v)

$f(x) = \ln(x^2-4)$

$f'(x) = \dfrac{1}{x^2-4} \cdot 2x$

For critical,

$f'(x) = 0$

$\dfrac{2x}{x^2-4} = 0$

$2x = 0$

$x = 0$

$f'(x)$ is undefined at 0 and ± 2.

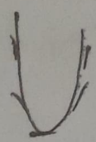
stationary point, $x=0$

⊛ First Derivative Test:

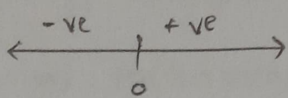
$f'(x)$ +ve | -ve, with 0 at the sign change

if x_0 is a critical point and $f'(x)$ changes sign +ve to -ve and that critical point then at $x = x_0$ will be maximum.

⊛

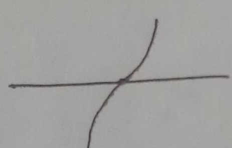


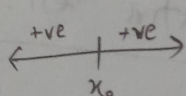
$f'(x)$



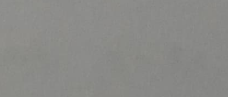
If, x_0 is a critical point and if $f'(x)$ changes sign from -ve to +ve, then on that critical point will be minimum.

⊛



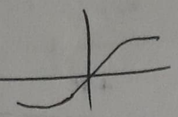


if x_0 is a critical points and $f'(x)$ not changes the sign. Then on that critical point neither maximum nor minimum.

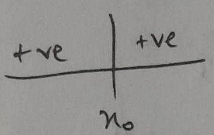
Same as Negative 

$f(x) = \sqrt[3]{x}$

$f'(x) =$ undefined



$x = x_0$ is a critical point but not stationary point.



same for $-\sqrt[3]{x}$

Using 1st Derivative test, find the relative extrema for the following functions.

i) $f(x) = 1 - 4x - x^2$ ii) $f(x) = x^3 - 3x^2 - 24x + 4$

Solution

i) $f(x) = 1 - 4x - x^2$

$f'(x) = -4 - 2x$

For critical points:

$f'(x) = 0$

$-4 - 2x = 0$

$x = -2$

Now make intervals using critical points,

sign of $f'(x)$: +ve | -ve, split at -2

$(-\infty, -2)$ $(-2, \infty)$

$x = -3$

$f'(-3) = -4 - 2(-3)$

$= +ve$

$x = 3$

$f'(3) = -ve$

Therefore,

at $x = -2$, there is maximum.

maximum value,

$f(-2) = 1 + 8 - 4$

$= 1 + 4$

$= 5$

ii) $f(x) = x^3 - 3x^2 - 24x + 4$

$f'(x) = 3x^2 - 6x - 24$

For critical points,

$$f'(x) = 0$$

$$3x^2 - 6x - 24 = 0$$

$$x^2 - 2x - 8 = 0$$

$$x^2 - 4x + 2x - 8 = 0$$

$$x(x-4) + 2(x-4) = 0$$

$$(x-4)(x+2) = 0$$

$$x = -2, 4$$

$f'(x)$:

$(-\infty, -2)$	$(-2, 4)$	$(4, \infty)$
+ve	-ve	+ve
$x = -3$	$x = 0$	$x = 5$
$f'(-3) = +ve$	$f'(0) = -ve$	$f'(5) = +ve$

Therefore,

at $x = -2$, $f'(x)$ changes sign from +ve to -ve.

$\therefore$ at $x = -2$, maximum

at $x = 4$, $f'(x)$ changes sign from -ve to +ve.

$\therefore$ at $x = 4$, minimum.

$\boxed{\text{Quizzes - 3} \Rightarrow \text{END}}$

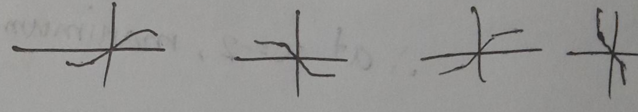
Second Derivative test!

Suppose f is twice differentiable at the point x_0

a) if $f'(x)=0$ and $f''(x_0) > 0$, then f has a relative minimum.

b) if $f'(x_0)=0$ and $f''(x_0) < 0$, then f has a relative maximum.

c) if $f'(x_0)=0$ and $f''(x_0)=0$, then the test is inconclusive.



Neither maximum nor minimum.

Find the relative extrema for $f(x) = 3x^5 - 5x^3$ using 2nd derivative test.

$\Rightarrow$ $f(x) = 3x^5 - 5x^3$

$f'(x) = 15x^4 - 15x^2$

$f''(x) = 60x^3 - 30x$

For critical points,

$f'(x) = 0$

$15x^4 - 15x^2 = 0$

$15x^2(x^2-1) = 0$

$x = 0, x = \pm 1$

Now,

$f''(x) = 60x^3 - 30x$

Here,

at $x = 1$,

$f''(1) = 60 - 30$

$= 30 > 0$

Therefore, at $x = 1$, there is minimum

minimum values,

$f(1) = 3 - 5 = -2$

point $(1, -2)$

at $x = -1$

$f''(-1) = -60 + 30$

$= -30 < 0$

Therefore, at $x = -1$, there is a maximum.

maximum values,

$f(-1) = -3 + 5 = 2$

point $(-1, 2)$

at $x = 0$,

$f''(0) = 0$

Therefore, at $x = 0$, neither maximum nor minimum.

⊛ Graphing Polynomial Function

i) End behavior of Polynomial Function.

ii) Find the critical points.

iii) Determine maximum/minimum by wing 1st or 2nd derivative test.

iv) Find the infection point and check the concavity

v) Find y intercepts and draw the graph.

⊛ Graph the function,

$$f(x) = 3x^4 - 4x^3 + 2$$

⇒

Here,

End behavior is $f(x) = 3x^4$

∴ graph will be up to the left and up to the right.

Now,

$f'(x) = 12x^3 - 12x^2$

For critical points,

$f'(x) = 0$

$12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$x = 0, 1$

Now,

$f''(x) = 36x^2 - 24x$

at $x = 0$

$f''(0) = 0$

i.e. inconclusives neither maximum nor minimum

point is, $x = 0$; $f(0) = 0 - 0 + 2$

$= 2$

at, $x = 1$

$f''(1) = 36 - 24$

$= 12 > 0$

i.e. at $x = 1$ minimum

$x = 1$; $f(1) = 3 - 4 + 2$

$= 1$

$\therefore$ point $(1, 1)$

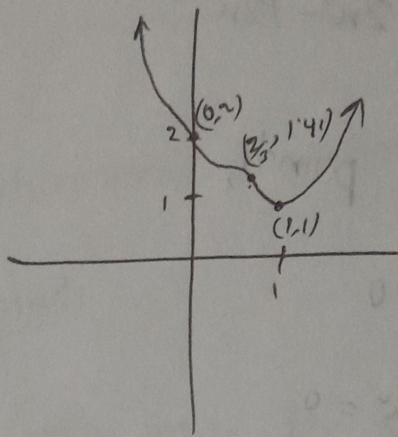
For infection point,

$f''(x) = 0$

$36x^2 - 24x = 0$

$12x(3x-2) = 0$

$x = 0, \; x = \frac{2}{3}$



$x = 0, \; f(0) = 2$; point $(0,2)$

$x = \frac{2}{3}, \; f\left(\frac{2}{3}\right) \Rightarrow (0.66, 1.44)$

Check concavity by using infection point,

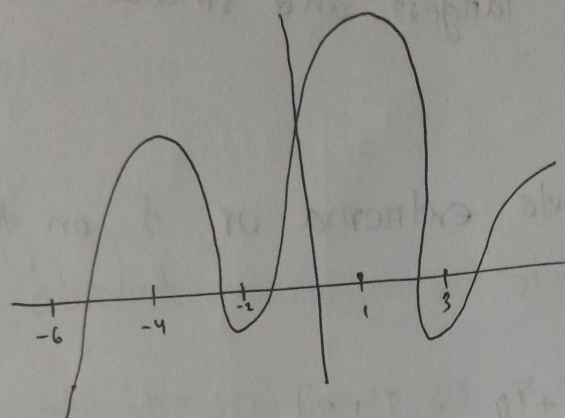
$$\begin{array}{ccc} +ve & -ve & +ve \\ (-\infty, 0) & \left(0, \frac{2}{3}\right) & \left(\frac{2}{3}, \infty\right) \end{array}$$

$x = -1$	$x = \frac{1}{2}$	$x = 1$
$f''(-1) = +ve$	$f''\left(\frac{1}{2}\right) = -ve$	$f''(1) = +ve$

L-18 / 10.08.2022 /

4.4

Absolute maximum and minimum (Absolute extrema)



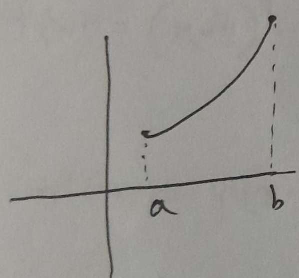
$\therefore$ absolute maximum, at $x = 1$

value $= f(1)$

$\therefore$ absolute minimum, at $x = -6$

value $= f(-6)$

⊛



Here,

absolute maximum $= b$

value $= f(b)$

absolute minimum $= a$

value $= f(a)$

⊛ Finding absolute extrema on finite closed interval.

Like $[a,b]$

Step-1: Find the critical points, with in (a,b)

Step-2: Evaluate the functional value for critical points and end points.

Step-3: Check the largest and smallest output (functional value)

⊛ Find the absolute extrema of f on the closed interval.

i) $f(x) = 4x^2 - 12x + 10$; $[1,3]$

$\Rightarrow$

$f'(x) = 8x - 12$

For critical points,

$f'(x) = 0$

$8x - 12 = 0$

$8x = 12$

$x = \frac{12}{8}$

$= \frac{3}{2}$

Here, critical point, $x = \frac{3}{2}$; with in $(1,3)$

Now,

$f(1) = 4-12+10 = 2$

$f(\frac{3}{2}) = 4\cdot(\frac{3}{2})^2 - 12\cdot\frac{3}{2} + 10 = 1$

$f(3) = 10$

Therefore,

absolute maximum at $x=3$ and value 10.

absolute minimum at $x=\frac{3}{2}$ and value 1.

ii) $f(x) = (x-2)^3$; $[1,4]$

$\Rightarrow$

$f'(x) = 3(x-2)^2\cdot 1$

For critical point,

$f'(x) = 0$

$3(x-2)^2 = 0$

$x = 2$

Here,

critical point $x=2$; with in $(1,4)$

Now, $f(1) = -1$

$f(2) = 0$

$f(4) = 8$

Therefore,

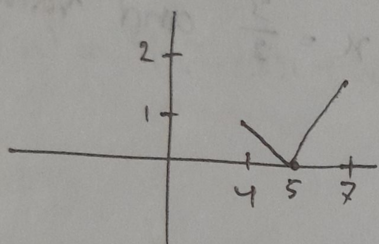
absolute maximum at $x=4$, value 8

absolute minimum at $x=1$, value -1

iii)

$f(x) = |x-5|$; $[4,7]$

Graph,



$$f(x) = \begin{cases} x-5 & ; x \geq 5 \\ -x+5 & ; x<5 \end{cases}$$

$$f'(x) = \begin{cases} 1 & ; x \geq 5 \\ -1 & ; x<5 \end{cases}$$

$Lf'(x) \neq Rf'(x)$

$\Rightarrow$ function is not ~~deri~~differentiable

$\therefore x=5$ is a critical point

at, $x=5$, there is a corner, so, the function

$f(x)$ is not differentiable; at $x=5$.

Therefore,

$x=5$ is a critical point, with in $(4,7)$

Now,

$f(4) = 1$

$f(5) = 0$

$f(7) = 2$

Therefore

absolute maximum at $x = 7$, value 2

absolute minimum at $x = 5$, value 0

iv) $f(x) = |6-4x|$; $[-3,3]$

$\Rightarrow$

$$f(x) = \begin{cases} 6-4x & ; \quad 6-4x \geq 0 \\ -6+4x & ; \quad 6-4x < 0 \end{cases}$$

$$= \begin{cases} 6-4x & ; \quad x \geq \frac{3}{2} \\ 4x-6 & ; \quad x < \frac{3}{2} \end{cases}$$

$$f'(x) = \begin{cases} -4 & ; \quad x \geq \frac{3}{2} \\ 4 & ; \quad x < \frac{3}{2} \end{cases}$$

$\therefore Lf'(x) \neq Rf'(x)$

$\therefore$ function is not differentiable at $x = \frac{3}{2}$

Therefore, $x = \frac{3}{2}$ is a critical point, with in $(-3,3)$

Now,

$f(-3) = 18$

$f(\frac{3}{2}) = 0$

$f(3) = 6$

Therefore,

absolute maximum at $x = -3$, value 18

absolute minimum at $x = \frac{3}{2}$, value 0

v) $f(x) = \sin x - \cos x$; $[0, \pi]$

$\Rightarrow$

$f'(x) = \cos x + \sin x$

For critical point,

$f'(x) = 0$

$\sin x + \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$

Therefore,

critical point $x = \frac{3\pi}{4}$ within $(0, \pi)$

Now,

$f(0) = -1$

$f\left(\frac{3\pi}{4}\right) = \sqrt{2}$

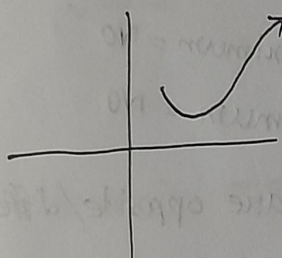
$f(\pi) = 1$

Therefore,

absolute maximum at $x = \frac{3\pi}{4}$, value $\sqrt{2}$

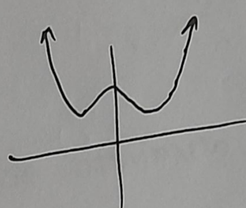
absolute minimum at $x = 0$, value -1.

⊛ Finding absolute extrema with in infinite $(-\infty, \infty)$ intervals.



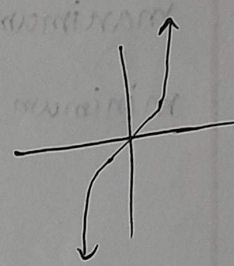
absolute maximum = NO

absolute minimum = YES



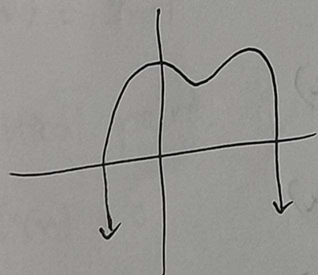
maximum = NO

minimum = YES



maximum = NO

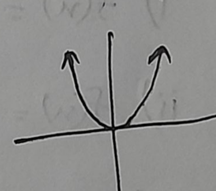
minimum = NO



maximum = YES

minimum = NO

⊛



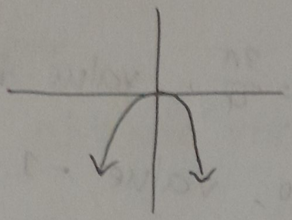
if, $\lim_{x \to +\infty} f(x) = +\infty$

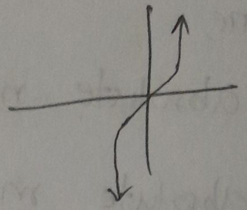
and, $\lim_{x \to -\infty} f(x) = +\infty$

Then,

maximum = NO

minimum = YES

⊛ 

⊛ 

if, $\lim_{x \to +\infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

maximum = YES

minimum = NO

if,

$\lim_{x \to +\infty} f(x) = +\infty$

$\lim_{x \to -\infty} f(x) = -\infty$

then,

maximum = NO

minimum = NO

they are opposite/different

⊛ Find the absolute extrema

i) $f(x) = x^2 - x - 2 \; ; \; (-\infty, \infty)$

ii) $f(x) = 4x^3 - 3x^4 \; ; \; (-\infty, \infty)$

iii) $f(x) = 2x^3 - 6x + 2 \; ; \; (-\infty, \infty)$

iv) $f(x) = \dfrac{x^2+1}{x+1} \; ; \; (-5, -1)$

Solution

i)

$f(x) = x^2 - x - 2$

Here,

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^2$$

$$= +\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^2$$

$$= +\infty$$

$\therefore$ there is only absolute minimum.

Here, absolute minimum could be occure at critical point.

$f'(x) = 2x - 1$

For critical point,

$f'(x) = 0$

$2x - 1 = 0$

$x = \frac{1}{2}$

$$\therefore f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2$$

$$= \frac{1}{4} - \frac{1}{2} - 2$$

$$= \frac{1-2-8}{4} = \frac{-9}{4}$$

So,

absolute minimum at, $x = \frac{1}{2}$; value $\frac{-9}{4}$.

ii) $f(x) = 4x^3 - 3x^4$

$\Rightarrow$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} -3x^4$$

$$= -\infty$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} -3x^4$$

$$= -\infty$$

Therefore,

no absolute minimum. Only absolute maximum here and could be occure at critical point.

$\therefore f'(x) = 12x^2 - 12x^3$

For critical point,

$$f'(x) = 0$$

$$12x^2 - 12x^3 = 0$$

$$12x^2(1-x) = 0$$

$$x = 0, 1$$

$\therefore f(0) = 0$

$\therefore f(1) = 4 - 3$

$= 1$

$\therefore$ absolute manimum at $x = 1$, value $= 1$.

iii)

No maximum.

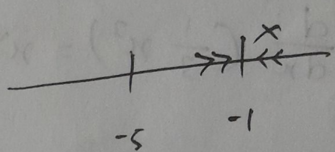
No minimum.

<u>4.1</u>

From trigonometric function, find out increasing, & decreasing and concavity.

iv)

$$f(x) = \frac{x^2+1}{x+1} \quad ; \quad (-5, -1)$$



$$\therefore \lim_{x \to -1} \frac{x^2+1}{x+1} = -\infty$$

Therefore,

no absolute minimum, only absolute manimum here and could be occure at critical point.

$$f'(x) = \frac{(x+1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x+1)}{(x+1)^2}$$

$$=$$

L-19 / 17.08.2023

5.2

Indefinite Integrals

Antiderivative:

If $\frac{d}{dx}[F(x)] = f(x)$

then,

F(x) is antiderivative of f(x)

* $\frac{d}{dx}(\frac{1}{3}x^3) = x^2$

Antiderivative of x^2 is $\frac{1}{3}x^3$

* $\frac{d}{dx}(\sin x) = \cos x$

Antiderivative of $\cos x$ is $\sin x$.

* $$\left.\begin{aligned} &\frac{d}{dx}(\frac{1}{3}x^3 - 10) = x^2 \\ &\frac{d}{dx}(\frac{1}{3}x^3 - \pi) = x^2 \\ &\frac{d}{dx}(\frac{1}{3}x^3 + c) = x^2 \end{aligned}\right\} \quad \frac{d}{dx}[F(x)] = f(x)$$

(*) <u>Indefinite Integration</u>: Process or finding antiderivative or antidifferentiation is called indefinite integration.

$$\int f(x)\,.\,dx = F(x) + C$$

Here, C is arbitrary constant.

$$\frac{d}{dx}[F(x)+C] = f(x)$$

(*) $\frac{d}{dx}(x) = 1 \Rightarrow \int 1.\,dx = x + C$

(*) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \; n \neq -1 \Rightarrow \int x^n.dx = \frac{x^{n+1}}{n+1} + c$

(*) $\int x^2 \, dx = \frac{x^{2+1}}{2+1} + c = \frac{1}{3}x^3 + c$

(*) $\int x^{-\frac{1}{2}} \, dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = 2x^{\frac{1}{2}} + c$

(*) $\int x^9 \, dx = \frac{x^{10}}{10} + c$

[Some Rule from Sheet]

Properties:

1. $\int c f(x)\, dx = c \int f(x)\, dx$

2. $\int (f(x) + g(x) + h(x)) = \int f(x)\, dx + \int g(x) dx + \int h(x)\, dx$

$= F(x) + G(x) + H(x) + C$

(Same for negatives.)

Find the following integrals,

i) $\int x(1+x^3) dx$ ii) $\int (2+y^2)^2 dy$

iii) $\int \sec x (\sec x + \tan x)\, dx$ iv) $\int \frac{\sec\theta}{\cos\theta}\, d\theta$

v) $\int \left(\frac{1}{2t} - \sqrt{2}\, e^t\right) dt$

Solution

i) $\int x(1+x^3)\, dx$

$= \int (x+x^4)\, dx$

$= \int x\, dx + \int x^4\, dx$

$= \frac{x^2}{2} + \frac{x^5}{5} + c$

ii) $\int (2+y^2)^2 dy$

$= \int (4+4y^2+y^4)\, dy$

$= \int 4\, dy + \int 4y^2 dy + \int y^4 dy$

$= 4y + \frac{4 \cdot y^3}{3} + \frac{y^5}{5} + C$

iii) $\int (\sec^2 x + \sec x \tan x)\, dx$

$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$

$= \tan x + \sec x + c$

iv) $\int \sec\theta \cdot \sec\theta\, d\theta$

$= \int \sec^2\theta\, d\theta$

$= \tan\theta + c$

v) $\int \frac{1}{2t}\, dt - \sqrt{2} \int e^t\, dt$

$= \frac{1}{2} \ln t - \sqrt{2}\, e^t + c$

(x) $\int \frac{3}{\sqrt{1-x^2}}\, dx$

$= 3 \sin^{-1} x + c$

(*) Indefinite integrals from the view point of Differential equation:

(*) Differential equation: If an equation involves a derivative then this equation is called differential equation.

$\frac{dy}{dx} = x^2$

$\frac{ds}{dt} - 3 = 0$

$\frac{dy}{dx} - 2x = 0$

* $\frac{dy}{dx} = x^2$ --- (1)

$\boxed{y = \frac{1}{3}x^3 + c}$ is a solution of (1)

→ Solution is not a number. its a function

family of a function/curve.

* $\frac{dy}{dx} = x^2$ → Passing through (2,1)

$y = \int x^2 dx$

$y = \frac{1}{3}x^3 + c$

$1 = \frac{1}{3} \cdot 2^3 + c$

$c = -\frac{5}{3}$

$\therefore y = \frac{1}{3}x^3 - \frac{5}{3}$

* $\frac{dy}{dx} - 2x = 0$

$\frac{dy}{dx} = 2x$

$y = \int 2x\, dx$

$\therefore y = x^2 + c$

* $\frac{ds}{dt} - 3 = 0$

$\frac{ds}{dt} = 3$

$s = \int 3\, dt$

$\therefore s = 3t + c$

* $\frac{dy}{dx} = f(x)$, $y(x_0) = y_0$ → $x_0 \doteq y_0$

* $\frac{dy}{dx} = f(x)$; $y(x_0) = y_0$

this type problem is called initial value problem (IVP).

* Solve the initial value problem.

i) $\frac{dy}{dt} = \sec^2 t - \sin t$; $y(\frac{\pi}{4}) = 1$

ii) $\frac{dy}{dx} = \frac{x^2-1}{x^2+1}$; $y(1) = \frac{\pi}{2}$

Solution

i) $\frac{dy}{dt} = \sec^2 t - \sin t$

$\therefore dy = (\sec^2 t - \sin t)\, dt$

$\int dy = \int (\sec^2 t - \sin t)\, dt$

$y = \tan t + \cos t + c$

Now,

$1 = \tan \frac{\pi}{4} + \cos \frac{\pi}{4} + c$

$\therefore c = -\frac{1}{\sqrt{2}}$

ii) $\frac{dy}{dx} = \frac{x^2-1}{x^2+1}$

$dy = \frac{x^2-1}{x^2+1} dx$

$\int dy = \int \frac{x^2-1}{x^2+1} dx$

$y = \int \left(1 - \frac{2}{x^2+1}\right) dx$

$y = \int 1 \, dx - \int \frac{2}{x^2+1} dx$

$y = x - 2 \tan^{-1} x + c$

Now,

if $x = 1$

$y = \frac{\pi}{2}$

$\therefore \frac{\pi}{2} = 1 - 2 \tan^{-1} 1 + c$

$\frac{\pi}{2} = 1 - 2 \cdot \frac{\pi}{4} + c$

$\therefore c = \pi - 1$

$\therefore y = x - 2 \tan^{-1} x + \pi - 1$

(*) A particle moves along an s axis with position function $s = s(t)$ and $v(t) = s'(t)$. Then $s(t)$ from the followings.

i) $v(t) = 32t$; $s(0) = 20$

ii) $v(t) = 3\sqrt{t}$; $s(4) = 1$

Solutions

i) Here,

$$v(t) = 32t$$

$$\frac{ds}{dt} = 32t$$

$$\therefore s(t) = \int 32t \, dt$$

$$= 32 \cdot \frac{t^2}{2} + c$$

$$s(t) = 16t^2 + c$$

Now, $s(0) = 20$

$t = 0$, then $s = 20$

$$20 = 16 \cdot 0 + c$$

$c = 20$ $\therefore s(t) = 16t^2 + 20$

5.3

Indefinite Integrals by using method of substitution:

(Undo of chain rule)

⊛ $\frac{d}{dx}(x^2+3x+2)^5$

$= 5 \cdot (x^2+3x+2)^4 \cdot (2x+3)$

⊛ $\int 5(x^2+3x+2)^4 (2x+3) = (x^2+3x+2)^5 + c$

Let,

$u = x^2+3x+2$

$\frac{du}{dx} = 2x+3$

$du = (2x+3)\, dx$

$\therefore \int 5 \cdot u^4 \cdot du$

$= 5\int u^4 \cdot du$

$= 5 \cdot \frac{u^5}{5} + c$

$= u^5 + c$

$= (x^2+3x+2)^5 + c$

* $\int \sin 5x \, dx$

Let,

$u = 5x$ $\quad = \int \sin u \frac{1}{5} du$

$\frac{du}{dx} = 5$ $\quad = \frac{1}{5} \int \sin u \, du$

$du = 5 dx$ $\quad = \frac{1}{5} (-\cos u) + c$

$dx = \frac{1}{5} du$ $\quad = -\frac{1}{5} \cos 5x + c$

* $\int \sin^2 x \cos x \, dx$

Let,

$u = \sin x$ $\quad = \int u^2 du$

$\frac{du}{dx} = \cos x$ $\quad = \frac{u^3}{3} + c$

$du = \cos x \, dx$ $\quad = \frac{\sin^3 x}{3} + c$

* $\int e^{\sin x} \cos x \, dx$

Let

* $\int (\sec 4x \tan 4x) dx$

Let,

$u = 4x$ $\quad = \int \sec u \tan u \frac{1}{4} du$

$\frac{du}{dx} = 4$ $\quad = \frac{1}{4} \int \sec u \tan u \, du$

$du = 4 dx$ $\quad = \frac{1}{4} \sec u + c$

$dx = \frac{1}{4} du$ $\quad = \frac{1}{4} \sec 4x + c$

⊛ $\int \frac{1}{1+16x^2}\,dx$

$= \int \frac{1}{1+(4x)^2}\,dx$

$= \int \frac{1}{1+u^2}\,\frac{1}{4}\,du$

Let,

$u = 4x$

$\frac{du}{dx} = 4$

$du = 4\,dx$

$dx = \frac{1}{4}\,du$

$= \frac{1}{4}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{4}\tan^{-1} u + c$

$= \frac{1}{4}\tan^{-1} 4x + c$

⊛ Evaluate,

$\int \frac{dx}{a^2+x^2}$ $\quad a \neq 0$

$= \int \frac{dx}{a^2\left(1+\frac{x^2}{a^2}\right)}$

$= \frac{1}{a^2}\int \frac{dx}{1+\frac{x^2}{a^2}}$

$= \frac{1}{a^2}\int \frac{a}{1+u^2}\,du$

$= \frac{1}{a}\int \frac{1}{1+u^2}\,du$

Let, $u = \frac{x}{a}$

$du = \frac{1}{a}\,dx$

$a\,du = dx$

$= \frac{1}{a}\tan^{-1} u + c$

$= \frac{1}{a}\tan^{-1} \frac{x}{a} + c$

Then,

Formula:

(*) $\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$

(*) $\int \frac{dx}{16+x^2} = \int \frac{dx}{4^2+x^2} = \frac{1}{4}\tan^{-1}\frac{x}{4} + c$

(*) $\int \frac{dx}{16+9x^2} = \int \frac{dx}{9(\frac{16}{9}+x^2)} = \frac{1}{9}\int \frac{dx}{(\frac{4}{3})^2+x^2}$

$= \frac{1}{9}\cdot\frac{3}{4}\cdot\tan^{-1}\frac{x}{4/3} + c$

$= \frac{1}{12}\tan^{-1}\frac{3x}{4} + c$

(*) $\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c}$

(*) $\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$

(*) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = \sin^{-1}\frac{x}{3} + c$

(*) $\int \frac{\cos 4\theta}{(1+2\sin 4\theta)} d\theta$

Let,

$u = 1 + 2\sin 4\theta$

$\frac{du}{d\theta} = 2\cos 4\theta \cdot 4 = 8\cos 4\theta$

$du = 8\cos 4\theta \, d\theta$

$\frac{1}{8} du = \cos 4\theta \, d\theta$

$= \int \frac{\frac{1}{8} du}{u}$

$= \frac{1}{8} \ln u + c$

$= \frac{1}{8} \ln(1+2\sin 4\theta) + c$

L-20/22.08.2022/

5.3

$$\int \frac{dx}{a^2+x^2}$$

$$= \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)} dx$$

$$= \frac{1}{a^2} \int \frac{1}{1+\left(\frac{x}{a}\right)^2} dx$$

Let,

$u = \frac{x}{a}$

$\frac{du}{dx} = \frac{1}{a}$

$dx = a\,du$

$$= \frac{1}{a^2} \int \frac{a\,du}{1+u^2}$$

$$= \frac{1}{a} \int \frac{1}{1+u^2} du$$

$$= \frac{1}{a} \tan^{-1} u + c$$

$$= \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= \int \frac{dx}{\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)}}$$

$$= \frac{1}{a} \int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$$

Let,

$u = \frac{x}{a}$

$\frac{du}{dx} = \frac{1}{a}$

$dx = a\,du$