6. $P(a)$ Modus Ponens of 4,5

7. $\neg B(a)$ Simplification of 3

8. $P(a) \land \neg B(a)$ Conjunction of 6,7

9. $\exists x\ P(x) \land \neg B(x)$ Existential generalization of 8.

Proof Techniques

Theorem: A statement that can be shown to be true.

Axioms: Statements that we assume to be true.

Lemma: Less important theorem that is helpful in the proof of other results.

Corrolary: Established directly from already proven theorem.

~~L-8~~ L-9 / 28.06.2022

Conjecture

Proposed as true statement based on partial evidence.

Proof Techniques → Direct
Proof Techniques → Indirect

Direct: Apply rules directly on the given statement.

⊛ Proof it

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

⇒

1. $p \vee q$
2. $\cancel{\neg q}$ $\neg p$
3. $\neg(\neg p) \vee q$ — Double negation of 1
4. $\neg p \rightarrow q$ — Defn of implication of 3
5. q — Modus Ponens of 2, 4

* Given, m is even and n is odd, their sum $(m+n)$ is always odd.

$\Rightarrow$

$m = 2k$; where k is an integer.

$n = 2j+1$; where j is an integer.

$$m+n = 2k + 2j + 1$$
$$= 2(k+j) + 1$$
$$= 2 \cdot \text{integer} + 1$$
$$= \text{odd}$$

(proved).

* Given m is even, show that m^2 is even.

$\Rightarrow$

* Given n is an integer and $3n+2$ is odd, show that n is odd.

$\Rightarrow$

$$3n+2 = 2k+1$$
$$3n = 2k-1$$
$$n = \frac{2k-1}{3}$$

→ Dead lock → Now apply indirect method

$\Rightarrow$ if $\underbrace{3n+2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

$p \to q \equiv \neg q \to \neg p \equiv$ contrapositive

proof by contraposition

If n is $\underbrace{\text{not odd}}_{\text{even}}$, then $3n+2$ is $\underbrace{\text{not odd}}_{\text{even}}$.

$n = 2k$

$$3n + 2 = 6k+2$$
$$= 2(3k+1)$$
$$= 2 \cdot \text{integer}$$
$$= \text{even}$$

⊛ Prove that, if $x^2 - 6x + 5$ is even, then x is odd.

$\Rightarrow$

Direct:

$$x^2 - 6x + 5 = 2k$$
$$x^2 - 5x - x + 5 = 2k$$
$$x(x-5) - 1(x-5) = 2k$$
$$(x-5)(x-1) = 2k \longrightarrow \text{Dead lock}$$

indirect:

if $\underbrace{x^2 - 6x + 5 \text{ is even}}_{p}$, then $\underbrace{x \text{ is odd}}_{q}$.

$$p \to q \equiv \neg q \to \neg p$$

$\neg q$: x is not odd. $\to$ even

$\neg p$: $x^2 - 6x + 5$ is not even. $\to$ odd

$x = 2k$; k is an integer.

$$x^2 - 6x + 5 = (2k)^2 - 6 \cdot 2k + 5$$

$$= 4k^2 - 12k + 4 + 1$$

$$= 2(2k^2 - 6k + 2) + 1$$

$$= 2 \cdot \text{integer} + 1$$

$$= \text{odd.}$$

(proved)

Proof by contradiction:

$\sqrt{2}$ is irrational number.

Assume it to be true

~~contr~~
contradiction $\rightarrow$ $\sqrt{2}$ is a rational number.

$$\sqrt{2} = \frac{p}{q} \quad ; \; p, q \text{ are integers and } q \neq 0$$

simplest form $\boxed{\gcd(p, q) = 1}$

$$\Rightarrow \frac{p^2}{q^2} = 2$$

$$p^2 = 2q^2$$
$$= 2 \cdot \text{integer}$$
$$= \text{even}$$

so, p is even

$\therefore p = 2k$ as p is even.

$$\therefore p^2 = 2q^2$$
$$2q^2 = 4k^2$$
$$q^2 = 2k^2$$
$$= 2 \cdot \text{integer}$$
$$= \text{even}$$

$\therefore q$ is even.

p, q, even so, $\boxed{gcd(p,q) > 1}$

→ contradiction.

So, $\sqrt{2}$ is not rational number

∴ ≡ $\sqrt{2}$ is irrational.

(proved)

L-10 / 03.07.2022 /

Proof Techniques

i) Direct

ii) Indirect → Proof by Contraposition
→ Proof by Contradiction
→ Proof by Cases.

Proof by Cases

$$\begin{array}{ll} p \to r & \text{case-1} \\ q \to r & \text{case-2} \\ p \vee q & \\ \hline \therefore r & \end{array}$$

(*) Given x is an integer.
show that x^2+x is even.

Case-1

If $\overbrace{x \text{ is even}}^{p}$, $\overbrace{x^2+x \text{ is even}}^{r}$

$x = 2k$ → integer

$$\begin{aligned} x^2+x &= 4k^2+2k \\ &= 2(2k^2+k) \\ &= 2\cdot\text{integer} \\ &= \text{even} \end{aligned}$$

Case-2

If $\overbrace{x \text{ is odd}}^{q}$, then $\overbrace{x^2+x}^{r}$ is even.

$x = 2k+1$

$$x^2+x = (2k+1)^2 + 2k+1$$
$$= 4k^2+4k+1+2k+1$$
$$= 4k^2+6k+2$$
$$= 2(2k^2+3k+1)$$
$$= 2 \cdot \text{integer}$$
$$= \text{even.}$$

So, $p \rightarrow r$

$q \rightarrow r$

$p \vee q$

$\therefore r$

(*) Given n is an integer show that if n is not divisible by 3 then $n^2 = 3k+1$. → integer

$\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$ $\frac{8}{3}$ $\frac{9}{3}$ ...

$n = 3m+1$

$n = 3m+2$

$n = 3m$

$n = 3m$

Case-1

$$n = 3m+1$$

$$n^2 = (3m+1)^2$$

$$= 9m^2 + 6m+1$$

$$= 3(3m^2+2m)+1$$

$$= 3 \cdot \text{integer} + 1$$

$$= 3k+1$$

Case-2

$$n = 3m+2$$

$$n^2 = (3m+2)^2$$

$$= 9m^2 + 12m + 3 + 1$$

$$= 3(3m^2+4m+1)+1$$

$$= 3 \cdot \text{integer} + 1$$

$$= 3k+1$$

L-11/ 05.07.2022/

Mathematical Induction

⊛ $1+2+3+4+5+ \cdots +n = \frac{n(n+1)}{2}$

⊛ $1^2+2^2+3^2+4^2+5^2+ \cdots +n^2 = \frac{n(n+1)(2n+1)}{6}$

⊛

Basic step:

$n=1$

$L.H.S. = 1$

$R.H.S = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

Inductive step:

Lets assume that the given statement is true for $n=k$. That is $p(k)$ is true.

Now, we have to show that the statement is true for $n=k+1$. That is we have to show that

$1+2+3+ \cdots +k+(k+1) = \frac{(k+1)(k+1+1)}{2} = \frac{(k+1)(k+2)}{2}$

$\boxed{P(k) \rightarrow P(k+1)}$

$$L.H.S. = 1+2+3+\dots \dots \dots + k + k+1$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+1+1)}{2} \equiv R.H.S.$$

* Use mathematical induction to show that,

$(6^n - 1)$ is divisible by 5, where $n \in N$

$\Rightarrow$

Basic step:

$n = 1$

$6^1 - 1 = 6 - 1 = 5$, which is divisible by 5.

Inductive step:

Lets assume that this statement is true for $n = k$.

That means that $(6^k - 1)$ is divisible by 5.

Now, we have to show that the statement is true for $(k+1)$. That means that $(6^{k+1} - 1)$ is divisible by 5.

$$6^{k+1}-1 = 6^k \cdot 6 - 1$$

$$= 6^k \cdot (5+1) - 1$$

$$= 5 \cdot 6^k + 6^k - 1$$

$$= 5 \cdot 6^k + 5m$$ m is an positive number

$$= 5(6^k + m)$$

$= 5 \cdot$ integer

which is divisible by 5.

Proof by Equivalence

If n is a positive integer, then $\underbrace{n \text{ is odd}}_{p}$

if and only if $\underbrace{n^2 \text{ is odd}}_{q}$.

$$p \longleftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$p \to q$: if n is odd, then n^2 is odd.

$q \to p$: if n^2 is odd, then n is odd.

$n = 2k+1$, k is an integer

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2(2k^2+2k)+1$$
$$= 2 \cdot \text{integer} + 1$$
$$= \text{odd}$$

$\neg p \to \neg q$

n is even,

$$n = 2k$$
$$n^2 = 4k^2$$
$$= 2 \cdot 2k^2$$
$$= 2 \cdot \text{integer}$$
$$= \text{even.}$$

Chapter-2

Set Function

Set: Collection of objects.

→ Names

→ Numbers

→ Anything and everything

→ List of Distinct elements

→ separated by comma (,)

$A = \{1, 2, 3, 5\}$

Cardinality : Total number of elements in a set.

→ Notation: $|A| = 4$

Singleton set: Set that have only one element.

$A = \{1\}$ $\quad B = \{5\}$ $\quad C = \{33\}$

$|A| = 1$ $\quad |B| = 1$ $\quad |K| = 1$

Empty set: $\phi = \{\}$

→ Null set

$|\phi| = 0$

$|\{\phi\}| = 1$

$|\{\phi, \{\phi\}\}| = 2$

Set Theory

↳ Cardinality ⇒ Number of elements.

$x \in A \Rightarrow \underbrace{x \text{ is in } A}_{\text{Set builder Notation}}$

Union

such that

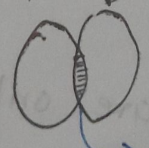
$x \in A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

is in

is in (belongs to)

* $x \in (A \cap B)$

$= \{x \mid x \in A \text{ and } x \in B\}$

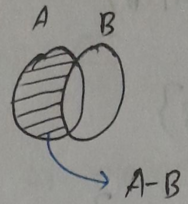


* $A = \{1, 2, 3, 4\}$

$B = \{1, 7, 5, 3\}$

$\therefore A \cap B = \{1, 3\}$

$$A \overset{\text{difference}}{-} B \equiv \{x \mid x \in A \text{ and } x \notin B\}$$



$A = \{1, 2, 7, 9\}$

$B = \{3, 5, 7, 9, 1\}$

$\therefore A - B = \{2\}$

$C = \{3, 2, 5, 7, 9, 1\}$

$\therefore A - C = \phi$ or $\{\}$

Set compliment

De Morgans Law

(i) $\overline{A \cap B} = \bar{A} \cup \bar{B}$

(ii) $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Prove - (i)/

$$L.H.S \equiv \{x \mid x \in \overline{A \cap B}\}$$

$$\equiv \{x \mid x \notin A \cap B\}$$

$$\equiv \{x \mid \neg (x \in A \cap B)\}$$

$$\equiv \{x \mid \neg (x \in A \text{ and } x \in B)\}$$

$$\equiv \{x \mid \neg (x \in A) \text{ or } \neg (x \in B)\}$$

$$\equiv \{x \mid x \notin A \text{ or } x \notin B\}$$

$$\equiv \{x \mid x \in \bar{A} \text{ or } x \in \bar{B}\}$$

$$\equiv \{x \mid x \in \bar{A} \cup \bar{B}\}$$

$$\equiv \bar{A} \cup \bar{B}$$

⊛ Power Set

→ Set of all the subset of a given set.

→ P(...) ⟶ Notation

⊛

$A = \{1, 2, 3\}$

$P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$

Proper Subset: $\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \phi\}$

→ Notation ⇒ $\subset$

→ Number of element will be strictly less than parent set.

$A \subseteq C \Rightarrow |A| \leq |C|$

$B \subset A \Rightarrow$ B is a subset of A.

⊛ Cartesian Product:

$\frac{A, B \text{ sets}}{}$

$A \times B \equiv \{(a,b) \mid a \in A \text{ and } b \in B\}$

cross

$B \times A \equiv \{(b,a) \mid b \in B \text{ and } a \in A\}$

⊛ $A = \{1, 4, 6\}$

$B = \{p, q\}$

$A \times B = \{(1,p), (1,q), (4,p), (4,q), (6,p), (6,q)\}$

$B \times A = \{(p,1), (p,4), (p,6), (q,1), (q,4), (q,6)\}$

$\therefore A \times B \neq B \times A$

⊛ $X = \{1, 2\}$

$$|P(X)| = 2^{|X|}$$
$$= 2^2 = 4$$

⊛ $|P(A \times B)|$

$$= 2^{|A \times B|}$$
$$= 2^{|A| \times |B|}$$
$$= 2^{3 \times 2}$$
$$= 2^6 = 64$$

⊛ $A = \{1, 4\}$

$B = \{p, q\}$

$C = \{*, \square\}$

$A \times B \times C = \{(a, b, c) \mid a \in A \text{ and } b \in B \text{ and } c \in C\}$

Set
- Finite ("Limited" number of elements)
- Infinite (that is not finite) → ex. $\mathbb{N}$: set of natural numbers.
  - Countable
  - Uncountable

L-13 / 12.07.2022 /

Quizzes-1 / Question /

1/

a)

$$(P \wedge Q \wedge \neg R \wedge \neg S) \vee (P \wedge \neg Q \wedge R \wedge \neg S) \vee (P \wedge \neg Q \wedge \neg R \wedge S)$$

$$\vee (\neg P \wedge Q \wedge R \wedge \neg S) \vee (\neg P \wedge Q \wedge \neg R \wedge S) \vee (\neg P \wedge \neg Q \wedge R \wedge S)$$

2/

$(\neg p \vee q) \rightarrow r$ Premises-1

$(s \rightarrow r) \vee \neg t$ Premises-2

$\neg s \wedge u$ Premises-3

$u \rightarrow t$ Premises-4

Conclusion $\Rightarrow$ p

Function

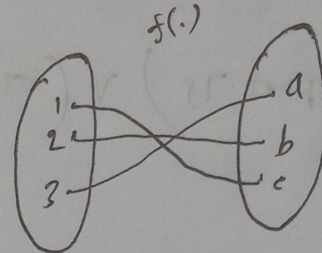
→ Relation between sets

→ Mapping

$f: A \to B$

Domain

Co - Domain



i) All the elements of set A must have an image B

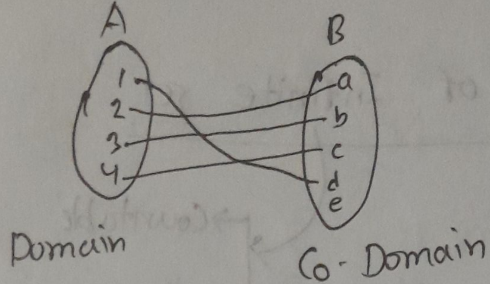
ii) Multiple images of single element of A are not possible.

⊛ There are three kind of function :

i) One - to - one / injective function

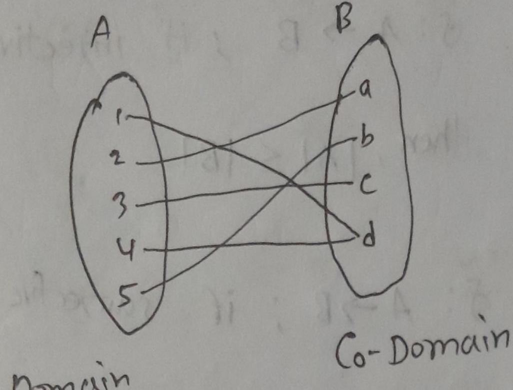
ii) Onto / surjective function

iii) One to one correspondence / Bijective function i)



$|A| = 4 \leq |B| = 4$

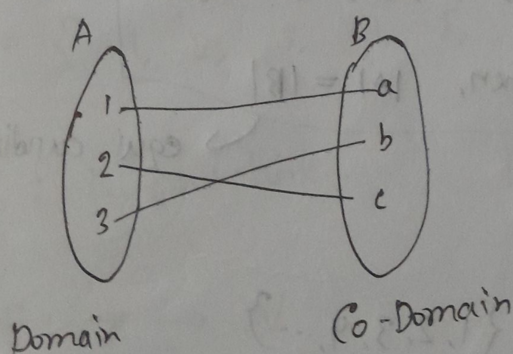
ii)



⊛ There cannot be domainless image.

$|A| = 5 \geq |B| = 5$

iii)



⊛ One-to-one + Onto (both)

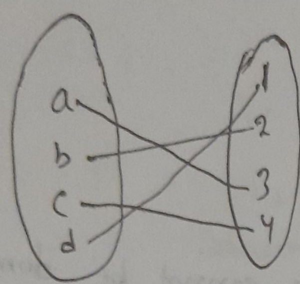
$|A| = |B|$

L-14 / 22.07.2022

Countability of Infinite set

→ Countable

→ uncountable



surjective

many to one

1. $f: A \to B$; if injective

then, $|A| \le |B|$

2. $f: A \to B$; if surjective

then, $|A| \ge |B|$

if exist then countable

if not exist, then uncountable

$$\boxed{\begin{array}{l} 3.\ f: A \to B \text{ ; if Bijective} \\ \text{then, } |A| = |B| \end{array}}$$

→ equi cardinality

Reference ← $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

countable ← $\mathbb{Z} = \{-\infty, \dots, \infty\}$

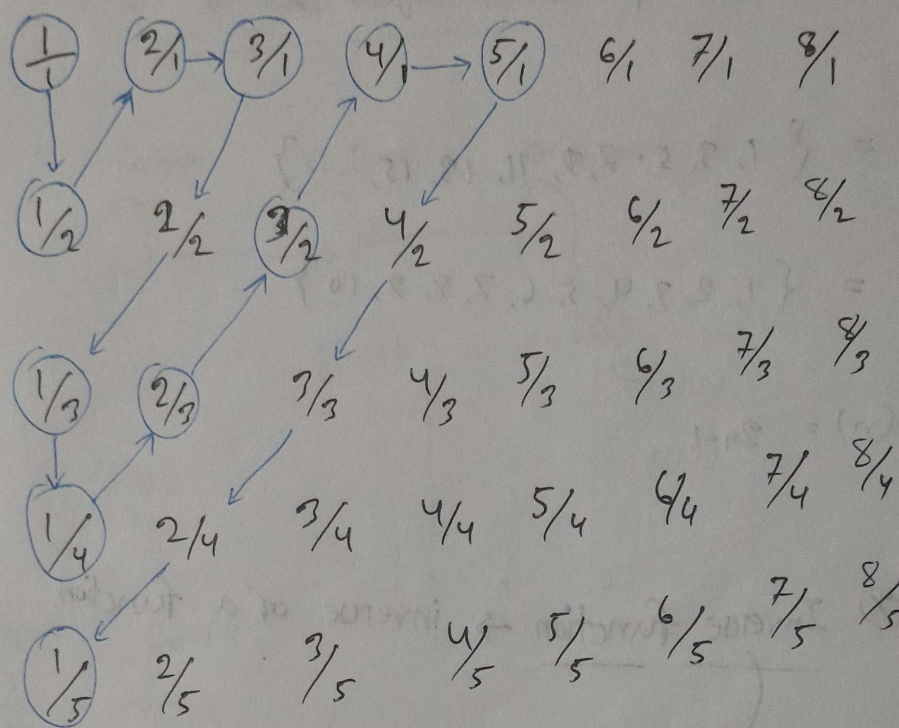
uncountable ← $\mathbb{R} = \{-\infty, \pm \text{integer}, \text{fraction}, \infty\}$

⊛ Q^+ = Set of all positive rational number.

→ Countable ?

or Uncountable?

$\Rightarrow$



$Q^+ = \{ 1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, 5, \frac{1}{5}, \ldots$

index $= \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots$

So, f is countable.

⊛ $f : \mathbb{Z} \to \mathbb{N}$

$-3, -2, -1, 0, 1, 2, 3$

$6, 3, 1, 2, 4, 5, 7$

⊛ Show that odd positive integers set is countable.

$\Rightarrow$

$$Z^{+}_{odd} = \{1, 3, 5, 7, 9, 11, 13, 15, \cdots\}$$

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

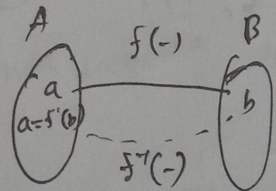
$$f(n) = 2n-1$$

⊛ Inverse function $\rightarrow$ inverse of a function.

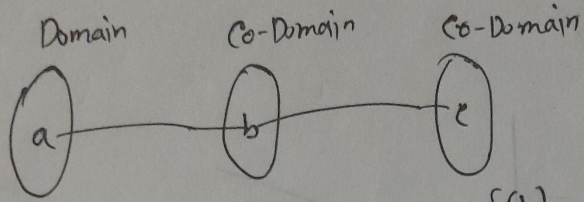
$\hookrightarrow$ Function needs to be a bijective.

$\hookrightarrow$ injective
$\hookrightarrow$ surjective } both

$f: A \rightarrow B$



⊛ Composit Function



$g(a) = b$

$c = f(b)$

$\hookrightarrow g(a)$

$= f(g(a))$

$g : A \to B$

⊛ $f(x) = 2x+3$

$g(x) = 3x+2$

$(f \circ g)(x) = f(g(x))$

$= f(3x+2)$

$= 2(3x+2)+3$

$= 6x+4+3$

$= 6x+7$

Ans

$(g \circ f)(x) = g(f(x))$

$= g(2x+3)$

$= 3(2x+3)+2$

$= 6x+9+2$

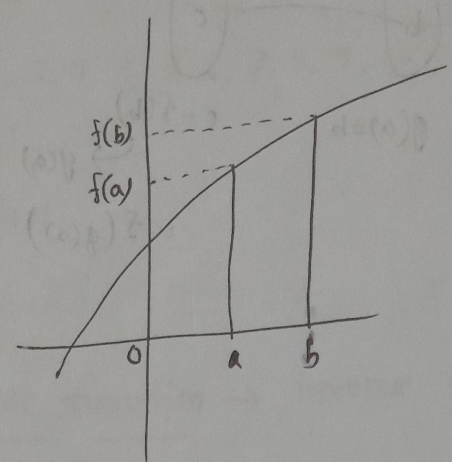
$= 6x+11$

Ans

|L-15| 24.07.2024|

⊛ Increasing & Decreasing Function

⊛ Increasing function:



$$\boxed{\forall a \forall b \,(a<b \rightarrow f(a) \leq f(b)}$$

without equal sign
strictly increasing

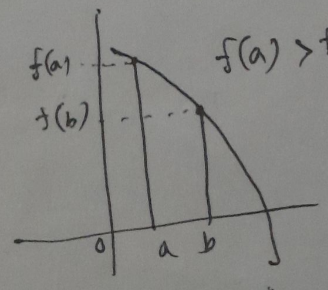
Activation function:

$e^x \rightarrow$ increasing

$e^{-x} \rightarrow$ decreasing

⊛ Decreasing function:

$f(a) > f(b)$



$a<b$

$$\boxed{\forall a \forall b \,(a<b \rightarrow f(a) \geq f(b))}$$

without equal sign, it is strictly decreasing.

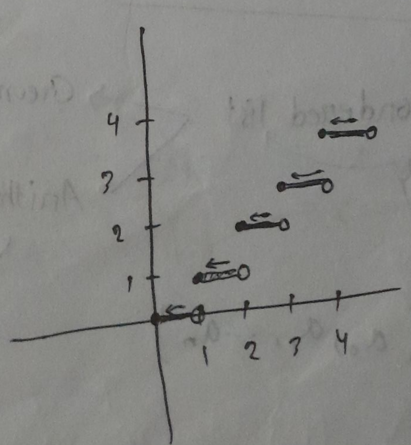
⊛ ~~Write a~~

⊛ Floor & Ceiling Function

⊛ **Floor :**

$\lfloor x \rfloor$ = Floor(x) — the greatest integer that is less than or equal to x.

⊛ $\lfloor 0.7 \rfloor = 0$ $\quad$ $\lfloor 3 \rfloor = 3$

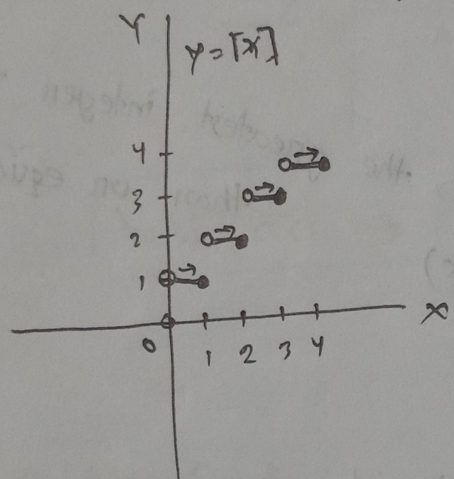


Ceiling function:

$\lceil x \rceil$ = ceil(x) — smallest integer that is largest than or equal to x.

$\lceil 0.001 \rceil = 1$ $\lceil 1.001 \rceil = 2$

$\lceil -0.5 \rceil = 0$



Sequence & Summations

An ordered list — $\{a_n\}$

An ordered list → Geometric sequence → গুণোত্তর ধারা

An ordered list → Arithmetic sequence → সমান্তর ধারা

$n = 0$
$n = 1$ $\{a_0, a_1, \ldots a_n\}$
$n = n$

Geometric Sequence: $ar^0, ar^1, ar^2, ar^3, \dots, ar^n$

$\frac{ar^1}{ar^0} = r$

$\frac{ar^2}{ar^1} = r$

$\frac{ar^n}{ar^{n-1}} = r$

common ratio.

Arithmetric:

$a, a+d, a+2d, a+3d, \dots, a+nd$

$a+d-a = d$

$a+2d-a-d = d$

$a+3d-a-2d = d$

common difference

※ Identify their types:

i) $\{S_n\} = -1+4n$

$\Rightarrow n=1 \Rightarrow -1, 3, 7, 11$

$\quad 4 \quad 4 \quad 4$ → common difference

$\boxed{-1+4n}$

ii) $\{b_n\} = (-1)^n$

$\Rightarrow$

$(-1)^0 = 1$

$(-1)^1 = -1$

$(-1)^2 = 1$

$\frac{-1}{1} = -1$

$\frac{1}{-1} = -1$

common ~~ration~~ ratio

ar^n

$= 1\,(-1)^n = (-1)^n$

Summation

Lower limit

Upper limit

$\boxed{1} + 2 + 3 + 4 + \cdots + \boxed{n}$

$$\sum_{i = \text{lower limit}}^{\text{upper limit}} \text{expression of } i$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

Midterm

\L-16 / 26.07.2022/

Stephen P Boyd

Summation

upper limit $\to$ + + +

$$\sum_{i = \text{Lower limit}}^{\text{upper limit}} 'i'$$

Upper limit $\to$ x x x

$$\prod_{i = \text{Lower limit}}^{\text{Upper limit}} 'i'$$

$$ar^0 + ar^1 + ar^2 + \dots + ar^n$$

$$\sum_{i=0}^{n} ar^i = \frac{ar^{n+1} - a}{r-1} \longrightarrow r \neq 1$$

$\Rightarrow$ Basic step:

$$n = 0$$

$$L.H.S. = ar^0 = a$$

$$R.H.S. = \frac{ar^{n+1} - a}{r-1}$$

$$= \frac{ar^1 - a}{r-1}$$

$$= \frac{a(r-1)}{r-1} = a = R.H.S.$$

Inductive Step:

Assume for $n=k$, statement is True

$\underbrace{P(1) \wedge \dots \wedge P(k-1) \wedge}_{\text{Strong induction}}$ $P(k)$ is true

$$\sum_{i=0}^{k} ar^i = a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

For,

$n = k+1$

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+1+1} - a}{r-1}$$

$$= \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + (r-1)\,ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1+1} - ar^{k+1}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1} = \text{R.H.S.}$$