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$$4y'' - 4y' + y = e^{x/2}\sqrt{1-x^2}$$

A.E. $\Rightarrow$ $4m^2 - 4m + 1 = 0$

$$(2m-1)^2 = 0$$

$$\therefore m = \frac{1}{2}, \frac{1}{2}$$

$$\therefore y_c = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{x}{2}}$$

$$\therefore y_1 = e^{x/2}$$

$$y_2 = x e^{x/2}$$

$$\therefore W = \begin{vmatrix} e^{x/2} & x e^{x/2} \\ \frac{1}{2}e^{x/2} & \frac{1}{2}x e^{x/2} + e^{x/2} \end{vmatrix}$$

$$= \frac{1}{2}x e^{x} + e^{x} - \frac{1}{2}x e^{x}$$

$$= e^{x} \neq 0$$

$$\therefore u_1 = -\int \frac{x e^{x/2} \cdot e^{x/2}\sqrt{1-x^2}}{e^x}\, dx$$

$$= -\int x\sqrt{1-x^2}\, dx$$

$$= \frac{1}{3}(1-x^2)^{3/2}$$

$u = 1-x^2$

$$\frac{du}{dx} = -2x\, dx$$

$$x\,dx = -\frac{1}{2}\, du$$

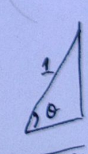
$$-\frac{1}{2}\int \sqrt{u}\, du$$

$$= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2}$$

$$= -\frac{1}{3}(1-x^2)^{3/2}$$

$$\therefore u_2 = \int \frac{e^{x/2}\cdot e^{x/2}\cdot\sqrt{1-x^2}}{e^x}\, dx$$

Let,
$x = \sin\theta$
$dx = \cos\theta\, d\theta$




$$= \int \sqrt{1-x^2}\, dx$$

$$= \int \cos\theta \cdot \cos\theta\, d\theta$$

$$= \frac{1}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\cdot\frac{\sqrt{1-x^2}}{1}$$

$$= \frac{1}{2}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{2}$$

$$\therefore y_p = \frac{1}{3}e^{x/2}(1-x^2)^{3/2} + \frac{1}{2}x e^{x/2}\sin^{-1}x + \frac{x^2 e^{x/2}\sqrt{1-x^2}}{2}$$

A

$y = y_c + y_p$

Ans

$$y_1 + y_1 = \frac{x}{2}\sin x + \frac{x}{2}\cos x$$

$$2y_1 = \frac{x}{2}\sin x + \frac{x}{2}\cos x$$

$$y_p = y_1 = \frac{x^2}{4}\sin x + \frac{x}{4}\cos x - \frac{x}{2}\sin x$$

$$I = \int e^x \sin x$$

$x = e^t \qquad t = \ln x \qquad \frac{dt}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$= \frac{1}{x}\cdot\frac{dy}{dt}$$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x}\frac{d^2y}{dt^2}\cdot\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt} \qquad D = \frac{d}{dt}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$D^2 = \frac{1}{x^2}\left(D^2 - D\right)$$

$$x^2 D^2 = D^2 - D$$

(*) $x^3 y''' = D(D-1)(D-2)y$

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x^2}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right)$$

$$= \frac{1}{x^2}\cdot\frac{d}{dx}\left(\frac{d^2y}{dt^2}\right) + \frac{-2}{x^3}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\cdot\frac{d}{dx}\frac{dy}{dt} - \frac{-2}{x^3}\cdot\frac{dy}{dt}$$

$$= \frac{1}{x^2}\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dx} - \frac{2}{x^3}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{2}{x^3}\frac{dy}{dt}$$

$$= \frac{1}{x^3}\frac{d^3y}{dt^3} - \frac{2}{x^3}\frac{d^2y}{dt^2} - \frac{1}{x^3}\frac{d^2y}{dt^2} + \frac{2}{x^3}\frac{dy}{dt}$$

$$\therefore x^3y''' = (D^3 - 2D^2 - D^2 + 2D)y$$

$$= (D^3 - 3D^2 + 2D)y$$

$$= D(D^2 - 3D + 2)y$$

$$= D(D^2 - 2D - D + 2)y$$

$$= D(D(D-2) - 1(D-2))y$$

$$= D(D-2)(D-1)y$$

(Proved)