NORTH SOUTH UNIVERSITY

Department of Mathematics & Physics

Assignment – 04

Name	: Joy Kumar Ghosh
Student ID	: 2211424 6 42
Course No.	: MAT 250
Course Title	: Calculus and Analytic Geometry IV
Section	: 16
Date	: 03 May, 2023

14·7

3|

Given that,

$$x = \sin u + \cos v$$

$$y = -\cos u + \sin v$$

Therefore,

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \cos u & -\sin v \\ \sin u & \cos v \end{vmatrix}$$

$$= \cos u \cos v + \sin u \sin v$$

$$= \cos(u-v).$$

Ans

7|

Given that,

$$u = x^2 - y^2$$

$$\Rightarrow x^2 = u + y^2$$

And, $$v = x^2 + y^2$$

$$\Rightarrow y^2 = v - x^2$$

$$\Rightarrow y^2 = v - u - y^2$$

$$\Rightarrow 2y^2 = v - u$$

$$\therefore y = \sqrt{\frac{v-u}{2}} = \frac{\sqrt{v-u}}{\sqrt{2}}$$

Therefore,

$$x^2 = u + y^2$$

$$\Rightarrow x^2 = u + \frac{v-u}{2}$$

$$\Rightarrow x^2 = \frac{2u + v - u}{2}$$

$$\Rightarrow x^2 = \frac{u+v}{2}$$

$$\therefore x = \sqrt{\frac{u+v}{2}} = \frac{\sqrt{u+v}}{\sqrt{2}}$$

Now,

$$\text{Jacobian} = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2\sqrt{2}\sqrt{u+v}} & \frac{1}{2\sqrt{2}\sqrt{u+v}} \\ -\frac{1}{2\sqrt{2}\sqrt{v-u}} & \frac{1}{2\sqrt{2}\sqrt{v-u}} \end{vmatrix}$$

$$= \frac{1}{2\sqrt{2}\sqrt{u+v}} \cdot \frac{1}{2\sqrt{2}\sqrt{v-u}} + \frac{1}{2\sqrt{2}\sqrt{v-u}} \cdot \frac{1}{2\sqrt{2}\sqrt{u+v}}$$

$$= \frac{1}{8\sqrt{u+v}\sqrt{v-u}} + \frac{1}{8\sqrt{u+v}\sqrt{v-u}}$$

$$= \frac{2}{8\sqrt{u+v}\cdot\sqrt{v-u}}$$

$$= \frac{1}{4\sqrt{v^2-u^2}}$$ Ans

11/

Given that,

$$u = xy$$

$$v = y$$

$$w = x + z$$

From here,

$$y = v$$

Then, $x = \frac{u}{y} = \frac{u}{v}$

And then, $z = w - x$

$$= w - \frac{u}{v}$$

$$= \cancel{\frac{vw-u}{v}}$$

Therefore,

$$\text{Jacobian} = \frac{\partial(x, y, z)}{\partial(u, v, w)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} & 0 \\ 0 & 1 & 0 \\ -\frac{1}{v} & \frac{u}{v^2} & 1 \end{vmatrix}$$

$$= \frac{1}{v}(1-0) + \frac{u}{v^2}(0-0) + 0$$

$$= \frac{1}{v}$$ Ans

16/

Given that,

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Then,

$$\text{Jacobian} = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta\left(0 + \rho\sin\phi\,\rho\sin\phi\cos\theta\right)$$
$$- \rho\cos\phi\cos\theta\left(0 - \cos\phi\,\rho\sin\phi\cos\theta\right)$$
$$- \rho\sin\phi\sin\theta\left(-\rho\sin\phi\sin\phi\sin\theta - \cos\phi\,\rho\cos\phi\sin\theta\right)$$

$$= \rho^2\sin^3\phi\cos^2\theta + \rho^2\cos^2\phi\cos^2\theta\sin\phi$$
$$+ \rho^2\sin^3\phi\sin^2\theta + \rho^2\sin\phi\sin^2\theta\cos^2\phi$$

$$= \rho^2\sin\phi\left(\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \sin^2\theta\cos^2\phi\right)$$

$$= \rho^2\sin\phi$$

Hence, given statement is True.

21|

Given that,

$$u = x - 2y$$

$$\Rightarrow x = u + 2y$$

and, $v = 2x + y$

$$\Rightarrow v = 2u + 4y + y$$

$$\Rightarrow 5y = v - 2u$$

$$\therefore y = \frac{v - 2u}{5}$$

then, $x = u + \frac{2}{5} \cdot (v - 2u)$

$$= u + \frac{2v - 4u}{5}$$

$$= \frac{5u + 2v - 4u}{5}$$

$$= \frac{u + 2v}{5}$$

Given rectangular region,

$x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, $2x + y = 3$

After transformation,

$u = 1$, $u = 4$, $v = 1$, $v = 3$

Therefore,

$$\text{Jacobian, } J = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{vmatrix}$$

$$= \frac{1}{25} + \frac{4}{25}$$

$$= \frac{5}{25}$$

$$= \frac{1}{5}$$

Now,

$$\iint_R \frac{x-2y}{2x+y}\, dA = \iint_S \frac{u}{v} \cdot J \cdot dA_{uv}$$

$$= \int_1^4 \int_1^3 \frac{u}{v} \cdot \frac{1}{5}\, dv\, du$$

$$= \frac{1}{5} \int_1^4 \int_1^3 \frac{u}{v}\, dv\, du$$

$$= \frac{1}{5} \int_1^4 u \cdot \ln|v| \Big]_1^3\, du$$

$$= \frac{1}{5}\int_1^4 u \cdot \ln 3 \, du$$

$$= \frac{1}{5} \cdot \ln 3 \int_1^4 u \, du$$

$$= \frac{1}{5} \cdot \ln 3 \cdot \frac{1}{2} \cdot u^2 \Big]_1^4$$

$$= \frac{1}{10} \cdot \ln 3 \cdot (16-1)$$

$$= \frac{15}{10} \ln 3$$

$$= \frac{3}{2} \ln 3$$ Ans

22)

Given that,

$$u = x + y$$

$$\Rightarrow x = u - y$$

and, $v = x - y$

$$\Rightarrow v = u - y - y$$

$$\Rightarrow 2y = u - v$$

$$\therefore y = \frac{u-v}{2}$$

$$\therefore x = u - \frac{u-v}{2}$$

$$= \frac{2u - u + v}{2}$$

$$= \frac{u+v}{2}$$

Given enclosed rectangular,

$$x+y=0,\quad x+y=1,\quad x-y=1,\quad x-y=4$$

after transformation,

$$u=0,\quad u=1$$

$$v=1,\quad v=4$$

Therefore,

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \iint_S v\,e^{uv}\cdot J\,dA_{uv}$$

Here,

$$\text{Jacobian, } J = \frac{\partial(x,y)}{\partial(u,v)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{4}-\frac{1}{4}$$

$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

$$\therefore \iint_S v e^{uv} \cdot J \, dA_{uv} = \int_1^4 \int_0^1 v e^{uv} \cdot \left|\frac{-1}{2}\right| du \, dv$$

$$= -\frac{1}{2} \int_1^4 \int_0^1 v \cdot e^{vu} \, du \, dv$$

$$= \frac{-1}{2} \int_1^4 \left. v \cdot \frac{1}{v} \cdot e^{uv} \right]_0^1 dv$$

$$= \frac{-1}{2} \int_1^4 \left(e^v - 1\right) dv$$

$$= -\frac{1}{2} \left[e^v - v\right]_1^4$$

$$= -\frac{1}{2} \left(e^4 - 4 - e^1 + 1\right)$$

$$= -\frac{1}{2} \left(e^4 - e - 3\right)$$

23)

Given that,

$$u = \frac{1}{2}(x+y)$$

$$\Rightarrow 2u = x+y$$

$$\therefore x = 2u - y$$

and

$$v = \frac{1}{2}(x-y)$$

$$\Rightarrow 2v = x-y$$

$$\Rightarrow 2v = 2u-y-y$$

$$\Rightarrow 2y = 2u-2v$$

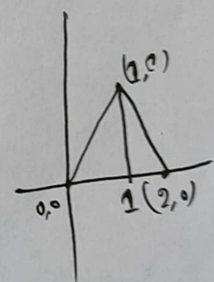
$$\therefore y = u-v$$

$$\therefore x = 2u-u+v$$

$$= u+v$$

Given enclosed ~~Rectangular~~, Triangular,

$(0,0)$, $(2,0)$, $(1,1)$



After transformation,

$$v=0, \quad v=u$$

$$u=1$$

Now,

Jacobian, $J = \dfrac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= -1 - 1 = -2$$

Therefore,

$$\iint_R \sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y)\, dA = 2\iint_S \sin u \cos v \, dA_{uv}$$

$$= 2\int_0^1\int_0^u \sin u \cos v \, dv\, du$$

$$= 2\int_0^1 \sin u \cdot \left[\sin v\right]_0^u du$$

$$= 2\int_0^1 \sin u \cdot \sin u \, du$$

$$= 2\int_0^1 \sin^2 u \, du$$

$$= 2\int_0^1 \tfrac{1}{2}(1-\cos 2u)\, du$$

$$= \left[u - \tfrac{1}{2}\sin 2u\right]_0^1$$

$$= 1 - \tfrac{1}{2}\sin 2$$

Ans

<u>24</u>

Given that,

$$u = \frac{y}{x}$$

$$\Rightarrow x = \frac{y}{u}$$

and, $v = xy$

$$\Rightarrow v = \frac{y}{u} y$$

$$\Rightarrow uv = y^2$$

$$\therefore y = \sqrt{uv}$$

$$\therefore x = \frac{\sqrt{uv}}{u} = \sqrt{\frac{v}{u}}$$

Given enclosed Region,

$y = x$ $\Rightarrow \frac{y}{x} = 1$ $\therefore u = 1$

$y = 3x$ $\Rightarrow \frac{y}{x} = 3$ $\therefore u = 3$

$xy = 1$ $\therefore v = 1$

$xy = 4$ $\therefore v = 4$

Now,

Jacobian, $j = \frac{\partial(x,y)}{\partial(u,v)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{2u}\sqrt{\frac{v}{u}} & \frac{1}{2\sqrt{uv}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{vmatrix}$$

$$= -\frac{1}{4u} - \frac{1}{4u}$$

$$= -\frac{1}{2u}$$

Therefore,

$$\iint_R xy^3 \, dA = \iint_R x^2y^2 \cdot \frac{y}{x} \, dA$$

$$= \iint_S uv^2 |J| \, dA_{uv}$$

$$= \int_1^4 \int_1^3 uv^2 \cdot \frac{1}{2u} \cdot du \, dv$$

$$= \frac{1}{2} \int_1^4 \int_1^3 v^2 \, du \, dv$$

$$= \frac{1}{2} \int_1^4 \left. v^2 u \right]_1^3 dv$$

$$= \frac{1}{2} \int_1^4 (3v^2 - v^2) \, dv$$

$$= \frac{1}{2} \int_{1}^{4} 2v^2 \, dv$$

$$= \frac{1}{2} \cdot \frac{2}{3} \left[v^3 \right]_1^4$$

$$= \frac{1}{3} \cdot (64 - 1)$$

$$= \frac{63}{3}$$

$$= 21 \quad \text{Ans}$$

25)

Given that,

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

$$\Rightarrow \frac{x^3}{3^2} + \frac{y^2}{4^2} = 1$$

Therefore

$$x = 3u \qquad \text{and} \qquad u = \frac{x}{3}$$

$$y = 4v \qquad\qquad\qquad v = \frac{y}{4}$$

Hence, circular region is,

$$u^2 + v^2 = 1$$

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= 12$$

$\therefore$ Jacobian, $J = |12| = 12$

Therefore

$$\iint_R \sqrt{16x^2+9y^2}\, dA = \iint_R 12\sqrt{\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2}\, dA$$

$$= \iint_S 12\sqrt{u^2+v^2}\,(12)\, dA_{uv}$$

$$= 144 \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta$$

$$= \frac{144}{3}\int_0^{2\pi} r^3\Big]_0^1\, d\theta$$

$$= \frac{144}{3}\int_0^{2\pi} 1\, d\theta$$

$$= \frac{144}{3}\,\theta\Big]_0^{2\pi} = \frac{144\cdot 2}{3}\pi = 96\pi$$

26/

Given ellipse,
$$\frac{x^2}{4} + y^2 = 1$$

$$\Rightarrow \frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$$

Therefore

$$x = 2u \qquad \text{and} \qquad u = \frac{x}{2}$$

$$y = v \qquad \qquad v = y$$

Hence,

Circular region,

$$u^2 + v^2 = 1$$

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 2$$

$\therefore$ Jacobian, $J = |2| = 2$

Therefore,

$$\iint_R e^{-(x^2+4y^2)}\, dA = \iint_S e^{-(4u^2+4v^2)} \cdot 2\, dA_{uv}$$

$$= 2\int_0^{2\pi}\int_0^1 r\, e^{-4r^2}\, dr\, d\theta$$

$$= 2\int_0^{2\pi} -\frac{1}{8} e^{-4r^2}\Big]_0^1\, d\theta$$

$$= -\frac{1}{4}\int_0^{2\pi} \left(e^{-4}-1\right) d\theta$$

$$= -\frac{1}{4}\left[e^{-4}\theta - \theta\right]_0^{2\pi}$$

$$= -\frac{1}{4}\left(e^{-4}\cdot 2\pi - 2\pi\right)$$

$$= -\frac{1}{4}\cdot 2\pi\left(e^{-4}-1\right)$$

$$= -\frac{\pi}{2}\left(e^{-4}-1\right)$$

Ans

27)

Let, S be the region in the uv-plane bounded by

$$u^2+v^2=1$$

Therefore,

$$u = 2x \qquad \text{and} \qquad x = \frac{u}{2}$$
$$v = 3y \qquad\qquad\qquad y = \frac{v}{3}$$

Now,
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{vmatrix}$$

$$= \frac{1}{6}$$

$\rightarrow$ Jacobian, $J = \left|\frac{1}{6}\right| = \frac{1}{6}$

Hence,

$$\iint_R \sin(4x^2+9y^2)\, dA = \iint_S \sin(u^2+v^2)\, \frac{1}{6} \cdot dA_{uv}$$

$$= \frac{1}{6}\int_0^{2\pi}\int_0^1 r \cdot \sin r^2 \, dr\, d\theta$$

$$= \frac{1}{6}\int_0^{2\pi} -\frac{1}{2}\cos r^2\Big]_0^1 \, d\theta$$

$$= -\frac{1}{12}\int_0^{2\pi} (\cos 1 - 1)\, d\theta$$

$$= -\frac{1}{12}\Big[\cos 1 \cdot \theta - \theta\Big]_0^{2\pi}$$

$$= -\frac{1}{12}\left(2\pi \cdot \cos 1 - 2\pi\right)$$

$$= -\frac{1}{12}\, 2\pi\,(\cos 1 - 1)$$

$$= -\frac{\pi}{6}(\cos 1 - 1)$$

For the first quandent,

integral will be $= -\frac{\pi}{6}\cdot\frac{1}{4}(\cos 1 - 1)$

$$= -\frac{\pi}{24}(\cos 1 - 1)$$

28|

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Therefore, $x = au$ and $u = \frac{x}{a}$

$y = bv$ $\quad v = \frac{y}{b}$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}$$

$$= ab$$

$\therefore$ Jacobian, $J = ab$

Now,

$$\text{Area, } A = \int_0^{2\pi}\int_0^1 r \cdot ab \, dr \, d\theta$$

$$= ab \int_0^{2\pi}\int_0^1 r \, dr \, d\theta$$

$$= \frac{ab}{2} \int_0^{2\pi} \left. r^2 \right]_0^1 d\theta$$

$$= \frac{ab}{2} \int_0^{2\pi} 1 \, d\theta$$

$$= \frac{ab}{2} \cdot \left. \theta \right]_0^{2\pi}$$

$$= \frac{ab}{2} \cdot 2\pi$$

$$= ab\pi$$

(Showed).

29/

Given ellipsoid,

$$9x^2+4y^2+z^2=36$$

∴ enclosed sphere, $u^2+v^2+w^2=36$

Therefore,

$u=3x$ and $x=\frac{u}{3}$

$v=2y$ $y=\frac{v}{2}$

$w=z$ $z=w$

$$\frac{\partial(x,y,z)}{\partial(u,v,z)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{3} & 0 & 0\\ 0 & \frac{1}{2} & 0\\ 0 & 0 & 1\end{vmatrix}$$

$$=\frac{1}{3}\cdot\frac{1}{2}$$

$$=\frac{1}{6}$$

∴ Jacobian $J=\frac{1}{6}$

Therefore,

$$\iiint_S \frac{u^2}{9}\cdot\frac{1}{6}\,dV_{uvw} = \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^{6} (\rho\sin\phi\cos\theta)^2\rho^2\sin\phi$$

$$= \frac{1}{54}\int_0^{2\pi}\int_0^{\pi}\int_0^{6} \rho^4\sin^3\phi\cos^2\theta\, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{54}\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}\sin^3\phi\cos^2\theta\Big]_0^6\, d\phi\, d\theta$$

$$= \frac{1}{54}\cdot\frac{7776}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\cos^2\theta\, d\phi\, d\theta$$

$$= \frac{144}{5}\int_0^{2\pi}\cos^2\theta\left[\frac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi} d\theta$$

$$= \frac{144}{5}\int_0^{2\pi}\cos^2\theta\left[-\frac{1}{3}+1-\frac{1}{3}+1\right] d\theta$$

$$= \frac{144}{5}\cdot\frac{4}{3}\int_0^{2\pi}\cos^2\theta\, d\theta$$

$$= \frac{192}{5}\cdot\frac{1}{2}\int_0^{2\pi}(\cos 2\theta + 1)\, d\theta$$

$$= \frac{96}{5}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{2\pi}$$

$$= \frac{96}{5}\cdot 2\pi$$

$$= \frac{192}{5}\pi$$

30/

Let, S be the region $u^2+v^2+w^2 \leq 1$, with

$$x = au$$
$$y = bv$$
$$z = cw$$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$$

$$= abc$$

$\therefore$ Jacobian, $J = abc$

Therefore,

$$\iiint_G (y^2+z^2)\, dV_{xyz} = \iiint_S (b^2v^2 + c^2w^2) \cdot abc \; dV_{uvw}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\,(b^2 \sin^2\phi \sin^2\theta + c^2\cos^2\phi)\, \rho^4 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= abc \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^5}{5} \sin\phi \left(b^2 \sin^2\phi \sin^2\theta + c^2 \cos^2\phi\right)\right]_0^1 d\phi \, d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi}\int_0^{\pi} \left(b^2 \sin^3\phi \sin^2\theta + c^2 \sin\phi \cos^2\phi\right) d\phi \, d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi}\int_0^{\pi} \left(b^2 \sin^3\phi \sin^2\theta + c^2 \sin\phi - c^2 \sin^3\phi\right) d\phi \, d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi}\int_0^{\pi} \left[c^2 \sin\phi + \sin^3\phi \left(b^2 \sin^2\theta - c^2\right)\right] d\phi \, d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi} \left[-c^2 \cos\phi + \left(b^2 \sin^2\theta - c^2\right)\left(\frac{1}{3}\cos^3\phi - \cos\phi\right)\right]_0^{\pi} d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi} \left[c^2 + \left(b^2 \sin^2\theta - c^2\right)\frac{2}{3} + c^2 - \left(b^2 \sin^2\theta - c^2\right)\left(-\frac{2}{3}\right)\right] d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi} \left[2c^2 + \frac{4}{3}\left(b^2 \sin^2\theta - c^2\right)\right] d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi} \left[2c^2 + \frac{4}{3} b^2 \sin^2\theta - \frac{4}{3} c^2\right] d\theta$$

$$= \frac{abc}{5} \int_0^{2\pi} \left[\frac{2}{3} c^2 + \frac{4}{3} b^2 \sin^2\theta\right] d\theta$$

$$= \frac{2abc}{15} \int_0^{2\pi} \left(c^2 + 2b^2 \sin^2\theta\right) d\theta$$

$$= \frac{2abc}{15}\left[c^2\theta + b^2\theta - \frac{b^2}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{2abc}{15}\left(2\pi c^2 + 2\pi b^2\right)$$

$$= \frac{2abc}{15} 2\pi \left(b^2+c^2\right)$$

$$= \frac{4}{15}\pi abc\left(b^2+c^2\right)$$ Ans

35)

Let,

$u = y - 4x$ and $v = y + 4x$

$\Rightarrow 4x = y - u$ $\Rightarrow v = y + y - u$

$\therefore x = \frac{y-u}{4}$ $\Rightarrow 2y = v + u$

$\therefore y = \frac{1}{2}(v+u)$

$$\therefore x = \frac{\frac{v}{2} + \frac{u}{2} - u}{4}$$

$$= \frac{\frac{v+u-2u}{2}}{4}$$

$$= \frac{v-u}{8}$$

$$= \frac{1}{8}(v-u)$$

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{16} - \frac{1}{16}$$

$$= -\frac{2}{16}$$

$$= -\frac{1}{8}$$

$\therefore$ Jacobian, $J = \left|-\frac{1}{8}\right| = \frac{1}{8}$

~~Therefore,~~

Given, enclosed region,

$y = 4x$	$y = 4x+2$	$y = 2-4x$	$y = 5-4x$
$\Rightarrow y - 4x = 0$	$\Rightarrow y-4x = 2$	$\Rightarrow y+4x = 2$	$\Rightarrow y+4x = 5$
$\therefore u = 0$	$\therefore u = 2$	$\therefore v = 2$	$\therefore v = 5$

Therefore,

$$\iint_R \frac{y-4x}{y+4x}\, dA = \iint_S \frac{u}{v} \cdot \frac{1}{8}\, dA_{uv}$$

$$= \frac{1}{8} \int_2^5 \int_0^2 \frac{u}{v}\, du\, dv$$

$$= \frac{1}{8} \int_2^5 \left. \frac{1}{v} \cdot \frac{u^2}{2} \right]_0^2 dv$$

$$= \frac{1}{8} \cdot \frac{1}{2} \int_2^5 \left. \frac{u^2}{v} \right]_0^2 dv$$

$$= \frac{1}{16} \int_2^5 \frac{4}{v}\, dv$$

$$= \frac{1}{4} \int_2^5 \frac{1}{v}\, dv$$

$$= \frac{1}{4} \cdot \left. \ln v \right]_2^5$$

$$= \frac{1}{4} (\ln 5 - \ln 2)$$

$$= \frac{1}{4} \ln \frac{5}{2}$$

Ans

36|

Let,

$u = x+y$ and $v = x-y$

$\Rightarrow x = \cancel{y}+u-y$ $\quad \Rightarrow v = -y+u-y$

$\Rightarrow 2y = u-v$

$\therefore y = \frac{1}{2}(u-v)$

$\therefore x = u - \frac{u-v}{2}$

$= \frac{2u-u+v}{2}$

$= \frac{1}{2}(u+v)$

$\therefore$ Now, $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$

$= -\frac{1}{4} - \frac{1}{4}$

$= -\frac{1}{2}$

$\therefore$ Jacobian, $J = \left|-\frac{1}{2}\right| = \frac{1}{2}$

Given enclosed region,

$y = -x$	$y = 1-x$	$y = x$	$y = x+2$
$\Rightarrow x+y = 0$	$\Rightarrow x+y = 1$	$\Rightarrow x-y = 0$	$\Rightarrow x-y = -2$
$\therefore u = 0$	$\therefore u = 1$	$\therefore v = 0$	$\therefore v = -2$

Therefore,

$$\iint_R (x^2-y^2)\, dA = \iint_R (x+y)(x-y)\, dA$$

$$= \iint_S uv \cdot \frac{1}{2}\, dA_{uv}$$

$$= \frac{1}{2}\int_{-2}^{0}\int_{0}^{1} uv\, du\, dv$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_{-2}^{0} vu^2\Big]_0^1\, dv$$

$$= \frac{1}{4}\int_{-2}^{0} v\, dv$$

$$= \frac{1}{8}\, v^2\Big]_{-2}^{0}$$

$$= \frac{1}{8}\cdot(-4)$$

$$= -\frac{1}{2}$$

Ans