L-15 / 01.08.2022 /

3.6

L' Hospital Rule: Indeterminate forms

[Finding Limit using derivative / Application of derivative]

Indeterminate forms, $\boxed{\frac{0}{0}, \frac{\infty}{\infty}}$, 0^0, ∞^0, 1^∞, $\infty-\infty$, $0 \cdot \infty$

(*) $\lim_{x\to a} \frac{f(x)}{g(x)}$

$= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

$= \frac{0}{0}$; indeterminate forms

(*) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

$= \frac{\infty}{\infty}$; indeterminate form

(*) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$= \frac{\lim_{x\to 2} x^2-4}{\lim_{x\to 2} x-2}$

$= \frac{0}{0}$; indeterminate form

(*) $\lim_{x\to \infty} \frac{e^x}{x^2} = \frac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty} x^2}$

$= \frac{\infty}{\infty}$; indeterminate form

⊛ L' Hospital Rule for $\frac{0}{0}$ & $\frac{\infty}{\infty}$ form:

Statement: Suppose that $f(x)$ and $g(x)$ are differentiable function and $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \text{so on} \ldots$$

$\lim_{x\to a^+} \frac{f(x)}{g(x)}$, $\lim_{x\to a^-} \frac{f(x)}{g(x)}$, $\lim_{x\to \pm\infty} \frac{f(x)}{g(x)}$ also applicable.

⊛ $\lim_{x\to 2} \frac{x^2-4}{x-2}$

$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)}$$

$$= \lim_{x\to 2} (x+2)$$

$$= 2+2$$

$$= 4$$

Ans

⊛ $\lim_{x\to 2} \frac{x^2-4}{x-2}$ [form $\frac{0}{0}$]

Now, using L' Hospital Rule

$$= \lim_{x\to 2} \frac{2x}{1}$$

$$= \lim_{x\to 2} 2x$$

$$= 2\cdot 2$$

$$= 4$$

Ans

⊛ $\lim_{x \to \infty} \frac{e^x}{x^2}$ [form $\frac{\infty}{\infty}$]

Using L' Hospital Rule.

$= \lim_{x \to \infty} \frac{e^x}{2x}$ [form $\frac{\infty}{\infty}$]

$= \lim_{x \to \infty} \frac{e^x}{2}$

$= \infty$

⊛ Find the following Limit:

i) $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ ii) $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ iii) $\lim_{x \to +\infty} \frac{\ln x}{x}$

iv) $\lim_{x \to +\infty} x e^{-x}$

$\Rightarrow$

i) $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$ [form $\frac{0}{0}$]

Now, using L' Hospital Rule

$= \lim_{x \to 0} \frac{e^x}{\cos x}$

$= \frac{e^0}{\cos 0}$

$= \frac{1}{1} = 1$

ii) $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$ [form $\frac{0}{0}$]

Now, using L' Hospital Rule,

$= \lim_{x \to \pi} \frac{\cos x}{1}$

$= \cos \pi$

$= -1$

iii) $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x}$ [form $\frac{\infty}{\infty}$]

Now, using L' Hospital Rule,

$= \lim\limits_{x \to +\infty} \dfrac{1/x}{1}$

$= \lim\limits_{x \to +\infty} \dfrac{1}{x}$

$= 0$

iv) $\lim\limits_{x \to +\infty} x e^{-x}$ [form $\infty \cdot 0$]

$= \lim\limits_{x \to +\infty} \dfrac{x}{e^x}$ [form $\frac{\infty}{\infty}$]

Now, using L' Hospital Rule:

$= \lim\limits_{x \to +\infty} \dfrac{1}{e^x}$

$= 0$

Ans

(x) $\lim\limits_{x \to 0^+} x \ln x$ [form $0 \cdot -\infty$]

$= \lim\limits_{x \to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ [form $\frac{\infty}{\infty}$]

Now using L' Hospital Rule,

$= \lim\limits_{x \to 0^+} \dfrac{1/x}{-1/x^2}$

$= \lim\limits_{x \to 0^+} -x$

$= 0$

Ans

⊛ Type of $(\infty - \infty)$

⊛ $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ [form $\infty - \infty$]

$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x}$ [form $\frac{0}{0}$]

Now, using L' Hospital Rule:

$= \lim_{x \to 0^+} \frac{\cos x - 1}{\sin x + x \cos x}$ [form $\frac{0}{0}$]

$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + \cos x - x \sin x}$

$= \frac{0}{1+1-0}$

$= 0$

⊛ Type of $\boxed{0^0, \infty^0, 1^\infty}$

→ Taking ln both side for converting $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

(*) Find Limit, $\lim_{x\to 0} (1+\sin x)^{\frac{1}{x}}$ $[1^{\infty} \text{ form}]$

Let,

$$y = (1+\sin x)^{\frac{1}{x}}$$

Now, taking ln both side

$$\Rightarrow \ln y = \ln (1+\sin x)^{\frac{1}{x}}$$

$$\Rightarrow \ln y = \frac{1}{x} \ln (1+\sin x)$$

$$\Rightarrow \ln y = \frac{\ln (1+\sin x)}{x}$$

Taking Limit both side,

$$\Rightarrow \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln (1+\sin x)}{x} \quad [\text{form } \tfrac{0}{0}]$$

Now, using L' Hospital Rule :

$$= \lim_{x\to 0} \frac{\frac{1}{1+\sin x}\cdot \cos x}{1}$$

$$= \lim_{x\to 0} \frac{\cos x}{1+\sin x}$$

$$= \frac{\cos 0}{1+\sin 0} = \frac{1}{1+0} = 1$$

$$\therefore \lim_{x\to 0} \ln y = 1$$

$$\ln \left(\lim_{x\to 0} y \right) = 1$$

$$\lim_{x\to 0} y = e^{1}$$

$$\therefore \lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} = e$$

⊛ Find Limit,

$$\lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x} \quad [\text{form } \infty^{0}]$$

Let,

$$y = (\tan x)^{\cos x}$$

Now, taking ln both side,

$$\ln y = \ln (\tan x)^{\cos x}$$

$$\ln y = \cos x . \ln (\tan x)$$

Taking Limit both side,

$$\lim_{x\to \frac{\pi}{2}} \ln y = \lim_{x\to \frac{\pi}{2}} \left(\cos x . \ln (\tan x) \right) \quad [\text{form } 0.\infty]$$

$$= \lim_{x\to \frac{\pi}{2}} \left[\frac{\ln (\tan x)}{\sec x} \right] \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{\sec x \tan x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$$

$$\therefore \lim_{x \to \frac{\pi}{2}} \ln y = \frac{0}{1} = 0$$

$$\ln\left(\lim_{x \to \frac{\pi}{2}} y\right) = 0$$

$$\lim_{x \to \frac{\pi}{2}} y = e^0$$

$$\lim_{x \to \frac{\pi}{2}} y = 1$$

$$\therefore \lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = 1$$

Chapter - 4

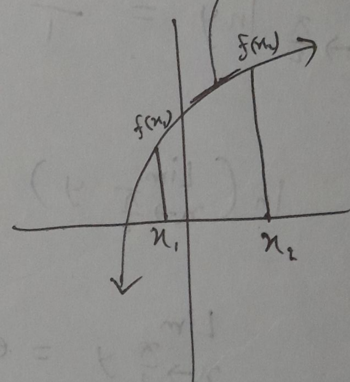
The Derivative in Graphing and Application

4.1

Analysis of function I:

* Increasing function:

slope of tangent line positive



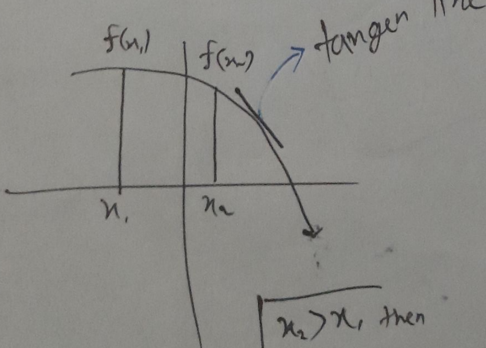
$x_2 > x_1$ then, $f(x_2) > f(x_1)$

Slope of tangent line +ve,

i.e. derivative +ve.

$f'(x) > 0$

* Decreasing function:



$x_2 > x_1$ then

$f(x_2) < f(x_1)$

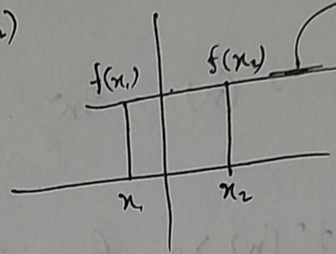
slope of tangent line -ve,

i.e. derivative -ve,

$f'(x) < 0$

⊛ Constant function:

| $x_2 > x_1$, but
| $f(x_1) = f(x_2)$



Slope of tangent line 0. (Horizontal)

i.e. derivative is 0.

$f'(x) = 0.$

⊛ Find the intervals on which the following function are increasing or decreasing.

(i) $f(x) = x^2 - 4x + 3$

$\Rightarrow f'(x) = 2x - 4$

for increasing,

$2x - 4 > 0$

$\therefore 2x > 4$

$x > 2$

therefore,
increasing interval $(2, \infty)$

for decreasing,

$2x - 4 < 0$

$2x < 4$

$x < 2$

therefore,
decreasing interval $(-\infty, 2)$

ii) $f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) > 0$ for all values of x

$\therefore f(x) = x^3$ is increasing function $(-\infty, \infty)$

iii) $f(x) = 3x^4 + 4x^3 - 12x^2 + \cancel{1}2$

$f'(x) = 12x^3 + 12x^2 - 24x$

$= 12x\,(x^2 + x - 2)$

$= 12x\,(x^2 + 2x - x - 2)$

$f'(x) = 12x\,(x+2)\,(x-1)$

Sign analysis of $f'(x)$

$-$ $\quad$ $+$ $\quad$ $-$ $\quad$ $+$

-2 $\quad$ 0 $\quad$ 1

$(-\infty, -2)$ $\quad$ $(-2, 0)$ $\quad$ $(0, 1)$ $\quad$ $(1, \infty)$

Let,

$x = -3$	$x = -1$	$x = \frac{1}{2}$	$x = 2$
$f'(-3) = -ve$	$f'(-1) = +ve$	$f'(\frac{1}{2}) = -ve$	$f'(2) = +ve$

∴ increasing: $(-2, 0)$, $(1, \infty)$

∴ decreasing: $(-\infty, -2)$, $(0, 1)$

(*) Concavity: $\rightarrow$ concave up
$\rightarrow$ concave down

concave up: 

concave down: 

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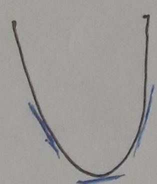
4.1

Concavity

Concave up:



Slope of a tangent line, $\boxed{-ve \rightarrow 0 \rightarrow +ve}$

↳ increasing

$f'(x)$ increasing

$\frac{d}{dx}(f'(x)) > 0$

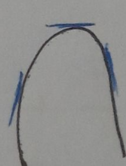
$\boxed{f''(x) > 0}$

⊛ if $f(x)$ increasing, then

$f'(x) > 0$

↳ rate of change of $f(x) > 0$

⊛ **Concave down:**

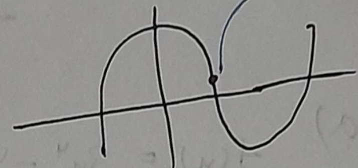


slope of tangent line $\boxed{+ve \rightarrow 0 \rightarrow -ve}$

↳ decreasing

∴ condition: $\boxed{f''(x) < 0}$

⊛ Infection point:



Transition point for a function from concave up to concave down or vice versa.

condition of infection point:

$$\boxed{f''(x) = 0}$$

⊛ Find

i) the intervals on which f(x) increasing.

ii) the intervals on which f(x) decreasing

iii) the intervals on which f(x) concave up.

iv) the intervals on which f(x) concave down.

v) x-co-ordinate of infection point.

a) $f(x) = 3x^4 - 4x^3$ b) $f(x) = \sqrt[3]{x^2+x+1}$ c) $f(x) = x^{4/3} - x^{1/3}$

d) $f(x) = \dfrac{x}{x^2+2}$ e) $f(x) = e^{-x^2/2}$

Solution

a) $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$= 12x^2(x-1)$

Now, sign analysis of $f'(x)$

sign of $f'(x)$

$(-\infty, 0)$	$(0,1)$	$(1, \infty)$
$-ve$	$-ve$	$+ve$

0   1

$x = -1$	$x = 0.5$	$x = 2$
$f'(-1) = -ve$	$f'(0.5) = -ve$	$f'(2) = +ve$

$\therefore$ increasing : $(1, \infty)$

$\therefore$ decreasing : $(-\infty, 0), (0, 1)$

Now, for concavity

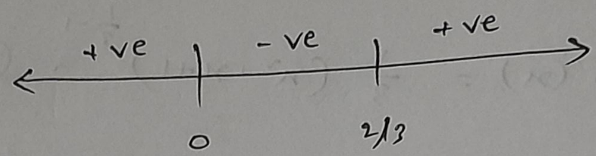
$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

$= 12x(3x-2)$

sign analysis of $f''(x)$

sign of $f''(x)$



$(-\infty, 0)$ $\quad$ $(0, \frac{2}{3})$ $\quad$ $(\frac{2}{3}, \infty)$

$x = -1$	$x = \frac{1}{3}$	$x = 2$
$f''(-1) = +ve$	$f''(\frac{1}{3}) = -ve$	$f''(2) = +ve$

concave up: $(-\infty, 0)$, $(\frac{2}{3}, \infty)$

concave down: $(0, \frac{2}{3})$

x- co-ordinate of infection point,

$$f''(x) = 0$$

$$12x(3x-2) = 0$$

$12x = 0$	$3x - 2 = 0$
$x = 0$	$x = \frac{2}{3}$

$\therefore$ infection point : 0 and $\frac{2}{3}$.

b)

$$f(x) = \sqrt[3]{x^2+x+1}$$

$$f'(x) = \frac{1}{3}(x^2+x+1)^{\frac{1}{3}-1} \cdot (2x+1)$$

$$= \frac{2x+1}{3(x^2+x+1)^{\frac{2}{3}}}$$

$$f''(x) = \frac{3(x^2+x+1)^{\frac{2}{3}} \cdot 2 - (2x+1) \cdot 3 \cdot \frac{2}{3}(x^2+x+1)^{\frac{2}{3}-1} \cdot (2x+1)}{\left(3(x^2+x+1)^{\frac{2}{3}}\right)^3}$$

$$= \frac{6(x^2+x+1)^{\frac{2}{3}} - 2 \cdot (2x+1)^2 (x^2+x+1)^{-\frac{1}{3}}}{9(x^2+x+1)^{4/3}}$$

$$= \frac{6(x^2+x+1)^{\frac{2}{3}} - \frac{2(2x+1)^2}{(x^2+x+1)^{1/3}}}{9(x^2+x+1)^{4/3}}$$

$$= \frac{6(x^2+x+1) - 2(2x+1)^2}{9(x^2+x+1)^{4/3} \cdot (x^2+x+1)^{1/3}}$$

$$= \frac{6x^2+6x+6-2(4x^2+4x+1)}{9(x^2+x+1)^{5/3}}$$

$$= \frac{6x^2+6x+6-8x^2-8x-2}{9(x^2+x+1)^{5/3}}$$

$$= \frac{-2x^2-2x+4}{9(x^2+x+1)^{5/3}}$$

$$f''(x) = \frac{-2(x^2+x-2)}{9(x^2+x+1)^{5/3}}$$

$$= \frac{-2(x^2+2x-x-2)}{9(x+x+1)^{5/3}}$$

$$\therefore f''(x) = \frac{-2(x+2)(x-1)}{9(x^2+x+1)^{5/3}}$$

For increasing,

$f'(x) > 0$

$$\frac{2x+1}{3(x^2+x+1)^{2/3}} > 0$$

$2x+1 > 0$

$x > -\frac{1}{2}$

$\therefore (-\frac{1}{2}, \infty)$

for decreasing

$f'(x) < 0$

$$\frac{2x+1}{3(x^2+x+1)^{2/3}} < 0$$

$2x+1 < 0$

$x < -\frac{1}{2}$

$\therefore (-\infty, -\frac{1}{2})$

for concave up and down

$f''(x)$

$$\begin{array}{ccccc} & -ve & & +ve & & -ve \\ \longleftarrow & & | & & | & \longrightarrow \\ & & -2 & & 1 & \end{array}$$

$(-\infty,-2)$ $\quad$ $(-2,1)$ $\quad$ $(1,\infty)$

$x=-3$	$x=0$	$x=2$
$f''(-3) = -ve$	$f''(0) = +ve$	$f''(2) = -ve$

concave up $= (-2,1)$

concave down $= (-\infty,-2), (1,\infty)$

for infection point:

$$-2(x+2)(x-1) = 0$$

$-2 = 0$	$x+2=0$	$x-1=0$
	$x=-2$	$x=1$

$\therefore$ infection point $x=1$ and -2

c) $f(x) = x^{4/3} - x^{1/3}$

$f'(x) = \frac{4}{3} x^{4/3-1} - \frac{1}{3} x^{1/3-1}$

$= \frac{4}{3} x^{1/3} - \frac{1}{3} x^{-2/3}$

$= \frac{4}{3} \cdot x^{1/3} - \frac{1}{3} \cdot \frac{1}{x^{2/3}}$

$= \frac{\frac{4}{3} \cdot x - \frac{1}{3}}{x^{2/3}}$

$= \frac{\frac{1}{3}(4x-1)}{x^{2/3}}$

$= \frac{4x-1}{3x^{2/3}}$

increasing:

$f'(x) > 0$

$\frac{4x-1}{3x^{2/3}} > 0$

$4x-1 > 0$

$x > \frac{1}{4}$

$\therefore \left(\frac{1}{4}, \infty\right)$

decreasing:

$f'(x) < 0$

$\frac{4x-1}{3x^{2/3}} < 0$

$4x-1 < 0$

$x < \frac{1}{4}$

$\therefore \left(-\infty, \frac{1}{4}\right)$

For concavity:

$$f'(x) = \frac{4x-1}{3x^{2/3}}$$

$$f''(x) = \cdots$$

$$= \frac{2(2x+1)}{9 \cdot x^{5/3}}$$

concave up:

$f''(x) > 0$

$\frac{2(2x+1)}{9x^{5/3}} > 0$

$2(2x+1) > 0$

$x > -\frac{1}{2}$

$\therefore \left(-\frac{1}{2}, \infty\right)$

concave down:

$f''(x) < 0$

$x < -\frac{1}{2}$

$\therefore \left(-\infty, -\frac{1}{2}\right)$

$\therefore$ infection point

$$f''(x) = 0$$

$$\frac{2(2x+1)}{9x^{5/3}} = 0$$

$$x = -\frac{1}{2}$$

A

d)

$$f(x) = \frac{x}{x^2+2}$$

$$f'(x) = \frac{(x^2+2)\cdot 1 - x\cdot 2x}{(x^2+2)^2}$$

$$= \frac{x^2+2-2x^2}{(x^2+2)^2}$$

$$f'(x) = \frac{-x^2+2}{(x^2+2)^2}$$

$$f''(x) = \frac{2x\,(x^2-6)}{(x^2+2)^3}$$

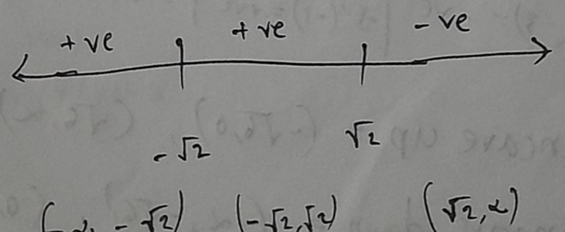
For increasing:

$$f'(x) > 0$$

$$\frac{-x^2+2}{(x^2+2)^2} > 0$$

$$2-x^2 > 0$$

$$(\sqrt{2}-x)(\sqrt{2}+x) > 0$$



$x=-2$	$x=0$	$x=2$
$f'(-2) = +ve$	$f'(0) = +ve$	$f'(2) = -ve$

$\therefore$ increasing: $(-\infty, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

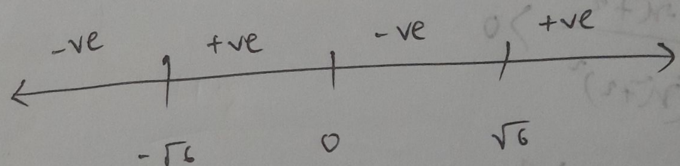
$\therefore$ decreasing: $(\sqrt{2}, \infty)$

For concavity:

$$f''(x) = 0$$

$$\frac{2x(x^2-6)}{(x^2+2)^3} = 0$$

$$2x(x-\sqrt{6})(x+\sqrt{6}) = 0$$



$x=-3$	$x=-1$	$x=2$	$x=3$
$f''(-3) = -ve$	$f''(-1) = +ve$	$f''(2) = -ve$	$f''(3) = +ve$

concave up $= (-\sqrt{6}, 0)$, $(\sqrt{6}, \infty)$

concave down $= (-\infty, -\sqrt{6})$, $(0, \sqrt{6})$

e)

$$f(x) = e^{-\frac{x^2}{2}}$$

$$f'(x) = e^{-\frac{x^2}{2}} \cdot \frac{-2x}{2}$$

$$= -x e^{-\frac{x^2}{2}}$$

$$f''(x) = -\left(e^{-x^2/2} + x \cdot e^{-x^2/2} \cdot \left(\frac{-2x}{2}\right)\right)$$

$$= -\left(e^{-x^2/2} - x^2 e^{-x^2/2}\right)$$

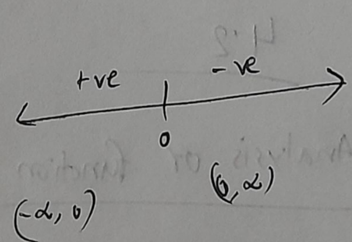
$$= e^{-\frac{x^2}{2}} (x^2 - 1)$$

if, $e^{-x^2/2} \neq 0$

$$f'(x) = 0$$

$$\boxed{-x} e^{-x^2/2} = 0$$

$\therefore x = 0$



$x = -1$	$x = 1$
$f'(-1) = +ve$	$f'(1) = -ve$

$\therefore$ increasing : $(-\infty, 0)$

$\therefore$ decreasing : $(0, \infty)$

For concavity:

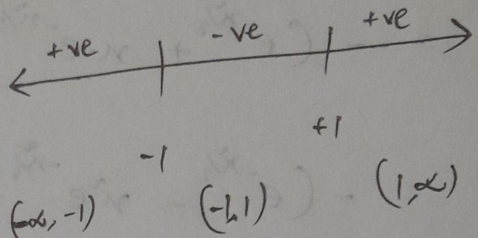
if $e^{-\frac{x^2}{2}} \neq 0$

$\therefore f''(x) = 0$

$e^{-x^2/2}(x^2-1) = 0$

$x^2 - 1 = 0$

$x = \pm 1$



$x = -2$	$x = 0$	$x = 2$
$f''(-2) = +ve$	$f''(0) = -ve$	$f''(2) = +ve$

concave up: $(-\infty, -1)$, $(1, \infty)$

concave down: $(-1, 1)$

$\therefore$ infection point: $x = -1, 1$

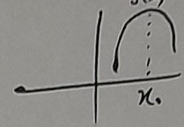
4.2

Analysis of function - II

Relative extrema

$\hookrightarrow$ Relative maximum and minimum.

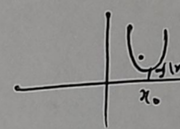
maximum:



if $f(x)$ has a maximum at $x = x_0$, then

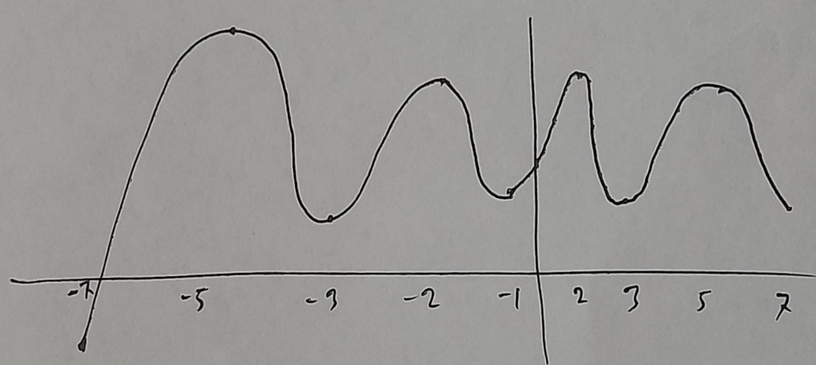
$$f(x_0) > f(x)$$

minimum:



if $f(x)$ has a minimum at $x = x_0$,

then, $f(x_0) < f(x)$



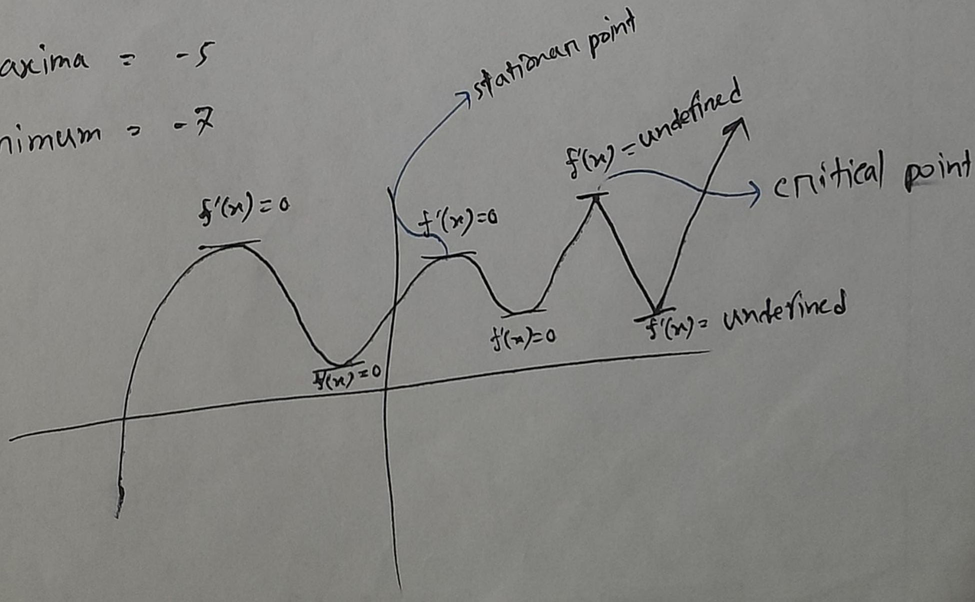
Relative maxima = 4 = −5, −2, 2, 5

Relative minimum = 5 = −7, −3, −1, 3, 7

Absolute maxima = −5

Absolute minimum = −7

(*)



MAT-120

L-17/ 08.08.2022

Critical Points:

Relative extremum points are called critical points.

Condition of critical points:

$f'(x) = 0$

and

$f'(x) = \text{undefined}$

Stationary Points:

Particular type of critical points where $f'(x) = 0$.

* Locate the critical points and identify which critical points are stationary points.

i) $x^3 - 3x + 1$

$\Rightarrow$ Here,

$f'(x) = 3x^2 - 3$

For critical points,

$f'(x) = 0$

$3x^2 - 3 = 0$

$3x^2 = 3$

$x = \pm 1$

Points are,

$x = 1,\ f(1) = -1 \Rightarrow (1, -1)$

$x = -1,\ f(-1) = 3 \Rightarrow (-1, 3)$

As this is polynomial function,

i.e. everywhere function is differentiable.

$\therefore$ All critical points are stationary points.

ii) $f(x) = \dfrac{x+1}{x^2+3}$

$$f'(x) = \frac{(x^2+3)\cdot\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x^2+3)}{(x^2+3)^2}$$

$$= \frac{x^2+3-2x^2-2x}{(x^2+3)^2}$$

$$= \frac{-x^2-2x+3}{(x^2+3)^2}$$

$\therefore$ For critical point,

$f'(x)=0$

$$\frac{-x^2-2x+3}{(x^2+3)^2} = 0$$

$x^2+2x-3=0$

$x^2+3x-x-3=0$

$(x+3)(x-1)=0$

$x=1,-3$

and,

here, $f'(x)$ nowhere undefined.

$\therefore$ stationary points and critical points are same.

iii)

$$f(x) = \frac{x+3}{x-2}$$

$$f'(x) = \frac{(x-2)\frac{d}{dx}(x+3) - (x+3)\frac{d}{dx}(x-2)}{(x-2)^2}$$

$$= \frac{(x-2)\cdot 1 - (x+3)\cdot 1}{(x-2)^2}$$

$$= \frac{x-2-x-3}{(x-2)^2}$$

$$= \frac{-5}{(x-2)^2}$$

For critical points,

$$f'(x) = 0$$

$$\frac{-5}{(x-2)^2} = 0$$

$-5 = 0$ (Not possible)

$f'(x)$ is undefined at $x=2$

Therefore, $x=2$ is a critical points and there is no stationary points.

iv) $f(x) = 3x^{5/3} - 15x^{2/3}$

$$f'(x) = \frac{5}{3} \cdot 3 \cdot x^{\frac{5}{3}-1} - 15 \cdot \frac{2}{3} \cdot x^{\frac{2}{3}-1}$$

$$= 5 \cdot x^{2/3} - 10 \cdot x^{-1/3}$$

$$= \frac{5 \cdot x^{2/3} \cdot x^{1/3} - 10}{x^{1/3}}$$

$$= \frac{5x-10}{x^{1/3}}$$

For critical points:

$f'(x) = 0$

$\frac{5x-10}{x^{1/3}} = 0$

$5x-10 = 0$

$5x = 10$

$x = 2$

and $f'(x)$ is undefined at $x=0$.

$\therefore$ critical points $x = 2, 0$ and stationary points $x=2$.