Chapter : 1

L-1 01.06.2022

Linear Equation:

$$y = mx + c$$

$$x + y = 7$$

⊛

$$x + 3y + 5z = 10$$
$$2x - y + 4z = -5$$
$$3x + 2y = 7$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 10 \\ 2 & -1 & 4 & -5 \\ 3 & 2 & 0 & 7 \end{array}\right]$$

co-efficient

constant part

Augmented Matrix

Target Matrix:

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow \text{One Solution}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \rightarrow \text{No solution}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{Many Solution}$$

L-2 06.06.2022

$$x + y + 2z = 9$$
$$2x + 4y - 3z = 1$$
$$3x + 6y - 5z = 0$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

$r_2' = r_2 - 2r_1$
$r_3' = r_3 - 3r_1$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$

$r_2' = \frac{1}{2} r_2$ $\Rightarrow$ $\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$

$r_3' = r_3 - 3r$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & \frac{-1}{2} & \frac{-3}{2} \end{bmatrix}$$

$r_3' = -2r_3$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{-7}{2} & \frac{-17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$ → in row-ichelon form

$\therefore \quad z = 3$

$\therefore \quad y + \frac{-7}{2} z = \frac{-17}{2}$

$\Rightarrow y + \frac{-7 \cdot 3}{2} = \frac{-17}{2}$

$\Rightarrow y = \frac{-17}{2} + \frac{21}{2}$

$\therefore y = \frac{-17+21}{2} = \frac{4}{2} = 2$

$\therefore \; x + y + 2z = 9$

$\Rightarrow \; x + 2 + 2 \cdot 3 = 9$

$\Rightarrow \; x = 9 - 8$

$\therefore x = 1$

$\therefore (x, y, z) = (1, 2, 3)$.

Ans

reduced row-echelon form

(*)

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

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a)

$2x = -1$

$-4x = -6$

$x = -1$

$3x = 0$

b)

$3x_2 - x_3 - x_4 = -1$

$5x_1 + 2x_2 \quad - 3x_4 = -6$

c)

$$x + 2y + 3z = 4$$
$$-4x - 3y - 2z = -1$$
$$5x - 6y + z = 1$$
$$-8x = 3$$

d)

$$3x_1 + x_3 - 4x_4 = 3$$
$$-4x_1 + 4x_3 + x_4 = -3$$
$$-x_1 + 3x_2 - 2x_4 = -9$$
$$-x_4 = -2$$

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$$y_1 = ax_1^2 + bx_1 + c$$
$$y_2 = ax_2^2 + bx_2 + c$$
$$y_3 = ax_3^2 + bx_3 + c$$

Augmented Matrix:

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$

L-3 / 08.06.2022

1·2

Example - 5:

The augmenet matrix for the system is,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow[R_3' = R_3 - 2R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow{R_2' = -R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$$\xrightarrow[r_4' = r_4 - 4r_2]{r_3' = r_3 - 5r_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow{r_3 \rightleftharpoons r_4} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3' = \frac{1}{6} r_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2' = r_2 - 3r_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_1' = r_1 + 2r_2} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \text{reduced row-echelon matrix}$$

$$\therefore \quad x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = \frac{1}{3}$$

$\therefore$ x_1, x_3, x_6 are leading variables.

$$\therefore \quad x_1 = -3x_2 - 4x_4 - 2x_5$$

$$\therefore x_3 = -2x_4$$

$$\therefore x_6 = \frac{1}{3}$$

$\therefore$ x_2, x_4, x_5 are free variables.

Take,

$x_2 = r$

$x_4 = s$

$x_5 = t$

$\therefore \; x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = \frac{1}{3}$

where, r, s, t can be any real numbers.

(*) $2x + y + 5z = 0$

$\longrightarrow$ linear homogenous equation.

P.T.O.

Example-6:

The augmented matrix for the system is,

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4' = R_4 - 2R_1]{R_2' = R_2 - 2R_1} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2' = -R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 0 & 0 & 4 & 8 & 0 & 18 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4' = R_4 - 4R_2]{R_3' = R_3 - 5R_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \rightleftarrows R_4} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4' = \frac{1}{6} R_4} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2' = R_2 - 3R_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1' = R_1 + 2R_2} \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \quad x_1 + 3x_2 + 4x_4 + 2x_5 = 0$

$x_3 + 2x_4 = 0$

$x_6 = 0$

$\therefore \quad x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = 0$

Let,

$x_2 = r$

$x_4 = s$

$x_5 = t$

$\therefore \quad x_1 = -3r - 4s - 2t$

$x_2 = r$

$x_3 = -2s$

$x_4 = s$

$x_5 = t$

$x_6 = 0$

Ans

L-4 / 13.06.2022

1.3

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}_{\substack{2\times 3 \\ r_1 \times c_1}} \quad \& \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}_{\substack{3\times 4 \\ r_2 \times c_2}}$$

$$c_1 = r_2$$

$$\therefore AB = \begin{bmatrix} 4+0+8 & 1-2+28 & 4+6+20 & 3+2+8 \\ 8+0+0 & 2-6+0 & 8+18+0 & 6+6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}_{2\times 4}$$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & -1 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & -1 \end{bmatrix}_{3\times 2}$$

23)

a) $A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & f \end{bmatrix}$

b) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & 0 & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & a_{66} \end{bmatrix}$

d) $A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$

24)

a) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$

L-5 / 15.06.2022 /

1.4 /

Theorem: 1.4.1 / Properties of Matrix Arithmetic

a) $A + B = B + A$

b) $A + (B + C) = (A + B) + C$

c) $A(BC) = (AB)C$

d) $A(B + C) = AB + AC$

e) $(B + C)A = BA + CA$

f) $A(B - C) = AB - AC$

g) $(B - C)A = BA - CA$

h) $a(B + C) = aB + aC$

i) $a(B - C) = aB - aC$

j) $(a + b)C = aC + bC$

k) $(a - b)C = aC - bC$

l) $a(bC) = (ab)C$

m) $a(BC) = (aB)C = B(aC)$

* Identity Matrix / Unit Matrix (Product)

* In general math: If product of two non-zero number is 1, then thy are inverse of each other.

* Square matrix whose determinate is zero is singular matrix

* Rules of inverse,

i. Square matrix

ii. Same size

iii. Non-singular

$AB = I = BA$

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(A) = ad - bc$

(*) $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow$

$$|A| = \begin{vmatrix} 2 & -5 \\ -1 & 3 \end{vmatrix} = 6 - 5 = 1$$

$$|B| = \begin{vmatrix} 3 & 5 \\ 1 & 2 \end{vmatrix} = 6 - 5 = 1$$

(*) $A = \begin{bmatrix} 1 & 4 \\ 2 & -3 \end{bmatrix}$

$$\det(A) = -3 - 8 = -11$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -3 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{4}{11} \\ \frac{2}{11} & -\frac{1}{11} \end{bmatrix}$$

(*) $B = \begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$

$$\det(B) = -5 - 6 = -11$$

$$\therefore B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} -1 & -2 \\ -3 & 5 \end{bmatrix} = \frac{1}{-11} \begin{bmatrix} -1 & -2 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{11} & \frac{2}{11} \\ \frac{3}{11} & \frac{-5}{11} \end{bmatrix}$$

(*)

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$$

$$\det(A) = 6 - 5 = 1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = B$$

$$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$\det(B) = 6 - 5 = 1$$

$$\therefore B^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = A$$

(*)

$$2x + 3y = 1$$
$$x - 5y = 7$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{-13}\begin{bmatrix} -5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ \frac{1}{13} & -\frac{2}{13} \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} \frac{5}{13} & \frac{3}{13} \\ \frac{1}{13} & -\frac{2}{13} \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{13} + \frac{21}{13} \\ \frac{1}{13} - \frac{14}{13} \end{bmatrix} = \begin{bmatrix} \frac{26}{13} \\ \frac{-13}{13} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\circledast$ $(AB)^{-1} = B^{-1}A^{-1}$

$$\Rightarrow (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= AIA^{-1}$$

$$= AA^{-1}$$

$$= I$$

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB$$

$$= B^{-1}B$$

$$= I$$

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1.4 /

(*) $(AB)^{-1} = B^{-1}A^{-1}$

$(A \pm B)^T = A^T \pm B^T$

$(AB)^T = B^T A^T$

(*) $A^{-2} = (A^2)^{-1} = (A^{-1})^2$

$A^2 \cdot A^3 = A^5$

$A^{-n} = (A^{-1})^n = (A^n)^{-1}$

$(A^{-1})^{-1} = A$

(*) Matrix Polynomial

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$P(x) = 2x^2 - 5x + 4$

$P(A)$ = matrix polynomial

$= 2A^2 - 5A + 4I$

$\Rightarrow$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12 & -6-3 \\ 8+4 & -12+1 \end{bmatrix} = \begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix}$$

$$P(A) = 2A^2 - 5A + 4I$$

$$= 2\begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix} - 5\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -18 \\ 24 & -22 \end{bmatrix} - \begin{bmatrix} 10 & -15 \\ 20 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -16-10+4 & -18+15+0 \\ 24+20+0 & -22-5+4 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & -3 \\ 44 & -23 \end{bmatrix}$$

(*) $(A^T)^{-1} = (A^{-1})^T$

$(A^T)(A^{-1})^T = I$

$\Rightarrow (A^{-1}A)^T = I$

$\Rightarrow I^T = I$

$\therefore I = I$

Sample Question: $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

verify that $(A^{-1})^T = (A^T)^{-1}$

(*) A square matrix A is said to be idempotent if $A^2 = A$.

54)

a) $(I-A)^2 = (I-A)(I-A)$

$= I^2 - IA - AI + A^2$

$= I - A - A + A$

$= (I-A)$

$\therefore (I-A)$ is idempotent.

b) $(2A-I)(2A-I) = 4A^2 - 2AI - 2I \cdot A + I^2$

$= 4A^2 - 2A - 2A + I$

$= 4A - 4A + I$

$= I$

So, $(2A-I)^{-1} = (2A-I)$

(Proved)

1.5

Elementary Matrix

Defination: An $n \times n$ matrix is called an elementary matrix if it can be obtained from the $n \times n$ identity matrix I_n by performing a single elementary row operation.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \rightleftarrows r_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \text{an elementary matrix}$$

$$\Big\downarrow r_3' = r_3 + 2r_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \rightarrow \text{an elementary matrix.}$$

Sample Question:

Find out the operation or an elementary matrix.

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 5 & 7 & 2 & 0 & 1 & 0 \\ 8 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

L-7 / 22.06.2022 /

1.5 /

Find A^{-1}

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & +2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = -R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\xrightarrow[R_2' = R_2 + 3R_3]{R_1' = R_1 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\xrightarrow{R_1' = R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$= \left[I \mid A^{-1} \right]$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

Example - 5

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3' = R_3 + R_1]{R_2' = R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2' = -\frac{1}{8} R_2} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{9}{8} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3' = R_3 - 8R_2} \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{9}{8} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

A is not ~~inter~~ invertseable.

1.6)

(*)

$$x_1 + 2x_2 + 3x_3 = 5$$
$$2x_1 + 5x_2 + 3x_3 = 3$$
$$x_1 + 8x_3 = 17$$

Here,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \rightarrow \text{coefficient matrix.}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

Logic!

$$AX = B$$
$$A^{-1}(AX) = A^{-1}B$$
$$X = A^{-1}B$$

$$\therefore A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 15 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} -200+48+153 \\ 65-15-51 \\ 25-6-17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Ans

L-8/ 27.06.2022/

1.7

Diagonal Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is a diagonal matrix.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-3) \cdot (-1) = 6 \neq 0$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(*) $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$D^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

(*)

$$A^4 = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Triangular Matrix

$$A = \begin{bmatrix} 16 & 0 & 0 \\ 5 & 1 & 0 \\ 2 & -3 & 81 \end{bmatrix}$$

$$\det(A) = 16 \cdot 1 \cdot 81 = 1296$$

Symmetric Matrix

A square matrix A is symmetric if,

$$A = A^T$$

$$\begin{bmatrix} 2 & -3 & 4 \\ -3 & 7 & -2 \\ 4 & -2 & -1 \end{bmatrix}$$

Here,

$a_{12} = a_{21}$

$a_{13} = a_{31}$

$a_{23} = a_{32}$

$a_{ij} = a_{ji}$

(*)

$$\begin{bmatrix} -5 & 6 & 3 \\ 6 & 1 & a \\ 3 & a & -3 \end{bmatrix}$$

a = any real number

$$(A^{-1})^T = (A^T)^{-1}$$

Chapter - 2

Determinants

2.1

$$A = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$$ is a matrix

$$\det(A) = |A| = \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} = -8+3 = -5$$

⊛ $$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Minor of 3: $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40-24 = 16$

$$M_{23} = \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 12-1 = 11$$

$$M_{21} = \begin{vmatrix} 1 & -4 \\ 4 & 8 \end{vmatrix} = 8+16 = 24$$

$$M_{22} = \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} = 24+4 = 28$$

⊛ Cofactor

$1+1=2 = \text{even} = (+)$

$C_{11} = (-1)^{(+)} M_{11}$

$2+1=3 = \text{odd} = (-1)$

$C_{21} = (-1) M_{21}$

$2+2=4 = \text{even} = (+1)$

$C_{22} = (+1) M_{22}$

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$C_{21} = -M_{21} = -24$

$C_{22} = M_{22} = 28$

$C_{23} = -M_{23} = -11$

$\det(A) = 2 \cdot C_{21} + 5 \cdot C_{22} + 6 \cdot C_{23}$

$= 2(-24) + 5(28) + 6(-11)$

$= 26$

Ans

- Cofactor expansion method to find det (A)

⊛ We can find det (A) by any row or collumn.

row × row cofactor

or

Collumn × Column cofactor

⊛

Cofactor of row-2 × entry of row 1 or 3 = 0.

L-9 / 29.06.2022 /

2.2 /

* Find determinant by row reduction:

$$A = \begin{bmatrix} 2 & 5 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = 2 \cdot 1 \cdot 6 = 12$$

*

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 1 \\ 3 & -4 & 2 \end{vmatrix} = - \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 4 \\ 3 & -4 & 2 \end{vmatrix}$$

*

$$\begin{vmatrix} 2 & 6 & 4 \\ 5 & 6 & 1 \\ 3 & -4 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 & 2 \\ 5 & 6 & 1 \\ 3 & -4 & 2 \end{vmatrix}$$

(*) $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

$\det(A) = ?$

2.3

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(AB) = \det(A) \cdot \det(B)$$

(*) $A = \begin{bmatrix} 1 & 3 \\ 5 & -4 \end{bmatrix}$

$$A^{-1} = -\frac{1}{19} \begin{bmatrix} -4 & -3 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{19} & \frac{3}{19} \\ \frac{5}{19} & -\frac{1}{19} \end{bmatrix}$$

$$\det(A^{-1}) =$$

(*) $AA^{-1} = I$

$\Rightarrow \det(AA^{-1}) = \det(I) = 1$

$\Rightarrow \det(A) \cdot \det(A^{-1}) = 1$

(*) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

matrix of cofactors $= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$

Transpose of the matrix $= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \text{adj}(A)$.

Now,

$A \cdot \text{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$

$= \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$

$$= \det(A) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \det(A) \cdot I$$

Now,

$$A \cdot \operatorname{adj}(A) = \det(A) \cdot I$$

$$\Rightarrow A \cdot \frac{\operatorname{adj}(A)}{\det(A)} = I$$

$$\therefore A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

$$\circledast \quad A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\text{Number of cofactors} = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

$$\det(A) = 3 \cdot 12 + 2 \cdot 6 + (-1)(-16)$$

$$= 36 + 12 + 16$$

$$= 64$$

25)

$$A = \begin{bmatrix} 4 & 5 & 0 \\ 4 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}$$

Replace with constant

$$A_2 = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 4 & 5 & 2 \\ 4 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

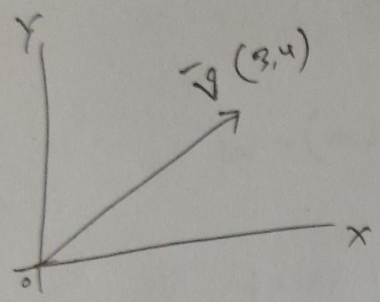
* Cramer's rule:

$$x = \frac{\det(A_1)}{\det(A)} \qquad y = \frac{\det(A_2)}{\det(A)}$$

$$z = \frac{\det(A_3)}{\det(A)}$$

L-10 / 04.07.2022

3.1 / Chapter - 3 / Vector /

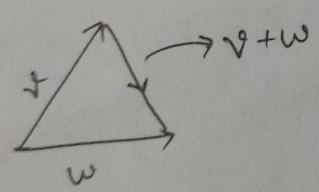


Magnitude / length / norm of the vector v : $\|v\|$

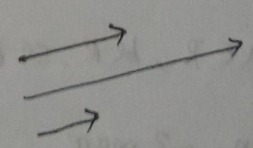
$\bar{v} = (1, -3, 4, -2)$

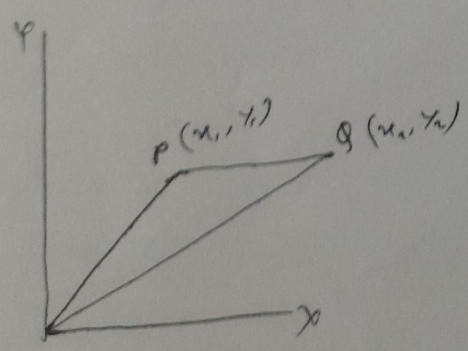
$\|v\| = \sqrt{1^2 + (-3)^2 + 4^2 + (-2)^2}$

$= \sqrt{1+9+16+4} = \sqrt{30}$



$v, 2v, \frac{1}{2}v \Rightarrow$ Parallel vector





$\bar{OP_1} = (x_1, y_1)$

$\bar{OP_2} = (x_2, y_2)$

$\bar{P_1P_2} = (x_2 - x_1, y_2 - y_1)$

(*)

$P_1\ (2,-1,4)$

$P_2\ (7,5,-8)$

$\vec{v} = \overrightarrow{P_1P_2}$

$v = (7-2,\ 5+1,\ -8-4)$

$= (5, 6, -12)$

(*) $\mathbb{R}$: set of all real numbers.

$X = \{a, b\}$

$Y = \{1, 2, 3\}$

$X \times Y = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2\ \{(a,b) : a \in \mathbb{R},\ b \in \mathbb{R}\}$

Euclidean 2 space.

$\mathbb{R}^3 = \{(a,b,c) : a \in \mathbb{R},\ b \in \mathbb{R},\ c \in \mathbb{R}\}$

Euclidean -3 space

. . .

$v = (1, 2, -3) \in R^3$

$v = (1, -2, 3, 5) \in R^4$

$w = (a, b, c, d) \in R^4$

$v = w$

$(a, b, c, d) = (1, -2, 3, 5)$

$a = 1$

$b = -2$

$c = 3$

$d = 5$

$v = (1, -2, 3, 5)$

$w = (-2, 5, 6, -2)$

$v + w = (-1, 3, 9, 3)$

$3v = (3, -6, 9, 15)$

3.2

Unit vector

$v = (5, -1, 3)$

$\|v\| = \sqrt{5^2 + (-1)^2 + 3^2} = \sqrt{25+1+9} = \sqrt{35}$

unit vector

$\hat{v} = \dfrac{v}{\|v\|} = \dfrac{5, -1, 3}{\sqrt{35}} = \left(\dfrac{5}{\sqrt{35}}, \dfrac{-1}{\sqrt{35}}, \dfrac{3}{\sqrt{35}}\right)$

Standard unit vector

In $\mathbb{R}^3$, $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$

are standard unit vector.

In $\mathbb{R}^4$, $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$

$e_4 = (0,0,0,1)$

are standard unit vector

u, v

$\|u - v\| = d(u, v)$

$u = (1, 3, -2, 7)$

$v = (0, 7, 2, 2)$

$\|u - v\| = \|(1, -4, -4, 5)\|$

$= \sqrt{1^2 + (-4)^2 + (-4)^2 + 5}$

$= \sqrt{1 + 16 + 16 + 25}$

Dot product / Eudiden inner product

$u \cdot v = \|u\| \, \|v\| \cos\theta$

$= u_1 v_1 + u_2 v_2$

$u = (u_1, u_2)$

$v = (v_1, v_2)$

$u = \cancel{(2, -3, 1)}\ (2, -3, 1)$

$v = (1, 4, -4)$

$u \cdot v = (2 \cdot -12 \cdot -4)$

$= 2 - 16 = -4$

$\cos\theta = ?$

$\|u\| = \sqrt{4+9+1} = \sqrt{14}$

$\|v\| = \sqrt{1^2+(-4)^2+4^2} = \sqrt{33}$

$\cos\theta = \dfrac{u \cdot v}{\|u\| \, \|v\|}$

$u \cdot v = 0 \Rightarrow u \perp v$

u orthogonal to v : $u \perp v$

L-11 / 06.07.2022 /

3.3 /

Orthogonality

$u \cdot v = 0 \Rightarrow$ Dot product

Euclidean inner product

$u = (-2, 3, 1, 4)$

$v = (1, 2, 0, -1)$

$u \cdot v = (-2 + 6 + 0 - 4)$

$= 0$

$\therefore u \perp v$

In $\mathbb{R}^3$

i, j, k are

$i = (1, 0, 0)$

$j = (0, 1, 0)$

$k = (0, 0, 1)$

$i \cdot j = (0 + 0 + 0) = 0$

$j \cdot k = 0$

$k \cdot i = 0$

$i \cdot i = 1$

$j \cdot j = 1$

$k \cdot k = 1$

vector component of u along a

$$\text{Proj}_a u = \frac{u \cdot a}{\|a\|^2} a$$

vector component of u orthogonal to u

$u - \text{Proj}_a u$

Example - 5

$u = (2,-1,3)$, $a = (4,-1,2)$

$$\text{Proj}_a u = \frac{u \cdot a}{\|a\|^2} a$$

$$= \frac{8+1+6}{4^2+(-1)^2+2^2} a$$

$$= \frac{15}{21} (4,-1,2)$$

$$= \frac{5}{7} (4,-1,2)$$

$$= \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)$$

vector component of u orthogonal to a

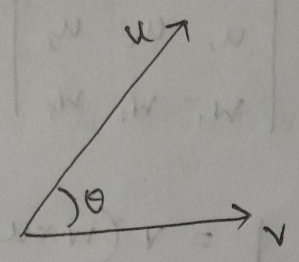
$$u - \text{Proj}_a u = (2,-1,3) - \left(\frac{20}{7}, \frac{-5}{7}, \frac{10}{7}\right)$$

$$= \left(2-\frac{20}{7}, -1+\frac{5}{7}, 3-\frac{10}{7}\right)$$

$$= \left(\frac{-6}{7}, \frac{-2}{7}, \frac{11}{7}\right)$$

3.5

Cross Product



$u \times v$ is a vector

$= \|u\| \, \|v\| \sin\theta \, \hat{n}$

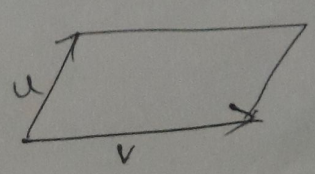
$\boxed{\begin{array}{l} u \perp (u \times v) \\ v \perp (u \times v) \end{array}}$ $\boxed{\begin{array}{l} u \cdot (u \times v) = 0 \\ v \cdot (u \times v) = 0 \end{array}}$

$u = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$

$v = (v_1, v_2, v_3) = v_1 i + v_2 j + v_3 k$

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

(*) $\|u \times v\|$ = area of the parallelogram determined by u and v



* u, v, w

$$u\cdot(v\times w) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = -\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = v\,(w\times u)$$

*

$$\begin{tikzcd} & u \arrow[dl] & \\ v \arrow[rr] & & w \arrow[ul] \end{tikzcd}$$

$$u\cdot(v\times w) = v\cdot(w\times u) = w\cdot(u\times v)$$

$|u\cdot(v\times w)|$ = volume of the parallelogram determined by u, v, w.

L-12/ 18.07.2022/

Electricity

For the point A,

$$I_1 = I_2 + I_3$$

$$7I_1 + 3I_3 - 30 = 0$$

$$11I_2 - 50 - 3I_3 = 0$$

1]

$\therefore I_1 = I_2 + I_3$

$\Rightarrow I_1 - I_2 - I_3 = 0$

$\therefore 13I_2 + 5I_1 + 8 = 0$

$\Rightarrow 5I_1 + 13I_2 = -8$

$\therefore -3 + 14I_3 - 13I_2 = 0$

$\Rightarrow -13I_2 + 14I_3 = 3$

$\therefore$ Augmented matrix:

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 5 & 13 & 0 & -8 \\ 0 & -13 & 14 & 3 \end{bmatrix}$$

2|

$\therefore I_3 = I_2 + I_1$

$\Rightarrow I_1 + I_2 - I_3 = 0$

$\therefore 6 - 2I_1 + 2I_2 = 0$

$\Rightarrow -2I_1 + 2I_2 = -6$

$\therefore 8 - 4I_3 - 2I_2 = 0$

$\Rightarrow -2I_2 - 4I_3 = -8$

$$\therefore \begin{bmatrix} 1 & 1 & -1 & 0 \\ -2 & 2 & 0 & -6 \\ 0 & -2 & -4 & -8 \end{bmatrix}$$

4.1

Vector Space

Let V be a non-empty set of objects.

Two operations, addition and scalar multiplication, are define in V.

$V = \mathbb{R}$ is a vector space

$V = \mathbb{R}^n$ is a vector space; $n \in \mathbb{N}$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$= M_{22} \longrightarrow$ Notation.

Exercise:

1] a) $u = (-1, 2)$

$v = (3, 4)$

$k = 3$

$u + v = (-1+3,\ 2+4) = (2, 6)$

$ku = (0, ku_2) = (0, 3 \cdot 2) = (0, 6)$

e)

$$1u = (1 \cdot u_1 , 1 \cdot u_2) = (0, u_2) \neq u$$

Axiom 10 is falls and hence that V is not a vector space.

L-13 / 20.07.2022

$M_{22} \Rightarrow$ 2 by 2 matrix of real numbers

$P_2 \Rightarrow$ all polynomial equation ; degree ≤ 2.

⊛ V, vector space,

$W \subset V$ is a subspace of V,

i) $u, v \in W \Rightarrow u+v \in W$

ii) k, u scalar, $u \in W \Rightarrow ku \in W$

⊛ x-axis is a subspace of W

⊛ $W = \{a + bx + cx^2 : a, b, c \in \mathbb{Z}\}$

$u = 2 + 3x + 7x^2 \in W$

rational

$\frac{1}{2}u = 1 + \frac{3}{2}x + \frac{7}{2}x^2 \notin W$

W is not subspace of P_2

* V is vector in V and

$$W = k_1 u_1 + k_2 u_2 + \dots + k_n u$$

where,

$u, u_2, \dots u_n \in V$ and $k_1 u_2 \dots k_n$ are scalar

* $V = \mathbb{R}^2$

$u = (1,-1)$, $v = (2,1) \in \mathbb{R}^2$

$\Rightarrow 2u + 3v$

$= 2(1,-1) + 3(2,1)$

$= (2,-2) + (6,3)$

$= (8,1) \in \mathbb{R}^3$

* $S = \{w_1, w_2, w_3\}$, $w_1, w_2, w_3 \in V$

W = set of all possible linear combination of the vector is S.

= spanned by S.

$W = \text{span}(S)$

* $V = \mathbb{R}^3$

standard unit vector in $\mathbb{R}^3$:

$$S = \{ e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1) \}$$

$$k_1 e_1 + k_2 e_2 + k_3 e_3$$

Let,

$$2e_1 + 3e_2 + 4e_3 = (2,3,4) \in \mathbb{R}^3$$

* P_2:

standard unit vector:

$$P_2 = \text{span}\{1, x, x^2\}$$

$$P_3 = \text{span}\{1, x, x^2, x^3\}$$

$$P_n = \text{span}\{1, x, x^2, x^3, \ldots, x^n\}$$

* Standard unit matrix

$$M_{22} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

L-14 / 27.07.2022

4.2

$v_1 = (1,1,2)$, $v_2 = (1,0,1)$, $v_3 = (2,1,3)$ span $\mathbb{R}^3$?

Take any vector $(a,b,c) \in \mathbb{R}^3$

Set,

$$(a,b,c) = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$= k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

$$= (k_1, k_1, 2k_1) + (k_2, 0, k_2) + (2k_3, k_3, 3k_3)$$

$$= (k_1 + k_2 + 2k_3,\ k_1 + 0 + k_3,\ 2k_1 + k_2 + 3k_3)$$

$\Rightarrow$

$$\left.\begin{aligned} k_1 + k_2 + 2k_3 &= a \\ k_1 \qquad + k_3 &= b \\ 2k_1 + k_2 + 3k_3 &= c \end{aligned}\right\} \text{System of linear equation}$$

Co-effecient matrix,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1(0-1) - 1(3-2) + 2(1-0)$$

$$= -1 - 1 + 2$$

$$= 0$$

det (A) is zero. So, system has no solution.

So, k_1, k_2, k_3 do not exist.

So, v_1, v_2, v_3 do not span $\mathbb{R}^3$.

4.3

Linear Independence

$v_1 = (2,3)$, $v_2 = (4,6)$

Relation:

$$v_2 = 2v_1$$

$$2v_1 - v_2 = (0,0)$$

Here, scalers,

$$2-1 = 1 \neq 0$$

So, vectors v_1, v_2 are dependent.

⊛ $W_1 = (2,3)$, $W_2 = (5,-1)$

$0 \cdot W_1 + 0 \cdot W_2 = (0,0)$

No, other scaler can't make it zero.

So, vector W_1 , W_2 are independent.

⊛ In $\mathbb{R}^3$

$e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$

Take $k_1 e_1 + k_2 e_2 + k_3 e_3 = 0$

$k_1 (1,0,0) + k_2 (0,1,0) + k_3 (0,0,1) = (0,0,0)$

$\Rightarrow (k_1, k_2, k_3) = (0,0,0)$

$\therefore \left.\begin{array}{l} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \end{array}\right\}$ only one solution is zero. So, they are linearly independent.

$\therefore \{e_1, e_2, e_3\}$ is a linear independent set.

* In P_2

$\{1, x, x^2\}$ is linearly independent.

$$k_1 + k_2 x + k_3 x^2 = 0$$

$$\Rightarrow k_1 = 0, \; k_2 = 0, \; k_3 = 0$$

* In homogenus system, if determinate is 0 then they are dependent, else independent.

* Linear homogenus system having equation more than variable, then system has only zero solution. So, they are linearly independent.

L-15 / 08.08.2022

$\boxed{4.4}$

S Basis for V

$\Rightarrow$ S is linearly independent

$\Rightarrow$ S spans V

(*) R^3

$S = \{v_1, v_2, v_3\}$ is independent

then it is spannis as well

and vice-versa

(*) R^3

standard unit basis for R^3 = $\{e_1, e_2, e_3\}$

standard basis for P_2 : $\{1, x, x^2\}$

M_{22} : $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Example-9/

$S = \{ v_1 = (1,2,1),\ v_2 = (2,9,0),\ v_3 = (3,3,4) \}$

is a basis for $\mathbb{R}^3$

$v = (5,-1,9) \quad (v)_S = ?$

Take, $v = c_1 v_1 + c_2 v_2 + c_3 v_3$

$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$

$= (c_1 + 2c_2 + 3c_3,\ 2c_1 + 9c_2 + 3c_3,\ c_1 + 4c_3)$

$\therefore\ c_1 + 2c_2 + 3c_3 = 5$

$2c_1 + 9c_2 + 3c_3 = -1$

$c_1 + 4c_3 = 9$

Augmented matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{bmatrix} \longrightarrow \text{Solve it.}$$

by solving,

$c_1 = 1$

$c_2 = -1$

$c_3 = 2$

$\therefore (v)_s = (1, -1, 2)$

(*) In $\mathbb{R}^3$

$v = (5, 2, -3)$

$S = \{ (1,0,0), (0,1,0), (0,0,1) \}$ basis for $\mathbb{R}^3$

$(v)_s = ?$

$(5, 2, -3) = c_1 (1,0,0) + c_2 (0,1,0) + c_3 (0,0,1)$

$= (c_1, c_2, c_3)$

$\therefore c_1 = 5$

$c_2 = 2$

$c_3 = -3$

$\therefore (v)_s = (5, 2, -3)$

$\therefore (v)_s = v$ (Theorem)

(proved)

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4.5

Dimension of a vector space

$\dim(\mathbb{R}^3) = 3$

dimension = Positive integers

$P_2 = \{1, x, x^2\}$

$\dim(P_2) = 3$

$\dim(M_{22}) = 4 = 2 \times 2$

$\dim(P_n) = n+1$

$= \{1, x, x^2, \ldots, x^n\}$

Theorem 4.5.2

Let V be a finite-dimensional vector space, and let $\{v_1, v_2, \ldots, v_n\}$ be any basis.

a) If a set has more than n ~~vec~~ vectors, then it is linearly dependent.

b) If a set has fewer than n vectors, then it does not span V.

Theorem 4.5.3

Let S be a nonempty set of vectors in a vector space V.

a) If S is a linearly independent set, and if v is a vector in V that is outside of $\text{span}(S)$, then the set $S \cup \{v\}$ that results by inserting v into S is still linearly independent.

b) If v is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{v\}$ denotes the set obtained by removing v from S, then $S - \{v\}$ span the same space; that is,

$$\text{span}(S) = \text{span}(S - \{v\})$$

⊛ Set of non zero solution of a homogenow system then the set is a subspace of this R^n space.

Example - 3

$$x_1 = -s-t$$
$$x_2 = s$$
$$x_3 = -t$$
$$x_4 = 0$$
$$x_5 = t$$

$$(x_1, x_2, x_3, x_4, x_5) = (-s-t, s, -t, 0, t)$$
$$= (-s, s, 0, 0, 0) + (-t, 0, -t, 0, t)$$
$$= s(-1, 1, 0, 0, 0) + t(-1, 0, -1, 0, 1)$$

$$(x_1, x_2, x_3, x_4, x_5) = s v_1 + t v_2$$

$\therefore$ v_1, v_2 span $\mathbb{R}^5$.

$\{v_1, v_2\}$ is a basis for the solution space and dimension for the solution space 2.

7)

a) The plane $3x - 2y + 5z = 0$

$$\Rightarrow \quad 3x = 2y - 5z$$

$$\Rightarrow \quad x = \frac{2}{3}y - \frac{5}{3}z$$

Take,

$$y = r, \quad z = s$$

$$\therefore \quad x = \frac{2}{3}r - \frac{5}{3}s$$

$$(x, y, z) = \left(\frac{2}{3}r - \frac{5}{3}s, r, s\right)$$

$$= \left(\frac{2}{3}r, r, 0\right) + \left(-\frac{5}{3}s, 0, s\right)$$

$$= r\left(\frac{2}{3}, 1, 0\right) + s\left(-\frac{5}{3}, 0, 1\right)$$

$$= rv_1 + sv_2$$

$\{v_1, v_2\}$ is a basis for the solution space of $\mathbb{R}^3$ and the dimension of the solution space $= 2$.

4.7

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 5 \end{bmatrix}$$

$r_1 = [2 \quad 3 \quad 5]$

are row vectors

$r_2 = [1 \quad -2 \quad 5]$

members of $\mathbb{R}^3$

$c_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$c_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ members of $\mathbb{R}^2$

$c_3 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

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|4.7|

⊛ Basis for the null space

Example - 1:

$$2x_1 + 2x_2 - x_3 + x_5 = 0$$
$$-x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0$$
$$x_1 + 2x_2 - x_3 - x_5 = 0$$
$$x_3 + x_4 + x_5 = 0$$

Co-effecient matrix,

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x_1 + x_2 + x_5 = 0$

$x_3 + x_5 = 0$

$x_4 = 0$

$\Rightarrow$

$x_1 = -x_2 - x_5$ $\Rightarrow$ $x_1 = -s - t$

$x_3 = -x_5$ $\quad$ $x_2 = s$

$x_3 = -t$

$x_4 = 0$ $\quad$ $x_4 = 0$

$x_5 = t$

$\Rightarrow$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix}$$

$$= s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = s v_1 + s v_2$$

$S = \{v_1, v_2\}$ is a basis for the null space of the matrix A

Dimension of the nullspace of A is called the nullity of A. and we write $\boxed{\text{nullity}(A) = 2}$

(*) Find ~~the~~ a basis for the row space, and column space of the matrix.

$$A = \begin{array}{c} \begin{array}{cccccc} c_1 & & c_3 & & c_5 & \end{array} \\ \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \end{array}$$

$\Rightarrow$

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow R_1 \\ \rightarrow R_2 \\ \rightarrow R_3 \\ \\ \end{array}$$

A basis for the row space of A is $S = \{R_1, R_2, R_3\}$

A basis for the column space of A is $S = \{C_1, C_3, C_5\}$

(*) Find the subset of the vectors,

$v_1 = (1,-2,0,3)$, $v_2 = (2,-5,-3,6)$, $v_3 = (0,1,3,0)$, $v_4 = (2,-1,4,-7)$,

$v_5 = (5,-8,1,2)$

$\Rightarrow$

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -5 & -3 & 6 \\ 2 & -1 & 4 & -7 \\ 5 & -8 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & \frac{13}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for the subspace spanned by the given vectors is

$$\left\{ (1, -2, 0, 3), (0, 1, 3, 0), (0\ 0\ 1\ \tfrac{13}{5}) \right\}$$

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4.8

Theorem 4.8.2

If A is a matrix with n columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

⊛ A is 5×3 matrix,

rank	nullity
3	0
2	1
1	2

4.9

$$R^n \rightarrow R^m$$

$$2x_1 + x_2 - 3x_3 = w_1$$

$$x_1 - x_2 + 5x_3 = w_2$$

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$\therefore \quad A \cdot x = W$

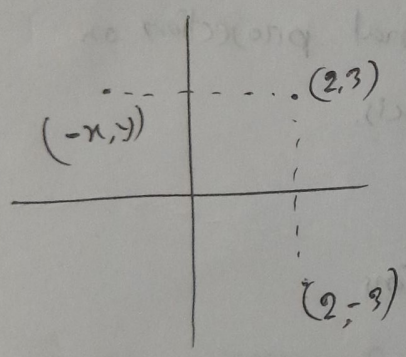
$x \in R^3 \quad , \quad W \in R^2$

$f : V \to W$

<u>linear transformation</u>

i) $T(A+B) = T(A) + T(B)$

ii) $T(kA) = kT(A)$



$T : R^2 \to R^2$

$T : (x,y) = (x,-y)$

reflection about x-axis.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

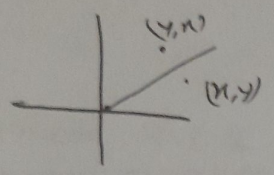
$w_1 = x$

$w_2 = -y$

$T : (x,y) = (-x,y)$

Reflection about y-axis.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



$T:(x,y) = (y,x)$

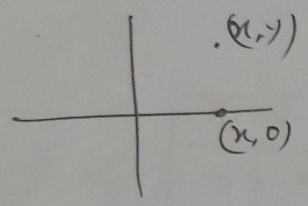
reflection about $y=x$ line

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(*)

$T: R^3 \to R^3$

(*)



$T: R^2 \to R^2$ defined,

$T: (x,y) = (x,0)$

an orthogonal projection on the x-axis.

(*)

$T(x) = kx$ → $k > 1$ Dilations

→ $0 < k < 1$ contraction

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5.1

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 3+0 \\ 8-2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 3x$$

$$\therefore Ax = 3x$$

Eigenvalues

Eigenvectors

A is $n \times n$, $0 \neq x \in \mathbb{R}^n$, such that

$Ax = \lambda x$; λ is a scalar

Then, x is Eigenvectors

and λ is Eigenvalues

⊛ Given matrix A, find out Eigenvectors and Eigenvalues

$\Rightarrow$

Take, a nonzero vector x such that,

$$Ax = \lambda x$$

$$\Rightarrow Ax - \lambda x = 0$$

$$\Rightarrow (A - \lambda I)x = 0$$

which is a linear homogenous system.

It passesses nonzero solution only if

$$\det(A - \lambda I) = 0$$

$\det(A - \lambda I) = 0$ is a polynomial equation in λ.

Also it is called the characteristic equation of A.

Example-2

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

Let,

x be a nonzero vector in $\mathbb{R}^2$ and λ is a scalar.

Then,

$$(A - \lambda I)x = 0 \quad \cdots (i)$$

and $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\therefore A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix}$$

$$\therefore \begin{vmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) - 0 = 0$$

$$(3-\lambda)(-1-\lambda) = 0$$

$$\therefore \lambda = 3, -1.$$

Case-1:

Put, $\lambda = 3$ in (i)

$$\begin{bmatrix} 0 & 0 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 4x_2 = 0$

$\Rightarrow x_1 = \frac{1}{2} x_2$

Let,
$x_2 = 2r$
$\therefore x_1 = r$

So, $\begin{bmatrix} r \\ 2r \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an independent Eigen Vectors of A for $\lambda = 3$.

Case-2

Put, $\lambda = -1$ in (i)

$$\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 4x_1 \\ 8x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 = 0$

and $x_2 = r$

$\therefore x = \begin{bmatrix} 0 \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is an independent Eigen vectors of A

for $\lambda = -1$.

5.2

$P = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ is invertable

$P^{-1} A P = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

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5.1

6|

d) $A = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

Let,

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a nonzero vector and a scalar λ...

then, $(A - \lambda I)x = 0$

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{bmatrix}$$

$$\therefore \begin{bmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots (i)$$

Now,

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ -1 & 3-\lambda & 0 \\ -4 & 13 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda)(-3+\lambda)(-1-\lambda) + 1\{-13+4(3-\lambda)\} = 0$$

$$\Rightarrow (-1-\lambda)(-3-3\lambda+\lambda+\lambda^2) - 13+12-4\lambda = 0$$

$$\Rightarrow (-1-\lambda)(\lambda^2-2\lambda-3) - 1 - 4\lambda = 0$$

$$\Rightarrow -\lambda^2 + 2\lambda + 3 - \lambda^3 + 2\lambda^2 + 3\lambda - 1 - 4\lambda = 0$$

$$\Rightarrow -\lambda^3 + \lambda^2 + \lambda + 2 = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - \lambda - 2 = 0 \longrightarrow \text{characteristic equation.}$$

$$\Rightarrow \lambda^2(\lambda-2) + \lambda(\lambda-2) + 1(\lambda-2) = 0$$

$$\Rightarrow (\lambda-2)(\lambda^2+\lambda+1) = 0$$

$$\lambda = 2, \quad \lambda = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore \lambda = 2, \quad \frac{-1 \pm i\sqrt{3}}{2}$$

Put, $\lambda = 2$ in (1) $\Rightarrow$

$$\begin{bmatrix} -3 & 0 & 1 \\ -1 & 1 & 0 \\ -4 & 13 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -1 & 1 & 0 \\ -3 & 0 & 1 \\ -4 & 13 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 9 & -3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $-x_1 + x_2 = 0$ | $x_1 = x_2$ | Let,

$x_2 - \frac{1}{3} x_3 = 0$ | $x_3 = 3x_2$ | $x_2 = r$

$x_1 = r$, $x_2 = r$

$x_3 = 2r$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} r \\ r \\ 3r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= r\, v_1$$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ is the independent basis eigenvector of A

corresponding to $\lambda = 2$

and $\{v_1\}$ is a basis for the eigenspaces of A.

corresponding to $\lambda = 2$

Dimension of the eigenspace is 1.

$$Ax = \lambda x$$

$$A(Ax) = A(\lambda x)$$

$$\Rightarrow A^2 x = \lambda (Ax)$$

$$= \lambda (\lambda x)$$

$$A^2 x = \lambda^2 x$$

$$A^3 x = A(\lambda^2 x)$$

$$= \lambda^2 (Ax)$$

$$= \lambda^2 (\lambda x)$$

$$= \lambda^3 x$$

$$\therefore A^k x = \lambda^k x$$

5.2

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix}$$

Now,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{-\lambda(3-\lambda)+2\} = 0$$

$$\Rightarrow (2-\lambda)(-3\lambda + \lambda^2 + 2) = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$\therefore \lambda = 1, 2, 2$$

$$\therefore \begin{bmatrix} -4 & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Put, $\lambda = 2$, in ①

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$\Rightarrow x_1 = -x_3$$

Let, $x_2 = r$, $x_3 = s$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$

$$= r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= r p_1 + s p_2$$

$P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are independent eigen vector of A. corresponding to $\lambda = 2$

Again,

put, $\lambda = 1$ in ①

$$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left.\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3 \end{aligned}$$

Let, $x_3 = k$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = r\, P_3$$

$P_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is the independent eigenvector of A corresponding to $\lambda = 1$.

$$P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$B = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(*)

$$|B| = |P^{-1}AP| = |P^{-1}|\,|A|\,|P|$$

$$= |P^{-1}| \cdot |P| \cdot |A|$$

$$= |A|$$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\det(A^{-1}) \cdot \det(A) = 1$

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(*)

Cherry stock = 130 pounds

Mints stock = 170 pounds

x_1 pound with the mixture of half cherry and half mints by weight.

x_2 pound with the mixture of $\frac{1}{3}$ cherry and $\frac{2}{3}$ mints by weight.

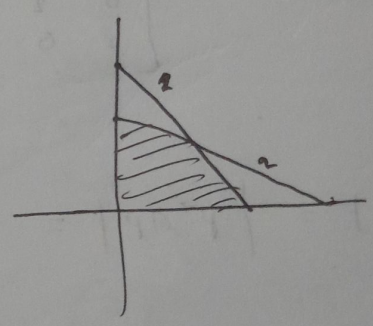
$Z = 2x_1 + 1.25x_2$

$\frac{1}{2}x_1 + \frac{1}{3}x_2 \leq 130$

$\frac{1}{2}x_1 + \frac{2}{3}x_2 \leq 170$

$x_1 \geq 0$

$x_2 \geq 0$



(*) $\frac{1}{2}x_1 + \frac{1}{3}x_2 \leq 130$

$\frac{x_1}{260} + \frac{x_2}{390} \leq 1 \quad \cdots (i)$

similarly $\frac{x_1}{340} + \frac{x_2}{255} \leq 1 \quad \cdots (ii)$

Entreame Point	Value of Z
(0,0)	00
(0,255)	518.75
(180,120)	510
(260,0)	520

Markov Process

State -1 : Sunny

State - 2 : Cloudy

State - 3 : Rainy

P_{ij} :

Previous state was in j now it will be i, the probality of being i.

$$\begin{array}{c} \text{Preceding State} \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

New state

'P' means Probality

Sum of a column = 1

Rented from location

$$\begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .8 & .3 & .2 \\ .1 & .2 & .6 \\ .1 & .5 & .2 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

Return to Location

Position $\boxed{23}$ → that means, rented from 3 and return to 2. Probality $.6 = 60\%$

(*)

State-1 : Alumni donate a year

State-2 : Alumni does not donate

$$\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \end{array}$$

→ Probability Matrix

Transition matrix

$\boxed{\text{END}}$