$$= \frac{1}{a}\int \frac{a\,du}{\sqrt{1-u^2}}$$

$$= \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1} u + c$$

$$= \sin^{-1} \frac{x}{a} + c$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$$

Find the followings.

i) $\int \frac{dx}{5+x^2}$

$$= \int \frac{dx}{(\sqrt{5})^2+x^2}$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{\sqrt{5}} + c$$

ii) $\int \frac{dx}{x\sqrt{x^2-\pi}}$

$$= \int \frac{dx}{x\sqrt{x^2-\pi}}$$

$$= \int \frac{dx}{x\sqrt{x^2-(\sqrt{\pi})^2}}$$

$$= \frac{1}{\sqrt{\pi}}\sec^{-1}\frac{x}{\sqrt{\pi}} + c$$

iii) $\int \frac{dy}{y\sqrt{5y^2-3}}$

$= \int \frac{dy}{y\sqrt{5(y^2-\frac{3}{5})}}$

$= \frac{1}{\sqrt{5}} \int \frac{dy}{y\sqrt{y^2-(\frac{\sqrt{3}}{\sqrt{5}})^2}}$

$= \frac{1}{\sqrt{5}} \cdot \frac{1}{\frac{\sqrt{3}}{\sqrt{5}}} \cdot \sec^{-1} \frac{y}{\frac{\sqrt{3}}{\sqrt{5}}} + C$

$= \frac{1}{\sqrt{3}} \sec^{-1} \frac{\sqrt{5}\, y}{\sqrt{3}} + C$

iv) $\int \frac{dx}{\sqrt{9-4x^2}}$

$= \int \frac{dx}{\sqrt{4(\frac{9}{4}-x^2)}}$

$= \frac{1}{2} \int \frac{dx}{(\frac{3}{2})^2 - x^2}$

$= \frac{1}{2} \cdot \sin^{-1} \frac{x}{3/2} + C$

$= \frac{1}{2} \sin^{-1} \frac{2x}{3} + C$

* $\int e^{-x}\, dx$

Let, $u = -x$

$\frac{du}{dx} = -1$

$\cancel{dx}\; dx = -du$

$\therefore \int e^u (-du)$

$= -\int e^u\, du$

$= -e^u$

$= -e^{-x} + C$

* $\int x\sqrt{4-x}\,dx$

Let,

$u = 4-x$

$\frac{du}{dx} = -1$

$dx = -du$

$$= \int (4-u)\cdot\sqrt{u}\,(-du)$$

$$= -\int (4u^{\frac{1}{2}} - u^{3/2})\,du$$

$$= -\int 4u^{\frac{1}{2}}\,du - \int u^{3/2}\,du$$

$\therefore$

$$= -4\cdot\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} + c$$

$$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2} + c$$

$$= -\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + c$$

* Solve the followings initial value problem (IVP).

i) $\frac{dy}{dx} = \sqrt{5x+1}$; $y(3) = -2$

$\Rightarrow$

$$\frac{dy}{dx} = \sqrt{5x+1}$$

$$dy = \sqrt{5x+1}\,dx$$

$$\int dy = \int \sqrt{5x+1}\,dx$$

$$y = \int \frac{\cancel{\quad}}{\sqrt{5x+1}}\,dx$$

Let,

$u = 5x+1$

$\frac{du}{dx} = 5$

$dx = \frac{1}{5}\,du$

$$= \int \sqrt{u} \cdot \frac{1}{5} \cdot du$$

$$= \frac{1}{5} \int u^{\frac{1}{2}} du$$

$$= \frac{1}{5} \cdot \frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{15} u^{3/2} + c$$

$$\therefore y = \frac{2}{15} (5x+1)^{3/2} + c$$

Now,

if, $x = 3$

$y = -2$

$$\therefore -2 = \frac{1}{15} (5 \cdot 3 + 1)^{3/2} + c$$

$$-2 = \frac{2}{15} (16)^{3/2} + c$$

$$c = -\frac{158}{15}$$

Hence, the particular solution.

$$\therefore y = \frac{2}{15} (5x+1)^{3/2} - \frac{158}{15}$$

ii) $\frac{dy}{dx} = \frac{1}{25+9x^2}$; $y(-\frac{5}{3}) = \frac{\pi}{30}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{25+9x^2}$

$dy = \frac{1}{25+9x^2} dx$

$\int dy = \int \frac{1}{9(\frac{25}{9}+x^2)} dx$

$y = \frac{1}{9}\int \frac{1}{(\frac{5}{3})^2+x^2} dx$

$= \frac{1}{9}\cdot\frac{1}{5/3}\tan^{-1}\frac{x}{5/3} + c$

$y = \frac{1}{15}\tan^{-1}\frac{3x}{5} + c$

Now,

if $x = -\frac{5}{3}$

$y = \frac{\pi}{30}$

$\therefore \frac{\pi}{30} = \frac{1}{15}\cdot\tan^{-1}\frac{3(-\frac{5}{3})}{5} + c$

$\therefore \frac{\pi}{30} = \frac{1}{15}\cdot\tan^{-1}(-1) + c$

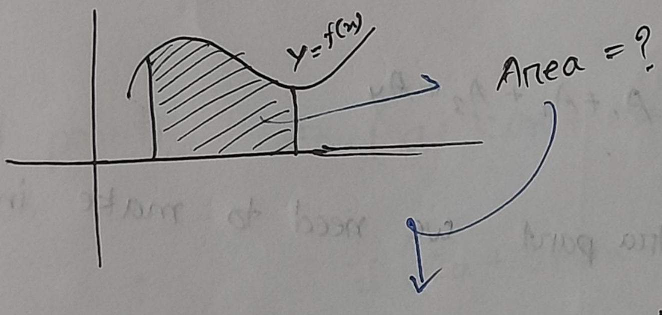
$\therefore \frac{\pi}{30} = \frac{1}{15}(-\frac{\pi}{4}) + c$

$\therefore C = \frac{\pi}{30} + \frac{\pi}{60} = \frac{\pi}{20}$

$\therefore y = \frac{1}{15} \tan^{-1} \frac{3t}{5} + \frac{\pi}{20}$

5.4

Finding Area Under the Curve



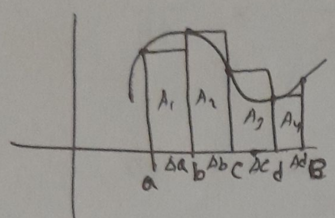
There are two methods for finding area under the curve.

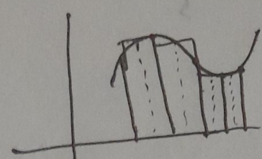
i) Rectangle method

ii) Anti derivative method

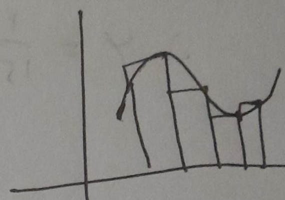
i) Rectangle method.

$\Rightarrow$





mid points



Right end points

Left end points

$A_1 = \Delta a \times f(a)$

$A_2 = \Delta b \times f(b)$

$A_3 = \Delta c \times f(c)$

$A_4 = \Delta d \times f(d)$

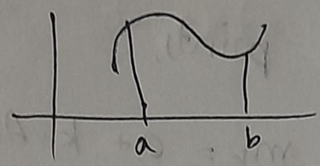
$\therefore$ Total Area $= A_1 + A_2 + A_3 + A_4$

$\therefore$ To remove extra part we need to make infinite sub intervals. Then we can get actual area.

If, you make nth sub intervals,

and $n = \infty$

Actual area $= \lim_{n \to \infty} (A_1 + A_2 + A_3 + \cdots + A_n)$



$\therefore$ n sub intervals

$\therefore$ each intervals length,

$$\Delta x = \frac{b-a}{n}$$

$\therefore$ Right end points $= a + k \cdot \Delta x$; $k = 1, 2, 3, 4 \ldots$

$\therefore$ Left end points $= a + (k-1)\Delta x$; $k = 1, 2, 3, 4 \ldots$

$\therefore$ Mid points $= (a + k\Delta x + a + (k-1)\Delta x)$

$$= \frac{1}{2}(2a + 2k\Delta x - \Delta x)$$

$$= a + (k - \tfrac{1}{2})\Delta x \,; \quad k = 1, 2, 3, \ldots$$

$\therefore$ Finally, area,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

If m_k is left ends points

$$m_k = a + (k-1)\Delta x$$

∴ Right end points.

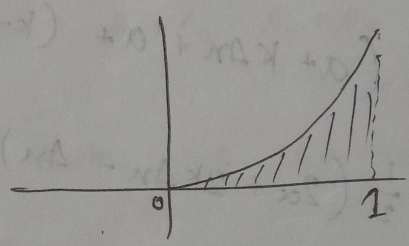
$$m_k = a + k \cdot \Delta x$$

∴ Mid point.

$$m_k = a + \left(k - \tfrac{1}{2}\right)\Delta x$$

⊛ Find the area, $f(x) = x^2$ on the interval $[0, 1]$ with right end points.

⇒



Now, we need to make n subintervals,

∴ each sub intervals length,

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Now, point, $m_k = a + k\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$

Now,

Area,

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n}$$

Here,

$f(x) = x^2$

$\therefore f\left(\frac{k}{n}\right) = \frac{k^2}{n^2}$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \left[1^2 + 2^2 + 3^2 + \dots + n^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n \cdot n\left(1+\frac{1}{n}\right) \cdot n\left(2+\frac{1}{n}\right)}{6}$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{6}(1+0)(2+0)$$

$$= \frac{1 \cdot 2}{6} = \frac{1}{3}$$

$\therefore$ Area $= \frac{1}{3}$

(*) Find the area, $f(x) = \frac{x}{2}$ on the interval $[1,4]$

with left end points.

$\Rightarrow$

We need to make n sub intervals

$\therefore$ each sub interval length,

$$\Delta x = \frac{b-a}{n} = \frac{4-1}{n} = \frac{3}{n}$$

points, $m_k = a + (k-1)\Delta x = 1 + (k-1)\Delta x = 1 + (k-1)\frac{3}{n}$

Now, area,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} f\left(1 + (k-1)\frac{3}{n}\right) \frac{3}{n}$$

Here,

$$f(x) = \frac{x}{2}$$

$$\therefore f\left(1+(k-1)\frac{3}{n}\right) = \frac{1+(k-1)\frac{3}{n}}{2}$$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(1+(k-1)\frac{3}{n}\right)\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\cdot\left(\frac{3}{n}\right) + \frac{1}{2}(k-1)\frac{9}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{2n}\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}(n+3k-3)$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[\sum_{k=1}^{n} n + \sum_{k=1}^{n} 3k - 3\sum_{k=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[n\sum_{k=1}^{n} 1 + 3\sum_{k=1}^{n} k - 3\sum_{k=1}^{n} 1\right]$$

$$= \lim_{n\to\infty} \frac{3}{2n^2}\left[n^2 + \frac{3}{2}\cdot\frac{n(n+1)}{2} - 3n\right]$$

$$= \lim_{n\to\infty} \frac{3}{2}\left[1 + \frac{3}{2}\left(1+\frac{1}{n}\right) - 3\frac{1}{n}\right]$$

$$= \frac{3}{2}\left[1+\frac{3}{2}(1+0) - 0\right]$$

$$= \frac{3}{2}\left[1+\frac{3}{2}\right] = \frac{3}{2}\cdot\frac{5}{2} = \frac{15}{4} \text{ Ans}$$

L-21 / 24.08.2024

5.5

The Definite Integral

Area under the curve by rectangle method,

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(m_k) \cdot \Delta x \quad \text{[Riemenn Sum]}$$

$n \to$ num of sub interval

$\Delta x \to$ width (each interval length)

if,

$n \to \infty$

$\Delta x \to 0$

$$\therefore A = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

Definite integral:

$$\int_{x=a}^{b} f(x) \cdot dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x = \text{area under the } \cancel{\text{sum}} \text{ curve}$$

Def^n: A function $f(x)$ is said to be Integrable on finite interval $[a,b]$.

if the limit $\lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$ exists, then we denote the limit by the symbol,

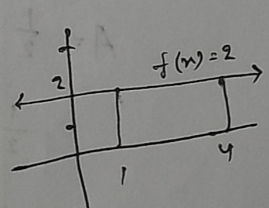
$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{k=1}^{n} f(m_k) \cdot \Delta x$$

Theorem: If a function f is continuous on $[a,b]$, then f is integrable on $[a,b]$ and then net signed area between the graph of f and interval $[a,b]$ is

$A = \int_a^b f(x)\,dx$.

⊛ Find the followings, definite integrals using appropriate formula from geometry.

a) $\int_1^4 2\,dx$



Here, $f(x) = 2$

$a = 1$

$b = 4$

This is rectangle

$\therefore$ area = length × width

$= (4-1) \times 2$

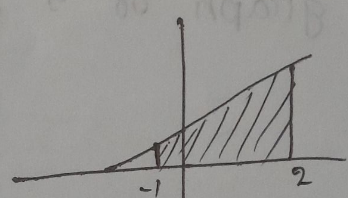
$= 3 \times 2$

$= 6$

Therefore, $\int_1^4 2 \, dx = 6$ Ans.

ii) $\int_{-1}^{2} (x+2) \, dx$

Here, $f(x) = x+2$; $[-1, 2]$



This is trapezoid,

We know that,

$A = \frac{1}{2}(b+b')h$

$= \frac{1}{2}(1+4)3$

$= \frac{1}{2} \cdot 5 \cdot 3$

$= \frac{15}{2}$

$b = f(-1) = 1$

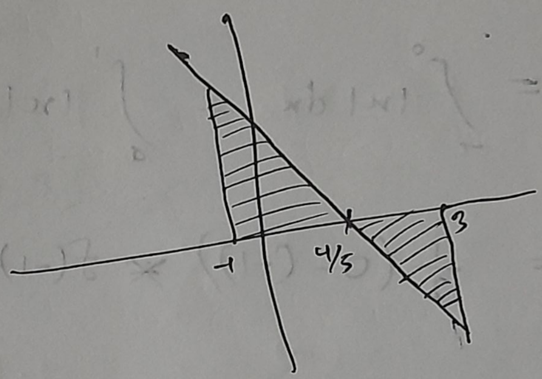
$b' = f(2) = 4$

$h = (2-(-1)) = 3$

$\therefore \int_{-1}^{2} (x+2)\,dx = \frac{15}{2}$

iii) $\int_{-1}^{3} (4-5x)\,dx$

Here, $f(x) = 4-5x$



$= \int_{-1}^{4/5} (4-5x)\,dx + \int_{4/5}^{3} (4-5x)\,dx$

$= \frac{1}{2} \times b \times h + \frac{1}{2} \times b \times h$

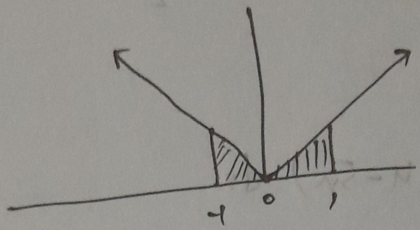
$= \frac{1}{2} \times (\frac{4}{5}+1) \times f(-1) + \frac{1}{2} \times (3-\frac{4}{5}) \times f(3)$

$= \frac{1}{2} \times \frac{9}{5} \times 9 + \frac{1}{2} \times \frac{11}{5} \times (-11)$

$= \frac{81}{10} - \frac{121}{10}$

$= \frac{-40}{10} = -4$

iv) $\int_{-1}^{1} |x|\, dx$



$= \int_{-1}^{0} |x|\, dx + \int_{0}^{1} |x|\, dx$

$= \frac{1}{2}(0-(-1)) \times f(-1) + \frac{1}{2} \times (1-0) \times f(1)$

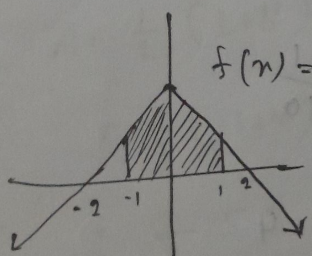
$= \frac{1}{2} \times 1 \times 1 + \frac{1}{2} \times 1 \times 1$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

$\therefore \int_{-1}^{1} |x|\, dx = 1$

v) $\int_{-1}^{1} (2-|x|)\, dx \; ; \; [-1, 1]$



$= \int_{-1}^{0} (2-|x|)\, dx + \int_{0}^{1} (2-|x|)\, dx$

$= \frac{1}{2}(b+b')h + \frac{1}{2}(b+b')h$

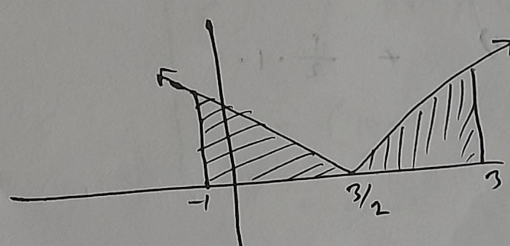
$= \frac{1}{2}(f(-1)+f(0))\cdot(0+1) + \frac{1}{2}(f(0)+f(1))(1-0)$

$= \frac{1}{2}(1+2)\cdot 1 + \frac{1}{2}(2+1)\,1$

$= \frac{3}{2} + \frac{3}{2}$

$= 3$ Ans

xi) $\int_{-1}^{2} |2x-3|\, dx \; ; \; [-1, 3]$



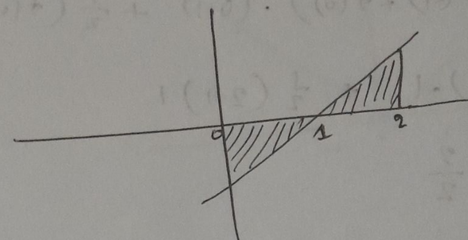
$= \int_{-1}^{3/2} |2x-3|\, dx + \int_{3/2}^{3} |2x-3|\, dx$

= Do yourself

(*) $\int_{0}^{3} |x-2|\, dx$

= Do yourself

⊛ $\int_0^2 (x-1)\,dx$



$$= \int_0^1 (x-1)\,dx + \int_1^2 (x-1)\,dx$$

$$= \frac{1}{2} \times (1-0) \cdot f(0) + \frac{1}{2}(2-1)\, f(2)$$

$$= \frac{1}{2} \times 1 \times (-1) + \frac{1}{2} \cdot 1 \cdot 1$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$= 0$$

⊛ Example:

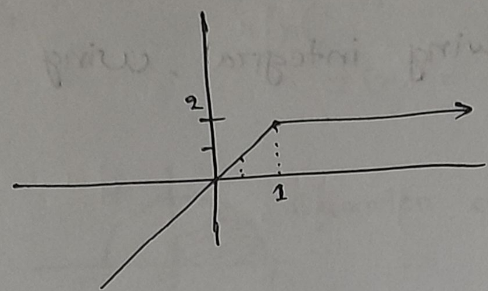
If $f(x) = \begin{cases} 2x & x \leq 1 \\ 2 & x > 1 \end{cases}$

then find,

a) $\int_0^1 f(x)\,dx$ b) $\int_{-}^{1} f(x)\,dx$ c) $\int_1^{10} f(x)\,dx$

d) $\int_{\frac{1}{2}}^{5} f(x)\,dx$

$\Rightarrow$



a) $\int_0^1 f(x)\,dx$

$= \frac{1}{2} \times \text{base} \times h$

$= \frac{1}{2}(1-0) \times f(1)$

$= \frac{1}{2} \times 1 \times 2$

$= 1$

b) $\int_{-1}^{1} f(x)\,dx$

$= 0$, (by symmetric)

c) $\int_1^0 f(x)\,dx$.

$= (10-1) \times f(1)$

$= 9 \times 2$

$= 18$ Ans

d) $\int_{\frac{1}{2}}^{5} f(x)\,dx$

$= \int_{\frac{1}{2}}^{1} f(x)\,dx + \int_1^5 f(x)\,dx$

$= \frac{1}{2}(b+b')h + (5-1) \times 2$

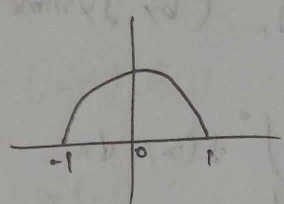
$= \frac{1}{2}\left(f(\tfrac{1}{2}) + f(1)\right) \times \frac{1}{2} + 8$

$= \frac{1}{2} \times \frac{1}{2} \times 3 + 8 = \frac{3}{4} + 8 = \frac{3+32}{4} = \frac{35}{4}$ Ans

⊛ Evaluate following integral, using appropriate formula from geometry.

i) $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$y = +\sqrt{1-x^2}$



This is upper of the circle whose center at (0,0) and radious 1.

$= \frac{1}{2} \pi r^2$

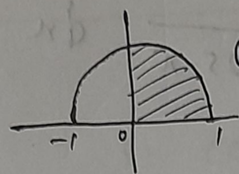
$= \frac{1}{2} \times \pi \times 1$

$= \frac{\pi}{2}$ Ans

$\therefore \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}.$

ii) $\int_0^1 \sqrt{1-x^2}\,dx$

$f(x) = +\sqrt{1-x^2}$



Quarter circle

$= \frac{1}{4}\pi r^2$

$= \frac{\pi}{4}$ Ans

$\therefore \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$

iii) $\int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}\pi r^2$

$= \frac{1}{4} \times \pi \times 2^2$

$= \pi$ Ans

$f(x) = \sqrt{4-x^2}$

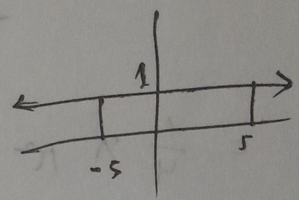
upper quarter of circle

centre, $(0,0)$

radius $= 2$

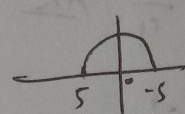
iv) $\int_{-5}^{5} \left(1+\sqrt{25-x^2}\right) dx$

$= \int_{-5}^{5} 1\, dx + \int_{-5}^{5} \sqrt{25-x^2}\, dx$



$= (5+5)\times 1 + \frac{1}{2}\times\pi r^2$

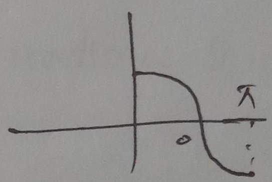
$= 10 + \frac{1}{2}\times\pi\times 25$



$= 10 + \frac{25\pi}{2}$ Ans

v) $\int_{0}^{\pi} \cos x\, dx$

$= 0$ (symmetric)



5.6

Fundamental Theorem of PART-I

<u>Theorem</u>: If f is continuous on $[a,b]$ and F is antiderivative of f on $[a,b]$ then,

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

$$\int_a^b f(x)\, dx = [F(x) + c]_a^b$$
$$= (F(b) + c) - (F(a) + c)$$
$$= F(b) + c - F(a) - c$$
$$= F(b) - F(a)$$

$$\int_1^4 2\, dx = (4-1) \times 2 = 6$$

$$\int_1^4 2\, dx = [2x]_1^4$$
$$= (2 \times 4) - (2 \times 1)$$
$$= 8 - 2 = 6 \text{ Ans}$$

$$\int_1^2 x^2\, dx$$

$$= \left.\frac{x^3}{3}\right]_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8-1}{3}$$

$$= \frac{7}{3} \text{ Ans}$$

Properties:

i) $\int_a^a f(x)\, dx = 0$

ii) $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$

iii) $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$

iv) $\int_a^b f(x) \pm g(x)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

v) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$; where c, within $[a, b]$

(*) Find the following using Fundamental theorem of calculus

Part - I

i) $\int_1^4 \frac{4}{x^2} dx$ ii) $\int_{-\frac{\pi}{4}}^{\pi/4} \cos x \, dx$ iii) $\int_{\ln 2}^{3} 5e^x \, dx$

iv) $\int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}}$ v) $\int_{-1}^{1} |2x-1| \, dx$ vi)

Solution

i) $\int_1^4 \frac{4}{x^2} dx$

$= 4 \int_1^4 \frac{1}{x^2} dx$

$= 4 \cdot \frac{x^{-1}}{-1} \Big]_1^4$

$= -\frac{4}{x} \Big]_1^4$

$= -\frac{4}{4} + \frac{4}{1}$

$= -1 + 4 = 3$ Ans

ii) $\sin x \Big]_{-\frac{\pi}{4}}^{\pi/4}$

$= \sin \frac{\pi}{4} - \sin(-\frac{\pi}{4})$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$ Ans

iii) $5e^x \Big]_{\ln 2}^{3}$

$= 5e^3 - 5e^{\ln 2}$

$= 5e^3 - 10$ Ans

iv) $\sin^{-1} x \Big]_0^{1/\sqrt{2}}$

$= \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1} 0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$. Δ

v) $f(x) = |2x-1|$

$= \begin{cases} 2x-1 & 2x-1 \geq 0 \\ -2x+1 & 2x-1 < 0 \end{cases}$

$= \begin{cases} 2x-1 & x \geq \frac{1}{2} \\ -2x+1 & x < \frac{1}{2} \end{cases}$

$\therefore \int_{-1}^{1} |2x-1|\, dx$

$= \int_{-1}^{\frac{1}{2}} |2x-1|\, dx + \int_{\frac{1}{2}}^{1} |2x-1|\, dx$

$= \int_{-1}^{\frac{1}{2}} (-2x+1)\, dx + \int_{\frac{1}{2}}^{1} (2x-1)\, dx$

$= \left[-2\frac{x^2}{2} + x\right]_{-1}^{\frac{1}{2}} + 2\cdot\frac{x^2}{2} - x\Big]_{\frac{1}{2}}^{1}$

$= -x^2+x\Big]_{-1}^{\frac{1}{2}} + x^2 - x\Big]_{\frac{1}{2}}^{1}$

$= \frac{5}{2}$ Δ

vi) $\int_1^4 \frac{|2-x|}{x}\, dx$

Here, $f(x) = \frac{|2-x|}{x}$

$= \begin{cases} \frac{2-x}{x} & 2-x \geq 0 \\ \frac{-2+x}{x} & 2-x < 0 \end{cases}$

$$= \begin{cases} \frac{2-x}{x} & 2 \geq x \,/\, x \leq 2 \\ \frac{x-2}{x} & 2 < x \,/\, x > 2 \end{cases}$$

$$= \int_1^2 \frac{2-x}{x}\,dx + \int_2^4 \frac{-2+x}{x}\,dx$$

$$= \int_1^2 \left(\frac{2}{x} - 1\right) dx + \int_2^4 \left(\frac{-2}{x} + 1\right) dx$$

$$= 2\ln x - x \Big]_1^2 + -2\ln x + x \Big]_2^4$$

$$= 4\ln 2 - 2\ln 4 + 1$$

⊛ Fundamental theorem of Part - II

$$\int_1^3 f(x)\,dx = \int_1^3 f(t)\,dt$$

⊛ $\int_1^3 t^2\,dt$

$$= \frac{t^3}{3}\Big]_1^3$$

$$= \frac{3^3}{3} - \frac{1^3}{3}$$

$$= \frac{27-1}{3}$$

$$= \frac{26}{3}$$

(*) $\int_1^x t^3 \, dt$

$= \frac{t^4}{4} \Big]_1^x$

$= \frac{x^4}{4} - \frac{1}{4}$

$= \frac{x^4 - 1}{4}$

$\boxed{\frac{d}{dx} \int_1^x t^3 \, dt = x^3}$

$\frac{d}{dx} \int_1^x t^3 \, dt = \frac{d}{dx} \left(\frac{x^4}{4} - \frac{1}{4} \right)$

$= \frac{1}{4} \cdot 4x^3 - 0$

$= x^3$

(*) $\int_3^x \sin y \, dy$

$= -\cos y \Big]_3^x$

$= -\cos x + \cos 3$

$\frac{d}{dx} \int_3^x \sin y \, dy = \frac{d}{dx} (-\cos x + \cos 3)$

$= \sin x$

Hence, the theorem state that,

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

$\frac{d}{dx}\int_1^x \sin(t^2)\,dt = \sin x^2$

$\frac{d}{dx}\int_x^0 t\sec t\,dt = -x\sec x$

$\Rightarrow -\frac{d}{dx}\int_0^x t\sec t\,dt$

$F(x) = \int_4^x \sqrt{t^2+9}\,dt$ find,

a) $F(4)$ b) $F'(4)$ c) $F''(4)$

Solution

a) $F(4) = \int_4^4 \sqrt{t^2+9}$

$= 0$

b) $F'(x) = \frac{d}{dx}\int_4^x \sqrt{t^2+9}$

$= \sqrt{x^2+9}$

$\therefore F'(4) = \sqrt{4^2+9} = \sqrt{16+9}$

$= \pm 5$

c) $F'(x) = \sqrt{x^2+9}$

$F''(x) = \dfrac{2x}{2\sqrt{x^2+9}}$

$F''(4) = \dfrac{2\cdot 4}{2\sqrt{4^2+9}}$

$= \dfrac{8}{10} = \dfrac{4}{5}$ Ans

If, $f(x) = \displaystyle\int_3^x \frac{\sin t}{t}\,dt$

then, find $f''(x)$

$f'(x) = \dfrac{d}{dx}\displaystyle\int_3^x \frac{\sin t}{t}\,dt$

$= \dfrac{\sin x}{x}$

$f''(x) = \dfrac{x\cos x - \sin x}{x^2}$

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5.9

Evaluating Definite

Integration by Substitution:

⊛ $\int_a^b f(g(x))\, g'(x)\, dx = \int_{x=a} f(u)\, du$

$u = g(x)$

$\frac{du}{dx} = g'(x)$

$du = g'(x)\, dx$

⊛ Evaluate the followings.

i) $\int_0^1 (2x+1)^3\, dx$

ii) $\int_0^{\ln 5} e^x (3-4e^x)\, dx$

iii) $\int_0^{1/\sqrt{3}} \frac{1}{1+9x^2}\, dx$

iv) $\int_{\ln 2}^{\ln(2/\sqrt{3})} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}$

v) $\int_1^3 \frac{x+2}{\sqrt{x^2+4x+7}}\, dx$

Solution

i) $\int_0^1 (2x+1)^3 \, dx$

Let,

$u = 2x+1$

$\frac{du}{dx} = 2$

$\frac{1}{2} du = dx$

if,

$x = 0$; $u = 2 \times 0 + 1 = 1$

$x = 1$, $u = 2 \times 1 + 1 = 3$

$\therefore \int_0^1 (2x+1)^3 \, dx = \frac{1}{2} \int_1^3 u^3 \, du$

$= \frac{1}{2} \left[\frac{u^4}{4} \right]_1^3$

$= \frac{1}{2} \left[\frac{3^4}{4} - \frac{1}{4} \right]$

$= \frac{1}{2} \left[\frac{81}{4} - \frac{1}{4} \right]$

$= 10$ Ans.

ii) $\int_0^{\ln 5} e^x (3 - 4e^x) \, dx$

$= \int_0^{\ln 5} (3 - 4e^x) e^x \, dx$

$u = 3 - 4e^x$

$\frac{du}{dx} = -4e^x$

$e^x dx = -\frac{1}{4} dx$

if,

$x = 0$, $u = 3 - 4e^0 = 3 - 4 = -1$

$x = \ln 5$, $u = 3 - 4e^{\ln 5} = -17$

$\therefore \int_{-1}^{-17} u \cdot \left(-\frac{1}{4}\right) du$

$= -\frac{1}{4} \int_{-1}^{-17} u\, du$

$= -\frac{1}{4} \left[\frac{u^2}{2}\right]_{-1}^{-17}$

$= -\frac{1}{4} \left[\frac{-17^2}{2} - \frac{-1^2}{2}\right]$

$= \cdot ---$

$= ---$

iii) $\int_0^{\frac{1}{\sqrt{3}}} \frac{1}{1+9x^2} dx$

$= \int_0^{1/\sqrt{3}} \frac{1}{1+(3x)^2} dx$

Let,

$u = 3x$

$\frac{du}{dx} = 3$

$dx = \frac{1}{3} du$

if,

$x = 0;\quad u = 3 \cdot 0 = 0$

$x = \frac{1}{\sqrt{3}};\quad u = 3 \cdot \frac{1}{\sqrt{3}} = \sqrt{3}$

$$= \frac{1}{3} \int_{0}^{\sqrt{3}} \frac{1}{1+u^2}\, du$$

$$= \frac{1}{3} \left[\tan^{-1} u \right]_{0}^{\sqrt{3}}$$

$$= \frac{1}{3} \left[\tan^{-1}\sqrt{3} - \tan^{-1} 0 \right]$$

$$= \frac{1}{3} \left[\frac{\pi}{3} - 0 \right]$$

$$= \frac{\pi}{9} \text{ Ans}$$

iv) $$\int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-x}}{\sqrt{1-e^{-2x}}}\, dx$$

$$= \int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-x}}{\sqrt{1-(e^{-x})^2}}\, dx$$