Serial = / 33 /

Need to Recap ⇒ MAT 130

↳ $\boxed{7.1,\ 7.2,\ 7.4,\ 7.5}$

L-01 / 23.07.2023 /

Differential Equation (DE)

→ contains derivatives $\left(\frac{d}{dx}\right)$

⊛ Two types :

1. Ordinary differential equation (ODE)

   i. $\frac{dy}{dx} = 0$

   ii. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \ln x$

   iii. $5y'' + 6y' + y = \tan x$

2. Partial differential equation (PDE) → Not in this course

   i. $\frac{\partial y}{\partial x} + \frac{\partial u}{\partial y} = 0$

   ii. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

   iii. $z_t = c\, z_{xx}\ ;\ c > 0$

⊛ ODE

⇒ Order of ODE

i. $\frac{dy}{dx} + y = 0 \Rightarrow 1^{st}$ order ODE

ii. $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0 \Rightarrow 2^{nd}$ order ODE

⇒ Degree of ODE

i. $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + xy = 5$

↳ degree = 3

$\int e^{x^2} dx = \times$

⊛ Formation of ODE

$y = f(x)$ = explicit function

we must form the function in explicit form

$y = c$

$\boxed{\frac{dy}{dx} = 0}$ DE of

$\frac{d^2y}{dx^2} = 0$

$\vdots$

⊛ we need to remove arbitary constant.

⊛ $y = A\cos x + B\sin x$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$= -(A\cos x + B\sin x)$$

$$= -y$$

$$\boxed{\frac{d^2y}{dx^2} + y = 0} \rightarrow \text{DE}$$

H.W. ⇒ $\boxed{\text{P.No.: 04} \Rightarrow 1.2 - 1.5}$

⊛ Variable Seperable method

$$\frac{dy}{dx} = f(x)\,g(y)$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

$$\frac{dy}{dx} = 0$$

$$dy = 0 \times dx$$

$$dy = 0$$

$$\int dy = 0$$

$$\boxed{y = c}$$

$$f(x)\,dy + g(y)\,dx = 0$$

Multiply and divide both

$$\Rightarrow \int \frac{f(x)\,dy}{f(x)g(y)} + \int \frac{g(y)\,dx}{f(x)\,g(y)} = 0$$

$$x\,dy + y\,dx = 0$$

$$\Rightarrow \int \frac{x\,dy}{xy} + \int \frac{y\,dx}{xy} = 0$$

$$\Rightarrow \int \frac{dy}{y} + \int \frac{dx}{x} = 0$$

$$\Rightarrow \ln y + \ln x = \ln C$$

$$\Rightarrow \ln(xy) = \ln C$$

$$\boxed{xy = C}$$

$$\sqrt{1-x^2}\,dy + (1+y^2)\,dx = 0$$

$$\Rightarrow \int \frac{dy}{1+y^2} + \int \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \boxed{\tan^{-1}y + \sin^{-1}x = C}$$

H. W. $\Rightarrow$ $\boxed{\text{P.N.: } 11 \Rightarrow 2.1 - 2.5}$

L-02/25.07.2023

Recap → L-01

PE

ODE

$$\frac{dy}{dx} + y = 0$$

PDE

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Order of ODE

$$\frac{dy}{dx} + y = 0 \rightarrow 1$$

$$\frac{d^2y}{dx^2} + y = \ln x \rightarrow 2$$

Degree of ODE

$$\left(\frac{dy}{dx}\right)^2 + y = 0 \rightarrow 2$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + y = 0 \rightarrow 2$$

⊛ Formation of ODE

$$y = A\cos x + B\sin x$$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x = -y$$

$$\boxed{\frac{d^2y}{dx^2} + y = 0}$$

⊛ Variable seperation method for solving ODE

⊛ $\frac{dy}{dx} = f(x)\,g(y)$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\,dx$$

⊛ $f(x)\,dy + g(y)\,dy = 0$

$$\Rightarrow \int \frac{f(x)\,dy}{f(x)\,g(y)} + \int \frac{g(y)\,dy}{f(x)\,g(y)} = 0$$

* Reduced to seperation form

* $\frac{dy}{dx} = (x+y)^2$

Let,

$x+y = u$

$1 + \frac{dy}{dx} = \frac{du}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{du}{dx} - 1 = u^2$

$\Rightarrow \frac{du}{dx} = (u^2+1)$

$\Rightarrow \int \frac{du}{u^2+1} = \int dx$

$\Rightarrow \tan^{-1} u = x + c$

$\Rightarrow \tan^{-1}(x+y) = x + c$

* $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

Let, $x+y = u$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{dy}{dx} = \sin(x+y)$

$\Rightarrow \frac{du}{dx} - 1 = \sin u$

$\Rightarrow \frac{du}{dx} = \sin u + 1$

$\Rightarrow \int \frac{du}{\sin u + 1} = \int dx$

$\Rightarrow \int \frac{1-\sin u}{1-\sin^2 u} du = x + c$

$\Rightarrow \int \frac{1-\sin u}{\cos^2 u} du = x + c$

$\Rightarrow \int (\sec^2 u - \sec u \tan u) du = x + c$

$\Rightarrow \tan u - \sec u = x + c$

Recap $\Rightarrow$ $\boxed{7.8 \Rightarrow \int \frac{1}{1+\tan x}\,dx \quad \& \quad \int \frac{\cos x}{\cos x+\sin x}\,dx}$

H.W $\Rightarrow$

1. $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$

2. $\frac{dy}{dx} = \sec(x+y)$

3. $\frac{dy}{dx} = \cos(x+y)$

4. $\frac{dy}{dx} = \frac{1}{1+\tan x}$

(*) $\frac{dy}{dx} = \frac{1-x-y}{x+y} = \frac{1-(x+y)}{(x+y)}$

Let, $x+y = u$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$\Rightarrow \frac{du}{dx} - 1 = \frac{1-u}{u}$

$\Rightarrow \frac{du}{dx} = 1 + \frac{1}{u} - 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{u}$

$\Rightarrow \int u\,du = \int dx$

$\Rightarrow \frac{u^2}{2} = x + c$

$\Rightarrow \frac{(x+y)^2}{2} = x + c$

$\therefore (x+y)^2 = 2x + c$ Ans

$$\circledast \quad \frac{dy}{dx} = \frac{x(2\ln x + 1)}{\sin y + y\cos y}$$

$$\Rightarrow \int (\sin y + y\cos y)\, dy = \int x\,(2\ln x + 1)\, dx$$

$$\Rightarrow \quad -\cos y + \cancel{\frac{y^2}{2}\sin y} = \int (2x\ln x + x)\, dx$$

$$\Rightarrow \quad -\cos y + \cancel{\frac{y^2}{2}\sin y} = \frac{x^2}{2} + \left[\ln x \int 2x\, dx - \int \left\{ \frac{d}{dx}(\ln x) \int 2x\, dx \right\} dx \right]$$

$$-\cos y + y\sin y + \cos y = \frac{x^2}{2} + \left[\ln x \cdot x^2 - \int \frac{1}{x} \cdot x^2\, dx \right]$$

$$y\sin y = \frac{x^2}{2} + x^2\ln x - \frac{x^2}{2} + C$$

$$\Rightarrow \cancel{-\cos y + \frac{y^2}{2}\sin y} = x^2 \ln x + C$$

$$y \sin y$$

Ans

$\circledast$ Homogeneous ODE

$$f(x,y)\,dy + g(x,y)\,dx = 0 \;\cdots (1)$$

$\Rightarrow$ if degree of $f(x,y)$ and $g(x,y)$ is same then (1) is homogeneous ODE.

Then if (1) is homogeneous, then we can write,

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \text{ form}$$

⊛ $(x^2+y^2)\,dy + 2xy\,dx = 0$

Here, $f(x,y) = x^2 + y^2 \Longrightarrow 2$

$g(x,y) = 2xy \Rightarrow 2$

Both same

⊛ $\frac{x^2}{y} = x^2 \cdot y^{-1} = \text{degree } 1$

⊛ $\underbrace{\underbrace{(x^{3}y^{1} + y^4)}_{4}\,dy + \underbrace{x^2y^2}_{4}\,dx = 0}_{\text{Same}}$

⊛ $(x^2+y^2)\,dy + 2xy\,dx = 0$

$$\frac{dy}{dx} = -\frac{2xy}{x^2+y^2}$$

$$= -\frac{2xy}{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$= -\frac{2\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2} = F\left(\frac{y}{x}\right)$$

realme Shot by Legend_T.JOY

$(x^3y + y^4)\, dy + x^2y^2\, dx = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2y^2}{x^3y+y^4}$$

Recap ⇛ integration by partial fraction

$$= -\frac{x^2y^2}{y(x^3+y^3)}$$

$$= -\frac{x^2y}{x^3+y^3}$$

$$= -\frac{\frac{y}{x}}{1+\frac{y^3}{x^3}}$$

$$= -\frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^3} = F\left(\frac{y}{x}\right)$$

$(x^2+y^2)\, dy + 2xy\, dx = 0$

$$\frac{dy}{dx} = \cancel{-\frac{2xy}{x^2+y^2}}\ \frac{x^2+y^2}{2xy}$$

Let,

$$v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

(1) $\Rightarrow$

$$v + x\frac{dv}{dx} = \cancel{-\frac{2 \cdot x \cdot vx}{x^2+(vx)^2}}\ \frac{x^2+(vx)^2}{2x\cdot vx}$$

$$= \cancel{-\frac{2v}{1+v^2}}\ \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \cancel{-\frac{2v}{1+v^2} - v}\ \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow (1-v^2)\,dx = 2vx\,dv$$

$$\Rightarrow \int \frac{1}{x}\,dx = \int \frac{2v}{1-v^2}\,dv$$

$$\Rightarrow \ln x = -\ln(1-v^2) + c$$

$$\Rightarrow \ln x + \ln(1-v^2) + c = 0$$

$$\Rightarrow \ln\left(x(1-v^2)\right) = \ln c$$

$$\Rightarrow x(1-v^2) = c$$

$$\Rightarrow x\left(1-\frac{y^2}{x^2}\right) = c$$

$$\therefore x - \frac{y^2}{x} = c$$

A

L-3/30.07.2023

Recap- Last Class

Recap - 7.5 (MAT130)

Homogeneous ODE

⊛ An ODE $f(x,y)\,dy + g(x,y)\,dx = 0 \cdots (i)$ is said to be homogeneous if its degree of $f(x,y)$ and $g(x,y)$ is same.

Then (i) can be written as,

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \text{ Form}$$

Ex.: $x^3 y\,dy + (x^4 + x^2y^2)\,dx = 0$

In this case we need to take,

Let, $\frac{y}{x} = v$

⊛ $xy\,dy - (x^2+y^2)\,dx = 0$

Let,

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{x \cdot vx}$$

$$= \frac{x^2(1+v^2)}{vx^2}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{1+v^2-v^2}{v} = \frac{1}{v}$$

$$\int v\,dv = \int \frac{1}{x}\,dx$$

This is fin, always $\frac{1}{x}$

$$\frac{v^2}{2} = \ln x + c$$

$$\frac{y^2}{x^2} = 2\ln x + c$$

Ans

(*) $(x+y)\,dy + (y-x)\,dx = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-(y-x)}{x+y} = \frac{x-y}{x+y}$$

Let,

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x-vx}{x+vx}$$

$$= \frac{x(1-v)}{x(1+v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-v}{1+v} - v = \frac{1-v-v(1+v)}{1+v}$$

$$= \frac{1-v-v-v^2}{1+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$$

$$\Rightarrow x(1+v)\,dv = (1-2v-v^2)\,dx$$

$$\Rightarrow \int \frac{1+v}{1-2v-v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\frac{1}{2}\int \frac{-2v-2}{1-2v-v^2}\,dv = \ln x + c$$

$\therefore -\frac{1}{2}\ln(1-2v-v^2) = \ln x + c$

$\therefore -\frac{1}{2}\ln\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right) = \ln x + c$

Ans

From MAT 116

$= -\int \frac{dv}{v^2+2v-1}$

$= -\int \frac{dv}{(v+\frac{1}{2}\cdot 2)^2-1-(1)^2}$

$= -\int \frac{dv}{(v+1)^2-(\sqrt{2})^2}$

$= -\frac{1}{2\cdot 1}\ln\left|\frac{(v+1)-\sqrt{2}}{(v+1)+\sqrt{2}}\right| + c$

$ax^2+bx+c.$

$= a[x^2+\frac{b}{a}x+\frac{c}{a}]$

$= a[(x+\frac{1}{2}\cdot\frac{b}{a})^2+\frac{c}{a}-(\frac{b}{2a})^2]$

$= a[(x+\frac{1}{2}\text{ coefficient of } x)^2+\frac{1}{a}-(\frac{1}{2}\text{ coefficient of } x)^2]$

⊛ Reduced to Homogeneous Form:

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

Let,

$x = \alpha + h$

$y = \beta + k$

$\frac{dx}{d\alpha} = 1 \Rightarrow dx = d\alpha$

$\frac{dy}{d\beta} = 1 \Rightarrow dy = d\beta$

$\therefore \frac{dy}{dx} = \frac{d\beta}{d\alpha}$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{a_1(\alpha+h) + b_1(\beta+k) + c_1}{a_2(\alpha+h) + b_2(\beta+k) + c_2}$$

$$= \frac{(a_1\alpha + b_1\beta) + (a_1 h + b_1 k + c_1)}{(a_2\alpha + b_2\beta) + (a_2 h + b_2 k + c_2)}$$

Now, we will choose h and k such a way that,

$$a_1 h + b_1 k + c_1 = 0$$
$$a_2 h + b_2 k + c_2 = 0$$

$$\frac{h}{b_1 c_2 - b_2 c_2} = \frac{k}{a_2 c_1 - a_1 c_2} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$h = \frac{b_1 c_2 - c_1 b_2}{a_1 b_2 - a_2 b_1} = P$$

$$k = \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} = Q$$

$$\boxed{\frac{d\beta}{d\alpha} = \frac{a_1\alpha + b_1\beta}{a_2\alpha + b_2\beta}} \rightarrow$$

Now solve it the replace variable

$x = \alpha + P \Rightarrow \alpha = x - P$

$y = \beta + Q \quad\quad \beta = y - Q$

Example:

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$$

Let,

$x = \alpha + h$

$y = \beta + k$

$$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{(\alpha+h)+(\beta+k)+1}{(\alpha+h)+(\beta+k)-1}$$

$$= \frac{(\alpha+\beta)+(h+k+1)}{(\alpha+\beta)+(h+k-1)}$$

Here,

$$\left.\begin{array}{l} h+k+1=0 \\ h+k-1=0 \end{array}\right. \Rightarrow \left.\begin{array}{l} h+k=-1 \\ h+k=1 \end{array}\right\} \text{Parallel line no solution}$$

- cannot makes zero, we can't solve using homogeneous.

We need to we reduced seperation form.

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$$

Let,

$u = x+y$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\frac{du}{dx} - 1 = \frac{u+1}{u-1}$$

$$\frac{du}{dx} = \frac{u+1}{u-1} + 1 = \frac{u+1+u-1}{u-1} = \frac{2u}{u-1}$$

$$\int \frac{u-1}{2u}\,du = \int dx$$

$$\Rightarrow \int \left(\frac{1}{2} - \frac{1}{2u}\cdot u'\right) du = x + c$$

$$\frac{1}{2}u - \frac{1}{2}\cdot \ln u = x + c$$

$$\therefore \frac{1}{2}(x+y) - \frac{1}{2}\ln(x+y) = x + c$$

Ans

⊛ $(3x - 7y - 3)\, dy = (3y - 7x + 7)\, dx$

$$\frac{dy}{dx} = \frac{3y - 7x + 7}{3x - 7y - 3}$$

Let,

$x = \alpha + h$

$y = \beta + k$

$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$

$$\frac{d\beta}{d\alpha} = \frac{-7(\alpha + h) + 3(\beta + k) + 7}{3(\alpha + h) - 7(\beta + k) - 3}$$

$$\frac{d\beta}{d\alpha} = \frac{(-7\alpha + 3\beta) + (-7h + 3k + 7)}{(3\alpha - 7\beta) + (3h - 7k - 3)}$$

Here,

$$-7h + 3k + 7 = 0$$
$$3h - 7k - 3 = 0$$

$$\Rightarrow \begin{aligned} -21h + 9k &= -21 \\ 21h - 49k &= 21 \\ \hline -40k &= 0 \end{aligned}$$

$$\therefore k = 0$$

$$\Rightarrow h = 1$$

$$x = \alpha + 1$$
$$y = \beta$$
$$\Rightarrow$$
$$\alpha = x - 1$$
$$\beta = y$$

Now,

$$\frac{d\beta}{d\alpha} = \frac{-7\alpha + 3\beta}{3\alpha - 7\beta}$$

Let,

$$v = \frac{\beta}{\alpha}$$

$$\beta = v\alpha$$

$$\frac{d\beta}{d\alpha} = v + \alpha \frac{dv}{d\alpha}$$

$$\Rightarrow v + \alpha \frac{dv}{d\alpha} = \frac{-7\alpha + 3v\alpha}{3\alpha - 7v\alpha}$$

$$= \frac{\alpha(-7+3v)}{\alpha(3-7v)}$$

$$\alpha \frac{dv}{d\alpha} = \frac{-7+3v}{3-7v} - v$$

$$= \frac{-7+3v-3v+7v^2}{3-7v}$$

$$\alpha \frac{dv}{d\alpha} = \frac{-7+7v^2}{3-7v}$$

$$\int \frac{3-7v}{-7+7v^2}\, dv = \int \frac{1}{\alpha}\, d\alpha$$

$$\Rightarrow -\frac{1}{2} \int \frac{14v-6}{7v^2-7}\, dv = \int \frac{1}{\alpha}\, d\alpha$$

$$\Rightarrow -\frac{1}{2}\int \left(\frac{14v}{7v^2-7} - \frac{6}{7v^2-7}\right) dv = \ln(\alpha) + c$$

$$\Rightarrow -\frac{1}{2}\left[\ln(7v^2-7) - \frac{6}{7}\int \frac{1}{v^2-1}\, dv\right] = \ln\alpha + c$$

$$\Rightarrow -\frac{1}{2}\left[\ln(7v^2-7) - \frac{6}{7}\cdot\frac{1}{2}\ln\left|\frac{v-1}{v+1}\right|\right] = \ln\alpha + c$$

$$-\frac{1}{2}\ln(7v^2-7) + \frac{6}{28}\ln\left|\frac{v-1}{v+1}\right| = \ln\alpha + c$$

$$-\frac{1}{2}\ln\left|7\frac{\beta^2}{\alpha^2} - 7\right| + \frac{6}{28}\ln\left|\frac{\frac{\beta}{\alpha}-1}{\frac{\beta}{\alpha}+1}\right| = \ln\alpha + c$$

$$-\frac{1}{2}\ln\left|7\frac{y^2}{(x-1)^2} - 7\right| + \frac{6}{28}\ln\left|\frac{\frac{y}{x-1}-1}{\frac{y}{x-1}+1}\right| = \ln(x-1) + c$$

Ans

H.W.

1. $(3x-2y)\,dy = (2x-3y)\,dx$
2. $x\,dy - y\,dx = 2\sqrt{y^2-x^2}\,dx$
3. $xy^2\,dy = (x^3+y^3)\,dx$
4. $(2\sqrt{xy} - y)\,dx - x\,dy = 0$

* 5. $-y\,dx + (x+\sqrt{xy})\,dy = 0$

* 6. $x\frac{dy}{dx} = y + \sqrt{x^2-y^2}$

R.H. form

1. $\frac{dy}{dx} = \frac{2x+2y-2}{3x+y-5}$
2. $\frac{dy}{dx} = \frac{x+2y+3}{2x+y+3}$
3. $\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$

L-4 / 01.08.2023 /

Recap

Reduced to Homogeneous Form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$ → 1st Degree

$x = \alpha + h$

$y = \beta + k$

$\therefore \frac{dy}{dx} = \frac{d\beta}{d\alpha}$

$$\frac{d\beta}{d\alpha} = \frac{a_1(\alpha+h) + b_1(\beta+k) + c_1}{a_2(\alpha+h) + b_2(\beta+k) + c_2}$$

(i) $$\boxed{\frac{d\beta}{d\alpha} = \frac{a_1\alpha + b_1\beta}{a_2\alpha + b_2\beta}}$$

here, $h = Q$

$k = P$

Let,

$\beta = v\alpha$

$$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha}$$

(ii) $$v + \alpha\frac{dv}{d\alpha} = \frac{a_1\alpha + b_1v\alpha}{a_2\alpha + b_2v\alpha}$$

$$v + \alpha\frac{dv}{d\alpha} = \frac{a_1 + b_1v}{a_2 + b_2v}$$

$$\boxed{\alpha\frac{dv}{d\alpha} = \frac{a_1 + b_1v}{a_2 + b_2v} - v}$$

⊛ Exact ODE

$$M = y \Rightarrow \frac{\partial M}{\partial y} = 1$$
$$N = x \Rightarrow \frac{\partial N}{\partial x} = 1$$

Rules Satisfied

⊛ $x\,dy + y\,dx = 0$

$$\Rightarrow \frac{d}{dx}(xy) = 0$$

$$x\frac{dy}{dx} + y \cdot 1 = 0$$
$$x\,dy + y\,dx = 0$$

$$d(xy) = 0 \cdot dx$$
$$\int d(xy) = 0$$
$$\boxed{xy = c}$$

⊛
$$\frac{y\,dx - x\,dy}{y^2} = 0$$
$$\frac{d}{dx}\left(\frac{x}{y}\right) = 0$$
$$\int d\left(\frac{x}{y}\right) = 0$$
$$\boxed{\frac{x}{y} = c}$$

$$\int d(x) = x + c$$
$$\int d(y) = y + c$$
$$\int d(NSV) = NSV + c$$

⊛ Sufficient condition for exactness:

An equation $M(x,y)\,dx + N(x,y)\,dy = 0$ ⓘ is said to exact ODE if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

integrating ⓘ,

$$\underset{y = const}{\int M(x,y)\,dx} + \underset{x \text{ free term}}{\int N(x,y)\,dy} = c$$

Example:

$$y\,dx + x\,dy = 0$$

$$\int_{y=const} y\,dx = c$$

$$xy = c$$

$$\frac{y\,dx - x\,dy}{y^2} = 0$$

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0$$

$$M = \frac{1}{y}, \quad N = -\frac{x}{y^2}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} \; ; \; \frac{\partial N}{\partial x} = -\frac{1}{y^2}$$

$$\int \frac{1}{y}dx = c$$

$$\frac{x}{y} = c$$

$$(2x+3y+4)\,dx + (3x-6y-5)\,dy = 0 \quad \text{......(i)}$$

$M = 2x+3y+4$ | $N = 3x-6y-5$

$\frac{\partial M}{\partial y} = 3$ | $\frac{\partial N}{\partial x} = 3$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [\text{Exact ODE}]$$

$$\int_{y=const} (2x+3y+4)\,dx + \int(-6y-5)\,dy = 0$$

$$\Rightarrow 2\cdot\frac{x^2}{2} + 3xy + 4x - 6\frac{y^2}{2} - 5y = c$$

$$\therefore x^2 + 3xy + 4x - 3y^2 - 5y = c$$

⊛ $(x^3+xy^2)\,dx + (y^3+x^2y)\,dy = 0$

$M = x^3 + xy^2$ | $N = y^3 + x^2y$

$\frac{\partial M}{\partial y} = 2xy$ | $\frac{\partial N}{\partial x} = 2xy$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [Exact ODE]

$$\underset{y=\text{const}}{\int (x^3+xy^2)\,dx} + \underset{x\text{ free}}{\int y^3\,dy} = 0$$

$$\Rightarrow \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4 = c$$

$$\therefore x^4 + 2x^2y^2 + y^4 = 4c = c$$

Ans

⊛ Integrating Factor

Non Exact ODE

if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ then $M\,dx + N\,dy = 0$ is not exact.

Rule-1: if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and $M\,dx + N\,dy = 0$ is a ~~ho~~ homogeneous ODE then, Integrating factor,

$$(I.F) = \frac{1}{Mx + Ny} \quad ; \quad Mx + Ny \neq 0$$

Example:

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0 \quad \ldots\ldots \text{(i)}$$

$M = x^2y - 2xy^2$ | $N = -x^3 + 3x^2y$

$\dfrac{\partial M}{\partial y} = x^2 - 4xy$ | $\dfrac{\partial N}{\partial x} = -3x^2 + 6xy$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{[Not exact]}$$

But homogeneous.

$$(I.F) = \frac{1}{Mx + Ny} = \frac{1}{x(x^2y - 2xy^2) - y(x^3 - 3x^2y)}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2}$$

$$= \frac{1}{x^2y^2}$$

(i) × (I.F),

$$\frac{1}{x^2y^2}(x^2y - 2xy^2)\,dx - \frac{1}{x^2y^2}(x^3 - 3x^2y)\,dy = 0$$

$$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \ldots \text{(ii)}$$

$M' = \dfrac{1}{y} - \dfrac{2}{x}$ | $N' = -\dfrac{x}{y^2} + \dfrac{3}{y}$

$\dfrac{\partial M'}{\partial y} = -\dfrac{1}{y^2}$ | $\dfrac{\partial N'}{\partial x} = -\dfrac{1}{y^2}$

$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ [Exact ODE]

integrading (ii)

$$\int \left(\frac{1}{y} - \frac{2}{x}\right) dx + \int \frac{3}{y}\, dy = 0$$

y = const

$$\Rightarrow \frac{x}{y} - 2\ln(x) + 3\ln(y) = c$$

Ans

H.W $\Rightarrow$ from page - 37

2·34
2·35
2·36 } Exact ODE

2·42 } Non Exact ODE

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Non Exact ODE

Rule-1: if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and $Mdx + Ndy = 0$ is homogeneous,

Then $I.F. = \frac{1}{Mx+Ny}$; $Mx+Ny \neq 0$

Rule-2 if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and not homogeneous but reduciable to

$y \cdot f(x,y)dx + x \cdot g(x,y)dy = 0$, then,

$$I.F. = \frac{1}{Mx-Ny}$$

Example:

$(y-xy^2)dx - (x+x^2y)dy = 0 \quad \cdots (i)$

$M = y-xy^2$

$\frac{\partial M}{\partial y} = 1-2xy$

$N = -x-x^2y$

$\frac{\partial N}{\partial x} = -1-2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [Not Exact ODE, not Homogeneous]

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{x(y-xy^2)-y(-x-x^2y)} = \frac{1}{xy-x^2y^2+xy+x^2y^2}$$

$$= \frac{1}{2xy}$$

(i) × I.F,

$$\frac{1}{2xy}(y - xy^2)\,dx - \frac{1}{2xy}(x + x^2y)\,dx = 0$$

$$\Rightarrow \left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\frac{1}{2y} + \frac{x}{2}\right)dy = 0 \quad \text{--- (ii)}$$

Here,

$M' = \frac{1}{2x} - \frac{y}{2}$ | $N' = -\frac{1}{2y} - \frac{x}{2}$

$\frac{\partial M'}{\partial y} = -\frac{1}{2}$ | $\frac{\partial N'}{\partial x} = -\frac{1}{2}$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad [\text{Exact ODE}]$$

integrating –(ii),

$$\int\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \int \frac{1}{2y}\,dy = 0$$

$$\Rightarrow \frac{1}{2}\ln(x) - \frac{1}{2}xy - \frac{1}{2}\ln(y) = c$$

Rule-3:/ if $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and fails to follow Rule-1 and 2.

(i) $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = f(x)$ only ; then I.F = $e^{\int f(x)\,dx}$

if y still exist then,

(ii) $\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = f(y)$ only : then I.F = $e^{\int f(y)\,dy}$

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Example:

$$(x^2+y^2+x)\,dx + xy\,dy = 0 \quad \cdots (i)$$

$$\frac{\partial M}{\partial y} = 2y \quad \Big| \quad \frac{\partial N}{\partial x} = y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ [Not Exact ODE, Not homogeneous, can't take common]

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x)$$

$$I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

(i) × I.F,

$$(x^3+xy^2+x^2)\,dx + x^2y\,dy = 0 \quad \cdots (ii)$$

$$\frac{\partial M'}{\partial y} = 2xy \quad \Big| \quad \frac{\partial N'}{\partial x} = 2xy$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$ [Exact ODE]

integrating (ii),

$$\int (x^3+xy^2+x^2)\,dx = 0$$

$$\Rightarrow \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{3}x^3 = c$$

Example: Solve

$$6xy\,dx + (4y + 9x^2)\,dy = 0 \quad \cdots (i)$$

Here,

$M = 6xy$ | $N = 4y + 9x^2$

$\frac{\partial M}{\partial y} = 6x$ | $\frac{\partial N}{\partial x} = 18x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [Not Exact, Not Homogeneous]

can't take common

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\frac{1}{N} = \frac{6x - 18x}{4y + 9x^2} = \frac{-12x}{4y + 9x^2} \quad [\text{Not } f(x)]$$

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\frac{1}{M} = \frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y} \quad (\cancel{f(x)}\ f(y))$$

$$\therefore I.F. = e^{\int \frac{2}{y} dy} = e^{2 \cdot \ln y} = y^2$$

(i) $\times$ I.F,

$$\therefore 6xy^3 dx + (4y^3 + 9x^2y^2)\,dy = 0 \quad \cdots (ii)$$

$\frac{\partial M'}{\partial y} = 18xy^2$ | $\frac{\partial N'}{\partial x} = 18xy^2$

$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$ [Exact ODE (ii)]

integrating (ii),

$$\int 6xy^3\,dx + \int 4y^3\,dy = 0$$

$$\therefore 3x^2y^3 + y^4 = c \quad \text{Ans}$$

$\boxed{\text{H.W.}} \Longrightarrow$

2·38
2·40
2·41
2·44
2·45

$\left.\begin{matrix}2\cdot38\\2\cdot40\\2\cdot41\\2\cdot44\end{matrix}\right\}$ M.k.C Books

Zillis' Books

Page-66 → Example – 2.3

Exercise – 2·4 ⇒ 6, 8, 11, 15, 17, 18, 25, 26, 34, 35, 36

Linear ODE

$$a_0(x)\frac{dy}{dx} + a_1(x)\,y = a_2(x) \quad \cdots \text{(i)}$$

it said to be linear, if it follows two conditions

i. power of the dependent variable ($\to y$) and its derivatives ($\to \frac{dy}{dx}$) is exactly 1.

ii. the co-officient of dependent variable and its derivatives should be function of x ($f(x)$) only or constant.

Example:

$x\frac{dy}{dx} + 10y = 5$

$\cos x\frac{dy}{dx} + (\sin x)y = \tan x$

$5\frac{dy}{dx} + 10y = 6$

$$\left.\begin{matrix} y\frac{dy}{dx} + xy = 5 \\ x\frac{dy}{dx} + xy\cdot\tan y = 5 \\ x\frac{dy}{dx} + xy^2 = 5x \\ x^2\frac{dy}{dx} + xy = \cos(2y) \end{matrix}\right\} \text{Non Linear}$$

* Now we need to convert into standard form.

* Co-efficient of leading derivatives will be positive 1.

$$\frac{dy}{dx} + \frac{a_1}{a_0} y = \frac{a_2}{a_0}$$

$$\boxed{\frac{dy}{dx} + P(x) y = Q(x)} \quad \text{--- (ii)}$$

Standard form of Linear ODE

Then, I.F. = $e^{\int P(x)\,dx}$

Now,

(ii) × I.F,

$$e^{\int P(x)dx} \frac{dy}{dx} + P(x) e^{\int P(x)dx} y = Q(x) e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx} (\text{I.F.} \times \text{dependent variable}) = Q(x) \cdot e^{\int P(x)dx}$$

$$\Rightarrow \frac{d}{dx} \left(e^{\int P(x)dx} \cdot y\right) = Q(x) e^{\int P(x)dx}$$

$$\Rightarrow \int d\left(e^{\int P(x)dx} \cdot y\right) = \int Q(x) e^{\int P(x)dx} \cdot dx$$

$$\therefore e^{\int P(x)dx} \cdot y = \text{-----}$$

integrating by parts, if needed

Example:

$$x \frac{dy}{dx} + 2y = x^5$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x} y = x^4 \quad \cdots (i)$$

Here,

$$P(x) = \frac{2}{x}$$

$$\therefore I.F. = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

(i) × I.F.

$$x^2 \frac{dy}{dx} + 2xy = x^6$$

$$\Rightarrow \frac{d}{dx}(x^2 y) = x^6$$

$$\Rightarrow \int d(x^2 y) = \int x^6 dx$$

$$\therefore x^2 y = \frac{x^7}{7} + c$$

Ans

(*)

$$\frac{dy}{dx} + \frac{3}{x} y = 2x \quad \cdots (i)$$

Here, $P(x) = \frac{3}{x}$

$$\therefore I.F. = e^{\int \frac{3}{x} dx} = e^{3 \ln x} = x^3$$

$\therefore$ (i) × I.F., $x^3 \frac{dy}{dx} + 3x^2 y = 2x^4$

$$\Rightarrow \frac{d}{dx}(x^3 y) = 2x^4$$

$$\Rightarrow \int d(x^3 y) = \int 2x^4\,dx$$

$$\therefore x^3 y = \frac{2x^5}{5} + c$$

Ans

(*) $$x\ln x \frac{dy}{dx} + y = 2\cdot \ln x$$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x\ln x} y = \frac{2}{x} \quad \text{--- (i)}$$

Here,

$$P(x) = \frac{1}{x\ln x}$$

$$\therefore I.F. = e^{\int \frac{1}{x\ln x}dx} = e^{\ln .\ln(x)}$$

$$= \ln x$$

$$\int \frac{1}{x\ln x}\,dx$$

let, $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\therefore \frac{1}{x}dx = du$$

$$\therefore \int \frac{1}{u}du = \ln u$$

$$= \ln(\ln x)$$

(i) $\times$ I.F,

$$\ln x \frac{dy}{dx} + \frac{1}{x} y = \frac{2\cdot \ln x}{x}$$

$$\Rightarrow \int d(\ln x\cdot y) = \int \frac{2\cdot \ln x}{x}\,dx$$

$$\Rightarrow y\ln x = 2\int \frac{\ln x}{x}\,dx$$

$$\therefore y\ln x = (\ln x)^2 + c$$

Ans

Let,

$u = \ln x$

$\frac{1}{x}dx = du$

$$\int u\,du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{(\ln x)^2}{2} + c$$

Next ~~Sat~~ Sunday Quiz-1

Homogeneous & Exact ODE

2 Question - 20 minutes

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L-6/08.08.2023/

Linear ODE

Recap

$$a_0 \frac{dy}{dx} + a_1 y = a_2$$

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)} \quad \text{--- (i)} \longrightarrow \text{Standard Linear ODE}$$

$$I.F. = e^{\int P(x)\, dx}$$

(i) × I.F.

$$e^{\int P(x) dx} \cdot \frac{dy}{dx} + P(x) \cdot e^{\int P(x) dx}\, y = Q(x) \cdot e^{\int P(x) dx}$$

$$\frac{d}{dx}\left(e^{\int P(x) dx} \cdot y\right) = Q(x) \cdot e^{\int P(x) dx}$$

Non-Linear ODE

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \; ; \quad n \neq 1,\; n \neq 0$$

divide by function of y from R.H.S.

$$\Rightarrow \quad y^{-n} \frac{dy}{dx} + P(x) \cdot y^{1-n} = Q(x) \quad \text{--- (i)}$$

Let, $y^{1-n} = u$

$$(1-n)\, y^{-n} \frac{dy}{dx} = \frac{du}{dx}$$

$$\text{(i)} \Rightarrow \frac{1}{1-n} \frac{du}{dx} + P(x)\, u = Q(x)$$

$$\Rightarrow \frac{du}{dx} + P_1(x)u = Q_1(x)$$ $\Rightarrow$ Now its Linear ODE solve it and replace the value of u.

Example:

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$$

$$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{1}{x^2} \quad \cdots (i)$$

Let,

$$y^{-1} = u$$

$$\Rightarrow -y^{-2}\frac{dy}{dx} = \frac{du}{dx}$$

$$(i) \Rightarrow -\frac{du}{dx} + \frac{1}{x}\cdot u = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx} - \frac{1}{x}u = -\frac{1}{x^2} \quad \cdots (ii)$$

Here,

$$p(x) = -\frac{1}{x}$$

$$I.F. = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

(ii) $\times$ I.F.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}u\right) = -\frac{1}{x^3}$$

$$\Rightarrow \int d\left(\frac{u}{x}\right) = \int -\frac{1}{x^3}\,dx$$