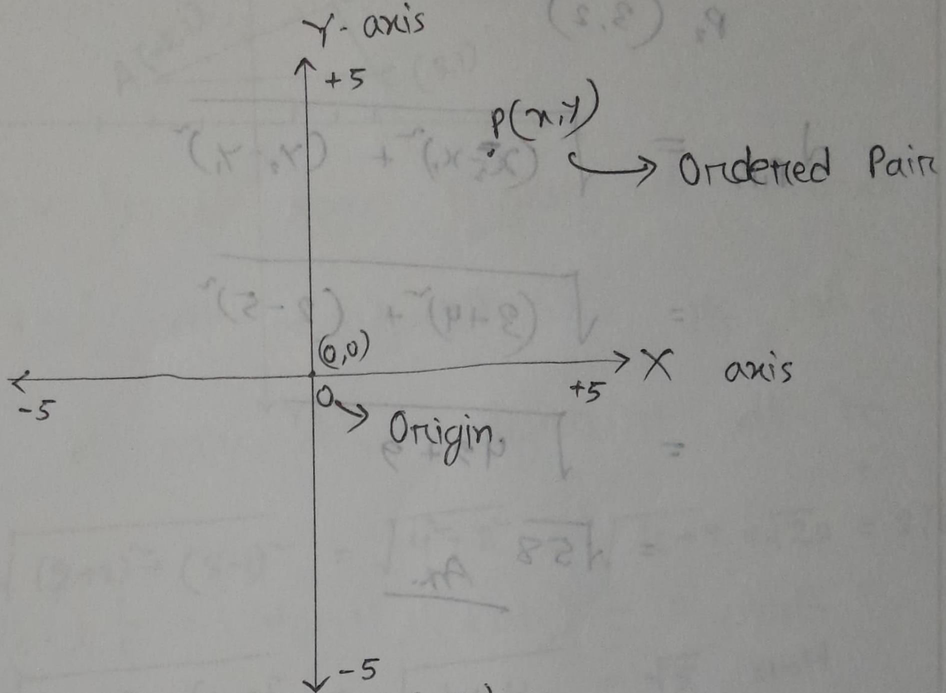
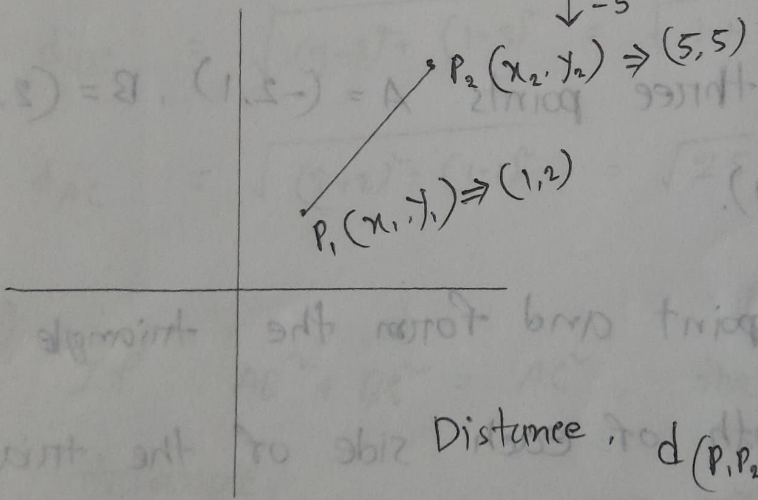
# Chapter - One

## Graphing




Distance, $d_{(P_1P_2)} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$= \sqrt{(5-1)^2 + (5-2)^2}$

$= \sqrt{4^2+3^2}$

$= \sqrt{25} = 5$ Ans

* Figure 13.

$P_1 (-4, 5)$

$P_2 (3, 2)$

$$d_{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(3+4)^2 + (2-5)^2}$$

$$= \sqrt{49 + 9}$$

$$= \sqrt{58}$$

Ans.

* Consider the three points $A = (-2, 1)$, $B = (2, 3)$ and $C = (3, 1)$.

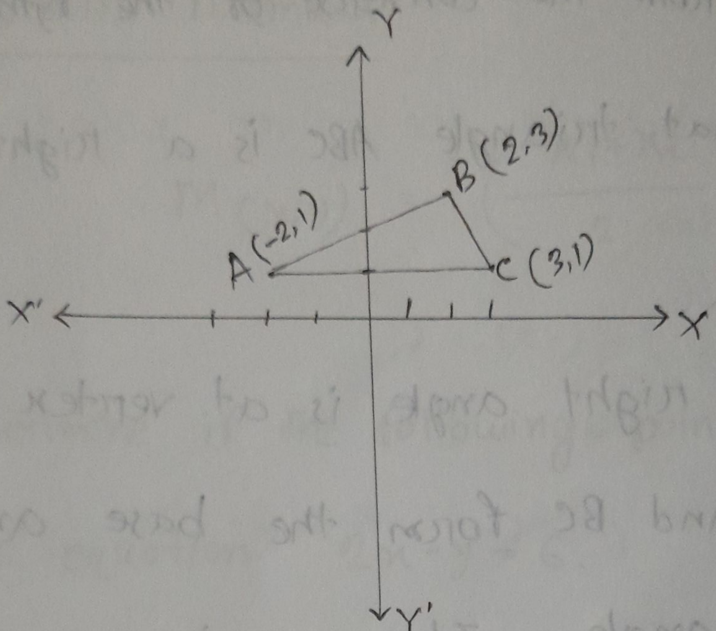
a) Plot each point and form the triangle ABC.

b) Find the length of each side of the triangle.

c) Verify that the triangle is a right triangle.

d) Find the area of the triangle.

a)




b)

$$d_{AB} = \sqrt{(2+2)^2 + (3-1)^2} = \sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5} \text{ unit}$$

$$d_{BC} = \sqrt{(3-2)^2 + (1-3)^2} = \sqrt{1+4} = \sqrt{5} \text{ unit}$$

$$d_{AC} = \sqrt{(3+2)^2 + (1-1)^2} = \sqrt{25} = 5 \text{ unit}$$

c)

If $AB^2 + BC^2 = AC^2$, then it will be a right triangle.

$$\begin{aligned} L.H.S. &= AB^2 + BC^2 \\ &= (d_{AB})^2 + (d_{BC})^2 \\ &= (2\sqrt{5})^2 + (\sqrt{5})^2 \\ &= 20 + 5 \\ &= 25 \\ &= 5^2 \\ &= (d_{AC})^2 \\ &= AC^2 \\ &= R.H.S. \end{aligned}$$

It follows from the converse of the Pythagorean Theorem that triangle ABC is a right triangle.

d)

Because the right angle is at vertex B, the sides AB and BC form the base and height of the triangle. Its area is

$$\text{Area} = \frac{1}{2}(\text{Base})(\text{Height}) = \frac{1}{2}(2\sqrt{5})(\sqrt{5})$$

$$= 5 \text{ square units.}$$

Midpoint Formula:

$$M(x,y) = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Determine if the following points are on the graph of the equation $2x - y = 6$.

a) $(2,3)$ b) $(2,-2)$

Solution

a) Let us substitute the given point in the above equation.

$2 \cdot 2 - 3 = 6$

$\Rightarrow 4 - 3 = 6$

$\Rightarrow 1 = 6$ ; this is not true.

Hence, the point $(2,3)$ is not on the graph of the above equation.

b) $(2,-2)$

Let us substitute the given point in the above equation.

$$2 \cdot 2 - (-2) = 6$$

$$\Rightarrow 4 + 2 = 6$$

$$\Rightarrow 6 = 6$$ ; this is true.

Hence, the given point $(2,-2)$ is on the graph of the above equation.

⊛ Find the $x$-intercepts and $y$-intercepts of the graph of $y = x^2 - 4$.

$\Rightarrow$

In order to find the $x$-intercept(s), let us put $y = 0$ in the above equation and solve it for $x$.

$$0 = x^2 - 4$$

$$\Rightarrow x^2 = 4$$

$$\therefore x = \pm 2$$

Hence, the x-intercepts are $-2$ and $2$.

In order to find the y-intercept(s), let us put $x=0$ in the above equation and solve it for y.

$$y = 0^2 - 4$$

$$\Rightarrow y = -4$$

Hence, the y-intercept is $-4$.

AYU - 1.1 (9th Edition)

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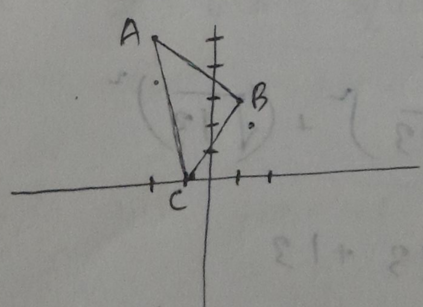
$A(-2,5)$, $B(1,3)$, $C(-1,0)$



To show that the triangle is a right triangle, we ~~we~~ need to show that the sum of the squares of the lengths of two of the sides equals the square of the length of the third side. Looking at figure, it seems reasonable to conjecture that the right angle is at vertex B. We shall check to see whether,

$$d_{AB} = \sqrt{(1+2)^2+(3-5)^2} = \sqrt{9+4} = \sqrt{13}$$

$$d_{BC} = \sqrt{(-1-1)^2+(0-3)^2} = \sqrt{4+9} = \sqrt{13}$$

$$d_{AC} = \sqrt{(-1+2)^2+(0-5)^2} = \sqrt{1+25} = \sqrt{26}$$

Now,

$$\therefore (d_{AB})^2 + (d_{BC})^2 = (\sqrt{13})^2 + (\sqrt{13})^2$$

$$= 13 + 13$$

$$= 26$$

And, $(d_{AC})^2 = (\sqrt{26})^2 = 26$

$\therefore (d_{AB})^2 + (d_{BC})^2 = (d_{AC})^2$

It follows from the converse of the Pythagorean Theorem that triangle ABC is a right triangle.

Because the right angle is at vertex B, the sides AB and BC form the base and height of the triangle. Its area is,

$$\text{Area} = \frac{1}{2}(\text{Base})(\text{Height})$$

$$= \frac{1}{2}(\sqrt{13})(\sqrt{13})$$

$$= \frac{1}{2} \cdot 13 = \frac{13}{2}$$ ~~square unit~~.

$$= 6.5$$ square unit.

## Lecture - 3

### Test an Equation for Symmetry

A graph is said to be symmetric with respect to the $x$-axis if, for every point $(x,y)$ on the graph, the point $(x,-y)$ is also on the graph.

A graph is said to be symmetric with respect to the $y$-axis if, for every point $(x,y)$ on the graph, the point $(-x,y)$ is also on the graph.

A graph is said to be symmetric with respect to the origin if, for every point $(x,y)$ on the graph, the point $(-x,-y)$ is also on the graph.

(*) For the equation $y = \frac{x^2-4}{x^2+1}$ :

a) find the intercepts

b) test for symmetry

Solution:

a)

In order to find the x-intercepts. Let us put $y=0$ in the given equation and solve it for $x$.

$0 = \frac{x^2-4}{x^2+1}$

$\Rightarrow x^2-4 = 0$

$\Rightarrow x^2 = 4$

$\therefore x = \pm 2$

Thus the x-intercepts are $+2$ and $-2$.

The points are $(2,0)$ and $(-2,0)$

2nd Part

In order to find the y-intercepts. Let us put $x=0$ in the given equation and solve it for $y$.

$y = \frac{0^2-4}{0^2+1}$

$\therefore y = -4$

Thus the y-intercept is $-4$.

The point is $(0,-4)$.

b)

In order to test the symmetry of the given equation let us replace $y$ by $-y$.

$$-y = \frac{x^2-4}{x^2+1}$$

$\Rightarrow y = \frac{-x^2+4}{x^2+1}$ ; which is not equivalent to the given equation.

Thus the given equation is not symmetric with respect to the $x$-axis.

In order to test the symmetry of the given equation, let us replace $x$ by $-x$.

$$y = \frac{(-x)^2-4}{(-x)^2+1}$$

$\Rightarrow y = \frac{x^2-4}{x^2+1}$ ; which is equivalent to the given equation.

Thus the given equation is symmetric with respect to the $y$-axis.

In onden to test the symmetry of the given equation let us replace $x$ by $-x$ and $y$ by $-y$.

$$-y = \frac{(-x)^2-4}{(-x)^2+1}$$

$$\Rightarrow y = \frac{-x^2+4}{x^2+1}$$ ; which is not equivalent to the given equation.

Thus the given equation is not symmetric with respect to the origin.

Home Work (9th Edt)

1/

$$2(x+3)-1=-7$$

$$\Rightarrow 2(x+3) = -6$$

$$\Rightarrow 2x+6 = -6$$

$$\Rightarrow 2x = -12$$

$$\therefore x = -6$$

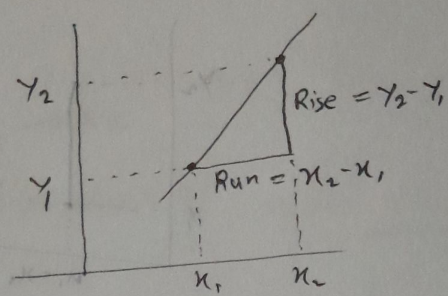
2/ $x^2-9=0$

$$\Rightarrow x^2 = 9$$

$$\Rightarrow x = \pm\sqrt{9}$$

$$\therefore x = \pm 3$$

# Slope

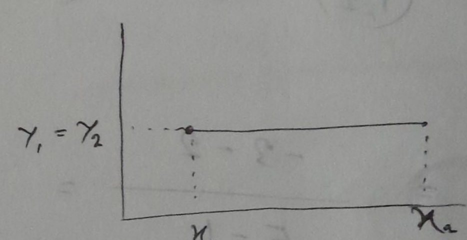



$$\text{Slope } m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{\text{change in } y}{\text{chang in } x}$$

* Slope of Horizontal Line.

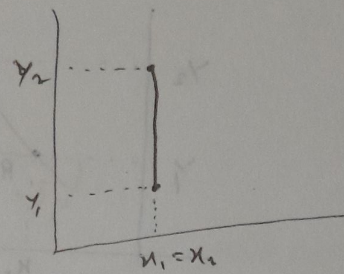



$$\text{Slope}, \ m = \frac{\text{Rise}}{\text{Run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{y_1 - y_1}{\Delta x}$$

$$= \frac{0}{\Delta x} = 0$$

Slope of Vertical Line.




$$\text{Slope, } m = \frac{\text{Rise}}{\text{Run}}$$

$$= \frac{\Delta y}{0}$$

$$= \text{undefined.}$$

⊛ Find the slope $m$ of the line containing the points $(1,2)$ and $(5,-3)$

$\Rightarrow$

$$m = \frac{-3-2}{5-1} = \frac{-5}{4} \text{ Ans.}$$

⊛ Draw a graph of the line that contains the point $(3,2)$ and has a slope of

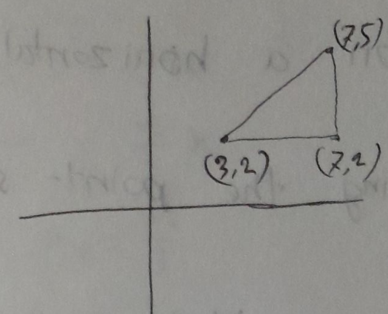
a) $\frac{3}{4}$ b) $-\frac{4}{5}$

$\Rightarrow$

a)

Given that

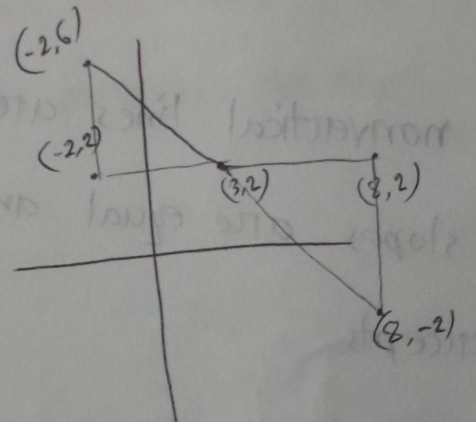
Slope, $m = \dfrac{3 \rightarrow \text{Rise}}{4 \rightarrow \text{Run}}$




b)

Given that

Slope, $m = -\dfrac{4}{5} = \dfrac{-4}{5} = \dfrac{4}{-5}$




$m = \dfrac{\text{Rise}}{\text{Run}} = \dfrac{y_2 - y}{x_2 - x_1}$

$\therefore \boxed{(y - y_1) = m\,(x - x_1)} \rightarrow$ Point-slope form of an equation.

Point $(3,2)$

for a horizontal line, $m=0$.

Using the point-slope form of an equation we have

$$y-y_1 = m(x-x_1)$$

$$\Rightarrow y-2 = 0(x-x_1)$$

$$\Rightarrow y = 2$$

Ans

Two nonvertical lines are parallel if and only if their slopes are equal and they have different y-intercepts.

Show that the lines given by the following equations are ~~patera~~ parallel:

$L_1 : 2x+3y = 5$

$L_2 : 4x+6y = 0$

$\Rightarrow$

and,

$2x + 3y = 6$ $\qquad$ $4x + 6y = 0$

$3y = -2x + 6$ $\qquad$ $6y = -4x$

$y = \frac{-2}{3}x + 2$ $\qquad$ $y = \frac{-2}{3}x$

$\therefore m_1 = \frac{-2}{3}$ $\qquad$ $\therefore m_2 = \frac{-2}{3}$

$\therefore m_1 = m_2 = \frac{-2}{3}$

Thus $L_1$ and $L_2$ are parallel to each other.

⊛ Find an equation for the line that contains the point $(2,-3)$ and is parallel to the line $2x + y = 6$

$\Rightarrow$

$2x + y = 6$

$y = -2x + 6$

$\therefore m = -2$

By using ~~slope~~ point-slope form,

$y + 3 = -2(x - 2)$

$y + 3 = -2x + 4$

$2x + y = 1$ $\quad$ Ans

⊛ Two nonvertical lines are perpendicular if and only if the product of their slopes is $-1$.

$\therefore m_1 m_2 = -1$.

⊛ Find an equation of the line that contains the point $(1,-2)$ and is perpendicular to the line $x+3y=6$.

$\Rightarrow$

$x + 3y = 6$

$\Rightarrow 3y = -x + 6$

$\therefore y = \frac{-1}{3}x + 6$

Hence, the slope $m_1 = \frac{-1}{3}$.

For perpendicular lines, we know that,

$m_1 m_2 = -1$

$-\frac{1}{3} \times m_2 = -1$

$\therefore m_2 = 3$

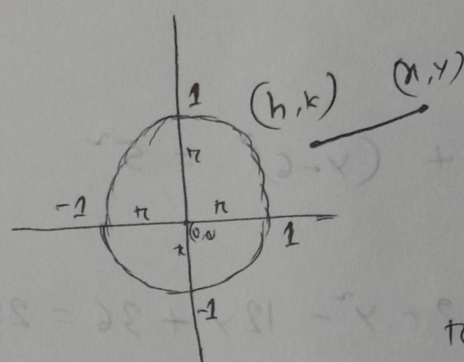
By using the point-slope form of an equation.

$$y + 2 = 3(x-1)$$

$$\Rightarrow y+2 = 3x-3$$

$$\therefore y = 3x-5 \quad \underline{\text{Ans}}$$

$x^2+y^2=1$




$$r = \sqrt{(x-h)^2+(y-k)^2}$$

$$r^2 = (x-h)^2 + (y-k)^2$$

$r \Rightarrow$ Radious of the circle, fixed.

$(h,k) \Rightarrow$ Center of the circle, fixed.

$$(x-h)^2 + (y-k)^2 = r^2 \quad \cdots\text{(i)}$$

If the center is brought at the origin

$$(x-0)^2 + (y-0)^2 = r^2$$

$$x^2+y^2 = r^2 \quad \cdots\text{(ii)}$$

If
$r = 1$;

$\Rightarrow x^2 + y^2 = 1$ ---(iii)

this is the equation of a unit circle.

* Write the standard form of the equation of the circle with radius 5 and center $(-3, 6)$.

$\Rightarrow$

$(x+3)^2 + (y-6)^2 = 5^2$

$x^2 + 6x + 9 + y^2 - 12y + 36 = 25$

$x^2 + y^2 + 6x - 12y = -20$

$x^2 + y^2 + 6x - 12y + 20 = 0$

Ans

* For the circle $(x+3)^2 + (y-2)^2 = 16$, find the intercepts, if any, of its graph.

$\Rightarrow$

The given circle is $(x+3)^2 + (y-2)^2 = 16 \quad \cdots (i)$

To find the $x$-intercepts let us put $y=0$ in equation (i) and solve it for $x$.

$(x+3)^2 + (0-2)^2 = 16$

$(x+3)^2 + 4 = 16$

$(x+3)^2 = 16-4$

$(x+3)^2 = 12$

$x+3 = \sqrt{12} = \pm 2\sqrt{3}$

$\therefore x = \pm 2\sqrt{3} - 3$

Hence the $x$-intercepts of the circle are $2\sqrt{3}-3$ and $-2\sqrt{3}-3$.

To find the $y$-intercepts let us put $x=0$ in the equation (i) and solve it for $y$.

$\Rightarrow (0+3)^2 + (y-2)^2 = 16$

$\Rightarrow (y-2)^2 = 16-9$

$\Rightarrow y-2 = \pm\sqrt{7}$

$\therefore y = \pm\sqrt{7}+2$

Hence, the y-intercepts of the circle are $\sqrt{7}+2$ and $-\sqrt{7}+2$.

⊛ General equation of a circle is,

$$x^2+y^2+ax+by+c=0$$

⊛ Find the general equation of the circle whose center is $(1,-2)$ and whose graph contains the point $(4,-2)$

$\Rightarrow$

By using distance formula,

$$r=\sqrt{(1-4)^2+(-2+2)^2}=3$$

Hence the equation of its circle is,

$$(x-1)^2+(y+2)^2=3^2$$

$$\Rightarrow x^2+y^2-2x+4y-4=0$$

# Chapter-2

## Function and Their Graphs

⊛ For the function $f$ defined by $f(x) = 2x^2 - 3x$, evalute

a) $f(3) = 2 \cdot 3^2 - 3 \cdot 3$
$= 2 \cdot 9 - 9$
$= 18 - 9$
$= 9$ Ans

b) $f(x) + f(3) = 2x^2 - 3x + 9$ Ans

c) $3f(x) = 3(\cancel{x}2x^2 - 3x)$
$= 6x^2 - 9x$ Ans

d) $f(-x) = 2 \cdot (-x)^2 - 3 \cdot (-x)$
$= 2x^2 + 3x$ Ans

e) $-f(x) = -(2x^2 - 3x)$
$= -2x^2 + 3x$ Ans

f) $f(3x) = 2 \cdot (3x)^2 - 3 \cdot (3x)$
$= 2 \cdot 9x^2 - 9x$
$= 18x^2 - 9x$ Ans

g) $f(x+3) = 2 \cdot (x+3)^2 - 3(x+3)$
$= 2 \cdot (x^2 + 6x + 9) - 3x - 9$
$= 2x^2 + 12x + 18 - 3x - 9$
$= 2x^2 + 9x + 9$
Ans

h)

$$\frac{f(x+h)-f(x)}{h} \; ; \; h \neq 0$$

$$= \frac{2(x+h)^2 - 3(x+h) - 2x^2 + 3x}{h}$$

$$= \frac{2x^2 + 4xh + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4xh + 2h^2 - 3h}{h}$$

$$= \frac{h(4x + 2h - 3)}{h}$$

$$= 4x + 2h - 3$$

Ans

b) $g(x) = \frac{3x}{x^2-4}$

Domain of $g(x)$ is

$\{x \mid x \neq -2, x \neq 2\}$

c) $h(t) = \sqrt{4-3t}$

$4-3t \geq 0$

$-3t \geq -4$

$t \leq \frac{4}{3}$

Domain of $h(t)$ is

$\{t \mid t \leq \frac{4}{3}\}$

$\Rightarrow (-\infty, \frac{4}{3}]$

d) $f(x) = \frac{\sqrt{3x+12}}{x-5}$

$3x+12 \geq 0$

$3x \geq -12$

$x \geq -4$

$x-5 \neq 0$

$x \neq 5$

$\therefore \{x \mid x \geq -4, x \neq 5\}$

$f(x) = x^2 + 9$ and $g(x) = 3x + 5$, then,

$$f(x) + g(x) = (x^2+9) + (3x+5)$$
$$= x^2 + 3x + 14$$

The new function $y = x^2 + 3x + 14$ is called the sum function $f+g$. Similarly,

$$f(x) \cdot g(x) = (x^2+9)(3x+5)$$
$$= 3x^3 + 5x^2 + 27x + 45$$

the new function $y = 3x^2 + 5x^2 + 27x + 45$ is called the product function $f \cdot g$.

(*)

If $f$ and $g$ are functions:

The sum $f+g$ is the function defined by

$$(f+g)(x) = f(x) + g(x)$$

the ~~d~~ domain of $f+g$ consists of the ~~d~~ numbers $x$ that are in the domains of both $f$ and $g$.

That is, domain of $f+g$ = domain of $f \cap$ domain of $g$.

The difference $f-g$ is the function defined by

$$(f-g)(x) = f(x) - g(x)$$

The domain of $f-g$ consists of the numbers $x$ that are in the domains of both $f$ and $g$. That is, domain of $f-g$ = domain of $f \cap$ domain of $g$.

The product $f \cdot g$ is the function defined by

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

The domain of $f \cdot g$ consists of the numbers $x$ that are in the domains of both $f$ and $g$. That is, domain of $f \cdot g$ = domain of $f \cap$ domain of $g$.

The quotient $\frac{f}{g}$ is the function defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad ; g(x) \neq 0$$

The domain of $\frac{f}{g}$ consists of the numbers $x$ for which $g(x) \neq 0$ and that are in the domains of both $f$ and $g$. That is,

domain of $\frac{f}{g} = \{x \mid g(x) \neq 0\} \cap$ domain of $f \cap$ domain of $g$.

$f(x) = \frac{1}{x+2}$ and $g(x) = \frac{x}{x-1}$

a) $f(x) + g(x) = (f+g)(x)$

$$= \frac{1}{x+2} + \frac{x}{x-1}$$

$$= \frac{x-1 + x^2+2x}{(x+2)(x-1)}$$

$$= \frac{x^2+3x-1}{(x+2)(x-1)}$$

$= \{x \mid x \neq -2, x \neq 1\}$

# Even and Odd Function

* A function $f$ is even, if and only if, whenever the point $(x, y)$ is on the graph of $f$ then the point $(-x, y)$ is also on the graph.

$$f(-x) = f(x)$$

* A function $f$ is odd, if and only if, whenever the point $(x, y)$ is on the graph of $f$ then the point $(-x, -y)$ is also on the graph.

$$f(-x) = -f(x)$$

## Theorem

A function is even if and only if its graph is symmetric with respect to the $y$-axis. A function is odd if and only if its graph is symmetric with respect to the origin.

⊛ Example,

$$g(x) = x^3 - 1$$

In order to test even and odd function, lets replace $x$ by $-x$.

$$g(-x) = (-x)^3 - 1$$
$$= -x^3 - 1$$
$$= -(x^3 + 1) \neq g(x)$$
$$\neq -g(x)$$

So, the function is neither even nor odd function.

Local Maximum and Local Minimum

Suppose $f$ is a function defined on an open interval containing $c$. If the value of $f$ at $c$ is greater than or equal to the values of $f$ on $I$, then $f$ has a local maximum at $c$.

If the value of f at c is less than or equal to the values of f on I, then f has a local minimum at c.

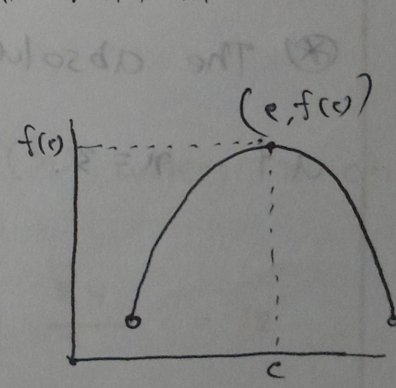
$\Rightarrow$

A function f has a local maximum at c if there is an open interval I containing c so that for all $x$ in I, $f(x) \leq f(c)$. We call $f(c)$ a local maximum value of f.

A function f has a local minimum at c if there is an open interval I containing c so that, for all $x$ in I, $f(x) \geq f(c)$. We call $f(c)$ a local minimum value of f.

<u>Local Maximum</u>

Open Interval




At c,

$f(x) = f(c)$

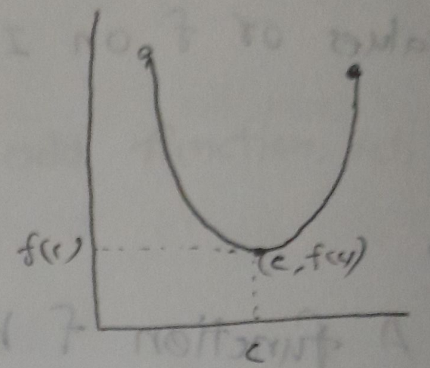
At all other points, except c,

$f(x) < f(c)$

$f(x) \leq f(c) \Rightarrow$ Local maximum.

## Local Minimum

Open Interval




At c,

$f(x) = f(c)$

At all other points, except c,

$f(x) > f(c)$

$f(x) \geq f(c) \Rightarrow$ Local minimum.

* The absolute minimum of the function occurs at $x = 0$. The value of the absolute minimum $f(0) = 1$.

* The absolute ~~ma~~ maximum of the function occurs at $x = 3$. The value of the absolute maximum $f(3) = 6$.

# Average Rate of Change of a Function.

If a ~~nd~~ and b, $a \neq b$, are ~~in~~ in the domain of a function $y = f(x)$, the average rate of change of f from a to b is defined as,

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} \; ; a \neq b$$

⊛ Find the average rate of change of $f(x) = 3x^2$.

a) From 1 to 3

b) From 1 to 5

c) From 1 to 7.

$\Rightarrow$

a) The average rate of change of $f(x)$ when x changes from 1 to 3 is,

$$\Delta f_1 = \frac{f(3) - f(1)}{3-1} = \frac{3 \cdot 3^2 - 3 \cdot 1^2}{3-1} = \frac{24}{2} = 12$$

b) from 1 to 5 is,

$$\Delta f_2 = \frac{f(5) - f(1)}{5-1} = \frac{3 \cdot 5^2 - 3 \cdot 1^2}{5-1} = \frac{72}{4} = 18$$

c)
from 1 to 7,

$$\Delta f_3 = \frac{f(7)-f(1)}{7-1} = \frac{3\cdot 7^2 - 3\cdot 1^2}{7-1} = \frac{144}{6} = 24$$

(*) Suppose that $g(x) = 3x^2 - 2x + 3$.

a) Find the average rate of change of $g$ from $-2$ to $1$.

b) Find an equation of the secant line containing $(-2, g(-2))$ and $(1, g(1))$.

$\Rightarrow$

a)

The average rate of change of $g$ from $-2$ to $1$.

$$\Delta g = \frac{g(1) - g(-2)}{1-(-2)} = \frac{[3\cdot 1^2 - 2\cdot 1 + 3] - [3\cdot(-2)^2 - 2(-2) + 3]}{1+2}$$

$$= \frac{4-19}{3}$$

$$= -5$$

$g(-2) = 19 = y_1$

$g(1) = 4 = y_2$

b) The average rate of change of y from -2 to 1 is equal to $m_{sec}$.

Thus, the slope of the secant line is

$$m_{sec} = -5$$

Using the point-slope form of a line, we get the equation of the secant line,

$$y - y_1 = m_{sec}(x - x_1)$$

$$y - 19 = -5(x - (-2))$$

$$y - 19 = -5(x + 2)$$

$$y - 19 = -5x - 10$$

$$y = -5x + 9$$

Ans