v)

$$f(x) = \ln(x^2-4)$$

$$f'(x) = \frac{1}{x^2-4} \cdot 2x$$

For critical,

$$f'(x) = 0$$

$$\frac{2x}{x^2-4} = 0$$

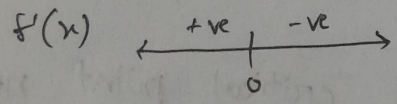
$$2x = 0$$

$$x = 0$$

$f'(x)$ is undefined at 0 and ± 2.

stationary point, $x=0$

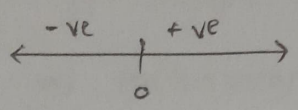
⊛ First Derivative Test:



if x_0 is a critical point and $f'(x)$ changes sign +ve to -ve and that critical point then at $x = x_0$ will be maximum.

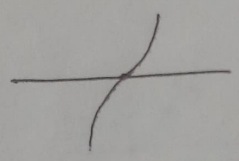
⊛

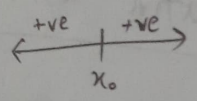
$f'(x)$



If, x_0 is a critical point and if $f'(x)$ changes sign from -ve to +ve, then on that critical point will be minimum.

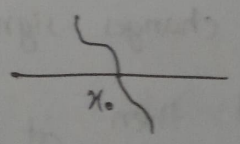
⊛





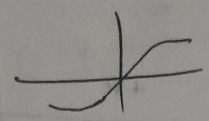
if x_0 is a critical points and $f'(x)$ not changes the sign. Then on that critical point neither maximum nor minimum.

Same as Negative

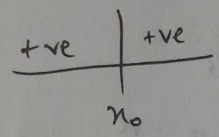


(*) $f(x) = \sqrt[3]{x}$

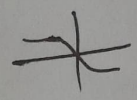
$f'(x) =$ undefined



$x = x_0$ is a critical point but not stationary point.



same for $-\sqrt[3]{x}$



(*) Using 1st Derivative test, find the relative extrema for the following functions.

i) $f(x) = 1 - 4x - x^2$ ii) $f(x) = x^3 - 3x^2 - 24x + 4$

Solution

i) $f(x) = 1 - 4x - x^2$

$f'(x) = -4 - 2x$

for critical points:

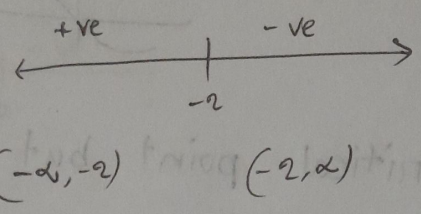
$f'(x) = 0$

$-4 - 2x = 0$

$x = -2$

Now make intervals using critical points,

sign of $f'(x)$:



$x = -3$

$f'(-3) = -4-2(-3)$

$= +ve$

$x = 3$

$f'(3) = -ve$

Therefore,

at $x = -2$, there is maximum.

maximum value,

$f(-2) = 1 + 8 - 4$

$= 1 + 4$

$= 5$

ii) $f(x) = x^3 - 3x^2 - 24x + 4$

$f'(x) = 3x^2 - 6x - 24$

For critical points,

$f'(x) = 0$

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

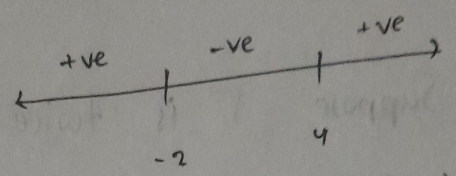
$x^2 - 4x + 2x - 8 = 0$

$x(x-4) + 2(x-4) = 0$

$(x-4)(x+2) = 0$

$x = -2, 4$

$f'(x)$:



$(-\infty, -2)$ $(-2, 4)$ $(4, \infty)$

$x = -3$	$x = 0$	$x = 5$
$f'(-3) = +ve$	$f'(0) = -ve$	$f'(5) = +ve$

Therefore,

at $x = -2$, $f'(x)$ changes sign from +ve to -ve.

$\therefore$ at $x = -2$, maximum

at $x = 4$, $f'(x)$ changes sign from -ve to +ve.

$\therefore$ at $x = 4$, minimum.

$\boxed{\text{Quizzes - 3} \Rightarrow \text{END}}$

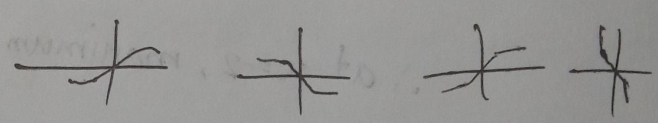
⊛ Second Derivative test!

Suppose f is twice differentiable at the point x_0

a) if $f'(x)=0$ and $f''(x_0)>0$, then f has a relative minimum.

b) if $f'(x_0)=0$ and $f''(x_0)<0$, then f has a relative maximum.

c) if $f'(x_0)=0$ and $f''(x_0)=0$, then the test is inconclusive.



Neither maximum nor minimum.

⊛ Find the relative extrema for $f(x)=3x^5-5x^3$ using 2nd derivative test.

$\Rightarrow$ $f(x) = 3x^5 - 5x^3$

$f'(x) = 15x^4 - 15x^2$

$f''(x) = 60x^3 - 30x$

For critical points,

$f'(x) = 0$

$15x^4 - 15x^2 = 0$

$15x^2(x^2-1) = 0$

$x = 0, x = \pm 1$

Now,

$f''(x) = 60x^3 - 30x$

Here,

at $x = 1$,

$f''(1) = 60 - 30$

$= 30 > 0$

Therefore, at $x = 1$, there is minimum

minimum value,

$f(1) = 3 - 5 = -2$

point $(1, -2)$

at $x = -1$

$f''(-1) = -60 + 30$

$= -30 < 0$

Therefore, at $x = -1$, there is a maximum.

maximum values,

$f(-1) = -3 + 5 = 2$

point $(-1, 2)$

at $x = 0$,

$f''(0) = 0$

Therefore at $x = 0$, neither maximum nor minimum.

⊛ Graphing Polynomial Function

i) End behavior of Polynomial Function.

ii) Find the critical points.

iii) Determine maximum/minimum by using 1st or 2nd derivative test.

iv) Find the infection point and check the concavity

v) Find y intercepts and draw the graph.

⊛ Graph the function,

$$f(x) = 3x^4 - 4x^3 + 2$$

$\Rightarrow$

Here,

End behavior is $f(x) = 3x^4$.

$\therefore$ graph will be up to the left and up to the right.

Now,

$f'(x) = 12x^3 - 12x^2$

For critical points,

$f'(x) = 0$

$12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$x = 0, 1$

Now,

$f''(x) = 36x^2 - 24x$

at $x = 0$

$f''(0) = 0$

i.e. inconclusives neither maximum nor minimum

point is, $x = 0$; $f(0) = 0 - 0 + 2$
$= 2$

at, $x = 1$

$f''(1) = 36 - 24$
$= 12 > 0$

i.e. at $x = 1$ minimum

$x = 1$; $f(1) = 3 - 4 + 2$
$= 1$

$\therefore$ point $(1, 1)$

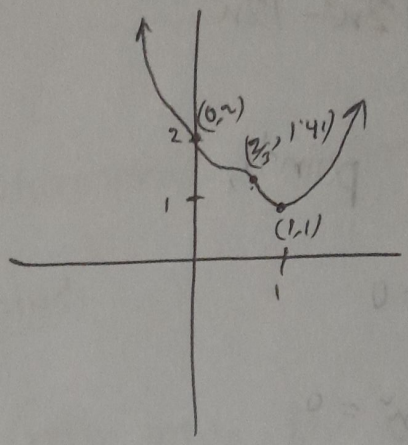
For infection point,

$f''(x) = 0$

$36x^2 - 24x = 0$

$12x(3x-2) = 0$

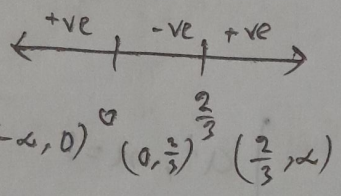
$x = 0, \; x = \frac{2}{3}$



$x = 0, \; f(0) = 2$; point $(0,2)$

$x = \frac{2}{3}, \; f\left(\frac{2}{3}\right) \Rightarrow (0.66, 1.44)$

check concavity by using infection point,



$x = -1$

$f''(-1) = +ve$

$x = \frac{1}{2}$

$f''\left(\frac{1}{2}\right) = -ve$

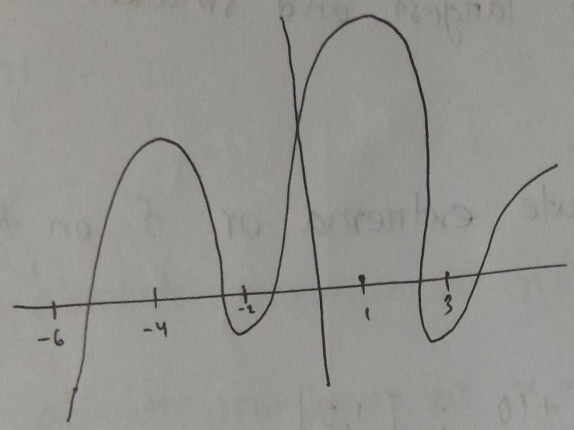
$x = 1$

$f''(1) = +ve$

L-18 / 10.08.2022 /

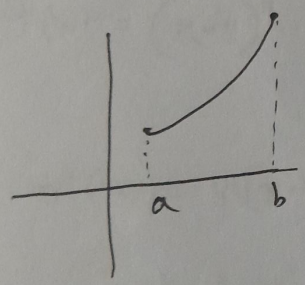
4.4

Absolute maximum and minimum (Absolute extrema)



$\therefore$ absolute maximum, at $x = 1$

value $= f(1)$



$\therefore$ absolute minimum, at $x = -6$

value $= f(-6)$

Here,

absolute maximum $= b$

value $= f(b)$

absolute minimum $= a$

value $= f(a)$

⊛ Finding absolute extrema on finite closed interval. Like $[a,b]$

Step-1: Find the critical points, with in (a,b)

Step-2: Evaluate the functional value for critical points and end points.

Step-3: Check the largest and smallest output (functional value)

⊛ Find the absolute extrema of f on the closed interval.

i) $f(x) = 4x^2 - 12x + 10$; $[1,3]$

$\Rightarrow$

$f'(x) = 8x - 12$

For critical points,

$f'(x) = 0$

$8x - 12 = 0$

$8x = 12$

$x = \frac{12}{8}$

$= \frac{3}{2}$

Here, critical point, $x = \frac{3}{2}$; with in $(1,3)$

Now,

$f(1) = 4-12+10 = 2$

$f(\frac{3}{2}) = 4\cdot(\frac{3}{2})^2 - 12\cdot\frac{3}{2} + 10 = 1$

$f(3) = 10$

Therefore,

absolute maximum at $x=3$ and value 10.

absolute minimum at $x = \frac{3}{2}$ and value 1.

ii) $f(x) = (x-2)^3$; $[1,4]$

$\Rightarrow$

$f'(x) = 3(x-2)^2 \cdot 1$

For critical point,

$f'(x) = 0$

$3(x-2)^2 = 0$

$x = 2$

Here,

critical point $x=2$; with in $(1,4)$

Now,

$f(1) = -1$

$f(2) = 0$

$f(4) = 8$

Therefore,

absolute maximum at $x = 4$, value 8

absolute minimum at $x = 1$, value -1

iii) $f(x) = |x-5|$; $[4,7]$

$$f(x) = \begin{cases} x-5 & ; x \geq 5 \\ -x+5 & ; x < 5 \end{cases}$$

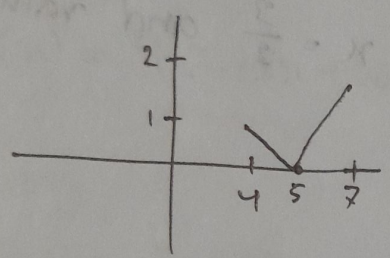
$$f'(x) = \begin{cases} 1 & ; x \geq 5 \\ -1 & ; x < 5 \end{cases}$$

$Lf'(x) \neq Rf'(x)$

$\therefore$ function is not ~~deri~~differentiable

$\therefore x = 5$ is a critical point

Graph,



at, $x = 5$, there is a corner, so, the function $f(x)$ is not differentiable, at $x = 5$.

Therefore,

$x = 5$ is a critical point, with in $(4,7)$

Now,

$f(4) = 1$

$f(5) = 0$

$f(7) = 2$

Therefore

absolute maximum at $x = 7$, value 2

absolute minimum at $x = 5$, value 0

iv) $f(x) = |6 - 4x| \quad ; [-3, 3]$

$\Rightarrow$

$$f(x) = \begin{cases} 6 - 4x & ; \quad 6 - 4x \geq 0 \\ -6 + 4x & ; \quad 6 - 4x < 0 \end{cases}$$

$$= \begin{cases} 6 - 4x & ; \quad x \geq \frac{3}{2} \\ 4x - 6 & ; \quad x < \frac{3}{2} \end{cases}$$

$$f'(x) = \begin{cases} -4 & ; \quad x \geq \frac{3}{2} \\ 4 & ; \quad x < \frac{3}{2} \end{cases}$$

$\therefore \; L f'(x) \neq R f'(x)$

$\therefore$ function is not differentiable at $x = \frac{3}{2}$

Therefore, $x = \frac{3}{2}$ is a critical point, with in $(-3, 3)$

Now,

$f(-3) = 18$

$f(\frac{3}{2}) = 0$

$f(3) = 6$

Therefore,

absolute maximum at $x = -3$, value 18

absolute minimum at $x = \frac{3}{2}$, value 0

v) $f(x) = \sin x - \cos x$; $[0, \pi]$

$\Rightarrow$

$f'(x) = \cos x + \sin x$

For critical point,

$f'(x) = 0$

$\sin x + \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$

Therefore,

critical point $x = \frac{3\pi}{4}$ within $(0, \pi)$

Now,

$f(0) = -1$

$f\left(\frac{3\pi}{4}\right) = \sqrt{2}$

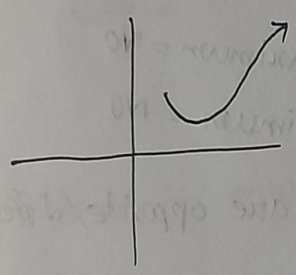
$f(\pi) = 1$

Therefore,

absolute maximum at $x = \frac{3\pi}{4}$, value $\sqrt{2}$

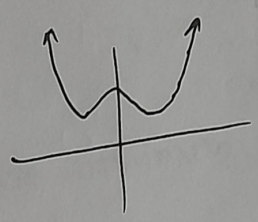
absolute minimum at $x = 0$, value -1.

⊛ Finding absolute extrema with in infinite $(-\alpha, \alpha)$ intervals.



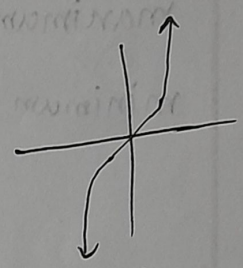
absolute maximum = NO

absolute minimum = YES



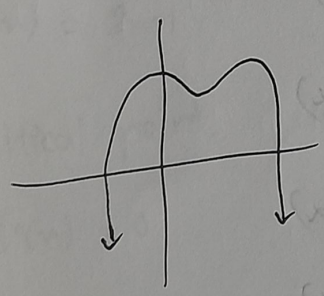
maximum = NO

minimum = YES



maximum = NO

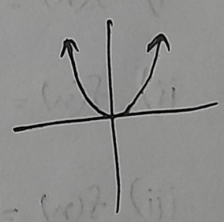
minimum = NO



maximum = YES

minimum = NO

⊛



if, $\lim_{x \to +\alpha} f(x) = +\alpha$

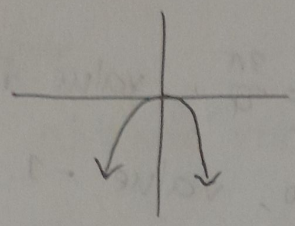
and, $\lim_{x \to -\alpha} f(x) = +\alpha$

Then,

maximum = NO

minimum = YES

(*)



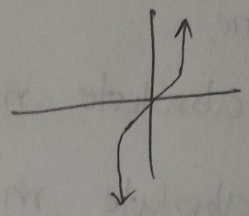
if, $\lim_{x \to +\alpha} f(x) = -\alpha$

$\lim_{x \to -\alpha} f(x) = -\alpha$

maximum = YES

minimum = NO

(*)



if,

$\lim_{x \to +\alpha} f(x) = +\alpha$

$\lim_{x \to -\alpha} f(x) = -\alpha$

then,

maximum = NO

minimum = NO

they are opposite/different

(*) Find the absolute extrema

i) $f(x) = x^2 - x - 2$; $(-\alpha, \alpha)$

ii) $f(x) = 4x^3 - 3x^4$; $(-\alpha, \alpha)$

iii) $f(x) = 2x^3 - 6x + 2$; $(-\alpha, \alpha)$

iv) $f(x) = \dfrac{x^2+1}{x+1}$; $(-5, -1)$

Solution

i)

$f(x) = x^2 - x - 2$

Here,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2$$

$$= +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2$$

$$= +\infty$$

$\therefore$ there is only absolute minimum.

Here, absolute minimum could be occure at critical point.

$f'(x) = 2x - 1$

For critical point,

$f'(x) = 0$

$2x - 1 = 0$

$x = \frac{1}{2}$

$$\therefore f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$$

$$= \frac{1}{4} - \frac{1}{2} - 2$$

$$= \frac{1 - 2 - 8}{4} = \frac{-9}{4}$$

So,

absolute minimum at, $x = \frac{1}{2}$; value $\frac{-9}{4}$.

ii) $f(x) = 4x^3 - 3x^4$

$\Rightarrow$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -3x^4$$

$$= -\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -3x^4$$

$$= -\infty$$

Therefore,

no absolute minimum. Only absolute maximum here and could be occure at critical point.

$\therefore f'(x) = 12x^2 - 12x^3$

For critical point,

$f'(x) = 0$

$12x^2 - 12x^3 = 0$

$12x^2(1-x) = 0$

$x = 0, 1$

$\therefore f(0) = 0$

$\therefore f(1) = 4 - 3$

$= 1$

$\therefore$ absolute maximum at $x = 1$, value $= 1$.

iii)

No maximum.

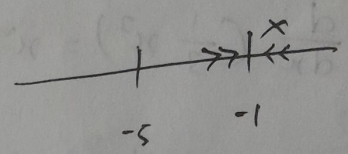
No minimum.

4.1

From trigonometric function, find out increasing, decreasing and concavity.

iv)

$$f(x) = \frac{x^2+1}{x+1} \quad ; \quad (-5, -1)$$



$$\therefore \lim_{x \to -1} \frac{x^2+1}{x+1} = -\infty$$

Therefore,

no absolute minimum, only absolute maximum here and could be occure at critical point

$$f'(x) = \frac{(x+1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$=$$

L-19/ 19.08.2023

5.2

Indefinite Integrals

Antiderivative:

If $\frac{d}{dx}[F(x)] = f(x)$

then,

$F(x)$ is antiderivative of $f(x)$

⊛ $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$

Antiderivative of x^2 is $\frac{1}{3}x^3$

⊛ $\frac{d}{dx}(\sin x) = \cos x$

Antiderivative of $\cos x$ is $\sin x$.

⊛ $\frac{d}{dx}\left(\frac{1}{3}x^3 - 10\right) = x^2$

$\frac{d}{dx}\left(\frac{1}{3}x^3 - \pi\right) = x^2$

$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2$

$\frac{d}{dx}[F(x)] = f(x)$

⊛ <u>Indefinite Integration</u>: Process of finding antiderivative or antidifferentiation is called indefinite integration.

$$\int f(x) \cdot dx = F(x) + C$$

Here, c is arbitary constant.

$$\frac{d}{dx}\left[F(x)+C\right] = f(x)$$

⊛ $\frac{d}{dx}(x) = 1 \Rightarrow \int 1 \cdot dx = x + C$

⊛ $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \; n \neq -1 \Rightarrow \int x^n \cdot dx = \frac{x^{n+1}}{n+1} + C$

⊛ $\int x^2 \, dx = \frac{x^{2+1}}{2+1} + C = \frac{1}{3}x^3 + C$

⊛ $\int x^{-\frac{1}{2}} \, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2x^{\frac{1}{2}} + C$

⊛ $\int x^9 \, dx = \frac{x^{10}}{10} + C$

[Some Rules from Sheet]

Properties:

1. $\int c\, f(x)\, dx = c \int f(x)\, dx$

2. $\int (f(x) + g(x) + h(x)) = \int f(x)\, dx + \int g(x)\, dx + \int h(x)\, dx$

$$= F(x) + G(x) + H(x) + c$$

(Same for negatives.)

Find the following integrals,

i) $\int x(1+x^3)\, dx$ ii) $\int (2+y^2)^2\, dy$

iii) $\int \sec x (\sec x + \tan x)\, dx$ iv) $\int \frac{\sec\theta}{\cos\theta}\, d\theta$

v) $\int \left(\frac{1}{2t} - \sqrt{2}\, e^t\right) dt$

Solution

i) $\int x(1+x^3)\, dx$

$= \int (x + x^4)\, dx$

$= \int x\, dx + \int x^4\, dx$

$= \frac{x^2}{2} + \frac{x^5}{5} + c$

ii) $\int (2+y^2)^2\, dy$

$= \int (4 + 4y^2 + y^4)\, dy$

$= \int 4\, dy + \int 4y^2\, dy + \int y^4\, dy$

$= 4y + \frac{4 \cdot y^3}{3} + \frac{y^5}{5} + c$

iii) $\int (\sec^2 x + \sec x \tan x)\, dx$

$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$

$= \tan x + \sec x + c$

iv) $\int \sec\theta \cdot \sec\theta\, d\theta$

$= \int \sec^2\theta\, d\theta$

$= \tan\theta + c$

v) $\int \frac{1}{2t}\, dt - \sqrt{2} \int e^t\, dt$

$= \frac{1}{2} \ln t - \sqrt{2}\, e^t + c$

(*) $\int \frac{3}{\sqrt{1-x^2}}\, dx$

$= 3 \sin^{-1} x + c$

(*) Indefinite integrals from the view point of Differential equation:

(*) Differential equation: If an equation involves a derivative then this equation is called differential equation.

$\frac{dy}{dx} = x^2$

$\frac{dy}{dx} - 2x = 0$

$\frac{ds}{dt} - 3 = 0$

(*) $\frac{dy}{dx} = x^2$ ... (1)

$\boxed{y = \frac{1}{3}x^3 + c}$ is a solution of (1)

Solution is not a number. its a function family of a function/curve.

(*) $\frac{dy}{dx} = x^2$ → Passing through (2,1)

$y = \int x^2 dx$

$y = \frac{1}{3}x^3 + c$

$1 = \frac{1}{3} \cdot 2^3 + c$

$c = -\frac{5}{3}$

$\therefore y = \frac{1}{3}x^3 - \frac{5}{3}$

(*) $\frac{dy}{dx} - 2x = 0$

$\frac{dy}{dx} = 2x$

$y = \int 2x\, dx$

$\therefore y = x^2 + c$

(*) $\frac{ds}{dt} - 3 = 0$

$\frac{ds}{dt} = 3$

$s = \int 3\, dt$

$\therefore s = 3t + c$

(*) $\frac{dy}{dx} = f(x)$; $y(x_0) = y_0$ → $x_0 = y_0$

* $\frac{dy}{dx} = f(x)$; $y(x_0) = y_0$

this type problem is called initial value problem (IVP).

* Solve the initial value problem.

i) $\frac{dy}{dt} = \sec^2 t - \sin t$; $y\left(\frac{\pi}{4}\right) = 1$

ii) $\frac{dy}{dx} = \frac{x^2-1}{x^2+1}$; $y(1) = \frac{\pi}{2}$

Solution

i) $\frac{dy}{dt} = \sec^2 t - \sin t$

$\therefore dy = (\sec^2 t - \sin t)\, dt$

$\int dy = \int (\sec^2 t - \sin t)\, dt$

$y = \tan t + \cos t + c$

Now,

$1 = \tan\frac{\pi}{4} + \cos\frac{\pi}{4} + c$

$\therefore c = -\frac{1}{\sqrt{2}}$

ii) $\frac{dy}{dx} = \frac{x^2-1}{x^2+1}$

$dy = \frac{x^2-1}{x^2+1} dx$

$\int dy = \int \frac{x^2-1}{x^2+1} dx$

$y = \int \left(1 - \frac{2}{x^2+1}\right) dx$

$y = \int 1 \, dx - \int \frac{2}{x^2+1} dx$

$y = x - 2 \tan^{-1} x + c$

Now,

if $x = 1$

$y = \frac{\pi}{2}$

$\therefore \frac{\pi}{2} = 1 - 2 \tan^{-1} 1 + c$

$\frac{\pi}{2} = 1 - 2 \cdot \frac{\pi}{4} + c$

$\therefore c = \pi - 1$

$\therefore y = x - 2 \tan^{-1} x + \pi - 1$

⊛ A particle moves along an s axis with position function $s = s(t)$ and $v(t) = s'(t)$. Then $s(t)$ from the followings.

i) $v(t) = 32t$; $s(0) = 20$

ii) $v(t) = 3\sqrt{t}$; $s(4) = 1$

Solutions

i) Here,

$$v(t) = 32t$$

$$\frac{ds}{dt} = 32t$$

$$\therefore s(t) = \int 32t \, dt$$

$$= 32 \cdot \frac{t^2}{2} + c$$

$$s(t) = 16t^2 + c$$

Now, $s(0) = 20$

$t = 0$, then $s = 20$

$$20 = 16 \cdot 0 + c$$

$$c = 20 \qquad \therefore s(t) = 16t^2 + 20$$

5.3

Indefinite Integrals by using method of substitution:

(Undo of chain rule)

⊛ $\frac{d}{dx}(x^2+3x+2)^5$

$= 5.(x^2+3x+2)^4.(2x+3)$

⊛ $\int 5(x^2+3x+2)^4(2x+3) = (x^2+3x+2)^5 + c$

Let,

$u = x^2+3x+2$

$\frac{du}{dx} = 2x+3$

$du = (2x+3)\,dx$

$\therefore \int 5.u^4.du$

$= 5\int u^4.du$

$= 5.\frac{u^5}{5} + c$

$= u^5 + c$

$= (x^2+3x+2)^5 + c$

(*) $\int \sin 5x \, dx$

Let,

$u = 5x$

$\frac{du}{dx} = 5$

$du = 5\,dx$

$dx = \frac{1}{5} du$

$= \int \sin u \, \frac{1}{5} du$

$= \frac{1}{5} \int \sin u \, du$

$= \frac{1}{5} (-\cos u) + c$

$= -\frac{1}{5} \cos 5x + c$

(*) $\int \sin^2 x \cos x \, dx$

Let,

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$= \int u^2 du$

$= \frac{u^3}{3} + c$

$= \frac{\sin^3 x}{3} + c$

(*) $\int e^{\sin x} \cos x \, dx$ Let

(*) $\int (\sec 4x \tan 4x) dx$

Let,

$u = 4x$

$\frac{du}{dx} = 4$

$du = 4\,dx$

$dx = \frac{1}{4} du$

$= \int \sec u \tan u \, \frac{1}{4} du$

$= \frac{1}{4} \int \sec u \tan u \, du$

$= \frac{1}{4} \sec u + c$

$= \frac{1}{4} \sec 4x + c$

(*) $\int \frac{1}{1+16x^2}\,dx$

$= \int \frac{1}{1+(4x)^2}\,dx$

Let,

$u = 4x$

$\frac{du}{dx} = 4$

$du = 4\,dx$

$dx = \frac{1}{4}\,du$

$= \int \frac{1}{1+u^2}\,\frac{1}{4}\,du$

$= \frac{1}{4}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{4}\tan^{-1} u + c$

$= \frac{1}{4}\tan^{-1} 4x + c$

(*) Evaluate,

$\int \frac{dx}{a^2+x^2} \quad a \neq 0$

$= \int \frac{dx}{a^2\left(1+\frac{x^2}{a^2}\right)}$

$= \frac{1}{a^2}\int \frac{dx}{1+\frac{x^2}{a^2}}$

Let, $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx$

$a\,du = dx$

$= \frac{1}{a^2}\int \frac{a}{1+u^2}\,du$

$= \frac{1}{a}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{a}\tan^{-1} u + c$

$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Then,

Formula:

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$\int \frac{dx}{16+x^2} = \int \frac{dx}{4^2+x^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$$

$$\int \frac{dx}{16+9x^2} = \int \frac{dx}{9\left(\frac{16}{9}+x^2\right)} = \frac{1}{9}\int \frac{dx}{\left(\frac{4}{3}\right)^2+x^2}$$

$$= \frac{1}{9}\cdot\frac{3}{4}\cdot\tan^{-1}\frac{x}{4/3} + c$$

$$= \frac{1}{12}\tan^{-1}\frac{3x}{4} + c$$

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$$

$$\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = \sin^{-1}\frac{x}{3} + c$$

(*) $\int \frac{\cos 4\theta}{(1+2\sin 4\theta)} d\theta$

Let,

$u = 1 + 2\sin 4\theta$

$\frac{du}{d\theta} = 2\cos 4\theta \cdot 4 = 8\cos 4\theta$

$du = 8\cos 4\theta \, d\theta$

$\frac{1}{8} du = \cos 4\theta \, d\theta$

$= \int \frac{\frac{1}{8} du}{u}$

$= \frac{1}{8} \ln u + c$

$= \frac{1}{8} \ln(1+2\sin 4\theta) + c$

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5.3

(*) $\int \frac{dx}{a^2+x^2}$

$= \int \frac{1}{a^2(1+\frac{x^2}{a^2})} dx$

$= \frac{1}{a^2} \int \frac{1}{1+(\frac{x}{a})^2} dx$

Let,

$u = \frac{x}{a}$

$\frac{du}{dx} = \frac{1}{a}$

$dx = a\,du$

$= \frac{1}{a^2} \int \frac{a\,du}{1+u^2}$

$= \frac{1}{a} \int \frac{1}{1+u^2} du$

$= \frac{1}{a} \tan^{-1} u + c$

$= \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

(*) $\int \frac{dx}{\sqrt{a^2-x^2}}$

$= \int \frac{dx}{\sqrt{a^2(1-\frac{x^2}{a^2})}}$

$= \frac{1}{a} \int \frac{dx}{\sqrt{1-(\frac{x}{a})^2}}$

Let,

$u = \frac{x}{a}$

$\frac{du}{dx} = \frac{1}{a}$

$dx = a\,du$

$$= \frac{1}{a}\int \frac{a\,du}{\sqrt{1-u^2}}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1}\frac{x}{a} + c$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$$

Find the followings.

i) $\int \frac{dx}{5+x^2}$

$$= \int \frac{dx}{(\sqrt{5})^2+x^2}$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{\sqrt{5}} + c$$

ii) $\int \frac{dx}{x\sqrt{x^2-7}}$

$$= \int \frac{dx}{x\sqrt{x^2-7}}$$

$$= \int \frac{dx}{x\sqrt{x^2-(\sqrt{7})^2}}$$

$$= \frac{1}{\sqrt{7}}\sec^{-1}\frac{x}{\sqrt{7}} + c$$

iii) $\int \frac{dy}{y\sqrt{5y^2-3}}$

$= \int \frac{dy}{y\sqrt{5(y^2-\frac{3}{5})}}$

$= \frac{1}{\sqrt{5}} \int \frac{dy}{y\sqrt{y^2-(\frac{\sqrt{3}}{\sqrt{5}})^2}}$

$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\frac{\sqrt{3}}{\sqrt{5}}} \cdot \sec^{-1} \frac{y}{\frac{\sqrt{3}}{\sqrt{5}}} + C$

$= \frac{1}{\sqrt{3}} \sec^{-1} \frac{\sqrt{5}\, y}{\sqrt{3}} + C$

iv) $\int \frac{dx}{\sqrt{9-4x^2}}$

$= \int \frac{dx}{\sqrt{4(\frac{9}{4}-x^2)}}$

$= \frac{1}{2} \int \frac{dx}{(\frac{3}{2})^2 - x^2}$

$= \frac{1}{2} \cdot \sin^{-1} \frac{x}{3/2} + C$

$= \frac{1}{2} \sin^{-1} \frac{2x}{3} + C$

(*) $\int e^{-x}\, dx$

Let, $u = -x$

$\frac{du}{dx} = -1$

$dx = -du$

$\therefore \int e^u (-du)$

$= -\int e^u\, du$

$= -e^u$

$= -e^{-x} + C$

⊛ $\int x\sqrt{4-x}\,dx$

Let,

$u = 4-x$

$\frac{du}{dx} = -1$

$dx = -du$

$\therefore$

$= \int (4-u)\cdot\sqrt{u}\,(-du)$

$= -\int (4u^{\frac{1}{2}} - u^{3/2})\,du$

$= -\int 4u^{\frac{1}{2}}\,du - \int u^{3/2}\,du$

$= -4\cdot\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$

$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c$

$= -\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + c$

⊛ Solve the followings initial value problem (IVP).

i) $\frac{dy}{dx} = \sqrt{5x+1}$; $y(3) = -2$

$\Rightarrow$

$\frac{dy}{dx} = \sqrt{5x+1}$

$dy = \sqrt{5x+1}\,dx$

$\int dy = \int \sqrt{5x+1}\,dx$

$y = \int \frac{\cancel{x}}{\sqrt{5x+1}}\,dx$

Let,

$u = 5x+1$

$\frac{du}{dx} = 5$

$dx = \frac{1}{5}du$

$$= \int \sqrt{u} \cdot \frac{1}{5} \cdot du$$

$$= \frac{1}{5} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{5} \cdot \frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{15} u^{3/2} + c$$

$$\therefore y = \frac{2}{15} (5x+1)^{3/2} + c$$

Now,

if, $x = 3$

$y = -2$

$$\therefore -2 = \frac{1}{15} (5 \cdot 3 + 1)^{3/2} + c$$

$$-2 = \frac{2}{15} (16)^{3/2} + c$$

$$c = -\frac{158}{15}$$

Hence, the particular solution.

$$\therefore y = \frac{2}{15} (5x+1)^{3/2} - \frac{158}{15}$$

ii) $\frac{dy}{dt} = \frac{1}{25+9t^2}$; $y(-\frac{5}{3}) = \frac{\pi}{30}$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{25+9t^2}$$

$$dy = \frac{1}{25+9t^2}\, dt$$

$$\int dy = \int \frac{1}{9(\frac{25}{9}+t^2)}\, dt$$

$$y = \frac{1}{9}\int \frac{1}{(\frac{5}{3})^2+t^2}\, dt$$

$$= \frac{1}{9}\cdot\frac{1}{5/3}\tan^{-1}\frac{t}{5/3} + c$$

$$y = \frac{1}{15}\tan^{-1}\frac{3t}{5} + c$$

Now,

if $t = -\frac{5}{3}$

$y = \frac{\pi}{30}$

$$\therefore \frac{\pi}{30} = \frac{1}{15}\cdot\tan^{-1}\frac{3(-\frac{5}{3})}{5} + c$$

$$\therefore \frac{\pi}{30} = \frac{1}{15}\cdot\tan^{-1}(-1) + c$$

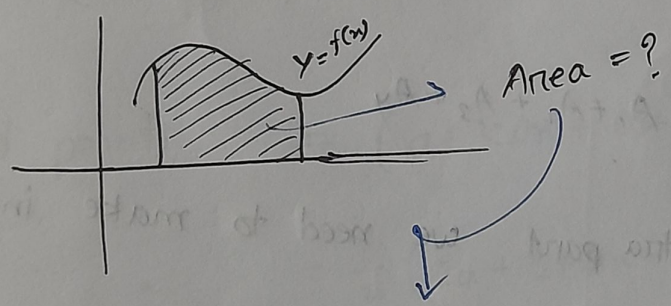
$$\therefore \frac{\pi}{30} = \frac{1}{15}\left(-\frac{\pi}{4}\right) + c$$

$$\therefore C = \frac{\pi}{30} + \frac{\pi}{60} = \frac{\pi}{20}$$

$$\therefore y = \frac{1}{15}\tan^{-1}\frac{3t}{5} + \frac{\pi}{20}$$

5.4

Finding Area Under the Curve



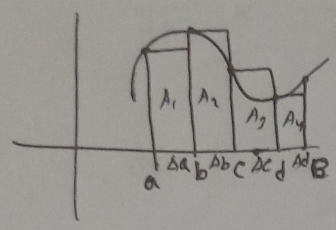
There are two methods for finding area under the curve.

i) Rectangle method

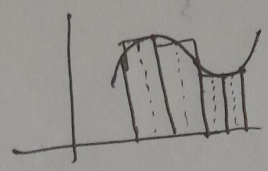
ii) Anti derivative method

i) Rectangle method.

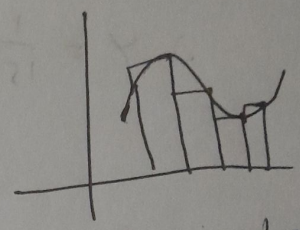
⇒



Left end points



mid points



Right end points

$A_1 = \Delta a \times f(a)$

$A_2 = \Delta b \times f(b)$

$A_3 = \Delta c \times f(c)$

$A_4 = \Delta d \times f(d)$

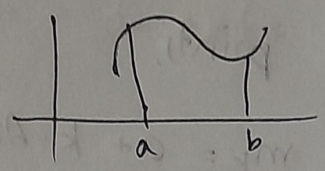
∴ Total Area $= A_1 + A_2 + A_3 + A_4$

∴ To remove extra part we need to make infinite sub intervalls. Then we can get actual area.

If, you make nth subintervals,

and $n = \alpha$

$$\text{Actual area} = \lim_{n \to \alpha} (A_1 + A_2 + A_3 + \cdots + A_n)$$



$\therefore$ n sub intervals

$\therefore$ each intervals length,

$$\Delta x = \frac{b-a}{n}$$

$\therefore$ Right end points $= a + k \cdot \Delta x$; $k = 1, 2, 3, 4 \ldots$

$\therefore$ Left end points $= a + (k-1)\Delta x$; $k = 1, 2, 3, 4 \ldots$

$\therefore$ Mid points $= (a + k\Delta x + a + (k-1)\Delta x)$

$= \frac{1}{2}(2a + 2k\Delta x - \Delta x)$

$= a + (k - \frac{1}{2})\Delta x$; $k = 1, 2, 3, \ldots$

$\therefore$ Finally, area,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

If m_k is left ends points

$m_k = a + (k-1)\Delta x$

∴ Right end points,

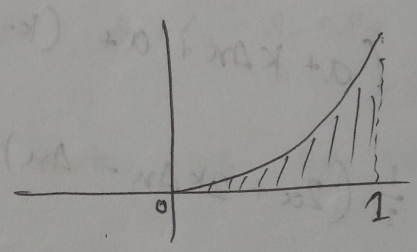
$$m_k = a + k \cdot \Delta x$$

∴ Mid point,

$$m_k = a + \left(k - \frac{1}{2}\right) \Delta x$$

(*) Find the area, $f(x) = x^2$ on the interval $[0, 1]$ with right end points.

⇒



Now, we need to make n subintervals,

∴ each sub intervals length,

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Now, point, $m_k = a + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$

Now,

Area,

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n}$$

Here,

$f(x) = x^2$

$\therefore f\left(\frac{k}{n}\right) = \frac{k^2}{n^2}$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left[1^2 + 2^2 + 3^2 + \dots + n^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n \cdot n\left(1+\frac{1}{n}\right) \cdot n\left(2+\frac{1}{n}\right)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}(1+0)(2+0)$$

$$= \frac{1 \cdot 2}{6} = \frac{1}{3}$$

$\therefore$ Area $= \frac{1}{3}$

* Find the area, $f(x) = \frac{x}{2}$ on the interval $[1,4]$ with left end points.

$\Rightarrow$

We need to make n sub intervals

$\therefore$ each sub interval length,

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

points, $m_k = a + (k-1)\Delta x = 1 + (k-1)\Delta x = 1 + (k-1)\frac{3}{n}$

Now, area,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f\left(1 + (k-1)\frac{3}{n}\right) \frac{3}{n}$$

Here,

$$f(x) = \frac{x}{2}$$

$$\therefore f\left(1+(k-1)\frac{3}{n}\right) = \frac{1+(k-1)\frac{3}{n}}{2}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(1+(k-1)\frac{3}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\cdot\frac{3}{n} + \frac{1}{2}(k-1)\frac{3}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{2n}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}(n+3k-3)$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[\sum_{k=1}^{n} n + \sum_{k=1}^{n} 3k - 3\sum_{k=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[n\sum_{k=1}^{n} 1 + 3\sum_{k=1}^{n} k - 3\sum_{k=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[n^2 + \frac{3}{2}\cdot\frac{n(n+1)}{2} - 3n\right]$$

$$= \lim_{n\to\infty} \frac{3}{2}\left[1 + \frac{3}{2}\left(1+\frac{1}{n}\right) - 3\frac{1}{n}\right]$$

$$= \frac{3}{2}\left[1 + \frac{3}{2}(1+0) - 0\right]$$

$$= \frac{3}{2}\left[1+\frac{3}{2}\right] = \frac{3}{2}\cdot\frac{5}{2} = \frac{15}{4}$$

L-21 / 24.08.2024

5.5

The Definite Integral

Area under the curve by rectangle method,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x \quad [\text{Riemann Sum}]$$

$n \to$ num of sub interval

$\Delta x \to$ width (each interval length)

if,

$n \to \infty$

$\Delta x \to 0$

$$A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

Definite integral:

$$\int_{x=a}^{b} f(x) \cdot dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x = \text{area under the } \cancel{\text{sum}} \text{ curve}$$

Def^n: A function $f(x)$ is said to be Integrable on finite interval $[a,b]$.

if the limit $\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$ exists, then we denote the limit by the symbol,

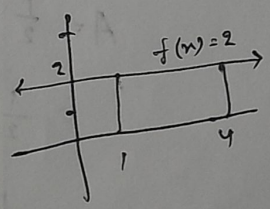
$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

Theorem: If a function f is continuous on $[a,b]$, then f is integrable on $[a,b]$ and then net signed area between the graph of f and interval $[a,b]$ is

$$A = \int_a^b f(x)\,dx.$$

(*) Find the followings, definite integrals using appropriate formula from geometry.

a) $\int_1^4 2\,dx$



Here, $f(x) = 2$

$a = 1$

$b = 4$

This is rectangle

$\therefore$ area = length $\times$ width

$= (4-1) \times 2$

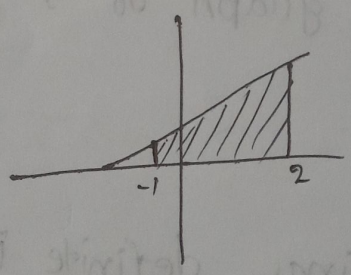
$= 3 \times 2$

$= 6$

Therefore, $\int_1^4 2\, dx = 6$ Ans.

ii) $\int_{-1}^{2} (x+2)\, dx$

Here, $f(x) = x+2$; $[-1, 2]$



This is trapezoid,

We know that,

$A = \frac{1}{2}(b+b')h$

$= \frac{1}{2}(1+4)3$

$= \frac{1}{2} \cdot 5 \cdot 3$

$= \frac{15}{2}$

$b = f(-1) = 1$

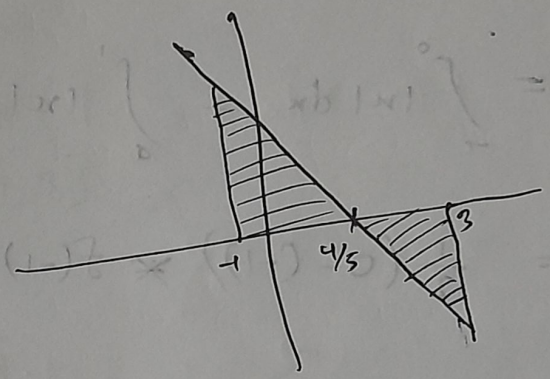
$b' = f(2) = 4$

$h = (2-(-1)) = 3$

$$\therefore \int_{-1}^{2} (x+2)\,dx = \frac{15}{2}$$

iii) $\int_{-1}^{3} (4-5x)\,dx$

Here, $f(x) = 4-5x$



$$= \int_{-1}^{4/5} (4-5x)\,dx + \int_{4/5}^{3} (4-5x)\,dx$$

$$= \frac{1}{2} \times b \times h + \frac{1}{2} \times b \times h$$

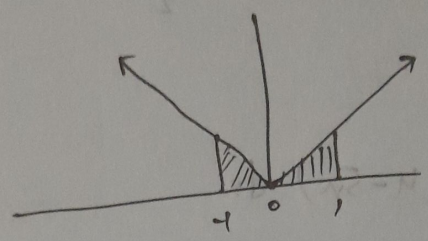
$$= \frac{1}{2} \times \left(\frac{4}{5}+1\right) \times f(-1) + \frac{1}{2} \times \left(3-\frac{4}{5}\right) \times f(3)$$

$$= \frac{1}{2} \times \frac{9}{5} \times 9 + \frac{1}{2} \times \frac{11}{5} \times (-11)$$

$$= \frac{81}{10} - \frac{121}{10}$$

$$= \frac{-40}{10} = -4$$

iv) $\int_{-1}^{1} |x|\, dx$



$$= \int_{-1}^{0} |x|\, dx + \int_{0}^{1} |x|\, dx$$

$$= \frac{1}{2}(0-(-1)) \times f(-1) + \frac{1}{2} \times (1-0) \times f(1)$$

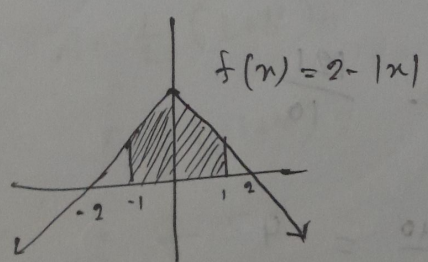
$$= \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 1 \times 1$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

$$\therefore \int_{-1}^{1} |x|\, dx = 1$$

v) $\int_{-1}^{1} (2-|x|)\, dx \; ; \; [-1, 1]$



$$= \int_{-1}^{0} (2-|x|)\, dx + \int_{0}^{1} (2-|x|)\, dx$$

$= \frac{1}{2}(b+b')h + \frac{1}{2}(b+b')h$

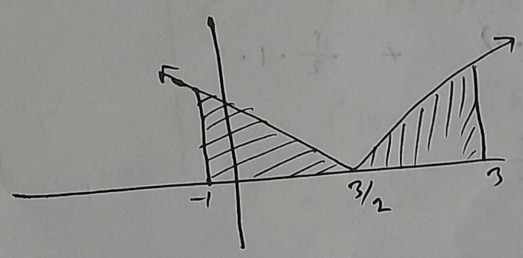
$= \frac{1}{2}\left(f(-1)+f(0)\right)\cdot(0+1) + \frac{1}{2}\left(f(0)+f(1)\right)(1-0)$

$= \frac{1}{2}(1+2)\cdot 1 + \frac{1}{2}(2+1)1$

$= \frac{3}{2} + \frac{3}{2}$

$= 3$ Ans

vi) $\int_{-1}^{2} |2x-3|\, dx \; ; \; [-1, 3]$



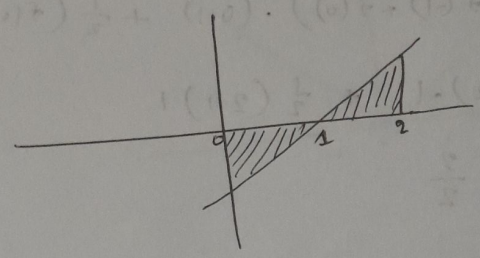
$= \int_{-1}^{3/2} |2x-3|\, dx + \int_{3/2}^{3} |2x-3|\, dx$

$=$ Do yourself

(*) $\int_{0}^{3} |x-2|\, dx$

$=$ Do yourself

⊛ $\int_0^2 (x-1)\,dx$



$$= \int_0^1 (x-1)\,dx + \int_1^2 (x-1)\,dx$$

$$= \frac{1}{2} \times (1-0) \cdot f(0) + \frac{1}{2}(2-1) f(2)$$

$$= \frac{1}{2} \times 1 \times (-1) + \frac{1}{2} \cdot 1 \cdot 1$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0$$

⊛ <u>Example :</u>

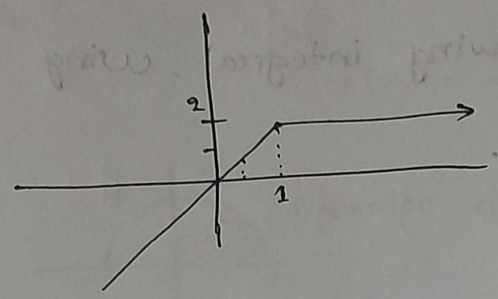
If $f(x) = \begin{cases} 2x & x \leq 1 \\ 2 & x > 1 \end{cases}$

then find,

a) $\int_0^1 f(x)\,dx$ b) $\int_{-1}^1 f(x)\,dx$ c) $\int_1^{10} f(x)\,dx$

d) $\int_{\frac{1}{2}}^5 f(x)\,dx$

$\Rightarrow$



a) $\int_0^1 f(x)\,dx$

$= \frac{1}{2} \times \text{base} \times h$

$= \frac{1}{2}(1-0) \times f(1)$

$= \frac{1}{2} \times 1 \times 2$

$= 1$

b) $\int_{-1}^{1} f(x)\,dx$

$= 0,$ (by symmetric)

c) $\int_1^{10} f(x)\,dx.$

$= (10-1) \times f(1)$

$= 9 \times 2$

$= 18$ Ans

d) $\int_{\frac{1}{2}}^{5} f(x)\,dx$

$= \int_{\frac{1}{2}}^{1} f(x)\,dx + \int_1^5 f(x)\,dx$

$= \frac{1}{2}(b+b')h + (5-1) \times 2$

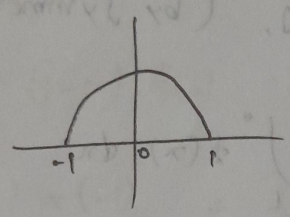
$= \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1)\right) \times \frac{1}{2} + 8$

$= \frac{1}{2} \times \frac{1}{2} \times 3 + 8 = \frac{3}{4} + 8 = \frac{3+32}{4} = \frac{35}{4}$ Ans

Evaluate following integral, using appropriate formula from geometry.

i) $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$$y = +\sqrt{1-x^2}$$



This is upper of the circle whose center at (0,0) and radious 1.

$= \frac{1}{2} \pi r^2$

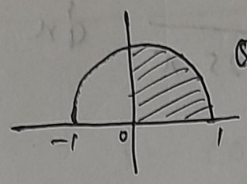
$= \frac{1}{2} \times \pi \times 1$

$= \frac{\pi}{2}$

$\therefore \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$.

ii) $\int_0^1 \sqrt{1-x^2}\, dx$

$f(x) = +\sqrt{1-x^2}$



Quanten cincle

$= \frac{1}{4} \pi r^2$

$= \frac{\pi}{4}$ Ans

$\therefore \int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$

ii) $\int_0^2 \sqrt{4-x^2}\, dx = \frac{1}{4} \pi r^2$

$= \frac{1}{4} \times \pi \times 2^2$

$= \pi$ Ans

$f(x) = \sqrt{4-x^2}$

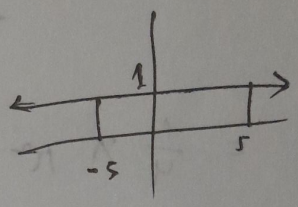
upper quanten of cincle

centre, $(0,0)$

radiow $= 2$

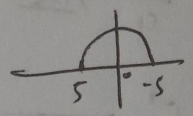
iv) $\int_{-5}^{5} \left(1+\sqrt{25-x^2}\right) dx$

$= \int_{-5}^{5} 1\, dx + \int_{-5}^{5} \sqrt{25-x^2}\, dx$



$= (5+5)\times 1 + \frac{1}{2}\times\pi r^2$

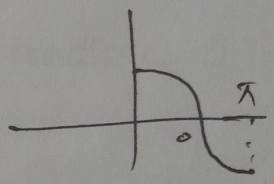
$= 10 + \frac{1}{2}\times\pi\times 25$



$= 10 + \frac{25\pi}{2}$ Ans

v) $\int_{0}^{\pi} \cos x\, dx$

$= 0$ (symmetric)



5.6

Fundamental Theorem of PART-I

Theorem: If f is continuous on $[a,b]$ and F is antiderivative of f on $[a,b]$ then.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$$\int_a^b f(x)\,dx = [F(x)+c]_a^b$$

$$= (F(b)+c) - (F(a)+c)$$

$$= F(b)+c - F(a) - c$$

$$= F(b) - F(a)$$

$$\int_1^4 2\,dx = (4-1)\times 2 = 6$$

$$\int_1^4 2\,dx = 2x\Big]_1^4$$

$$= (2\times 4) - (2\times 1)$$

$$= 8-2 = 6 \text{ Ans}$$

⊛ $\int_1^2 x^2 \, dx$

$= \left. \frac{x^3}{3} \right]_1^2$

$= \frac{2^3}{3} - \frac{1^3}{3}$

$= \frac{8-1}{3}$

$= \frac{7}{3}$ Ans

⊛ Properties :

i) $\int_a^a f(x) \, dx = 0$

ii) $\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx$

iii) $\int_a^b c f(x) \, dx = c \int_a^b f(x) \, dx$

iv) $\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

v) $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_a^b f(x) \, dx$; where c, within $[a, b]$

(*) Find the following using Fundamental theorem of calculus

Part-I

i) $\int_1^4 \frac{4}{x^2}\,dx$ ii) $\int_{-\frac{\pi}{4}}^{\pi/4} \cos x\,dx$ iii) $\int_{\ln 2}^{3} 5e^x\,dx$

iv) $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$ v) $\int_{-1}^{1} |2x-1|\,dx$ vi)

Solution

i) $\int_1^4 \frac{4}{x^2}\,dx$

$= 4\int_1^4 \frac{1}{x^2}\,dx$

$= 4 \cdot \left.\frac{x^{-1}}{-1}\right]_1^4$

$= -\left.\frac{4}{x}\right]_1^4$

$= -\frac{4}{4} + \frac{4}{1}$

$= -1 + 4 = 3$ Ans

ii) $\left.\sin x\right]_{-\frac{\pi}{4}}^{\pi/4}$

$= \sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$ Ans

iii) $\left.5e^x\right]_{\ln 2}^{3}$

$= 5e^3 - 5e^{\ln 2}$

$= 5e^3 - 10$ Ans

iv) $\sin^{-1} x \Big]_0^{1/\sqrt{2}}$

$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ Ans

v) $f(x) = |2x-1|$

$= \begin{cases} 2x-1 & 2x-1 \geq 0 \\ -2x+1 & 2x-1 < 0 \end{cases}$

$= \begin{cases} 2x-1 & x \geq \frac{1}{2} \\ -2x+1 & x < \frac{1}{2} \end{cases}$

$\therefore \int_{-1}^{1} |2x-1|\, dx$

$= \int_{-1}^{\frac{1}{2}} |2x-1|\, dx + \int_{\frac{1}{2}}^{1} |2x-1|\, dx$

$= \int_{-1}^{\frac{1}{2}} (-2x+1)\, dx + \int_{\frac{1}{2}}^{1} (2x-1)\, dx$

$= \left[-2\frac{x^2}{2} + x\right]_{-1}^{\frac{1}{2}} + \left[2\cdot\frac{x^2}{2} - x\right]_{\frac{1}{2}}^{1}$

$= -x^2 + x\Big]_{-1}^{\frac{1}{2}} + x^2 - x\Big]_{\frac{1}{2}}^{1}$

$= \frac{5}{2}$ Ans

vi) $\int_1^4 \frac{|2-x|}{x}\, dx$

Here, $f(x) = \frac{|2-x|}{x}$

$= \begin{cases} \frac{2-x}{x} & 2-x \geq 0 \\ \frac{-2+x}{x} & 2-x < 0 \end{cases}$

$$= \begin{cases} \frac{2-x}{x} & 2 \geq x \,/\, x \leq 2 \\ \frac{x-2}{x} & 2 < x \,/\, x > 2 \end{cases}$$

$$= \int_1^2 \frac{2-x}{x}\, dx + \int_2^4 \frac{-2+x}{x}\, dx$$

$$= \int_1^2 \left(\frac{2}{x} - 1\right) dx + \int_2^4 \left(\frac{-2}{x} + 1\right) dx$$

$$= 2 \ln - x \Big]_1^2 + -2\ln x + x \Big]_2^4$$

$$= 4 \ln 2 - 2\ln 4 + 1$$

⊛ Fundamental theorem of Part - II

$$\int_1^3 f(x)\, dx = \int_1^3 f(t)\, dt$$

⊛ $\int_1^3 t^2\, dt$

$$= \frac{t^3}{3} \Big]_1^3$$

$$= \frac{3^3}{3} - \frac{1^3}{3}$$

$$= \frac{27-1}{3}$$

$$= \frac{26}{3}$$

(*) $\int_1^x t^3 \, dt$

$= \left. \frac{t^4}{4} \right]_1^x$

$= \frac{x^4}{4} - \frac{1}{4}$

$= \frac{x^4 - 1}{4}$

$$\boxed{\frac{d}{dx} \int_1^x t^3 \, dt = x^3}$$

$\frac{d}{dx} \int_1^4 t^3 \, dt = \frac{d}{dx} \left(\frac{x^4}{4} - \frac{1}{4} \right)$

$= \frac{1}{4} \cdot 4x^3 - 0$

$= x^3$

(*) $\int_3^x \sin y \, dy$

$= \left. -\cos y \right]_3^x$

$= -\cos x + \cos 3$

$\frac{d}{dx} \int_3^x \sin y \, dy = \frac{d}{dx} \left(-\cos x + \cos 3 \right)$

$= \sin x$

Hence, the theorem state that,

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

* $\frac{d}{dx}\int_1^x \sin(t^2)\,dt = \sin x^2$

* $\frac{d}{dx}\int_x^0 t \sec t \, dt = -x \sec x$

$\Rightarrow -\frac{d}{dx}\int_0^x t \sec t \, dt$

* $F(x) = \int_4^x \sqrt{t^2+9}\,dt$ find,

a) $F(4)$ b) $F'(4)$ c) $F''(4)$

<u>Solution</u>

a) $F(4) = \int_4^4 \sqrt{t^2+9}$

$= 0$

b) $F'(x) = \frac{d}{dx}\int_4^x \sqrt{t^2+9}$

$= \sqrt{x^2+9}$

$\therefore F'(4) = \sqrt{4^2+9} = \sqrt{16+9}$

$= \pm 5$

e) $F'(x) = \sqrt{x^2+9}$

$$F''(x) = \frac{2x}{2\sqrt{x^2+9}}$$

$$F''(4) = \frac{2 \cdot 4}{2\sqrt{4^2+9}}$$

$$= \frac{8}{10} = \frac{4}{5}$$

8

If, $f(x) = \int_3^x \frac{\sin t}{t}\, dt$

then, find $f''(x)$

$$f'(x) = \frac{d}{dx} \int_3^x \frac{\sin t}{t}\, dt$$

$$= \frac{\sin x}{x}$$

$$f''(x) = \frac{x \cos x - \sin x}{x^2}$$

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5.9

Evaluating Definite

Integration by Substitution:

(*) $\int_a^b f(g(x))\, g'(x)\, dx = \int_{x=a} f(u)\, du$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\, dx$

(*) Evaluate the followings.

i) $\int_0^1 (2x+1)^3\, dx$

ii) $\int_0^{\ln 5} e^x (3 - 4e^x)\, dx$

iii) $\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\, dx$

iv) $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}$

v) $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\, dx$

Solution

i) $\int_0^1 (2x+1)^3 \, dx$

Let,

$u = 2x+1$

$\frac{du}{dx} = 2$

$\frac{1}{2} du = dx$

if,

$x = 0 \; ; \; u = 2 \times 0 + 1 = 1$

$x = 1 \; , \; u = 2 \times 1 + 1 = 3$

$$\therefore \int_0^1 (2x+1)^3 \, dx = \frac{1}{2} \int_1^3 u^3 \, du$$

$$= \frac{1}{2} \left[\frac{u^4}{4} \right]_1^3$$

$$= \frac{1}{2} \left[\frac{3^4}{4} - \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{81}{4} - \frac{1}{4} \right]$$

$$= 10$$

Ans.

ii) $\int_0^{\ln 5} e^x (3 - 4e^x) \, dx$

$$= \int_0^{\ln 5} (3 - 4e^x) e^x \, dx$$

$u = 3 - 4e^x$

$\frac{du}{dx} = -4e^x$

$e^x dx = -\frac{1}{4} dx$

If.

$x = 0, \quad u = 3 - 4e^0 = 3 - 4 = -1$

$x = \ln 5, \quad u = 3 - 4e^{\ln 5} = -17$

$$\therefore \int_{-1}^{-17} u \cdot \left(-\frac{1}{4}\right) du$$

$$= -\frac{1}{4} \int_{-1}^{-17} u \, du$$

$$= -\frac{1}{4} \left[\frac{u^2}{2} \right]_{-1}^{-17}$$

$$= -\frac{1}{4} \left[\frac{-17^2}{2} - \frac{-1^2}{2} \right]$$

$$= \ldots$$

$$= \ldots$$

iii) $$\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2} dx$$

$$= \int_0^{1/\sqrt{3}} \frac{1}{1+(3x)^2} dx$$

Let,

$u = 3x$

$\frac{du}{dx} = 3$

$dx = \frac{1}{3} du$

if,

$x = 0; \quad u = 3 \cdot 0 = 0$

$x = \frac{1}{\sqrt{3}}; \quad u = 3 \cdot \frac{1}{\sqrt{3}} = \sqrt{3}$

$$= \frac{1}{3} \int_0^{\sqrt{3}} \frac{1}{1+u^2} du$$

$$= \frac{1}{3} \left[\tan^{-1} u \right]_0^{\sqrt{3}}$$

$$= \frac{1}{3} \left[\tan^{-1}\sqrt{3} - \tan^{-1} 0 \right]$$

$$= \frac{1}{3} \left[\frac{\pi}{3} - 0 \right]$$

$$= \frac{\pi}{9}$$ Ans

iv) $$\int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-x}}{\sqrt{1-e^{-2x}}} dx$$

$$= \int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}} dx$$

Let,

$u = e^{-x}$

$\frac{du}{dx} = -e^{-x}$

$e^{-x}\,dx = -du$

$x = \ln 2$; $u = e^{-\ln 2} = \frac{1}{2}$

$x = \ln\left(\frac{2}{\sqrt{3}}\right)$; $u = e^{-\ln(2/\sqrt{3})} = \frac{\sqrt{3}}{2}$

$$= -\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du$$

$$= -\left[\sin^{-1} u\right]_{\frac{1}{2}}^{\sqrt{3}/2}$$

$$= -\left[\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{2}\right]$$

$= \ldots\ldots$

$= \ldots\ldots$

v) $$\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\,dx$$

Let,

$u = x^2 + 4x + 7$

$\frac{du}{dx} = 2x + 4 = 2(x+2)$

$(x+2)\,dx = \frac{1}{2}\,du$

$x = 1$; $u = 1^2 + 4\cdot 1 + 7$

$= 12$

$x = 3$; $u = 3^2 + 4\cdot 3 + 7$

$= 9 + 12 + 7$

$= 28$

$$\frac{1}{2}\int_{12}^{28} \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{2}\int_{12}^{28} u^{-\frac{1}{2}}\, du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_{12}^{28}$$

$$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_{12}^{28}$$

$$= \left[u^{\frac{1}{2}}\right]_{12}^{28}$$

$$= \sqrt{28} - \sqrt{12}$$

$$\circledast \int_{-1}^{1} \sqrt{2+|x|}\, dx$$

$$= \int_{-1}^{0} \sqrt{2-x}\, dx + \int_{0}^{1} \sqrt{2+x}\, dx$$

$$\circledast \int_{-1}^{1} |e^x - 1|\, dx$$

$$= \int_{-1}^{0} -(e^x - 1)\, dx + \int_{0}^{1} (e^x - 1)\, dx$$

END