(*) The function f is defined as

$$f(x) = \begin{cases} -2x+1 & \text{if } -3 \le x < 1 \\ 2 & \text{if } x = 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

a) Find $f(-2)$, $f(1)$ and $f(2)$.

b) Determine the domain of f.

c) ~~Use~~ Locate any intercepts.

d) Graph f.

e) Use the graph to find the range of f.

f) Is f continuous on its domain?

~~Soltt~~ Solution

a) To find $f(-2)$, observe that when $x = -2$ the equation for f is given by $f(x) = -2x+1$. So,

$$f(-2) = -2(-2) + 1$$
$$= 5$$

when, $x=1$, the equation for f is $f(x) = ~~x~~ 2$ so,

$$\cancel{f(2) = 2^2 = 4}$$

$$\therefore f(1) = 2$$

when $x=2$ the equation for f is $f(x) = x^2$, so.

$$f(2) = 2^2$$
$$= 4$$

b)

To find the domain of f, look at its definition. Since f is defined for all x greater than or equal to -3. the domain of f is $\{x \mid x \geq -3\}$, on the interval $[-3, \infty)$.

c) Let us put $x=0$, and ~~sto~~ solve the equation for y.

$$f(0) = -2 \cdot 0 + 1$$
$$= 1$$

$\therefore$ y-intercept is $(0,1)$

Let us put $y=0$ and solve the equation for x.

$f(x)=0=-2x+1$	$f(x)=0=2$	$f(x)=0=x^2$
$2x=1$	$\therefore 0=2$	$\therefore x=0$; $x>1$
$\therefore x=\frac{1}{2}$	No solution	Not acceptable.

$\therefore$ x-intercepts is ~~are~~ $(\frac{1}{2}, 0)$.

P.NO.: 99

Example - 4

a)

Charge $= \$5.50 + \$(0.064471 \times 300)$

$= \$24.84$

b) Charge $= \$5.50 + \$(0.064471 \times 1000) + \$(0.078391 \times 500)$

$= \$109.17$

e)

$C(x) = \$5.50 + \$(0.064471 \times 1000) + \$(0.078391 \times (x-1000))$

$= 6.9971 + (0.078391(x-1000))$

$= 0.07831x - 8.42$; if $x > 1000$

if x is between 0 to 1000.

$f(x) = \$5.50 + 0.064471x$ if $0 \leq x \leq 1000$

$$\rightarrow C(x) = \begin{cases} 0.064471x + 5.50 & ; \text{ if } 0 \leq x \leq 1000 \\ 0.078391x - 8.42 & ; \text{ if } x > 1000 \end{cases}$$

Chapter - 3 : Linear and Quadratic Function

Linear function.

$$\cancel{f(x)\, y = mx + b}$$

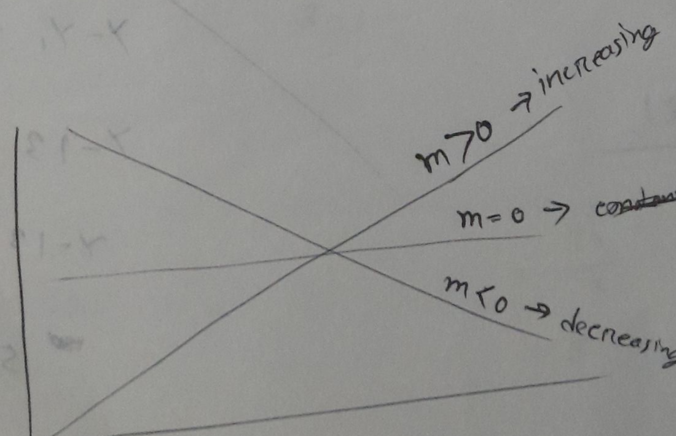
$$f(x) = y = mx + b$$

(*) Average rate of change in ~~lee~~ linear function is constant.

(*) Linear function have a constant average rate of change. That is, the average rate of change of a linear function $f(x) = mx + b$ is,

$$\frac{\Delta y}{\Delta x} = m$$

(*)



Theorem:

A linear function $f(x) = mx + b$ is increasing over ~~the~~ its domain if its slope, m, is positive. It is decreasing over its domain if its slope, m, is negative. It is constant over its domain if its slope, m, is zero.

Quadratic Function

$F(x) = 3x^2 - 5x + 1$ $g(x) = -6x^2 + 1$ $H(x) = \frac{1}{2}x^2 + \frac{2}{3}x$

A quadratic function is a function of the form

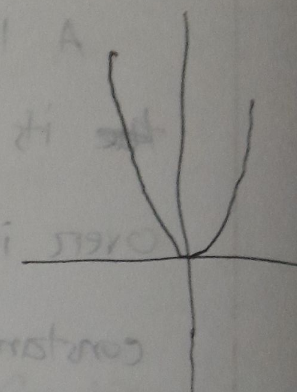
$$f(x) = ax^2 + bx + c \quad ; \quad a \neq 0$$

where, a, b and c are real numbers and $a \neq 0$. The domain of a quadratic function is the set of all real numbers.

$$f(x) = y = 2x^2 + 8x + 5$$

$$y = 2(x^2 + 4x + 4) - 3$$
$$= 2x^2 + 8x + 8 - 3$$
$$= 2x^2 + 8x + 5$$

$$y = x^2$$
$$\downarrow$$
$$y = 2x^2$$
$$\downarrow$$
$$y = 2(x+2)^2$$
$$\downarrow$$
$$y = 2(x+2)^2 - 3$$
$$\updownarrow$$
$$y = 2(x-h)^2 + k$$
$$= a(x-h)^2 + k$$



$h = -2$

$k = -3$

$(h, k) = (-2, -3)$ Vertex of the function.

The axis of symmetry is

$$x = h = -2.$$

$$f(x) = y = ax^2 + bx + c$$

$$= a\left[x^2 + \frac{b}{a}x\right] + c$$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right] + c - \frac{b^2}{4a}$$

$$= a\left[x + \frac{b}{2a}\right]^2 + \left[\frac{4ac - b^2}{4a}\right]$$

$$\updownarrow$$

$$= a(x-h)^2 + k$$

Here, $h = \frac{-b}{2a}$

$k = \frac{4ac - b^2}{4a}$

$$\boxed{f(x) = ax^2 + bx + c = a(x-h)^2 + k}$$

$$\text{vertex} = (h, k)$$

$$= (x, y)$$

$$= \left\{\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right\}$$

$f(x) = 2x^2 + 8x + 5$ | $a = 2$, $b = 8$, $c = 5$

$h = \frac{-b}{2a} = \frac{-8}{2 \cdot 2} = -2$

$k = f(-\frac{b}{2a}) = f(-2) = 2 \cdot (-2)^2 + 8(-2) + 5$

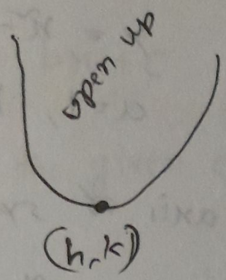
$= 2 \cdot 4 - 16 + 5$

$= 8 - 16 + 5$

$= -3$

vertex, $(h, k) = (-2, -3)$.

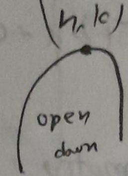
The graph of $f(x) = a(x-h)^2 + k$ is the parabola $y = ax^2$ shifted horizontally h units (replace x by $x-h$) and vertically k units (add k). As a result, the vertex is at (h, k), and the graph opens up if $a > 0$ and down if $a < 0$. The axis of symmetry is the vertical line $x = h$.



is the vertex

It is the minimum point

$a > 0$



is the vertex which is the maximum point

$a < 0$

It is easier to obtain the vertex of a quadratic function f by remembering that its x-coordinate is $h = -\frac{b}{2a}$. The y-coordinate k can then be found by ~~every~~ evaluating f at $-\frac{b}{2a}$. That is, $k = f\left(-\frac{b}{2a}\right)$.

The graph of the parabola represented by the quadratic function $y = ax^2 + bx + c$ has an axis of symmetry represented by the equation of the vertical line $x = \frac{-b}{2a}$.

$y = ax^2 + bx + c$

axis of symmetry: $x = \frac{-b}{2a}$

$y = x^2 - 4x + 3$

$a = 1,\ b = -4,\ c = 3$

axis of symmetry: $x = 2$

$x = \frac{-b}{2a} = \frac{-(-4)}{2(1)}$

$= 2$

$y = a(x - r_1)(x - r_2)$

axis of symmetry: $x = \frac{r_1 + r_2}{2}$

$y = a(x-1)(x-3)$

$a = 1,\ r_1 = 1,\ r_2 = 3$

$x = \frac{r_1 + r_2}{2}$

$= \frac{1+3}{2} = 2$

axis of symmetry: $x = 2$.

Without graphing, locate the vertex and axis of symmetry of the parabola defined by

$f(x) = -3x^2 + 6x + 1$. Does it open up or down.

$a = -3,\ b = 6,\ c = 1$

$a < 0 \Rightarrow$ open down

$x = \frac{-b}{2a} = \frac{-6}{2 \cdot (-3)} = \frac{-6}{-6} = 1$

vertex, $(1, f(1)) = (1, 4)$

$\Rightarrow$ highest point.

$f(x) = ax^2 + bx + c$

Set, $x=0$ in the given equation and solve it for y.

$f(0) = a \cdot 0^2 + b \cdot 0 + c$

$f(0) = c$

Set. $y=0$ in the given equation and solve it for x.

$f(x) = 0 = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

here, $b^2 - 4ac = \Delta$; is called discriminant.

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

If, $\Delta > 0$; if Δ is positive then,

$$x = \frac{-b + \sqrt{\Delta}}{2a} \;;\; x = \frac{-b - \sqrt{\Delta}}{2a}$$

Two real solutions

~~Two~~ Two x- intercepts.

Here,

$a > 0$, hence the parabola opens up.

vertex (h, k)

$h = -\frac{b}{2a}$

$k = f(\frac{-b}{2a})$

The axis of symmetry

$x = h = -\frac{b}{2a}$

The y-intercept is c.

If $\Delta = 0$, then

$$x = \frac{-b}{2a}$$

If $\Delta < 0$; if Δ is negative then,

$$x = \frac{-b \pm \sqrt{-\Delta}}{2a}$$

$$x = -\frac{b}{2a} + i\frac{\sqrt{\Delta}}{2a} \quad ; \quad x = -\frac{b}{2a} - i\frac{\sqrt{\Delta}}{2a}$$

Two complex roots. No real solutions.

⊛ Graph $f(x) = -3x^2 + 6x + 1$ using its properties.

Determine the domain and the range of f.

Determine where f is increasing and where it is decreasing.

$\Rightarrow$

$f(x) = -3x^2 + 6x + 1$

Here,

$a = -3$

$b = 6$

$c = 1$

$h = -\frac{6}{2(-3)} = 1$

$k = f(h) = f(-\frac{b}{2a}) = f(1) = 4$

Set, $x = 0$ in $f(x)$

$f(0) = y = 1$

Set, $y = 0$ in $f(x)$ and solve for x.

$f(x) = -3x^2 + 6x - 1 = 0$

$\Delta = b^2 - 4ac = 48 > 0$

Here, $a < 0$, hence the parabola opens down.

Vertex $(h, k) = \left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$

$= 1, 4$

$h = -\frac{b}{2a}$

$k = f(-\frac{b}{2a})$

The axis of symmetry

$x = h = -\frac{b}{2a} = 1$

The y-intercept is 1

The x-intercept are

-0.15 and 2.15

Domain $= (-\infty, \infty)$

Range $= (-\infty, 4]$

Increasing $= (-\infty, 1)$

Decreasing $= (1, \infty)$

The descriminant is positive, Hence the function has two real solution and two x-intercepts.

$$x = \frac{-6 \pm \sqrt{48}}{-6}$$

$\therefore x = -0.15$ and $x = 2.15$.

8)

a) Graph $f(x) = x^2 - 6x + 9$ by determining whether the graph opens up or down and by finding its vertex, axis of symmetry, y-intercept, and x-intercept, if any.

b) Determine the domain and the range of f.

c) Determine where f is increasing and where it is decreasing.

$\Rightarrow$

a)

$f(x) = x^2 - 6x + 9$

Here, $a = 1$

$b = -6$

$c = 9$

$\therefore b^2 - 4ac = (-6)^2 - 4 \cdot 1 \cdot 9$

$= 36 - 36$

$= 0$

$h = -b/2a = -\frac{-6}{2 \cdot 1} = \frac{6}{2} = 3$

Here, $a > 0$, hence the parabola opens up.

vertex, $(h, k) = \{-\frac{b}{2a}, f(-\frac{b}{2a})\}$

$= (3, 0)$

$h = -b/2a$

$k = f(-b/2a)$

The axis of symmetry

$x = h = \frac{-b}{2a} = 3$

y-intercept is 9

x-intercept is 3

Thus descriminant is equal to zero, hence it has one real solution.

Set, $x=0$ in $f(x)$

$$f(0) = 0^2 - 6\cdot 0 + 9$$
$$= 0 - 0 + 9$$
$$= 9$$

Set, $y=0$ in $f(x)$ and solve for x.

$$f(x) = x^2 - 6x + 9 = 0$$

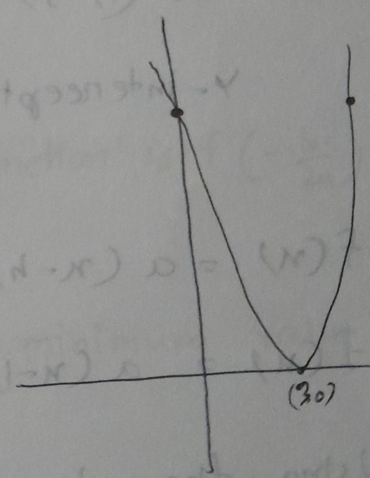
$$\cancel{x^2 - 3x - 3x + 9}$$

$$x^2 - 2\cdot x\cdot 3 + 3^2 = 0$$

$$(x-3)^2 = 0$$

$$x - 3 = 0$$

$$x = 3$$



$$f(3) = 3^2 - 6\cdot 3 + 9$$
$$= 9 - 18 + 9$$
$$= 0$$

b) Domain $= (-\infty, \infty)$

Range $= (0, \infty)$

c)

increasing = ~~$(-\alpha, 3)$~~ $(3, \alpha)$

Decreasing = $(-\alpha, 3)$

Determine the quadratic function whose vertex is $(1, -5)$ and whose y-intercept is -3.

$\Rightarrow$

vertex $(1, -5) = (h, k)$

y-intercept $= -3$

$h = 1$

$k = -5$

$f(0) = -3$

$f(x) = a(x-h)^2 + k$

$f(x) = a(x-1)^2 - 5 \quad \ldots (i)$

Using the value of y-intercept we get

$f(0) = -3.$

$\Rightarrow a(0-1)^2 - 5 = -3$

$\Rightarrow a - 5 = -3$

$\Rightarrow a = -3+5$

$\therefore a = 2$

Substitute the value of 'a' in equation ①.

$$f(x) = 2(x-1)^2 - 5$$

$$= 2(x^2 - 2x + 1) - 5$$

$$= 2x^2 - 4x + 2 - 5$$

$$\therefore f(x) = 2x^2 - 4x - 3$$

⊛ Theorem.

1. If $a < 0$, the function has a maximum value (highest point)

The maximum value of the function is $f\left(-\frac{b}{2a}\right)$

2. If $a > 0$, the function has a minimum value (lowest point)

The minimum value of the function is $f\left(-\frac{b}{2a}\right)$

⊛ Determine whether the quadratic function

$$f(x) = x^2 - 4x - 5$$

has a maximum or minimum value. Then find the maximum or minimum value.

$\Rightarrow$

$f(x) = x^2 - 4x - 5$

$a = 1 > 0$

Since $a > 0$, the function has the minimum value.

$$\frac{-b}{2a} = \frac{4}{2\cdot 1} = 2$$

$$f(2) = 2^2 - 4\cdot 2 - 5$$
$$= 4 - 8 - 5$$
$$= -9$$

So, the minimum value of the function is -9.

H.W | 9th Ed

3.1

12

⊗ $R = xp$ ---- (i) ⇒ Demand Equation

$x = 21000 - 150p$ ---- (ii)

$R = (21000 - 150p)p$

$R = 21000p - 150p^2$ ---- (iii)

Here, x is the total number of calculators sold at a price P.

P is price per calculator.

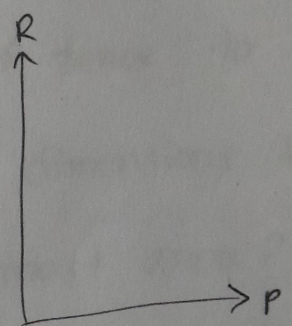
R is revenue earn by the company.

Here,

$a = 150$

$b = 21000$

$c = R$



b)

$P > 0$

and $x \geq 0$

$\Rightarrow 21000 - 150p \geq 0$

$P \leq 140$

Combining the above condition, we get the domain of R is

$\{P \mid 0 < P \leq 140\}$

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c) Since, $a<0$, the vertex is the highest point, hence the parabola opens down.

$$h = -\frac{b}{2a} = \frac{-21000}{2\times(-150)}$$

$$= 70$$

d) So, the ~~b~~ maximum will be ~~R = 21000 × 70~~ -150

$$R = 21000 \times 70 - 150 \times (70)^2$$

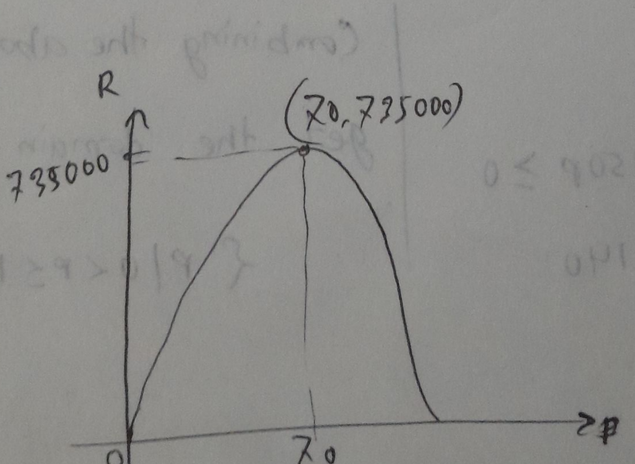
$$= 735000$$

e)

$$x = 21000 - 150(70)$$

$$x = 10500$$

f)



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g)

$R = 675000 = 21000p - 150p^2$

$\Rightarrow p^2 - 140P + 4500 = 0$

$(P-50)(P-90) = 0$

$P = 50$ and $P = 90$.

(*) A farmer has 2000 yards of fence to enclose a rectangular field. What are the dimensions of the rectangular that encloses the most area?

$\Rightarrow$

Perimeter $= 2x + 2w = 2000$

$\Rightarrow x + w = 1000$

$w = 1000 - x$ --(i)

Area, $A = xw$

$= x(1000-x)$

$A = 1000x - x^2$ ---(ii)

Here,

$a = -1$

$b = 1000$

$x = h = \frac{-b}{2a} = -\frac{1000}{2(-1)} = 500$

Area, $A = 1000x - x^2$

$= (1000 \times 500) - (500)^2$

$= 500000 - 250000$

$= 250000$ sq yards.

So dimension is (500×500).

(20) A projectile is fired from a cliff 500 feet above the water at an inclination of 45° to the horizontal, with a muzzle velocity of 400 feet per second. In physics, it is established that height h of the projectile above the water can be modeled by

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500$$

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Where

a) Find the maximum height of the projectile.

$\Rightarrow$

The height of the projectile is given by a quadratic function.

$$h(x) = \frac{-32x^2}{(400)^2} + x + 500$$

$$= \frac{-1}{5000}x^2 + x + 500$$

We are looking for the maximum value of h. Since $a < 0$, the maximum value is obtained at the vertex, whose x-coordinate is.

$$x = -\frac{b}{2a} = -\frac{1}{2\left(-\frac{1}{5000}\right)} = \frac{5000}{2} = 2500$$

The maximum height of the projectile is

$$h(2500) = \frac{-1}{5000}(2500)^2 + 2500 + 500$$

$$= -1250 + 2500 + 500$$

$$= 1750 \text{ ft}$$

b) How far from the base of the cliff will the projectile strike the water?

$\Rightarrow$

The projectile will strike the water when the height is zero. To find the distance x traveled, solve the equation.

$$h(x) = \frac{-1}{5000}x^2 + x + 500 = 0$$

The discriminant of this quadratic equation is

$$b^2 - 4ac = 1^2 - 4\left(\frac{-1}{5000}\right)(500) = 1.4$$

Then

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1.4}}{2\left(-\frac{1}{5000}\right)}$$

$$= \begin{cases} -458 \\ 5458 \end{cases}$$

Discard the negative solution. The projectile will strike the water at a distance of about 5458 feet from the base of the cliff.

Polynomial and Rational Functions.

General Equation of a Polymomial function:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Power Function

A power function of degree n is a monomial function of the form.

$$f(x) = ax^n$$

Where, a is a real number, $a \neq 0$, and $n > 0$ is an integer.

* If f is a function and r is a real number for which $f(r) = 0$, then r is called a real zero of f.

As a consequence of this definition, the following statements are equivalent.

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1. r is a real zero of a polynomial function f.

2. r is an x-intercept of the graph of f.

3. $x-r$ is a factor of f.

4. r is a solution to the equation $f(x)=0$.

⊛ Find a polynomial f of degree 3 whose zeros are -3, 2 and 5.

$\Rightarrow$

$$f(x) = a(x+3)(x-2)(x-5)$$

⊛ If $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a zero of multiplicity m of f.

⊛ Identifying Zeros and Their Multiplicities.

For the polynomial function.

$$f(x) = 5(x-2)(x+3)^2\left(x-\frac{1}{2}\right)^4$$

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⇒ There are three real zeros. These are, $2, -3, \frac{1}{2}$.

The multiplicity of 2 is 1

The multiplicity of -3 is 2

The multiplicity of $\frac{1}{2}$ is 4

⊛ If r is a zero of Even multiplicity:

⇒ Sign of $f(x)$ does not change from one side of r to the other side of r.

Graph touches x-axis at r.

⊛ If r is a zero of odd multiplicity

⇒ Sign of $f(x)$ changes from one side of r to the other side of r.

Graph crosses x-axis at r.

⊛ Turning Points.

⇒ Points on the graph where the graph changes from an increasing function to a decreasing function, or vice versa, are called turning points.

⊛ Theorem:

If f is a polynomial function of degree n, then f has at most n-1 turning points.

If the graph of a polynomial function f has n-1 turning points, the degree of f is at least n.

⇒ Based on the first part of the theorem, a polynomial function of degree 5 will have at most 5-1 = 4 turning points. Based on the second part of the theorem, if a polynomial function has 3 turning points, then its degree must be at least 4.

⊛ End Behaviour:

⇒ For very large values of x, either positive or negative, the graph of $f(x) = (x+1)^2 (x-2)$ looks lik the graph of $y = x^3$.

⊛ Theorem:

For large values of x, either positive or negative, the graph of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

$$y = a_n x^n .$$

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(8) Analysing the graph of a polynomial function.

⇒

Step 1: Determine the end behavior of the graph of the function.

Step 2: Find the x- and y-intercepts of the graph of the function.

Step 3: Determine the zeros of the function and their multiplicity. Use this information to determine whether the graph crosses or touches the x-axis at each x-intercept.

Step 4: Use a graphing utility to graph the function

Step 5: Approximate the turning points of the graph.

Step 6: Use the information in steps 1 through 5 to draw a complete graph of the function by hand.

Step 7: Find the domain and the range of the function.

Step 8: Use the graph to determine where the function is increasing and where it is decreasing.

⊛ Remainder Theorem:

Let f be a polynomial function. If $f(x)$ is divided by $x-c$, then the remainder is $f(c)$.

⊛ Factor Theorem:

⇒ An important and useful consequence of the remainder theorem is called the factor theorem.

Let f be a polynomial function. Then $x-c$ is a factor of $f(x)$ if and only if $f(c)=0$.

The Factor Theorem actually consists of two separate statements.

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1. If $f(c)=0$, then $x-c$ is a factor of $f(x)$.

2. If $x-c$ is a factor of $f(x)$, then $f(c)=0$.

(*) Theorem:

A polynomial function cannot have more real zeros than its degree.

Proof: The proof is based on the Factor Theorem.
If r is a real zero of a polynomial function f, then $f(r)=0$ and, hence, $x-r$ is a factor of $f(x)$. Each real zero corresponds to a factor of degree 1. Because f cannot have more first-degree factors than its degree, the result follows.

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⊛ Rational Zeros Theorem

General equation,

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0$$

Here, $a_n \neq 0$, $a_0 \neq 0$}

where each coefficient is an integer. If $\frac{p}{q}$, in lowest terms, is a rational zeros of f, then p must be a factor of a_0 and q must be a factor of a_n.

⊛ List the

$$f(x) = 2x^3 + 11x^2 - 7x - 6$$

Here,

$a_0 = -6$

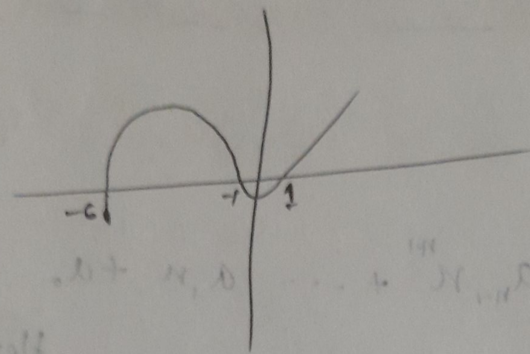
$a_3 = 2$

Factors of a_0 ; $p = \pm 1, \pm 2, \pm 3, \pm 6$

Factors of a_n ; $q = \pm 1, \pm 2$

$$p/q = \pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 2, \pm 3, \pm 6$$

Graph:



From the graph, -6 is real zeros

$(x+6)$ is a factor of the function.

$\therefore f(x) = 2x^3 + 11x^2 - 7x - 6$

$= (x+6)(2x^2 - x - 1)$

Now,

$2x^2 - x - 1 = 0$

$ax^2 + bx + c = 0$

$2x^2 - 2x + x - 1 = 0$

$2x(x-1) + 1(x-1) = 0$

$(x-1)(2x+1) = 0$

$\therefore x - 1 = 0$ and $2x + 1 = 0$

$x = 1$ $\quad$ $x = -\frac{1}{2}$

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$\Rightarrow f(x) = 2x^3 + 11x^2 - 7x - 6$

$= (x+6)(x-1)(2x+1)$

$f(-6) = 0$

$f(-\frac{1}{2}) = 0$

$f(1) = 0$

$\Rightarrow \left. \begin{array}{l} x = -6 \\ x = -\frac{1}{2} \\ x = 1 \end{array} \right\}$ Real zeros of the given function.

(*) Steps for Finding the real zeros of a Polynomial function.

Step 1: Use the degree of the polynomial function to determine the maximum number of zeros.

Step 2: If the polynomial function has integer coefficients, use the rational zeros theorem to identify those rational numbers that potentially can be zeros.

Step 3: Graph the polynomial function.