31.05.2022

Tuesday

Lecture - 1

Chapter - 1

The Foundations: Logic and Proofs

Propositional Logic

Argument

i. Man is mortal.

ii. Socrates is a human.

⇒ So, Socrates will die

→ Propositions → Special Statement

⊛ Propositions : statements that have definitive truth values.

⊛

× 1. Where is my book?

2. $2+7 = 33$

3. CSE 173 is a bad course.

× 4. $x + 25 = 1$ → Here, we don't know is it true or false because x is unknown

Transformation of Propositions.

Process :

i. Negation ($\neg$, $\sim$ → Logical Operator

① It is not the case ~~of~~ that p.

p : Man is mortal → Variable

$\neg p$: Man is not mortal → Negation of p.

Truth table:

p	$\neg p$
T	F
F	T

⊛ Compound Propositions:

Conjunction

"Today is Tuesday. → p

AND ⇒ ∧ → Conjuction

I eat rice." → q

⇒ $p \wedge q$ ⇒ p conjuction q.

Truth Table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

Given that, p, q are propositions.

Then their disjunction denoted as $p \vee q$. → or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

By default disjunction is inclusive or.

Exclusive or:

p or q, but not both

⇒ In $p \oplus q$, one of p and q must be true, but not both.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement

Implication ⇒ "$\longrightarrow$"

If p and q are prepositions, then $p \to q$ is a conditional statement or implication which is read as "if p, then q".

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Different ways of expressing "$p \to q$"

⇒ if p, then q

⇒ if p, q

⇒ q unless $\neg p$

⇒ q if p

⇒ q whenever p

⇒ q follows from p

⇒ p implies q

⇒ p only if q

⇒ q when p

⇒ p is sufficient for q

⇒ q is necessary for p

⇒ a necessary condition for p is q

⇒ a sufficient condition for q is p.

Converse of $p \to q \Rightarrow q \to p$

Contrapositive of $p \to q \Rightarrow \neg q \to \neg p$

inverse of $p \to q \Rightarrow \neg p \to \neg q$

~~Bic~~ Biconditional statement

"$\leftrightarrow$" bi-implication

If p and q are propositions. ~~then~~ then we can form the biconditional proposition $p \leftrightarrow q$, read as "p if and only if q."

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

when two values are same

$(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p)$

Logically equivalent

⊗ $\neg(p \leftrightarrow q) \equiv p \oplus q$

⊛

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$p \to q \equiv \neg p \vee q$

⊛ some alternative ways "p if and only if q". is expressed in English:

⇒ p is necessary and sufficient for q

⇒ if p then q, and conversely

⇒ p iff q.

L-3

07.06.2022

⊛

If today is Friday, then $2+3=6$.

p q

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

→ Vacuously true.

⊛ If I stay at home, then I am sick

I stay at home only if I am sick.

⊛ $p \oplus q = (p \wedge \neg q) + (q \wedge \neg p)$ → $\vee$

⊛ Demorgans Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

⊛ $\neg(p_1 \vee p_2 \ldots \vee \ldots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n$

⊛ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ ⟶ Associative Law

$\neg$	High
$\wedge$	
$\vee$	
$\rightarrow$	
$\leftrightarrow$	Low

⊛ Distributive Law:

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$

$(p \wedge q) \vee (r \wedge s) \equiv R \vee (r \wedge s) \equiv (R \vee r) \wedge (R \vee s)$

Types of Compound Propositions

⊛ Tautology : When all the outpots/truth values are "true."

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

⊛ Contradiction : When all the outputs/truth values are "false."

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

⊛ Contrigency : Outputs True/False mixed.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Transformation of $p \to q$ into other conditional forms.

(i) Converse of $p \to q$: $q \to p$

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$\therefore p \to q \not\equiv q \to p$

(ii) inverse of $p \to q$: $\neg p \to \neg q$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

$\therefore p \to q \not\equiv \neg p \to \neg q$

(iii) contrapositive of $p \to q$ is $\neg q \to \neg p$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$\therefore p \to q \equiv \neg q \to \neg p$

proof technique.

L-4 / 12.06.2022 /

(*) Show that below expression is a Tautology:

$(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

$$\underbrace{(p \wedge q)}_{R} \rightarrow \underbrace{(p \vee q)}_{S}$$

$\equiv R \rightarrow S$

$\equiv \neg R \vee S$ Def^n of implication

$\equiv \neg(p \wedge q) \vee (p \vee q)$ $\because R = p \wedge q, \because S = p \vee q$

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ Demorgan's Law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ Associative Law

$\equiv T \vee T$ Negation Law

$\equiv T$

⊛ $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$: check if tautology

Predicate

All students from CSE173, SEC-1 are good.

$P(x)$: $\underbrace{x}_{\text{Subject}}$ $\underbrace{\text{from CSE 173, SEC-1 is good}}_{\text{Predicate}}$

→ Propositional function

Predicated statement $\begin{cases} P(x): \\ x \in \text{SEC-1} \end{cases}$ → Domain

For all → Universal

There's at least one → Existential

Quantification process:

→ Evaluates if predicated statement is True or False

Quantifier

→ Universal → All, each, everybody

→ Existential → There's at least one, some, a few, some of them

Universal Quantifier:

For All $\xrightarrow{\text{Notation}}$ $\forall x \; P(x)$

$x \in$ CSE173, SEC-1

⊛ There exists at least $\overbrace{\text{one student}}^{x}$ in CSE173, SEC-1 who ~~is from~~ plays football.

$P(x)$: x in CSE173 .. plays football

Existential:

$\exists x \; P(x)$

p	q	r	$\neg p$	$(p \vee q)$	$(\neg p \vee r)$	$(q \vee r)$	$((p \vee q) \wedge (\neg p \vee r))$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	T	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	T	T	F	T
F	F	F	T	F	T	F	F	T

$\therefore ((p \vee q) \wedge (\neg p \vee r)) \longrightarrow (q \vee r) \equiv T$

L-5 / 14.06.2022

⊛ Quantification

→ Quantifier

→ Universal → $\forall x$ (For all)

→ Existential → $\exists x$ (There exist)

$\forall x \; P(x)$ T/F?

- Predicate
- Propositional function

$x \in$ Domain

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \ldots \wedge (Px_n) \equiv T$$

$\exists x \; P(x)$ T/F?

$$P(x_1) \vee P(x_2) \ldots \vee P(x_n)$$

$\downarrow$

$T \equiv T$

⊛ $\forall x\, P(x)$, $P(x) = x^{100} \geq x^{20}$

$x \in \mathbb{N} \rightarrow T$

$x \in \mathbb{R}^{+} \rightarrow F \longrightarrow x = 0.5$ → Counter example

$0.5^{100} < 0.5^{20}$

$\exists x\, P(x)$, $P(x) = \frac{1}{x} > x$

$x \in \mathbb{N} \rightarrow F$

$x \in \mathbb{Z} \rightarrow T \longrightarrow x = -2$ → Counter Example

$\frac{1}{-2} = -0.5 > -2$

⊛

$\neg(\forall x\, p(x)) \equiv ? \equiv \exists x\, \neg p(x)$

$\neg(\exists x\, p(x)) \equiv ? \equiv \forall x\, \neg p(x)$

⊛ $\underbrace{\text{Every students}}_{\forall x}$ $\underbrace{\text{from SEC-1 has done CSE 115.}}_{P(x)}$

↓

Negation

It is not the case that every student from SEC-1 has done CSE 115.

So, there exists at least one student who has not done CSE115.

$\exists x \neg p(x)$.

⊛ $\neg(p(x_1) \wedge p(x_2) \ldots \wedge p(x_n))$

$\equiv \neg p(x_1) \vee \neg p(x_2) \vee \ldots \neg p(x_n)$

$\equiv \exists x \neg p(x)$.

⊛ All Birds can fly.

$p(x)$: x can fly

$x \in$ All Birds $\longrightarrow \forall x P(x)$

$x \in$ Animal kingdom $\Rightarrow$

$p(x)$: x is Bird

$q(x)$: x can fly

if x is a bird, it can fly

$\forall x \, (p(x) \rightarrow q(x))$

⇒ All student from SEC-1 have taken CSE115

$x \in NSU$

$P(x)$: Student x is from SEC-1

$q(x)$: Student x has taken CSE 115

$$\forall x \ (P(x) \rightarrow q(x))$$

⇒ There exists at least one student in SEC-1 who has done CSE 2115

$\exists x \ P(x) \Rightarrow x \in$ SEC-1

$P(x)$: x is from SEC-1

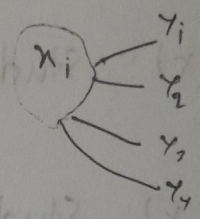
$q(x)$: x has done CSE 215

$\exists x \ (P(x) \wedge q(x))$

L-6 / 19.06.2022

Nested Quantifiers

$\forall x \; \forall y$



$x, y \in \mathbb{Z}$

$\forall x \; \exists y \quad x+y=0 \rightarrow T$

2

$-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$

$2-2=0 \rightarrow T$

(*) $\forall x \; \forall y \quad x+y=0 \rightarrow F$

$x, y \in \mathbb{Z}$

(*)

$\forall x \; \exists y \quad x+y=0 \rightarrow F$

$x \in \mathbb{Z}, \; y \in \mathbb{R}^+$

(*) $\forall x \; \exists y \quad x \cdot y = 0 \rightarrow F$

$x \in \mathbb{Z}, \; y \in \mathbb{R}^+$

(*) $\forall x \; \exists y \quad x \cdot y = 0 \rightarrow T$

$x \in \mathbb{Z}, \; y \in \mathbb{R}^+ \cup \{0\}$

⊛ $\forall x \, \exists y \;\; x \cdot y = 1 \longrightarrow F$

$x \in \mathbb{Z}, \; y \in \mathbb{R}^+ \cup \{0\}$

$x \in \mathbb{Z}, \; y \in \mathbb{R}^+ \cup \{0\}$

⊛ $\longrightarrow T$

$\forall x \, \exists y \;\; x \cdot y = 0$

⊛ $\exists x \, \forall y \;\; x \cdot y = 0 \longrightarrow T$

⊛ $\exists x \, \forall y \;\; x \cdot y = 0 \longrightarrow F$

$x \in \mathbb{Z} - \{0\}, \; y \in \mathbb{R}^+ \cup \{0\}$

⊛ Every real number except zero has a multiplicative inverse.

$$\forall x \left(x \neq 0 \longrightarrow \exists y \, (xy = 1) \right)$$

$$\neg \forall x \underbrace{\exists y \ (xy = 1)}_{Q(x,y)}$$

$$\begin{aligned}
\neg \forall x \, Q(x,y) &\equiv \exists x \ \neg Q(x,y) \\
&\equiv \exists x \ \neg \exists y \ (xy = 1) \\
&\equiv \exists x \ \forall y \ \neg (xy = 1) \\
&\equiv \exists x \ \forall y \quad xy \neq 1
\end{aligned}$$

⊛ There does ~~not~~ not exists a $\underbrace{\text{person}}_{w}$ who has taken a ~~fly~~ $\underbrace{\text{flight}}_{f}$ on $\underbrace{\text{every}}_{\forall}$ $\underbrace{\text{airline}}_{a}$ in the world.

$\exists$

$\Rightarrow$

$p(w,f)$: Person "w" has taken a flight "f".

$q(f,a)$: flight "f" is in airline "a".

$$\neg\left(\exists w \ \forall a \ \exists f \ \left(p(w,f) \wedge q(f,a)\right)\right)$$

(*) $\neg(\exists w\, \forall a\, \exists f\, \underbrace{(p(w,f) \wedge q(f,a))}_{Q})$

$$\neg(\exists w\, Q) \equiv \forall w\, \neg Q$$

$$= \forall w\, \neg(\underbrace{\forall a\, \exists f\, (p(w,f) \wedge q(f,a))}_{M})$$

$$= \forall w\, (\neg \forall a\, M)$$

$$= \forall w\, \exists a\, \neg M$$

$$= \forall w\, \exists a\, \neg(\exists f\, (p(w,f) \wedge q(f,a)))$$

$$= \forall w\, \exists a\, \forall f\, \neg(p(w,f) \wedge q(f,a))$$

$$= \forall w\, \exists a\, \forall f\, (\neg p(w,f) \vee \neg q(f,a))$$

⊛ Inference (Process)

→ Argument

↓

Sequence of statement that with conclusions.

Argument:

Man is mortal

Socrates is a human

So, socrates will die. → Conclusion

Valid Invalid

⊛ If you have an RFID, then you get attendance.

You have a RFID

therefore

You get attendance.

Argument form:

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

$p \rightarrow q$, p } Premises

$\therefore q$ → Conclusion

L-7 / 21.06.2022

Inference

to infer

Process to deduce the conclusion from collected information

If you have an RFID, you get attendance.

p q

$\Rightarrow$ You have an RFID

p

therefore / so / thus

You get attendance.

q

$\Rightarrow$ Premises $\begin{cases} p \to q \\ p \end{cases}$

conclusion $\to \therefore q$

$[((p \to q) \wedge p) \to q]$

should be tautology

Argument

- Valid
- Invalid

Valid → When all the premises are true, the conclusion must be true

→ How do we check it?

- Tautology Approach
- Critical row approach

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Tautology

Critical row approach:

1. $p \vee q$
2. $p \to r$
3. $q \to r$

$\therefore r$

$$\left[\left((p \vee q) \wedge (p \to r) \wedge (q \to r)\right) \to r\right]$$

should be true.

p	q	r	$p \vee q$	$p \to r$	$q \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	T	T	T
T	F	F	T	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	F	T	T
F	F	F	F	T	T

$\Rightarrow$ Tautology / True

Rule

⊛ Modus Ponens (Mode that affirms)

1. $p \to q$
2. p

$\therefore q$

⊛ Modus Tollens (Mode that denies)

1. $p \to q$
2. $\neg q$

$\therefore \neg p$

⊛ Conjunction Rule:

p
q

$p \wedge q$

⊛
1. $p \wedge q \to r$
2. p
3. q
4. $? = p \wedge q$ (Conjunction Rule)
5. $? = r$ (Modus Ponens)

⊛ Simplification :

$$\frac{1.\ p \wedge q}{p, q}$$

⊛ Addition Rule :

$$\frac{p}{\therefore p \vee \text{"anything"}}$$

⊛ $(p \vee s) \rightarrow q$

$$\frac{\begin{array}{c} p \\ r \end{array}}{p \vee s}$$

⊛ Disjunctive Syllogism :

$$\frac{\begin{array}{c} p \vee q \\ \neg p \end{array}}{\therefore q}$$

⊛ Resolution :

$$\frac{\begin{array}{c} p \vee q \\ \neg p \vee r \end{array}}{\therefore q \vee r}$$

⊛ Hypothetical Syllogism :

$$\frac{\begin{array}{c} p \rightarrow q \\ q \rightarrow r \end{array}}{\therefore p \rightarrow r}$$

L-8/26.06.2022/

(*)

It is not sunny this afternoon, and it is colder than yesterday. We will go for swimming only if it is sunny. If we do not go for swimming, then we will take a canoe trip. If we take a canoe trip, then we will be ~~to~~ home by sunset.

$\neg p$ q r p $\neg r$ s s t

Conclusion: We will reach home by sunset.

$\Rightarrow$

1. $\neg p \wedge q$ Premise 1
2. $r \rightarrow p$ Premise 2
3. $\neg r \rightarrow s$ Premise 3
4. $s \rightarrow t$ Premise 4
5. $\neg p$ Simplification of 1
6. $\neg r$ Modus Tollens 2, 5

7. s Modus Ponens 3,6

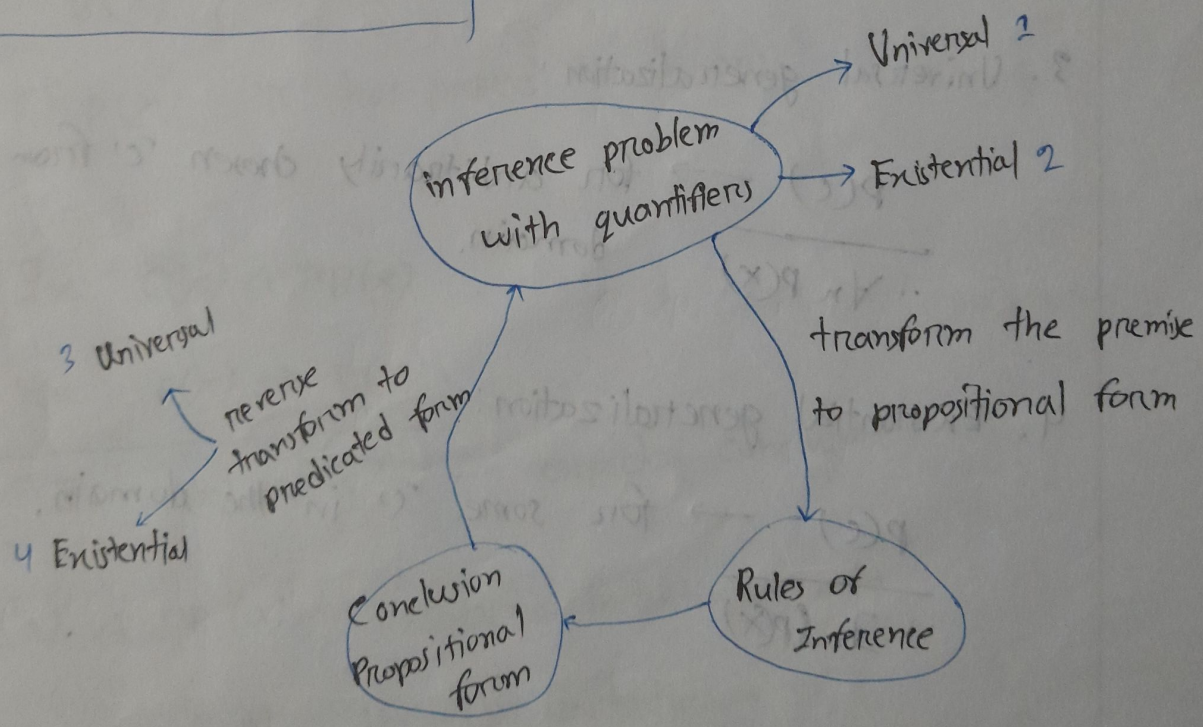
8. t Modus Ponens 4,7

Quantifiers

→ Rules of Inference

$$\frac{p \to q \quad p}{\therefore q} \qquad \frac{p \to q \quad q \to r}{\therefore p \to r}$$

→ Propositional form



Rules

1. Universal instantiation:

$$\frac{\forall x\ P(x)\ T}{\therefore P(c)}$$

$\rightarrow$ for any randomly chosen specific 'c'.

2. Existential instantiation:

$$\frac{\exists x\ P(x)\ T}{\therefore P(c)}$$

$\rightarrow$ specific 'c'

3. Universal generalization:

$$\frac{P(c)}{\therefore \forall x\ P(x)}$$

$\rightarrow$ for arbitrarily chosen 'c' from the domain.

4. Existential generalization:

$$\frac{P(c)}{\therefore \exists x\ P(x)}$$

$\rightarrow$ for some 'c' in the domain.

⊛ A student in this class has not read the book.

Everyone from this class passed the first exam.

Therefore,

Someone, who passed the first exam has not read the book.

$\Rightarrow$ $C(x)$: 'x' is in this class.

$B(x)$: 'x' has read the book.

$P(x)$: 'x' has passed the exam.

1. $\exists x \; C(x) \wedge \neg B(x)$

2. $\forall x \; C(x) \rightarrow P(x)$

$\exists x \; P(x) \wedge \neg B(x) \rightarrow$ expected conclusion.

3. $C(a) \wedge \neg B(a)$ Existential instantiation of 1

4. $C(a) \rightarrow P(a)$ Universal instantiation of 2

5. $C(a)$ Simplification of 3

6. $P(a)$ Modus Ponens of 4,5

7. $\neg B(a)$ Simplification of 3

8. $P(a) \wedge \neg B(a)$ Conjunction of 6,7

9. $\exists x \; P(x) \wedge \neg B(x)$ Existential generalization of 8.

Proof Techniques

Theorem: A statement that can be shown to be true.

Axioms: Statements that we assume to be true.

Lemma: Less important theorem that is helpful in the proof of other results.

Corrolary: Established directly from already proven theorem.

~~L-8~~ L-9 / 28.06.2022

Conjecture

Proposed as true statement based on partial evidence.

Proof Techniques → Direct
Proof Techniques → Indirect

Direct: Apply rules directly on the given statement.

⊛ Proof it

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

⇒

1. $p \vee q$
2. ~~$\neg q$~~ $\neg p$
3. $\neg(\neg p) \vee q$ — Double negation of 1
4. $\neg p \rightarrow q$ — Def^n of implication of 3
5. q — Modus Ponens of 2, 4

⊛ Given, m is even and n is odd, their sum $(m+n)$ is always odd.

$\Rightarrow$

$m = 2k$; where k is an integer.

$n = 2j+1$; where j is an integer.

$$m+n = 2k + 2j + 1$$
$$= 2(k+j)+1$$
$$= 2 \cdot \text{integer} + 1$$
$$= \text{odd}$$ (proved).

⊛ Given m is even, show that m^2 is even.

$\Rightarrow$

⊛ Given n is an integer and $3n+2$ is odd, show that n is odd.

$\Rightarrow$

$$3n + 2 = 2k+1$$
$$3n = 2k-1$$
$$n = \frac{2k-1}{3}$$ → Deadlock → Now apply indirect method

$\Rightarrow$ if $\underbrace{3n+2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

$$p \to q \equiv \neg q \to \neg p \equiv \text{contrapositive}$$

proof by contraposition

If n is $\underbrace{\text{not odd}}_{\text{even}}$, then $3n+2$ is $\underbrace{\text{not odd}}_{\text{even}}$.

$n = 2k$

$$\begin{aligned} 3n+2 &= 6k+2 \\ &= 2(3k+1) \\ &= 2 \cdot \text{integer} \\ &= \text{even} \end{aligned}$$

⊛ Prove that, if $x^2 - 6x + 5$ is even, then x is odd.

$\Rightarrow$

<u>Direct</u>:

$$\begin{aligned} x^2 - 6x + 5 &= 2k \\ x^2 - 5x - x + 5 &= 2k \\ x(x-5) - 1(x-5) &= 2k \\ (x-5)(x-1) &= 2k \longrightarrow \text{Deadlock} \end{aligned}$$

indirect:

if $\underbrace{x^2-6x+5 \text{ is even}}_{p}$, then $\underbrace{x \text{ is odd}}_{q}$.

$$p \to q \equiv \neg q \to \neg p$$

$\neg q$: x is not odd. $\to$ even

$\neg p$: x^2-6x+5 is not even. $\to$ odd

$x = 2k$; k is an integer.

$$x^2-6x+5 = (2k)^2 - 6 \cdot 2k + 5$$
$$= 4k^2 - 12k + 4 + 1$$
$$= 2(2k^2 - 6k + 2) + 1$$
$$= 2 \cdot \text{integer} + 1$$
$$= \text{odd}.$$

(proved)

⊛ Proof by contradiction:

⊛ $\sqrt{2}$ is irrational number.

Assume it to be true

~~Corrt~~

contradiction $\rightarrow$ $\sqrt{2}$ is a rational number.

$\sqrt{2} = \frac{p}{q}$; p, q are integers and $q \neq 0$

simplest form $\boxed{\gcd(p,q) = 1}$

$$\Rightarrow \frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$

$$= 2 \cdot \text{integer}$$

$$= \text{even}$$

so, p is even

$\therefore p = 2k$ as p is even.

$$\therefore p^2 = 2q^2$$

$$2q^2 = 4k^2$$

$$q^2 = 2k^2$$

$$= 2 \cdot \text{integer}$$

$$= \text{even}$$

$\therefore q$ is even.

p, q, even so, $\boxed{\gcd(p,q) > 1}$

contradiction.

So, $\sqrt{2}$ is not rational number

$\therefore$ $\sqrt{2}$ is irrational.

(proved)

L-10 / 03.07.2022 /

Proof Techniques

i) Direct

ii) Indirect → Proof by Contraposition
→ Proof by Contradiction
→ Proof by Cases.

Proof by Cases

$p \to r$ case-1

$q \to r$ case-2

$p \vee q$

$\therefore r$

(*) Given x is an integer.

show that x^2+x is even.

Case-1

If $\overbrace{x \text{ is even}}^{p}$, $\overbrace{x^2+x \text{ is even}}^{r}$

$x = 2k$ → integer

$x^2+x = 4k^2+2k$

$= 2(2k^2+k)$

$= 2 \cdot \text{integer}$

$= \text{even}$

Case-2

If $\overbrace{x \text{ is odd}}^{q}$, then $\overbrace{x^2+x \text{ is even.}}^{r}$

$x = 2k+1$

$$x^2+x = (2k+1)^2 + 2k+1$$
$$= 4k^2 + 4k + 1 + 2k + 1$$
$$= 4k^2 + 6k + 2$$
$$= 2(2k^2 + 3k + 1)$$
$$= 2 \cdot \text{integer}$$
$$= \text{even.}$$

So, $p \to r$

$q \to r$

$p \vee q$

$\therefore r$

(*) Given n is an integer show that if n is not divisible by 3 then $n^2 = 3k+1$. (integer)

$\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$ $\frac{8}{3}$ $\frac{9}{3}$ ....

$n = 3m+1$ $n = 3m+2$ $n = 3m$ $n = 3m$

Case-1

$n = 3m+1$

$n^2 = (3m+1)^2$

$= 9m^2 + 6m + 1$

$= 3(3m^2+2m)+1$

$= 3 \cdot \text{integer} + 1$

$= 3k+1$

Case-2

$n = 3m+2$

$n^2 = (3m+2)^2$

$= 9m^2 + 12m + 3 + 1$

$= 3(3m^2+4m+1)+1$

$= 3 \cdot \text{integer} + 1$

$= 3k+1$

L-11 / 05.07.2022

Mathematical Induction

* $1+2+3+4+5+\cdots+n = \frac{n(n+1)}{2}$

* $1^2+2^2+3^2+4^2+5^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

*

Basic step:

$n=1$

L.H.S. $= 1$

R.H.S $= \frac{1(1+1)}{2} = \frac{2}{2} = 1$

Inductive step:

Lets assume that the given statement is true for $n=k$. That is $p(k)$ is true.

Now, we have to show that the statement is true for $n=k+1$. That is we have to show that,

$$1+2+3+\cdots+k+(k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\boxed{P(k) \rightarrow P(k+1)}$$

$$L.H.S = 1+2+3+\dots\dots\dots\dots + k + k+1$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2} \equiv R.H.S.$$

⊛ Use mathematical induction to show that,

$(6^n - 1)$ is divisible by 5. where $n \in N$

$\Rightarrow$

Basic step:

$n = 1$

$6^1 - 1 = 6 - 1 = 5$, which is divisible by 5.

Inductive step:

Lets assume that this statement is true for $n = k$.

That means that $(6^k - 1)$ is divisible by 5.

Now, we have to show that, the statement is true for $(k+1)$. That means that $(6^{k+1} - 1)$ is divisible by 5.

$$6^{k+1} - 1 = 6^k \cdot 6 - 1$$

$$= 6^k \cdot (5+1) - 1$$

$$= 5 \cdot 6^k + 6^k - 1$$

$$= 5 \cdot 6^k + 5m$$ → m is a positive number

$$= 5(6^k + m)$$

$$= 5 \cdot \text{integer}$$

$\Rightarrow$ which is divisible by 5.

Proof by Equivalence

⊛ If n is a positive integer, then $\underbrace{n \text{ is odd}}_{p}$

if and only if $\underbrace{n^2 \text{ is odd}}_{q}$.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$p \to q$: if n is odd, then n^2 is odd.

$q \to p$: if n^2 is odd, then n is odd.

$n = 2k+1$, k is an integer

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$
$$= 2 \cdot \text{integer} + 1$$
$$= \text{odd}$$

$\neg p \to \neg q$

n is even,

$$n = 2k$$
$$n^2 = 4k^2$$
$$= 2 \cdot 2k^2$$
$$= 2 \cdot \text{integer}$$
$$= \text{even.}$$

Chapter - 2

Set Function

Set: Collection of objects.

- List of Distinct elements
  - separated by comma (,)
- Names
- Numbers
- Anything and everything

$A = \{1, 2, 3, 5\}$

Cardinality : Total number of elements in a set.

- Notation : $|A| = 4$

Singleton set : Set that have only one element.

$A = \{1\}$ $\quad B = \{5\}$ $\quad C = \{33\}$

$|A| = 1$ $\quad |B| = 1$ $\quad |K| = 1$

Empty set : $\phi = \{\ \}$

- Null set

$|\phi| = 0$

$|\{\phi\}| = 1$

$|\{\phi, \{\phi\}\}| = 2$

Set Theory

↳ Cardinality ⇒ Number of elements.

$x \in A \Rightarrow \underbrace{x \text{ is in } A}_{\text{Set builder Notation}}$

Union

such that

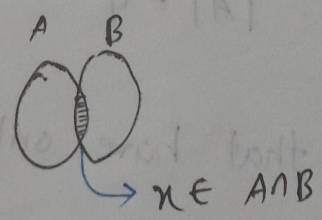
$x \in A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

is in

is in (belongs to)

⊛ $x \in (A \cap B)$

$= \{x \mid x \in A \text{ and } x \in B\}$



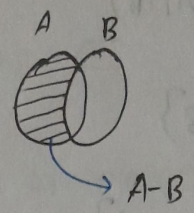
⊛ $A = \{1, 2, 3, 4\}$

$B = \{1, 7, 5, 3\}$

$\therefore A \cap B = \{1, 3\}$

difference

(*) $A - B \equiv \{x / x \in A \text{ and } x \notin B\}$



(*) $A = \{1, 2, 7, 9\}$

$B = \{3, 5, 7, 9, 1\}$

$\therefore A - B = \{2\}$

$C = \{3, 2, 5, 7, 9, 1\}$

$\therefore A - C = \phi$ or $\{\}$

(*) Set compliment

De Morgans Law

(i) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

(ii) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Prove - (i)

$$L.H.S \equiv \{x \mid x \in \overline{A \cap B}\}$$

$$\equiv \{x \mid x \notin A \cap B\}$$

$$\equiv \{x \mid \neg(x \in A \cap B)\}$$

$$\equiv \{x \mid \neg(x \in A \text{ and } x \in B)\}$$

$$\equiv \{x \mid \neg(x \in A) \text{ or } \neg(x \in B)\}$$

$$\equiv \{x \mid x \notin A \text{ or } x \notin B\}$$

$$\equiv \{x \mid x \in \bar{A} \text{ or } x \in \bar{B}\}$$

$$\equiv \{x \mid x \in \bar{A} \cup \bar{B}\}$$

$$\equiv \bar{A} \cup \bar{B}$$

(*) Power Set

→ Set of all the subset of a given set.

P(...) → Notation

(*)

$A = \{1, 2, 3\}$

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$

Proper Subset : $\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \phi\}$

Notation $\Rightarrow \subset$

Number of elements will be strictly less than parent set.

$A \subseteq C \Rightarrow |A| \leq |C|$

$B \subset A \Rightarrow$ B is a subset of A.

Cartesian Product :

A, B sets

cross

$A \times B \equiv \{(a,b) \mid a \in A \text{ and } b \in B\}$

$B \times A \equiv \{(b,a) \mid b \in B \text{ and } a \in A\}$

$A = \{1, 4, 6\}$

$B = \{p, q\}$

$A \times B = \{(1,p), (1,q), (4,p), (4,q), (6,p), (6,q)\}$

$B \times A = \{(p,1), (p,4), (p,6), (q,1), (q,4), (q,6)\}$

$\therefore A \times B \neq B \times A$

⊛ $X = \{1, 2\}$

$|P(X)| = 2^{|X|}$

$= 2^2 = 4$

⊛ $|P(A \times B)|$

$= 2^{|A \times B|}$

$= 2^{|A| \times |B|}$

$= 2^{3 \times 2}$

$= 2^6 = 64$

⊛ $A = \{1, 4\}$

$B = \{p, q\}$

$C = \{*, \square\}$

$A \times B \times C = \{(a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$

Set
- Finite ("Limited" number of elements)
- Infinite (that is not finite)
  - ex. $\mathbb{N}$: set of natural numbers.
  - Countable
  - Uncountable

L-13/12.07.2022/

Quizzes-1 /Question/

1/

a)

$$(P \wedge Q \wedge \neg R \wedge \neg S) \vee (P \wedge \neg Q \wedge R \wedge \neg S) \vee (P \wedge \neg Q \wedge \neg R \wedge S)$$
$$\vee (\neg P \wedge Q \wedge R \wedge \neg S) \vee (\neg P \wedge Q \wedge \neg R \wedge S) \vee (\neg P \wedge \neg Q \wedge R \wedge S)$$

2/

$(\neg p \vee q) \rightarrow r$ Premises-1

$(s \rightarrow r) \vee \neg t$ Premises-2

$\neg s \wedge u$ Premises-3

$u \rightarrow t$ Premises-4

Conclusion $\Rightarrow p$

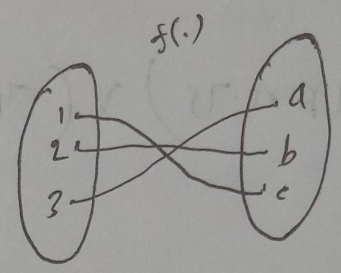
Function

- Relation between sets
- Mapping

$f: A \rightarrow B$

A → Domain

B → Co-Domain



i) All the elements of set A must have an image B

ii) Multiple images of single element of A are not possible.

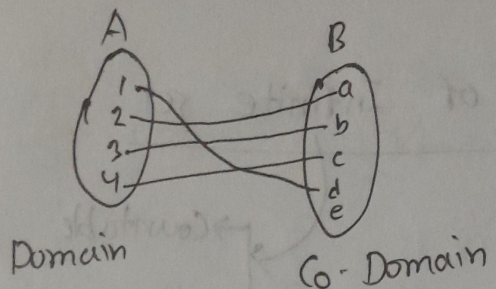
* There are three kind of function:

i) One-to-one / injective function

ii) Onto / surjective function

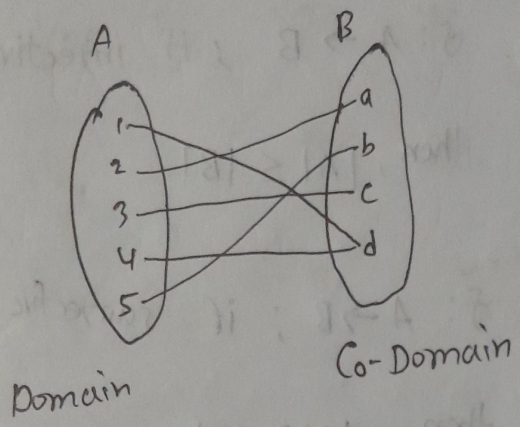
iii) One to one correspondence / Bijective function

i)



$|A| = 4 \leq |B| = 4$

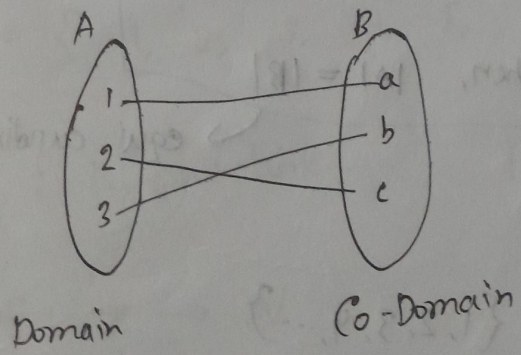
ii)



(*) There cannot be domainless image.

$|A| = 5 \geq |B| = 5$

iii)



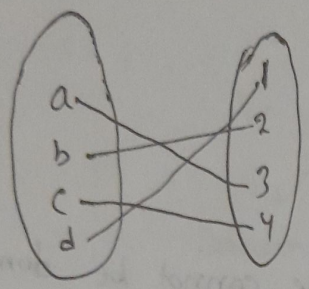
(*) One-to-one + Onto (both)

$|A| = |B|$

L-14 | 22.07.2022

Countability of Infinite set

→ Countable
→ uncountable



Surjective

many to one

1. $f: A \to B$; if injective

then, $|A| \leq |B|$

2. $f: A \to B$; if surjective

then, $|A| \geq |B|$

3. $f: A \to B$; if Bijective

then, $|A| = |B|$

→ equi cardinality

if exist then countable

if not exist, then uncountable

Reference ← $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

countable ← $\mathbb{Z} = \{-\infty, \ldots, \infty\}$

uncountable ← $\mathbb{R} = \{-\infty, \pm \text{integer}, \text{fraction}, \infty\}$

⊛ Q^+ = Set of all positive rational number.

→ Countable ?

or Uncountable ?

$\Rightarrow$

$$
\begin{array}{cccccccc}
1/1 & 2/1 & 3/1 & 4/1 & 5/1 & 6/1 & 7/1 & 8/1 \\
1/2 & 2/2 & 3/2 & 4/2 & 5/2 & 6/2 & 7/2 & 8/2 \\
1/3 & 2/3 & 3/3 & 4/3 & 5/3 & 6/3 & 7/3 & 8/3 \\
1/4 & 2/4 & 3/4 & 4/4 & 5/4 & 6/4 & 7/4 & 8/4 \\
1/5 & 2/5 & 3/5 & 4/5 & 5/5 & 6/5 & 7/5 & 8/5
\end{array}
$$

$$Q^+ = \{ 1, \tfrac{1}{2}, 2, 3, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, \tfrac{4}{1}, 5, \tfrac{1}{5}, \cdots$$

$$\text{index} = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \cdots$$

So, f is countable.

⊛ $f : \mathbb{Z} \to \mathbb{N}$

$-3, -2, -1, 0, 1, 2, 3$

$6, 3, 1, 2, 4, 5, 7$

⊛ Show that odd positive integers set is countable.

$\Rightarrow$

$$Z^{+}_{odd} = \{1, 3, 5, 7, 9, 11, 13, 15, \dots\}$$

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

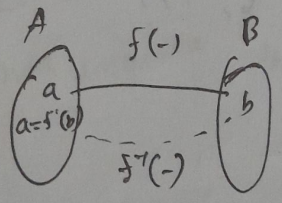
$$f(n) = 2n-1$$

⊛ Inverse function → inverse of a function.

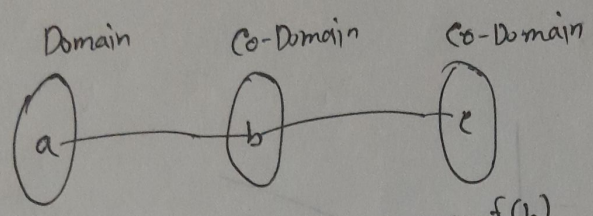
↳ Function needs to be a bijective.

↳ injective
↳ surjective } both

$f: A \to B$



⊛ Composit Function



$g(a) = b$

$c = f(b)$

$\hookrightarrow g(a)$

$= f(g(a))$

$g : A \to B$

⊛ $f(x) = 2x+3$

$g(x) = 3x+2$

$(f \circ g)(x) = f(g(x))$

$= f(3x+2)$

$= 2(3x+2)+3$

$= 6x+4+3$

$= 6x+7$

Ans

$(g \circ f)(x) = g(f(x))$

$= g(2x+3)$

$= 3(2x+3)+2$

$= 6x+9+2$

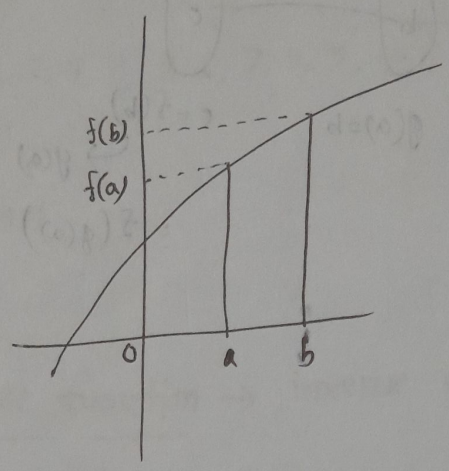
$= 6x+11$

Ans

L-15 / 24.07.2022

⊛ Increasing & Decreasing Function

⊛ Increasing function:



$$\boxed{\forall a \forall b \, (a < b \rightarrow f(a) \leq f(b))}$$

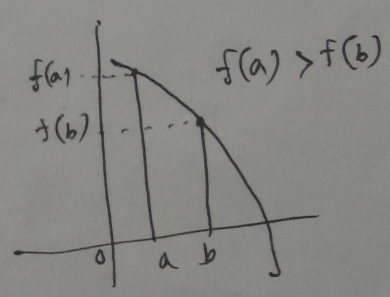
without equal sign
strictly increasing

Activation function:

$e^x \rightarrow$ increasing

$e^{-x} \rightarrow$ decreasing

⊛ Decreasing function:



$a < b$

$$\boxed{\forall a \forall b \, (a<b \rightarrow f(a) \geq f(b))}$$

without equal sign, it is strictly decreasing.

~~Write a~~

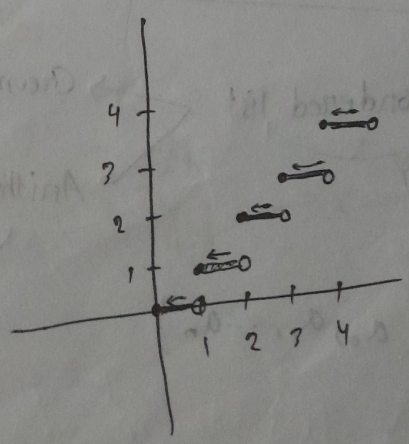
Floor & Ceiling Function

Floor :

$$\lfloor x \rfloor = \text{Floor}(x)$$

the greatest integer that is less than or equal to x.

$\lfloor 0.7 \rfloor = 0$ $\qquad$ $\lfloor 3 \rfloor = 3$



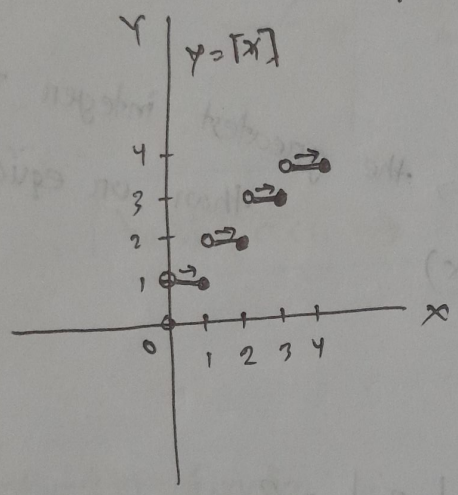
⊛ Ceiling function:

smallest integer that is largest than or equal to x.

$\lceil x \rceil = \text{ceil}(x)$

⊛ $\lceil 0.001 \rceil = 1$ $\lceil 1.001 \rceil = 2$

$\lceil -0.5 \rceil = 0$



⊛ Sequence & Summations

An ordered list $\{a_n\}$

- Geometric sequence → গুণোত্তর ধারা
- Arithmetic sequence → সমান্তর ধারা

$n=0$
$n=1$ $\{a_0, a_1, \ldots a_n\}$
$n=n$

Geometric Sequence : $ar^0, ar^1, ar^2, ar^3, \ldots, ar^n$

$$\frac{ar^1}{ar^0} = r$$

$$\frac{ar^2}{ar^1} = r$$

$$\frac{ar^n}{ar^{n-1}} = r$$

common ratio.

Arithmatric :

$$a, a+d, a+2d, a+3d, \ldots, a+nd$$

$$a+d-a = d$$

$$a+2d-a-d = d$$

$$a+3d-a-2d = d$$

common difference

Identify their types :

i) $\{S_n\} = -1+4n$

$\Rightarrow$ $n = 1$ $\Rightarrow$ $-1, 3, 7, 11$

4 4 4

common difference

$\boxed{-1+4n}$

ii) $\{b_n\} = (-1)^n$

$\Rightarrow$

$(-1)^0 = 1$

$(-1)^1 = -1$

$(-1)^2 = 1$

$\frac{-1}{1} = -1$

$\frac{1}{-1} = -1$

common ~~[illegible]~~ ratio

ar^n

$= 1(-1)^n = (-1)^n$

Summation

Lower limit

Upper limit

$\boxed{1} + 2 + 3 + 4 + \cdots + \boxed{n}$

$$\sum_{i=\text{lower limit}}^{\text{upper limit}} \text{expression of } i$$

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n$$

$\boxed{\text{Midterm}}$

L-16 / 26.07.2022

Stephen P Boyd

Summation

$$\sum_{i=\text{Lower limit}}^{\text{upper limit}} \text{'i'} \rightarrow + + +$$

$$\prod_{i=\text{Lower limit}}^{\text{Upper limit}} \text{'i'} \rightarrow \times \times \times$$

(*)

$$ar^0 + ar^1 + ar^2 + \dots + ar^n$$

$$\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1} \longrightarrow r \neq 1$$

$\Rightarrow$ Basic step:

$n = 0$

$$L.H.S. = ar^0 = a$$

$$R.H.S. = \frac{ar^{n+1} - a}{r-1}$$

$$= \frac{ar^1 - a}{r-1}$$

$$= \frac{a(r-1)}{r-1} = a = R.H.S.$$

Inductive Step:

Assume for $n=k$, statement is True

$\underbrace{P(1) \wedge \dots \wedge P(k-1) \wedge}_{\text{Strong induction}}$ $P(k)$ is true

$$\sum_{i=0}^{k} ar^i = a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

For,

$n = k+1$

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1+1} - a}{r-1}$$

$$= \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + (r-1)ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1+1} - ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1} = R.H.S.$$

⊛ $1+2+\cdots+n = \dfrac{n(n+1)}{2}$

⊛ $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6} \Rightarrow \sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$

⊛ $1^3+2^3+3^3+\cdots+n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

⊛

$$\sum_{k=50}^{100} k^2 = ?$$

$$\Rightarrow \frac{100(100+1)(2\cdot 100+1)}{6} - \frac{49(49+1)(2\cdot 49+1)}{6}$$

$$1^2+2^2+\cdots+50^2+\cdots+100^2 = ? \quad \text{— (i)}$$

$$1^2+2^2+\cdots+49^2 = ? \quad \text{— (ii)}$$

$$\boxed{\text{(i)} - \text{(ii)}}$$

⊛ $\sum_{i=1}^{4}\sum_{j=1}^{3} ij$

$$= \sum_{i=1}^{4}(i+2i+3i) = \sum_{i=1}^{4} 6i = 6\sum_{i=1}^{4} i = 6\cdot 10 = 60$$

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij+2$$

$$\sum_{i=1}^{5} 2$$

H.W.

Recursion

$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

$(n-1)! = (n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

$n! = n(n-1)!$

an object that is defined by the "object itself".

$a_n = 2^n, \quad n = 0, 1, 2, 3$

$a_0 = 1$

$a_1 = 2 = 2 \times a_0$

$a_2 = 4 = 2 \times a_1$

$a_3 = 8 = 2 \times a_2$

$a_4 = 16 = 2 \times a_3$

$a_n = 2 \times a_{n-1}$ → Recursive definction

L-17 / 29.07.2022

Inference

Premises

1. $(a \wedge e) \rightarrow (c \vee d)$
2. $b \rightarrow (f \wedge e)$
3. $f \rightarrow a$
4. $\neg d$
5. b
6. $f \wedge e$ modus ponens of 2,5
7. f, e simplify of 6
8. a modus ponens of 3,7
9. $a \wedge e$ conjunction of 7,8
10. $c \vee d$ modus ponens of 1,9
11. c disjunctive sylolism of 4,10

⊛ Recursion

Some recursive statement:

(i) $n! = n\,(n-1)!$

(ii) $f(n) = f(n-1) + f(n-2)$

(iii) $a_n = 2^n, \quad n = 0, 1, 2, \cdots$

$\rightarrow a_n = 2a_{n-1}$

⊛ $f(0) = 3, \quad f(n+1) = 2\,f(n) + 3$

$f(4) = ?$

$\Rightarrow$

$f(4) = 2 \cdot f(3) + 3 = 2 \cdot 45 + 3 = 93$

$f(3) = 2 \cdot f(2) + 3 = 2 \cdot 21 + 3 = 45$

$f(2) = 2 \cdot f(1) + 3 = 2 \cdot 9 + 3 = 21$

$f(1) = 2 \cdot f(0) + 3 = 2 \cdot 3 + 3 = 9$

$f(0) = 3$

(*)

Given,

$$a_1 = 2, \quad a_k = 5a_{k-1}, \quad k \geq 2$$

It is claimed that terms of the sequence satisfy the equation $2 \times 5^{n-1}$, for $n \geq 1$

$\Rightarrow$

		Term
L.H.S =	$a_1 = 2$	1
	$a_2 = 5a_1 = 5 \times 2 = 10$	2
	$a_3 = 5a_2 = 50$	3
	$a_4 = 5a_3 = 250$	4
	$a_5 = 5a_4 = 1250$	5
	...	...

Basic step:

$n = 1$

$$R.H.S = 2 \times 5^{n-1} = 2 \times 5^{1-1} = 2 = L.H.S.$$

Inductive steps:

Let's assume that for $n = k$,

the kth term satisfies the given equation,

$$a_k = 2 \times 5^{k-1}$$

Now,

We have to show that the given equation is satisfied by the a_{k+1} term that is

$$a_{k+1} = 2 \times 5^{k+1-1} = 2 \times 5^{k}$$

$$a_{k+1} = 5a_k$$

$$= 5 \times 2 \times 5^{k-1}$$

$$= 2 \times 5^{k-1+1}$$

$$= 2 \times 5^{k}$$

(proved)

L-18 / 22.07.2022

Recursive Set

* Set of integers that is divisible by 3.

$\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, \ldots\}$

$\Rightarrow$

Basic Step:

$S = \{3\}$

Recursive step:

if $x \in S$, $y \in S$, then

$x + y \in S$

iteration 1: $S = 3$

$x \in S$, $x = 3$

$y \in S$, $y = 3$

$x + y = 6 \in S$

$S_1 = \{3\} \cup \{6\} = \{3, 6\}$

iteration 2:

$x \to 3, 6$

$y \to 3, 6$

	3	6
3	6	9
6	9	12

$S_2 = S_1 \cup \{9, 12\}$

$= \{3, 6, 9, 12\}$

iteration 3:

	3	6	9	12
3	6	9	12	15
6	9	12	15	18
9	12	15	18	21
12	15	18	21	24

$S_3 = S_2 \cup \{15, 18, 21, 24\}$

$= \{3, 6, 9, 12, 15, 18, 21, 24\}$

relation

$aRb \equiv (a, b) \in R$

Relation

Relate two or more sets

Basic form of relation is to form ordered pair between the sets.

A, B are sets.

Binary relation form A to B is defined, as the

$\boxed{\text{subset of } A \times B}$ → Cartesian product.

$R \subseteq A \times B$

⊛

$A = \{0, 1, 2\}$

$B = \{p, q\}$

How many relations are possible from A to B and B to A.

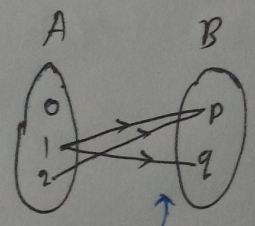
⇒ Here

$|A \times B| = 6$

subset number of $A \times B = 2^6 = 64$

⊛ Representation of Relation:

1. Arrow diagram



2. Tables form

for $\{(1,p), (1,q), (2,p)\}$

	p	q
0		
1	x	x
2	x	

3. Matrix notation

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

⊛ Matrix notation:

m = row

n = column

$$\begin{bmatrix} & \\ & \end{bmatrix}_{m \times n}$$

$m \to$ 1st set

$n \to$ 2nd set

⊛

$A = \{0, 1, 2\}$

$B = \{p, q\}$

⇒ Previous relation:

$$\begin{array}{c|cc} & p & q \\ \hline 0 & 0 & 0 \\ & (0,p) & (0,q) \\ 1 & 1 & 1 \\ & (1,p) & (1,q) \\ 2 & 1 & 0 \\ & (2,p) & (2,q) \end{array} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$= M_R$ → Relation of matrix

⊛

Given

$A = \{1, 2, 3, 4, 5\}$

$B = \{1, 2, 3, 4, 5\}$

Draw the M_R, where

Relation from A to B

$R = \{(a,b) \mid a \leq b,\ a \in A,\ b \in B\}$

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & (1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\ 2 & \cancel{(2,1)} & (2,2) & (2,3) & (2,4) & (2,5) \\ 3 & \cancel{(3,1)} & \cancel{(3,2)} & (3,3) & (3,4) & (3,5) \\ 4 & \cancel{(4,1)} & \cancel{(4,2)} & \cancel{(4,3)} & (4,4) & (4,5) \\ 5 & \cancel{(5,1)} & \cancel{(5,2)} & \cancel{(5,3)} & \cancel{(5,4)} & (5,5) \end{array} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

L-19 | 02. 08.2022

Relation

→ between sets

→ Subset of $A \times B$

→ $2^{|A \times B|} = 2^{|A| \cdot |B|}$

Relation from A to B is the subset of $A \times B$

Relation from A to A is the subset of $A \times A$

on the set A

Types of Relation:

⊛ Reflexive Relation:

a Relation R is Reflexive if ordered pair (a, a) is in R for all $a \in A$

$A = \{1, 2, 3\}$

number of Relation: 2^9

$R_1 = \{(1,2), (2,3), (3,3)\}$

is it reflexive ? ⇒ No

$R_2 = \{(1,1), (2,2), (3,3)\}$

it is Reflexive

$\Delta = \{(1,1), (2,2)\}$

$R \cup \Delta \equiv$ Reflexive closure of R_1

$R_1 = \{(1,2),(2,3),(3,3)\}$

$\Delta = \{(1,1),(2,2)\}$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & \emptyset 1 & 1 & 0 \\ 2 & 0 & \emptyset 1 & 1 \\ 3 & 0 & 0 & 1 \end{array}$$

Sum = Trace = 3 ≡ number of rows or collumn

* Is the divides relation on the set of positive integers reflexive? ⇒ Yes.

* Is the divides relation on the set of integers reflexive? ⇒ No

because (0,0) is undefined.

* <u>Symmetric Relation:</u>

if $(a,b) \in R$

then, (b,a) should be in R as well.

$A = \{1,2,3\}$

R: Relation on the set A

$\hookrightarrow \{(1,2), (1,1), (1,3)\} \rightarrow$ Not symmetric

Because,

$(2,1) \notin R$

$(3,1) \notin R$

$\Delta = \{(2,1), (3,1)\}$

$R \cup \Delta \equiv$ Symmetric closure of R

(*)

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & \cancel{0}1 & 0 & 0 \\ 3 & \cancel{0}1 & 0 & 0 \end{array}$$

Foldable/matrix is symmetric

or $A = A^T$

L-20 / 05. 08. 2022 /

Relation

- → Matrix Notation
- → Reflexive
- → Anti- Symmetric

$A = \{1, 2, 3\}$

$R_1 = \{(1,2), (2,1), (2,3), (3,2)\}$ → Symmetric

$R_2 = \{(1,2), (2,3)\}$ → Anti- Symmetric (No symmetric trace available)

$R_3 = \{(1,2), (2,1), (2,3)\}$ → Not Symmetric
Not Anti-Symmetric

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \end{array}$$

≡ 1 superimpose with 1

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$

≡ 1 superimpose with 0

* A relation R is anti-symmetric if for $(a,b) \in R$ there's $(b,a) \in R$ then,

$$a = b$$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

→ Reflexive

→ Symmetric

→ Anti-Symmetric

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

→ Symmetric

→ Anti-Symmetic

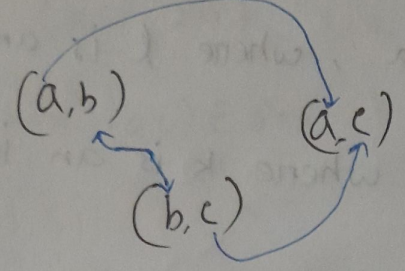
→ Innocent until proven guilty

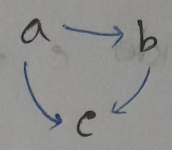
* Is the divider relation on the set of positive integers Symmetric / Anti-Symmetric or both?

⇒ Anti-Symmetric.

Transitive

$\forall a \forall b \forall c \left(\left((a,b) \in R \cap (b,c) \in R \right) \rightarrow (a,c) \in R \right)$





A Relation R on the set A is called transitive whenever $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$

$\forall a,b,c \in A$

$A = \{1,2,3,4\}$

$R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

→ transitive

For this ⇒ counter example

$R = \{(1,1), (1,2), (2,1)\}$

→ not transitive

* divides relation is transitive.

* a divides b $\Rightarrow$ $b = la$, where l is an integer

b divides c $\Rightarrow$ $c = kb$, where k is an integer

$\therefore$ $c = kb$ → product of two integer

$= kla$

$= ja$ → where $j = kl$

$c = ja$

$\Rightarrow$ c is a multiple of a

$\Rightarrow$ "a divides c"

* Composite Relation:

→ R is a relation from A to B

S is a relation from B to C

Then a composit relation formed by R and S is

$S \circ R$ where,

$(a,b) \in R$

$(b,c) \in S$

then $(a,c) \in S \circ R$

$R \circ R = R^2$

$R^3 = R^2 \circ R$

$R^{n+1} = R^n \circ R$

$\{1,2,3\}$ to $\{1,2,3,4\}$

$R = \{(1,1), (1,4), (2,3), (3,1), (3,4)\}$

$\{1,2,3,4\}$ to $\{0,1,2\}$

$S = \{(1,0), (2,0), (3,1), (3,2), (4,0)\}$

$S \circ R = \{(2,1), (3,0), (1,0), (2,2)\}$

Bullian Product

$$S \circ R = \begin{bmatrix} M_R \end{bmatrix} \cdot \begin{bmatrix} M_S \end{bmatrix}$$

$M_{R^2} = M_R \cdot M_R$

L-21 / 07.08.2022

Counting

Sum Rule & Product Rule

Task 1

sub1		sub2
n_1 ways		n_2 ways

$n_1 \times n_2$ ways to finish task 1

List1 $\cap$ List 2

$= |\phi| = 0$

Lis 1 $\cap$ List 2

$= \phi$

Inclusion - Exclusion

Principle

L-22 / 26.08.2022 /

Counting

Sum Rule:

$A \cap B = \phi$

$|A \cap B| = |\phi| = 0$

Project

List 1 — List 2

$A = \{1, \ldots 15\}$

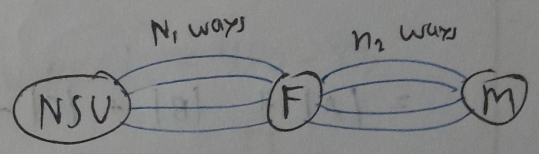
$B = \{a_1 \ldots a_{17}\}$

Number of possible ways to chose a project $= |A| + |B|$

Project Rule

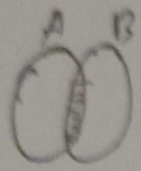
Task

Part A n_1 — Part B n_2



$n_1 \times n_2$ ways to travel from NSU to M.

Inclusion. Exclusion. Principle



$\longrightarrow A \cup B$

$$|A \cup B| = \underbrace{|A| + |B|}_{\text{inclusion}} - \underbrace{|A \cap B|}_{\text{exclusion}}$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$= |A \cup (B \cup C)| = |A \cup M|$$

$$= |A| + |M| - |A \cap M|$$

$$= |A| + |B \cup C| - |A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - (A \cap (B \cup C)|$$

$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$

$$= |A| + |B| + |C| - |B \cap C| - \{ |A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|$$

$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

~~450 Application~~

450 Applicants

$|CSE| = 275$

$|BBA| = 150$

$|CSE \cap BBA| = 30$

How many job applicants are neither from CSE nor from BBA.

$\Rightarrow$ $|CSE \cup BBA| = |CSE| + |BBA| - |CSE \cap BBA|$

$= 275 + 150 - 30$

$= 395$

$\therefore$ All applicant $- |CSE \cup BBA| = 450 - 395$

$= 55$

Spanish : 1232

French : 879

Russian : 114

$S \cap F = 103$

$S \cap R = 23$

$F \cap R = 14$

2092 students are learning one of the three languages. $\longrightarrow$ $S \cup F \cup R = 2092$

Q. How many students are learning all the three languages. $S \cap F \cap R = ?$

$\Rightarrow$

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|$$

$$|S \cap F \cap R| = 2092 - 1232 - 879 - 114 + 103 + 23 + 14$$

$$= 2232 - 2225$$

$$= 7$$ Ans.

⊛ Permutation & Combination

Permutation

a	b	c
a	c	b
b	a	c
b	c	a
c	a	b
c	b	a

$^3P_3 = 6$

$n! = 3! = 6$

$^3C_3 = 1$

combination $\Rightarrow 1$

$$\boxed{^nC_r \times r! = {}^nP_r}$$

$$\boxed{⊛ \quad {}^nP_r = \frac{n!}{(n-r)!}}$$

$$\boxed{⊛ \quad {}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{^nP_r}{r!}}$$

⊛ ABCDEFG

How many permutations are possible using the given letters that contain

① String ABC and CEF $\longrightarrow$ ABCEF D G $\quad {}^3P_3 = 6$

② BDE and DAC $\longrightarrow$ Not possible $\Rightarrow 0$.

L-23 / 28.08.2022 /

Counting

$^nP_r = {}^nC_r \cdot r!$

⊛ How many strings of five ASCII characters contains at least one @ ?

$\rightarrow 128^5 - 127^5$

with everything

$\underline{128}\ \underline{128}\ \underline{128}\ \underline{128}\ \underline{128}$

without @

$\underline{127}\ \underline{127}\ \underline{127}\ \underline{127}\ \underline{127}$

$\therefore 128^5 - 127^5$

* $^nC_r = {}^nC_{n-r}$

L.H.S. $= \dfrac{n!}{(n-r)!\,r!}$ R.H.S. $= \dfrac{n!}{(n-(n-r))!\,(n-r)!}$

$= \dfrac{n!}{(n-n+r)!\,(n-r)!}$

$= \dfrac{n!}{(n-r)!\,r!}$

$=$ L.H.S.

(Proved)

* How many poker hands of five can be dealt from standard deck of 52 cards?

$\Rightarrow C(52,5) = {}^{52}C_5$

* How many ways are there to select 47 cards from the same deck of 52

${}^{52}P_{47}$

Binomial Expansion

$C(n,r) \equiv {}^nC_r$ act as the coefficient of Binomial expansion.

$(a+b)^n = ?$

$(x+y)^n = ?$

$(x+y)^1 =$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + \cdots + y^4$

$$\binom{3}{0} x^{n-j} \cdot y^j + \binom{3}{1} x^{3-1} y^1 + \binom{3}{2} x^{3-2} y^2 + \binom{3}{3} x^{3-3} y^3$$

$$1 \cdot x^3 + 3x^2y + 3xy^2 + 1 \cdot y^3$$

$$\sum_{j=0}^{n=3} \binom{n}{j} x^{n-j} y^j = (x+y)^3$$

$$\sum_{j=0}^{n=4} \binom{n}{j} x^{n-j} y^{j} = (x+y)^4 = ?$$

$\Rightarrow$

$$(x+y)^4 = \binom{4}{0} x^{4-0} y^0 + \binom{4}{1} x^{4-1} y^1 + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} x^{4-4} y^4$$

$$= 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$ Ans.

What is the co-efficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$$^{25}C_{13} \cdot (2x)^{25-13} \cdot (-3y)^{13}$$

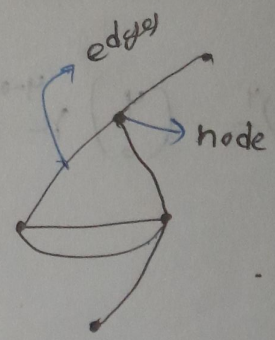
$$= -\,^{25}C_{13} \cdot 2^{12} \cdot 3^{13}$$

Graphs

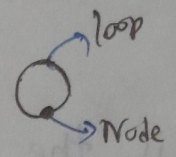
node — edge

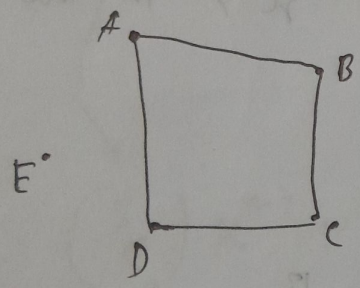
$G \equiv (V, E)$

- V → Vertices → nodes
- E → Edges
- non empty



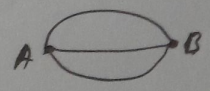
Simple Graph





i. No loop exist

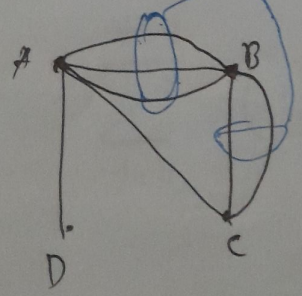
ii. No multiple edges.



Multi Graph

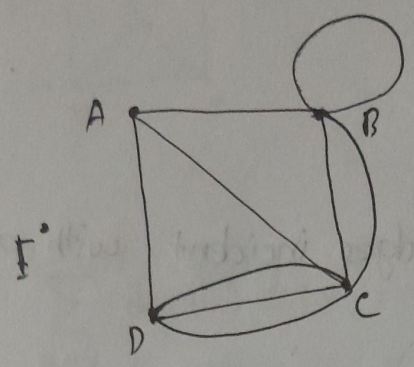
edges of multiplicity of 3 between (A-B)

& 2 between (B-C)



Allows multiple edges.

⊛ Pseudo - Graph



⊛ Allows loop

⊛ multi edges are allowed

deg (A) = 3

deg (B) = 5

Degree of a node

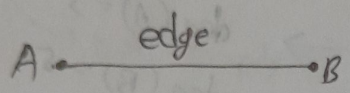
Number of edges attached to a node.

L-24 / 30.08.2022 /

Undirected Graph

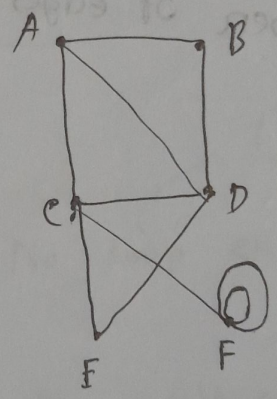
Degree of a node:

"number of edges incident with a node."



• R

$\deg(R) = 0$



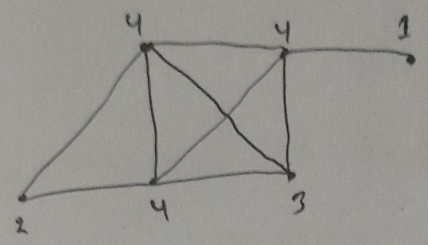
$\deg(F) = 5$

$\deg(D) = 4$

⊛ Handshaking Theorem:

$$\sum_{u \in V} \deg(u) = 2 \times e$$

→ number of edges



9 edges

$$\sum_{u \in V} \deg(u) = 18 = 2 \times e$$

$$\therefore e = 9$$

⊛ How many edges are there in a graph with 10 vertices each of degree 6?

$$10 \times 6 = 60 = 2 \times 30$$

→ edges

$$\sum_{u \in V} \deg(u) = 10 \times 6 = 60 = 2 \times 30$$

$$\therefore e = 30$$

⊛ "An undirected graph has an even number of vertices of odd degree."

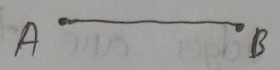
(Prove it)

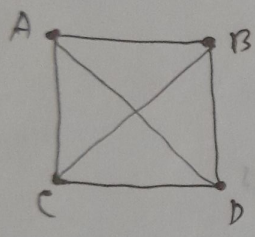
Simple Graph

i. Complete Graph : $\rightarrow K_n$

Any node of the graph should be completed with the rest of the nodes.

K_2 $e=1$ $^2C_2 = 1$



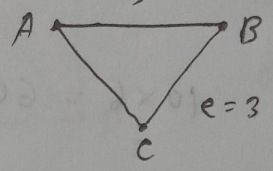


$e=6$

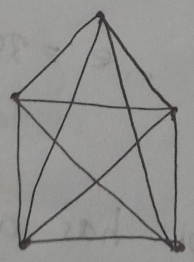
$^4C_2 = 6$

K_4

K_3



$e=3$ $^3C_2 = 3$



$^5C_2 = 10$

$e=10$

K_5

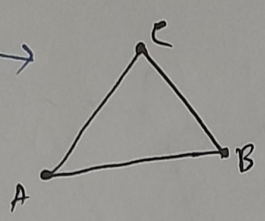
$^nC_2$ = number of edges we need for a complete graph of size K_n

$n \rightarrow$ number of nodes

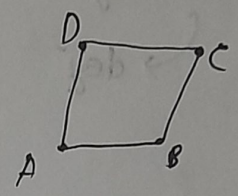
2. Cycle :

required at least 3 nodes.

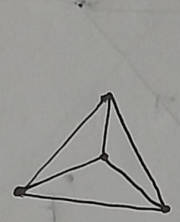
C_3 = Cycle of 3 nodes.



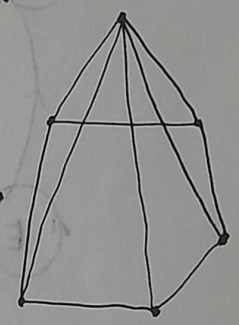
$C_4 \Rightarrow$



3. Wheel :



W_3



W_5

⊛ Directed Graph (Digraph)

$G = (V, E)$

edges are directed

represent as an ordered pair of vertices.

edge associated with (u, v) suggest at that the

(u → starts, v → end)

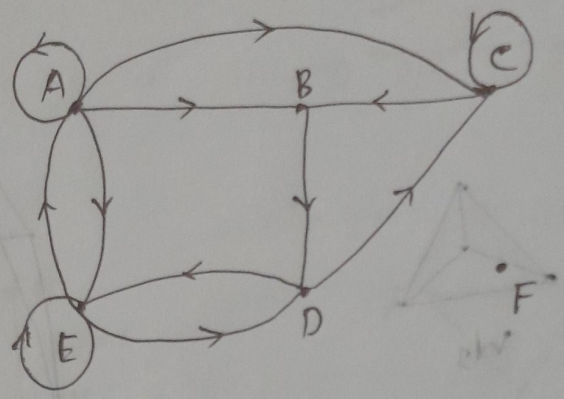
edge starts u and ends with v.

$u \longrightarrow v$

Degree of Directed Graph

Out-degree → deg^+

In-degree → deg^-



total edges = 12

$deg^-(A) = 2$ $deg^-(C) = 3$ $deg^-(F) = 0$

$deg^+(A) = 4$ $deg^+(C) = 2$ $deg^+(F) = 0$

$deg^-(B) = 2$ $deg^-(D) = 2$ $deg^-(E) = 3$

$deg^+(B) = 1$ $deg^+(D) = 2$ $deg^+(E) = 3$

$deg^- = 12$

$deg^+ = 12$

$e = deg^- = deg^+$

END