$\boxed{\text{H.W.}} \Rightarrow$

MKC Books

112 ⇒ Example 4.13

122 ⇒ 7-12, 14*, 16*

Zillis Book

155 ⇒ Exercis - 4.5 ⇒

5-8, 54, 57, 58

* $(D^2+1)y = x\cos x - \cos x$

$$y_p = \frac{1}{(D^2+1)} x\cos x - \frac{1}{(D^2+1)} \cos x$$

$$y = \underbrace{c_1\cos x + c_2\sin x}_{y_c} + \underbrace{\frac{1}{4}x\cos x - \frac{1}{2}x\sin x + \frac{1}{4}x^2\sin x}_{y_p}$$

* Variation of parameter method (VPM)

(Not suitable for inverse operator method)

* $a_2y'' + a_1y' + a_0y = g(x)$

$$\Rightarrow y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = \frac{g(x)}{a_2}$$

$$\Rightarrow \boxed{y'' + Py' + Qy = f(x)} \rightarrow \text{Standard form}$$

A.E. ⇒ $m^2 + Pm + Q = 0$

$m = m_1, m_2$

$y_c = c_1e^{m_1x} + c_2e^{m_2x}$

where, $y_1 = e^{m_1x}$, $y_2 = e^{m_2x}$

Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Let,

$$y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$u_1 = -\int \frac{y_2 f(x)}{W}\, dx$$

Recap Creamary Rule

$$u_2 = \int \frac{y_1 f(x)}{W}\, dx$$

⊛ Example:

$$4y'' + 36y = \operatorname{cosec} 3x$$

$$\Rightarrow y'' + 9y = \frac{1}{4} \operatorname{cosec} 3x = f(x)$$

A.E. $\Rightarrow$

$$m^2 + 9 = 0$$

$$m = \pm 3i$$

$$\therefore y_c = A\cos 3x + B \sin 3x$$

$$\therefore y_1 = \cos 3x$$

$$y_2 = \sin 3x$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3 \neq 0$$

$$u_1 = -\int \frac{y_2 f(x)}{W}\, dx = -\int \frac{\sin 3x \cdot \frac{1}{4} \operatorname{cosec} 3x}{3}$$

$$= -\frac{1}{12}\int dx = -\frac{1}{12}x$$

$$u_2 = \int \frac{y_1 f(x)}{W}\, dx = \int \frac{\cos 3x \cdot \frac{1}{4}\operatorname{cosec} 3x}{3}\, dx = \frac{1}{12}\int \frac{\cos 3x}{\sin 3x}\, dx$$

$$= \frac{1}{36}\ln(\sin 3x)$$

$\therefore Y_p = u_1 Y_1 + u_2 Y_2$

$= -\frac{1}{12} x \cos 3x + \frac{1}{36} \ln(\sin 3x) \sin 3x$

$\therefore Y = Y_c + Y_p = A\cos 3x + B\sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} \ln(\sin 3x) \sin 3x$ Ans

(*)

$y'' + y = \tan x = f(x)$

A.E. $\Rightarrow$ $m^2+1 = 0$

$m = \pm i$

$\therefore Y_c = A\cos x + B\sin x$

$\therefore Y_1 = \cos x$

$Y_2 = \sin x$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1 \neq 0$

$\therefore u_1 = -\int \frac{Y_2 f(x)}{W} dx = -\int \frac{\sin x \cdot \tan x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx$

$= -\int \frac{1-\cos^2 x}{\cos x} dx = -\int (\sec x - \cos x) dx$

$= -\ln(\sec x + \tan x) + \sin x$

$\therefore u_2 = \int \frac{Y_1 f(x)}{W} = \int \frac{\cos x \cdot \tan x}{1} dx$

$= \int \sin x \, dx$

$= -\cos x$

$\therefore y = Y_c + Y_p$

$\therefore Y_p = u_1 Y_1 + u_2 Y_2$

$= -\cos x \cdot \ln(\sec x + \tan x) + \sin x \cos x - \sin x \cos x$

L-13 / 05.07.2023 /

Method of variation of parameter

Recap

$$a_2y'' + a_1y' + a_0y = g(x)$$

$$y'' + Py' + Qy = f(x)$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

A.E. $\Rightarrow$ $m^2 + pm + q = 0$

$m = m_1, m_2$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 y_1 + c_2 y_2$

$$u_1 = -\int \frac{y_2 f(x)}{W} dx$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx$$

$$\boxed{y_p = u_1 y_1 + u_2 y_2}$$

#

$$y'' + Py' + Qy = f(x) \quad \cdots \text{(i)}$$

A.E. $\Rightarrow$ $m^2 + pm + q = 0$

$m = m_1, m_2$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 y_1 + c_2 y_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Let,

$$y_p = u_1 y_1 + u_2 y_2 \quad \cdots \text{(ii)}$$

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

Let, $\boxed{u_1' y_1 + u_2' y_2 = 0} \quad \cdots \text{(iii)}$

$$y_p' = u_1 y_1' + u_2 y_2' \quad \cdots \text{(iv)}$$

$\Rightarrow Y_p'' = u_1 Y_1'' + u_1' Y_1' + u_2 Y_2'' + u_2' Y_2'$

Now substituting Y_p, Y_p', Y_p'' in (i)

$$\Rightarrow u_1 Y_1'' + u_1' Y_1' + u_2 Y_2'' + u_2' Y_2' + P(u_1 Y_1' + u_2 Y_2') + Q(u_1 Y_1 + u_2 Y_2) = f(x)$$

$$\Rightarrow u_1 \overbrace{\underbrace{(Y_1'' + PY_1' + QY_1)}_{Y_1=0}}^{0} + u_2 \overbrace{\underbrace{(Y_2'' + PY_2' + QY_2)}_{Y_2=0}}^{0} + u_1' Y_1' + u_2' Y_2' = f(x)$$

$$\Rightarrow u_1' Y_1' + u_2' Y_2' = f(x) \quad \cdots (v)$$

Now,

$$\left.\begin{aligned} (iii) &\Rightarrow u_1' Y_1 + u_2' Y_2 = 0 \\ (v) &\Rightarrow u_1' Y_1' + u_2' Y_2' = f(x) \end{aligned}\right\}$$ Linear System

Use Creamers Rule

$$D = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = W$$

$$D_1 = \begin{vmatrix} 0 & Y_2 \\ f(x) & Y_2' \end{vmatrix} = -Y_2 f(x)$$

$$D_2 = \begin{vmatrix} Y_1 & 0 \\ Y_1' & f(x) \end{vmatrix} = Y_1 f(x)$$

$$\therefore u_1' = \frac{D_1}{W} = \frac{-Y_2 f(x)}{W} \Rightarrow u_1 = -\int \frac{Y_2 f(x)}{W}\,dx$$

$$\therefore u_2' = \frac{D_2}{W} = \frac{Y_1 f(x)}{W} \Rightarrow u_2 = \int \frac{Y_1 f(x)}{W}\,dx$$

Assignment ⇒ 15.09.2023

* $y''' + Py'' + Qy' + Ry = f(x)$

$y_c = c_1y_1 + c_2y_2 + c_3y_3$

Let,

$y_p = u_1y_1 + u_2y_2 + u_3y_3$

Now Derive the formula for u_1, u_2, u_3

And hence solve 25-28 (Page 162, Zill's Book)

$\boxed{\text{H.W.}}$ ⇒ Page 160-162

Exercise 4.6 ⇒ 1, 4, 6, 11-18 } mid must 1

Higher Order Linear ODE with variable coefficient

* Cauchy Euler Equation:

can be any constant

$ax^2y'' + xy' + y = g(x)$

derivatives and power same

$D = \frac{d}{dx}$

$y' = Dy = \frac{dy}{dx}$

$\Rightarrow (x^2D^2 + xD + 1)y = g(x)$

Let, $x = e^t \Rightarrow t = \ln x$

$\frac{dt}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}\cdot\frac{dy}{dt}}$$

$$x\frac{dy}{dx} = \frac{dy}{dt}$$

$$\Rightarrow x\frac{dy}{dx} = xy' = xDy = \frac{dy}{dt}$$

(ii)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x}\frac{d^2y}{dt^2}\cdot\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dt}$$

$$xy = Dy \qquad D = \frac{d}{dt}$$

$$x^2y'' = (D^2 - D)y$$

$$\therefore x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\therefore x^2y'' = \frac{dy}{dt} = Dy$$

$$\therefore x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt} = (D^2 - D)y = D(D-1)y$$

where, $D = \frac{d}{dt}$

Similarly,

$\boxed{\text{H.W.}} \Rightarrow x^3y''' = D(D-1)(D-2)y$ | $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$

* $x^2y'' - 2xy' + 2y = x^3 \quad \ldots(i)$

$(x+2)^2y'' - 2xy' + 2y = x^3$

$e^t = x+2$

Let, $x = e^t \Rightarrow t = \ln x$

we know,

$$xy' = Dy$$

$$x^2y'' = D(D-1)y$$

$$(i) \Rightarrow D(D-1)y - 2Dy + 2y = e^{3t}$$

$$\Rightarrow D^2y - Dy - 2Dy + 2y = e^{3t}$$

$$= D^2y - 3Dy + 2y = e^{3t}$$

$$\Rightarrow (D^2 - 3D + 2)y = e^{3t} \quad \cdots \text{(ii)}$$

A.E. $\Rightarrow$

$$m^2 - 3m + 2 = 0$$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow m(m-2) - 1(m-2) = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\therefore m = 1, 2$$

$$\therefore y_c = c_1 e^t + c_2 e^{2t} = c_1 e^{\ln x} + c_2 e^{2\ln x} = c_1 x + c_2 x^2$$

$$y_p = \frac{1}{D^2 - 3D + 2} e^{3t}$$

$$= \frac{1}{3^2 - 3 \cdot 3 + 2} e^{3t}$$

$$= \frac{1}{2} e^{3t}$$

$$= \frac{1}{2} e^{3\ln x} = \frac{1}{2} x^3$$

$$\therefore \text{G.S.} \Rightarrow y = y_c + y_p = c_1 x + c_2 x^2 + \frac{1}{2} x^3$$

Ans.

Midterm Syllabus End Here

L-14 / 10.09.2023 /

Higher Order ODE with Variable Coeffient

Recap

$x^2 y'' + xy' + y = g(x)$

$\Rightarrow (x^2 D^2 + xD + 1)y = g(x)$

Here, $D = \frac{d}{dx}$

Let, $x = e^t \Rightarrow t = \ln x$

$\therefore \frac{dt}{dx} = \frac{1}{x}$

$\left.\begin{aligned} \therefore xy' &= Dy \\ x^2 y'' &= D(D-1)y \\ x^3 y''' &= D(D-2)(D-1)y \end{aligned}\right\} D = \frac{d}{dt}$

⊛ $x^2 y'' - 3xy' + 3y = 2x^4 e^x \quad \dots \text{(i)}$

Let, $x = e^t$

$\therefore t = \ln x$

$\left.\begin{aligned} xy' &= Dy \\ x^2 y'' &= D(D-1)y \end{aligned}\right\} D = \frac{d}{dt}$

(i) $\Rightarrow \left(D(D-1) - 3D + 3\right)y = 2e^{4t} e^{e^t}$

$\Rightarrow (D^2 - 4D + 3)y = 2e^{4t} e^{e^t} \quad \dots \text{(ii)}$

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A.E. $\Rightarrow$

$$m^2 - 4m + 3 = 0$$

$$\therefore m = 1, 3$$

$$\therefore y_c = c_1 e^{t} + c_2 e^{3t}$$

$$= c_1 x + c_2 x^3$$

$$y_p = \frac{1}{D^2 - 4D + 3} 2e^{4t} e^{e^t}$$

$\Rightarrow$ Not applicable for inverse operator method

$\Rightarrow$ Not applicable for integration

$\Rightarrow$ Not applicable for variation parameter method

$\Rightarrow$ But we can solve in terms of x using variation parameter method.

$$\therefore W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 3x^3 - x^3 = 2x^3 \neq 0$$

(i) $\Rightarrow$ $y'' - \frac{3}{x} y' + \frac{3}{x^2} y = 2x^2 e^x = f(x)$

$$\therefore u_1 = -\int \frac{x^3 \cdot 2x^2 e^x}{2x^3} dx$$

$$= -\int x^2 e^x \, dx$$

$$= -x^2 e^x + 2x e^x - 2e^x$$

x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

$$\therefore u_2 = \int \frac{x \, 2x^2 e^x}{2x^3} dx$$

$$= \int e^x \, dx$$

$$= e^x$$

$\therefore Y_p = u_1 Y_1 + u_2 Y_2$

$= -x^3 e^x + 2x^2 e^x - 2xe^x + x^3 e^x$

$= 2x^2 e^x - 2xe^x$

$= 2xe^x (x-1)$

$\therefore Y = Y_c + Y_p = c_1 x + c_2 x^3 + 2xe^x (x-1)$

Ans

(*) $x^2 y'' - xy' + y = \ln x$ .... (i)

Let, $x = e^t$

$\therefore t = \ln x$

Here, $\left. \begin{array}{l} x^2 y'' = D(D-1)y \\ xy' = Dy \end{array} \right\} D = \frac{d}{dt}$

(i) $\Rightarrow$ $D(D-1)y - Dy + y = \ln(e^t)$

$\Rightarrow D^2 y - Dy - Dy + y = t$

$\Rightarrow (D^2 - 2D + 1) y = t$ .... (ii)

A.E. $\Rightarrow$ $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\therefore m = 1, 1$

$\therefore Y_c = c_1 e^t + c_2 t e^t$

$= c_1 x + c_2 x \ln x$

$$\therefore y_p = \frac{1}{D^2 - 2D + 1} t$$

$$= \frac{1}{(1-D)^2} t$$

$$= (1-D)^{-2} t$$

$$= (1 + 2D + \cdots) t$$

$$= t + 2$$

$$= \ln x + 2$$

$$\therefore y = y_c + y_p$$

$$= c_1 x + c_2 x \ln x + \ln x + 2$$ Ans

(*) $(x+2)^2 y'' + (x+2) y' + y = 3x + 4 \quad \cdots \text{(i)}$

Let, $x + 2 = e^t$

$\therefore x = e^t - 2$

$\therefore t = \ln(x+2)$

$\therefore \frac{dx}{dt} = e^t$

$\therefore \frac{dt}{dx} = e^{-t}$

$$\therefore y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

$$\therefore y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right)$$

$$= \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx} = \left(e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right)e^{-t}$$

$$= e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

$$\therefore (x+2)y' = e^t y' = e^t \cdot e^{-t}\frac{dy}{dt} = \frac{dy}{dt} = Dy$$

$$① \Rightarrow e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right) - e^t \cdot e^{-t}\frac{dy}{dt} + y = 3(e^t-2)+4$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + y = 3e^t - 6 + 4$$

$$(D^2 - 2D + 1)y = 3e^t - 2 \quad \cdots\cdots ⓘⓘ$$

$\therefore$ A.E. $\Rightarrow$

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\therefore m = 1, 1$$

$$\therefore y_c = c_1 e^t + c_2 t e^t$$

$$= c_1(x+2) + c_2(x+2)\ln(x+2)$$

$$\therefore y_p = \frac{1}{D^2 - 2D + 1} 3e^t - \frac{1}{D^2 - 2D + 1} 2$$

$$= \frac{t}{2D - 2} 3e^t - (1-D)^{-2} \cdot 2$$

$$= \frac{t^2}{2} 3e^t - (1 + \cdots) 2$$

$$= \frac{3}{2} t^2 e^t - 2 \quad ? \quad \cancel{\frac{3}{2} t \, 2\ln(x+2) \, (x+2)}$$

$$\cancel{= \frac{3}{2} x^2 (x+2) - 2}$$

$$\therefore \cancel{y = c_1(x+2) + c_2(x+2)\ln(x+2)} + \frac{3}{2} x^2 (x+2) - 2$$

An

$$(1+x)^2 y'' + (1+x)y' + y = \sin\{2\ln(1+x)\}$$

Let,

$1+x = e^t$ $\quad \therefore \dfrac{dx}{dt} = e^t$

$\Rightarrow x = e^t - 1$ $\quad \therefore \dfrac{dt}{dx} = e^{-t}$

$\therefore t = \ln(1+x)$

$$\therefore y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right)$$
$$= \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx}$$
$$= \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t}$$
$$= \left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right)e^{-t}$$
$$= e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

(i) $\Rightarrow e^{2t}\left(e^{-2t}\dfrac{d^2y}{dt^2} - e^{-2t}\dfrac{dy}{dt}\right) + e^t\left(e^{-t}\dfrac{dy}{dt}\right) + y = \sin\{2\ln(e^t)\}$

$\Rightarrow D^2y - Dy + Dy + y = \sin 2t$

$\therefore (D^2+1)y = \sin 2t$

$\therefore$ A.E. $\Rightarrow m^2+1 = 0$

$m = \pm i$

$\therefore y_c = A\cos t + B\sin t$

$= A\cos\{\ln(1+x)\} + B\sin\{\ln(1+x)\}$

$$\therefore y_p = \frac{1}{D^2+1} \sin 2t$$

$$= \frac{1}{-2^2+1} \sin 2t$$

$$= -\frac{1}{5} \sin 2t$$

$$= -\frac{1}{5} \sin\{2\ln(1+x)\}$$

H.W. ⇒

MKC Book

Example : 4·19 – 4·21, 4·24*

Page -122:

Exercise : 20 - 22, 26

Zilli's Book

Page -168, Exercise 4·7 ⇒ 19, 21-24, 34, 36

Midterm Suggestion

Inverse Operator (R1-5) ⇒ 1Q

Wronskian / Variation Parameter ⇒ 1Q

Application of 1^{st} Order ⇒ 1Q

1^{st} Order ⇒ 3Q

L-15/12.02.2023

System of Linear ODE : Method of Elimination

$$\left.\begin{aligned} \frac{dx}{dt} &= 3y \\ \frac{dy}{dt} &= 2x \end{aligned}\right\} \text{System of Linear ODE}$$

$\Rightarrow \quad Dx - 3y = 0 \qquad \Rightarrow \quad Dx - 3y = 0 \quad \cdots \text{(i)}$

$\quad\quad Dy - 2x = 0 \qquad\qquad -2x + Dy = 0 \quad \cdots \text{(ii)}$

Operating (i) by D and multiplying (ii) by 3

$$D^2x - 3Dy = 0$$

$$-6x + 3Dy = 0$$

$(+)$

$$D^2x - 6x = 0$$

$\therefore$ A.E. $\Rightarrow \quad m^2 - 6 = 0$

$$m = \pm\sqrt{6}$$

$\therefore \quad x(t) = c_1 e^{-\sqrt{6}t} + c_2 e^{\sqrt{6}t}$ Ans

$\therefore \quad \frac{dx}{dt} = -\sqrt{6}\, c_1 e^{-\sqrt{6}t} + \sqrt{6}\, c_2 e^{\sqrt{6}t}$

(i) $\Rightarrow \quad -\sqrt{6}\, c_1 e^{-\sqrt{6}t} + \sqrt{6}\, c_2 e^{\sqrt{6}t} - 3y = 0$

$\Rightarrow \quad -3y = \sqrt{6}\, c_1 e^{-\sqrt{6}t} - \sqrt{6}\, c_2 e^{\sqrt{6}t}$

$\therefore \quad y = -\frac{\sqrt{6}}{3} c_1 e^{-\sqrt{6}t} + \frac{\sqrt{6}}{3} c_2 e^{\sqrt{6}t}$ Ans

$$\frac{dx}{dt} + 2x - 3y = t$$

$$\frac{dy}{dt} - 3x + 2y = e^{2t}$$

$$\Rightarrow \quad (D+2)x - 3y = t \quad \text{--- (i)}$$

$$-3x + (D+2)y = e^{2t} \quad \text{..... (ii)}$$

Operating (i) by $(D+2)$ and multiplying (ii) by 3

$$\Rightarrow \quad (D+2)^2 x - 3(D+2)y = (D+2)t$$

$$-9x + 3(D+2)y = 3e^{2t}$$

$$(+) \quad (D+2)^2 x - 9x = (D+2)t + 3e^{2t}$$

$$\Rightarrow (D^2+4D+4)x - 9x = 1+2t + 3e^{2t}$$

$$\Rightarrow (D^2+4D-5)x = (1+2t) + 3e^{2t}$$

A.E. $\Rightarrow$

$$m^2 + 4m - 5 = 0$$

$$m = 1, -5$$

$$\therefore x_c(t) = c_1 e^{-5t} + c_2 e^{t}$$

$$\therefore x_p(t) = \frac{1}{D^2+4D-5}(1+2t) + 3\frac{1}{D^2+4D-5}e^{2t}$$

$$= -\frac{1}{5}\left(1 - \frac{D^2+4D}{5}\right)^{-1}(1+2t) + 3\frac{1}{4+8-5}e^{2t}$$

$$= -\frac{1}{5}\left(1 + \frac{D^2+4D}{5} + \cdots\right)(1+2t) + \frac{3}{7}e^{2t}$$

$$= -\frac{1}{5}\left(1 + 2t + \frac{8}{5}\right) + \frac{3}{7}e^{2t}$$

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$$= -\frac{1}{5}\left(2t + \frac{13}{5}\right) + \frac{3}{7}e^{2t}$$

$$= -\frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$x = x_c(t) + x_p(t)$$

$$x(t) = c_1 e^{-5t} + c_2 e^{t} - \frac{2}{5}t + \frac{3}{7}e^{2t} - \frac{13}{25}$$

$$\frac{dx}{dt} = -5c_1 e^{-5t} + c_2 e^{t} - \frac{2}{5} + \frac{6}{7}e^{2t}$$

$$\therefore -3y = t - \frac{dx}{dt} - 2x$$

$$\Rightarrow y = -\frac{1}{3}t + \frac{1}{3}\cdot\frac{dx}{dt} + \frac{2}{3}x$$

$$= -\frac{1}{3}t + \frac{1}{3}\left(-5c_1 e^{-5t} + c_2 e^{t} - \frac{2}{5} + \frac{6}{7}e^{2t}\right) + \frac{2}{3}\left(c_1 e^{-5t} + c_2 e^{t} - \frac{2}{5}t + \frac{3}{7}e^{2t} - \frac{13}{25}\right)$$

$$= -\frac{1}{3}t - \frac{5}{3}c_1 e^{-5t} + \frac{1}{3}c_2 e^{t} - \frac{2}{15} + \frac{6}{21}e^{2t} + \frac{2}{3}c_1 e^{-5t} + \frac{2}{3}c_2 e^{t} - \frac{4}{15}t + \frac{6}{21}e^{2t} - \frac{26}{75}$$

$$y = -c_1 e^{-5t} + c_2 e^{t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$ Ans

H.W. $\Rightarrow$

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Recap

System of Linear ODE

$$\begin{cases} Dx + 2y = t \quad \cdots (i) \\ -x + Dy = e^t \quad \cdots (ii) \end{cases} \qquad D = \frac{d}{dt}$$

Seriese Solution of ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad \cdots (i)$$

$$\Rightarrow y'' + P(x)y' + Q(x)y = 0 \Rightarrow \text{Do in Rough}$$

(i) if $P(x)$ and $Q(x)$ are defined at $x=0$, then (i) is an ordinary point problem. ⇒ One seriese solution will be found.

(ii) if $P(x)$ or $Q(x)$ are undefined at $x=0$, then (i) is a singular point problem. ⇒ Two seriese solution will be found.

⊛ Solve by the seriese solution technique.

⊛ $y'' + xy = 0 \quad \cdots (i)$

$P(x) = 0$
$Q(x) = x$ } Both defined

∴ Ordinary point problem

∴ It has single seriese solution.

$\rightarrow c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + \dots + c_nx^n + \dots$

Let,

$y = \sum_{n=0}^{\infty} c_n x^n$ be the solution of (1)

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting y'', y', y in (1),

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = 0$$ [Now multiply coefficient into Σ]

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$ [Now extract the minimum power of x]

$$\Rightarrow 2\cdot(2-1)\cdot c_2 \cdot x^0 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$ [Now we need to connect these seriese in common and combine]

Let,

$k = n-2$

$n = k+2$

if, $n=3$, $k=1$

Let,

$k = n+1$

$n = k-1$

if, $n=0$; $k=1$

$$\Rightarrow 2c_2x^0 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$\Rightarrow 2c_2x^0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} + c_{k-1} \right] x^k = 0\cdot x^0 + 0\cdot x^k$$

Equating coefficient of the lite terms.

$x^0 : 2C_2 = 0$

$C_2 = 0$

$x^k : (k+2)(k+1)C_{k+2} + C_{k-1} = 0 \quad ; k = 1,2,3,4,\dots$

$$C_{k+2} = -\frac{1}{(k+2)(k+1)} \cdot C_{k-1}$$

Now,

if, $k = 1$,

$$C_3 = -\frac{1}{2\cdot3} C_0$$

$k = 2$,

$$C_4 = -\frac{C_1}{3\cdot4}$$

$k = 3$,

$$C_5 = \frac{-C_2}{4\cdot5} = 0$$

$k = 4$,

$$C_6 = -\frac{C_3}{5\cdot6} = \frac{1}{2\cdot3\cdot5\cdot6} C_0$$

$k = 5$,

$$C_7 = -\frac{C_4}{7\cdot6} = \frac{C_1}{3\cdot4\cdot6\cdot7}$$

$k = 6$,

$$C_8 = -\frac{C_5}{8\cdot7} = 0$$

$k = 7$,

$$C_9 = -\frac{C_6}{9\cdot8} = -\frac{C_0}{2\cdot3\cdot5\cdot6\cdot8\cdot9}$$

$k = 8$,

$$C_{10} = -\frac{C_7}{10\cdot9} = -\frac{1}{3\cdot4\cdot6\cdot7\cdot9\cdot10} C_1$$

$k = 9$,

$$C_{11} = -\frac{C_8}{11\cdot10} = 0$$

Here, the solution is,

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + \cdots$$

$$= c_0 + c_1 x + 0 + \frac{-c_0}{2\cdot 3} x^3 + \frac{-c_1}{3\cdot 4} x^4 + 0 + \frac{c_0}{2\cdot 3\cdot 5\cdot 6} x^6$$

$$+ \frac{c_1}{3\cdot 4\cdot 6\cdot 7} x^7 + 0 + \cdots$$

$$\therefore y = \left(1 - \frac{1}{2\cdot 3} x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6} x^6 - \cdots\right) c_0$$

$$+ \left(x - \frac{1}{3\cdot 4} x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7} x^7 - \cdots\right) c_1$$

Ans

$$y'' + \frac{x}{x^2+1} y' - \frac{1}{x^2+1} y = 0$$

$\therefore$ Finite / defined

$$(x^2+1) y'' + x y' - y = 0 \quad \cdots \text{(i)}$$

Let,

$y = \sum_{n=0}^{\infty} c_n x^n$ be the seriese solution of (i)

$$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

H.W

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Learn Seriese

Exercise ⇒ 6.2 (246)

↳ 9, 13, 17, 23

Example - 8 (245)

Now, substitute y'', y', y in the equation (i)

$$(x^2+1) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + x \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)\, c_n x^n + \sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} + \sum_{n=1}^{\infty} n\, c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2(2-1)c_2 x^0 - c_0 x^0 + \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$\Rightarrow \cancel{2c_2 x^0 - c_0 x^0}$$

Let,

$k = n$	$k = n-2$	$k = n$	$k = n$
$n = k$	$n = k+2$	$n = k$	$n = k$
if, $n=2$, $k=2$	if, $n=3$, $k=1$	if, $n=1$, $k=1$	if, $n=1$, $k=1$

$$\Rightarrow \cancel{2c_0 x^0 - c_0 x^0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=1}^{\infty} k(k-1)c}$$

$$= 2c_2 x^0 - c_0 x^0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=1}^{\infty} c_k x^k = 0$$

$$= 2c_2 - c_0 + (1+2)(1+1)c_{1+2} x^1 + 1 \cdot c_1 \cdot x^1 - c_1 x^1 + \sum_{k=2}^{\infty} k(k-1) c_k x^k$$

$$+ \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=2}^{\infty} k c_k x^k - \sum_{k=2}^{\infty} c_k x^k = 0$$

$$= 2c_2 - c_0 + 6c_3 x + c_1 x - c_1 x + \sum_{k=2}^{\infty} \left[\{k(k-1)c_k\} + \{(k+2)(k+1)c_{k+2}\} + k c_k - c_k \right] x^k = 0$$

$$= (2c_2 - c_0) + 6c_3 x + \sum_{k=2}^{\infty} \left[k(k-1)c_k + (k+2)(k+1)c_{k+2} + k c_k - c_k \right] x^k = 0$$

∴ Equating Coefficient of little teanms,

x^0: $\quad 2c_2 - c_0 = 0$

$\Rightarrow \quad c_2 = \frac{1}{2} c_0$

x: $\quad 6c_3 x = 0$

$\therefore \; c_3 = 0$

x^k :- $\quad k(k-1)c_k + (k+2)(k+1)c_{k+2} + k c_k - c_k = 0$

$\Rightarrow \quad \cancel{\{k(k-1) + (k+2)(k+1) + k - 1\}c_k + (k+2)(k+1)c_{k+2}} = 0$

$\Rightarrow \quad \cancel{(k^2 - k + (k+2)(k+1) + k - 1)c_k}$

$\Rightarrow \quad \cancel{k^2 - 1 + k^2}$

$\Rightarrow \quad (k^2 - k + k - 1)c_k + (k+2)(k+1)c_{k+2} = 0$

$\Rightarrow \quad (k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0$

$\therefore k = 2, 3, 4, \ldots$

$\Rightarrow \quad (k+2)(k+1)c_{k+2} = -(k+1)(k-1)c_k$

$\therefore \quad c_{k+2} = -\frac{(k+1)(k-1)}{(k+2)(k+1)} c_k$

Now,

if, $k=2$,

$$C_4 = -\frac{3\cdot 1}{4\cdot 3} C_2 = -\frac{3}{12}\cdot\frac{1}{2}\cdot C_0 = -\frac{3}{24} C_0 = -\frac{1}{8} C_0$$

if, $k=3$,

$$C_5 = -\frac{4\cdot 2}{5\cdot 4} C_3 = 0$$

if,

$k=4$,

$$C_6 = -\frac{5\cdot 3}{6\cdot 5} C_4 = \frac{15}{30}\cdot\frac{1}{8} C_0 = \frac{1}{16} C_0$$

if,

$k=5$,

$$C_7 = -\frac{6\cdot 4}{7\cdot 6} C_5 = 0$$

$$\vdots$$

Here, the solution is,

$$y = \sum_{n=0}^{\infty} c_n x^n +$$

$$= C_0 x^0 + C_1 x^1 + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + C_6 x^6 + C_7 x^7 + \cdots$$

$$= C_0 + C_1 x + \frac{1}{2} x^2 C_0 + 0 - \frac{1}{8} C_0 x^4 + 0 + \frac{1}{16} \times C_0 x^6 + 0 + \cdots$$

$$= C_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{8} x^4 + \frac{1}{16} x^6 - \cdots\right] + C_1 x$$