = 1$	

(*) $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$

$\frac{d}{dx}(y^4) = 4y^3 \cdot \frac{dy}{dx}$

(*) $\frac{d}{dx}(\sin x) = \cos x \cdot \frac{dx}{dy}$

$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$

(*) $x^2y + 3xy^3 - x = 3$

the $\frac{dy}{dx} = ?$

$\Rightarrow$

Here,

$$x^2y + 3xy^3 - x = 3$$

differentiate both side w.r.t. x.

$$\frac{d}{dx}(x^2y + 3xy^3 - x) = \frac{d}{dx}(3)$$

$$\Rightarrow x^2 \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x^2) + 3x \cdot \frac{d}{dx}(y^3) + y^3 \cdot \frac{d}{dx}(3x) - 1 = 0$$

$$\Rightarrow x^2 \cdot \frac{dy}{dx} + y \cdot 2x + 3x \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 3 - 1 = 0$$

Now solve for $\frac{dy}{dx}$

$$(x^2 + 9xy^2)\frac{dy}{dx} = 1 - 3y^3 - 2xy$$

$$\therefore \frac{dy}{dx} = \frac{1 - 3y^3 - 2xy}{x^2 + 9xy^2}$$

$x^2+y^2=1$

$y = \pm\sqrt{1-x}$

$y = +\sqrt{1-x^2} \Rightarrow fn$

$y = -\sqrt{1-x^2} \Rightarrow fn$

$x^2+y^2=1$

$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$

$(2x+2y)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\frac{dy}{dx} = -\frac{x}{y}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-1}{1} = -1$

Use implicit derivative to find $\frac{dy}{dx}$,

i) $5y^2 + \sin y = x^2$

ii) $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$

iii) $\cos(xy^2) = y$

Solution:

i) $5y^2 + \sin y = x^2$

$\Rightarrow \frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$

$\Rightarrow 10y \cdot \frac{d}{dx}y + \cos y \cdot \frac{dy}{dx} = 2x$

$\Rightarrow (10y + \cos y)\frac{dy}{dx} = 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{10y + \cos y}$

ii) $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$

$\frac{d}{dx}\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right) = \frac{d}{dx}(1)$

$\frac{d}{dx}\left(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}\right) = 0$

$-\frac{1}{2}x^{-\frac{1}{2}-1} - \frac{1}{2}y^{-\frac{1}{2}-1} \cdot \frac{d}{dx}y = 0$

$x^{-3/2} + y^{-3/2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^{-3/2}}{y^{-3/2}} = -\frac{y^{3/2}}{x^{3/2}} = \frac{-y\sqrt{y}}{x\sqrt{x}}$

iii)

$$\cos(xy^2) = y$$

$$\frac{d}{dx}\left(\cos(xy^2)\right) = \frac{dy}{dx}$$

$$-\sin xy^2 \cdot \frac{d}{dx} xy^2 = \frac{dy}{dx}$$

$$-\sin xy^2 \cdot \left[x \cdot \frac{d}{dx} y^2 + y^2 \frac{d}{dx} x\right] = \frac{dy}{dx}$$

$$-\sin xy^2 \left[x \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 1\right] = \frac{dy}{dx}$$

$$-2xy \sin xy^2 \frac{dy}{dx} - y^2 \sin xy^2 = \frac{dy}{dx}$$

$$\left(-2xy \sin xy^2 - 1\right) \frac{dy}{dx} = y^2 \sin xy^2$$

$$\frac{dy}{dx} = \frac{y^2 \sin xy^2}{-(1 + 2xy \sin xy^2)}$$

(*) Find the equation of a tangent line to the Folium of Descantes $\boxed{x^3 + y^3 = 3xy}$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

$\Rightarrow$

$$x^3 + y^3 = 3xy$$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy)$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(3x)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \cdot \frac{dy}{dx} + 3y$$

$$3x \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 3x^2 - 3y$$

$$(3x - 3y^2) \frac{dy}{dx} = 3x^2 - 3y$$

$$\frac{dy}{dx} = \frac{3x^2 - 3y}{3x - 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = \frac{3 \cdot \left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2}}{3 \cdot \frac{3}{2} - 3 \cdot \left(\frac{3}{2}\right)^2}$$

$$= \frac{\frac{27}{4} - \frac{9}{2}}{\frac{9}{2} - \frac{27}{4}}$$

$$= -1$$

Therefore, equation of tangent line,

$$y - \frac{3}{2} = -1\left(x - \frac{3}{2}\right)$$

$$y - \frac{3}{2} = -x + \frac{3}{2}$$

$$\boxed{y + x = 3}$$

(*) If, $y + \sin y = x$

find, $\dfrac{d^2y}{dx^2} = ?$

$$y + \sin y = x$$

$$\frac{d}{dx}(y + \sin y) = \frac{d}{dx}x$$

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{1+\cos y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{1+\cos y}\right)$$

$$\cancel{\frac{d^2y}{dx^2} = \frac{(1+\cos y)\frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(1+\cos y)}{(1+\cos y)^2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+\cos y)^{-1}$$

$$= -1(1+\cos y)^{-2} \cdot \frac{d}{dx}(1+\cos y)$$

$$= -(1+\cos y)^{-2} \cdot \left(0 - \sin y \frac{dy}{dx}\right)$$

$$= -(1+\cos y)^{-2} \cdot \left(-\sin y \frac{dy}{dx}\right)$$

$$= \frac{-1}{(1+\cos y)^2} \cdot \left(-\sin y \frac{dy}{dx}\right)$$

$$= \frac{\sin y \cdot \frac{1}{1+\cos y}}{(1+\cos y)^2}$$

$$= \frac{\sin y}{(1+\cos y)^3}$$

$$\therefore \frac{d^2x}{dx^2} = \frac{\sin y}{(1+\cos y)^3}$$ Ans

3.2

$$f(x) = \ln x$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\ln x)$$

$$= \frac{1}{x}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

= . . . . . .

$$= \frac{1}{x}$$

⊛ $\frac{d}{dx}(\log x) = ?$

$\log \to$ base 10

⊛ $\frac{d}{dx}(\log_a x)$

$= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$

$= \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x)$

$= \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$

⊛ $\frac{d}{dx}(\log_2 x)$

$= \frac{1}{x \ln 2}$

⊛ If, $y = x \log x$

$\frac{dy}{dx} = \frac{d}{dx}(x \log x)$

$= x \cdot \frac{d}{dx}(\log x) + \log x \cdot 1$

$= x \cdot \frac{1}{x \ln 10} + \log x$

$= \log x + \frac{1}{\ln 10}$

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3·2/

Derivative of Logarithmic function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{h}$$

Let,

$h = vx$

$\therefore \frac{h}{x} = v$

if, $h \to 0$, then $v \to 0$

$$= \lim_{v\to 0} \frac{\ln(1+v)}{vx}$$

$$= \frac{1}{x} \cdot \lim_{v\to 0} \frac{1}{v} \cdot \ln(1+v)$$

$$= \frac{1}{x} \lim_{v\to 0} \ln(1+v)^{\frac{1}{v}}$$

$$= \frac{1}{x} \cdot \ln\left(\lim_{v\to 0} (1+v)^{\frac{1}{v}}\right)$$

$$= \frac{1}{x} \cdot \ln e$$

$$= \frac{1}{x} \cdot 1 = \frac{1}{x}.$$

⊛ $f(x) = \ln x$

$f'(x) = \frac{d}{dx}(\ln x)$

$= \frac{1}{x}$

⊛ $y = \log x$

$\frac{dy}{dx} = \frac{1}{x \ln 10}$

⊛ $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x \ln 3}$

⊛ If $f(x) = \ln x$, is there any horizontal tangent line?

$\Rightarrow \quad f'(x) = \frac{d}{dx}(\ln x)$

$= \frac{1}{x}$

For horizontal tangent line,

$f'(x) = 0$

$\frac{1}{x} = 0$

$1 = 0$ (Not possible)

So, there is no horizontal tangent line for $f(x) = \ln x$.

⊛ Find the following derivatives.

i) $y = \ln \frac{x}{3}$ ii) $y = x^3 \ln x$ iii) $y = \ln(\tan x)$

iv) $y = \ln(\ln x)$ v) $y = \log(\sin^2 x)$

Solutions:

i) $y = \ln \frac{x}{3}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln \frac{x}{3}\right)$$

$$= \frac{1}{x/3} \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{x} \cdot \frac{3}{x} \cdot \frac{d}{dx} x$$

$$= \frac{1}{x} \cdot 1$$

$$= \frac{1}{x}$$

ii) $y = x^3 \ln x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \ln x)$$

$$= x^3 \cdot \frac{1}{x} + \ln x \cdot 3x^2$$

$$= x^2 + 3x^2 \ln x$$

iii) $$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$$

$$= \frac{\sec^2 x}{\tan x}$$

iv) $$\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{d}{dx} \cancel{(\sin^2 x)} (\ln x)$$

$$\cancel{= \frac{2 \sin x \cdot \cos x}{\sin^2 x \cdot \ln 10}}$$

$$\cancel{= \frac{2 \cos x}{\sin x \ln 10}}$$

$$= \frac{1}{x \ln x}$$

v) $\frac{dy}{dx} = \frac{1}{\sin^2 x \ln 10} \cdot \frac{d}{dx}(\sin^2 x)$

$= \frac{2 \sin x \cdot \cos x}{\sin^2 x \cdot \ln 10}$

$= \frac{2 \cos x}{\sin x \ln 10}$

* $y = \ln |x|$

$x > 0$

$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$x < 0$

$y = \ln(-x)$

$\frac{dy}{dx} = \frac{1}{-x} \cdot \frac{d}{dx}(-x)$

$= \frac{+1}{+x} = \frac{1}{x}$

$\therefore y = \ln |x|$

$\frac{dy}{dx} = \frac{1}{x}$

$$y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x$$

$$y = \ln\left|\frac{1+x}{1-x}\right|$$

$$\frac{dy}{dx} = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1-x}{1+x} \cdot \frac{(1-x)\cdot 1 - (1+x)\cdot -1}{(1-x)^2}$$

$$= \text{simplify}$$

$$y = \sqrt[5]{\frac{x-1}{x+1}}$$

taking ln both side,

$$\ln y = \ln\left(\frac{x-1}{x+1}\right)^{\frac{1}{5}}$$

$$\ln y = \frac{1}{5}\ln\left(\frac{x-1}{x+1}\right)$$

$$\ln y = \frac{1}{5}\cdot\left(\ln(x-1) - \ln(x+1)\right)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{1}{5}\cdot\left(\ln(x-1) - \ln(x+1)\right)\right)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{5} \cdot \left[\frac{1}{x-1} \cdot 1 - \frac{1}{x+1} \cdot 1 \right]$$

$$\frac{dy}{dx} = y \cdot \frac{1}{5} \left[\frac{x+1-x+1}{(x-1)(x+1)} \right]$$

$$= \frac{1}{5} \cdot y \cdot \frac{2}{x^2-1}$$

$$= \frac{1}{5} \sqrt[5]{\frac{x-1}{x+1}} \cdot \frac{2}{x^2-1}$$

Find the following derrivatives.

i) $y = \log_x e$ ii) $y = \log_x 2$ iii) $y = \log_{\frac{1}{x}} e$

Solutions:

1) $y = \log_x e$

$y = \frac{\ln e}{\ln x}$

$y = \frac{1}{\ln x}$

$$\frac{dy}{dx} = \frac{\ln x \cdot 0 - 1 \cdot \frac{1}{x}}{(\ln x)^2}$$

$$= \frac{-\frac{1}{x}}{(\ln x)^2}$$

$$= -\frac{1}{x(\ln x)^2}$$

ii)

$$y = \log_x 2$$

$$y = \frac{\ln 2}{\ln x}$$

$\cdots$

iii) $y = \log_{\frac{1}{x}} e$

$$= \frac{\ln e}{\ln \frac{1}{x}}$$

$$= \frac{1}{\ln 1 - \ln x}$$

$$= \frac{1}{0 - \ln x}$$

$$= -\frac{1}{\ln x}$$

$\cdots$

*

$$y = \ln (x^2 - 2)$$

$$\frac{dy}{dx} = \frac{1}{x^2-2} \cdot \frac{d}{dx}(x^2-2)$$

$$= \frac{1}{x^2-2} \cdot 2x$$

$$= \frac{2x}{x^2-2}$$ Ans

3.3

⊛ Derivative of exponential function.

⊛ $y = e^x$

$\frac{dy}{dx} = e^x$

⊛ $y = 2^x$

$\frac{dy}{dx} = 2^x \cdot \ln 2$

⊛ $y = b^x$

$x = \log_b y$

$x = \frac{\ln y}{\ln b}$

$\ln y = x \ln b$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln b)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln b \cdot 1$

$\therefore \frac{dy}{dx} = y \ln b$

$= b^x \ln b$

$\therefore y = b^x$

$\boxed{\frac{dy}{dx} = b^x \ln b}$

if, $b = e$

$y = e^x$

$\frac{dy}{dx} = e^x \cdot \ln e$

$\frac{dy}{dx} = e^x$

⊛ $y = 10^x$

$\frac{dy}{dx} = 10^x \cdot \ln 10$

⊛ $y = e^{\cos x}$

$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\cos x}\right)$

$= e^{\cos x} \cdot \frac{d}{dx}(\cos x) = -\sin e^{\cos x}$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}(e^{-x})$$

$$= e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= -e^{-x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})\cdot(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

= .... simplify ...

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$y = e^{x\tan x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x\tan x}\right)$$

$$= e^{x\tan x} \cdot \frac{d}{dx}(x\tan x)$$ → uv rules

$$= e^{x\tan x} \cdot (x\sec^2 x + \tan x)$$

⊛ Derivative of inverse trig. function

i) $y = \sin^{-1} x$ ii) $y = \cos^{-1} x$ iii) $y = \tan^{-1} x$

$\Rightarrow$

i) $y = \sin^{-1} x$

$\sin y = x$

differentiate w.r.t. x

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \sin^{-1} x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \cos^{-1} x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \tan^{-1}$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

⊛ $y = \sec^{-1} x$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

⊛ $y = \operatorname{cosec}^{-1} x$

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}$$

⊛ $y = \cot^{-1} x$

$$\frac{dy}{dx} = \frac{1}{-(1+x)^2}$$

⊛ $y = \sin^{-1} x^3$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^6}} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1-x^6}} \cdot 3x^2$$

⊛ $y = \sec^{-1} e^x$

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{e^{2x}-1}} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x$$

$$= \frac{1}{\sqrt{e^{2x}-1}} \quad \text{Ans}$$

$$x^3 + x\tan^{-1}y = e^y$$

$$\frac{d}{dx}\left(x^3 + x\tan^{-1}y\right) = \frac{d}{dx}\left(e^y\right)$$

$$3x^2 + \left\{\tan^{-1}y + x\cdot\frac{1}{1+y^2}\cdot\frac{dy}{dx}\right\} = e^y\cdot\frac{dy}{dx}$$

$$\left(\frac{x}{1+y^2} - e^y\right)\frac{dy}{dx} = 3x^2 - \tan^{-1}y$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - \tan^{-1}y}{\frac{x}{1+y^2} - e^y}$$

$$\sin^{-1}(xy) = \cos^{-1}(x-y)$$

$$\frac{d}{dx}\left(\sin^{-1}xy\right) = \frac{d}{dx}\left(\cos^{-1}(x-y)\right)$$

$$\frac{1}{\sqrt{1-(xy)^2}}\cdot\frac{d}{dx}(xy) = \frac{1}{-\sqrt{1-(x-y)^2}}\cdot\frac{d}{dx}(x-y)$$

$$\frac{1}{\sqrt{1-(xy)^2}}\cdot\left(y + x\frac{dy}{dx}\right) = \frac{1}{-\sqrt{1-(x-y)^2}}\cdot\left(1 - \frac{dy}{dx}\right)$$

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3.4

Related Rates

Working Step:

(i) Draw a picture and name the variable, use 't' for time. Assume that all variables are differentiable w.r.t. 't'.

(ii) Write down the numerical information.

(iii) Write what are you asked to find.

(iv) Write the equation that relate the variables.

(v) Differentiate w.r.t. 't'

(vi) Evaluate using known values to find unknown.

(8) Assume that oil spilled ~~form~~ from a rapture tanker spread in a circular pattern whose radius increased at a constant rate 2 ft/sec. How fast is the are of the spill increasing when the radius is 60 ft.

⇒

We know that, area of the circle

$$A = \pi r^2$$

Here,
A is area
r is radius

Here given that,

$$\frac{dr}{dt} = 2 \text{ ft/sec}$$

$$\frac{dA}{dt} = ?$$

when $r = 60$ ft.

$A = \pi r^2$

differentiate both side w.r.t. 't'

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2 \cdot 60 \cdot 2 \ ft^2/sec$$

$$\therefore \frac{dA}{dt} = 240\pi \ ft^2/sec$$

increasing : $\frac{dA}{dt} = 240\pi \ ft^2/sec$

(*) Let 'l' be the length of the diagonal of a rectangle whose sides have length x and y and assume that x and y vary with time.

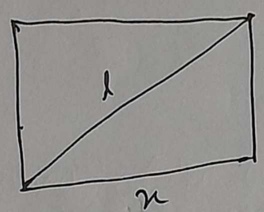
i) How are $\frac{dl}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$ related?

ii) If x increases at a constant rate $\frac{1}{2}$ ft/sec. and y decreases at a constant rate $\frac{1}{4}$ ft/sec. How fast the diagonal changing when $x=3$ and $y=4$?

Is the diagonal increasing or decreasing at that instant?

$\Rightarrow$

(i)



Here, x, y and l are changing with time.

therefore we know that, from figure, $x^2+y^2=l^2$

differentiate both side w.r.t. 't'.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(l^2)$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2l \cdot \frac{dl}{dt}$$

$$\boxed{\therefore \; x\frac{dx}{dt} + y\frac{dy}{dt} = l\frac{dl}{dt}}$$

relation

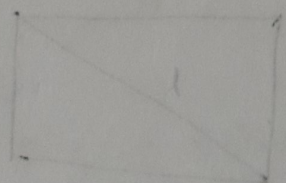
ii)

Here,

$$\frac{dx}{dt} = \frac{1}{2} \text{ ft/sec}$$

$$\frac{dy}{dt} = -\frac{1}{4} \text{ ft/sec}$$

$x=3$
$y=4$ / at $x=3$, $y=4$, l will be $= \sqrt{3^2+4^2} = 5$



from (i)

$$x\frac{dx}{dt} + y\frac{dy}{dt} = l\frac{dl}{dt}$$

$$5 \cdot \frac{dl}{dt} = 3 \cdot \frac{1}{2} + 4\left(-\frac{1}{4}\right)$$

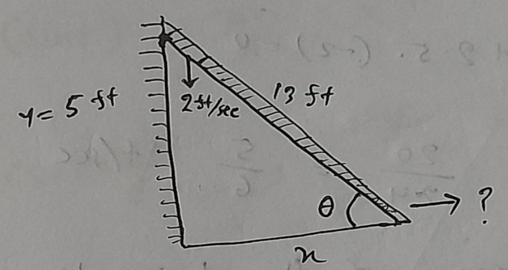
$$5 \cdot \frac{dl}{dt} = \frac{3}{2} - 1$$

$$\therefore \frac{dl}{dt} = \frac{3-2}{2 \cdot 5} = \frac{1}{10} \text{ ft/sec}$$

as, $\frac{dl}{dt}$ is positive, so, diagonal is increasing.

(*) A 13 ft. ladder leaning against a wall. If the top of the ladder slip down the wall at a rate 2 ft/sec. How fast will the foot of the ladder moving away from the wall when the top is 5 ft above the ground?

$\Rightarrow$



Let,

base $= x$

wall height $= y$

Given that,

$$\frac{dy}{dt} = -2 \text{ ft/sec}$$

$$\frac{dx}{dt} = ?$$

$$y = 5 \text{ ft}$$

if $y = 5$

then, $x^2 + y^2 = 13^2$

$\therefore x = 12$

from figure,

$$x^2+y^2=13^2$$

differentiate w.r.t. 't'

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(13^2)$$

$$2x\cdot\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2\cdot 12\frac{dx}{dt} + 2\cdot 5\cdot(-2) = 0$$

$$\frac{dx}{dt} = \frac{20}{24} = \frac{5}{6} \text{ ft/sec}$$

ii) How fast the area of the triangle changing?

⇒ area, $A = \frac{1}{2}\cdot x\cdot y$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2}xy\right)$$

$$\frac{dA}{dt} = \frac{1}{2}x\cdot\frac{dy}{dt} + y\frac{d}{dt}\left(\frac{1}{2}x\right)$$

$$= \frac{1}{2}\cdot 12(-2) + 5\cdot\frac{1}{2}\cdot\frac{5}{6}$$

$$= -12 + \frac{25}{12}$$

$$= \frac{-119}{12} \text{ ft}^2/\text{sec}$$

(iii) How fast the θ changing?

$\Rightarrow$

$$\cos\theta = \frac{x}{13} = \frac{12}{13}$$

Now, $\sin\theta = \frac{y}{13}$

differentiate w.r.t. t

$$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{13}\right)$$

$$\cos\theta \frac{d\theta}{dt} = \frac{1}{13} \cdot \frac{dy}{dt}$$

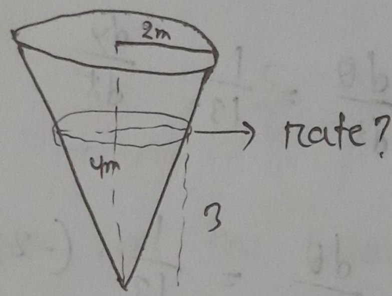
$$\frac{12}{13} \cdot \frac{d\theta}{dt} = \frac{1}{13} \cdot (-2)$$

$$\frac{d\theta}{dt} = \frac{-2\times 13}{13\times 12} = -\frac{1}{6} \text{ degree/sec}$$

decreasing

A water tank has the shape of an inverted circular cone with base radius 2m. and height 4m. If water is being pump into the tank at the rate of $2m^3/min$. Find the rate at which the water lebel is rising, when the water is 3m. deep.

⇒



Changing,

Volume of water, V

Radius of lebel, r

Height of the lebel, h

Given,

$h = 3m$

$\frac{dv}{dt} = 2m^3/min$ / $\frac{dh}{dt} = ?$

$\frac{2}{4} = \frac{r}{h}$

$2h = 4r$

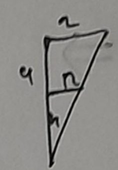
$r = \frac{1}{2}h$

We know that,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$



by similar triangle

$$\frac{4}{2} = \frac{h}{r}$$

$$\frac{h}{r} = 2$$

$$r = \frac{1}{2}h = \frac{h}{2}$$

differentiate w.r.t. t

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right)$$

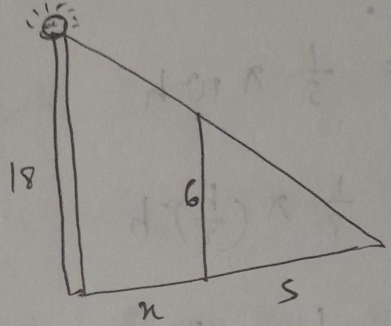
$$\frac{dV}{dt} = \frac{1}{12}\cdot\pi\cdot 3h^2 \frac{dh}{dt}$$

$$2 = \frac{1}{12}\cdot\pi\cdot 3\cdot 3^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2\times \cancel{12}^{4}}{\pi\cdot\cancel{3}\cdot 3^2} = \frac{8}{9\pi} \text{ m/min}$$

* If at night a 6 ft. tall man is walking away at 3 ft/sec from a 18 ft. tall lampost. How fast is the length of his shadow changing?

$\Rightarrow$



Let,

x = distance of the man from the lamp post at time t.

$$\frac{dx}{dt} = 3 \text{ ft/sec.}$$

Let,

s = length of the shadow at time t.

$$\frac{ds}{dt} = ?$$

Now,

from figure by similar triangle,

$$\frac{18}{x+s} = \frac{6}{s}$$

$$18s = 6x + 6s$$

$$12s = 6x$$

$$s = \frac{1}{2}x$$

differentiate w.r.t. t

$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2} \times 3 = \frac{3}{2} \text{ ft/sec.}$$