Chapter - 1

1.2/

a) $y = ax + a^2$

$\Rightarrow \frac{dy}{dx} = a$

$\therefore \quad y = \frac{dy}{dx}x + \left(\frac{dy}{dx}\right)^2$

Here,

order = 1

degree = 2

b) $y = a\cos x + b\sin x$

$\Rightarrow \frac{dy}{dx} = -a\sin x + b\cos x$

$\Rightarrow \frac{d^2y}{dx^2} = -a\cos x - b\sin x$

$= -(a\cos x + b\sin x)$

$= -y$

$\therefore \quad \frac{d^2y}{dx^2} + y = 0$

Here,

order = 2

degree = 1

c)

$$y = e^x (a\cos x + b\sin x)$$

$$\Rightarrow \frac{dy}{dx} = e^x(-a \cancel{\cos x} \sin x + b\cos x) + e^x(a\cos x + b\sin x)$$

$$= e^x(-a\sin x + b\cos x) + y$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-a\cos x - b\sin x) + e^x(-a\sin x + b\cos x)$$

$$= \frac{dy}{dx} + e^x(-(a\cos x + b\sin x)) + e^x(-a\sin x + b\cos x)$$

$$= \frac{dy}{dx} - y + e^x(-a\sin x + b\cos x)$$

$$= \frac{dy}{dx} - y + \frac{dy}{dx} - y$$

$$= 2\frac{dy}{dx} - 2y$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Here, order = 2

degree = 1

1.3/

$$y = ae^{3x} + be^{x}$$

$$\Rightarrow \frac{dy}{dx} = 3ae^{3x} + be^{x} \quad \cdots \text{(i)}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 9ae^{3x} + be^{x} \quad \cdots \text{(ii)}$$

(ii) - (i)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 6ae^{3x}$$

$$\Rightarrow \frac{1}{6}e^{-3x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) = a$$

substitute a in (ii),

$$be^{x} = \frac{d^2y}{dx^2} - 9e^{3x}\,\frac{1}{6}e^{-3x}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$$

$$= \frac{d^2y}{dx^2} - \frac{3}{2}\frac{d^2y}{dx^2} + \frac{3}{2}\frac{dy}{dx}$$

$$= -\frac{1}{2}\frac{d^2y}{dx^2} + \frac{3}{2}\frac{dy}{dx}$$

$$\therefore b = \left(-\frac{1}{2}\frac{d^2y}{dx^2} + \frac{3}{2}\frac{dy}{dx}\right)e^{-x}$$

Therefore,

$$y = ae^{3x} + be^{x}$$

$$y = \frac{1}{6}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \left(-\frac{1}{2}\frac{d^2y}{dx^2} + \frac{3}{2}\frac{dy}{dx}\right)$$

$$y = -\frac{1}{3}\frac{d^2y}{dx^2} + \frac{4}{3}\frac{dy}{dx}$$

$$\Rightarrow 3y = -\frac{d^2y}{dx^2} + 4\frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$ Ans

1.4/

$$y = e^{mx}(A\cos nx + B\sin nx)$$

$$\frac{dy}{dx} = e^{mx}(-An\sin nx + Bn\cos nx) + me^{mx}(A\cos nx + B\sin nx)$$

$$\frac{dy}{dx} = e^{mx}(-An\sin nx + Bn\cos nx) + my$$

$$\frac{d^2y}{dx^2} = m\frac{dy}{dx} + e^{mx}(-An^2\cos nx - Bn^2\sin nx) + me^{mx}(-An\sin nx + Bn\cos nx)$$

$$\frac{d^2y}{dx^2} = m\frac{dy}{dx} - n^2y + \left(\frac{dy}{dx} - my\right)m$$

$$\frac{d^2y}{dx^2} = m\frac{dy}{dx} - n^2y + m\frac{dy}{dx} - m^2y$$

$$= 2m\frac{dy}{dx} - y(m^2+n^2)$$

$$\therefore \frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + y(m^2+n^2) = 0$$

Ans

15

$$y = e^{x/a} - e^{x/a}$$

$$\frac{dy}{dx} = \frac{1}{a}e^{x/a} + \frac{1}{a}e^{-x/a}$$

$$= \frac{1}{a}\left(e^{x/a} + e^{-x/a}\right)$$

$$= \frac{1}{a}\sqrt{\left(e^{x/a} - e^{-x/a}\right)^2 + 4\cdot e^{x/a}\cdot e^{-x/a}}$$

$$\frac{dy}{dx} = \frac{1}{a}\sqrt{y^2+4}$$

$$a\cdot\frac{dy}{dx} = \sqrt{y^2+4}$$

$$a = \frac{\sqrt{y^2+4}}{\frac{dy}{dx}}$$

$$\therefore y = e^{x/\frac{\sqrt{y^2+4}}{\frac{dy}{dx}}} - e^{-x/\frac{\sqrt{y^2+4}}{\frac{dy}{dx}}}$$

$$= e^{\frac{x\frac{dy}{dx}}{\sqrt{y^2+4}}} - e^{-\frac{x\frac{dy}{dx}}{\sqrt{y^2+4}}}$$

Ans

Chapter-2

2.1/

$$y\sqrt{1+x^2}\,dy - x\sqrt{1+y^2}\,dx = 0$$

Let,

$$u = 1+y^2$$

$$\frac{du}{dy} = 2y$$

$$y\,dy = \frac{1}{2}du$$

$$\int \frac{1}{\sqrt{u}}\,\frac{1}{2}du$$

$$= \frac{1}{2}\,2\sqrt{u} + c$$

$$= \sqrt{1+y^2}$$

$$\Rightarrow \frac{y\,dy}{\sqrt{1+y^2}} - \frac{x\,dx}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \int \frac{y\,dy}{\sqrt{1+y^2}} - \int \frac{x\,dx}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \sqrt{1+y^2} - \sqrt{1+x^2} + c = 0$$

Ans

2.2/

$$3e^{2x}\sec^2 3y\,dy + 2(e^{2x}-1)\tan 3y\,dx = 0$$

$$\Rightarrow 3e^{2x}\sec^2 3y\,dy = -2(e^{2x}-1)\tan 3y\,dx$$

$$\Rightarrow \int \frac{3\sec^2 3y}{\tan 3y}\,dy = \int \frac{-2(e^{2x}-1)}{e^{2x}}\,dx$$

$$\Rightarrow \ln(\tan 3y) = \int \frac{-2e^{2x}+2}{e^{2x}}\,dx$$

$$\Rightarrow \ln(\tan 3y) = \int \left(-2 + 2e^{-2x}\right) dx$$

$$\therefore \ln(\tan 3y) = -2x - e^{-2x} + C$$

Ans

Let,

$$u = \tan 3y$$

$$\frac{du}{dy} = \sec^2 3y \cdot 3$$

$$\sec^2 3y = \frac{1}{3}\,du$$

$$\int \frac{\cancel{3}}{u}\cdot\frac{1}{\cancel{3}}\,du$$

$$\int \frac{1}{u}\,du$$

$$= \ln u + c$$

$$= \ln(\tan 3y) + c$$

2.3)

$$xy^4\,dx + (y^2+2)e^{-3x}\,dy = 0$$

$$\Rightarrow \frac{x\,dx}{e^{-3x}} + \frac{(y^2+2)\,dy}{y^4} = 0$$

$$\cancel{\Rightarrow \int \frac{x}{e^{-3x}}\,dx =}$$

$$\begin{array}{ccc} x & + & e^{3x} \\ 1 & - & \frac{e^{3x}}{3} \\ 0 & & \frac{e^{3x}}{9} \end{array}$$

$$\Rightarrow \int \left(y^{-2} + 2y^{-4}\right) dy = -\int x e^{3x}\,dx$$

$$\Rightarrow \frac{y^{-1}}{-1} + 2\,\frac{y^{-3}}{-3} = -\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c\right]$$

$$\therefore -\frac{1}{y} - \frac{2}{3y^3} = -\frac{1}{3}xe^{3x} + \frac{1}{9}e^{3x} + c$$

Ans

$\underline{2.4}$

$$(x+y)^2\left(x\frac{dy}{dx}+y\right) = xy\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \int \frac{x\frac{dy}{dx}+y}{xy}\,dx = \int \frac{1+\frac{dy}{dx}}{(x+y)^2}\,dx$$

let,

$u = x+y$

$\frac{du}{dx} = 1+\frac{dy}{dx}$

$\cancel{\int \frac{\frac{du}{dx}}{u^2}}$

$\left(1+\frac{dy}{dx}\right)dx = du$

$$\Rightarrow \ln(xy) = \int \frac{du}{u^2}$$

$$= \int u^{-2}\,du$$

$$= -\frac{1}{u}+c$$

$$\therefore \ln(xy) = -\frac{1}{x+y}+c$$

Ans

$\underline{2.5}$

$y \xrightarrow{+} \sin y$

$1 \xrightarrow{-} -\cos y$

0

$\cos y$

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$$

$$\Rightarrow \int(\sin y + y\cos y)\,dy = \int x(2\log x+1)\,dx$$

$$\Rightarrow -\cos y + y\sin y + \cos y = \int (2x\log x + x)\,dx$$

$$\Rightarrow y\sin y = \left[\log x\int 2x\,dx - \int\left\{\frac{d}{dx}(\log x)\int 2x\,dx\right\}dx\right]$$

$$+ \frac{x^2}{2} -$$

$\cancel{\Rightarrow y\sin y} = \log x \cdot x^2 - \int$

2.5|

$$\frac{dy}{dx} = \frac{x(2\ln x+1)}{\sin y + y\cos y}$$

$$\Rightarrow \int(\sin y + y\cos y)\,dy = \int x(2\log x+1)\,dx$$

$$\Rightarrow -\cos y + y\sin y + \cos y = \int(2x\ln x + x)\,dx$$

$$\Rightarrow y\sin y = \left[\ln x\int 2x\,dx - \int\left\{\frac{d}{dx}(\ln x)\int 2x\,dx\right\}dx\right] + \frac{x^2}{2}$$

$$= \left[x^2\ln x - \int\frac{1}{x}x^2\,dx\right] + \frac{x^2}{2}$$

$$= x^2\ln x - \frac{x^2}{2} + \frac{x^2}{2} + c$$

$$\therefore y\sin y = x^2\ln x + c$$

H.W - from Lecture-2

1.

$$\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$$

Let,

$u = x+y$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx} - 1$

$$\Rightarrow \frac{du}{dx} - 1 = \cos u + \sin u$$

$$\Rightarrow \frac{du}{dx} = \cos u + \sin u + 1$$

$$\Rightarrow \int \frac{du}{\cos u + \sin u + 1} = \int dx$$

$$\Rightarrow \int \frac{du}{\frac{1-\tan^2\frac{u}{2}}{1+\tan^2\frac{u}{2}} + \frac{2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}} + 1} = x + c$$

$$\Rightarrow \int \frac{du}{\frac{1-\tan^2\frac{u}{2} + 2\tan\frac{u}{2} + 1 + \tan^2\frac{u}{2}}{1+\tan^2\frac{u}{2}}} = x + c$$

$$\Rightarrow \int \frac{du}{\frac{2+2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}}} = x + c$$

$$\Rightarrow \int \frac{1+\tan^2\frac{u}{2}}{2(1+\tan\frac{u}{2})}\,du = x+c$$

Let,

$v = \tan\frac{u}{2}$

$\frac{u}{2} = \tan^{-1}(v)$

$\frac{1}{2}\cdot\frac{du}{dv} = \frac{1}{v^2+1}$

$\therefore du = \frac{2}{v^2+1}\,dv$

$$\Rightarrow \int \frac{(1+v^2)}{2(1+v)}\cdot\frac{2}{(v^2+1)}\,dv = x+c$$

$$\Rightarrow \int \frac{1}{1+v}\,dv = x+c$$

$$\Rightarrow \ln(1+v) = x+c$$

$$\Rightarrow \ln\left(1+\tan\frac{u}{2}\right) = x+c$$

$$\therefore \ln\left(1+\tan\frac{x+y}{2}\right) = x+c$$

2)

$$\frac{dy}{dx} = \sec(x+y)$$

Let,

$u = x+y$

$\frac{du}{dx} = 1+\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{du}{dx}-1$

$$\Rightarrow \frac{du}{dx}-1 = \sec u$$

$$\Rightarrow \frac{du}{dx} = \sec u+1$$

$$\Rightarrow \int \frac{du}{1+\sec u} = \int dx$$

$$\Rightarrow \int \frac{du}{1+\frac{1}{\cos u}} = x+c$$

$$\Rightarrow \int \frac{du}{\frac{\cos u+1}{\cos u}} = x+c$$

$$\cos u+1 \overline{\big)\ \cos u\ \big(\ 1}$$
$$\frac{\underset{(-)}{\cos u} \underset{(-)}{+1}}{-1}$$

$$\Rightarrow \int \frac{\cos u}{\cos u+1}\, du = x+c$$

$$= 1-\frac{1}{\cos u+1}$$

$$\Rightarrow \int \left(1-\frac{1}{\cos u+1}\right) du = x+c$$

$$\Rightarrow \int \left(1-\frac{1}{2\cos^2\frac{u}{2}}\right) du = x+c$$

$$\Rightarrow \int \left(1-\frac{1}{2}\sec^2\frac{u}{2}\right) du = x+c$$

$$\Rightarrow u-\frac{1}{2}\tan\frac{u}{2}\cdot 2 = x+c$$

$$\Rightarrow u-\tan\frac{u}{2} = x+c$$

$$\Rightarrow (x+y)-\tan\left(\frac{x+y}{2}\right) = x+c$$

Ans

3)

$$\frac{dy}{dx} = \cos(x+y)$$

Let,

$$u = x+y$$

$$\frac{du}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{du}{dx} - 1$$

$$\Rightarrow \frac{du}{dx} - 1 = \cancel{\cot}\ \cos u$$

$$\Rightarrow \frac{du}{dx} = \cos u + 1$$

$$\Rightarrow \int \frac{du}{\cos u + 1} = \int dx$$

$$\Rightarrow \int \frac{1}{2\cos^2\frac{u}{2}}\, du = x + c$$

$$\Rightarrow \int \frac{1}{2} \sec^2\frac{u}{2}\, du = x + c$$

$$\Rightarrow \frac{1}{2} \tan\frac{u}{2} \cdot 2 = x + c$$

$$\Rightarrow \tan\frac{u}{2} = x + c$$

$$\therefore \tan\frac{x+y}{2} = x + c$$

Ans

4|

$$\frac{dy}{dx} = \frac{1}{1+\tan x}$$

$$\Rightarrow dx = (1+\tan x)\, dy$$

$$\Rightarrow \int \frac{dx}{1+\tan x} = \int dy$$

Let,

$$u = \tan x$$

$$x = \tan^{-1} u$$

$$\frac{dx}{du} = \frac{1}{1+u^2}$$

$$\Rightarrow dx = \frac{1}{1+u^2}\, du$$

$$\Rightarrow \int \frac{\frac{1}{1+u^2}}{1+u}\, du = y+c$$

$$\Rightarrow \int \frac{1}{(1+u)(1+u^2)}\, du = y+c$$

Here

$$\frac{1}{(1+u)(1+u^2)} = \frac{A}{1+u} + \frac{Bu+c}{1+u^2}$$

$$\therefore A(1+u^2) + (Bu+c)(1+u) = 1$$

$$A + u^2A + Bu + c + Bu^2 + cu = 1$$

$$u^2(A+B) + u(B+c) + (A+c) = 1$$

$$\therefore A+B = 0 \Rightarrow A = -B \Rightarrow A = \frac{1}{2}$$

$$B+c = 0 \Rightarrow B+1+B = 0$$

$$\Rightarrow 2B = -1$$

$$\therefore B = -\frac{1}{2}$$

$$A+c = 1$$

$$\Rightarrow -B+c = 1$$

$$\Rightarrow c = 1+B$$

$$\therefore c = \frac{1}{2}$$

$$\Rightarrow \int \left(\frac{\frac{1}{2}}{1+u} + \frac{-\frac{1}{2}u + \frac{1}{2}}{1+u^2} \right) du = y+c$$

$$\Rightarrow \frac{1}{2}\ln(1+u) + \int \frac{-\frac{1}{2}u}{1+u^2}\, du + \int \frac{\frac{1}{2}}{1+u^2}\, du = y+c$$

$$\Rightarrow \frac{1}{2}\ln(1+u) - \frac{1}{4}\int \frac{2u}{1+u^2}\, du + \frac{1}{2}\tan^{-1}u = y+c$$

$$\Rightarrow \frac{1}{2}\ln(1+u) - \frac{1}{4}\ln(1+u^2) + \frac{1}{2}\tan^{-1}u = y+c$$

$$\therefore \frac{1}{2}\ln(1+\tan x) - \frac{1}{4}\ln(1+\tan^2 x) + \frac{1}{2}\tan^{-1}(\tan x) = y+c$$

$\therefore \frac{1}{2}\ln(1+\tan x) - \frac{1}{4}\ln(1+\tan^2 x) + \frac{x}{2} = y + c$

H.W. from Lecture-3

Homogeneous ODE

1) $(3x-2y)\,dy = (2x-3y)\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{2x-3y}{3x-2y}$

Let, $v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{2x-3vx}{3x-2vx}$

$= \frac{x(2-3v)}{x(3-2v)}$

$\Rightarrow x\frac{dv}{dx} = \frac{2-3v}{3-2v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{2-3v-v(3-2v)}{3-2v} = \frac{2-3v-3v+2v^2}{3-2v}$

$$\Rightarrow x\frac{dv}{dx} = \frac{2-6v+2v^2}{3-2v}$$

$$\Rightarrow \int \frac{3-2v}{2-6v+2v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \int \frac{3-2v}{2v^2-6v+2}\,dv = \ln x + c$$

$$\Rightarrow \frac{1}{2}\int \frac{3-2v}{v^2-3v+1}\,dv = \ln x + c$$

$$\Rightarrow -\frac{1}{2}\int \frac{2v-3}{v^2-3v+1}\,dv = \ln x + c$$

$$\Rightarrow -\frac{1}{2}\ln\left(v^2-3v+1\right) = \ln x + c$$

$$\therefore -\ln\left(\frac{y^2}{x^2} - 3\frac{y}{x} + 1\right) = 2\ln x + c$$

Ans

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$$x\,dy - y\,dx = 2\sqrt{y^2-x^2}\,dx$$

$$\Rightarrow x\,dy = 2\sqrt{y^2-x^2}\,dx + y\,dx$$

$$\Rightarrow x\,dy = \left(2\sqrt{y^2-x^2}+y\right)dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sqrt{y^2-x^2}+y}{x}$$

Let,

$$v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2\sqrt{v^2x^2-x^2}+vx}{x}$$

$$= \frac{2\sqrt{x^2(v^2-1)}+vx}{x}$$

$$= \frac{x\left(2\sqrt{v^2-1}+v\right)}{x}$$

$$x\frac{dv}{dx} = 2\sqrt{v^2-1}+v-v$$

$$x\frac{dv}{dx} = 2\sqrt{v^2-1}$$

$$\Rightarrow \int \frac{1}{\sqrt{v^2-1}}\,dv = 2\int \frac{1}{x}\,dx$$

$$\Rightarrow \ln\left(v+\sqrt{v^2-1}\right) = 2\ln x + c$$

$$\therefore \ln\left(\frac{y}{x}+\sqrt{\frac{y^2}{x^2}-1}\right) = 2\ln x + c$$

3)

$$xy^2\,dy = (x^3+y^3)\,dx$$

Let,

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{x^3+v^3x^3}{xv^2x^2}$$

$$= \frac{x^3(1+v^3)}{x^3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^3}{v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^3-v^3}{v^2} = \frac{1}{v^2}$$

$$\Rightarrow \int v^2\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \frac{v^3}{3} = \ln x + c$$

$$\therefore \frac{y^3}{x^3} = 3\ln x + c$$

Ans

41)

$$\left(2\sqrt{xy}-y\right)dx - x\,dy = 0$$

$$\Rightarrow \quad x\,dy = \left(2\sqrt{xy}-y\right)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\sqrt{xy}-y}{x}$$

Let,

$$v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{2\sqrt{xvx}-vx}{x}$$

$$= \frac{\cancel{2x}\,2x\sqrt{v}-vx}{x}$$

$$v + x\frac{dv}{dx} = \frac{x\left(2\sqrt{v}-v\right)}{x}$$

$$\Rightarrow \quad x\frac{dv}{dx} = 2\sqrt{v}-v-v = 2\sqrt{v}-2v = 2\left(\sqrt{v}-v\right)$$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{v}-v} = 2\int \frac{1}{x}\,dx$$

Let,

$$u = \sqrt{v}$$

$$\frac{du}{dv} = \frac{1}{2\sqrt{v}}$$

$$dv = 2\sqrt{v}\,du$$

$$= 2u\,du$$

$$\Rightarrow \quad \int \frac{2u}{u-u^2}\,du = 2\ln x + c$$

$$\Rightarrow \quad 2\int \frac{u}{-u(u-1)}\,du = 2\ln x + c$$

$$\Rightarrow \quad -2\int \frac{1}{u-1}\,du = 2\ln x + c$$

$\Rightarrow -2\ln(u-1) = 2\ln x + c$

$\Rightarrow -2\ln(\sqrt{v}-1) = 2\ln x + c$

$\therefore -2\ln\left(\sqrt{\frac{y}{x}}-1\right) = 2\ln x + c$

Ans

5|

$-y\,dx + (x+\sqrt{xy})\,dy = 0$

$\Rightarrow (x+\sqrt{xy})\,dy = y\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

Let,

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{vx}{x+\sqrt{xvx}}$

$= \frac{vx}{x+x\sqrt{v}}$

$= \frac{vx}{x(1+\sqrt{v})}$

$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+\sqrt{v}} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{v-v-v\sqrt{v}}{1+\sqrt{v}} = \frac{-v\sqrt{v}}{1+\sqrt{v}}$

$$\Rightarrow \int \frac{1+\sqrt{v}}{-v\sqrt{v}}\, dv = \int \frac{1}{x}\, dx$$

$v^1 \cdot v^{\frac{1}{2}}$

$= v^{\frac{3}{2}}$

$= v^{-3/2}$

$$\Rightarrow \int \left(\frac{1}{-v\sqrt{v}} + \frac{1}{-v} \right) dv = \ln x + c$$

$$\Rightarrow -\int \left(\frac{1}{v\sqrt{v}} + \frac{1}{v} \right) dv = \ln x + c$$

$$\Rightarrow -\int \left(v^{-3/2} + \frac{1}{v} \right) dv = \ln x + c$$

$$\Rightarrow -\frac{v^{-\frac{1}{2}}}{-\frac{1}{2}} + \ln v = \ln x + c$$

$$\Rightarrow 2\frac{1}{\sqrt{v}} + \ln v = \ln x + c$$

$$\Rightarrow 2\frac{1}{\sqrt{\frac{y}{x}}} + \ln \frac{y}{x} = \ln x + c$$

Ans

6)

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2}$$

$$\Rightarrow x\, dy = \left(y + \sqrt{x^2 - y^2} \right) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 - v^2x^2}}{x}$$

Let,

$v = \frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{vx + x\sqrt{1-v^2}}{x}$$

$$\cancel{= v + \sqrt{1-v^2}}$$

$$\Rightarrow x\frac{dv}{dx} = v + \sqrt{1-v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1-v^2}$$

$$\Rightarrow \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{1}{x}\,dx$$

$$\Rightarrow \sin^{-1} v = \ln x + c$$

$$\therefore \sin^{-1}\frac{y}{x} = \ln x + c$$

Ans

R.H. form

Let,

$x = \alpha + h$

$y = \beta + k$

$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$

1) $$\frac{dy}{dx} = \frac{2x + 2y - 2}{3x + y - 5}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{2(\alpha+h) + 2(\beta+k) - 2}{3(\alpha+h) + (\beta+k) - 5}$$

$$= \frac{(2\alpha + 2\beta) + (2h + 2k - 2)}{(3\alpha + \beta) + (3h + k - 5)}$$

Now,

$$2h + 2k - 2 = 0 \quad \Rightarrow \quad h + k = 1$$
$$3h + k - 5 = 0 \qquad\quad 3h + k = 5$$
$$(-)\ \ (-)\ \ (-)$$
$$-2h = -4$$
$$\therefore h = 2$$
$$\therefore k = -1$$

Then, $x = \alpha + 2$

$\Rightarrow \alpha = x - 2$

$y = \beta - 1$

$\Rightarrow \beta = y + 1$

Now,

$$\frac{d\beta}{d\alpha} = \frac{2\alpha + 2\beta}{3\alpha + \beta}$$

Let,

$$v = \frac{\beta}{\alpha}$$
$$\beta = v\alpha$$
$$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha}$$

$$\Rightarrow v + \alpha\frac{dv}{d\alpha} = \frac{2\alpha + 2v\alpha}{3\alpha + v\alpha}$$

$$= \frac{\alpha(2 + 2v)}{\alpha(3 + v)}$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{2 + 2v}{3 + v} - v$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{2 + 2v - 3v - v^2}{3 + v} = \frac{2 - v - v^2}{3 + v}$$

$$\Rightarrow \int \frac{3+v}{2-v-v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\int \frac{v+3}{v^2+v-2}\,dv = \ln x + c$$

$$\Rightarrow -\int \frac{v+3}{(v+2)(v-1)}\,dv = \ln x + c$$

$$\cancel{\Rightarrow -\int \frac{1+3}{(v+2)}}$$

$$\Rightarrow -\int \frac{-2+3}{(v+2)(-2-1)}\,dv - \int \frac{1+3}{(1+2)(v-1)}\,dv = \ln x + c$$

$$\Rightarrow \frac{1}{3}\int \frac{1}{v+2}\,dv - \frac{4}{3}\int \frac{1}{v-1}\,dv = \ln x + c$$

$$\Rightarrow \frac{1}{3}\ln(v+2) - \frac{4}{3}\ln(v-1) = \ln x + c$$

$$\Rightarrow \therefore \ln\left(\frac{y}{x}+2\right) - 4\ln\left(\frac{y}{x}-1\right) = 3\ln x + c$$

2]

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+3}$$

Let,

$x = \alpha + h$

$y = \beta + k$

$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{(\alpha+h)+2(\beta+k)+3}{2(\alpha+h)+(\beta+k)+3}$$

$$= \frac{(\alpha+2\beta)+(h+2k+3)}{(2\alpha+\beta)+(2h+k+3)}$$

Here,

$h+2k+3=0$

$2h+k+3=0$

$$\Rightarrow \begin{array}{rcl} 2h+4k &=& -6 \\ 2h+k &=& -3 \\ (-)\quad(-) & & (+) \\ \hline 3k &=& -3 \end{array}$$

$\therefore k = -1$

$\therefore h = -1$

Then,

$x = \alpha - 1$

$\therefore \alpha = x+1$

$y = \beta - 1$

$\therefore \beta = y+1$

Now,

$$\frac{d\beta}{d\alpha} = \frac{\alpha+2\beta}{2\alpha+\beta}$$

Let,

$v = \frac{\beta}{\alpha}$

$\beta = v\alpha$

$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha}$

$$\Rightarrow v+\alpha\frac{dv}{d\alpha} = \frac{\alpha+2v\alpha}{2\alpha+v\alpha}$$

$$= \frac{\alpha(1+2v)}{\alpha(2+v)}$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v} = \frac{1-v^2}{2+v}$$

$$\Rightarrow \int \frac{2+v}{1-v^2}\, dv = \int \frac{1}{\alpha}\, d\alpha$$

$$\Rightarrow -\int \frac{v+2}{v^2-1}\, dv = \int \frac{1}{\alpha}\, d\alpha$$

$$\Rightarrow -\int \frac{v+2}{(v+1)(v-1)}\, dv = \ln\alpha + c$$

$$\Rightarrow -\int \left(\frac{-1+2}{(v+1)(-1-1)} + \frac{1+2}{(1+1)(v-1)} \right) dv = \ln\alpha + c$$

$$\Rightarrow -\int \left(\frac{1}{-2(v+1)} + \frac{3}{2(v-1)} \right) dv = \ln\alpha + c$$

$$\Rightarrow \frac{1}{2}\ln(v+1) - \frac{3}{2}\ln(v-1) = \ln\alpha + c$$

$\cancel{\therefore \ln\left(\frac{y}{x}+1\right) - 3\ln\left(\frac{y}{x}-1\right) = 2\ln x +}$

$$\Rightarrow \ln\left(\frac{\beta}{\alpha}+1\right) - 3\ln\left(\frac{\beta}{\alpha}-1\right) = 2\ln\alpha + c$$

$$\therefore \ln\left(\frac{y+1}{x+1}+1\right) - 3\ln\left(\frac{y+1}{x+1}-1\right) = 2\ln(x+1) + c$$

Ans

3|

$$\frac{dy}{dx} = \frac{6x-2y-7}{2x+3y-6}$$

Let,

$x = \alpha + h$

$y = \beta + k$

$$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{6(\alpha+h)-2(\beta+k)-7}{2(\alpha+h)+3(\beta+k)-6}$$

$$= \frac{(6\alpha-2\beta)+(6h-2k-7)}{(2\alpha+3\beta)+(2h+3k-6)}$$

Here,

$6h-2k-7=0$

$2h+3k-6=0$

$$\Rightarrow \begin{array}{l} 12h-4k=14 \\ 12h+18k=36 \\ (-)\quad(-)\quad(-) \\ \hline -22k=-22 \end{array}$$

$\therefore k = 1$

$\therefore h = \frac{6-3\cdot 1}{2}$

$= \frac{3}{2}$

Then,

$x = \alpha + \frac{3}{2}$

$\therefore \alpha = x - \frac{3}{2}$

$y = \beta + 1$

$\therefore \beta = y - 1$

~~Now, $d\beta$~~

Now,

$$\frac{d\beta}{d\alpha} = \frac{6\alpha - 2\beta}{2\alpha + 3\beta}$$

Let,

$$v = \frac{\beta}{\alpha}$$

$$\beta = v\alpha$$

$$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha}$$

$$\Rightarrow v + \alpha\frac{dv}{d\alpha} = \frac{6\alpha - 2v\alpha}{2\alpha + 3v\alpha}$$

$$= \frac{2\alpha(3-v)}{\alpha(2+3v)}$$

$$\alpha\frac{dv}{d\alpha} = \cancel{2(3-v)}\ \frac{6-2v}{2+3v} - v$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{6-2v-2v-3v^2}{2+3v} = \frac{6-4v-3v^2}{3v+2}$$

$$\Rightarrow -\int\frac{3v+2}{3v^2+4v-6}\,dv = \int\frac{1}{\alpha}\,d\alpha$$

$$\Rightarrow -\frac{1}{2}\int\frac{6v+4}{3v^2+4v-6}\,dv = \ln\alpha + c$$

$$\Rightarrow -\frac{1}{2}\ln(3v^2+4v-6) = \ln\alpha + c$$

$$\Rightarrow -\frac{1}{2}\ln\left(3\frac{\beta^2}{\alpha^2} + 4\frac{\beta}{\alpha} - 6\right) = \ln\alpha + c$$

$$\Rightarrow -\ln\left(3\frac{(y-1)^2}{(x-\frac{3}{2})^2} + 4\frac{y-1}{x-\frac{3}{2}} - 6\right) = 2\ln\left(x-\frac{3}{2}\right) + c$$

Extra from previous semester!

5|

$$2y^3\,dx + (x^2-3y^2)x\,dy = 0$$

$$\Rightarrow (x^3-3xy^2)\,dy = -2y^3\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2y^3}{(x^3-3xy^2)}$$

Let,

$$v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{-2v^3x^3}{x^3-3xv^2x^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-2v^3}{1-3v^2} - v$$

$$= \frac{-2v^3 - v + 3v^3}{1-3v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^3-v}{1-3v^2}$$

$$\Rightarrow \int \frac{1-3v^2}{v^3-v}\,dv = \int \frac{1}{x}\,dx$$

$$\cancel{\Rightarrow \int \frac{1-3v^2}{v(v^2-1)}\,dv = \ln x + c}$$

$$\Rightarrow -\int \frac{3v^2-1}{v^3-v}\,dv = \ln x + c$$

$$\Rightarrow -\ln(v^3 - v) = \ln x + C$$

$$\Rightarrow -\ln\left(\frac{y^3}{x^3} - \frac{y}{x}\right) = \ln x + C$$

6|

$$(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0$$

$$\Rightarrow (y^2 + 2xy - x^2)\,dy = -(x^2 + 2xy - y^2)\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$$

Let,

$$v = \frac{y}{x}$$

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2x^2 - 2xvx - x^2}{v^2x^2 + 2xvx - x^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 2v - 1}{v^2 + 2v - 1} - v$$

$$= \frac{v^2 - 2v - 1 - v^3 - 2v^2 + v}{v^2 + 2v - 1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-v^3 - v^2 - v - 1}{v^2 + 2v - 1}$$

$$\Rightarrow -\int \frac{v^2 + 2v - 1}{v^3 + v^2 + v + 1}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow -\int \frac{v^2+2v-1}{(v^2+1)(v+1)}\,dv = \ln x + c$$

$$\Rightarrow -\int\left(\frac{2v}{v^2+1} - \frac{1}{v+1}\right)dv = \ln x + c$$

$$\Rightarrow -\ln(v^2+1) + \ln(v+1) = \ln x + c$$

$$\Rightarrow -\ln\left(\frac{y^2}{x^2}+1\right) + \ln\left(\frac{y}{x}+1\right) = \ln x + c$$

$$\int \frac{1}{x^2-a^2}\,dx = \int \frac{1}{(x+a)(x-a)}\,dx$$

$$= \int\left(\frac{1}{(x+a)(-a-a)} + \frac{1}{(a+a)(x-a)}\right)dx$$

$$= \int\left(\frac{1}{2a(x-a)} - \frac{1}{2a(x+a)}\right)dx$$

$$= \frac{1}{2a}\ln(x-a) - \frac{1}{2a}\ln(x+a) + c$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

(*) $\int \frac{1}{\sqrt{x^2 \pm a^2}}\, dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + c$

H.W. from Lecture - 4

Exact ODE

2.34/

$$(2x^3 + 4y)\, dx + (4x + y - 1)\, dy = 0$$

Let,

$M = 2x^3 + 4y$ | $N = 4x + y - 1$

$\frac{\partial M}{\partial y} = 4$ | $\frac{\partial N}{\partial x} = 4$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [Exact ODE]

Therefore,

$$\int_{y = const} (2x^3 + 4y)\, dx + \int (y - 1)\, dy = 0$$

$$\Rightarrow \frac{1}{2}x^4 + 4xy + \frac{1}{2}y^2 - y = c$$

Ans

2.35/

$$(e^{2y} - y\cos xy)\,dx + (2xe^{2y} - x\cos xy + 2y)\,dy = 0$$

Let,

$$M = e^{2y} - y\cos xy$$

$$\frac{\partial M}{\partial y} = 2e^{2y} - (y\cdot(-\sin xy)\cdot x + \cos xy)$$

$$= 2e^{2y} + xy\sin xy - \cos xy$$

and,

$$N = 2xe^{2y} - x\cos xy + 2y$$

$$\frac{\partial N}{\partial x} = 2e^{2y} - (x\cdot(-\sin xy)\cdot y + \cos xy)$$

$$= 2e^{2y} + xy\sin xy - \cos xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad [\text{Exact ODE}]$$

Therefore

$$\int_{y=\text{const.}} (e^{2y} - y\cos xy)\,dx + \int 2y\,dy = 0$$

$$\Rightarrow \quad xe^{2y} - \sin xy + y^2 = C$$

An

2.36/

$$(y^3 - y^2 \sin x - x)\,dx + (3xy^2 + 2y\cos x)\,dy = 0$$

Let,

$M = y^3 - y^2 \sin x - x$

$\frac{\partial M}{\partial y} = 3y^2 - 2y\sin x$

$N = 3xy^2 + 2y\cos x$

$\frac{\partial N}{\partial x} = 3y^2 - 2y\sin x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [Exact ODE]

Therefore,

$$\int_{y=\text{const.}} (y^3 - y^2 \sin x - x)\,dx = 0$$

$$\Rightarrow \quad xy^3 + y^2\cos x - \frac{1}{2}x^2 = C$$

Ans

2.42/

$$(x^4 + y^4)\,dx - xy^3\,dy = 0 \quad \text{.....(i)}$$

Let,

$M = x^4 + y^4$

$\frac{\partial M}{\partial y} = 4y^3$

$N = -xy^3$

$\frac{\partial N}{\partial x} = -y^3$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ [Not Exact ODE but Homogeneous]

$\therefore$ I.F $= \dfrac{1}{Mx + Ny} = \dfrac{1}{x(x^4+y^4) - y(xy^3)}$

$= \dfrac{1}{x^5 + xy^4 - xy^4}$

$= \dfrac{1}{x^5}$

(i) × I.F.,

$$\frac{1}{x^5}(x^4+y^4)\,dx - \frac{1}{x^5}(xy^3)\,dy = 0$$

$$\Rightarrow \left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx - \frac{y^3}{x^4}\,dy = 0 \quad \text{(ii)}$$

Let,

$M' = \dfrac{1}{x} + \dfrac{y^4}{x^5}$

$\therefore \dfrac{\partial M'}{\partial y} = 4\dfrac{y^3}{x^5}$

$N' = -\dfrac{y^3}{x^4} = -y^3 \cdot x^{-4}$

$\dfrac{\partial N'}{\partial x} = 4\dfrac{y^3}{x^5}$

$$\therefore \frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} \quad \text{[Exact ODE]}$$

Therefore, $\displaystyle\int_{y=\text{const.}} \left(\frac{1}{x} + \frac{y^4}{x^5}\right)dx = 0$

$$\Rightarrow \ln(x) - \frac{y^4}{4x^4} = c$$

Ans