H. W. ⇒ from Lecture - 14

MKC Book

Example - 4.19/

$$(x^2D^2 - 2xD + 2)y = x^3 \quad \cdots (i)$$

Let,

$$x = e^t$$

$$\therefore t = \ln x$$

Now,

$$\cancel{xy' = Dy} \qquad xD = Dy$$

$$\cancel{x^2y'' =} \qquad x^2D^2 = D(D-1)y$$

Right. $D = \frac{d}{dt}$

$(i) \Rightarrow$

$$D(D-1)y - 2Dy + 2y = e^{3t}$$

$$\Rightarrow (D^2 - D - 2D + 2)y = e^{3t}$$

$$\Rightarrow (D^2 - 3D + 2)y = e^{3t}$$

$\frac{1}{x+2} \cdot 1$

$\therefore$ A.E. $\Rightarrow$

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\therefore m = 1, 2$$

$$\therefore y_c = c_1 e^{t} + c_2 e^{2t}$$

$$= c_1 x + c_2 x^2$$

$$\therefore y_p = \frac{1}{D^2 - 3D + 2} e^{3t}$$

$$= \frac{1}{9 - 9 + 2} e^{3t}$$

$$= \frac{1}{2} e^{3t}$$

$$= \frac{1}{2} x^3$$

$\therefore$ G.S $\Rightarrow$ $y = y_c + y_p = c_1 x + c_2 x^2 + \frac{1}{2} x^3$

Example: 4.20

$$(x^2D^2 - xD + 4)y = \cos(\ln x) + x\sin(\ln x) \quad \text{--- (i)}$$

Let,

$$x = e^t$$

$$\therefore t = \ln x$$

Then,

$$xD = Dy$$

$$x^2D^2 = D(D-1)y$$

Right, $D = \frac{d}{dt}$

(i) $\Rightarrow$ $D(D-1)y - Dy + 4y = \cos t + e^t \sin t$

$$\Rightarrow (D^2 - D - D + 4)y = \cos t + e^t \sin t \qquad 3.4$$

$$\Rightarrow (D^2 - 2D + 4)y = \cos t + e^t \sin t \quad \text{--- (ii)}$$

A.E. $\Rightarrow$

$$m^2 - 2m + 4 = 0$$

$$\therefore m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= 1 \pm \frac{2}{2}\sqrt{3}\,i$$

$$= 1 \pm \sqrt{3}\,i$$

$$\therefore y_c = e^t\left[A\cos\sqrt{3}t + B\sin\sqrt{3}t\right]$$

$$= x\left[A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right]$$

$$\therefore y_p = \frac{1}{D^2-2D+4}\cos t + \frac{1}{D^2-2D+4} e^t \sin t$$

$$= \frac{1}{-1-2D+4}\cos t + e^t \frac{1}{(D+1)^2-2(D+1)+4}\sin t$$

$$= \frac{1}{3-2D}\cos t + e^t \frac{1}{D^2+2D+1-2D-2+4}\sin t$$

$$= \frac{3+2D}{9-4D^2}\cos t + e^t \frac{1}{D^2+3}\sin t$$

$$= \frac{3+2D}{9+4\cdot 1}\cos t + e^t \frac{1}{-1+3}\sin t$$

$$= \frac{1}{13}(3\cos t - 2\sin t) + \frac{1}{2}e^t \sin t$$

$$= \frac{1}{13}(3\cos|\ln x| - 2\sin|\ln x|) + \frac{1}{2} x \sin(\ln x)$$

$\therefore$ G.S. $\Rightarrow$

$$y = y_c + y_p$$

$$= x\left[A\cos(\sqrt{3}\ln x) + B\sin(\sqrt{3}\ln x)\right] + \frac{x}{2}\sin(\ln x)$$

$$+ \frac{1}{13}\left[3\cos(\ln x) - 2\sin(\ln x)\right]$$

Ans

Example: 4.21

$$(x^3D^3 + 2x^2D^2 + 2)y = \left(x + \frac{1}{x}\right) \cdots \text{(i)}$$

Let,

$$x = e^t$$

$$\therefore t = \ln x$$

Now,

$$\left.\begin{aligned} x^2D^2 &= D(D-1)y \\ x^3D^3 &= D(D-2)(D-1)y \end{aligned}\right\} \text{Right, } D = \frac{d}{dt}$$

(i) $\Rightarrow$

$$D(D-2)(D-1)y + 2D(D-1)y + 2y = e^t + e^{-t}$$

$$\Rightarrow ((D^2-2D)(D-1) + 2D^2 - 2D + 2)y = e^t + e^{-t}$$

$$\Rightarrow (D^3 - 2D^2 - D^2 + 2D + 2D^2 - 2D + 2)y = e^t + e^{-t}$$

$$\Rightarrow (D^3 - D^2 + 2)y = e^t + e^{-t} \quad \cdots \text{(ii)}$$

$\therefore$ A. E. $\Rightarrow$

$$m^3 - m^2 + 2 = 0$$

$$m^3 + m^2 - 2m^2 - 2m + 2m + 2 = 0$$

$$m^2(m+1) - 2m(m+1) + 2(m+1) = 0$$

$$(m+1)(m^2 - 2m + 2) = 0$$

$m+1=0$

$\Rightarrow m=-1$

$m^2-2m+2=0$

$\therefore m = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2}$

$= 1 \pm i$

$\therefore Y_c = C_1 e^{-t} + e^{t}\left[A\cos t + B \sin t\right]$

$= C_1 \dfrac{1}{x} + x\left[A\cos(\ln x) + B\sin(\ln x)\right]$

$\therefore Y_p = \dfrac{1}{D^3-D^2+2} e^{t} + \dfrac{1}{D^3-D^2+2} e^{-t}$

$= \dfrac{1}{1-1+2} e^{t} + \dfrac{t}{3D^2-2D} e^{-t}$

$= \dfrac{1}{2} e^{t} + \dfrac{t}{3+2} e^{-t}$

$= \dfrac{1}{2} e^{t} + \dfrac{1}{5} t \cdot e^{-t}$

$= \dfrac{1}{2} x + \dfrac{1}{5} \cdot \ln x \cdot \dfrac{1}{x}$

$= \dfrac{x}{2} + \dfrac{1}{5x} \ln x$

$\therefore$ G.S. $\Rightarrow$

$y = Y_c + Y_p$

$= \dfrac{C_1}{x} + x\left[A\cos(\ln x) + B\sin(\ln x)\right] + \dfrac{x}{2} + \dfrac{\ln x}{5x}$

Example-4.24

$$\left[(3x+2)^2D^2 + 3(3x+2)D - 36\right]y = x^2+x \quad \text{--- (i)}$$

Let,

$$3x+2 = e^t$$

$$\therefore x = \frac{e^t-2}{3}$$

$$\therefore t = \ln|3x+2|$$

Then,

$$\left.\begin{aligned}(3x+2)D &= Dy \\ (3x+2)^2D^2 &= D(D-1)y\end{aligned}\right\} \quad \text{Right, } D = \frac{d}{dt}$$

(i) $\Rightarrow$

$$\left(D(D-1) + 3D - 36\right)y = \left(\frac{e^t-2}{3}\right)^2 + \frac{e^t-2}{3}$$

$$\Rightarrow \left(D^2 - D + 3D - 36\right)y = \frac{e^{2t} - 4e^t + 4 + 3e^t - 6}{9}$$

$$\Rightarrow \left(D^2 + 2D - 36\right)y =$$

Example: 4·24

$$\left[9(3x+2)^2D^2 + 9(3x+2)D - 36\right]y = x^2+x \quad \cdots\text{(i)}$$

Let,

$$3x+2 = e^t$$

$$\therefore x = \frac{e^t-2}{3}$$

$$\therefore t = \ln|3x+2|$$

Then,

$$\left.\begin{aligned}(3x+2)D &= Dy\\ (3x+2)^2D^2 &= D(D-1)y\end{aligned}\right\} \text{Right, } D=\frac{d}{dt}$$

(i) $\Rightarrow$

$$\{9D(D-1)+9D-36\}y = \left(\frac{e^t-2}{3}\right)^2 + \frac{e^t-2}{3}$$

$$(9D^2-9D+9D-36)y = \frac{e^{2t}-4e^t+4+3e^t-6}{9}$$

$$(D^2-4)y = \frac{e^{2t}-e^t-2}{81} \quad \cdots\text{(ii)}$$

A.E. $\Rightarrow$

$$m^2-4=0$$

$$(m+2)(m-2)=0$$

$$\therefore m = \pm 2$$

$$\therefore y_c = c_1e^{2t} + c_2e^{-2t}$$

$$= c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\therefore Y_p = \frac{1}{D^2-4} \cdot \frac{e^{2t}}{81} - \frac{1}{D^2-4}\frac{e^t}{81} - \frac{1}{D^2-4}\frac{2}{81}$$

$$= \frac{t}{2D}\frac{e^{2t}}{81} - \frac{1}{1-4}\cdot\frac{e^t}{81} + \frac{1}{4}\left(1-\frac{D^2}{4}\right)^{-1}\frac{2}{81}$$

$$= \frac{t e^{2t}}{4\cdot 81} + \frac{e^t}{3\cdot 81} + \frac{1}{2\cdot 81}$$

$$= \frac{1}{81}\left(\frac{1}{4}\cdot t e^{2t} + \frac{1}{3}e^t + \frac{1}{2}\right)$$

$$= \frac{1}{81}\left(\frac{1}{4}\cdot \ln(3x+2)(3x+2)^2 + \frac{1}{3}(3x+2) + \frac{1}{2}\right)$$

$$= \frac{1}{972}\left(3(3x+2)^2\ln(3x+2) + 4(3x+2) + 6\right)$$

$\therefore$ G.S. $\Rightarrow$

$$y = Y_c + Y_p$$

$$= c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{972}\left[3(3x+2)^2\ln(3x+2) + 4(3x+2)+6\right]$$

Ans

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20)

$$(x^2D^2 - xD - 3)y = x^2 \ln x \quad \cdots \text{(i)}$$

Let,

$$x = e^t$$

$$\therefore t = \ln x$$

Hence,

$$xD = Dy$$

$$x^2D^2 = D(D-1)y$$

Right, $D = \frac{d}{dt}$

(i) $\Rightarrow$

$$(D(D-1) - D - 3)y = e^{2t} \cdot t$$

$$\Rightarrow (D^2 - D - D - 3)y = te^{2t}$$

$$\Rightarrow (D^2 - 2D - 3)y = te^{2t} \quad \cdots \text{(ii)}$$

A.E. $\Rightarrow$

$$m^2 - 2m - 3 = 0$$

$$\Rightarrow m^2 - 3m + m - 3 = 0$$

$$\Rightarrow m(m-3) + 1(m-3) = 0$$

$$\Rightarrow (m-3)(m+1) = 0$$

$$\therefore m = -1, 3$$

$$\therefore y_c = c_1 e^{-t} + c_2 e^{3t} = c_1 x^{-1} + c_2 x^3$$

$$\therefore Y_P = \frac{1}{D^2-2D-3}\, t\, e^{2t}$$

$$= e^{2t} \frac{1}{(D+2)^2-2(D+2)-3}\, t$$

$$= e^{2t} \frac{1}{D^2+4D+4-2D-4-3}\, t$$

$$= e^{2t} \frac{1}{D^2+2D-3}\, t$$

$$= e^{2t} \frac{1}{-3} \cdot \left(1 - \frac{D^2+2D}{3}\right)^{-1} t$$

$$= -\frac{e^{2t}}{3} \left(1 + \frac{D^2+2D}{3} + \cdots\right) t$$

$$= -\frac{e^{2t}}{3} \left(t + \frac{2}{3}\right)$$

$$= -\frac{x^2}{3} \left(\ln x + \frac{2}{3}\right)$$

$\therefore$ G.S. $\Rightarrow$

$$y = Y_c + Y_p$$

$$= \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}\left(\ln x + \frac{2}{3}\right)$$

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$$\left(D^2+\frac{1}{x}D\right)y = \frac{12\ln x}{x^2}$$

$$\Rightarrow \left(x^2D^2+xD\right)y = 12\ln x \quad \cdots\cdots (i)$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$$\left.\begin{aligned} xD &= Dy \\ x^2D^2 &= D(D-1)y \end{aligned}\right\} \text{ Right, } D = \frac{d}{dt}$$

(i) $\Rightarrow$

$$\left(D(D-1)+D\right)y = 12t$$

$$\Rightarrow \left(D^2-D+D\right)y = 12t$$

$$\Rightarrow D^2y = 12t \quad \cdots (ii)$$

$\therefore$ A.E. $\Rightarrow$ $m^2 = 0$

$\therefore y_c = c_1+c_2$

$\therefore y_p = \frac{1}{D^2}12t$

$= \frac{1}{D}6t^2$

$= 2t^3$

$= 2(\ln x)^3$

$= 2\ln^3 x$

$\therefore$ G.S. $\Rightarrow$

$y = y_c+y_p$

$= c_1+c_2+2\ln^3 x$

Ans

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$$(x^2D^3 + 3xD^2 + D)y = x^2 \ln x$$

$$\Rightarrow (x^3D^3 + 3x^2D^2 + Dx)y = x^3 \ln x \quad \cdots \text{(i)}$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xD = Dy$

$x^2D^2 = D(D-1)y$

$x^3D^3 = D(D-2)(D-1)y$

Right, $D = \frac{d}{dt}$

$$\text{(i)} \Rightarrow \{D(D-2)(D-1) + 3D(D-1) + D\}y = e^{3t}\, t$$

$$\Rightarrow ((D^2-2D)(D-1) + 3D^2 - 3D + D)y = te^{3t}$$

$$\Rightarrow (D^3 - 2D^2 - D^2 + 2D + 3D^2 - 3D + D)y = te^{3t}$$

$$\Rightarrow D^3y = te^{3t} \quad \cdots \text{(ii)}$$

A.E $\Rightarrow$ $m^3 = 0$

$m = 0,0,0$

$\therefore y_c = c_1 + c_2 + c_3$

$$\therefore y_p = \frac{1}{D^3} te^{3t}$$

$$= e^{3t} \frac{1}{(D+3)^3} t$$

$$= e^{3t} \frac{1}{D^3 + 3D^2 \cdot 3 + 3 \cdot D \cdot 9 + 27} t$$

$$= e^{3t} \frac{1}{D^3 + 9D^2 + 27D + 27} t$$

$$= \frac{e^{3t}}{27}\left(1 + \frac{D^3 + 9D^2 + 27D}{27}\right)^{-1} t$$

$$= \frac{e^{3t}}{27}\left(1 - \frac{D^3 + 9D^2 + 27D}{27} + \cdots\right) t$$

$$= \frac{e^{3t}}{27}\left(t - \frac{27}{27}\right)$$

$$= \frac{e^{3t}}{27}(t-1)$$

$$= \frac{x^3}{27}(\ln x - 1)$$

$\therefore$ G.S. $\Rightarrow$

$$y = y_c + y_p$$

$$= c_1 + c_2 + c_3 + \frac{x^3}{27}(\ln x - 1)$$

Ans

26| $(x^2D^2 + xD + 1)y = \sin(\ln x^2)$ ... (i)

Let,

$x = e^t$

$\therefore t = \ln x$

Hence

$xD = Dy$

$x^2D^2 = D(D-1)y$

Right, $D = \frac{d}{dt}$

(i) $\Rightarrow$

$$\left(D(D-1) + D + 1\right)y = \sin(2t)$$

$$\Rightarrow (D^2 - D + D + 1)y = \sin(2t)$$

$$\Rightarrow (D^2+1)y = \sin(2t) \quad \text{... (ii)}$$

$\therefore$ A.E. $\Rightarrow$

$m^2 + 1 = 0$

$\therefore m = \pm i$

$\therefore y_c = A\cos t + B\sin t = A\cos(\ln x) + B\sin(\ln x)$

$$\therefore y_p = \frac{1}{D^2+1} \sin(2t)$$

$$= \frac{1}{-2^2+1} \sin(2t)$$

$$= -\frac{1}{3} \sin(2t)$$

$$= -\frac{1}{3} \sin(2\ln x)$$

$\therefore$ G.S. $\Rightarrow$

$$y = y_c + y_p$$

$$= A\cos(\ln x) + B\sin(\ln x) - \frac{1}{3}\sin(2\ln x)$$

Ans

Zill's Book

Exercise - 4.7

19|

$$xy'' - 4y' = x^4$$

$$\Rightarrow x^2y'' - 4xy' = x^5 \quad \ldots\ldots (i)$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy' = Dy$

$x^2y'' = D(D-1)y$

Here, $D = \frac{d}{dt}$

$$(i) \Rightarrow \big(D(D-1) - 4D\big)y = \cancel{x}\ e^{5t}$$

$$\Rightarrow (D^2 - D - 4D)y = e^{5t}$$

$$\Rightarrow (D^2 - 5D)y = e^{5t} \quad \cdots (ii)$$

$\therefore$ A.E. $\Rightarrow$

$$m^2 - 5m = 0$$

$$m(m-5) = 0$$

$$\therefore m = 0, 5$$

$$\therefore y_c = c_1 + c_2 e^{5t}$$

$$= c_1 + c_2 x^5$$

$$\therefore y_p = \frac{1}{D^2 - 5D} e^{5t}$$

$$= \frac{t}{2D - 5} e^{5t}$$

$$= \frac{t}{10 - 5} e^{5t}$$

$$= \frac{1}{5} t e^{5t}$$

$$= \frac{x^5}{5} \ln(x)$$

$\therefore$ G.S. $\Rightarrow$

$$y = y_c + y_p$$

$$= c_1 + c_2 x^5 + \frac{x^5}{5} \ln(x)$$

Ans

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$$x^2 y'' - xy' + y = 2x \quad \cdots (i)$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy = Dy$

$x^2 y'' = D(D-1)y$

$\Bigg\} \; D = \frac{d}{dt}$

$$(i) \Rightarrow (D^2 - D - D + 1)y = 2e^t$$

$$\Rightarrow (D^2 - 2D + 1)y = 2e^t \quad \cdots (ii)$$

A.E. $\Rightarrow$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$\therefore m = 1, 1$

$\therefore y_c = c_1 e^t + c_2 t e^t$

$= c_1 x + c_2 x \ln x$

$\therefore y_p = \frac{1}{D^2 - 2D + 1} 2e^t$

$= \frac{t}{2D - 2} 2e^t$

$= \frac{t^2}{2} 2e^t$

$= t^2 e^t$

$= x \ln^2 x$

$\therefore$ G.S. $\Rightarrow$

$y = y_c + y_p$

$= c_1 x + c_2 x \ln x + x \ln^2 x$ Ans.

22| $x^2 y'' - 2xy' + 2y = x^4 e^x \quad \ldots (i)$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy' = Dy$

$x^2 y'' = D(D-1)y$

$\Big\} \; D = \frac{d}{dt}$

(i) $\Rightarrow$

$(D^2 - D - 2D + 2)y = e^{4t} e^{e^t}$

$\Rightarrow (D^2 - 3D + 2)y = e^{4t} e^{e^t}$

A.E. $\Rightarrow$

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-2)(m-1) = 0$

$\therefore m = 1, 2$

$\therefore y_c = c_1 e^t + c_2 e^{2t}$

$= c_1 x + c_2 x^2$

$$\therefore W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

$$= 2x^2 - x^2$$

$$= x^2 \neq 0$$

x	$\xrightarrow{+}$	e^x
1	$\xrightarrow{-}$	e^x
0		e^x

(i) $\Rightarrow$

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = x^2 e^x$$

$$\therefore u_1 = -\int \frac{x^2 \cdot x^2 e^x}{x^2}\,dx \qquad \therefore u_2 = \int \frac{x \cdot x^2 e^x}{x^2}\,dx$$

$$= -\int x^2 e^x\,dx \qquad = \int x e^x\,dx$$

$$= -x^2 e^x + 2x e^x - 2e^x \qquad = x e^x - e^x$$

$$\therefore y_p = u_1 y_1 + u_2 y_2$$

$$= -x^3 e^x + 2x^2 e^x - 2x e^x + x^3 e^x - x^2 e^x$$

$$= e^x\left(-x^3 + 2x^2 - 2x + x^3 - x^2\right)$$

$$= e^x\left(x^2 - 2x\right)$$

$$\therefore y = y_c + y_p$$

$$= c_1 x + c_2 x^2 + e^x\left(x^2 - 2x\right)$$

Ans

23)

$$x^2y'' + xy' - y = \ln x \quad \cdots \text{(i)}$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy' = Dy$

$x^2y'' = D(D-1)y$

$\Big\} D = \frac{d}{dt}$

(i) $\Rightarrow$

$$(D^2 - D + D - 1)y = t$$

$$\Rightarrow (D^2-1)y = t \quad \cdots \text{(ii)}$$

A.E. $\Rightarrow$

$m^2 - 1 = 0$

$\therefore m = \pm 1$

$\therefore y_c = c_1 e^t + c_2 e^{-t}$

$= c_1 x + \frac{c_2}{x}$

$\therefore y_p = \frac{1}{D^2-1} t$

$= -(1-D^2)^{-1} t$

$= -(1 + D^2 + \cdots) t$

$= -t$

$= -\ln x$

$\therefore y = y_c + y_p$

$= c_1 x + \frac{c_2}{x} - \ln x$

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$$x^2y'' + xy' - y = \frac{1}{x+1} \quad \cdots \text{(i)}$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy' = Dy$

$x^2y'' = D(D-1)y$

$D = \frac{d}{dt}$

(i) $\Rightarrow$

$$(D^2 - D + D - 1)y = \frac{1}{e^t+1}$$

$$\Rightarrow (D^2-1)y = \frac{1}{e^t+1} \quad \cdots \text{(ii)}$$

A.E. $\Rightarrow$

$m^2 - 1 = 0$

$m = \pm 1$

$\therefore y_c = c_1 e^t + c_2 e^{-t}$

$= c_1 x + c_2 x^{-1}$

$$\therefore W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}$$

$= -x^{-1} - x^{-1}$

$= -2x^{-1} \neq 0$

(i) $\Rightarrow$ $$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x^2(x+1)}$$

$$\therefore u_1 = -\int \frac{\frac{1}{x^3+x^2} \cdot x^{-1}}{-2x^{-1}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^3+x^2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x^2(x+1)}\,dx$$

$$\frac{1}{x^2(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{x+1}$$

$\Rightarrow (Ax+B)(x+1) + Cx^2 = 1$

$\Rightarrow Ax^2 + Bx + Ax + B + Cx^2 = 1$

$\Rightarrow x^2(A+C) + x(A+B) + B = 1$

x^{-2}

x^{-1}

$$= \frac{1}{2}\int\left(\frac{-x+1}{x^2} + \frac{1}{x+1}\right)dx$$

$\therefore B = 1$

$\therefore A+B = 0$

$A = -1$

$\therefore A+C = 0$

$C = 1$

$$= \frac{1}{2}\int\left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx$$

$$= \frac{1}{2}\left[-\ln x - \frac{1}{x} + \ln(x+1)\right]$$

$$= -\frac{1}{2}\ln x - \frac{1}{2}x^{-1} + \frac{1}{2}\ln(x+1)$$

$$\frac{x^2-1}{2x}$$

$$\therefore u_2 = \int \frac{x \cdot \frac{1}{x^2(x+1)}}{-2x^{-1}}$$

$$= -\frac{1}{2}\int \frac{1}{\cancel{x}(x+1)}\cdot dx$$

$$= -\frac{1}{2}\ln(x+1)$$

$$\therefore y_p = u_1y_1 + u_2y_2$$

$$= -\frac{1}{2}x\ln x - \frac{1}{2} + \frac{1}{2}x\ln(x+1) - \frac{1}{2}x^{-1}\ln(x+1)$$

$$= -\frac{1}{2}x\ln x + \left(\frac{1}{2}x - \frac{1}{2}x^{-1}\right)\ln(x+1) - \frac{1}{2}$$

$$\therefore y = y_c + y_p$$

$$= c_1x + c_2x^{-1} - \frac{1}{2}x\ln x + \left(\frac{x}{2} - \frac{1}{2x}\right)\ln(x+1) - \frac{1}{2}$$

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$x^2y'' - 4xy' + 6y = \ln x^2 \quad \cdots\cdots (i)$

Let,

$x = e^t$

$t = \ln x$

Hence,

$xy' = Dy$

$x^2y'' = D(D-1)y$

$\Big\} \; D = \frac{d}{dt}$

$(i) \Rightarrow$

$(D^2 - D - 4D + 6)y = 2t$

$(D^2 - 5D + 6)y = 2t \quad \cdots\cdots (ii)$

A.E. $\Rightarrow$

$m^2 - 5m + 6 = 0$

$m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$(m-3)(m-2) = 0$

$\therefore m = 2, 3$

$\therefore Y_c = C_1 e^{2t} + C_2 e^{3t}$

$= C_1 x^2 + C_2 x^3$

$\therefore Y_p = \frac{1}{D^2 - 5D + 6} 2t$

$= \frac{1}{6}\left(1 + \frac{D^2 - 5D}{6}\right)^{-1} 2t$

$= \frac{1}{6}\left(1 - \frac{D^2 - 5D}{6} + \cdots\right) 2t$

$= \frac{1}{6}\left(2t + \frac{10}{6}\right)$

$= \frac{1}{3}t + \frac{5}{18}$

$= \frac{1}{3}\ln x + \frac{5}{18}$

$\therefore y = Y_c + Y_p$

$= C_1 x^2 + C_2 x^3 + \frac{1}{3}\ln x + \frac{5}{18}$

Ans

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$$x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3 \quad \cdots \text{(i)}$$

Let,

$x = e^t$

$\therefore t = \ln x$

Hence,

$xy' = Dy$

$x^2y'' = D(D-1)y$

$x^3y''' = D(D-2)(D-1)y$

$D = \frac{d}{dt}$

(i) $\Rightarrow$

$$\left(D(D-2)(D-1) - 3D(D-1) + 6D - 6\right)y = 3 + 3t$$

$$\Rightarrow \left((D^2-2D)(D-1) - 3D^2 + 3D + 6D - 6\right)y = 3t + 3$$

$$\Rightarrow \left(D^3 - 2D^2 - D^2 + 2D - 3D^2 + 9D - 6\right)y = 3t + 3$$

$$\Rightarrow \left(D^3 - 6D^2 + 11D - 6\right)y = 3t + 3 \quad \cdots \text{(ii)}$$

A.E. $\Rightarrow$

$$m^3 - 6m^2 + 11m - 6 = 0$$

$$\therefore m = 1, 2, 3$$

$$\therefore y_c = c_1e^t + c_2e^{2t} + c_3e^{3t}$$

$$= c_1x + c_2x^2 + c_3x^3$$

$$\therefore Y_p = \frac{1}{D^3-6D^2+11D-6} 3t+3$$

$$= \frac{1}{-6}\left(1-\frac{D^3-6D^2+11D}{6}\right)^{-1}(3t+3)$$

$$= -\frac{1}{6}\left(1+\frac{D^3-6D^2+11D}{6}+\cdots\right)(3t+3)$$

$$= -\frac{1}{6}\left(3t+3+\frac{33}{6}\right)$$

$$= -\frac{1}{6}\left(3t+\frac{17}{2}\right)$$

$$= -\frac{1}{6}\left(3\ln x+\frac{17}{2}\right)$$

$$\therefore Y = Y_c + Y_p$$

$$= c_1x + c_2x^2 + c_3x^3 - \frac{1}{6}\left(3\ln x + \frac{17}{2}\right)$$

Ans

H.W. $\Rightarrow$ from Lecture - 15

MKC Book (157)

Example - 6.21

$$(D-1)x + (D+1)y = 0 \quad \cdots\cdots (i)$$

$$(2D+2)x + (2D-2)y = t \quad \cdots\cdots (ii)$$

Operating (i) by $(2D-2)$ and operating (ii) by $(D+1)$,

$$(D-1)(2D-2)x + (D+1)(2D-2)y = 0$$

$$\underset{(-)}{(2D+2)(D+1)x} \underset{(-)}{+} (2D-2)(D+1)y = \underset{(-)}{(D+1)t}$$

$$(2D^2-2D-2D+2-2D^2-2D-2D-2)x = -(1+t)$$

$$\Rightarrow (-8D)x = -t-1$$

$$\Rightarrow (8D)x = t+1$$

$$\Rightarrow Dx = \frac{t}{8} + \frac{1}{8}$$

$$\therefore x(t) = \frac{t^2}{16} + \frac{t}{8} + c_1 \quad \text{A}$$

$$\frac{dx}{dt} = \frac{t}{8} + \frac{1}{8}$$

(i) $\Rightarrow$

$$(D-1)x + (D+1)y = 0$$

$$(D+1)y = -Dx + x$$

$$= \frac{t^2}{16} + \frac{t}{8} + c_1 - \frac{t}{8} - \frac{1}{8}$$

$$(D+1)y = \frac{t^2}{16} - \frac{1}{8} + c_1$$

$\therefore$ A.E. $\Rightarrow$ $m+1=0$ $\therefore y_p = \frac{1}{1+D}\left(\frac{t^2}{16} - \frac{1}{8} + c_1\right)$

$\therefore m = -1$

$$= (1 - D + D^2 - \cdots)\left(\frac{t^2}{16} - \frac{1}{8} + c_1\right)$$

$\therefore y_c = c_2 e^{-t}$

$$= \frac{t^2}{16} - \frac{1}{8} + c_1 - \frac{t}{8} + \frac{1}{8}$$

$$= \frac{t^2}{16} - \frac{t}{8} + c_1$$

$$\therefore y(t) = c_2 e^{-t} + \frac{t^2}{16} - \frac{t}{8} + c_1$$

$$\therefore \frac{dy}{dt} = -c_2 e^{-t} + \frac{t}{8} - \frac{1}{8}$$

(ii) $\Rightarrow$

$$2Dx + 2x + 2Dy - 2y = t$$

$$\Rightarrow 2\left(\frac{t}{8} + \frac{1}{8}\right) + 2\left(\frac{t^2}{16} + \frac{t}{8} + c_1\right) + 2\left(-c_2 e^{-t} + \frac{t}{8} - \frac{1}{8}\right) - 2\left(c_2 e^{-t} + \frac{t^2}{16} - \frac{t}{8} + c_1\right) = t$$

$$\Rightarrow \frac{t}{4} + \frac{1}{4} + \frac{t^2}{8} + \frac{t}{4} + 2c_1 - 2c_2 e^{-t} + \frac{t}{4} - \frac{1}{4} - 2c_2 e^{-t} - \frac{t^2}{8} + \frac{t}{4} - 2c_1 = t$$

$$\Rightarrow \cancel{t} - 4c_2 e^{-t} = t$$

$$\therefore c_2 e^{-t} = 0$$

$$\therefore y(t) = \frac{t^2}{16} - \frac{t}{8} + c_1$$

Ans

Example- 6.3 /

$$(D+3)x + (D+1)y = e^t \quad \cdots \text{(i)}$$

$$(D+1)x + (D-1)y = t \quad \cdots \text{(ii)}$$

Operating (i) by $(D-1)$ and operating (ii) by $(D+1)$,

$$(D+3)(D-1)x + (D+1)(D-1)y = (D-1)e^t$$

$$(D+1)(D+1)x + (D-1)(D+1)y = (D+1)t$$

$$(-) \qquad (-) \qquad (-)$$

$$(D^2 - D + 3D - 3 - D^2 - 2D - 1)x = e^t - e^t - 1 - t$$

$$\Rightarrow -4x = -t - 1$$

$$\therefore x(t) = \frac{t}{4} + \frac{1}{4}$$

Ans.

$$\therefore \frac{dx}{dt} = \frac{1}{4}$$

$$\text{(i)} \Rightarrow Dx + 3x + (D+1)y = e^t$$

$$\Rightarrow (D+1)y = e^t - \frac{1}{4} - \frac{3}{4}t - \frac{3}{4}$$

$$\Rightarrow (D+1) = e^{t} * \frac{3}{4}t * 1$$

$$\therefore y_c = c_1 e^{-t}$$

$$\cancel{\therefore y_p = (1+D)^{-1} (e^{t}}$$

$$\therefore y_p = \frac{1}{D+1} e^{t} - (1+D)^{-1} \cancel{\left(\frac{3}{4}t+1\right)} \left(\frac{3}{4}t + 1\right)$$

$$= \frac{1}{2} e^{t} * (1 - D + \cdots)\left(\frac{3}{4}t + 1\right)$$

$$= \frac{1}{2} e^{t} * \frac{3}{4}t * 1 + \frac{3}{4}$$

$$= \frac{1}{2} e^{t} * \frac{3}{4}t * \frac{1}{4}$$

$$\therefore y(t) = c_1 e^{-t} + \frac{1}{2} e^{t} * \frac{3}{4}t * \frac{1}{4}$$

$$\therefore \frac{dy}{dt} = -c_1 e^{-t} + \frac{1}{2} e^{t} - \frac{3}{4}$$

(ii) $\Rightarrow$

$$(D+1)x + (D-1)y = t$$

$$Dx + x + Dy - y = t$$

$$\frac{1}{4} + \frac{t}{4} + \frac{1}{4} - c_1 e^{-t} + \frac{1}{2} e^{t} - \frac{3}{4} - c_1 e^{-t} - \frac{1}{2} e^{t} + \frac{3}{4}t + \frac{1}{4} = t$$

$$\Rightarrow t - 2c_1 e^{-t} = t$$

$$\therefore c_1 e^{-t} = 0$$

$\Rightarrow y(t) = \frac{1}{2}e^{t} - \frac{3}{4}t - \frac{1}{4}$ Ans

Example :- 6.4/

$$(D-3)x + 2(D+2)y = 2\sin t \quad \cdots (i)$$

$$2(D+1)x + (D-1)y = \cos t \quad \cdots (ii)$$

Operating (i) by $(D-1)$ and operating (ii) by $2(D+2)$

$$(D-3)(D-1)x + 2(D+2)(D-1)y = (D-1)2\sin t$$

$$2(D+1)2(D+2)x + (D-1)\cdot 2(D+2)y = 2(D+2)\cos t$$

$$(-) \quad (-) \quad (-)$$

$$(D^2 - D - 3D + 3 - 4D^2 - 8D - 4D - 8)x = 2\cos t - 2\sin t + 2\sin t - 4\cos t$$

$$\Rightarrow (-3D^2 - 16D - 5)x = -2\cos t$$

$$\Rightarrow (3D^2 + 16D + 5)x = 2\cos t$$

A.E. $\Rightarrow$ $3m^2 + 16m + 5 = 0$

$\therefore m = -\frac{1}{3}, -5$

$\therefore x_c = C_1 e^{-5t} + C_2 e^{-t/3}$

$$\therefore x_p = \frac{1}{3D^2+16D+5}\, 2\cos t$$

$$= \frac{1}{16D+2}\, 2\cos t$$

$$= \frac{1}{8D+1}\cos t$$

$$= \frac{8D-1}{64D^2-1}\cos t$$

$$= \frac{8D-1}{-65}\cos t$$

$$= -\frac{1}{65}(-8\sin t - \cos t)$$

$$= \frac{8}{65}\sin t + \frac{1}{65}\cos t$$

$$\therefore x(t) = c_1 e^{-5t} + c_2 e^{-t/3} + \frac{1}{65}(8\sin t + \cos t)$$

$$\therefore \frac{dx}{dt} = -5c_1 e^{-5t} - \frac{1}{3}c_2 e^{-t/3} + \frac{1}{65}(8\cos t - \sin t)$$

$$\cancel{-5c_1 e^{-5t} - \frac{1}{3}c_2 e^{-t/3} + \frac{1}{65}(}$$

(i) $\Rightarrow$

$$2(D+2)y = 2\sin t - Dx + 3x$$

$$\Rightarrow 2(D+2)y = 2\sin t + 5c_1 e^{-5t} + \frac{1}{3}c_2 e^{-t/3} - \frac{1}{65}(8\cos t - \sin t)$$

$$+ 3c_1 e^{-5t} + 3c_2 e^{-t/3} + \frac{3}{65}(8\sin t + \cos t)$$

$$= 8c_1 e^{-5t} + \frac{10}{3} c_2 e^{-t/3} + 2\sin t - \frac{8}{65}\cos t + \frac{1}{65}\sin t$$

$$+ \frac{24}{65}\sin t + \frac{3}{65}\cos t$$

$$= 8c_1 e^{-5t} + \frac{10}{3} c_2 e^{-t/3} + \frac{5}{13}\sin t - \frac{1}{13}\cos t$$

$$\Rightarrow \quad (D+2)y = 4c_1 e^{-5t} + \frac{5}{3} c_2 e^{-t/3} + \frac{5}{26}\sin t - \frac{1}{26}\cos t$$

$$\therefore \quad y_c = c_3 e^{-2t}$$

$$\therefore \quad y_p = \frac{1}{D+2} 4c_1 e^{-5t} + \frac{1}{D+2}\frac{5}{3} c_2 e^{-\frac{1}{3}t} + \frac{1}{D+2}\cdot\frac{5}{26}\sin t - \frac{1}{D+2}\cdot\frac{1}{26}\cos t$$

$$= \frac{4}{-3} c_1 e^{-5t} + c_2 e^{-\frac{1}{3}t} + \frac{D-2}{D^2-4}\cdot\frac{5}{26}\sin t - \frac{D-2}{D^2-4}\cdot\frac{1}{26}\cos t$$

$$= -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-\frac{1}{3}t} + \frac{D-2}{-5}\cdot\frac{5}{26}\sin t - \frac{D-2}{-5}\cdot\frac{1}{26}\cos t$$

$$= -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-\frac{1}{3}t} - \frac{1}{26}\cos t + \frac{1}{13}\sin t - \frac{1}{130}\sin t - \frac{1}{65}\cos t$$

$$= -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-t/3} + \frac{9}{130}\sin t - \frac{7}{130}\cos t$$

$$y\,(\cancel{D+2}) = -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-t/3} + c_3 e^{-2t} + \frac{9}{130}\sin t - \frac{7}{130}\cos t$$

$$\therefore \quad \frac{dy}{dt} = \frac{20}{3} c_1 e^{-5t} - \frac{1}{3} c_2 e^{-t/3} - 2c_3 e^{-2t} + \frac{9}{130}\cos t + \frac{7}{130}\sin t$$

(ii) $\Rightarrow$

$$2Dx + 2x + Dy - y = \cos t$$

$$\Rightarrow -10c_1 e^{-5t} - \frac{2}{3} c_2 e^{-t/3} + \frac{16}{65} \cos t - \frac{2}{65} \sin t + 2c_1 e^{-5t} + 2c_2 e^{-t/3}$$

$$+ \frac{16}{65} \sin t + \frac{2}{65} \cos t + \frac{20}{3} c_1 e^{-5t} - \frac{1}{3} c_2 e^{-t/3} - 2c_3 e^{-2t}$$

$$+ \frac{2}{130} \cos t + \frac{7}{130} \sin t + \frac{4}{3} c_1 e^{-5t} - c_2 e^{-t/3} - c_3 e^{-2t}$$

$$- \frac{9}{130} \sin t + \frac{7}{130} \cos t = \cos t$$

$$\Rightarrow -3c_3 e^{-2t} + \frac{1}{5} \sin t + \frac{2}{5} \cos t = \cos t$$

$$\Rightarrow -3c_3 e^{-2t} = -\frac{1}{5} \sin t + \cos t - \frac{2}{5} \cos t$$

$$\therefore c_3 e^{-2t} = \frac{1}{15} \sin t - \frac{1}{5} \cos t$$

$$\therefore y(t) = -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-t/3} + \frac{9}{130} \sin t - \frac{7}{130} \cos t + \frac{1}{15} \sin t - \frac{1}{5} \cos t$$

$$= -\frac{4}{3} c_1 e^{-5t} + c_2 e^{-t/3} + \frac{53}{390} \sin t - \frac{33}{130} \cos t$$

Example - 6.5/

$$(3D^2+3D)x = 4t-3 \quad \dots (i)$$

$$(D+1)x - D^2y = t^2 \quad \dots (ii)$$

From (i)

A.E. $\Rightarrow$ $3m^2+3m=0$

$3m(m+1)=0$

$\therefore m = 0, -1$

$\therefore x_c = c_1 + c_2 e^{-t}$

$\therefore x_p = \frac{1}{3D^2+3D}(4t-3)$

$= \frac{1}{3D}(1+D)^{-1}(4t-3)$

$= \frac{1}{3D}(1-D+\cdots)(4t-3)$

$= \frac{1}{3D}(4t-3-4)$

$= \frac{1}{3}\cdot\frac{1}{D}(4t-7)$

$= \frac{1}{3}\cdot\frac{4t^2}{2} - \frac{1}{3}\cdot 7t$

$= \frac{2}{3}t^2 - \frac{7}{3}t$

$\therefore x = c_1 + c_2 e^{-t} + \frac{2}{3}t^2 - \frac{7}{3}t$

$\therefore \frac{dx}{dt} = -c_2 e^{-t} + \frac{4}{3}t - \frac{7}{3}$

From (ii)

$$-D^2y = t^2 - Dx + x$$

$$\Rightarrow D^2y = Dx + x - t^2$$

$$= -c_2e^{-t} + \frac{4}{3}t - \frac{7}{3} + c_1 + c_2e^{-t} + \frac{2}{3}t^2 + \frac{7}{3}t - t^2$$

$$\Rightarrow D^2y = c_1 - \frac{1}{3}t^2 - t - \frac{7}{3}$$

$$\Rightarrow Dy = c_1t - \frac{1}{9}t^3 - \frac{t^2}{2} - \frac{7}{3}t + c_3$$

$$\therefore y = \frac{1}{2}c_1t^2 - \frac{1}{36}t^4 - \frac{1}{6}t^3 - \frac{7}{6}t^2 + c_4$$

$$D^2y = -c_1 - 2c_2e^{-t} - \frac{5}{3}t^2 + \frac{11}{3}t - \frac{7}{3}$$

$$Dy = -c_1t + 2c_2e^{-t} - \frac{5}{9}t^3 + \frac{11}{6}t^2 - \frac{7}{3}t + c_3$$

~~y = c_1t~~

$$y = -\frac{1}{2}c_1t^2 - 2c_2e^{-t} + c_3t + c_4 - \frac{5}{36}t^4 + \frac{11}{18}t^3 - \frac{7}{6}t^2$$

Example - 6.6/

$$(D+1)x + (D-1)y = e^t \quad \cdots (i)$$

$$(D^2+D+1)x + (D^2-D+1)y = t^2 \quad \cdots (ii)$$

Operating (i) by (D^2-D+1) and operating (ii) by $(D-1)$,

$$(D+1)(D^2-D+1)x + (D-1)(D^2-D+1)y = (D^2-D+1)e^t$$

$$(D^2+D+1)(D-1)x + (D^2-D+1)(D-1)y = (D-1)t^2$$

$$(-) \qquad (-) \qquad (-)$$

$$(D^3-D^2+D+D^2-D+1-D^3-D^2-D+D^2+D+1)x = e^t - e^t + e^t - 2t + t^2$$

$$2x = e^t - 2t + t^2$$

$$\therefore x = \frac{1}{2}e^t - t + \frac{1}{2}t^2 \quad \text{Ans}$$

$$\frac{dx}{dt} = \frac{1}{2}e^t - 1 + t$$

(i) $\Rightarrow$

$$Dx + x + (D-1)y = e^t$$

$$\Rightarrow (D-1)y = e^t - Dx - x$$

$$= e^t - \frac{1}{2}e^t + 1 - t - \frac{1}{2}e^t + t - \frac{1}{2}t^2$$

$$(D-1)y = 1 - \frac{1}{2}t^2$$

$$\therefore y = c_1e^t - (1-D)^{-1}\left(1 - \frac{1}{2}t^2\right)$$