H.W. from Lecture-5

Exact ODE

2.38/

$$(x^2+y^2+x)\,dx + xy\,dy = 0 \quad \cdots (i)$$

Here,

$M = x^2+y^2+x$ | $N = xy$

$\frac{\partial M}{\partial y} = 2y$ | $\frac{\partial N}{\partial x} = y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [Not Exact, Not homogenous]

$$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} \; [f(x)]$$

$$\therefore I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$\therefore$ (i) $\times$ I.F,

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0 \quad \cdots (ii)$$

Let,

$M' = x^3+xy^2+x^2$ | $N' = x^2y$

$\frac{\partial M'}{\partial y} = 2xy$ | $\frac{\partial N'}{\partial x} = 2xy$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ [(ii) Exact ODE]

$\therefore$ integrating (ii)

$$\int_{y=\text{const.}} (x^3 + xy^2 + x^2)\, dx = 0$$

$$\Rightarrow \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$$

Ans

2.40/

$$(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0 \quad \text{(i)}$$

Let,

$M = 2xy^4e^y + 2xy^3 + y$

$\frac{\partial M}{\partial y} = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$

$N = x^2y^4e^y - x^2y^2 - 3x$

$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ [Not Exact]}$$

$$\therefore \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{\cancel{2xy^4e^y} + 8xy^3e^y + 6xy^2 + 1 - \cancel{2xy^4e^y} + 2xy^2 + 3}{x^2y^4e^y - x^2y^2 - 3x}$$

$$= \frac{8xy^3e^y + 8xy^2 + 4}{x^2y^4e^y - x^2y^2 - 3x} \quad [f(x,y)]$$

$$\therefore \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = \frac{\cancel{2xy^4e^y} - 2xy^2 - 3 - \cancel{2xy^4e^y} - 8xy^3e^y - 6xy^2 - 1}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-4\left(2xy^3e^y + 2xy^2 + 1\right)}{y\left(2xy^3e^y + 2xy^2 + 1\right)}$$

$$= -\frac{4}{y}$$

$$\therefore \text{I.F.} = e^{\int -\frac{4}{y}dy} = e^{-4\ln y} = e^{\ln y^{-4}} = \frac{1}{y^4}$$

$\therefore$ (i) $\times$ I.F.,

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0 \quad \text{--- (ii)}$$

Let,

$$M' = 2xe^y + \frac{2x}{y} + \frac{1}{y^3} \qquad N' = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$$

$$\frac{\partial M'}{\partial y} = 2xe^y - \frac{2x}{y^2} - 3\frac{1}{y^4} \qquad \frac{\partial N'}{\partial x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad \text{[(ii) Exact ODE]}$$

integrating (ii),

$$\int_{y=const.} \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right) dx = 0$$

$$\Rightarrow \quad x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

Ans

2.41/

$$y^2 dy + y dx - x dy = 0$$

$$\Rightarrow \quad y dx + (y^2 - x) dy = 0 \quad \text{---- (i)}$$

Let,

$M = y$ | $N = y^2 - x$

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = -1$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ [Not Exact]}$$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{1+1}{y^2-x} = \frac{2}{y^2-x} \; [f(x,y)]$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = \frac{-1-1}{y} = \frac{-2}{y} \; [f(y)]$$

$$\therefore I.F. = e^{\int -\frac{2}{y} dy} = e^{-2\ln y} = \frac{1}{y^2}$$

(i) × I.F.

$$\frac{1}{y}\,dx + \left(1 - \frac{x}{y^2}\right) dy = 0 \quad \cdots (ii)$$

Let,

$M' = \frac{1}{y}$ | $N' = 1 - \frac{x}{y^2}$

$\frac{\partial M'}{\partial y} = -\frac{1}{y^2}$ | $\frac{\partial N'}{\partial x} = -\frac{1}{y^2}$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad \left[\text{(ii) Exact ODE}\right]$$

integrating (ii),

$$\int_{y=\text{const.}} \frac{1}{y}\,dx + \int dy = 0$$

$$\therefore \frac{x}{y} + \cancel{x}y = c$$

Ans

2·44)

$$y(2xy+1)\,dx + x(1 + 2xy - x^3y^3)\,dy = 0 \quad \cdots (i)$$

$$I.F. = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2 + \cancel{2}xy - xy - 2x^2y^2 + x^4y^4}$$

$$= \frac{1}{x^4y^4}$$

(i) $\times$ I.F.

$$\left(\frac{2xy^2}{x^4y^4} + \frac{y}{x^4y^4}\right)dx + \left(\frac{x}{x^4y^4} + \frac{2x^2y}{x^4y^4} - \frac{x^4y^3}{x^4y^4}\right)dy = 0$$

$$\Rightarrow \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}\right)dy = 0 \quad \text{--- (ii)}$$

Let,

$$M' = \frac{2}{x^3y^2} + \frac{1}{x^4y^3} \qquad N' = \frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}$$

$$\frac{\partial M'}{\partial y} = -\frac{4}{x^3y^3} - \frac{3}{x^4y^4} \qquad \frac{\partial N'}{\partial x} = -\frac{3}{x^4y^4} - \frac{4}{x^3y^3}$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad \text{[(ii) Exact ODE]}$$

integrating (ii),

$$\int_{y=\text{const.}} \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \int -\frac{1}{y}\,dy = 0$$

$$\Rightarrow -\frac{1}{x^2y^2} - \frac{1}{3x^3y^3} - \ln y = C$$

Ans

2.45/

$$(2y + 3xy^2)\,dx + (x + 2x^2y)\,dy = 0 \quad \cdots \text{(i)}$$

Let,

$$M = 2y + 3xy^2 \qquad \Big| \qquad N = x + 2x^2y$$

$$\frac{\partial M}{\partial y} = 2 + 6xy \qquad \Big| \qquad \frac{\partial N}{\partial x} = 1 + 4xy$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad [\text{(i) Not Exact}]$$

$$\therefore I.F. = \frac{1}{Mx - Ny} = \frac{1}{2xy + 3x^2y^2 - xy - 2x^2y^2}$$

$$= \frac{1}{xy + x^2y^2}$$

$$= \frac{1}{xy(1+xy)}$$

(i) $\times$ I.F.,

$$\left(\frac{2}{x(1+xy)} + \frac{3y}{1+xy}\right)dx + \left(\frac{1}{y(1+xy)} + \frac{2x}{1+xy}\right)dy = 0 \quad \cdots \text{(ii)}$$

Let,

$$\cancel{M' = x + x^2y} \qquad \Big| \qquad N' =$$

$$\cancel{\frac{\partial M'}{\partial y} = x^2}$$

Let,

$$\cancel{M' = \frac{2}{x + x^2 y} + \frac{3y}{1+xy}}$$

$$\cancel{\frac{\partial m'}{\partial y} =}$$

Let,

$$M' = \frac{2}{x(1+xy)} + \frac{3y}{1+xy}$$

$$\frac{\partial m'}{\partial y} = -\frac{2}{x} \cdot \frac{1}{(1+xy)^2} \cdot x + \frac{(1+xy)3 - 3yx}{(1+xy)^2}$$

$$= \frac{-2}{(1+xy)^2} + \frac{3xy + 3 - 3xy}{(1+xy)^2}$$

$$= \frac{-2+3}{(1+xy)^2}$$

$$= \frac{1}{(1+xy)^2}$$

$$N' = \frac{1}{y(1+xy)} + \frac{2x}{1+xy}$$

$$\frac{\partial N'}{\partial x} = -\frac{1}{y} \cdot \frac{1}{(1+xy)^2} \cdot y + \frac{(1+xy)2 - 2xy}{(1+xy)^2}$$

$$= \frac{-1}{(1+xy)^2} + \frac{2 + 2xy - 2xy}{(1+xy)^2}$$

$$= \frac{1}{(1+xy)^2}$$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ [(ii) Exact ODE]

integrating (ii),

$$\int_{y=const.} \left(\frac{2}{x(1+xy)} + \frac{3y}{1+xy} \right) dx =$$

$$\Rightarrow \int \left(\frac{2}{x(1+0)} + \frac{2}{-\frac{1}{y}(1+xy)} + \frac{3y}{1+xy} \right) dx$$

$$\Rightarrow \int \left(\frac{2}{x} - \frac{2y}{1+xy} + \frac{3y}{1+xy} \right) dx$$

$$= 2\ln x - 2\ln(1+xy) + 3\ln(1+xy) + c$$

$$= 2\ln x + \ln(1+xy) + c$$

Again,

$$\frac{1}{y(1+xy)} + \frac{2x}{1+xy} = \frac{1+2xy}{y(1+xy)} = \frac{1+xy+xy}{y(1+xy)}$$

$$= \frac{1}{y} + \frac{x}{1+xy}$$

$$\therefore \int \frac{1}{y}\, dy = \ln y + c$$

$\therefore$ ~~Solution: $2\ln x - 2\ln(1+xy) + 3\ln(1+xy) + \ln y = c$~~

ii) Solution : $2\ln x + \ln(1+xy) + \ln y = \ln C$

$\therefore \; x^2 y(1+xy) = C$

Ans

from- Zill's Book

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Example: 2:

$$\left(e^{2y} - y\cos xy\right)dx + \left(2xe^{2y} - x\cos xy + 2y\right)dy = 0 \quad \cdots (i)$$

Let,

$$M = e^{2y} - y\cos xy$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - \left(-y\sin xy \cdot x + \cos xy\right)$$

$$= 2e^{2y} + xy\sin xy - \cos xy$$

$$N = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - \left(-x\sin xy \cdot y + \cos xy\right)$$

$$= 2e^{2y} + xy\sin xy - \cos xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [(i) \text{ Exact ODE}]$$

integrating (i),

$$\int_{y=\text{Const.}} (e^{2y} - y\cos xy)\,dx + \int 2y\,dy = 0$$

$$\Rightarrow xe^{2y} - \sin xy + y^2 = C$$

Ans

Example-3:

$$\frac{dy}{dx} = \frac{xy^2 - \cos x \sin x}{y(1-x^2)}$$

$$\Rightarrow y(1-x^2)\,dy = (xy^2 - \cos x \sin x)\,dx \quad \cdots (i)$$

$$\Rightarrow (\cos x \sin x - xy^2)\,dx + y(1-x^2)\,dy = 0$$

Let,

$M = \cos x \sin x - xy^2$

$\frac{\partial M}{\partial y} = -2xy$

$N = y(1-x^2) = y - x^2 y$

$\frac{\partial N}{\partial x} = -2xy$

Let,

$\sin x = u$

$\frac{du}{dx} = \cos x$

$\cos x\,dx = du$

$\int u\,du$

$= \frac{u^2}{2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [(i) Exact ODE]

integrating (i),

$$\int (\cos x \sin x - xy^2)\,dx + \int y\,dy = 0$$

$$\Rightarrow \frac{\sin^2 x}{2} - \frac{x^2y^2}{2} + \frac{y^2}{2} = C$$

$\therefore \sin^2 x - x^2y^2 + y^2 = c \quad \cdots (ii)$

Given,

$y(0) = 2$

$\therefore (ii) \Rightarrow \quad c = \sin^2 0 - 0 \cdot 2^2 + 4$

$= 4$

$\therefore \sin^2 x - x^2y^2 + y^2 = 4$

2.4

6)

$$2y - \frac{1}{x} + \cos 3x \frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y \sin 3x = 0$$

$$\Rightarrow \cos 3x \frac{dy}{dx} = 4x^3 - 2y + \frac{1}{x} - \frac{y}{x^2} - 3y \sin 3x$$

$$\Rightarrow \cos 3x \frac{dy}{dx} = \frac{4x^5 - 2x^2y + x - y - 3x^2y \sin 3x}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^5 - 2x^2y + x - y - 3x^2y \sin 3x}{x^2 \cos 3x}$$

$$\Rightarrow (x^2 \cos 3x)\, dy = (4x^5 - 2x^2y + x - y - 3x^2y \sin 3x)\, dx$$

$$\therefore (3x^2y \sin 3x - 4x^5 + 2x^2y - x + y)\, dx + (x^2 \cos 3x)\, dy = 0 \quad \cdots (i)$$

Let,

$$M = 3x^2y\sin 3x - 4x^5 + 2x^2y - x + y$$

$$\frac{\partial M}{\partial y} = 3x^2\sin 3x + 2x^2 + 1$$

$$N = x^2\cos 3x$$

$$\frac{\partial N}{\partial x} = -x^2\cdot\sin 3x\cdot 3 + \cos 3x\cdot 2x$$

$$= -3x^2\sin 3x + 2x\cos 3x$$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [(i) Not Exact]

8|

$$\left(1 + \ln x + \frac{y}{x}\right)dx = (1 - \ln x)\,dy$$

$$\cancel{\frac{y}{x}\,dx - (1 - \ln x)\,dy = -(1 + \ln x)}$$

$$\Rightarrow \left(1 + \ln x + \frac{y}{x}\right)dx + (\ln x - 1)\,dy = 0 \quad \text{---(i)}$$

Let,

$M = 1 + \ln x + \frac{y}{x}$ | $N = \ln x - 1$

$\frac{\partial M}{\partial y} = \frac{1}{x}$ | $\frac{\partial N}{\partial x} = \frac{1}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [(i) Exact ODE] integrating ①,

$$\int_{y=\text{const}} \left(1 + \ln x + \frac{y}{x}\right) dx + \int - dy = 0$$

$$\Rightarrow x + x\ln x - x + y\ln x - y = c$$

$$\therefore x\ln x + y\ln x - y = c$$

6)

$$\left(2y - \frac{1}{x} + \cos 3x\right)\frac{dy}{dx} + \frac{y}{x^2} - 4x^3 + 3y\sin 3x = 0$$

$$\Rightarrow \left(2y - \frac{1}{x} + \cos 3x\right) dy = \left(-\frac{y}{x^2} + 4x^3 - 3y\sin 3x\right) dx$$

$$\Rightarrow \left(\frac{y}{x^2} - 4x^3 + 3y\sin 3x\right) dx + \left(2y - \frac{1}{x} + \cos 3x\right) dy = 0 \quad \text{--- (7)}$$

Let,

$$M = \frac{y}{x^2} - 4x^3 + 3y\sin 3x \qquad N = 2y - \frac{1}{x} + \cos 3x$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2} + 3\sin 3x \qquad \frac{\partial N}{\partial x} = \frac{1}{x^2} - 3\sin 3x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ [Not Exact]}$$

11)

$$\left(y\ln y - e^{-xy}\right)dx + \left(\frac{1}{y} + x\ln y\right)dy = 0$$

Let,

$$M = y\ln y - e^{-xy} \qquad N = \frac{1}{y} + x\ln y$$

$$\frac{\partial M}{\partial y} = y\cdot\frac{1}{y} + \ln y - e^{-xy}\cdot(-x) \qquad \frac{\partial N}{\partial x} = \ln y$$

$$= 1 + \ln y + xe^{-xy}$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{[Not Exact]}$$

15)

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)\frac{dx}{dy} + x^3y^2 = 0$$

$$\Rightarrow \left(x^2y^3 - \frac{1}{1+9x^2}\right)dx = -x^3y^2\,dy$$

$$\Rightarrow x^3y^2\,dx + \left(x^2y^3 - \frac{1}{1+9x^2}\right)dy = 0$$

Let,

$$M = x^3y^2 \qquad N = x^2y^3 - \frac{1}{1+9x^2}$$

$$\frac{\partial M}{\partial y} = 2x^3y \qquad \frac{\partial N}{\partial x} = 2xy^3$$

15|

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)\frac{dx}{dy} + x^3y^2 = 0$$

$$\left(x^2y^3 - \frac{1}{1+9x^2}\right)dx + x^3y^2\,dy = 0 \quad \cdots (i)$$

Let,

$$M = x^2y^3 - \frac{1}{1+9x^2} \qquad N = x^3y^2$$

$$\frac{\partial M}{\partial y} = 3x^2y^2 \qquad \frac{\partial N}{\partial x} = 3x^2y^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [(i) \text{ Exact ODE}]$$

integrating (i),

$$\int_{y=\text{const.}} \left(x^2y^3 - \frac{1}{1+9x^2}\right)dx = 0$$

$$\Rightarrow \frac{x^3y^3}{3} - \frac{1}{9}\cdot 3\,\tan^{-1}\frac{x}{1/3} = c$$

$$\cancel{\therefore x^3y^3 - \frac{1}{9}\tan^{-1}3x = c}$$

$$\therefore x^3y^3 - \tan^{-1}3x = c$$

17|

$(\tan x - \sin x \sin y)\,dx + \cos x \cos y\,dy = 0$ --- (i)

Let,

$M = \tan x - \sin x \sin y$ | $N = \cos x \cos y$

$\frac{\partial M}{\partial y} = -\sin x \cos y$ | $\frac{\partial N}{\partial x} = -\sin x \cos y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [(i) Exact ODE]

integrating (i),

$\int(\tan x - \sin x \sin y)\,dx = 0$

$\therefore \ln(\sec x) + \cos x \sin y = c$

Ans

18|

$(2y \sin x \cos x - y + 2y^2 e^{xy^2})\,dx = (x - \sin^2 x - 4xye^{xy^2})\,dy$

$\Rightarrow (2y \sin x \cos x - y + 2y^2 e^{xy^2})\,dx + (\sin^2 x - x + 4xye^{xy^2})\,dy = 0$ --- (i)

Let,

$M = 2y \sin x \cos x - y + 2y^2 e^{xy^2}$

$\frac{\partial M}{\partial y} = 2 \sin x \cos x - 1 + 2(y^2 \cdot e^{xy^2} \cdot 2xy + e^{xy^2} \cdot 2y)$

$= 2 \sin x \cos x - 1 + 4xy^3 e^{xy^2} + 4y e^{xy^2}$

Let

$N = \sin^2 x - x + 4xye^{xy^2}$

$\frac{\partial N}{\partial x} = 2\sin x\cos x - 1 + 4y\left(x \cdot e^{xy^2} \cdot y^2 + e^{xy^2} \cdot 1\right)$

$= 2\sin x\cos x - 1 + 4xy^3 e^{xy^2} + 4ye^{xy^2}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [ⓘ Exact ODE]

integrating ⓘ,

$\int\limits_{y=const.} \left(2y\sin x\cos x - y + 2y^2 e^{xy^2}\right) dx = 0$

$\Rightarrow \int \left(y\sin 2x - y + 2y^2 e^{xy^2}\right) dx = 0$

$\Rightarrow -y \cdot \frac{1}{2}\cos 2x - xy + 2y^2 \cdot \frac{1}{y^2} \cdot e^{xy^2} = c$

$\Rightarrow -\frac{y\cos 2x}{2} - xy + \frac{2e^{xy^2}}{1} = c$

$\therefore y\cos 2x + 2xy - 4e^{xy^2} = c$

Ans

25)

$$(y^2\cos x - 3x^2y - 2x)\,dx + (2y\sin x - x^3 + \ln y)\,dy = 0 \quad \cdots (i)$$

Let,

$M = y^2\cos x - 3x^2y - 2x$

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2$

$N = 2y\sin x - x^3 + \ln y$

$\frac{\partial N}{\partial x} = 2y\cos x - 3x^2$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [(i) Exact ODE]

integrating (i),

$$\int_{y=const.} (y^2\cos x - 3x^2y - 2x)\,dx + \int \ln y\,dy = 0$$

$$\Rightarrow y^2\sin x - x^3y - x^2 + y\ln y - y = c$$

Given, $y(0) = e$

$$\therefore c = e^2.\sin 0 - 0.e - 0 + e.\ln e - e$$

$$= 0$$

$$\therefore y^2\sin x - x^3y - x^2 + y\ln y - y = 0 \quad \text{Ans}$$

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$$\left(\frac{1}{1+y^2} + \cos x - 2xy\right)\frac{dy}{dx} = y(y+\sin x)$$

$$\Rightarrow \quad y(y+\sin x)\,dx + \left(2xy - \cos x - \frac{1}{1+y^2}\right)dy = 0 \quad \cdots \text{(i)}$$

Let,

$M = y(y+\sin x)$

$M = y^2 + y\sin x$

$\therefore \frac{\partial M}{\partial y} = 2y + \sin x$

$N = 2xy - \cos x - \frac{1}{1+y^2}$

$\frac{\partial N}{\partial x} = 2y + \sin x$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [\text{(i) Exact ODE}]$$

integrating (i),

$$\int_{y=\text{const.}} y(y+\sin x)\,dx - \int \frac{1}{1+y^2}\,dy = 0$$

$$\Rightarrow \quad xy^2 - y\cos x - \tan^{-1} y = c$$

Given, $y(0) = 1$

$$\therefore c = 0 - 1\cdot 1 - \tan^{-1}(1)$$

$$= -1 - \frac{\pi}{4}$$

$$\therefore xy^2 - y\cos x - \tan^{-1} y = -\left(1 + \frac{\pi}{4}\right)$$

Ans

34|

$\cos x\, dx + \left(1+\frac{2}{y}\right)\sin x\, dy = 0 \quad \cdots (i)$

Let,

$M = \cos x$ | $N = \left(1+\frac{2}{y}\right)\sin x = \cancel{\sin x + \frac{2\sin x}{y}}$

$\frac{\partial M}{\partial y} = \cancel{-\sin x}\ 0$ | $\frac{\partial N}{\partial x} = \left(1+\frac{2}{y}\right)\cos x = \cos x + \frac{2}{y}\cos x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [(i) Not Exact]

$\cancel{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{-\sin x - \cos x - \frac{2}{y}\cos x}{\left(1+\frac{2}{y}\right)\sin x}}$ $\cancel{[f(x,y)]}$

$\cancel{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = \frac{\left(1+\frac{2}{y}\right)\cos x + \sin x}{\cos x}}$

$=$

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{-\left(1+\frac{2}{y}\right)\cos x}{\left(1+\frac{2}{y}\right)\sin x} = -\cot x \quad [f(x)]$$

$$\therefore I.F. = e^{\int -\cot x\, dx} = e^{-\ln(\sin x)} = \sin^{-1} x = \operatorname{cosec} x$$

(i) $\times$ I.F,

$$\cos x \operatorname{cosec} x\, dx + \left(1+\frac{2}{y}\right)\sin x \operatorname{cosec} x\, dy = 0$$

$$\therefore \cot x\, dx + \left(1+\frac{2}{y}\right) dy = 0 \quad \cdots (ii)$$

Let,

$M' = \cot x$ | $N' = 1 + \frac{2}{y}$

$\frac{\partial M'}{\partial y} = 0$ | $\frac{\partial N'}{\partial x} = 0$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ [(ii) Exact ODE]

integrating (ii),

$$\int \cot x \, dx + \int \left(1 + \frac{2}{y}\right) dy$$

$$\therefore \ln(\sin x) + y + 2 \ln y = c$$

35)

$$(10 - 6y + e^{-3x})\, dx - 2\, dy = 0 \quad \text{--- (i)}$$

Let,

$M = 10 - 6y + e^{-3x}$ | $N = -2$

$\frac{\partial M}{\partial y} = -6$ | $\frac{\partial N}{\partial x} = 0$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [Not Exact]

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{-6-0}{-2} = 3$$

$$\therefore I.F. = e^{\int 3\,dx} = e^{3x}$$

(i) × I.F,

$$\left(10e^{3x} - 6ye^{3x} + e^{-3x}\cdot e^{3x}\right)dx - 2e^{3x}\,dy = 0 \quad \text{--- (ii)}$$

$$\therefore \left(10e^{3x} - 6ye^{3x} + 1\right)dx - 2e^{3x}\,dy = 0$$

Let,

$$M' = 10e^{3x} - 6ye^{3x} + 1 \qquad N' = -2e^{3x}$$

$$\frac{\partial M'}{\partial y} = -6e^{3x} \qquad \frac{\partial N'}{\partial x} = -2\cdot e^{3x}\cdot 3 = -6e^{3x}$$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad \text{[(ii) Exact ODE]}$$

integrating (ii),

$$\int\left(10e^{3x} - 6ye^{3x} + 1\right)dx = 0$$

$$\therefore \frac{10e^{3x}}{3} - 2ye^{3x} + x = c$$

$$\therefore 10e^{3x} - 6ye^{3x} + x = c$$

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$$(y^2+xy^3)\,dx + (5y^2 - xy + y^3\sin y)\,dy = 0 \quad \cdots \text{(i)}$$

Let,

$M = y^2+xy^3$ | $N = 5y^2 - xy + y^3\sin y$

$\frac{\partial M}{\partial y} = 2y + 3xy^2$ | $\frac{\partial N}{\partial x} = -y$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad [\text{Not Exact}]$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = \frac{-y - 2y - 3xy^2}{y^2+xy^3}$$

$$= \frac{-3y - 3xy^2}{y^2+xy^3}$$

$$= \frac{-3y(1+xy)}{y^2(1+xy)}$$

$$= -\frac{3}{y} \quad [f(y)]$$

$$\therefore \text{I.F.} = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = \frac{1}{y^3}$$

(i) × I.F.,

$$\left(\frac{1}{y} + x\right)dx + \left(\frac{5}{y} - \frac{x}{y^2} + \sin y\right)dy = 0 \quad \cdots \text{(ii)}$$

Let $M' = \frac{1}{y} + x$ | $N' = \frac{5}{y} - \frac{x}{y^2} + \sin y$

$\frac{\partial M'}{\partial y} = -\frac{1}{y^2}$ | $\frac{\partial N'}{\partial x} = -\frac{1}{y^2}$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad [\text{(ii) Exact ODE}]$$

integrating (ii),

$$\int \left(\frac{1}{y} + x\right) dx + \int \left(\frac{5}{y} + \sin y\right) dy = 0$$

$$\therefore \frac{x}{y} + \frac{x^2}{2} + 5 \ln y - \cos y = c$$

H.W. ⇒ from Lecture - 6

Linear ODE

1)

$$x \ln \frac{dy}{dx} + y = 2 \ln x$$

$$\frac{dy}{dx} + \frac{y}{x \ln} = \frac{2 \ln x}{x \ln}$$

2) ~~$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$~~

~~$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}$~~

from Zill's book

2.3

9) $x\frac{dy}{dx} - y = x^2 \sin x$

$\frac{dy}{dx} - \frac{1}{x}\cdot y = x\sin x \quad \cdots (i)$

Here, $P(x) = -\frac{1}{x}$

$\therefore$ I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

(i) × I.F.

$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}y = \sin x$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}\cdot y\right) = \sin x$

$\Rightarrow \int d\left(\frac{y}{x}\right) = \int \sin x\,dx$

$\therefore \frac{y}{x} = -\cos x + c$

|2|

$$(1+x)\frac{dy}{dx} - xy = x+x^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1+x}\cdot y = \frac{x(1+x)}{(1+x)}$$

$$\therefore \frac{dy}{dx} - \frac{x}{1+x}\cdot y = x \quad \text{---(i)}$$

Here,

$$p(x) = -\frac{x}{1+x}$$

$$\therefore I.F. = e^{\int -\frac{x}{1+x}dx} = e^{-1-x+\ln(1+x)} = e^{\ln(1+x)-x-1}$$

$$= \frac{e^{\ln(1+x)}}{e^x \cdot e^1}$$

$$= \frac{\ln(1+x)}{e^{x+1}}$$

$\int -\frac{x}{1+x}dx$

Let,

$u = 1+x \Rightarrow x = u-1$

$\frac{du}{dx} = 1$

$dx = du$

$-\int \frac{u-1}{u}du$

$-\int(1-\frac{1}{u})du$

$= -u + \ln u$

$= -1-x+\ln(1+x)$

$+c$

(i) $\times$ I.F,

$$e^{\ln(1+x)-x-1}\cdot\frac{dy}{dx} - e^{\ln(1+x)-x-1}\cdot\frac{x}{1+x}\cdot y = xe^{\ln(1+x)-x-1}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-1-x+\ln(1+x)}\cdot y\right) = xe^{\ln(1+x)-x-1}$$

$$\Rightarrow \int d\left(e^{\ln(1+x)-x-1}\cdot y\right) = \int xe^{\ln(1+x)-x-1}\cdot dx$$

$$\Rightarrow ye^{\ln(1+x)-x-1} = \int x(1+x)\cdot e^{-(x+1)}dx$$

Let,

$u = 1+x \Rightarrow x = u-1$

$\frac{du}{dx} = 1$

$du = dx$

$\int (u-1)ue^{-u}du$

$= \int(u^2e^{-u} - ue^{-u})du$

$=$

u^2 +	e^{-u}
$2u$ −	$-e^{-u}$
2 +	e^{-u}
0	$-e^{-u}$

u +	e^{-u}
1 −	$-e^{-u}$
0	e^{-u}

$$\Rightarrow \frac{(1+x)y}{e^{x+1}} = -u^2e^{-u} - 2ue^{-u} - 2e^{-u} + ue^{-u} + e^{-u}$$

$$\therefore \frac{(1+x)y}{e^{x+1}} = -(1+x)^2e^{-(1+x)} - 2(1+x)e^{-(1+x)} - 2e^{-(1+x)} + (1+x)e^{-(1+x)} + e^{-(1+x)}$$

13)

$$x^2 \frac{dy}{dx} + x(x+2)y = e^x$$

$$\frac{dy}{dx} + \frac{x+2}{x} \cdot y = \frac{e^x}{x^2} \quad \text{---} (1)$$

Here $\quad p(x) = \frac{x+2}{x} = 1 + \frac{2}{x}$

$$\therefore I.F. = e^{\int (1+\frac{2}{x}) dx} = e^{x+2\ln x} = e^x \cdot e^{2\ln x} = x^2 e^x$$

(1) $\times$ I.F.,

$$x^2 e^x \frac{dy}{dx} + x(x+2)e^x y = e^{2x}$$

$$\Rightarrow \frac{d}{dx}(x^2 e^x \cdot y) = e^{2x}$$

$$\Rightarrow \int d(x^2 e^x \cdot y) = \int e^{2x} dx$$

$$\therefore x^2 y e^x = \frac{1}{2} e^{2x} + c$$

14)

$$x \frac{dy}{dx} + (1+x)y = e^{-x} \sin 2x$$

$$\therefore \frac{dy}{dx} + \frac{1+x}{x} \cdot y = \frac{e^{-x} \sin 2x}{x} \quad \text{---} (1)$$

Here, $\quad p(x) = \frac{1+x}{x} = \frac{1}{x} + 1$

$$\therefore I.F. \; e^{\int (1+\frac{1}{x}) dx} = e^{x+\ln x} = e^x \cdot e^{\ln x} = x e^x$$

(i) × I.F,

$$x e^{x} \frac{dy}{dx} + (1+x)e^{x} \cdot y = \sin 2x$$

$$\Rightarrow \frac{d}{dx}\left(x e^{x} y\right) = \sin 2x$$

$$\Rightarrow \int d\left(x e^{x} y\right) = \int \sin 2x \, dx$$

$$\therefore \quad x e^{x} y = -\frac{1}{2}\cos 2x + c$$

|6|

$$y \, dx = \left(y e^{y} - 2x\right) dy$$

$$\cancel{\frac{dy}{dx} = \frac{y e^{y} - 2x}{y} \quad \frac{y}{y e^{y} - 2x}}$$

$$\frac{dx}{dy} = \frac{y e^{y} - 2x}{y}$$

$$= e^{y} - \frac{2x}{y}$$

$$\therefore \quad \frac{dx}{dy} + \frac{2}{y} \cdot x = e^{y} \quad \text{--- (i)}$$

$$\frac{dx}{dy} = \frac{y e^{y} - 2x}{y}$$

$$= e^{y} - \frac{2x}{y}$$

$$\frac{dx}{dy} + \frac{2}{y} \cdot x = e^{y}$$

Here,

$$P(y) = \frac{2}{y}$$

$$\text{I.F.} = e^{\int \frac{2}{y} dy} = e^{2\ln y} = y^{2}$$

(ii)

① × I.F.,

$$y^2 \frac{dx}{dy} + 2yx = y^2 e^y$$

$$\Rightarrow \frac{d}{dy}(y^2 \cdot x) = y^2 e^y$$

$$\Rightarrow \int d(xy^2) = \int y^2 e^y \, dy$$

$$\therefore xy^2 = y^2 e^y - 2y e^y + 2e^y + c$$

$y^2 \; + \; e^y$
$2y \; - \; e^y$
$2 \; + \; e^y$
$0 \quad e^y$

17|

$$\cos x \frac{dy}{dx} + (\sin x) y = 1$$

$$\therefore \frac{dy}{dx} + \tan x \, y = \sec x \quad \text{—①}$$

$-\int \frac{\sin x}{\cos x} dx$

$-\ln(\cos x)$

$\ln(\sec x)$

Hens $\quad p(x) = \tan x$

$$\therefore I.F. = e^{\int \tan x \, dx} = e^{\ln(\sec x)} = \sec x$$

① × I.F.,

$$\sec x \frac{dy}{dx} + \tan x \cdot \sec x \, y = \sec^2 x$$

$$\frac{d}{dx}(y \sec x) = \sec^2 x$$

$$\int d(y \sec x) = \int \sec^2 x \, dx$$

$$\therefore y \sec x = \tan x + c$$

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$$\cos^2 x \sin x \frac{dy}{dx} + (\cos^3 x) y = 1$$

$$\therefore \frac{dy}{dx} + (\cot x) y = \sec^2 x \operatorname{cosec} x \quad \cdots (i)$$

Here

$$P(x) = \cot x$$

$$\therefore I.F. \; e^{\int \cot x \, dx} = e^{\ln \sin x} = \cancel{\operatorname{cosec} x} \sin x$$

(i) × I.F.,

$$\sin x \frac{dy}{dx} + (\cot x \sin x) y = \sec^2 x$$

$$\Rightarrow \frac{d}{dx} (\sin x \cdot y) = \sec^2 x$$

$$\Rightarrow \int d(y \sin x) = \int \sec^2 x \, dx$$

$$\therefore y \sin x = \tan x + c$$

21|

$$\frac{dr}{d\theta} + r\sec\theta = \cos\theta \quad \text{(i)}$$

Here,

$$p(\theta) = \sec\theta$$

$$\therefore \text{I.F.} \quad e^{\int \sec\theta \, d\theta} = e^{\ln(\sec\theta + \tan\theta)} = \sec\theta + \tan\theta$$

(i) × I.F.,

$$(\sec\theta + \tan\theta)\frac{dr}{d\theta} + (\sec^2\theta + \sec\theta\tan\theta)r = \sec\theta\cdot\cos\theta + \tan\theta\cdot\cos\theta$$

$$\Rightarrow \frac{d}{d\theta}\left((\sec\theta + \tan\theta)r\right) = 1 + \sin\theta$$

$$\Rightarrow \int d(\sec\theta + \tan\theta)r = \int (1 + \sin\theta)\,d\theta$$

$$\therefore (\sec\theta + \tan\theta)r = \theta - \cos\theta + c$$

Ans

33| 34|

Non Linear ODE

1|

$$\frac{dy}{dx} + \frac{x}{1-x^2} y = x\sqrt{y}$$

Solved in class Notes:

2|

$$(1-x^2)\frac{dy}{dx} + xy = xy^2$$

$$\Rightarrow (1-x^2)y^{-2}\frac{dy}{dx} + xy^{-1} = x \quad \dots (i)$$

Let,

$$y^{-1} = u$$

$$-y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

$$(i) \Rightarrow -(1-x^2)\frac{du}{dx} + xu = x$$

$$\therefore \frac{du}{dx} + \frac{x}{x^2-1}u = \frac{x}{x^2-1} \quad \dots (ii)$$

Here,

$$P(x) = \frac{x}{x^2-1} = \frac{1}{2}\cdot\frac{2x}{(x^2-1)}$$

$$\therefore I.F. = e^{\int\left(\frac{1}{2}\cdot\frac{2x}{x^2-1}\right)dx} = e^{\frac{1}{2}\ln(x^2-1)} = \sqrt{x^2-1}$$

(ii) × I.F,

$$\sqrt{x^2-1}\,\frac{du}{dx} + \frac{x}{(x^2-1)}\cdot\sqrt{x^2-1}\;u = \frac{x}{(x^2-1)}\cdot\sqrt{x^2-1}$$

$$\Rightarrow \frac{d}{dx}\left(\sqrt{x^2-1}\cdot u\right) = \frac{x}{\sqrt{x^2-1}}$$

$$\Rightarrow \int d\left(u\sqrt{x^2-1}\right) = \int \frac{x}{\sqrt{x^2-1}}\,dx$$

$$\Rightarrow u\sqrt{x^2-1} = \sqrt{x^2-1} + c$$

Ans

Let,

$u = x^2-1$

$\frac{du}{dx} = 2x$

$x\,dx = \frac{1}{2}du$

$\int \frac{\frac{1}{2}du}{\sqrt{u}}$

$= \frac{1}{2}\int u^{-\frac{1}{2}}\,du$

$= \frac{1}{2}\cdot 2\sqrt{u} + c$

3||

$$\frac{dy}{dx} + (2x\tan^{-1}y - x^3)(1+y^2) = 0$$

$$\cancel{\Rightarrow \frac{dy}{dx} + 2x\tan^{-1}y - x^3 + 2xy^2\tan^{-1}y - x^3y^2 = 0}$$

$$\cancel{\Rightarrow \frac{dy}{dx} + y^2(2x\tan^{-1}y - x^3) = x^3 - 2x\tan^{-1}y}$$

$$\cancel{\Rightarrow \frac{dy}{dx} + (2x\tan^{-1}y - x^3)\,y^2 = x(x^2 - 2\tan^{-1}y)}$$

$$\Rightarrow \frac{1}{1+y^2}\frac{dy}{dx} + 2x\tan^{-1}y = x^3 \quad \text{--- (i)}$$

let, $\tan^{-1}y = u$

$$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{du}{dx}$$

(i) $\Rightarrow$ $\frac{du}{dx} + 2xu = x^3$ .. (ii)

Here,

$p(x) = 2x$

$\therefore I.F. = e^{\int 2x\,dx} = e^{x^2}$

(ii) $\times$ I.F.,

$$e^{x^2}\frac{du}{dx} + 2xe^{x^2}u = x^3 e^{x^2}$$

$$\Rightarrow \frac{d}{dx}(e^{x^2}.u) = x^3 e^{x^2}$$

$$\Rightarrow \int d(u.e^{x^2}) = \int x^3 e^{x^2}\,dx$$

Let,

$x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{1}{2}dz$

$$\Rightarrow u.e^{x^2} = \int x^2.x e^{x^2}\,dx$$

$$= \frac{1}{2}\int z.e^z\,dz$$

Dif	int
z	$+\ e^z$
1	$-\ e^z$
0	e^z

$$u.e^{x^2} = \frac{1}{2}(ze^z - e^z) + c$$

$$\tan^{-1}y\, e^{x^2} = \frac{1}{2}(x^2e^{x^2} - e^{x^2}) + c$$

Ans