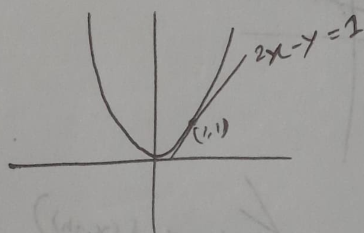
* Slope of a tangent line.

i.e. Slope at a single point.

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

* Find the equation of a tangent line to the parabola $y = x^2$ at a point $(1,1)$.

$\Rightarrow$



Now, we know that, slope of a tangent line,

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Here,

$x_0 = 1$

$$\therefore m_{tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^2 - f(1)}{h}$$

Here,

$f(x) = x^2$

$f(1) = 1$

$f(1+h) = (1+h)^2$

$$= \lim_{h \to 0} \frac{1+2h+h^2-1}{h}$$

$$= \lim_{h \to 0} \frac{2h+h}{h}$$

$$= \lim_{h \to 0} (2+h)$$

$$= 2$$

$\therefore m_{tan} = 2$

and point $(1,1)$.

$\therefore$ equation of tangent line,

$$(y-y_1) = m(x-x_1)$$

$$y-1 = 2(x-1)$$

$$y-1 = 2x-2$$

$$2x-y = 1$$

Find the equation of a tangent line to the parabola

$f(x) = \frac{1}{x}$ at $x = 2$.

$\Rightarrow$

Now, we know that, slope of a tangent line,

$$m_{tan} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Here,
$f(x) = \frac{1}{x}$
$f(2+h) = \frac{1}{2+h}$
$f(2) = \frac{1}{2}$

$$= \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2-2-h}{2(2+h)}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{2(2+h) \cdot h}$$

$$= -\frac{1}{2(2+0)} = \frac{-1}{4}$$

$m_{tan} = -\frac{1}{4}$

point $(2, \frac{1}{2})$

equation,

$$y - \frac{1}{2} = -\frac{1}{4}(x-2)$$

$$y - \frac{1}{2} = -\frac{1}{4}x + \frac{1}{2}$$

$$y = -\frac{1}{4}x + 1$$

$$4y = -x + 4$$

$$\therefore x + 4y = 4$$ Ans

(*) velocity = $\dfrac{\text{tatal distance}}{\text{tatal time}}$

$$s = f(t)$$

$$V_{ag} = \frac{s_2 - s_1}{t_2 - t_1}$$

$$= \frac{f(t_0+h) - f(t_0)}{t_0 + h - t_0}$$

$$= \frac{f(t_0+h) - f(t_0)}{h}$$

Velocity

Average Velocity

$$\frac{f(t_0+h) - f(t_0)}{h}$$

Instanteneous velocity

$$\lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h}$$

⊛ A particle moves on a line away from its initial position so that after t hours it is

$s = 3t^2 + t$ mile.

a) Find the average velocity over $t=1$ to $t=3$ hour

b) Find the instantneous velocity at $t=1$ hour.

$\Rightarrow$

a)

at $t=1$, $s(1) = 3+1 = 4$

$t=3$, $s(3) = 3\times3^2+3 = 30$

$$\text{Average Velocity} = \frac{s(3)-s(1)}{3-1}$$

$$= \frac{30-4}{2} = 13 \text{ mile/hour.}$$

b)

Now, intanteneous velocity,

$$V_{is} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{3(1+h)^2 + (1+h) - 4}{h}$$

$$= \lim_{h \to 0} \frac{3(1+2h+h^2) + 1 + h - 4}{h}$$

$$= \lim_{h \to 0} \frac{6h + 3h^2 + h}{h}$$

$$= \lim_{h \to 0} 7 + 3h$$

$= 7$ mile/hour

Ans

$\boxed{\text{H.W.}}$

* ~~so~~ $s(t) = 1 + 5t - 2t^2$

a) $t = 1$ to $t = 3$

b) $t = 1$?

$t = 2$?

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2·2/

Average rate of change = $\dfrac{f(x_0+h)-f(x_0)}{h}$

$P(x_0, f(x_0))$

$Q(x_0+h, f(x_0+h))$

Instantaneous rate of change,

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

⊛ $\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$

i) Instantaneous rate of change at $x = x_0$

ii) Slope at a single point at $x = x_0$

iii) Slope of tangent line at $x = x_0$

iv) Also this limit has special name, i.e. called derivative at $x = x_0$, and define by

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

⊛ Generally for any values of x, we will define,

Derivative of function $f(x)$

$$\hookrightarrow f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $x = 2$,

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

For $x = -1$

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

⊛ Find the derivative of $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$= \lim_{h\to 0} \dfrac{1}{\sqrt{x+h} + \sqrt{x}}$

$= \dfrac{1}{\sqrt{x}+\sqrt{x}}$

$= \dfrac{1}{2\sqrt{x}}$ Ans

(*) Find the tangent line to the curve $f(x) = \sqrt{x}$ ~~of~~ at $x=4$.

$\Rightarrow$ at $x=4$,

$f(x) = \sqrt{x}$

$f(4) = \sqrt{4} = 2$

$\therefore$ point $(4,2)$

$\therefore$ slope at $x=4$,

$f'(x) = \dfrac{1}{2\sqrt{x}}$ from previous question

$f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{2\cdot 2}$

$= \dfrac{1}{4}$

Therefore,

equation of a tangent line,

$$y-2 = \frac{1}{4}(x-4)$$

$$y-2 = \frac{1}{4}x-1$$

$$y = \frac{1}{4}x+1$$

Ans

Differentiability:

If this limit $f'(x_0) = \lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exists, then we would say that the function $f(x)$ is differentiable at a point $x=x_0$.

Lef hand derivative (L.H.D.)

$$Lf'(x_0) = \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

Right hand derivative (R.H.D)

$$Rf'(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

If, L.H.D = R.H.D,

$$\lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}, \text{ then}$$

$\lim_{h\to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ exist and then function is differentiable at $x=x_0$.

⊛ Show that, $f(x) = |x|$ is not differentiable at $x=0$.

$f(x) = |x|$

$= \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

Now,

L.H.D, at $x=4$,

$Lf'(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0^-} \frac{-h-0}{h}$

$= \lim_{h \to 0^-} -1$

$= -1$

Here,
$f(x) = -x$
$f(0+h) = -(0+h) = -h$
$f(0) = 0$

R.H.D at $x=0$,

$Rf'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0^+} \frac{h-0}{h}$

$= \lim_{h \to 0^+} 1$

$= 1$

Here,
$f(x) = x$
$f(0+h) = h$
$f(0) = 0$

$\therefore$ L.H.D $\neq$ R.H.D

Hence, Limit doesn't exists.

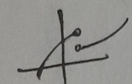
Therefore, the function is not differentiable at $x=0$.

(proved)

* When does a function not have derivative at a point.

i) A corner 'V' or 'Λ'

ii) A cusp 

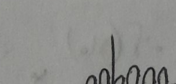
iii) A discontinuity 

iv) Vertical tangency.

(tangent line will be vertical)

$\therefore$ slope of tangent line = undefined.

i.e. Derivative = undefined

v) $\sin \frac{1}{x}$ 

⊛ Relation between differentiability and continuity:

<u>Theorem:</u> If a function differentiable, then the function is continuous at $x = x_0$.

For continuity, we have to show,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

$$\lim_{x \to x_0} f(x) = f(x_0) = 0$$

let,

$h = x - x_0 \quad \therefore x = x_0 + h$

if $x \to x_0$

then $h \to 0$

$$\Rightarrow \lim_{h \to 0} \left[f(x_0+h) - f(x_0) \right] = 0$$

$$\Rightarrow \lim_{h \to 0} \left[\frac{f(x_0+h) - f(x_0)}{h} \times h \right]$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \times \lim_{h \to 0} h$$

$$= f'(x_0) \times 0 = 0$$

if a function differentiable, then $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

if a function differentiable then,

$$\lim_{h \to 0} \left[f(x_0+h) - f(x_0) \right] = 0$$

$$\therefore \lim_{h \to 0} f(x_0+h) = f(x_0)$$

* Show that $f(x) = \begin{cases} x & 0 \le x < \frac{1}{2} \\ 1-x & \frac{1}{2} \le x \le 2 \end{cases}$ is continuous but not differentiable at $x = \frac{1}{2}$. Also draw the graph.

$\Rightarrow$ Check the continuity at $x = \frac{1}{2}$.

1st check

$f(x) = 1-x$

$\therefore f\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

2nd check

$$\lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} x$$

$$= \frac{1}{2}$$

$$\lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} 1-x$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore$ L.H.L = R.H.L.

$\therefore$ Limit exist

3rd check

$$\lim_{x \to \frac{1}{2}} f(x) = f(x)$$

Therefore, this function is continuous at $x = \frac{1}{2}$.

2nd Part

Now for derivative :

L.H.D at $x = \frac{1}{2}$,

$$Lf'(\tfrac{1}{2}) = \lim_{h \to 0^-} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h}$$

Here,
$f(x) = x$
$f(\frac{1}{2}+h) = \frac{1}{2}+h$
$f(\frac{1}{2}) = \frac{1}{2}$

$$= \lim_{h \to 0^-} \frac{\frac{1}{2}+h-\frac{1}{2}}{h}$$

$$= 1$$

R.H.D at $x = \frac{1}{2}$

$$Rf'(\tfrac{1}{2}) = \lim_{h \to 0^+} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h}$$

Here,
$f(x) = 1-x$
$f(1/2+h) = 1-\frac{1}{2}-h$
$= \frac{1}{2}-h$
$f(\frac{1}{2}) = 1-\frac{1}{2} = \frac{1}{2}$

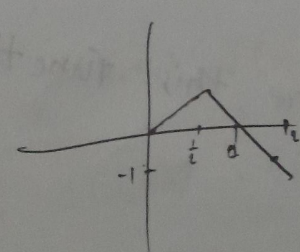
$$= \lim_{h \to 0^+} \frac{\frac{1}{2}-h-\frac{1}{2}}{h}$$

$$= -1$$

$\therefore$ L.H.D $\neq$ R.H.D

Hences function is not differentiable at $x = \frac{1}{2}$.

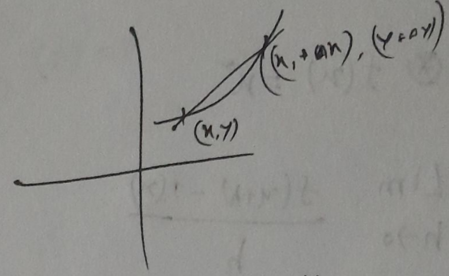
Graph :



⊛ Show that $f(x) = \begin{cases} x^2+2 & x \le 1 \\ -x+4 & x > 1 \end{cases}$ check at $x=1$

⊛ $f(x) = \begin{cases} x^2+1 & x \le 1 \\ 2x & x > 1 \end{cases}$ check at $x=1$

⊛ Derivative of $f(x)$,



$f'(x)$

$\frac{dy}{dx}$

y_1

y'

$$\lim_{\Delta x \to 0} \frac{y+\Delta y - y}{x+\Delta x - x}$$

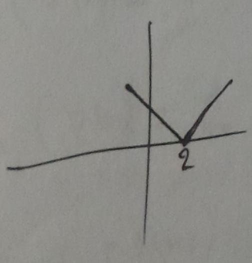
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$f'(x) = \frac{dy}{dx}$$

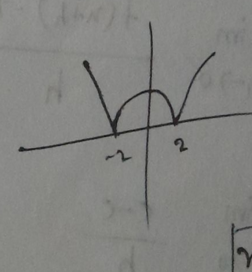
⊛ Find all points where the following function fails to differentiable.

a) $f(x) = |x-2|$

b) $f(x) = |x^2-4|$



$\boxed{x=2}$



$\boxed{x=-2,2}$

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2.3/

⊛ Technique of Derivative:

⊛ $f(x) = x^2$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2h+h^2-x^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$= 2x+0 = 2x$$

⊛ if $f(x) = x^2$

$$\frac{d}{dx}(x^2) = 2x.$$

⊛ $f(x) = c$, $f(x+h) = c$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0} \frac{c-c}{h}$$

$$= 0$$

i) Derivative of constant function:

$f(x) = c,$

i.e. $f'(x) = \frac{d}{dx}(c) = 0$

$\frac{d}{dx}(1) = 0$

$\frac{d}{dx}(5) = 0$

$\frac{d}{dx}(-\sqrt{2}) = 0$

ii) Derivative of power function

$f(x) = x^n$

$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(x^2) = 2x^{2-1} = 2x$

$\frac{d}{dx}(x^4) = 4x^3$

$\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}(x^{-8}) = -8x^{-8-1}$

$= -8x^{-9}$

⊛ $\frac{d}{dx}(f(x) \pm g(x))$

$= \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

⊛ $\frac{d}{dx}(c.f(x))$

$= c.\frac{d}{dx}(f(x))$

⊛ If $f(x) = 8x^5 + 7x^3 + 3x - 1$

$$f'(x) = \frac{d}{dx}(8x^5 + 7x^3 + 3x - 1)$$

$$= 5.8x^4 + 7.3.x^2 + 3.1.x^0 - 0$$

$$= 40x^4 + 21x^2 + 3$$

⊛ If $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$

$$f'(x) = \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$$ Ans

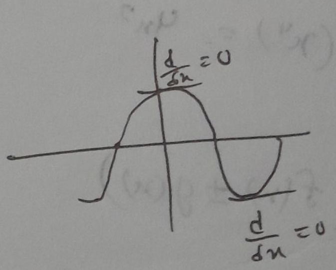
⊛ <u>Horizontal tangent line:</u>

⇒ tangent line is horizontal

⇒ slope = 0

⇒ $\frac{dy}{dx} = 0$

⇒ $f'(x) = 0$



⊛ At what points the function $f(x) = x^3 - 3x + 4$ have horizontal tangent line.

$\Rightarrow$

Here, $f(x) = x^3 - 3x + 4$

$\therefore f'(x) = 3x^2 - 3$

For horizontal tangent line, $f'(x) = 0$

$$3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

$x = 1$, $f(1) = 1 - 3 + 4 = 2$; point $(1, 2)$

$x = -1$, $f(-1) = -1 + 3 + 4 = 6$; point $(-1, 6)$

⊛ Higher derivative,

$y = f(x)$

$\frac{dy}{dx} = f'(x)$

$\frac{d^2y}{dx^2} = f''(x)$

$\frac{d^3y}{dx^3} = f'''(x)$

. . .

. . .

$\frac{d^ny}{dx^n} = f^n(x)$

⊛ $y = x^3 + 2x^2 - 1$

$$\frac{dy}{dx} = 3x^2 + 4x$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 + 4x)$$

$$\frac{d^2y}{dx^2} = 6x + 4$$

$$\frac{d^3y}{dx^3} = 6x$$

$$\frac{d^4y}{dx^4} = 0$$

⊛ if $y = 4x^4 + 2x^3 + 3$

find $y_3(0)$ / $\left.\frac{d^3y}{dx^3}\right|_{x=0}$ / $f'''(0)$ / $y'''(0)$

$$y = 4x^4 + 2x^3 + 3$$

$$\frac{dy}{dx} = 16x^3 + 6x^2$$

$$\frac{d^2y}{dx^2} = 48x^2 + 12x$$

$$\frac{d^3y}{dx^3} = 96x + 12$$

$$y_3|_{x=0} = 96 \cdot 0 + 12 = 12$$ Ans

⊛ $\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{d}{dx}\left(f(x)\right) \pm \frac{d}{dx}\left(g(x)\right)$

2.4

⊛ Product and Quotient Rule

⊛ Product Rule

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right)$$

$$= f(x) \cdot \frac{d}{dx}\left(g(x)\right) + g(x) \cdot \frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

⊛ Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$$

$$= \frac{g(x) \cdot \frac{d}{dx}\left(f(x)\right) - f(x) \cdot \frac{d}{dx}\left(g(x)\right)}{\left(g(x)\right)^2}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

$$\boxed{\frac{L.D.H - H.D.L}{L^2}}$$

$f(x) = (x^2+1)(x^2-1)$

$f'(x) = \frac{d}{dx}\{(x^2+1)(x^2-1)\}$

$= (x^2+1)(2x) + (x^2-1)(2x)$

$= 2x^3 + 2x + 2x^3 - 2x$

$= 4x^3$

Ans

If $y = \frac{2x-1}{2x+3}$, find $\left.\frac{dy}{dx}\right|_{x=1}$

$\Rightarrow$

$y = \frac{2x-1}{2x+3}$

$\frac{dy}{dx} = \frac{dy}{dx}\frac{2x-1}{2x+3}$

$= \frac{(2x+3)\cdot\frac{dy}{dx}(2x-1) - (2x-1)\cdot\frac{dy}{dx}(2x+3)}{(2x+3)^2}$

$= \frac{(2x+3)\cdot 2 - (2x-1)\cdot 2}{(2x+3)^2}$

$= \frac{4x+6-4x+2}{(2x+3)^2}$

$= \frac{8}{(2x+3)^2}$

$\therefore \frac{dy}{dx} = \frac{8}{(2x+3)^2}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{8}{(2+3)^2}$

$= \frac{8}{25}$ Ans

Find all values of x at which the tangent line to the curve is horizontal, $y = \frac{x^2-1}{x+2}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2-1}{x+2}\right)$$

$$= \frac{(x+2)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x+2)}{(x+2)^2}$$

$$= \frac{(x+2)\cdot 2x - (x^2-1)\cdot 1}{(x+2)^2}$$

$$= \frac{2x^2+4x-x^2+1}{(x+2)^2}$$

$$= \frac{x^2+4x+1}{(x+2)^2}$$

Now, for horizontal tangent line,

$$\frac{dy}{dx} = 0$$

$$\frac{x^2+4x+1}{(x+2)^2} = 0$$

$$x^2+4x+1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{16-4}}{2\cdot 1}$$

$$= \frac{-4 \pm \sqrt{12}}{2}$$

$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

$\therefore x = -2+\sqrt{3}, -2-\sqrt{3}$

Ans

(*) Find all values of x at which the tangent line to the curve $y = \frac{x^2+1}{x-1}$ is Parallel to the line $y = x$

$\Rightarrow$

Here,
$$y = \frac{x^2+1}{x-1}$$

we know that slope of tangent line is

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{(x-1)\cdot 2x - (x^2+1)\cdot 1}{(x-1)^2} = \frac{2x^2-2x-x^2-1}{(x-1)^2}$$

$$= \frac{x^2-2x-1}{(x-1)^2}$$

$\therefore$ slope of tangent line,

$$m_1 = \frac{dy}{dx} = \frac{x^2-2x-1}{(x-1)^2}$$

and slope of the line $y = x$ is

$$m_2 = 1$$

Now, if two lines are parallel, then,

$$m_1 = m_2$$

$$\frac{x^2-2x-1}{(x-1)^2} = 1$$

$$x^2-2x-1 = (x-1)^2$$

$$x^2-2x-1 = x^2-2x+1$$

$-1 = 1$ (Not possible)

$\therefore$ points are none.

(*) $y = \frac{12}{x}$; parallel to the line $3x + y = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{12}{x}\right)$$

$$= \frac{x \cdot \frac{d}{dx}(12) - 12 \cdot \frac{d}{dx} \cdot x}{x^2}$$

$$= \frac{0-12}{x^2}$$

$$= \frac{-12}{x^2}$$

$$\therefore m_1 = \frac{-12}{x^2}$$

slope of the line $3x + y = 0$

$$\Rightarrow y = -3x$$

$$m_2 = -3$$

Now,

$$m_1 = m_2$$

$$\frac{-12}{x^2} = -3$$

$$-12 = -3x^2$$

$$x^2 = 4$$

$$x = \pm 2$$

$x = 2$, $y = \frac{12}{2} = 6 \Rightarrow (2, 6)$

$x = 2$, $y = \frac{12}{-2} = -6 \Rightarrow (-2, -6)$

⊛ $y = 2x^2 - 4x$; perpendicular to the line $y = \frac{1}{4}x + 5$

$\Rightarrow$

$y = 2x^2 - 4x$

$\frac{dy}{dx} = 4x - 4$

$m_1 = 4x - 4$ / $m_2 = \frac{1}{4}$

$\therefore (4x-4)\frac{1}{4} = -1$

$4x - 4 = -4$

$4x = 0.$

$x = 0$

$\therefore x = 0$

$y = 2 \cdot 0 - 4 \cdot 0 = 0$

$\therefore$ point $(0,0)$.

2·5

Derivative of Trig. Fn.

$y = \sin x$

$\frac{dy}{dx} = \frac{d}{dx}(\sin x)$

$= \cos x$

$y = \cos x$

$\frac{dy}{dx} = \frac{d}{dx}(\cos x)$

$= -\sin x$

$y = \tan x$

$\frac{dy}{dx} = \frac{d}{dx}(\tan x)$

$= \sec^2 x$

$y = \sec x$

$\frac{dy}{dx} = \frac{d}{dx}(\sec x)$

$= \sec x \tan x$

$y = \operatorname{cosec} x$

$\frac{dy}{dx} = \frac{d}{dx}(\operatorname{cosec} x)$

$= -\operatorname{cosec} x \cot x$

$y = \cot x$

$\frac{dy}{dx} = \frac{d}{dx}(\cot x)$

$= -\operatorname{cosec}^2 x$

⊛ $\frac{d}{dx}(x^2 + x\cos x)$

$= 2x + \frac{d}{dx}(x\cos x)$

$= 2x + x(-\sin x) + \cos x \cdot 1$

$= 2x - x\sin x + \cos x$ Ans

⊛ $f(x) = \sec x - \sqrt{2}\tan x$

$f'(x) = \frac{d}{dx}(\sec x - \sqrt{2}\tan x)$

$= \sec x \tan x - \sqrt{2}\sec^2 x$

(*) $f(x) = \dfrac{\sin x \sec x}{1+ s \tan x}$; find $f'(x) = ?$

$\Rightarrow$

$$f(x) = \frac{\sin x \sec x}{1+ x \tan x}$$

$$= \frac{\sin x \cdot \frac{1}{\cos x}}{1+ x \cdot \frac{\sin x}{\cos x}}$$

$$= \frac{\sin x}{\cos x + x \sin x}$$

$$f'(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x + x \sin x} \right)$$

$$= \frac{(\cos x + x\sin x) \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x + x \sin x)}{(\cos x + x \sin x)^2}$$

$$= \frac{(\cos x + x \sin x) \cdot \cos x - \sin x\,(-\sin x + x\cos x + \sin x)}{(\cos x + x \sin x)^2}$$

$$= \frac{\cos^2 x}{(\cos x + x \sin x)^2}$$

(*) Find the all values of x in the interval $[-2\pi, 2\pi]$ at which the graph of ~~f~~ f has horizontal tangent line.

i) $f(x) = \sin x$ ii) $f(x) = x + \cos x$ iii) $f(x) = \tan x$

=>

i) $f(x) = \sin x$

$f'(x) = \cos x$

For H. tangent line

$f'(x) = 0$

$\therefore \cos x = 0$

$x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}$

ii) $f(x) = x + \cos x$

$f'(x) = 1 - \sin x$

for, H. tangent line,

$f'(x) = 0$

$1 - \sin x = 0$

$\therefore \sin x = 1$

$\therefore x = -\frac{3\pi}{2}, \frac{\pi}{2}$

iii) $f(x) = \tan x$

$f'(x) = \sec^2 x$

for H. tangent line,

$f'(x) = 0$

$\sec^2 x = 0$

$\frac{1}{\cos^2 x} = 0$

$1 = 0$ [Not possible]

$\therefore$ No points

(*) $y = x \sin x - x \cos x$

Find $\frac{d^2y}{dx^2} = ?$

(*) Find the tangent line for, $f(x) = \tan x$ at, $x=0$ and $x = \frac{\pi}{4}$

$\Rightarrow$

Here,

$f(x) = \tan x$

$f'(x) = \sec^2 x$

$\therefore$ slope at $x=0$

$f'(0) = \sec^2 0 = 1$

if $x=0$, $y = f(0) = \tan 0 = 0$

point $(0,0)$

tangent line,

$y - 0 = 1(x - 0)$

$y = x$

slope at $x = \frac{\pi}{4}$

$f'(\frac{\pi}{4}) = \sec^2 \frac{\pi}{4} = 2$

L-12/ 18.07.2022/

2.6/

The chain Rule

$$\frac{d}{dx}(x\cos x) = x\frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}x$$

$$= -x\sin x + \cos x$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1})$$

$$= -1x^{-2} = -x^{-2}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{x\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{-1}{x^2} = -x^{-2}$$

Derivative of composite function

$$y = f(g(x))$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(f(g(x))\right)$$

$$= f'(g(x)) \cdot \frac{d}{dx}(g(x))$$

⊛ $y = x^4$

$$\frac{dy}{dx} = 4x^3$$

⊛ $y = (3x+1)^7$

$$\frac{dy}{dx} = 7(3x+1)^6 \cdot \frac{d}{dx}(3x+1)$$

$$= 7(3x+1)^6 \cdot 3$$

$$= 21(3x+1)^6$$

⊛ $y = 7x^6 + 2$

$$\frac{dy}{dx} = 7 \cdot 6x^5 + 0$$

$$= 42x^5$$

⊛ $y = \sqrt{7x^6+2} = (7x^6+2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}(7x^6+2)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(7x^6+2)$$

$$= \frac{1}{2}(7x^6+2)^{-\frac{1}{2}} \cdot 42x^5$$

$$= 21x^5(7x^6+2)^{-\frac{1}{2}}$$

⊛ If $y = \cos^2 x$

$$y = (\cos x)^2$$

$$\frac{dy}{dx} = 2(\cos x)^{2-1} \cdot \frac{d}{dx}(\cos x)$$

$$= 2\cos x \cdot (-\sin x)$$

$$= -2\cos x \sin x$$

⊛ $y = \sin^5 x$

$$y = (\sin x)^5$$

$$\frac{dy}{dx} = 5(\sin x)^4 \cdot \frac{d}{dx}(\sin x)$$

$$= 5\sin^4 x \cdot \cos x$$

⊛ $f(x) = \sqrt{x^3 - 2x + 5}$

$$f(x) = (x^3 - 2x + 5)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(x^3 - 2x + 5)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^3 - 2x + 5)$$

$$= \frac{1}{2}(x^3 - 2x + 5)^{-\frac{1}{2}} \cdot (3x^2 - 2)$$

$y = \sin 5x$

$$\frac{dy}{dx} = \cos 5x \cdot \frac{d}{dx}(5x)$$

$$= 5 \cos 5x$$

$y = (x + \tan 7x)^3$

$$\frac{dy}{dx} = 3(x + \tan 7x)^2 \cdot \frac{d}{dx}(x + \tan 7x)$$

$$= 3(x + \tan 7x)^2 \cdot \left(1 + \sec^2 7x \cdot \frac{d}{dx}(7x)\right)$$

$$= 3(x + \tan 7x)^2 \cdot (1 + 7\sec^2 7x)$$

Find $\frac{dy}{dx}$ from the followings.

i) $y = \sin(3x^2)$ ii) $y = \cos(\cos x)$ iii) $y = \sqrt{x^3 + \operatorname{cosec} x}$

Solve /

i) $y = \sin(3x^2)$

$$\frac{dy}{dx} = \cos 3x^2 \cdot \frac{d}{dx}(3x^2)$$

$$= \cos 3x^2 \cdot 6x$$

$$= 6x \cdot \cos 3x^2$$

ii) $y = \cos(\cos x)$

$$\frac{dy}{dx} = -\sin(\cos x) \cdot \frac{d}{dx}(\cos x)$$

$$= -\sin(\cos x) \cdot (-\sin x)$$

$$= \sin(\cos x) \cdot \sin x$$

iii) $$\frac{dy}{dx} = \frac{1}{2}(x^3 + \operatorname{cosec} x)^{-\frac{1}{2}} \cdot \frac{d}{dx}(x^3 + \operatorname{cosec} x)$$

$$= \frac{3x^2 - \operatorname{cosec} \cot x}{2\sqrt{x^3 + \operatorname{cosec} x}}$$

If $y = \cot^3(\pi - \theta)$. find $\frac{dy}{d\theta}$

$$\Rightarrow \frac{dy}{dx} = 3\cot^2(\pi-\theta) \cdot \frac{d}{d\theta}(\cot(\pi-\theta))$$

$$= 3\cot^2(\pi-\theta) \cdot -\operatorname{cosec}^2(\pi-\theta) \cdot \frac{d}{d\theta}(\pi-\theta)$$

$$= 3\cot^2(\pi-\theta) \cdot -\operatorname{cosec}^2(\pi-\theta) - 1$$

$$= 3\cot^2(\pi-\theta) \cdot \operatorname{cosec}^2(\pi-\theta)$$

$$y = (1+\cos 2x)^2$$

$$\frac{dy}{dx} = \frac{d}{dx}(1+\cos 2x)^2$$

$$= 2(1+\cos 2x)\cdot(-\sin 2x \cdot 2)$$

Chapter - 3

3.1
Implicit Differentiation

Implicit Function	Explicit Function
	$y = f(x)$
$x^2 + 2xy + y^2 = 0$	$y = x^2 + 2x + 1$
$xy = 1$	$y = \frac{1}{x}$
$xy + x = 2$	$y = \frac{2-x}{x}$
$f(x,y) = g(x,y)$	$y = \pm\sqrt{1-x^2}$
$x^2 + y^2 = 1$	

$$\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(y^4) = 4y^3 \cdot \frac{dy}{dx}$$

$$\frac{d}{dx}(\sin x) = \cos x \cdot \frac{dx}{dy}$$

$$\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$$

(*) $x^2y + 3xy^3 - x = 3$

the $\frac{dy}{dx} = ?$

$\Rightarrow$

Here,

$$x^2y + 3xy^3 - x = 3$$

differentiate both side w.r.t. x.

$$\frac{d}{dx}(x^2y + 3xy^3 - x) = \frac{d}{dx}(3)$$

$$\Rightarrow x^2 \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x^2) + 3x \cdot \frac{d}{dx}(y^3) + y^3 \cdot \frac{d}{dx}(3x) - 1 = 0$$

$$\Rightarrow x^2 \cdot \frac{dy}{dx} + y \cdot 2x + 3x \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 3 - 1 = 0$$

Now solve for $\frac{dy}{dx}$

$$(x^2 + 9xy^2)\frac{dy}{dx} = 1 - 3y^3 - 2xy$$

$$\therefore \frac{dy}{dx} = \frac{1 - 3y^3 - 2xy}{x^2 + 9xy^2}$$

$$x^2+y^2=1$$

$$y= \pm\sqrt{1-x}$$

$$y=+\sqrt{1-x^2} \Rightarrow fn$$

$$y=-\sqrt{1-x^2} \Rightarrow fn$$

$$x^2+y^2=1$$

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(1)$$

$$(2x+2y)\cdot\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{2y}$$

$$\frac{dy}{dx}=-\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)}=\frac{-1}{1}=-1$$

Use implicit derivative to find $\frac{dy}{dx}$,

i) $5y^2+\sin y=x^2$

ii) $\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}=1$

iii) $\cos(xy^2)=y$

Solution:

i) $5y^2 + \sin y = x^2$

$\Rightarrow \frac{d}{dx}(5y^2 + \sin y) = \frac{d}{dx}(x^2)$

$\Rightarrow 10y \cdot \frac{d}{dx} y + \cos y \cdot \frac{dy}{dx} = 2x$

$\Rightarrow (10y + \cos y) \frac{dy}{dx} = 2x$

$\therefore \frac{dy}{dx} = \frac{2x}{10y + \cos y}$

ii) $\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = 1$

$\frac{d}{dx}\left(\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\right) = \frac{d}{dx}(1)$

$\frac{d}{dx}(x^{-\frac{1}{2}} + y^{-\frac{1}{2}}) = 0$

$-\frac{1}{2} x^{-\frac{1}{2}-1} - \frac{1}{2} y^{-\frac{1}{2}-1} \cdot \frac{d}{dx} y = 0$

$x^{-3/2} + y^{-3/2} \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-x^{-3/2}}{y^{-3/2}} = -\frac{y^{3/2}}{x^{3/2}} = \frac{-y\sqrt{y}}{x\sqrt{x}}$

iii)

$$\cos(xy^2) = y$$

$$\frac{d}{dx}\left(\cos(xy^2)\right) = \frac{dy}{dx}$$

$$-\sin xy^2 \cdot \frac{d}{dx} xy^2 = \frac{dy}{dx}$$

$$-\sin xy^2 \cdot \left[x \cdot \frac{d}{dx} y^2 + y^2 \frac{d}{dx} x\right] = \frac{dy}{dx}$$

$$-\sin xy^2 \left[x \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 1\right] = \frac{dy}{dx}$$

$$-2xy \sin xy^2 \frac{dy}{dx} - y^2 \sin xy^2 = \frac{dy}{dx}$$

$$\left(-2xy \sin xy^2 - 1\right) \frac{dy}{dx} = y^2 \sin xy^2$$

$$\frac{dy}{dx} = \frac{y^2 \sin xy^2}{-(1 + 2xy \sin xy^2)}$$

Find the equation of a tangent line to the Folium of Descantes $\boxed{x^3 + y^3 = 3xy}$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

$\Rightarrow$

$$x^3 + y^3 = 3xy$$

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3xy)$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 3x \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}(3x)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \cdot \frac{dy}{dx} + 3y$$

$$3x \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 3x^2 - 3y$$

$$(3x - 3y^2) \frac{dy}{dx} = 3x^2 - 3y$$

$$\frac{dy}{dx} = \frac{3x^2 - 3y}{3x - 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2}, \frac{3}{2}\right)} = \frac{3 \cdot \left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2}}{3 \cdot \frac{3}{2} - 3 \cdot \left(\frac{3}{2}\right)^2}$$

$$= \frac{\frac{27}{4} - \frac{9}{2}}{\frac{9}{2} - \frac{27}{4}}$$

$$= -1$$

Therefore, equation of tangent line,

$$y - \frac{3}{2} = -1 \left(x - \frac{3}{2}\right)$$

$$y - \frac{3}{2} = -x + \frac{3}{2}$$

$$\boxed{y + x = 3}$$

(*) If, $y + \sin y = x$

find, $\dfrac{d^2y}{dx^2} = ?$

$$y + \sin y = x$$

$$\frac{d}{dx}(y + \sin y) = \frac{d}{dx} x$$

$$\frac{dy}{dx} + \cos y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{1+\cos y}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{1+\cos y}\right)$$

$$\boxed{\cancel{\frac{d^2y}{dx^2} = \frac{(1+\cos y)\frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(1+\cos y)}{(1+\cos y)^2}}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(1+\cos y)^{-1}$$

$$= -1(1+\cos y)^{-2} \cdot \frac{d}{dx}(1+\cos y)$$

$$= -(1+\cos y)^{-2} \cdot \left(0 - \sin y \frac{dy}{dx}\right)$$

$$= -(1+\cos y)^{-2} \cdot \left(-\sin y \frac{dy}{dx}\right)$$

$$= \frac{-1}{(1+\cos y)^2} \cdot \left(-\sin y \frac{dy}{dx}\right)$$

$$= \frac{\sin y \cdot \frac{1}{1+\cos y}}{(1+\cos y)^2}$$

$$= \frac{\sin y}{(1+\cos y)^3}$$

$$\therefore \frac{d^2x}{dx^2} = \frac{\sin y}{(1+\cos y)^3} \quad \text{Ans}$$

3.2

$$f(x) = \ln x$$

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\ln x)$$

$$= \frac{1}{x}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \cdots$$

$$= \frac{1}{x}$$

⊛ $\frac{d}{dx}(\log x) = ?$

$\log x \to$ base 10

⊛ $\frac{d}{dx}(\log_a x)$

$= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$

$= \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x)$

$= \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a}$

⊛ $\frac{d}{dx}(\log_2 x)$

$= \frac{1}{x \ln 2}$

⊛ If, $y = x \log x$

$\frac{dy}{dx} = \frac{d}{dx}(x \log x)$

$= x \cdot \frac{d}{dx}(\log x) + \log x \cdot 1$

$= x \cdot \frac{1}{x \ln 10} + \log x$

$= \log x + \frac{1}{\ln 10}$

L-13/ 20.07.2022/

3.2/

Derivative of Logarithmic function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\ln\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{v \to 0} \frac{\ln(1+v)}{vx}$$

Let,

$h = vx$

$\therefore \frac{h}{x} = v$

if, $h \to 0$, then $v \to 0$

$$= \frac{1}{x} \cdot \lim_{v \to 0} \frac{1}{v} \cdot \ln(1+v)$$

$$= \frac{1}{x} \lim_{v \to 0} \ln(1+v)^{\frac{1}{v}}$$

$$= \frac{1}{x} \cdot \ln\left(\lim_{v \to 0} (1+v)^{\frac{1}{v}}\right)$$

$$= \frac{1}{x} \cdot \ln e$$

$$= \frac{1}{x} \cdot 1 = \frac{1}{x}.$$

⊛ $f(x) = \ln x$

$f'(x) = \frac{d}{dx}(\ln x)$

$= \frac{1}{x}$

⊛ $y = \log x$

$\frac{dy}{dx} = \frac{1}{x \ln 10}$

⊛ $y = \log_3 x$

$\frac{dy}{dx} = \frac{1}{x \ln 3}$

⊛ If $f(x) = \ln x$, is there any horizontal tangent line?

$\Rightarrow$ $f'(x) = \frac{d}{dx}(\ln x)$

$= \frac{1}{x}$

For horizontal tangent line,

$f'(x) = 0$

$\frac{1}{x} = 0$

$1 = 0$ (Not possible)

So, there is no horizontal tangent line for $f(x) = \ln x$.

(*) Find the following derivatives.

i) $y = \ln \frac{x}{3}$ ii) $y = x^3 \ln x$ iii) $y = \ln(\tan x)$

iv) $y = \ln(\ln x)$ v) $y = \log(\sin^2 x)$

Solutions:

i) $y = \ln \frac{x}{3}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\ln \frac{x}{3}\right)$$

$$= \frac{1}{x/3} \cdot \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{x} \cdot \frac{3}{x} \cdot \frac{d}{dx} x$$

$$= \frac{1}{x} \cdot 1$$

$$= \frac{1}{x}$$

ii) $y = x^3 \ln x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \ln x)$$

$$= x^3 \cdot \frac{1}{x} + \ln x \cdot 3x^2$$

$$= x^2 + 3x^2 \ln x$$

iii) $\frac{dy}{dx} = \frac{1}{\tan x} \cdot \frac{d}{dx}(\tan x)$

$$= \frac{\sec^2 x}{\tan x}$$

iv) $\frac{dy}{dx} = \frac{1}{\ln x} \cdot \frac{d}{dx} \cancel{(\sin^2 x)} (\ln x)$

$$\cancel{= \frac{2\sin x \cdot \cos x}{\sin^2 x \cdot \ln 10}}$$

$$\cancel{= \frac{2\cos x}{\sin x \ln 10}}$$

$$= \frac{1}{x \ln x}$$

v) $\frac{dy}{dx} = \frac{1}{\sin^2 x \ \ln 10} \cdot \frac{d}{dx}(\sin^2 x)$

$= \frac{2 \sin x \cdot \cos x}{\sin^2 x \cdot \ln 10}$

$= \frac{2 \cos x}{\sin x \ \ln 10}$

(*) $y = \ln |x|$

$x > 0$

$y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$x < 0$

$y = \ln(-x)$

$\frac{dy}{dx} = \frac{1}{-x} \cdot \frac{d}{dx}(-x)$

$= \frac{+1}{+x} = \frac{1}{x}$

$\therefore y = \ln |x|$

$\frac{dy}{dx} = \frac{1}{x}$

$$y = \ln|\sin x|$$

$$\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x$$

$$y = \ln\left|\frac{1+x}{1-x}\right|$$

$$\frac{dy}{dx} = \frac{1}{\frac{1+x}{1-x}} \cdot \frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1-x}{1+x} \cdot \frac{(1-x)\cdot 1-(1+x)\cdot -1}{(1-x)^2}$$

$= \text{simplify}$

$$y = \sqrt[5]{\frac{x-1}{x+1}}$$

taking ln both side,

$$\ln y = \ln\left(\frac{x-1}{x+1}\right)^{\frac{1}{5}}$$

$$\ln y = \frac{1}{5}\ln\left(\frac{x-1}{x+1}\right)$$

$$\ln y = \frac{1}{5}\cdot\left(\ln(x-1) - \ln(x+1)\right)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(\frac{1}{5}\cdot\left(\ln(x-1) - \ln(x+1)\right)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{5}\cdot\left[\frac{1}{x-1}\cdot 1 - \frac{1}{x+1}\cdot 1\right]$$

$$\frac{dy}{dx} = y\cdot\frac{1}{5}\left[\frac{x+1-x+1}{(x-1)(x+1)}\right]$$

$$= \frac{1}{5}\cdot y\cdot\frac{2}{x^2-1}$$

$$= \frac{1}{5}\sqrt[5]{\frac{x-1}{x+1}}\cdot\frac{2}{x^2-1}$$

* Find the following derrivatives.

i) $y = \log_x e$ ii) $y = \log_x 2$ iii) $y = \log_{\frac{1}{x}} e$

Solutions:

i)

$$y = \log_x e$$

$$y = \frac{\ln e}{\ln x}$$

$$y = \frac{1}{\ln x}$$

$$\frac{dy}{dx} = \frac{\ln x\cdot 0 - 1\cdot\frac{1}{x}}{(\ln x)^2}$$

$$= \frac{-\frac{1}{x}}{(\ln x)^2}$$

$$= -\frac{1}{x(\ln x)^2}$$

ii) $Y = \log_x 2$

$Y = \dfrac{\ln 2}{\ln x}$

iii) $Y = \log_{\frac{1}{x}} e$

$= \dfrac{\ln e}{\ln \frac{1}{x}}$

$= \dfrac{1}{\ln 1 - \ln x}$

$= \dfrac{1}{0 - \ln x}$

$= -\dfrac{1}{\ln x}$

$Y = \ln(x^2 - 2)$

$\dfrac{dy}{dx} = \dfrac{1}{x^2-2} \cdot \dfrac{d}{dx}(x^2-2)$

$= \dfrac{1}{x^2-2} \cdot 2x$

$= \dfrac{2x}{x^2-2}$ Ans

3.3

⊛ Derivative of exponential function.

⊛ $y = e^x$

$\frac{dy}{dx} = e^x$

⊛ $y = 2^x$

$\frac{dy}{dx} = 2^x \cdot \ln 2$

⊛ $y = b^x$

$x = \log_b y$

$x = \frac{\ln y}{\ln b}$

$\ln y = x \ln b$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln b)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln b \cdot 1$

$\therefore \frac{dy}{dx} = y \ln b$

$= b^x \ln b$

$\therefore y = b^x$

$\boxed{\frac{dy}{dx} = b^x \ln b}$

if, $b = e$

$y = e^x$

$\frac{dy}{dx} = e^x \cdot \ln e$

$\frac{dy}{dx} = e^x$

⊛ $y = 10^x$

$\frac{dy}{dx} = 10^x \cdot \ln 10$

⊛ $y = e^{\cos x}$

$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x})$

$= e^{\cos x} \cdot \frac{d}{dx}(\cos x) = -\sin e^{\cos x}$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{d}{dx}(e^{-x})$$

$$= e^{-x} \cdot \frac{d}{dx}(-x)$$

$$= -e^{-x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x}) \cdot (e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

= .... simplify ...

$$= \frac{4}{(e^x + e^{-x})^2}$$

$$y = e^{x\tan x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x\tan x}\right)$$

$$= e^{x\tan x} \cdot \frac{d}{dx}(x\tan x)$$ → uv rules

$$= e^{x\tan x} \cdot (x\sec^2 x + \tan x)$$

Derivative of inverse trig. function

i) $y = \sin^{-1} x$ ii) $y = \cos^{-1} x$ iii) $y = \tan^{-1} x$

$\Rightarrow$

i) $y = \sin^{-1} x$

$\sin y = x$

differentiate w.r.t. x

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}(x)$$

$$\cos y \cdot \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \sin^{-1}x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \cos^{-1}x$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

⊛ $y = \tan^{-1}$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

⊛ $y = \sec^{-1}x$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$$

⊛ $y = \operatorname{cosec}^{-1}x$

$$\frac{dy}{dx} = \frac{1}{-x\sqrt{x^2-1}}$$

⊛ $y = \cot^{-1}x$

$$\frac{dy}{dx} = \frac{1}{-(1+x)^2}$$

⊛ $y = \sin^{-1}x^3$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^6}} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1-x^6}} \cdot 3x^2$$

⊛ $y = \sec^{-1} e^x$

$$\frac{dy}{dx} = \frac{1}{e^x\sqrt{e^{2x}-1}} \cdot \frac{d}{dx}(e^x)$$

$$= \frac{1}{e^x\sqrt{e^{2x}-1}} \cdot e^x$$

$$= \frac{1}{\sqrt{e^{2x}-1}} \quad \text{Ans}$$

$x^3 + x\tan^{-1}y = e^y$

$$\frac{d}{dx}\left(x^3 + x\tan^{-1}y\right) = \frac{d}{dx}\left(e^y\right)$$

$$3x^2 + \left\{\tan^{-1}y + x \cdot \frac{1}{1+y^2} \cdot \frac{dy}{dx}\right\} = e^y \cdot \frac{dy}{dx}$$

$$\left(\frac{x}{1+y^2} - e^y\right)\frac{dy}{dx} = 3x^2 - \tan^{-1}y$$

$$\therefore \frac{dy}{dx} = \frac{3x^2 - \tan^{-1}y}{\frac{x}{1+y^2} - e^y}$$

$\sin^{-1}(xy) = \cos^{-1}(x-y)$

$$\frac{d}{dx}\left(\sin^{-1}xy\right) = \frac{d}{dx}\left(\cos^{-1}(x-y)\right)$$

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot \frac{d}{dx}(xy) = \frac{1}{-\sqrt{1-(x-y)^2}} \cdot \frac{d}{dx}(x-y)$$

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot \left(y + x\frac{dy}{dx}\right) = \frac{1}{-\sqrt{1-(x-y)^2}} \cdot \left(1 - \frac{dy}{dx}\right)$$

|L-14 / 27.07.2022|

3.4

Related Rates

Working Step:

(i) Draw a picture and name the variable, use 't' for time. Assume that all variables are differentiable w.r.t. 't'.

(ii) Write down the numerical information.

(iii) Write what are you asked to find.

(iv) Write the equation that relate the variables.

(v) Differentiate w.r.t. 't'

(vi) Evaluate using known values to find unknown.

(8) Assume that oil spilled ~~form~~ from a rapture tanker spread in a circular pattern whose radius increased at a constant rate 2 ft/sec. How fast is the are of the spill increasing when the radius is 60 ft.

⇒

We know that, area of the circle

$A = \pi r^2$

Here,
A is area
r is radius

Here given that,

$\frac{dr}{dt} = 2$ ft/sec

$\frac{dA}{dt} = ?$

when $r = 60$ ft.

$$A = \pi r^2$$

differentiate both side w.r.t. 't'

$$\frac{d}{dt}(A) = \frac{d}{dt}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2 \cdot 60 \cdot 2 \ ft^2/sec$$

$$\therefore \frac{dA}{dt} = 240\pi \ ft^2/sec$$

increasing : $\frac{dA}{dt} = 240\pi \ ft^2/sec$

(8) Let 'l' be the length of the diagonal of a rectangle whose sides have length x and y and assume that x and y vary with time.

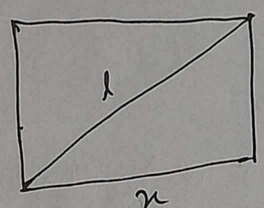
i) How are $\frac{dl}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dt}$ related?

ii) If x increases at a constant rate $\frac{1}{2}$ ft/sec. and y decreases at a constant rate $\frac{1}{4}$ ft/sec. How fast the diagonal changing when $x=3$ and $y=4$?

Is the diagonal increasing or decreasing at that instant?

$\Rightarrow$

(i)



Here, x, y and l are changing with time.

therefore we know that, from figure, $x^2+y^2=l^2$

differentiate both side w.r.t. 't'.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(l^2)$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 2l \cdot \frac{dl}{dt}$$

$$\boxed{\therefore \; x\frac{dx}{dt} + y\frac{dy}{dt} = l\frac{dl}{dt}}$$

relation

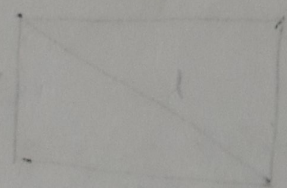
ii)

Here,

$$\frac{dx}{dt} = \frac{1}{2} \text{ ft/sec}$$

$$\frac{dy}{dt} = -\frac{1}{4} \text{ ft/sec}$$

$x = 3$
$y = 4$ / at $x=3$, $y=4$, l will be $= \sqrt{3^2+4^2} = 5$



from ①

$$x\frac{dx}{dt} + y\frac{dy}{dt} = l\frac{dl}{dt}$$

$$5 \cdot \frac{dl}{dt} = 3 \cdot \frac{1}{2} + 4\left(-\frac{1}{4}\right)$$

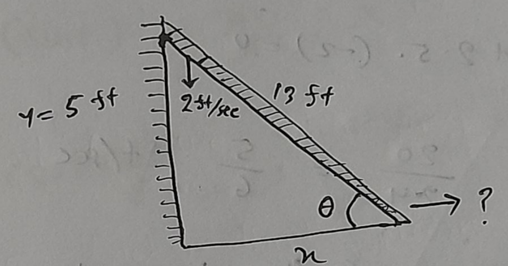
$$5 \cdot \frac{dl}{dt} = \frac{3}{2} - 1$$

$$\therefore \frac{dl}{dt} = \frac{3-2}{2 \cdot 5} = \frac{1}{10} \text{ ft/sec}$$

as, $\frac{dl}{dt}$ is positive, so, diagonal is increasing.

(*) A 13 ft. ladder leaning against a wall. If the top of the ladder slip down the wall at a rate 2 ft/sec. How fast will the foot of the ladder moving away from the wall when the top is 5 ft above the ground?

$\Rightarrow$



Let,

base $= x$

wall height $= y$

Given that,

$$\frac{dy}{dt} = -2 \text{ ft/sec}$$

$$\frac{dx}{dt} = ?$$

$$y = 5 \text{ ft}$$

if $y = 5$

then, $x^2 + y^2 = 13^2$

$\therefore x = 12$

from figure,

$$x^2+y^2=13^2$$

differentiate w.r.t. 't'

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(13^2)$$

$$2x \cdot \frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2 \cdot 12\frac{dx}{dt} + 2 \cdot 5 \cdot (-2) = 0$$

$$\frac{dx}{dt} = \frac{20}{24} = \frac{5}{6} \text{ ft/sec}$$

ii) How fast the area of the triangle changing?

⇒ area, $A = \frac{1}{2} \cdot x \cdot y$

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\frac{1}{2}xy\right)$$

$$\frac{dA}{dt} = \frac{1}{2}x \cdot \frac{dy}{dt} + y\frac{d}{dt}\left(\frac{1}{2}x\right)$$

$$= \frac{1}{2} \cdot 12(-2) + 5 \cdot \frac{1}{2} \cdot \frac{5}{6}$$

$$= -12 + \frac{25}{12}$$

$$= \frac{-119}{12} \text{ ft}^2\text{/sec}$$

(iii) How fast the θ changing?

$\Rightarrow$

$$\cos\theta = \frac{x}{13} = \frac{12}{13}$$

Now, $\sin\theta = \frac{y}{13}$

differentiate w.r.t. t

$$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{13}\right)$$

$$\cos\theta \frac{d\theta}{dt} = \frac{1}{13}\cdot\frac{dy}{dt}$$

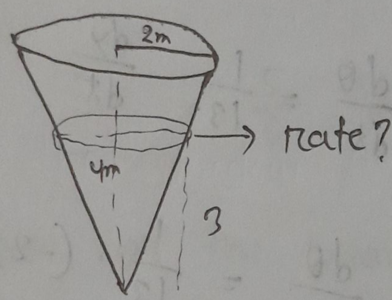
$$\frac{12}{13}\cdot\frac{d\theta}{dt} = \frac{1}{13}\cdot(-2)$$

$$\frac{d\theta}{dt} = \frac{-2\times 13}{13\times 12} = -\frac{1}{6} \text{ degree/sec}$$

$\rightarrow$ decreasing

A water tank has the shape of an inverted circular cone with base radius 2m. and height 4m. If water is being pump into the tank at the rate of $2m^3/min$. Find the rate at which the water lebel is rising, when the water is 3m. deep.

⇒



Changing,

Volume of water, V

Radius of lebel, r

Height of the lebel, h

Given,

$h = 3m$

$\frac{dv}{dt} = 2m^3/min$ / $\frac{dh}{dt} = ?$

$\frac{2}{4} = \frac{r}{h}$

$2h = 4r$

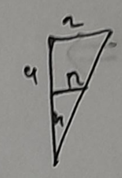
$r = \frac{1}{2}h$

We know that,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$



by similar triangle

$$\frac{4}{2} = \frac{h}{r}$$

$$\frac{h}{r} = 2$$

$$r = \frac{1}{2}h = \frac{h}{2}$$

differentiate w.r.t. t

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right)$$

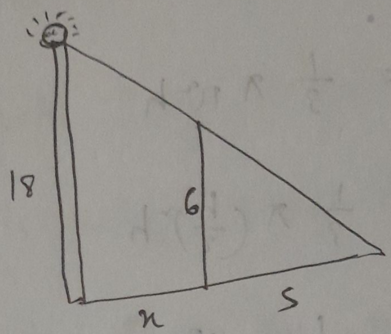
$$\frac{dV}{dt} = \frac{1}{12}\cdot\pi\cdot 3h^2\frac{dh}{dt}$$

$$2 = \frac{1}{12}\cdot\pi\cdot 3\cdot 3^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2\times \cancel{12}^{4}}{\pi\cdot\cancel{3}\cdot 3^2} = \frac{8}{9\pi}\ \text{m/min}$$

⊛ If at night a 6 ft. tall man is walking away at 3 ft/sec from a 18 ft. tall lampost. How fast is the length of his shadow changing?

$\Rightarrow$



Let,

x = distance of the man from the lamp post at time t.

$$\frac{dx}{dt} = 3 \text{ ft/sec.}$$

Let,

s = length of the shadow at time t.

$$\frac{ds}{dt} = ?$$

Now,

from figure by similar triangle,

$$\frac{18}{x+s} = \frac{6}{s}$$

$$18s = 6x + 6s$$

$$12s = 6x$$

$$s = \frac{1}{2}x$$

differentiate w.r.t. t

$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}\times 3 = \frac{3}{2} \text{ ft/sec.}$$

L-15/ 01.08.2022/

3.6

L' Hospital Rule: Indeterminate forms

[Finding Limit using derivative / Application of derivative]

Indeterminate forms, $\boxed{\frac{0}{0}, \frac{\infty}{\infty}}$, 0^0, ∞^0, 1^∞, $\infty-\infty$, $0\cdot\infty$

(*) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$

$= \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

$= \dfrac{0}{0}$; indeterminate forms

(*) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$

$= \dfrac{\infty}{\infty}$; indeterminate form

(*) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

$= \dfrac{\lim\limits_{x\to 2} x^2-4}{\lim\limits_{x\to 2} x-2}$

$= \dfrac{0}{0}$; indeterminate form

(*) $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \dfrac{\lim\limits_{x\to\infty} e^x}{\lim\limits_{x\to\infty} x^2}$

$= \dfrac{\infty}{\infty}$; indeterminate form

⊛ L' Hospital Rule for $\frac{0}{0}$ & $\frac{\infty}{\infty}$ form:

Statement: Suppose that $f(x)$ and $g(x)$ are differentiable function and $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = \text{so on} \cdots$$

$\lim_{x \to a^+} \frac{f(x)}{g(x)}$, $\lim_{x \to a^-} \frac{f(x)}{g(x)}$, $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ also applicable.

⊛ $\lim_{x \to 2} \frac{x^2-4}{x-2}$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x \to 2} (x+2)$$

$$= 2+2$$

$$= 4$$

Ans

⊛ $\lim_{x \to 2} \frac{x^2-4}{x-2}$ [form $\frac{0}{0}$]

Now, using L' Hospital Rule

$$= \lim_{x \to 2} \frac{2x}{1}$$

$$= \lim_{x \to 2} 2x$$

$$= 2 \cdot 2$$

$$= 4$$

Ans

$$\lim_{x \to \infty} \frac{e^x}{x^2} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

Using L' Hospital Rule.

$$= \lim_{x \to \infty} \frac{e^x}{2x} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \infty} \frac{e^x}{2}$$

$$= \infty$$

Find the following Limit:

i) $\lim_{x \to 0} \dfrac{e^x - 1}{\sin x}$ ii) $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$ iii) $\lim_{x \to +\infty} \dfrac{\ln x}{x}$

iv) $\lim_{x \to +\infty} x e^{-x}$

$\Rightarrow$

i) $\lim_{x \to 0} \dfrac{e^x - 1}{\sin x} \quad \left[\text{form } \frac{0}{0}\right]$

Now, using L' Hospital Rule

$$= \lim_{x \to 0} \frac{e^x}{\cos x}$$

$$= \frac{e^0}{\cos 0}$$

$$= \frac{1}{1} = 1$$

ii) $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi} \quad \left[\text{form } \frac{0}{0}\right]$

Now, using L' Hospital Rule,

$$= \lim_{x \to \pi} \frac{\cos x}{1}$$

$$= \cos \pi$$

$$= -1$$

iii) $\lim\limits_{x\to+\infty} \frac{\ln x}{x}$ [form $\frac{\infty}{\infty}$]

Now, using L' Hospital Rule,

$= \lim\limits_{x\to+\infty} \frac{1/x}{1}$

$= \lim\limits_{x\to+\infty} \frac{1}{x}$

$= 0$

iv) $\lim\limits_{x\to+\infty} x e^{-x}$ [form $\infty \cdot 0$]

$= \lim\limits_{x\to+\infty} \frac{x}{e^{x}}$ [form $\frac{\infty}{\infty}$]

Now, using L' Hospital Rule:

$= \lim\limits_{x\to+\infty} \frac{1}{e^{x}}$

$= 0$

Ans

(v) $\lim\limits_{x\to0^+} x \ln x$ [form $0 \cdot -\infty$]

$= \lim\limits_{x\to0^+} \frac{\ln x}{\frac{1}{x}}$ [form $\frac{\infty}{\infty}$]

Now using L' Hospital Rule,

$= \lim\limits_{x\to0^+} \frac{1/x}{-1/x^2}$

$= \lim\limits_{x\to0^+} -x$

$= 0$

Ans

⊛ Type of $(\infty-\infty)$

⊛ $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$ [form $\infty-\infty$]

$$= \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} \quad [\text{form } \tfrac{0}{0}]$$

Now, using L' Hospital Rule:

$$= \lim_{x\to 0^+} \frac{\cos x - 1}{\sin x + x\cos x} \quad [\text{form } \tfrac{0}{0}]$$

$$= \lim_{x\to 0^+} \frac{-\sin x}{\cos x + \cos x - x\sin x}$$

$$= \frac{0}{1+1-0}$$

$$= 0$$

⊛ Type of $\boxed{0^0, \infty^0, 1^\infty}$

→ Taking ln both side for converting $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

(*) Find Limit, $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ [1^∞ form]

Let,

$$y = (1+\sin x)^{\frac{1}{x}}$$

Now, taking ln both side

$$\Rightarrow \ln y = \ln (1+\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \ln y = \frac{1}{x} \ln (1+\sin x)$$

$$\Rightarrow \ln y = \frac{\ln (1+\sin x)}{x}$$

Taking Limit both side,

$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln (1+\sin x)}{x} \quad \left[\text{form } \frac{0}{0}\right]$$

Now, using L' Hospital Rule :

$$= \lim_{x\to 0} \frac{\frac{1}{1+\sin x}\cdot \cos x}{1}$$

$$= \lim_{x\to 0} \frac{\cos x}{1+\sin x}$$

$$= \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0} = 1$$

$\therefore \lim_{x\to 0} \ln y = 1$

$\ln \left(\lim_{x\to 0} y \right) = 1$

$\lim_{x\to 0} y = e^1$

$\therefore \lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e$

⊛ Find Limit,

$$\lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x} \quad [\text{form } \infty^0]$$

Let,

$y = (\tan x)^{\cos x}$

Now, taking ln both side,

$\ln y = \ln (\tan x)^{\cos x}$

$\ln y = \cos x \cdot \ln (\tan x)$

Taking Limit both side,

$$\lim_{x\to \frac{\pi}{2}} \ln y = \lim_{x\to \frac{\pi}{2}} \left(\cos x \cdot \ln (\tan x) \right) \quad [\text{form } 0 \cdot \infty]$$

$$= \lim_{x\to \frac{\pi}{2}} \left[\frac{\ln (\tan x)}{\sec x} \right] \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \ln y = \frac{0}{1} = 0$$

$$\ln \left(\lim_{x \to \frac{\pi}{2}} y \right) = 0$$

$$\lim_{x \to \frac{\pi}{2}} y = e^0$$

$$\lim_{x \to \frac{\pi}{2}} y = 1$$

$$\therefore \lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = 1$$

Chapter - 4

The Derivative in Graphing and Application

4.1

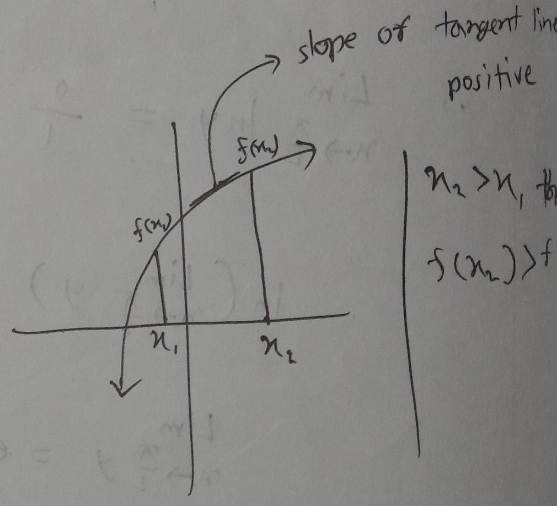
Analysis of function I:

* Increasing function:

Slope of tangent line +ve,

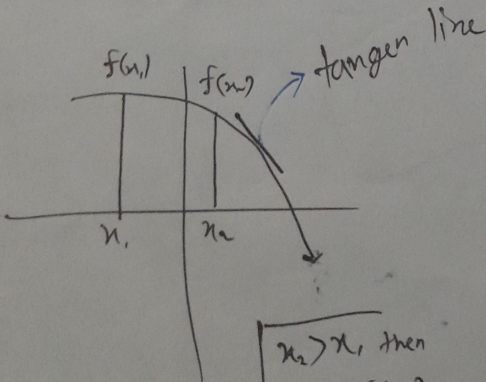
i.e. derivative +ve.

$f'(x) > 0$



$x_2 > x_1$ then,
$f(x_2) > f(x_1)$

* Decreasing function:



$x_2 > x_1$ then
$f(x_2) < f(x_1)$

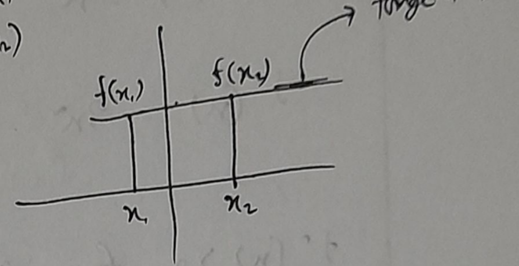
slope of tangent line -ve,

i.e. derivative -ve,

$f'(x) < 0$

⊛ Constant function:

$x_2 > x_1$, but
$f(x_1) = f(x_2)$



Slope of tangent line 0. (Horizontal)

i.e. derivative is 0.

$f'(x) = 0.$

⊛ Find the intervals on which the following function are increasing or decreasing.

(i) $f(x) = x^2 - 4x + 3$

$\Rightarrow f'(x) = 2x - 4$

for increasing,

$2x - 4 > 0$

$\therefore 2x > 4$

$x > 2$

therefore,

increasing interval $(2, \infty)$

for decreasing,

$2x - 4 < 0$

$2x < 4$

$x < 2$

therefore,

decreasing interval $(-\infty, 2)$

ii) $f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) > 0$ for all values of x

$\therefore f(x) = x^3$ is increasing function $(-\infty, \infty)$

iii) $f(x) = 3x^4 + 4x^3 - 12x^2 + \cancel{1}2$

$f'(x) = 12x^3 + 12x^2 - 24x$

$= 12x(x^2 + x - 2)$

$= 12x(x^2 + 2x - x - 2)$

$f'(x) = 12x(x+2)(x-1)$

Sign analysis of $f'(x)$

	$-$		$+$		$-$		$+$
		-2		0		1	
	$(-\infty, -2)$		$(-2, 0)$		$(0, 1)$		$(1, \infty)$

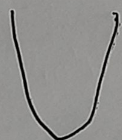
Let,

$x = -3$	$x = -1$	$x = \frac{1}{2}$	$x = 2$
$f'(-3) = -ve$	$f'(-1) = +ve$	$f'(\frac{1}{2}) = -ve$	$f'(2) = +ve$

∴ increasing : $(-2, 0)$, $(1, \infty)$

∴ decreasing : $(-\infty, -2)$, $(0, 1)$

(*) Concavity : → concave up
→ concave down

concave up : 

concave down : 

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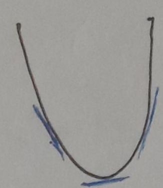
4.1

Concavity

Concave up:



Slope of a tangent line, $\boxed{-ve \to 0 \to +ve}$

↳ increasing

$f'(x)$ increasing

$\frac{d}{dx}(f'(x)) > 0$

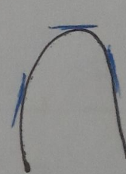
$\boxed{f''(x) > 0}$

⊛ if $f(x)$ increasing, then

$f'(x) > 0$

↳ rate of change of $f(x) > 0$

⊛ Concave down:

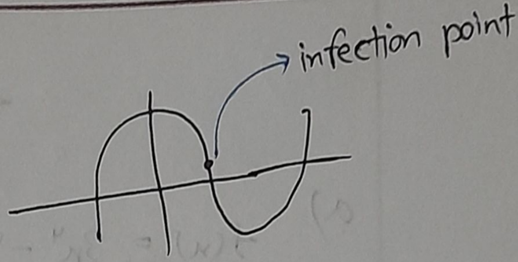


slope of tangent line $\boxed{+ve \to 0 \to -ve}$

↳ decreasing

∴ condition: $\boxed{f''(x) < 0}$

⊛ Infection point:



Transition point for a function from concave up to concave down or vice versa.

condition of infection point:

$$\boxed{f''(x) = 0}$$

⊛ Find

i) the intervals on which f(x) increasing.

ii) the intervals on which f(x) decreasing

iii) the intervals on which f(x) concave up.

iv) the intervals on which f(x) concave down.

v) x-co-ordinate of infection point.

a) $f(x) = 3x^4 - 4x^3$

b) $f(x) = \sqrt[3]{x^2+x+1}$

c) $f(x) = x^{4/3} - x^{1/3}$

d) $f(x) = \frac{x}{x^2+2}$

e) $f(x) = e^{-x^2/2}$

Solution

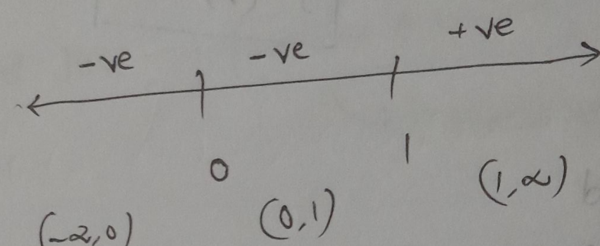
a) $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x-1)$

Now, sign analysis of $f'(x)$

sign of $f'(x)$



$x = -1$	$x = 0.5$	$x = 2$
$f'(-1) = -ve$	$f'(0.5) = -ve$	$f'(2) = +ve$

$\therefore$ increasing : $(1, \infty)$

$\therefore$ decreasing : $(-\infty, 0)$, $(0,1)$

Now, for concavity

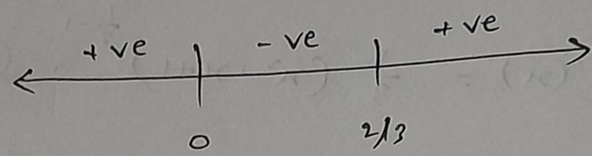
$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

$= 12x(3x-2)$

sign analysis of $f''(x)$

sign of $f''(x)$



$(-\alpha, 0)$ $\quad$ $(0, \frac{2}{3})$ $\quad$ $(\frac{2}{3}, \alpha)$

$x = -1$	$x = \frac{1}{3}$	$x = 2$
$f''(-1) = +ve$	$f''(\frac{1}{3}) = -ve$	$f''(2) = +ve$

concave up: $(-\alpha, 0)$, $(\frac{2}{3}, \alpha)$

concave down: $(0, \frac{2}{3})$

x- co-ordinate of infection point,

$$f''(x) = 0$$

$$12x(3x-2) = 0$$

$12x = 0$	$3x - 2 = 0$
$x = 0$	$x = \frac{2}{3}$

$\therefore$ infection point : 0 and $\frac{2}{3}$.

b)

$$f(x) = \sqrt[3]{x^2+x+1}$$

$$f'(x) = \frac{1}{3}(x^2+x+1)^{\frac{1}{3}-1} \cdot (2x+1)$$

$$= \frac{2x+1}{3(x^2+x+1)^{\frac{2}{3}}}$$

$$f''(x) = \frac{3(x^2+x+1)^{\frac{2}{3}} \cdot 2 - (2x+1) \cdot 3 \cdot \frac{2}{3}(x^2+x+1)^{\frac{2}{3}-1} \cdot (2x+1)}{\left(3(x^2+x+1)^{\frac{2}{3}}\right)^3}$$

$$= \frac{6(x^2+x+1)^{\frac{2}{3}} - 2 \cdot (2x+1)^2 (x^2+x+1)^{-\frac{1}{3}}}{9(x^2+x+1)^{4/3}}$$

$$= \frac{6(x^2+x+1)^{\frac{2}{3}} - \frac{2(2x+1)^2}{(x^2+x+1)^{1/3}}}{9(x^2+x+1)^{4/3}}$$

$$= \frac{6(x^2+x+1) - 2(2x+1)^2}{9(x^2+x+1)^{4/3} \cdot (x^2+x+1)^{1/3}}$$

$$= \frac{6x^2+6x+6-2(4x^2+4x+1)}{9(x^2+x+1)^{5/3}}$$

$$= \frac{6x^2+6x+6-8x^2-8x-2}{9(x^2+x+1)^{5/3}}$$

$$= \frac{-2x^2-2x+4}{9(x^2+x+1)^{5/3}}$$

$$f''(x) = \frac{-2(x^2+x-2)}{9(x^2+x+1)^{5/3}}$$

$$= \frac{-2(x^2+2x-x-2)}{9(x+x+1)^{5/3}}$$

$$\therefore f''(x) = \frac{-2(x+2)(x-1)}{9(x^2+x+1)^{5/3}}$$

For increasing,

$f'(x) > 0$

$\frac{2x+1}{3(x^2+x+1)^{2/3}} > 0$

$2x+1 > 0$

$x > -\frac{1}{2}$

$\therefore (-\frac{1}{2}, \alpha)$

for decreasing

$f'(x) < 0$

$\frac{2x+1}{3(x^2+x+1)^{2/3}} < 0$

$2x+1 < 0$

$x < -\frac{1}{2}$

$\therefore (-\alpha, -\frac{1}{2})$

for concave up and down

$f''(x)$

$(-\infty,-2)$	$(-2,1)$	$(1,\infty)$
$-ve$	$+ve$	$-ve$

$-2 \qquad 1$

$x=-3$	$x=0$	$x=2$
$f''(-3) = -ve$	$f''(0) = +ve$	$f''(2) = -ve$

concave up $= (-2,1)$

concave down $= (-\infty,-2), (1,\infty)$

for infection point:

$$-2(x+2)(x-1) = 0$$

$-2 = 0$	$x+2=0$	$x-1=0$
	$x=-2$	$x=1$

$\therefore$ infection point $x=1$ and -2

c) $f(x) = x^{4/3} - x^{1/3}$

$f'(x) = \frac{4}{3} x^{4/3-1} - \frac{1}{3} x^{1/3-1}$

$= \frac{4}{3} x^{1/3} - \frac{1}{3} x^{-2/3}$

$= \frac{4}{3} \cdot x^{1/3} - \frac{1}{3} \cdot \frac{1}{x^{2/3}}$

$= \frac{\frac{4}{3} \cdot x - \frac{1}{3}}{x^{2/3}}$

$= \frac{\frac{1}{3}(4x-1)}{x^{2/3}}$

$= \frac{4x-1}{3x^{2/3}}$

increasing:

$f'(x) > 0$

$\frac{4x-1}{3x^{2/3}} > 0$

$4x-1 > 0$

$x > \frac{1}{4}$

$\therefore (\frac{1}{4}, \infty)$

decreasing:

$f'(x) < 0$

$\frac{4x-1}{3x^{2/3}} < 0$

$4x-1 < 0$

$x < \frac{1}{4}$

$\therefore (-\infty, \frac{1}{4})$

For concavity:

$$f'(x) = \frac{4x+1}{3x^{2/3}}$$

$$f''(x) = \ldots\ldots$$

$$= \frac{2(2x+1)}{9 \cdot x^{5/3}}$$

concave up:

$f''(x) > 0$

$\frac{2(2x+1)}{9x^{5/3}} > 0$

$2(2x+1) > 0$

$x > -\frac{1}{2}$

$\therefore \left(-\frac{1}{2}, \infty\right)$

concave down:

$f''(x) < 0$

$x < -\frac{1}{2}$

$\therefore \left(-\infty, -\frac{1}{2}\right)$

$\therefore$ infection point

$$f''(x) = 0$$

$$\frac{2(2x+1)}{9x^{5/3}} = 0$$

$$x = -\frac{1}{2}$$

A

d)

$$f(x) = \frac{x}{x^2+2}$$

$$f'(x) = \frac{(x^2+2)\cdot 1 - x\cdot 2x}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2}$$

$$f'(x) = \frac{-x^2+2}{(x^2+2)^2}$$

$$f''(x) = \frac{2x\,(x^2-6)}{(x^2+2)^3}$$

For increasing:

$$f'(x) > 0$$

$$\frac{-x^2+2}{(x^2+2)^2} > 0$$

$$2-x^2 > 0$$

$$(\sqrt{2}-x)\,(\sqrt{2}+x) > 0$$

+ve | + ve | - ve

$-\sqrt{2}$ | $\sqrt{2}$

$(-\infty, -\sqrt{2})$ $(-\sqrt{2}, \sqrt{2})$ $(\sqrt{2}, \infty)$

$x = -2$ | $x = 0$ | $x = 2$
$f'(-2) = +ve$ | $f'(0) = +ve$ | $f'(2) = -ve$

$\therefore$ increasing: $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

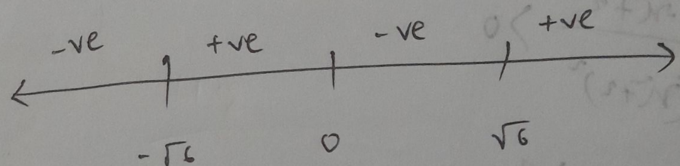
$\therefore$ decreasing: $(\sqrt{2}, \infty)$

For concavity:

$$f''(x) = 0$$

$$\frac{2x(x^2-6)}{(x^2+2)^3} = 0$$

$$2x(x-\sqrt{6})(x+\sqrt{6}) = 0$$



$x = -3$ | $x = -1$ | $x = 2$ | $x = 3$
$f''(-3) = -ve$ | $f''(-1) = +ve$ | $f''(2) = -ve$ | $f''(3) = +ve$

concave up = $(-\sqrt{6}, 0)$, $(\sqrt{6}, \infty)$

concave down = $(-\infty, -\sqrt{6})$, $(0, \sqrt{6})$

e) $f(x) = e^{-\frac{x^2}{2}}$

$f'(x) = e^{-\frac{x^2}{2}} \cdot \frac{-2x}{2}$

$= -x e^{-\frac{x^2}{2}}$

$f''(x) = -\left(e^{-x^2/2} + x \cdot e^{-x^2/2} \cdot \left(\frac{-2x}{2}\right)\right)$

$= -\left(e^{-x^2/2} - x^2 e^{-x^2/2}\right)$

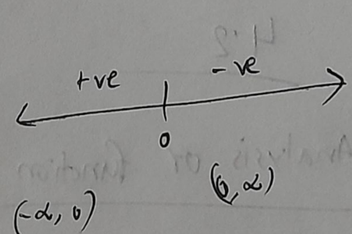
$= e^{-\frac{x^2}{2}} (x^2-1)$

if, $e^{-x^2/2} \neq 0$

$f'(x) = 0$

$\boxed{-x} e^{-x^2/2} = 0$

$\therefore x = 0$



$x = -1$ | $x = 1$
$f'(-1) = +ve$ | $f'(1) = -ve$

$\therefore$ increasing : $(-\alpha, 0)$

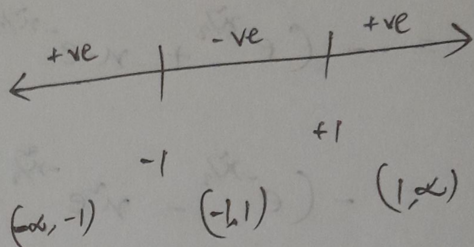
$\therefore$ decreasing : $(0, \alpha)$

For concavity:

if $e^{-\frac{x^2}{2}} \neq 0$

$\therefore f''(x) = 0$

$e^{-x^2/2}(x^2-1) = 0$



$x^2 - 1 = 0$

$x = \pm 1$

$x = -2$	$x = 0$	$f = 2$
$f''(-2) = +ve$	$f''(0) = -ve$	$f''(2) = +ve$

concave up: $(-\infty, -1)$, $(1, \infty)$

concave down: $(-1, 1)$

$\therefore$ infection point : $x = -1, 1$

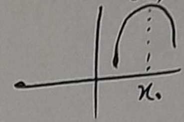
4·2

Analysis of function - II

Relative extrema

↳ Relative maximum and minimum.

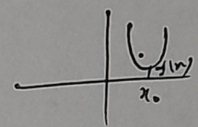
maximum:



if $f(x)$ has a maximum at $x = x_0$, then

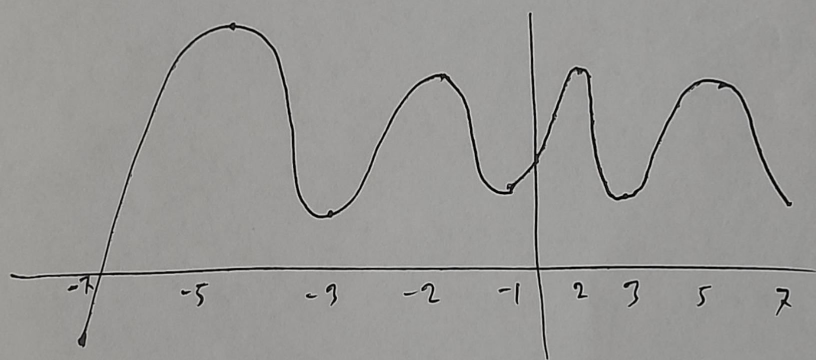
$$f(x_0) > f(x)$$

minimum:



if $f(x)$ has a minimum at $x = x_0$,

then, $f(x_0) < f(x)$

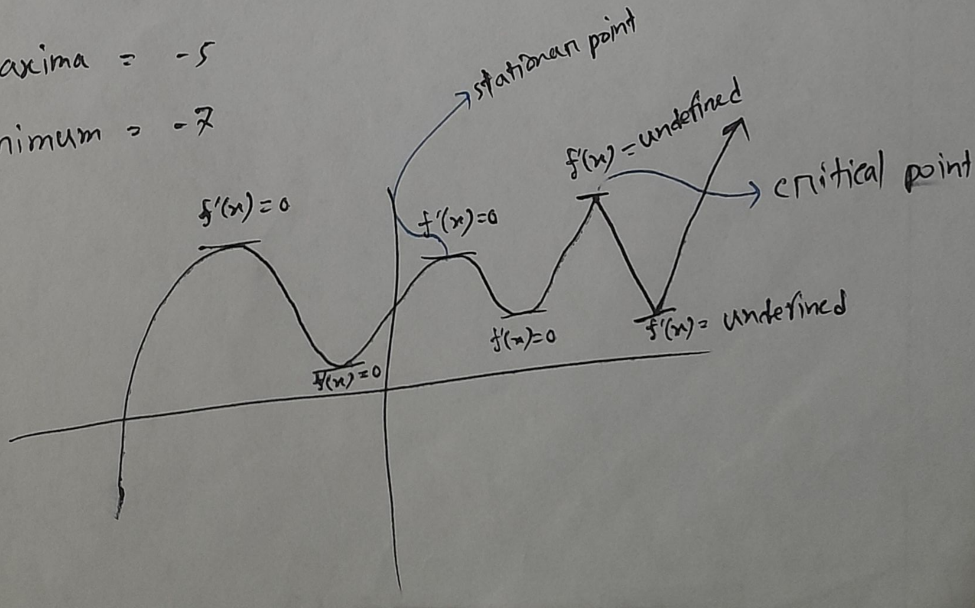


Relative maxima = 4 = −5, −2, 2, 5

Relative minimum = 5 = −7, −3, −1, 3, 7

Absolute maxima = −5

Absolute minimum = −7



MAT-120

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Critical Points:

Relative extremum points are called critical points.

Condition of critical points:

$f'(x) = 0$

and

$f'(x) = \text{undefined}$

Stationary Points:

Particular type of critical points where $f'(x) = 0$.

(*) Locate the critical points and identify which critical points are stationary points.

i) $x^3 - 3x + 1$

$\Rightarrow$ Here,

$f'(x) = 3x^2 - 3$

For critical points,

$f'(x) = 0$

$3x^2 - 3 = 0$

$3x^2 = 3$

$x = \pm 1$

Points are,

$x = 1,\ f(1) = -1 \Rightarrow (1, -1)$

$x = -1,\ f(-1) = 3 \Rightarrow (-1, 3)$

As this is polynomial function,

i.e. everywhere function is differentiable.

$\therefore$ All critical points are stationary points.

ii) $f(x) = \dfrac{x+1}{x^2+3}$

$$f'(x) = \frac{(x^2+3)\cdot\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{x^2+3-2x^2-2x}{(x^2+3)^2}$$

$$= \frac{-x^2-2x+3}{(x^2+3)^2}$$

$\therefore$ For critical point,

$f'(x)=0$

$$\frac{-x^2-2x+3}{(x^2+3)^2} = 0$$

$x^2+2x-3=0$

$x^2+3x-x-3=0$

$(x+3)(x-1)=0$

$x=1,-3$

and,

here, $f'(x)$ nowhere undefined.

$\therefore$ stationary points and critical points are same.

iii)

$$f(x) = \frac{x+3}{x-2}$$

$$f'(x) = \frac{(x-2)\frac{d}{dx}(x+3) - (x+3)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$= \frac{(x-2)\cdot 1 - (x+3)\cdot 1}{(x-2)^2}$$

$$= \frac{x-2-x-3}{(x-2)^2}$$

$$= \frac{-5}{(x-2)^2}$$

For critical points,

$$f'(x) = 0$$

$$\frac{-5}{(x-2)^2} = 0$$

$$-5 = 0 \text{ (Not possible)}$$

$f'(x)$ is undefined at $x=2$

Therefore, $x=2$ is a critical points and there is no stationary points.

iv) $f(x) = 3x^{5/3} - 15x^{2/3}$

$$f'(x) = \frac{5}{3} \cdot 3 \cdot x^{\frac{5}{3}-1} - 15 \cdot \frac{2}{3} \cdot x^{\frac{2}{3}-1}$$

$$= 5 \cdot x^{2/3} - 10 \cdot x^{-1/3}$$

$$= \frac{5 \cdot x^{2/3} \cdot x^{1/3} - 10}{x^{1/3}}$$

$$= \frac{5x-10}{x^{1/3}}$$

For critical points:

$f'(x) = 0$

$\frac{5x-10}{x^{1/3}} = 0$

$5x - 10 = 0$

$5x = 10$

$x = 2$

and $f'(x)$ is undefined at $x=0$.

$\therefore$ critical points $x = 2, 0$ and stationary points $x=2$.