L-17/ 19.09.2023/

Midterm Exam

L-18/24.09.2023/

Seriese Solution ODE

Recap

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$$\Rightarrow y'' + P(x)y' + Q(x)y = 0$$

(i) $P(x)$ and $Q(x)$ are defined at $x=0$, then ordinary point problem.

(ii) $P(x)$ or $Q(x)$ is undefined at $x=0$, then singular point problem.

(*) Solve,

$$3xy'' + y' - y = 0 \quad \cdots \text{(i)}$$

$$P(x) = \frac{1}{3x}, \quad Q(x) = -\frac{1}{3x}$$

So, (i) is a singular point problem.

unknown constant (two value & two seriese)

Let,

$$y = \sum_{n=0}^{\infty} c_n x^n \cdot x^r = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$\cancel{y'' = \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1}}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2}$$

Substituting y'', y', y, in (i)

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1)\, c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 3(n+r)(n+r-1)\, c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} (3n+3r-3+1) - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} (3n+3r-2) - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow x^r \left[\sum_{n=0}^{\infty} (n+r)(3n+3r-2)\, c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right] = 0$$

$$\Rightarrow r(3r-2)x^{-1} + \sum_{n=1}^{\infty} (n+r)(3n+3r-2)\, c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Let,

$k = n-1$	$k = n$
$n = k+1$	$n = k$
if, $n=1$, $k=0$	if $n=0$, $k=0$

$$\Rightarrow r(3r-2)x^{-1} + \sum_{k=0}^{\infty} (k+r+1)(3k+3+3r-2)\, c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\Rightarrow r(3r-2)x^{-1} + \sum_{k=0}^{\infty} \left[(k+r+1)(3k+3r+1)\, c_{k+1} - c_k \right] x^k = 0 \quad \cdots \text{(ii)}$$

Now equating the coefficient of lite terms,

x^{-1}: $r(3r-2) = 0$

$\Rightarrow r = 0, \frac{2}{3}$

x^k: $(k+r+1)(3k+3r+1)C_{k+1} - C_k = 0$

$$\therefore C_{k+1} = \frac{1}{(k+r+1)(3k+3r+1)} C_k \; ; \; k = 0,1,2,3,\dots$$

$\underline{r = 0,}$

$$C_{k+1} = \frac{1}{(k+1)(3k+1)} \cdot C_k$$

$\underline{k=0,}$

$$C_1 = \frac{C_0}{1\cdot 1} = C_0$$

$\underline{k=1,}$

$$C_2 = \frac{C_1}{2\cdot 4} = \frac{C_0}{8}$$

$\underline{k=2,}$

$$C_3 = \frac{C_2}{3\cdot 7} = \frac{C_0}{168}$$

$\vdots$

$\underline{r = \frac{2}{3},}$

$$C_{k+1} = \frac{1}{(k+\frac{5}{3})(3k+3)} C_k$$

$$= \frac{C_k}{(k+1)(3k+5)}$$

$\underline{k=0,}$

$$C_1 = \frac{C_0}{1\cdot 5} = \frac{C_0}{5}$$

$\underline{K=1,}$

$$C_2 = \frac{C_1}{2\cdot 8} = \frac{C_0}{80}$$

$\underline{k=2,}$

$$C_3 = \frac{C_2}{3\cdot 11} = \frac{C_0}{2640}$$

$\vdots$

$\underline{r=0,}$

$$Y_1 = \sum_{n=0}^{\infty} C_n x^n = C_0 x^0 + C_1 x^1 + C_2 x^2 + C_3 x^3 + \dots$$

$$= C_0 + C_0 x + \frac{C_0}{8} x^2 + \frac{C_0}{168} x^3 + \dots$$

$r = \frac{2}{3}$,

$$Y_2 = x^{2/3}\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right]$$

$$= x^{2/3}\left[c_0 + \frac{c_0}{5}x + \frac{c_0}{80}x^2 + \frac{c_0}{2640}x^3 + \cdots\right]$$

$\therefore$ G.S. $\Rightarrow$

$$Y = C_1 Y_1 + C_2 Y_2$$

$$= C_1\left[c_0 + c_0 x + \frac{c_0}{8}x^2 + \frac{c_0}{168}x^3 + \cdots\right] + C_2 \cdot x^{2/3}\left[c_0 + \frac{c_0}{5}x + \frac{c_0}{80}x^2 + \frac{c_0}{2640}x^3 + \cdots\right]$$

$$= C_3\left(1 + x + \frac{1}{8}x^2 + \frac{1}{168}x^3 + \cdots\right) + C_4 x^{2/3}\left(1 + \frac{1}{5}x + \frac{1}{80}x^2 + \frac{1}{2640}x^3 + \cdots\right)$$

Ans

Next Sunday
Quiz-2
System & Seriese

H.W. $\Rightarrow$

from Zill's Book

Exercise- 6.3 $\Rightarrow$ 17, 18, 21

&

$$2x\, y'' + (1+x)\, y' + y = 0$$

L-19/26.09.2023/

Laplace Transform

LT (Deffination): Let f be a function defined for $t \geq 0$. Then the integral $\int_0^\infty e^{-st} f(t)\, dt$ is said to be the Laplace Transform of $f(t)$. This LT is denoted by

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s) \; ; \quad s>0$$

(*) $L\{1 = t^0\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

$$= \lim_{b\to\infty} \int_0^b e^{-st}\, dt$$

$$= \lim_{b\to\infty} \left[\frac{e^{-st}}{-st} \right]_0^b$$

$$= \lim_{b\to\infty} \left(-\frac{1}{s}\right) \left[e^{-st} - e^0 \right]$$

$$= -\frac{1}{s}\left(e^{-\infty} - 1\right) = \frac{1}{s}$$

(*) $L\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$

$$= \left[t \cdot \frac{e^{-st}}{-s} \right]_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\, dt$$

$$= \frac{1}{s} \int_0^\infty e^{-st}\, dt = \frac{1}{s} \cdot L\{1\} = \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}$$

Altarnative Tabular method

$$= \left[-\frac{e^{-st}}{-s} t - \frac{e^{-st}}{s^2} \right]_0^\infty = \frac{1}{s^2}$$

$$\circledast \; L\{t^2\} = \int_0^\infty e^{-st} t^2 \, dt$$

$$= \left[\frac{e^{-st}}{-s} t^2 - \frac{e^{-st}}{s^2} \cdot 2t - \frac{e^{-st}}{s^3} \cdot 2 \right]_0^\infty$$

$$= \frac{2}{s^3}$$

t^2	$\xrightarrow{+}$	e^{-st}
$2t$	$\xrightarrow{-}$	$\frac{e^{-st}}{-s}$
2	$\xrightarrow{+}$	$\frac{e^{-st}}{s^2}$
0		$\frac{e^{-st}}{-s^3}$

Similarly: $L\{t^3\} = \frac{\lfloor 3}{s^{3+1}}$

$$L\{t^{100}\} = \frac{\lfloor 100}{s^{100+1}}$$

positive integer

$$\boxed{L\{t^n\} = \frac{\lfloor n}{s^{n+1}}}$$

$$\circledast \; L\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at} \, dt$$

$$= \int_0^\infty e^{-st+at} \, dt$$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= -\frac{1}{s-a} (e^{-\infty} - e^0)$$

$$\boxed{= \frac{1}{s-a}}$$

Similarly:

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

$$L\{e^{3t}\} = \frac{1}{s-3}$$

$$L\{e^{-2t}) = \frac{1}{s+2}$$

$$L\{e^{\sqrt{7}t}\} = \frac{1}{s-\sqrt{7}}$$

$$L\{\sin kt\} = \int_0^\infty e^{-st} \sin kt \, dt$$

$$= \left[\cancel{\sin kt \frac{e^{-st}}{-s}}\right]_0^\infty \xrightarrow{} 0 \; - \int_0^\infty k \cos kt \frac{e^{-st}}{-s} \, dt$$

$$= \frac{k}{s} \left\{ \left[\cos kt \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty -k \sin kt \frac{e^{-st}}{-s} \, dt \right\}$$

$$= \frac{k}{s} \left\{ \left(0 + \frac{1}{s}\right) - \frac{k}{s} \int_0^\infty e^{-st} \sin kt \, dt \right\}$$

$$L\{\sin kt\} = \frac{k}{s^2} - \frac{k^2}{s^2} L\{\sin kt\}$$

$$\Rightarrow L\{\sin kt\}\left(1 + \frac{k^2}{s^2}\right) = \frac{k}{s^2}$$

$$\therefore \boxed{L\{\sin kt\} = \frac{k}{s^2} \cdot \frac{s^2}{s^2+k^2}} = \frac{k}{s^2+k^2}$$

Similarly: $L\{\sin 3t\} = \frac{3}{s^2+9}$

$$L\{\sin \sqrt{7}t\} = \frac{\sqrt{7}}{s^2+7}$$

$$L\{\cos kt\} = ? = \int_0^\infty e^{-st} \cos kt \, dt$$

$$= \cos kt \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty -k \sin kt \frac{e^{-st}}{-s} \, dt$$

$$= \left(0 + \frac{1}{s}\right) - \frac{k}{s} \int_0^\infty e^{-st} \sin kt \, dt$$

$$= \frac{1}{s} - \frac{k}{s} \left\{ \sin kt \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty k \cos kt \frac{e^{-st}}{-s} \, dt \right\}$$

$$= \frac{1}{s} - \frac{k}{s} \left\{ \frac{k}{s} \int_0^\infty e^{-st} \cos kt \, dt \right\}$$

$$L\{\cos kx\} = \frac{1}{s} - \frac{k^2}{s^2}\left(L\{\cos kx\}\right)$$

$$\left(1+\frac{k^2}{s^2}\right) L\{\cos kx\} = \frac{1}{s}$$

$$\therefore L\{\cos kx\} = \frac{1}{s}\cdot\frac{s^2}{s^2+k^2} = \frac{s}{s^2+k^2}$$

Similarly:

$$L\{\cos 7x\} = \frac{s}{s^2+49}$$

$$\boxed{\begin{aligned}\therefore L\{\sin kx\} &= \frac{k}{s^2+k^2}\\ \therefore L\{\cos kx\} &= \frac{s}{s^2+k^2}\end{aligned}}$$

$$e^x = \frac{e^x+e^{-x}}{2} + \frac{e^x-e^{-x}}{2}$$

$$= \cosh x + \sinh x$$

$$\boxed{\begin{aligned}\therefore L\{\sinh kx\} &= \frac{k}{s^2-k^2}\\ \therefore L\{\cosh kx\} &= \frac{s}{s^2-k^2}\end{aligned}}$$

$$L\{\sinh kx\} = L\left\{\frac{e^{kx}-e^{-kx}}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right)$$

$$= \frac{1}{2}\left(\frac{s+k-s+k}{s^2-k^2}\right)$$

$$= \frac{1}{2}\cdot\frac{2k}{s^2-k^2}$$

$$= \frac{k}{s^2-k^2}$$

$$L\{\cosh kx\} = L\left\{\frac{e^{kx}+e^{-kx}}{2}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right)$$

$$= \frac{1}{2}\left(\frac{s+k+s-k}{s^2-k^2}\right)$$

$$= \frac{1}{2}\cdot\frac{2s}{s^2-k^2}$$

$$= \frac{s}{s^2-k^2}$$

(Proved)

L-20 / 01.10.2023 /

Recap

Laplace Transformation

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s) \;;\quad s>0$$

LT

Inverse LT (L^{-1})

1. $L\{t^n\} = \frac{\lfloor n}{s^{n+1}} \longrightarrow L^{-1}\left\{\frac{\lfloor n}{s^{n+1}}\right\} = t^n$

2. $L\{e^{at}\} = \frac{1}{s-a} \longrightarrow L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

3. $L\{\sin kt\} = \frac{k}{s^2+k^2} \longrightarrow L^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

4. $L\{\cos kt\} = \frac{s}{s^2+k^2} \longrightarrow L^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

5. $L\{\sinh kt\} = \frac{k}{s^2-k^2} \longrightarrow L^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh kt$

6. $L\{\cosh kt\} = \frac{s}{s^2-k^2} \longrightarrow L^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh kt$

⊛ $L^{-1}\left\{\frac{1}{s^{11}}\right\} = \frac{1}{\lfloor 10} L^{-1}\left\{\frac{\lfloor 10}{s^{10+1}}\right\}$

$= \frac{1}{\lfloor 10} t^{10}$

⊛ $L^{-1}\left\{\frac{1}{s+7}\right\} = e^{-7t}$

⊛ $L^{-1}\left\{\frac{1}{s^2+16}\right\} = \frac{1}{4} L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$

$= \frac{1}{4} \sin 4t$

⊛ $L^{-1}\left\{\frac{5s}{s^2-7}\right\}$

$= 5 L^{-1}\left\{\frac{s}{s^2-(\sqrt{7})^2}\right\}$

$= 5 \cosh \sqrt{7}\, t$

⊛ Find $L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$

$$= L^{-1}\left\{\frac{-2s}{s^2+4}\right\} + L^{-1}\left\{\frac{6}{s^2+4}\right\} = (-2)\, L^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 3L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= -2\cos 2t + 3\sin 2t$$

⊛ Evaluate, $L^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$

$$= L^{-1}\left\{\frac{1+6+9}{(s-1)(1-2)(1+4)} + \frac{4+12+9}{(s-2)(2-1)(2+4)} + \frac{16-24+9}{(s+4)(-4-1)(-4-2)}\right\}$$

$$= L^{-1}\left\{\frac{16}{-5(s-1)} + \frac{25}{6(s-2)} + \frac{1}{30(s+4)}\right\}$$

$$= -\frac{16}{5} L^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6} L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30} L^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{16}{5} e^{t} + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

Ans

⊛ $y = f(t)$

$y' = f'(t)$

$y'' = f''(t)$

$$L\{y = f(t)\} = \boxed{\int_0^\infty e^{-st}\{y(t) = f(t)\}\, dt = F(s) =} Y(s)$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt$$

$$= \left[e^{-st} \cdot f(t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st} f(t)\, dt$$

$$= \{0 - f(0)\} + s\int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore L\{f'(t)\} = sF(s) - f(0)$$

$$\therefore L\{y'(t)\} = sY(s) - y(0)$$

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \left[e^{-st} \cdot f'(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f'(t)\,dt$$

$$= \{0 - f'(0)\} + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$\therefore L\{f''(t)\} = s^2F(s) - s \cdot f(0) - f'(0)$$

$$\therefore L\{y''(t)\} = s^2Y(s) - s \cdot y(0) - y'(0)$$

$$L\{f'''(t)\} = \int_0^\infty e^{-st} f'''(t)\,dt$$

$$= \left[e^{-st} \cdot f''(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} \cdot f''(t)\,dt$$

$$= [0 - f''(0)] + s\int_0^\infty e^{-st} \cdot f''(t)\,dt$$

$$\therefore L\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\therefore L\{y'''(t)\} = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

$$\therefore L\{f^{iv}(t)\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

$$\therefore L\{y^{iv}(t)\} = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$$

※ $y'' - 3y' + 2y = 0$; $y(0) = 1,\ y'(0) = 5$

⇒

Applying LT,

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = 0$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = 0$$

Same as A.E.

$$\Rightarrow (s^2 - 3s + 2)Y(s) - s\cdot 1 - 5 + 3\cdot 1 = 0$$

$$\Rightarrow (s^2 - 3s + 2)\,Y(s) = s + 2$$

$$\Rightarrow Y(s) = \frac{s+2}{(s-1)(s-2)}$$

Applying inverse LT,

$$\Rightarrow L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s+2}{(s-1)(s-2)}\right\}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{1+2}{(s-1)(1-2)} + \frac{2+2}{(s-2)(2-1)}\right\}$$

$$= L^{-1}\left\{\frac{3}{-(s-1)} + \frac{4}{(s-2)}\right\}$$

$$= -3L^{-1}\left\{\frac{1}{s-1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\therefore y(t) = \underbrace{-3}_{c_1}e^{t} + \underbrace{4}_{c_2}e^{2t}$$

Ans

H.W

Zill's Book

Exercise - 7.2 (288)

⇒ 7, 17, 21, 22, 24–26

L-21/03.10.2023

Recap

LT

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

$$L\{f'(t)\} = L\{y'(t)\} = sY(s) - y(0)$$

$$L\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

$$L\{y'''(t)\} = s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0)$$

(*) $y'' - 3y' + 2y = e^{-4t}$; $y(0) = 1$, $y'(0) = 5$

Applying Laplace Transform,

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = L\{e^{-4t}\}$$

$$\Rightarrow s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2) Y(s) - s\cdot 1 - 5 + 3\cdot 1 = \frac{1}{s+4}$$

$$\Rightarrow (s^2 - 3s + 2) Y(s) = (s+2) + \frac{1}{s+4}$$

$$\Rightarrow Y(s) = \frac{s+2}{(s-1)(s-2)} + \frac{1}{(s+4)(s-1)(s-2)} = \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}$$

Applying inverse LT,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)}\right\}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{-16/5}{s-1}\right\} + L^{-1}\left\{\frac{25/6}{s-2}\right\} + L^{-1}\left\{\frac{1/30}{s+4}\right\}$$

$$\therefore y(t) = \underbrace{-\frac{16}{5}e^{t}}_{c_1} + \underbrace{\frac{25}{6}e^{2t}}_{c_2} + \underbrace{\frac{1}{30}e^{-4t}}_{y_p}$$

⊛ Perfect Square

ax^2+bx+c

$= a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$

$= a\left[(x+\frac{1}{2}\times \text{co-efficient of } x)^2 + \frac{c}{a} - (\frac{1}{2}\times\text{coefficient of } x)^2\right]$

$= a\left[(x+\frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$

⊛ **1st translation theorem:** If $L\{f(t)\} = F(s)$ and a is any real number then,

$$L\{e^{at}f(t)\} = F(s-a)$$

⊛ $L\{e^{2t}t\} = \frac{1}{(s-2)^2}$

$L\{t\} = \frac{\lfloor 1}{s^{1+1}} = \frac{1}{s^2}$

⊛ $L\{e^{-5t}\cos 3t\} = \frac{s+5}{(s+5)^2+9}$

$L\{\cos 3t\} = \frac{s}{s^2+3^2}$

⊛ $L\{e^{3t}\sinh 2t\} = \frac{2}{(s-3)^2-4}$

$L\{\sinh 2t\} = \frac{2}{s^2-2^2}$

⊛ $L^{-1}\left\{\frac{1}{(s-3)^3}\right\} = e^{3t}\cdot\frac{t^2}{2}$

$L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{\lfloor 2}L^{-1}\left\{\frac{\lfloor 2}{s^{2+1}}\right\}$

$= \frac{1}{2}\cdot t^2 = \frac{t^2}{2}$

⊛ Find $L^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\}$

need to be carefull about variable

$$\frac{s}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+4}$$

$$\Rightarrow \quad s = A(s^2+4) + (Bs+C)(s+2) \quad \cdots (i)$$

$$= As^2 + 4A + Bs^2 + 2Bs + Cs + 2C$$

$$s = (A+B)s^2 + (2B+C)s + (4A+2C) \quad \cdots (ii)$$

Putting $s=-2$ in (i),

$$-2 = A(4+4)+0$$

$$\therefore A = -\frac{2}{8} = -\frac{1}{4}$$

Equating the coefficient of s^2

$$0 = A+B$$

$$\therefore B = -A = \frac{1}{4}$$

Equating the coefficient of s,

$$1 = 2B+C$$

$$\therefore C = 1-2B = 1-2\cdot\frac{1}{4} = \frac{1}{2}$$

$$\therefore L^{-1}\left\{\frac{s}{(s+2)(s^2+4)}\right\} = L^{-1}\left\{\frac{-1/4}{s+2}\right\} + L^{-1}\left\{\frac{\frac{1}{4}s+\frac{1}{2}}{s^2+4}\right\}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{4}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= -\frac{1}{4}e^{-2t} + \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t$$

Ans

(*) Solve the IVP by using LT

$$y'' - 6y' + 9y = t^2 e^{3t} \; ; \; y(0)=2, \; y'(0)=17$$

Applying LT,

$$s^2 Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s^2 - 6s + 9) Y(s) - s.2 - 17 + 6.2 = \frac{2}{(s-3)^3}$$

$$\Rightarrow (s-3)^2 Y(s) = (2s+5) + \frac{2}{(s-3)^3}$$

$$\Rightarrow Y(s) = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

Applying inverse LT

$$\Rightarrow L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} + L^{-1}\left\{\frac{2}{(s-3)^5}\right\}$$

Here,

$$L^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$$

$$= L^{-1}\left\{\frac{2(s-3) + 5+6}{(s-3)^2}\right\}$$

$$= L^{-1}\left\{\frac{2(s-3)}{(s-3)^2}\right\} + L^{-1}\left\{\frac{11}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11e^{3t}.t$$

$$L^{-1}\left\{\frac{2}{(s-3)^5}\right\}$$

$$= \frac{1}{12} t^4 e^{3t}$$

$$L^{-1}\left\{\frac{2}{s^5}\right\}$$

$$= \frac{1}{12} L^{-1}\left\{\frac{\lfloor 4}{s^{4+1}}\right\}$$

$$= \frac{1}{12} t^4$$

$$\therefore y(t) = \underbrace{2e^{3t}}_{y_{c_1}} + \underbrace{11te^{3t}}_{y_{c_2}} + \underbrace{\frac{1}{12}e^{3t}.t^4}_{y_p}$$ Ans

H.W.

Zillis Book

7.2 ⇒ 36-40 (IVP)

7.3 ⇒ 21, 22, 24-26, 28, 29

(*) $y'' + 4y' + 6y = 1 + e^{-t}$

$y(0) = y'(0) = 0$

Quiz-3

15.10.2023

Laplace Transform

L-22/08.10.2023/

Laplace Transform

Find,

$$L^{-1}\left\{\frac{\frac{s}{2}+\frac{5}{3}}{s^2+4s+6}\right\}$$

$$= L^{-1}\left\{\frac{\frac{1}{2}s}{s^2+4s+6}\right\} + L^{-1}\left\{\frac{5/3}{s^2+4s+6}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+(\sqrt{2})^2}\right\} + \frac{5}{3\sqrt{2}}L^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+(\sqrt{2})^2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{(s+2)}{(s+2)^2+(\sqrt{2})^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{2}{(s+2)^2+(\sqrt{2})^2}\right\} + \frac{5}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}t$$

$$= \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t + \frac{5}{3\sqrt{2}}e^{-2t}\sin\sqrt{2}t$$

Ans.

$$s^2+4s+6$$
$$=(s+\tfrac{1}{2}\cdot 4)^2+6-4$$
$$=(s+2)^2+(\sqrt{2})^2$$

$$y''-y'=e^t\cos t \quad ; \quad y(0)=y'(0)=0$$

Applying LT,

$$s^2Y(s)-sy(0)-y'(0)-[sY(s)-y(0)] = \frac{s-1}{(s-1)^2+1}$$

$$\Rightarrow (s^2-s)Y(s) - 0 = \frac{s-1}{(s-1)^2+1}$$

$$\therefore Y(s) = \frac{s-1}{s((s-1)^2+1)(s-1)} = \frac{1}{s((s-1)^2+1)}$$

$$\therefore Y(s) = \frac{1}{s\{(s-1)^2+1\}}$$

$$\therefore Y(s) = \frac{1/2}{s} + \frac{-\frac{1}{2}s + 1}{(s-1)^2+1}$$

Applying LT^{-1},

$$y(t) = \frac{1}{2} - \frac{1}{2} L^{-1}\left\{\frac{s}{(s-1)^2+1}\right\} + L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= \frac{1}{2} - \frac{1}{2} L^{-1}\left\{\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}\right\} + e^t \sin t$$

$$= \frac{1}{2} - \frac{1}{2} \cos t \, e^t - \frac{1}{2} e^t \sin t + e^t \sin t$$

$$= \frac{1}{2} - \frac{1}{2} e^t \cos t + \frac{1}{2} e^t \sin t$$

Ans

$$\frac{1}{s\{(s-1)^2+1\}} = \frac{A}{s} + \frac{Bs+C}{(s-1)^2+1}$$

$$A(s^2-2s+2) + s(Bs+C) = 1$$

$$As^2 - 2As + 2A + Bs^2 + Cs = 1$$

$$(A+B)s^2 + (C-2A)s + 2A = 1$$

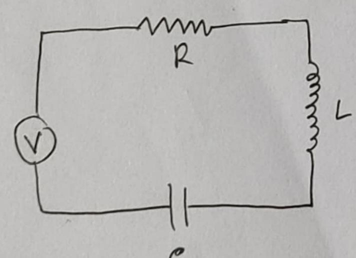
$$\therefore A = \frac{1}{2}$$

$$B = -A = -\frac{1}{2}$$

$$C = 2A = 1$$

A battery of 150 volt is connected to an LRC circuit in which the inductance is 1 henry and resistance is 20 Ω and capacitance is 0·005 Farad. Use laplace transform to find $q(t)$ and $i(t)$ if the initial charge and current are zero.

⇒



Applying KVL,

$$L\frac{di}{dt} + iR + \frac{q}{C} = 150$$

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 150$$

$$\Rightarrow \frac{d^2q}{dt^2} + 20\frac{dq}{dt} + 200q = 150$$

$$\Rightarrow q'' + 20q' + 200q = 150$$

Applying LT,

$$s^2Q(s) - sq(0) - q'(0) + 20[sQ(s) - q(0)] + 200Q(s) = 150\frac{1}{s}$$

$$\Rightarrow (s^2+20s+200)Q(s) - 0 = \frac{150}{s}$$

$$\therefore Q(s) = \frac{150}{s(s^2+20s+200)}$$

Here,

$$\frac{150}{s(s^2+20s+200)} = \frac{A}{s} + \frac{Bs+C}{s^2+20s+200}$$

$$\therefore A(s^2+20s+200) + s(Bs+C) = 150$$

$$\Rightarrow As^2 + 20As + 200A + Bs^2 + Cs = 150$$

$$\Rightarrow (A+B)s^2 + (20A+C)s + 200A = 150$$

$$\therefore A = \frac{150}{200} = \frac{3}{4}$$

$$\therefore B = -A = -\frac{3}{4}$$

$$\therefore C = -20A = -15$$

$$\therefore Q(s) = \frac{3/4}{s} + \frac{-3/4\, s - 15}{s^2+20s+200}$$

$$= \frac{3/4}{s} + \frac{-3/4\, s - 15}{(s+10)^2+(10)^2}$$

$$s^2 + 20s + 200$$

$$= \left(s+\tfrac{1}{2}\cdot 20\right)^2 + 200 - 100$$

$$= (s+10)^2 + 100$$

$$= (s+10)^2 + \cancel{(\sqrt{100})^2}\ (10)^2$$

Applying LT^{-1},

$$q(t) = \frac{3}{4} + L^{-1}\left\{ \frac{-3/4\, s - \frac{15}{2} - \frac{15}{2}}{(s+10)^2 + (10)^2} \right\}$$

$$= \frac{3}{4} - \frac{3}{4} L^{-1}\left\{ \frac{(s+10)}{(s+10)^2+(10)^2} \right\} - \frac{15}{2} L^{-1}\left\{ \frac{1}{(s+10)^2+(10)^2} \right\}$$

$$= \frac{3}{4} - \frac{3}{4} e^{-10t}\cos 10t - \frac{15}{20}\cdot e^{-10t}\sin 10t$$

$$\therefore q(t) = \frac{3}{4} - \frac{3}{4} e^{-10t}\cos 10t - \frac{3}{4} e^{-10t}\sin 10t$$

Ans

$$\therefore i(t) = q'(t) = 15\, e^{-10t}\sin 10t$$

Ans

L-23/10.10.2023/

Numerical Solution of IVP (ODE + IC)

IC → initial condition

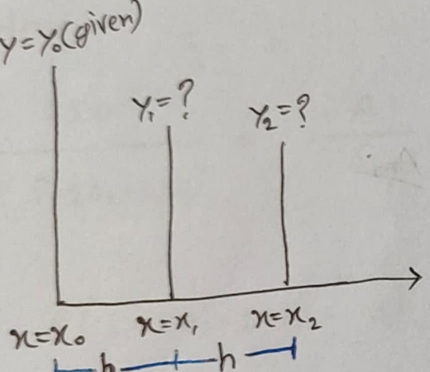
ODE : $\frac{dy}{dx} = f(x,y)$

I.C. : $y(x_0) = y_0$

$\Big\}$ IVP → initial value problem

⊛ Euler's method:

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)} \;;\; n = 0, 1, 2, 3, \cdots$$



$x_i = x_0 + ih$

$i = 1$

$x_1 = x_0 + h$

$h = x_1 - x_0$

$i = 2$

$x_2 = x_0 + 2h$

$2h = x_2 - x_0$

$= x_0 + 2h - x_0 = 2h$

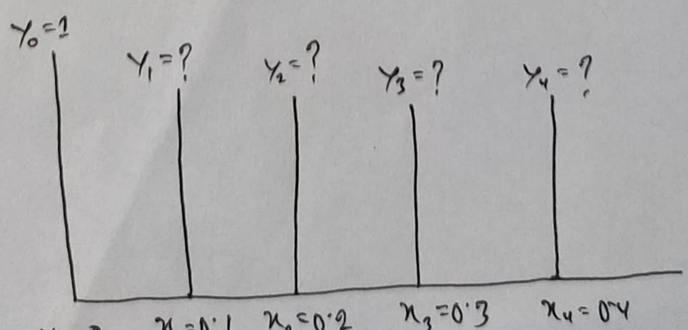
$\therefore h = h$

⊛ Solve,

$\frac{dy}{dx} = -y$

$y(0) = 1$

$\Big\}$ Find $y(0.4)$ if, $h = 0.1$



We know,

$$y_{n+1} = y_n + h f(x_n, y_n) \quad \cdots \text{(i)}$$

$n = 0, 1, 2, 3, \cdots$

Here $f(x_n, y_n) = -y_n$

For $n=0$,

(i) $\Rightarrow y_1 = y_0 + h f(x_0, y_0)$

$= 1 + 0.1 \times (-1)$

$= 0.9$

for $n = 1$,

(i) $\Rightarrow y_2 = y_1 + h f(x_1, y_1)$

$= 0.9 + 0.1(-0.9)$

$= 0.81$

for $n=2$,

(i) $\Rightarrow y_3 = y_2 + h f(x_2, y_2) = y_2 + h(-y_2)$

$= 0.81 + 0.1 \times (-0.81)$

$= 0.729$

for $n=3$,

$y_4 = y_3 + h f(x_3, y_3)$

$= y_3 + h(-y_3)$

$= 0.729 + 0.1(-0.729)$

$= 0.6561$

Exact Result:

$$\frac{dy}{dx} = -y$$

$\Rightarrow \int \frac{dy}{y} = \int -dx$

$\Rightarrow \ln y = -x + c_1$

$\Rightarrow y = e^{-x+c_1} = e^{c_1} e^{-x} = ce^{-x}$

$\boxed{\therefore y = ce^{-x}}$

at, $x = 0, y = 1$

$\therefore 1 = ce^0$

$\therefore c = 1$

$\therefore y = e^{-x}$

$y(0.1) = e^{-0.1} = 0.9048$

$y(0.2) = e^{-0.2} = 0.8187$

$y(0.3) = e^{-0.3} = 0.7408$

$y(0.4) = e^{-0.4} = 0.6703$

Runge-kutta (R-K) 2nd Order method:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h, y_n + k_1)$$

For $n=0$,

$k_1 = h \times (-y_0) = -0.1$

$k_2 = h \times \{-(y_0 + k_1)\} = 0.1 \times \{-(1 - 0.1)\} = -0.09$

$\therefore y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(-0.1 - 0.09)$

$= 0.905$

For $n=1$,

$k_1 = h \times (-y_1) = 0.1 \times (-0.905) = 0.0905$

$k_2 = h \times \{-(y_1 + k_1)\} = 0.1 \times \{-(0.905 - 0.0905)\}$

$= -0.08145$

$\therefore y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 0.905 + \frac{1}{2}(-0.0905 - 0.08145)$

$= 0.819025$

For $n=2$,

$k_1 = h \times (-y_2) = 0.1 \times (-0.819025) = -0.0819025$

$k_2 = h \times \{-(y_2 + k_1)\} = 0.1 \times \{-(0.819025 - 0.0819025)\}$

$= -0.07371225$

$\therefore y_3 = y_2 + \frac{1}{2}(k_1 + k_2) = 0.741217125$

For $n=3$,

$$k_1 = h \times (-y_3) = 0.1 \times (-0.741217625) = -0.0741217625$$

$$k_2 = h \times \{-(y_2 + k_1)\} = 0.1 \times (-(0.741217625 - 0.0741217625)$$

$$= -0.06670958625$$

$$\therefore y_4 = y_3 + \frac{1}{2}(k_1 + k_2)$$

$$= 0.741217625 + \frac{1}{2}(-0.0741217625 - 0.06670958625)$$

$$= 0.67080195 06$$

Ans

$\boxed{\text{H.W.}} \Rightarrow$

Zilli's Book - Page 367

Exercise - 9.1 : 1, 3, 4, 7, 9

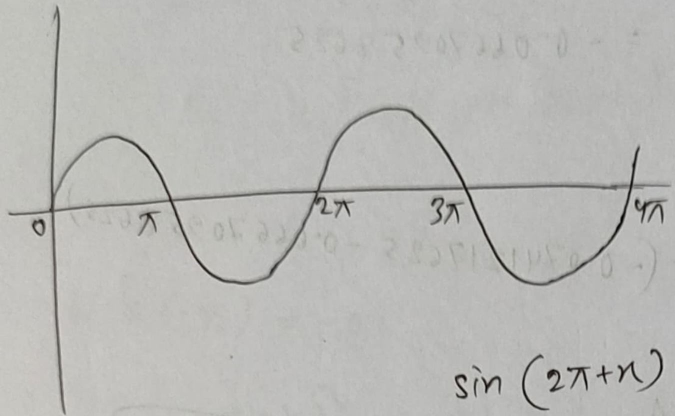
by using Euler + R.K

and show in table

n	y	Euler	R-K

L-24/ 15.10.2023/

Fourier Series



$$\sin(2\pi + x) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

Trigonometric Series:

The trigonometric defines by the following form as,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx);$$ where a_0, a_n and b_n are the unknown constant.

Fourier Series:

The trigonometric series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cdots (i)$$

is said to be a fourier series if its coefficient a_0, a_n and b_n are calculated by the following formulas,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Limit as given in Question

But difference must be 2π

$\cos 2n\pi = 1$

$\cos n\pi = (-1)^n$

$\sin 2n\pi = 0$

$\sin n\pi = 0$

✻ Find the fourier series of $f(x) = e^{-x}$ for $0 \leq x \leq 2\pi$

$\Rightarrow$

We know,

$$f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \cdots (1)$$

Now,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi} = -\frac{1}{\pi}\left(e^{-2\pi} - e^0\right)$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(e^{-x}\cos nx\right)dx$$

$$= \frac{1}{\pi}\left\{\left[\cos nx \frac{e^{-x}}{-1}\right]_0^{2\pi} - \int_0^{2\pi} -(\sin nx)\cdot n\cdot\frac{e^{-x}}{-1}\,dx\right\}$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right) - \frac{n}{\pi}\int_0^{2\pi} \sin nx\, e^{-x}\,dx$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right) - \frac{n}{\pi}\left\{\left[\sin nx \frac{e^{-x}}{-1}\right]_0^{2\pi} - \int_0^{2\pi}(\cos nx)\,n\cdot\frac{e^{-x}}{-1}\,dx\right\}$$

$$= \frac{1}{\pi}\left(1 - e^{-2\pi}\right) - \frac{n^2}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$a_n = \frac{1}{\pi}(1-e^{-2\pi}) - n^2 a_n$$

$$\Rightarrow a_n + n^2 a_n = \frac{1}{\pi}(1-e^{-2\pi})$$

$$\Rightarrow a_n = \frac{1}{\pi(1+n^2)}(1-e^{-2\pi})$$

Similarly,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\sin nx\,\frac{e^{-x}}{-1}\right]_0^{2\pi} - \int_0^{2\pi} n\cdot\cos nx\,\frac{e^{-x}}{-1}\,dx\right\}$$

$$= \frac{n}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{n}{\cancel{\pi}}\cdot\frac{1}{\pi(1+n^2)}(1-e^{-2\pi})$$

$$= \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})$$

(i) becomes,

$$e^{-x} = \frac{1}{2\pi}(1-e^{-2\pi}) + \sum_{n=1}^{\infty}\frac{1}{\pi}\,\frac{1-e^{-2\pi}}{1+n^2}\cos nx + \frac{1}{\pi}\cdot\frac{n}{(1+n^2)}(1-e^{-2\pi})\sin nx$$

$$= \frac{1}{\pi}(1-e^{-2\pi})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos nx + \frac{n}{1+n^2}\sin nx\right]$$

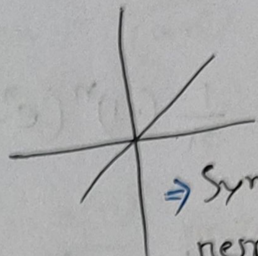
$$= \frac{1}{\pi}(1-e^{-2\pi})\left[\frac{1}{2} + \frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots + \frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \cdots\right]$$

(*) Odd function! $\longrightarrow$ $a_0 = a_n = 0$

$f(-x) = -f(x)$

$f(x) = \sin x$

$f(x) = x$



(*) Even function! $\longrightarrow$ $b_n = 0$

$f(-x) = f(x)$

$f(x) = \cos x$

$f(x) = x^2$

⇒ Symmetric with respect to origin

⇒ Then we can calculate ~~half~~ by applying half limit.

(*) Find the fourier series of $f(x) = e^x$; $-\pi \leq x \leq \pi$

⇒

We know,

$$f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \quad \cdots (i)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \, dx = \frac{1}{\pi} \left[e^x \right]_{-\pi}^{\pi} = \frac{1}{\pi} \left(e^{\pi} - e^{-\pi} \right) = \frac{\left(e^{\pi} - e^{-\pi} \right)}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(e^x \cos nx \right) dx$$

$$= \frac{1}{\pi} \left\{ \left[\cos nx \, e^x \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -(\sin nx) \cdot n \cdot e^x \, dx \right\}$$

$$\cancel{= \frac{1}{\pi} \quad (-1)^n \quad (-1)^n \quad (-1)^n e^{\pi} - (-1)^n e^{-\pi}}$$

$$= \frac{1}{\pi}(-1)^n(e^{\pi}-e^{-\pi}) + \frac{n}{\pi}\int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi}(-1)^n(e^{\pi}-e^{-\pi}) + \frac{n}{\pi}\left\{ \left[\sin nx \, e^x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx \cdot n \cdot e^x \, dx \right.$$

$$a_n = \frac{1}{\pi}(-1)^n(e^{\pi}-e^{-\pi}) - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$$a_n = \frac{1}{\pi}(-1)^n(e^{\pi}-e^{-\pi}) - n^2 a_n$$

$$\therefore a_n = \frac{1}{\pi(1+n^2)}(-1)^n(e^{\pi}-e^{-\pi})$$

$$= \frac{(-1)^n}{(1+n^2)} \cdot \frac{(e^{\pi}-e^{-\pi})}{\pi}$$

Similarly,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{ \left[\sin nx \, e^x\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx \cdot n \cdot e^x \, dx \right\}$$

$$= -\frac{n}{\pi}\int_{-\pi}^{\pi} e^x \cos nx \, dx$$

$$= -\frac{n(-1)^n}{(1+n^2)} \cdot \frac{(e^{\pi}-e^{-\pi})}{\pi}$$

① becames,

$\cancel{e^{x} \; e^{x}}$

$$e^{x} = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{(-1)^n}{(1+n^2)} \cdot \frac{e^{\pi} - e^{-\pi}}{\pi} \cos nx - \frac{n(-1)^n}{(1+n^2)} \cdot \frac{e^{\pi} - e^{-\pi}}{\pi} \sin nx$$

$$= \frac{e^{\pi} - e^{-\pi}}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} \cos nx - \frac{n(-1)^n}{1+n^2} \sin nx \right]$$

$$= \frac{e^{\pi} - e^{-\pi}}{\pi} \left[\frac{1}{2} - \frac{1}{2} \cos x + \frac{1}{5} \cos 2x - \frac{1}{10} \cos 3x + \cdots + \frac{1}{2} \sin x - \frac{2}{5} \sin 2x + \frac{3}{10} \sin 3x - \cdots \right]$$

Ans

L-25/17.10.2023

The Fourier Cosine and Sine Series

Even Function: If $f(-x) = f(x)$, then $f(x)$ is an even function.

for even function, $\boxed{b_n = 0}$ & function is symmetric with respect to y-axis.

Then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ → known as half range cosine series.

multiply by 2, because of half range

where,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

Odd Function: If $f(-x) = -f(x)$, then $f(x)$ is an odd function and symmetric with respect to origin.

for odd function,

$$\boxed{a_0 = a_n = 0}$$

$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx$ → half range sine series.

where,

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x)\, \cancel{\sin x}\, \sin nx\,dx$$

(*) Find the fourier series of $f(x) = x^2$; $-\pi \leq x \leq \pi$

$\Rightarrow$

Here, $f(x)$ is an even function and symmetric with respect to y-axis.

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \cancel{dx}$$

Here,

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2}{3} (\pi)^2$$

Diff.	Intr.
x^2	$\cos nx$
$2x$	$\frac{\sin nx}{n}$
2	$-\frac{\cos nx}{n^2}$
0	$-\frac{\sin nx}{n^3}$

+ − +

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cdot \cos nx \, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{n} \sin nx + \frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx \right]_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{2\pi}{n^2} (-1)^n$$

$$= \frac{4}{n^2} (-1)^n$$

$$\therefore f(x) = \frac{2}{3} (\pi)^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

$$= \frac{2}{3} \pi^2 + \left[-4\cos x + \cos 2x - \frac{4}{9} \cos 3x + \cdots \right)$$

Ans

⊛ Change of interval in the Fourier Series:

If $f(x)$ is defined in $(-c, c)$, then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) + b_n \sin\left(\frac{n\pi x}{c}\right)$$

where,

$$a_0 = \frac{1}{c} \int_{-c}^{c} f(x)\, dx$$

$$a_n = \frac{1}{c} \int_{-c}^{c} f(x) \cos\left(\frac{n\pi x}{c}\right) dx$$

$$b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx$$

⊛ Expands $f(x) = x$ as a half range sine series in $0 \leq x \leq 2$

$\Rightarrow$ $f(x) = x$ is an odd function.

$\therefore$ $a_0 = a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

Here,

$$b_n = \frac{2}{2} \int_{0}^{2} x \sin\left(\frac{2\pi x}{2}\right) dx$$

$$= \left[-\frac{2x}{n\pi} \cos\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2 \sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{4}{n\pi} (-1)^n$$

x		$\sin\frac{n\pi x}{2}$
	$+$	
1		$\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$
	$-$	
0		$\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$

$$\therefore f(x) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{2}\right)$$

$$= \frac{4}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin \pi x + \frac{4}{3\pi}\sin\frac{3\pi x}{2} - \cdots\cdots$$

(*) Find fourier series of $f(x) = |x|$; $-2 \leq x \leq 2$

$\Rightarrow$ Here, $f(x)$ is an even function and symmetric with respect to y-axis.

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad ; \quad n = 1, 2, 3, \cdots$$

Here,

$$a_0 = \frac{2}{2}\int_0^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= 2$$

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{2x}{n\pi}\sin\frac{n\pi x}{2} + \left(\frac{2}{n\pi}\right)^2\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \left(\frac{2}{n\pi}\right)^2(-1)^n - \left(\frac{2}{n\pi}\right)^2$$

$$= \left(\frac{2}{n\pi}\right)^2\left\{(-1)^n - 1\right\}$$

x	$\cos\frac{n\pi x}{2}$
1	$\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}$
0	$\frac{-\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$

$$\therefore f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right)^2 \left\{(-1)^n - 1\right\} \cos\left(\frac{n\pi x}{2}\right)$$

$$= 1 + \left[-\frac{8}{\pi^2} \cos\frac{\pi x}{2} + 0 - \frac{8}{9\pi^2} \cos\frac{3\pi x}{2} + 0 - \cdots \right]$$

Ans

Assignment - 2

* All math done in class (L-25 ⇒ 17.10.2023)

⇒ $f(x) = \begin{cases} 0 & ; -\pi \le x < 0 \\ h & ; 0 \le x \le \pi \end{cases}$

⇒ $f(x) = \begin{cases} 0 & ; -\pi \le x < 0 \\ \sin x & ; 0 \le x \le \pi \end{cases}$

⇒ $f(x) = x + x^2 \;;\; -\pi \le x \le \pi$

⇒ $f(x) = \begin{cases} \frac{1}{4} - x & ; 0 \le x < \frac{1}{2} \\ x - \frac{3}{4} & ; \frac{1}{2} \le x \le 1 \end{cases}$ find the half range sine series.