|L-26/22.10.2023|

Fourier Series

⊛ $f(t) = \begin{cases} -\sin \omega t & ; -\pi < \omega t < 0 \\ \sin \omega t & ; 0 \leq \omega t < \pi \end{cases}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t\right)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t)\, d\omega t$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin \omega t \, d\omega t$$

$$= -\frac{2}{\pi} \cos \omega t \Big]_0^{\pi}$$

$$= -\frac{2}{\pi} (-1-1)$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos n\omega t \; d\omega t$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin \omega t \cos n\omega t \; d\omega t$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \cdot \sin \omega t \cos n\omega t \; d\omega t$$

$$= \frac{1}{\pi} \int_0^{\pi} \left\{ \sin(n+1)\omega t - \sin(n-1)\omega t \right\} d\omega t$$

$$= \frac{1}{\pi} \left\{ \left[\frac{-\cos(n+1)\omega t}{n+1} \right]_0^{\pi} + \left[\frac{\cos(n-1)\omega t}{n-1} \right]_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1 - \cos(n+1)\pi}{n+1} + \frac{-1 + \cos(n-1)\pi}{n-1} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1 + \cos n\pi}{n+1} + \frac{-1 - \cos n\pi}{n-1} \right\}$$

$$= \frac{1}{\pi} \cdot \frac{(n-1)(1+(-1)^n) + (n+1)(1+(-1)^n)}{n^2-1}$$

$$= \frac{1}{\pi} \cdot \frac{\{1+(-1)^n\}\{(n-1)+(n+1)\}}{n^2-1}$$

$$= \frac{1}{\pi} \cdot \frac{\cancel{2n(1} \; 2n + 2n(-1)^n}{n^2-1}$$

$$= \frac{1}{\pi} \cdot \frac{(n-1)(1+\cos n\pi) + (n+1)(-1-\cos n\pi)}{n^2-1}$$

$$= \frac{1}{\pi} \cdot \frac{n + n\cos n\pi - 1 - \cos n\pi - n - n\cos n\pi - 1 - \cos n\pi}{n^2-1}$$

$$= \frac{1}{\pi} \cdot \frac{-2 - 2\cos n\pi}{n^2-1}$$

$$= \frac{-2}{\pi} \cdot \frac{\cos n\pi + 1}{n^2-1} = -\frac{2}{\pi} \cdot \frac{1+(-1)^n}{n^2-1}$$

L-27/ 31.10.2027

Gamma, Bita Function

* **Gamma function:** The integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ defined for $n>0$ is called gamma function and is denoted by Γn (read as gamma n) that is,

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx$$

* **Bita function:** The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ defined for $m>0$, $n>0$ is called Beta function m and n denoted by $B(m,n)$, that is,

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Therefore,

$$\boxed{\Gamma_{n+1} = \underline{\lfloor n}}$$

$$\boxed{B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}}$$

Example:

$$\Gamma 7 = \lfloor 6$$

$$\Gamma\frac{7}{2} = \lfloor\frac{5}{2} = \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times \lfloor\frac{1}{2}-1$$

$$= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\lfloor-\frac{1}{2}$$ → reverse

$$= \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2} \quad \left[\because \Gamma n+1 = \lfloor n\right]$$

Now,

$\Gamma\frac{1}{2} = ?$ | $\boxed{\text{Here, } \Gamma\frac{1}{2} = \sqrt{\pi}}$ → Prove it

We know,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

putting $m=n=\frac{1}{2}$, we get,

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \int_0^1 \frac{dx}{\sqrt{x}\sqrt{1-x}}$$

let,
$x = \sin^2\theta$
$dx = 2\sin\theta\cos\theta\,d\theta$

Limit,
$x=0,\ \theta=0$
$x=1,\ \theta=\frac{\pi}{2}$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} = \int_0^{\pi/2}\frac{2\sin\theta\cos\theta}{\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \int_0^{\pi/2}\frac{2\sin\theta\cos\theta}{\sin\theta\cdot\cos\theta}\,d\theta$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\lfloor 0} = 2\int_0^{\pi/2} d\theta$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = 2 \left[\theta\right]_0^{\pi/2}$$

$$= 2 \cdot \frac{\pi}{2}$$

$$\Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\therefore \Gamma\frac{1}{2} = \sqrt{\pi} \quad \text{(Proved)}$$

$$\therefore \Gamma\frac{7}{2} = \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$$

$$\Gamma\frac{11}{2} = \lfloor\frac{9}{2} = \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}$$

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

$$\beta(2,3) = \frac{\Gamma 2 \, \Gamma 3}{\Gamma 2+3} = \frac{\lfloor 1 \, \lfloor 2}{\lfloor 4} = \frac{2}{24} = \frac{1}{12}$$

$$\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{\Gamma\frac{5}{2} \, \Gamma\frac{3}{2}}{\Gamma \frac{5}{2}+\frac{3}{2}} = \frac{\frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{2} \times \sqrt{\pi}}{\Gamma 4}$$

$$= \frac{\frac{3}{8} \times \pi}{\lfloor 3} = \frac{3 \times \pi}{8 \times 6}$$

$$= \frac{\pi}{16}$$

$$\int_0^1 x^7 (1-x)^3 \, dx = \int_0^1 x^{8-1} (1-x)^{4-1} \, dx$$

$$= \beta(8,4)$$

$$= \frac{\Gamma 8 \, \Gamma 4}{\Gamma 8+4} = \frac{\lfloor 7 \, \lfloor 3}{\lfloor 11} = \frac{1}{1320} \text{ Ans}$$

$$\int_0^1 x^{7/2} (1-x)^{5/2} \, dx = \int_0^1 x^{9/2-1} (1-x)^{7/2-1} \, dx$$

$$= \beta\left(\frac{9}{2}, \frac{7}{2}\right)$$

$$= \frac{\Gamma \frac{9}{2} \, \Gamma \frac{7}{2}}{\Gamma \frac{9}{2}+\frac{7}{2}}$$

$$= \frac{\lfloor \frac{7}{2} \, \lfloor \frac{5}{2}}{\Gamma 8} = \frac{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{\lfloor 7}$$

$$= \frac{1575 \times \pi}{128 \times 5040}$$

$$= \frac{5\pi}{2048} \text{ Ans}$$

formula was only x

(*) $\int_0^\alpha x^5 e^{-4x} dx$

$= \frac{1}{4}\int_0^\alpha \left(\frac{u}{4}\right)^5 e^{-u} du$

$= \frac{1}{4^6}\int_0^\alpha u^5 e^{-u} du$

$= \frac{1}{4096}\int_0^\alpha u^{6-1} e^{-u} du$

$= \frac{1}{4096}\Gamma 6 = \frac{\lfloor 5}{4096} = \frac{15}{512}$ Ans

Let,

$4x = u$

$4\,dx = du \Rightarrow dx = \frac{1}{4} du$

$x = \frac{u}{4}$

Limit,

$x=0,\ u=0$

$x\to\alpha,\ u=\alpha$

(*) $\int_0^\alpha e^{-y^2} y^5 dy$

$= \frac{1}{2}\int_0^\alpha e^{-u} (\sqrt{u})^5 \frac{1}{\sqrt{u}} du$

$= \frac{1}{2}\int_0^\alpha e^{-u} (\sqrt{u})^4 du$

$= \frac{1}{2}\int_0^\alpha e^{-u} u^2 du$

$= \frac{1}{2}\int_0^\alpha u^{3-1} e^{-u} du$

$= \frac{1}{2}\Gamma 3 = \frac{1}{2}\times \lfloor 2 = 1$

Ans

Let,

$y^2 = u$

$2y\,dy = du$

$dy = \frac{1}{2y} du$

$= \frac{1}{2\sqrt{u}} du$

$y = \sqrt{u}$

Limit will be same

⊛ $\int_0^1 x^{5/2}(1-x)^{3/2}\,dx$

$= \beta\left(\frac{7}{2}, \frac{5}{2}\right)$

$= \dfrac{\Gamma\frac{7}{2}\,\Gamma\frac{5}{2}}{\Gamma\left(\frac{7}{2}+\frac{5}{2}\right)} = \dfrac{\lfloor\frac{5}{2}\,\lfloor\frac{3}{2}}{\Gamma 6}$

$= \dfrac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}}{\lfloor 5}$

$= \dfrac{45\times\pi}{32\times 120}$

$= \dfrac{3\pi}{256}$ Ans

⊛ Prove that,

$$\boxed{\int_0^{\pi/2} \sin^m x\,\cos^n x\,dx = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}}$$

Let,

$$I = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\pi/2} \sin^{m-1} x \cos^{n-1} x \sin x \cos x \, dx$$

$$= \int_0^{\pi/2} (\sin^2 x)^{\frac{m-1}{2}} (\cos^2 x)^{\frac{n-1}{2}} \sin x \cos x \, dx$$

$$\Rightarrow I = \int_0^{\pi/2} (\sin^2 x)^{\frac{m-1}{2}} (1-\sin^2 x)^{\frac{n-1}{2}} \sin x \cos x \, dx$$

Let, $\sin^2 x = u$

$2 \sin x \cos x \, dx = du$

Limit,

$x = 0 \; ; \; u = 0$

$x = \frac{\pi}{2} \; ; \; u = 1$

$$\therefore I = \int_0^1 u^{\frac{m-1}{2}} (1-u)^{\frac{n-1}{2}} \frac{1}{2} du$$

$$= \frac{1}{2} \int_0^1 u^{\frac{m+1}{2}-1} (1-u)^{\frac{n+1}{2}-1} du$$

$$= \frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2} + \frac{n+1}{2}\right)}$$

$$\boxed{\therefore \int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\,\Gamma\left(\frac{m+n+2}{2}\right)}}$$

(Proved)

$$\int_0^{\pi/2} \sin^2 x \cos^4 x \, dx$$

$$= \frac{\Gamma\left(\frac{2+1}{2}\right) \Gamma\left(\frac{4+1}{2}\right)}{2\,\Gamma\left(\frac{2+4+2}{2}\right)} = \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{5}{2}\right)}{2\,\Gamma(4)}$$

$$= \frac{\frac{1}{2} \times \sqrt{\pi} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}{2 \times 3!}$$

$$= \frac{3\pi}{8 \times 2 \times 6}$$

$$= \frac{\pi}{32} \text{ Ans}$$

$$\int_0^{\pi/2} \cos^5 x \, dx = \frac{\Gamma\left(\frac{0+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{2\,\Gamma\left(\frac{0+5+2}{2}\right)} \qquad \begin{array}{|l} m=0 \\ n=5 \end{array}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(3)}{2\,\Gamma\left(\frac{7}{2}\right)}$$

$$= \frac{\sqrt{\pi} \times 2}{2 \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi}}$$

$$= \frac{8}{15} \quad \text{Ans}$$

H.W. ⇒

1. $\int_0^a y^7 \sqrt{a^4 - y^4}\, dy$; Hints, $y^4 = a^4 x$
   $\frac{a^{10}}{15}$

2. $\int_0^4 y \sqrt[3]{64 - y^3}\, dy$; Hints $y^3 = 64x$
   $\frac{128}{9\sqrt{3}}$

3. $\int_0^{\pi/8} \sin^2 4x \cos^5 4x \, dx$; Hints $4x = u$
   $\frac{2}{105}$

Optional

4. Prove $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

5. Prove, $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Final Syllabus

35 marks

5 out of 6 (Can be contain a,b,c)

Causly Euler/Variable Coefficient ⇒ 1

System of Linear ODE ⇒ 1

Seriese Solution of ODE ⇒ 1

Laplace Transform ⇒ 1

Fourier Series ⇒ 1

Numerical Solution of IVP (ODE + IC)
Gamma, Beta Function
} any 1