$$\Rightarrow \frac{u}{x} = -\frac{x^{-2}}{-2} + c$$

$$\Rightarrow \frac{u}{x} = \frac{1}{2x^2} + c$$

$$\therefore \frac{1}{xy} = \frac{1}{2x^2} + c$$

Ans

(*) Solve, $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\sqrt{y}$

$$\Rightarrow y^{-\frac{1}{2}} \frac{dy}{dx} + \frac{x}{1-x^2} \cdot y^{\frac{1}{2}} = x \quad \cdots (i)$$

Let, $y^{\frac{1}{2}} = u$

$$\frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx} = \frac{du}{dx}$$

$$(i) \Rightarrow 2\frac{du}{dx} + \frac{x}{1-x^2} u = x$$

$$\Rightarrow \frac{du}{dx} + \frac{x}{2(1-x^2)} u = \frac{x}{2} \quad \cdots (ii)$$

Here, $P(x) = \frac{x}{2(1-x^2)}$

$$I.F. = e^{\int \frac{x}{2(1-x^2)} dx}$$

$$= e^{\ln (1-x^2)^{-\frac{1}{4}}}$$

$$= \frac{1}{(1-x^2)^{1/4}}$$

$$\int \frac{x}{2(1-x^2)}$$

$$= -\frac{1}{2} \cdot \frac{1}{2} \int \frac{-2x}{-x^2+1}$$

$$= -\frac{1}{4} \ln (1-x^2)$$

(ii) × I.F

$$\frac{1}{(1-x^2)^{1/4}} \cdot \frac{du}{dx} + \frac{x}{2(1-x^2)} \cdot \frac{1}{(1-x^2)^{1/4}} \cdot u = \frac{x}{2} \cdot \frac{1}{(1-x^2)^{1/4}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{(1-x^2)^{1/4}} \cdot u \right) = \frac{x}{2(1-x^2)^{1/4}}$$

$$\Rightarrow \int d\left(\frac{u}{(1-x^2)^{1/4}} \right) = \int \frac{x}{2(1-x^2)^{1/4}} dx$$

$$\Rightarrow \frac{u}{(1-x^2)^{1/4}} = -\frac{1}{3}(1-x^2)^{3/4} + c$$

$$\Rightarrow \frac{y^{\frac{1}{2}}}{(1-x^2)^{1/4}} = -\frac{1}{3}(1-x^2)^{3/4} + c$$

Ans

Let,

$v = 1-x^2$

$\frac{dv}{dx} = -2x$

$x\,dx = -\frac{1}{2} dv$

$\int \frac{1}{2v^{1/4}} \cdot \frac{1}{-2} dv$

$= -\frac{1}{4} \int v^{-1/4} dv$

$= -\frac{1}{4} \cdot \frac{4}{3} \cdot v^{3/4}$

$= -\frac{1}{3} v^{3/4}$

$= -\frac{1}{3}(1-x^2)^{3/4}$

(*) $\frac{dy}{dx} + x\sin(2y) = x^3\cos^2 y$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + \frac{x \cdot 2\sin y \cdot \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x\tan y = x^3 \quad \text{--- (i)}$$

Let,

$\tan y = u$

$\sec^2 y \frac{dy}{dx} = \frac{du}{dx}$

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(i) $\Rightarrow \dfrac{du}{dx} + 2x\,u = x^3$ --- (ii)

Here, $p(x) = 2x$

$$I.F. = e^{\int 2x\,dx} = e^{x^2}$$

(ii) $\times$ I.F.,

$$e^{x^2}\cdot\frac{du}{dx} + 2x e^{x^2} u = x^3 e^{x^2}$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2}\cdot u\right) = x^3\cdot e^{x^2}$$

$$\Rightarrow \int d\left(e^{x^2}\cdot u\right) = \int x^3 e^{x^2}\cdot dx$$

$$\Rightarrow e^{x^2}\cdot u = \int x^2\cdot x e^{x^2}\,dx$$

$$= \frac{1}{2}\int z\cdot e^z\,dz$$

$$\Rightarrow e^{x^2}\cdot u = \frac{1}{2}\left(z e^z - e^z\right) + c$$

$$\Rightarrow e^{x^2}\cdot u = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$$

$$\therefore e^{x^2}\cdot \tan y = \frac{1}{2}\left(x^2 e^{x^2} - e^{x^2}\right) + c$$

Ans

Let,

$x^2 = z$

$2x\,dx = dz$

$x\,dx = \frac{1}{2}\,dz$

Table method

Diff.		Inte.
		e^z
z	$+$	e^z
1	$-$	e^z
0		

(*) $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$

$$\Rightarrow e^{-y}\frac{dy}{dx} + \frac{1}{x}\cdot e^{-y} = \frac{1}{x^2} \quad \text{--- (i)}$$

Let, $e^{-y} = u$

$$-e^{-y}\frac{dy}{dx} = \frac{du}{dx}$$

$$① \Rightarrow \quad -\frac{du}{dx} + \frac{1}{x}\cdot u = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{du}{dx} - \frac{1}{x}\cdot u = -\frac{1}{x^2} \quad \cdots (ii)$$

Here,

$$P(x) = -\frac{1}{x}$$

$$I.F. = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

(ii) × I.F.,

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}\cdot u\right) = -\frac{1}{x^3}$$

$$\Rightarrow \int d\left(\frac{u}{x}\right) = \int -\frac{1}{x^3}dx$$

$$\Rightarrow \frac{u}{x} = -\frac{x^{-2}}{-2} + c$$

$$\Rightarrow \frac{u}{x} = \frac{1}{2x^2} + c$$

$$\therefore \frac{e^{-y}}{x} = \frac{1}{2x^2} + c$$

Ans

H.W.:

from slide uploaded today linear ODE

Page-30: MKC 4-5x

Recap- for next Class!

Karchof' Law

Voltage and current Law

Cooling and warming by Newton's Law

L-7/13.08.2023/

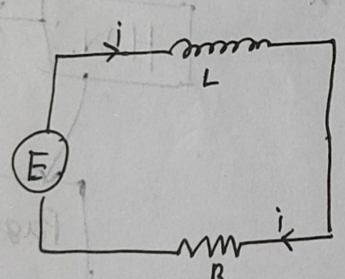
So Far Done — 1st Order 1st Degree ODE

1. Various Sepherable method → reducable Sepherable.
2. Homogeneous ODE → reducable to homogeneous.
3. Exact ODE → Non Exact ODE
4. Linear ODE → Non Linear ODE

Application of 1st Order 1st Degree ODE ← Chapter 3.1 Zill's Book

IVP = Initial Value Problem

⊛ RL Circuit



R = Resistance

L = Inductance

voltage drop accross the inductor $= L \frac{di}{dt}$

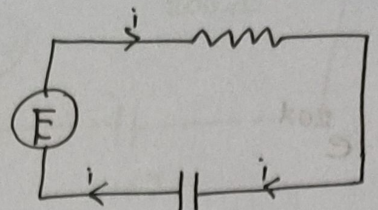
" " " " resistor $= iR$

Apply Voltage, $= E(t)$

according to kirchoff's voltage drops law,

$$\boxed{L \frac{di}{dt} + iR = E}$$ → ODE → Modeling

RC Circuit



Voltage drop accross the capacitor with capacitance of c $= \frac{q(t)}{c}$

Modeling Part

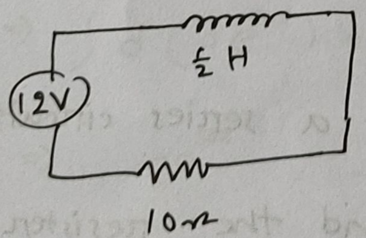
$$\boxed{iR + \frac{q}{c} = E} \longrightarrow \text{Not ODE}$$

$$\Rightarrow \boxed{R\frac{dq}{dt} + \frac{q}{c} = E} \longrightarrow \text{ODE}$$

⊛ A 12 Volt battery is connected to a series circuits in which the inductance is $\frac{1}{2}$ henry and the resistance is 10 ohms. Determine the current i if the initial current is zero.

→ IVP

→ $\begin{cases} t=0 \\ i=0 \end{cases} \Rightarrow i(0)=0$



$$L\frac{di}{dt} + iR = E$$

$$\frac{1}{2}\frac{di}{dt} + 10i = 12 \longrightarrow \text{Linear ODE}$$

$$\frac{di}{dt} + 20i = 24 \quad \cdots (i)$$

Here,

$$P(t) = 20$$

$$\therefore I.F. = e^{\int 20\,dt} = e^{20t}$$

① × I.F.,

$$e^{20t}\frac{di}{dt} + 20e^{20t}\,i = 24e^{20t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{20t}\cdot i\right) = 24e^{20t}$$

$$\Rightarrow \int d\left(ie^{20t}\right) = \int 24e^{20t}\,dt$$

$$\Rightarrow ie^{20t} = \frac{24}{20}e^{20t} + c$$

$$\Rightarrow ie^{20t} = \frac{6}{5}e^{20t} + c$$

$$i = \frac{6}{5} + ce^{-20t}$$

Given, $i(0) = 0$

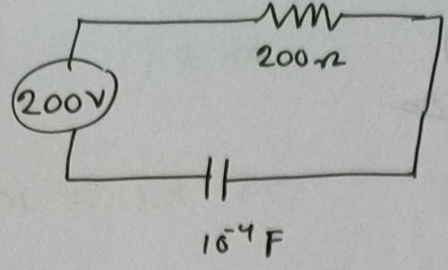
then, $0 = \frac{6}{5} + ce^{0}$

$c = -\frac{6}{5}$

$$\therefore i = \frac{6}{5}\left(1 - e^{-20t}\right)$$ Ans

⊛ A 200 volt battery is connected to a series circuit in which the capacitance is 10^{-4} farad and the resister is 200 ohm. Determine the charge if $q(0) = 0$. also determine the current.

$\Rightarrow$



Here,

$$R\frac{dq}{dt} + \frac{q}{c} = E$$

$$\Rightarrow 200\frac{dq}{dt} + 10^{4} q = 200 \longrightarrow \text{Linear ODE}$$

$$\frac{dq}{dt} + \frac{10000}{200} q = 1 \quad \text{--- (i)}$$

Here

$$P(t) = \frac{1000}{200} = 50$$

$$\therefore I.F. = e^{\int 50 dt} = e^{50t}$$

(i) $\times$ IF,

$$e^{50t}\frac{dq}{dt} + 50e^{50t} q = e^{50t}$$

$$\Rightarrow \frac{d}{dt}(e^{50t} \cdot q) = e^{50t}$$

$$\Rightarrow \int d(qe^{50t}) = \int e^{50t} dt$$

$$\Rightarrow qe^{50t} = \frac{e^{50t}}{50} + c$$

$$\therefore q = \frac{1}{50} + ce^{-50t}$$

Given, $q(0) = 0$

then, $0 = \frac{1}{50} + c \cdot e^{0}$

$$\therefore c = -\frac{1}{50}$$

$$\therefore q = \frac{1}{50}\left(1 - e^{-50t}\right)$$ Ans

We know,

$$i = \frac{dq}{dt} = e^{-50t}$$ Ans

⊛ Growth and decay problem

⊛ The number of population or amount of carbon increases or decreases at a rate, ~~of~~ proportional to the number of population or amount of carbon present at any time t, that is,

$$\frac{dx}{dt} \propto x \quad \rightarrow \text{amount of carbon any time } t$$

$$\Rightarrow \boxed{\frac{dx}{dt} = kx} \rightarrow \text{ODE}$$

⊛ The population of a country is known to increase at a rate proportional to the people at any time.
If the population has doubled in 30 years, how long will it take to triple?

$\Rightarrow$ Let x be the number of people at time t.
Let, x_0 be the initial number of people at $t=0$.
according to the growth and decay problem, we know,

$$\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx$$

$$\Rightarrow \int \frac{dx}{x} = \int k\, dt$$

$$\Rightarrow \ln x = kt + c_1$$

$$\Rightarrow x = e^{kt+c_1} = e^{kt} \cdot e^{c_1} = ce^{kt}$$

$$\therefore x(t) = ce^{kt} \quad \ldots (i)$$

at,

$t = 0, \; x = x_0$

$$\therefore x_0 = ce^0$$

$$\therefore c = x_0$$

$$\therefore (i) \Rightarrow x(t) = x_0 e^{kt} \quad \ldots (ii)$$

at,

$t = 30$ years, $x = 2x_0$

$$\therefore 2x_0 = x_0 e^{30k}$$

$$e^{30k} = 2$$

$$30k = \ln 2$$

$$k = \frac{\ln 2}{30}$$

$$(ii) \Rightarrow x(t) = x_0 e^{\left(\frac{\ln 2}{30}\right)t} \quad \ldots (iii)$$

Let,

t^* be the time for which $x = 3x_0$

$$(iii) \Rightarrow 3x_0 = x_0 e^{\left(\frac{\ln 2}{30}\right)t^*}$$

$$3 = e^{\left(\frac{\ln 2}{30}\right)t^*}$$

$$\frac{\ln 2}{30} t^* = \ln 3$$

$$t^* = \frac{\ln 3}{\ln 2} \times 30$$

$$\cong 47.55 \text{ years}$$

Newton's Law of Cooling / Warming

$$\frac{dT}{dt} \propto (T - T_m)$$

average temperature

$$\frac{dT}{dt} = k(T - T_m)$$

H.W. ⇒

Zill's Book

Page - 84 ⇒ Example ⇒ 1, 2, 3

Example ⇒ 7

Page - 90

Exercise 3.1 ⇒ 1–6 ⇒ Growth and decay

29–33 ⇒ series circuits

L-8/20.08.2023 /

Newton's Law of Cooling

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\frac{dT}{dt} = k(T - T_m)$$

⊛ The body of a murdered victim was discovered at 11:00 pm. The doctor took the temperature of the body at 11:30 pm which was 94·6°F. He again took the temperature after 1 hour when showed 93·4°F and noticed that the temperature of room was 70°F. Estimate the time of death. Normal temperature of the human body is 98·6° F.

$\Rightarrow$

$$\frac{dT}{dt} = k(T - 70)$$

$$\int \frac{dT}{T-70} = \int k\,dt$$

$$\ln(T-70) = kt + c_1$$

$$T - 70 = e^{kt+c_1} = e^{kt} \cdot e^{c_1} = ce^{kt}$$

$$\boxed{T = 70 + ce^{kt}} \quad \text{---(i)}$$

Let, $t = 0$, $T = 94·6°F$

(i) $\Rightarrow 94·6 = 70 + ce^{0}$

$c = 94·6 - 70 = 24·6$

$\therefore$ (i) $\Rightarrow$ $\boxed{T = 70 + 24·6\, e^{kt}}$ ---(ii)

Let,

$t = 1$ hour, $T = 93.4°F$

(ii) $\Rightarrow$ $93.4 = 70 + 24.6\, e^{k}$

$\Rightarrow$ $e^{k} = \dfrac{93.4 - 70}{24.6} = \dfrac{23.4}{24.6}$

$k = \ln\left(\dfrac{23.4}{24.6}\right)$

(ii) $\Rightarrow$ $\boxed{T = 70 + 24.6\, e^{\ln\left(\frac{23.4}{24.6}\right)t}}$ --- (iii)

Let,

t^{*} is the time when the temperature was $98.6°F$

(iii) $\Rightarrow$ $98.6 = 70 + 24.6\, e^{\ln\left(\frac{23.4}{24.6}\right)t^{*}}$

$\Rightarrow$ $e^{\ln\left(\frac{23.4}{24.6}\right)t^{*}} = \dfrac{98.6 - 70}{24.6} = \dfrac{28.6}{24.6}$

$\ln\left(\dfrac{23.4}{24.6}\right)t^{*} = \ln\left(\dfrac{28.6}{24.6}\right)$

$t^{*} \cong -3.013$ hour

$\therefore$ Time to death $= 11.30 - 3.013 \cong 8:30$ pm.

$\boxed{\text{H.W.}}$ $\Rightarrow$ Page - 91 Exercise 3.1 $\Rightarrow$ 13-17

(*) Application of ODE in Midterm must.

Higher Order Linear ODE with constant coefficient

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$$\left(a_2 \frac{d^2}{dx^2} + a_1 \frac{d}{dx} + a_0\right) y = 0$$

$$(a_2 D^2 + a_1 D + a_0) y = 0$$

Homogeneous form

$y = 0$; its trivial solution.

a_i = known constant

Solution is

$y = e^{mx}$ be the trial solution of all ~~kind~~ equation mentioned above.

⊛ Operator Notation,

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2$$

$$\frac{d^3}{dx^3} = D^3 \; \ldots \text{ etc.}$$

Lets satisfy,

$$y' = Dy = \frac{dy}{dx} = me^{mx}$$

$$y'' = D^2 y = \frac{d^2y}{dx^2} = m^2 e^{mx}$$

Then,

$$a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$$

can not be zero

$$e^{mx} (a_2 m^2 + a_1 m + a_0) = 0$$

known as Auxiliary equation (AE)

Since $e^{mx} \neq 0$ then, $\boxed{a_2 m^2 + a_1 m + a_0 = 0}$ → Now solve for m.

$m = m_1$ & m_2

i.e.: $(2D^2 + 5D + 10)y = 0$

$\therefore$ AE ⇒ $2m^2 + 5m + 10 = 0$

$\therefore y_1 = e^{m_1 x}$ & $y_2 = e^{m_2 x}$

2nd order = 2 arbitrary constant

3rd order = 3 " "

... ...

⊛ Linear Combination Recap for next class (MAT 125)

L-9/22.08.2023

Higher Order ODE with constant Coefficient

$$(a_2 D^2 + a_1 D + a_0) y = 0 \quad \cdots (i)$$

when,

$D = \frac{d}{dx}$

$D^2 = \frac{d^2}{dx^2}$

$D^3 = \frac{d^3}{dx^3}$

Let,

$y = e^{mx}$ is the trial solution

$Dy = me^{mx}$

$D^2 y = m^2 e^{mx}$

$$(i) \Rightarrow a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$$

Since, $e^{mx} \neq 0$; $\boxed{a_2 m^2 + a_1 m + a_0 = 0}$ which is A.E.

$m = m_1, m_2$

$y_1 = e^{m_1 x}$

$y_2 = e^{m_2 x}$

General Solution (G.S.)

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case-1: if m_1 and m_2 are real and $m_1 \neq m_2$, then

$$G.S. \Rightarrow y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

Example:

$$(D^2 + 5D + 6)y = 0$$

$$A.E. \Rightarrow m^2 + 5m + 6 = 0$$

$$m = -3, -2$$

$$G.S. \Rightarrow y = c_1 e^{-3x} + c_2 e^{-2x}$$

Case-2: if m_1 and m_2 are real and $m_1 = m_2$, then

$$G.S. \Rightarrow y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

[if it is 3rd Order and same value, then next one will be $c_3 x^2 e^{m_1 x}$]

Example: $(D^2 + 4D + 4)y = 0$

$$A.E \Rightarrow m^2 + 4m + 4 = 0$$

$$m = -2, -2$$

$$G.S. \Rightarrow y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Another way solutions:

$$(D+2)(D+2)y = 0 \quad \ldots (i)$$

$$\text{Let, } (D+2)y = v \quad \ldots (ii)$$

$$(i) \Rightarrow (D+2)v = 0$$

$$\Rightarrow \frac{dv}{dx} + 2v = 0$$

$$\Rightarrow \frac{dv}{dx} = -2v$$

$$\Rightarrow \int \frac{dv}{v} = -2\int dx$$

$$\Rightarrow \ln v = -2x + c$$

$$v = e^{-2x} \cdot e^{c} = c_2 e^{-2x}$$

(ii) $\Rightarrow$ $\frac{dy}{dx} + 2y = c_2 e^{-2x}$ ... (iii)

Here, $P = 2$

$\therefore I.F. = e^{\int 2\,dx} = e^{2x}$

(iii) $\times$ I.F.

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = c_2$$

$$\frac{d}{dx}(e^{2x}.y) = c_2$$

$$\int d(ye^{2x}) = \int c_2\,dx$$

$$ye^{2x} = c_2 x + c_1$$

Here, x comes for 2nd Order diff.

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Case-3: if $m = \alpha \pm i\beta$

G.S. $\Rightarrow$ $y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}$

$$= c_1 e^{\alpha x}.e^{i\beta x} + c_2 e^{\alpha x}.e^{-i\beta x}$$

$$= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$= e^{\alpha x}\left[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(c_1 + c_2)\cos\beta x + (c_1 i - c_2 i)\sin\beta x\right]$$

$$\therefore y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$$

Eulen form of Complen number

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{2ix} = \cos 2x + i\sin 2x$

$\underbrace{e^{x} = \cos x + \sin x}$

in terms of seriese.

Enample: $(D^2 - 4D + 13)y = 0$

A.E. $\Rightarrow m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$

G.S. $\Rightarrow y = e^{2x}[A\cos 3x + B\sin 3x]$

⊛ $Y = A\cos x + B\sin x$

$\Rightarrow \frac{d^2y}{dx^2} + y = 0 \Rightarrow (D^2+1)y = 0$

A.E $\Rightarrow m^2+1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1} = \pm i$

G.S. $\Rightarrow y = e^{0x}[A\cos x + B\sin x]$

$= A\cos x + B\sin x$

realme Shot by Legend T-JOY

⊛ $(D^2+2D+4)y=0$

A.E $\Rightarrow$ $m^2+2m+4=0$

$$m=\frac{-2\pm\sqrt{4-16}}{2}=\frac{-2\pm\sqrt{-12}}{2}=\frac{-2\pm2\sqrt{3}i}{2}$$

$$=-1\pm\sqrt{3}i$$

$\therefore$ G.S. $\Rightarrow$ $y=e^{-x}\left[A\cos(\sqrt{3}x)+B\sin(\sqrt{3}x)\right]$

⊛ $(D^3-6D^2+9D)y=0$

A.E. $\Rightarrow$ $m^3-6m^2+9m=0$

$\Rightarrow m(m^2-6m+9)=0$

$\Rightarrow m(m-3)^2=0$

$\Rightarrow m(m-3)(m-3)=0$

$\therefore m=0,3,3$

$\therefore$ G.S. $\Rightarrow$ $y=c_1e^{0x}+c_2e^{3x}+c_3xe^{3x}$

$\boxed{\text{H.W.}}\Longrightarrow$

1. $(D^2-2D-3)y=0$
2. $(D^3-D)y=0$
3. $(D^2+1)y=0$
4. $(D^4-1)y=0$
5. $(D^3-5D^2+7D-2)y=0$
6. $(D^4+2D^2+1)y=0$

L-10 / 27.08.2023

Recap

Higher Order ODE with Constant Coefficient

$(a_2 D^2 + a_1 D + a_0) y = 0 \longrightarrow$ Homogeneous Form

A.E. $\Rightarrow$ $a_2 m^2 + a_1 m + a_0 = 0$

$\therefore m = m_1, m_2$

Case-1: if $m_1 \neq m_2$ and real

G.S. $\Rightarrow$ $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case-2: if $m_1 = m_2$ and real

G.S. $\Rightarrow$ $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$

Case-3: if m is not real, $m = \alpha \pm j\beta$

G.S. $\Rightarrow$ $y = e^{\alpha x}\left[A \cos\beta x + B \sin\beta x\right]$

Now, Not Homogeneous Form

$(a_2 D^2 + a_1 D + a_0) y = R(x)$

G.S. $\Rightarrow$ $y = y_c + y_p$

G.S. of Homogeneous Form

y_c = complementary function

y_p = particular integral

for y_p,

inverse operator method:

$$D = \frac{d}{dx} \Rightarrow \frac{1}{D} = D^{-1} = \int dx$$

$$D^2 = \frac{d^2}{dx^2} \Rightarrow \frac{1}{D^2} = D^{-2} = \frac{1}{D}\int dx$$

R.H.S.

$R(x) \Rightarrow$ (constant, polynomial of x), $\boxed{e^{ax}}$, ($\sin bx$ or $\cos bx$)

$\Rightarrow$ Then we can apply inverse operator method.

Rule-1: for $R(x)$ = constant or polynomial of x

1. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \cdots$
2. $(1+D)^{-1} = 1 - D + D^2 - D^3 + \cdots$
3. $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \cdots$
4. $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \cdots$

Theorem :

$$\frac{d^{n+1}}{dx^{n+1}}(x^n) = 0$$

Example : $(D^2 + D - 6)y = x \quad \cdots (i)$

A.E. $\Rightarrow$ $m^2 + m - 6 = 0$

$m = -3, 2$

$\therefore y_c = c_1 e^{-3x} + c_2 e^{2x}$

(i) $\Rightarrow$ $y_p = \dfrac{1}{D^2 + D - 6} x$ always in right

$= \dfrac{1}{-6\left(1 - \frac{D^2+D}{6}\right)} x$ [minimum power of D will be common]

$= -\frac{1}{6}\left(1 - \frac{D^2+D}{6}\right)^{-1} x$ No need (Here, minimum power of D is 2)

$= -\frac{1}{6}\left[1 + \frac{D^2+D}{6} + \left(\frac{D^2+D}{6}\right)^2 + \cdots\right] x$ [we should consider power of D as the power of x]

minimum power of D is 1 = power of x

$$= -\frac{1}{6}\left[x + \frac{0+1}{6} + 0\right]$$

$$= -\frac{1}{6}\left(x + \frac{1}{6}\right)$$

$$y_p = -\frac{x}{6} - \frac{1}{36}$$

$\therefore$ G.S. $\Rightarrow$ $y = y_c + y_p = c_1 e^{-3x} + c_2 e^{2x} - \frac{x}{6} - \frac{1}{36}$

⊛ $(D^2 - 6D + 9)y = x^2 + x + 1$

A.E. $\Rightarrow$ $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\therefore m = 3, 3$

$x^2 + x + 1$

$D = 2x + 1$

$D^2 = 2$

$(D^2 - 6D)^2$

$= D^4 - 12D^3 + 36D^2$

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$y_p = \frac{1}{D^2 - 6D + 9} \cdot (x^2 + x + 1)$$

$$= \frac{1}{9\left(1 + \frac{D^2 - 6D}{9}\right)} (x^2 + x + 1)$$

$$= \frac{1}{9}\left(1 + \frac{D^2 - 6D}{9}\right)^{-1} (x^2 + x + 1)$$

$$= \frac{1}{9}\left[1 - \frac{D^2 - 6D}{9} + \left(\frac{D^2 - 6D}{9}\right)^2 - \dots\right] (x^2 + x + 1)$$

$$= \frac{1}{9}\left[x^2 + x + 1 - \frac{2 - 6(2x+1)}{9} + \frac{0 - 0 + 36 \cdot 2}{81}\right]$$

$$= \frac{1}{9}\left(x^2 + x + 1 - \frac{2 - 12x - 6}{9} + \frac{72}{81}\right]$$

$$= \frac{1}{9}\left(x^2 + x + 1 + \frac{4 + 12x}{9} + \frac{72}{81}\right)$$

$$Y_p = \frac{x^2}{9} + \frac{x}{9} + \frac{1}{9} + \frac{4}{81} + \frac{12x}{81} + \frac{72}{81\cdot 9}$$

$$= \frac{x^2}{9} + \frac{7x}{27} + \frac{7}{27}$$

$\therefore$ G.S. $\Rightarrow$ $Y = Y_c + Y_p$

$$= c_1 e^{m_1 x} + c_2 x e^{m_1 x} + \frac{x^2}{9} + \frac{7x}{27} + \frac{7}{27}$$

Rule-2:

for $R(x) = e^{ax}$

$$\therefore Y_p = \frac{1}{f(D)} e^{ax}$$

Case-1: successful case

$$Y_p = \frac{1}{f(a)} e^{ax} \; ; \text{ if } f(a) \neq 0$$

Example :

$$(D^2 + 3D + 4)y = e^{2x}$$

A.E. $\Rightarrow$ $m^2 + 3m + 4 = 0$

$$m = \frac{-3 \pm \sqrt{9-16}}{2} = \frac{-3 \pm \sqrt{-7}}{2}$$

$$= -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\therefore Y_c = e^{-\frac{3}{2}x} \left[A \cos \frac{\sqrt{7}}{2} x + B \sin \frac{\sqrt{7}}{2} x \right]$$

$$Y_p = \frac{1}{(D^2+3D+4)} e^{2x}$$

$$= \frac{1}{2^2+3\cdot 2+4} \cdot e^{2x}$$

$$= \frac{1}{14} e^{2x}$$

$\therefore$ G.S. $\Rightarrow$ $y = Y_c + Y_p$

$$= e^{-\frac{3}{2}x}\left[A\cos\frac{\sqrt{7}}{2}x + B\sin\frac{\sqrt{7}}{2}x\right] + \frac{1}{14}e^{2x}$$

Case-2: failure case

$$Y_p = \frac{1}{f(a)} e^{ax} \ ; \text{ if } f(a) = 0$$

Example:

$$(D^2 - 3D + 2)y = e^x$$

A.E $\Rightarrow$ $m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m = 1, 2$

$\therefore Y_c = c_1 e^x + c_2 e^{2x}$

Short Method

$$Y_p = \frac{1}{(D^2-3D+2)} e^x$$

$$= \frac{x}{2D-3} \cdot e^x$$

[if $f(a) = 0$, until we get $f(a) \neq 0$, we need to diff. $f(D)$ wrt D and multiply by a x, everytime.]

$= \frac{x}{-1} \cdot e^x$

$= -xe^x$

$\therefore$ G.S. $\Rightarrow$ $y = y_c + y_p$

$= c_1 e^x + c_2 e^{2x} - xe^x$

Elaborate method:

$$y_p = \frac{1}{D^2 - 3D + 2} e^x$$

$$= e^x \frac{1}{(D+1)^2 - 3(D+1) + 2} \cdot 1$$

$$= e^x \frac{1}{D^2 + 2D + 1 - 3D + 3 + 2} \cdot 1$$

$$= e^x \frac{1}{D^2 - D} 1$$

$$= e^x \left(-\frac{1}{D}\right)(1-D)^{-1} \cdot 1$$

$$= e^x \left(-\frac{1}{D}\right)(1 + D + \cdots) 1$$

$$= e^x \left(-\frac{1}{D}\right)(1+0)$$

integral

$$= e^x(-x)$$

$\boxed{\text{H.W.}} \Rightarrow$ From MKC Book

Page - 108: Example : 4·5 – 4·7

- $(D^2 + 4D - 2)y = 2x^2 - 3x + 6$

  Ans.: $y = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$

- $(D^2 - 5D + 4)y = 8e^x$

L-11 / 29.08.2023

Recap

Higher Order Linear ODE with constant coefficient

* Inverse operator method

$R(x) = $ (const., poly), e^{ax}, ($\cos bx$ or $\sin bx$)

G.S. $\Rightarrow y = y_c + y_p$

Rule-1:

$R(x) = $ (constant, polynomial)

i. $(1-D)^{-1} = 1 + D + D^2 + \cdots$

ii. $(1+D)^{-1} = 1 - D + D^2 - \cdots$

iii. $(1-D)^{-2} = 1 + 2D + 3D^2 + \cdots$

iv. $(1+D)^{-2} = 1 - 2D + 3D^2 - \cdots$

Rule-2: $R(x) = \frac{1}{f(D)} e^{ax}$

i. $y_p = \frac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

ii. $y_p = \frac{1}{f(a)} e^{ax}$; $f(a) = 0$

$= \frac{x}{f'(D)} e^{ax}$

Rule-3:

$R(x) = \sin bx$ or $\cos bx$

$$y_p = \frac{1}{f(D^2)} (\sin bx \text{ or } \cos bx)$$

Case-i: successful case :

$$y_p = \frac{1}{f(-b^2)} (\sin bx \text{ or } \cos bx) \text{ ; if } f(-b^2) \neq 0$$

Example: $(D^2+1)y = \cos 2x$

A.E. $\Rightarrow m^2+1 = 0$

$m = \pm i$

$y_c = A\cos x + B\sin x$

$$y_p = \frac{1}{(D^2+1)} \cos 2x$$

$$= \frac{1}{-2^2+1} \cos 2x$$

$$= -\frac{1}{3} \cos 2x$$

$\therefore$ G.S. $\Rightarrow$ $y = y_c + y_p = A\cos x + B\sin x - \frac{1}{3}\cos 2x$

Ans

Case-2: Failure Case:

$$y_p = \frac{1}{f(-b^2)} (\sin bx \text{ or } \cos bx) \; ; \text{ if } f(-b^2) = 0$$

$$= \frac{x}{2} \int (\sin bx \text{ or } \cos bx)\, dx$$

$\sin 2x$

Example:

$$(D^2+1)y = \sin x$$

$D^2 + 4$

$a + 4$

$$y_c = A\cos x + B\sin x$$

$$y_p = \frac{1}{D^2+1} \sin x$$

$$= \frac{1}{-1^2+1} \sin x \; [\text{failure case}]$$

$$= \frac{x}{2} \int \sin x \, dx$$

$$= -\frac{x}{2} \cos x$$

$\therefore$ G.S. $\Rightarrow$

$y = y_c + y_p$

$= A\cos x + B\sin x - \frac{x}{2}\cos x$

Ans

$$(D^2 - 3D + 2)y = \sin 3x \quad \cdots (i)$$

A.E. $\Rightarrow$

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$\Rightarrow m = 1, 2$$

$$\therefore y_c = c_1 e^x + c_2 e^{2x}$$

(i) $\Rightarrow$ $y_p = \dfrac{1}{D^2 - 3D + 2} \sin 3x$

$$= \frac{1}{-3^2 - 3D + 2} \sin 3x$$

if only D left, then do the integral

$$= -\frac{1}{7 + 3D} \sin 3x$$

$$= -\frac{7 - 3D}{7^2 - (3D)^2} \sin 3x$$

$$= \frac{3D - 7}{49 - 9D^2} \sin 3x$$

$$= \frac{3D - 7}{49 - 9(-3^2)} \sin 3x$$

$$= \frac{3D - 7}{49 + 81} \sin 3x$$

normal diff.

$$= \frac{3D - 7}{130} \sin 3x$$

$$= \frac{1}{130} (3 \cos 3x \cdot 3 - 7 \sin 3x)$$

$$= \frac{1}{130} (9 \cos 3x - 7 \sin 3x)$$

$\therefore$ G.S. $\Rightarrow$

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{2x} + \frac{1}{130} (9 \cos 3x - 7 \sin 3x)$$

Ans

$$\frac{1}{49 - 9D^2} \int (3D - 7) \sin 3x$$

$$\frac{x}{2} \int (3D - 7)(\sin 3x)$$

$$(D^2-3D+4)y = \cos(4x+5)$$

A.E. $\Rightarrow$ $m^2 - 3m + 4 = 0$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm \sqrt{-7}}{2}$$

$$= \frac{3}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\therefore y_c = e^{\frac{3}{2}x}\left[A\cos\frac{\sqrt{7}}{2}x + B\sin\frac{\sqrt{7}}{2}x\right]$$

$$y_p = \frac{1}{D^2-3D+4}\cos(4x+5)$$

$$= \frac{1}{-4^2-3D+4}\cos(4x+5)$$

$$= -\frac{1}{12+3D}\cos(4x+5)$$

$$= -\frac{12-3D}{(12)^2-(3D)^2}\cos(4x+5)$$

$$= \frac{3D-12}{144-9D^2}\cos(4x+5)$$

$$= \frac{3D-12}{144-9(-4^2)}\cos(4x+5)$$

$$= \frac{3D-12}{144+144}\cos(4x+5)$$

$$= \frac{1}{288}\left(-3\sin(4x+5)\cdot 4 - 12\cos(4x+5)\right)$$

$$= \frac{-12\sin(4x+5) - 12\cos(4x+5)}{288} = -\frac{1}{24}\left(\sin(4x+5) + \cos(4x+5)\right)$$

Rule-4:

$$Y_p = \frac{1}{f(D)} e^{ax} \cdot M(x)$$ → Polynomial or $\sin x$ or $\cos x$

$$= e^{ax} \frac{1}{f(D+a)} M(x)$$ ⇒ $M(x)$ ⇒ Polynomial, then rule-1

$M(x)$ ⇒ ($\sin x$ or $\cos x$), then rule-3

Example:

$$(D^2-4D+4)y = x^2 e^{2x}$$

A.E. ⇒ $m^2-4m+4=0$

$m = 2,2$

$\therefore Y_c = c_1 e^{2x} + c_2 x e^{2x}$

$$Y_p = \frac{1}{D^2-4D+4} \cdot e^{2x} \cdot x^2$$

$$= e^{2x} \frac{1}{(D+2)^2-4(D+2)+4} \cdot x^2$$ [Now apply Rule-1]

$$= e^{2x} \frac{1}{D^2+4D+4-4D-8+4} x^2$$

$$= e^{2x} \frac{1}{D^2} x^2$$

$$= e^{2x} \frac{1}{D} \cdot \frac{x^3}{3}$$

$$= e^{2x} \cdot \frac{x^4}{12}$$

$\therefore$ G.S. ⇒

$y = Y_c + Y_p$

$= c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{12} x^4 e^{2x}$

Ans

$(D^2-2D+1)y = e^x \sin x$

A.E. $\Rightarrow$

$m^2-2m+1=0$

$(m-1)^2=0$

$m = 1,1$

$\therefore y_c = c_1e^x + c_2xe^x$

$$y_p = \frac{1}{D^2-2D+1} e^x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+1} \sin x$$

$$= e^x \frac{1}{D^2+2D+1-2D-2+1} \sin x$$

$= e^x \frac{1}{D^2} \sin x$ [Rule-3]

$= e^x \frac{1}{-1^2} \sin x$

$= -e^x \sin x$

$\therefore$ G.S. $\Rightarrow$

$y = y_c + y_p$

$= c_1e^x + c_2xe^x - e^x \sin x$ Ans

⊛ $(D^3 - D)y = e^x x$

A.E ⇒

$m^3 - m = 0$

$m(m^2-1) = 0$

$m = 0, \pm 1$

$\therefore y_c = c_1 e^0 + c_2 e^{-x} + c_3 e^x$

$$y_p = \frac{1}{D^3 - D} e^x . x$$

$\boxed{\text{H.W.}}$ ⇒ MKC Book

Page : 109

Example: 4.8 – 4.11

4.15 – 4.17

$$= e^x \frac{1}{(D+1)^3 - (D+1)} x$$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 - D - 1} x$$

$$= e^x \frac{1}{D^3 + 3D^2 + 2D} x \quad \text{[Rule-1]}$$

$$= e^x \frac{1}{2D\left(1 + \frac{D^2+3D}{2}\right)} x$$

$$= e^x \frac{1}{2D}\left(1 - \frac{D^2+3D}{2} + \dots\right) x$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left(x - \frac{0+3}{2}\right)$$

$$= \frac{e^x}{2}\left(\frac{x^2}{2} - \frac{3}{2}x\right)$$

$\therefore$ G.S. ⇒

$$y = c_1 + c_2 e^{-x} + c_3 e^x + \frac{e^x}{2}\left(\frac{x^2}{2} - \frac{3}{2}x\right)$$

Ans

L-12 / 03.09.2023

Higher Order ODE with constant coefficient

Recap

Rule-3:

(i) $y_p = \frac{1}{f(D^2)}(\sin bx \text{ or } \cos bx)$

$= \frac{1}{f(-b^2)}(\sin bx \text{ or } \cos bx)$

$f(-b^2) \neq 0$

(ii) $y_p = \frac{x}{2}\int(\sin bx \text{ or } \cos bx)\,dx$

$f(-b^2) = 0$

Rule-4:

$y_p = \frac{1}{f(D)} M(x) e^{ax}$ → must contain e^{ax}

$= e^{ax} \frac{1}{f(D+a)} M(x)$ → Rule 1 or 3

Rule-5:

$y_p = \frac{1}{f(D)} x^n (\sin bx \text{ or } \cos bx)$; $n > 1$

$e^{\pm ix} = \cos x \pm i \sin x$

$e^{\pm ibx} = \cos bx \pm i \sin bx$

Example:

$(D^2+1)y = x^2 \cos x$ ..(i)

A.E. ⇒ $m^2+1 = 0$

$m = \pm i$

$y_c = A\cos x + B\sin x$

(i) ⇒ $y_p = \frac{1}{D^2+1} x^2 \cos x$

$= \text{R.P. of } \frac{1}{D^2+1} x^2 e^{ix}$

$= R.P. \text{ of } e^{ix} \dfrac{1}{(D+i)^2+1} x^2$

$= R.P. \text{ of } e^{ix} \dfrac{1}{D^2+2iD-1+1} x^2$

$= \text{"} \quad e^{ix} \dfrac{1}{D^2+2iD} x^2$

$= \text{"} \quad e^{ix} \dfrac{1}{2iD} \left(1+\dfrac{D}{2i}\right)^{-1} x^2$

$= \text{"} \quad \dfrac{e^{ix}}{2i} \cdot \dfrac{1}{D} \left(1 - \dfrac{D}{2i} + \dfrac{D^2}{4(-1)} - \cdots\right) x^2$

$= \text{"} \quad \dfrac{e^{ix}}{2}(-i) \cdot \dfrac{1}{D} \left[x^2 - \dfrac{2xi}{-2} + \dfrac{2}{-4} - 0\right]$

$= \text{"} \quad \dfrac{e^{ix}}{2}(-i) \cdot \dfrac{1}{D} \left(x^2 + xi - \dfrac{1}{2}\right)$

$= \text{"} \quad \dfrac{e^{ix}}{2}(-i) \left(\dfrac{x^3}{3} + \dfrac{x^2}{2}i - \dfrac{1}{2}x\right)$

$= \text{"} \quad \dfrac{1}{2}(\cos x + i \sin x) \left(\dfrac{-ix^3}{3} + \dfrac{x^2}{2} + i\dfrac{1}{2}x\right)$

$= \text{"} \quad \dfrac{1}{2}\left(-i\dfrac{x^3}{3}\cos x + \dfrac{x^2}{2}\cos x + \dfrac{i}{2}x\cos x + \dfrac{x^3}{3}\sin x + i\dfrac{x^2}{2}\sin x - \dfrac{1}{2}x\sin x\right)$

$y_p = \dfrac{1}{2}\left(\dfrac{x^2}{2}\cos x + \dfrac{x^3}{3}\sin x - \dfrac{1}{2}x\sin x\right)$

✱ if given function is $\sin bx$ then,

R.P. of ⇒ I.P. of

Cofficient of i only

Special Formula of Rule-5 when, $n = 1$

$$y_p = \frac{1}{f(D)} x^1 (\cos bx \text{ or } \sin bx)$$

$$= \frac{1}{f(D)} \cdot x V$$

$$= x \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$$

Example:

$$(D^2 - 2D + 1) y = x \sin x$$

$$y_p = \frac{1}{(D^2 - 2D + 1)} x \sin x$$

$$= x \frac{1}{D^2 - 2D + 1} \sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2} \sin x$$

$$= x \frac{1}{-1^2 - 2D + 1} \sin x - \frac{2D - 2}{(-1^2 - 2D + 1)^2} \sin x$$

$$= x \frac{1}{-2D} \sin x - \frac{2D - 2}{(-2D)^2} \sin x$$

$$= \frac{x}{-2} (-\cos x) - \frac{2D - 2}{4(-1)} \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2} (D - 1) \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2} (\cos x - \sin x)$$