$$= c_1 e^t - (1 + D + D^2 + \cdots)\left(1 - \frac{1}{2}t^2\right)$$

$$= c_1 e^t - \left(1 - \frac{1}{2}t^2 - t - 1\right)$$

$$y = c_1 e^t + \frac{1}{2}t^2 + t$$

~~from (ii)~~, $\frac{dy}{dt} = c_1 e^t + t + 1$

from (i),

$$Dx + x + Dy - y = e^t$$

$$\frac{1}{2}e^t - 1 + t + \frac{1}{2}e^t - t + \frac{1}{2}t^2 + c_1 e^t + t + 1 - c_1 e^t - \frac{1}{2}t^2 - t = e^t$$

from (ii),

$$(D^2 + D + 1)x + (D^2 - D + 1)y = t^2$$

$$D^2x + Dx + x + D^2y - Dy + y = t^2$$

$$\Rightarrow \frac{1}{2}e^t - t + \frac{1}{2}t^2 + \frac{1}{2}e^t - 1 + t + \frac{1}{2}e^t + 1$$

$$\underline{c_1 e^t} + \frac{1}{2}t^2 + t - \underline{c_1 e^t} - t - 1 + c_1 e^t + 1 = t^2$$

$$\Rightarrow e^t\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + c_1 - c_1 + c_1\right) + \underline{t^2\left(\frac{1}{2} + \frac{1}{2}\right)} + t(-1 + 1 + 1 - 1) = \underline{t^2}$$

$$\frac{3}{2} + c_1 = 0$$

$$c_1 = -\frac{3}{2}$$

$$\therefore y = \frac{1}{2}t^2 + t - \frac{3}{2}e^t$$

Ans

Example-6.7

$$\frac{d^2x}{dt^2} - 3x - 4y = 0$$

$$\frac{d^2y}{dt^2} + x + y = 0$$

$\Rightarrow$

$$D^2x - 3x - 4y = 0 \quad \cdots \text{(i)}$$

$$D^2y + x + y = 0 \quad \cdots \text{(ii)}$$

$\Rightarrow$

$$(D^2-3)x - 4y = 0 \quad \cdots \text{(i)}$$

$$x + (D^2+1)y = 0 \quad \cdots \text{(ii)}$$

Operating (i) by (D^2+1) and multiplying (ii) by 4,

$$(D^2-3)(D^2+1)x - 4(D^2+1)y = 0$$

$$4x + 4(D^2+1)y = 0$$

$$(D^4 + D^2 - 3D^2 - 3 + 4)x = 0$$

$$(D^4 - 2D^2 + 1)x = 0$$

$\therefore$ A.E. $\Rightarrow$ ~~m^4 $2D^2+1$~~

$$m^4 - 2m^2 + 1 = 0$$

$$m = -1, -1, 1, 1$$

$$\therefore x(t) = c_1e^t + c_2te^t + c_3e^{-t} + c_4te^{-t}$$

$$\therefore \frac{dx}{dt} = c_1 e^t + c_2 t e^t + c_2 e^t - c_3 e^{-t} - c_4 t e^{-t} + c_4 e^{-t}$$

From (i),

$$\frac{d^2x}{dt^2} = c_1 e^t + c_2 t e^t + c_2 e^t + c_2 e^t + c_3 e^{-t} + c_4 t e^{-t} - c_4 e^{-t} - c_4 e^{-t}$$

$$= (c_1 + c_2 t + c_2 + c_2) e^t + (c_3 + c_4 t - c_4 - c_4) e^{-t}$$

$$= (c_1 + c_2 t + 2c_2) e^t + (c_3 + c_4 t - 2c_4) e^{-t}$$

From (i),

$$D^2x - 3x - 4y = 0$$

$$4y = D^2x - 3x$$

$$= (c_1 + c_2 t + 2c_2) e^t + (c_3 + c_4 t - 2c_4) e^{-t} - 3(c_1 + c_2 t) e^t - 3(c_3 + c_4 t) e^{-t}$$

$$= (c_1 + c_2 t + 2c_2 - 3c_1 - 3c_2 t) e^t + (c_3 + c_4 t - 2c_4 - 3c_3 - 3c_4 t) e^{-t}$$

$$= (2c_2 - 2c_1 - 2c_2 t) e^t + (-2c_3 - 2c_4 - 2c_4 t) e^{-t}$$

$$\therefore y = \frac{1}{2}(c_2 - c_1 - c_2 t)e^t + \frac{1}{2}(-c_3 - c_4 - c_4 t)e^{-t}$$

$$= \frac{1}{2}(c_2 - c_1 - c_2 t)e^t - \frac{1}{2}(c_3 + c_4 + c_4 t)e^{-t}$$

Ans

Example - 6.8/

$$\frac{dx}{dt} + 4x + 3y = t \qquad \Rightarrow \qquad (D+4)x + 3y = t \quad \cdots (i)$$

$$\frac{dy}{dt} + 2x + 5y = e^t \qquad\qquad 2x + (D+5)y = e^t \quad \cdots (ii)$$

Operating (i) by $(D+5)$ and multiplying (ii) by 3,

$$(D+4)(D+5)x + 3(D+5)y = (D+5)t$$

$$6x + 3(D+5)y = 3e^t$$

$$(-) \qquad (-) \qquad (-)$$

$$(D^2 + 5D + 4D + 20 \cancel{+6})x = 1 + 5t - 3e^t$$

$$\Rightarrow (D^2 + 9D + 14)x = 5t + 1 - 3e^t$$

A.E $\Rightarrow$

$$m^2 + 9m + 14 = 0$$

$$\cancel{m = -\frac{9}{2} \pm \frac{\sqrt{23}}{2}\,i}$$

$$m = -2, -7$$

$\therefore \cancel{x} =$

$\therefore x_c = c_1 e^{-2t} + c_2 e^{-7t}$

$\therefore x_p = \dfrac{1}{D^2+9D+14}(5t+1) - \dfrac{1}{D^2+9D+14} 3e^t$

$= \dfrac{1}{14}\left(1+\dfrac{D^2+9D}{14}\right)^{-1}(5t+1) - \dfrac{1}{1+9+14} 3e^t$

$= \dfrac{1}{14}\left(1-\dfrac{D^2+9D}{14}+\cdots\right)(5t+1) - \dfrac{3}{24} e^t$

$= \dfrac{1}{14}\left(5t+1-\dfrac{45}{14}\right) - \dfrac{1}{8} e^t$

$= \cancel{\dfrac{5}{14}t + \dfrac{1}{14}} -$

$= \dfrac{5}{14}t - \dfrac{31}{196} - \dfrac{1}{8} e^t$

$\therefore x = c_1 e^{-2t} + c_2 e^{-7t} + \dfrac{5}{14}t - \dfrac{1}{8}e^t - \dfrac{31}{196}$

Ans

$\therefore \dfrac{dx}{dt} = -2c_1 e^{-2t} - 7c_2 e^{-7t} + \dfrac{5}{14} - \dfrac{1}{8}e^t$

From (i),

$Dx + 4x + 3y = t$

$3y = t - Dx - 4x$

$$= t + 2c_1 e^{-2t} + 7c_2 e^{-7t} - \frac{5}{14} + \frac{1}{8} e^{t} - 4c_1 e^{-2t} - 4c_2 e^{-7t}$$

$$- \frac{20}{14} t + \frac{4}{8} e^{t} + \frac{124}{196}$$

$$= -2c_1 e^{-2t} + 3c_2 e^{-7t} - \frac{3}{7} t + \frac{27}{98} + \frac{5}{8} e^{t}$$

$$\therefore y = -\frac{2}{3} c_1 e^{-2t} + c_2 e^{-7t} - \frac{1}{7} t + \frac{9}{98} + \frac{5}{24} e^{t}$$

Ans

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3)

$$(D^2-2)x - 3y = e^{2t} \quad \cdots (i)$$

$$x + (D^2+2)y = 0 \quad \cdots (ii)$$

Operating (i) by (D^2+2) and multiplying (ii) by 3,

$$(D^2-2)(D^2+2)x - 3(D^2+2)y = (D^2+2)e^{2t}$$

$$3x + 3(D^2+2)y = 0$$

$$(D^4 - 4 + 3)x = 4e^{2t} + 2e^{2t}$$

$$(D^4 - 1)x = 6e^{2t}$$

A.E. $\Rightarrow$ $m^4 - 1 = 0$

$m^4 = 1$

$\therefore m = \pm 1, \pm 1$

$$\therefore x_c = c_1 e^{t} + c_2 t e^{t} + c_3 e^{-t} + c_4 t e^{-t}$$

$$= (c_1 + c_2 t)e^{t} + (c_3 + c_4 t)e^{-t}$$

$\therefore x_p = \frac{1}{D^4-1} 6e^{2t}$

$= \frac{1}{16-1} 6e^{2t}$

$= \frac{6}{15} e^{2t}$

$= \frac{2}{5} e^{2t}$

$\therefore x = (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t} + \frac{2}{5}e^{2t}$

$\frac{dx}{dt} = (c_1 + c_2 t + c_2)e^t + (-c_3 + c_4 - c_4 t)e^{-t}$

$+ \frac{4}{5}e^{2t}$

$\therefore \frac{d^2x}{dt^2} = (c_1 + c_2 + c_2 t + c_2)e^t + (c_3 - c_4 - c_4 + c_4 t)e^{-t}$

$+ \frac{8}{5}e^{2t}$

$\therefore$ from (i),

$D^2x - 2x - 3y = e^{2t}$

$\Rightarrow$ e^{2t} =

$\Rightarrow -3y = e^{2t} - D^2x - 2x$

$\Rightarrow 3y = D^2x + 2x - e^{2t}$

$= (c_1 + 2c_2 + c_2 t)e^t + (c_3 - 2c_4 + c_4 t)e^{-t} + \frac{8}{5}e^{2t}$

$+ (c_1 + c_2 t)e^t + (c_3 + c_4 t)e^{-t} + \frac{2}{5}e^{2t} - e^{2t}$

$= (2c_1 + 2c_2 + 2c_2 t)e^t + (2c_3 - 2c_4 + 2c_4 t)e^{-t} + e^{2t}$

$\therefore y = \frac{1}{3}[\quad]$

6)

$(D+1)x + (2D+7)y = e^t + 2 \cdots (i)$

$-2x + (D+3)y = e^t - 1 \cdots (ii)$

Multiplying (i) by -2 and operating (ii) by $(D+1)$,

$-2(D+1)x - 2(2D+7)y = -2e^t - 4$

$-2(D+1)x + (D+3)(D+1)y = (D+1)(e^t - 1)$

$(+) \qquad (-) \qquad (-)$

$(-4D - 14 - D^2 - D - 3D - 3)y = -2e^t - 4 - e^t - e^t + 1$

$(-D^2 - 8D - 17)y = -4e^t - 3$

$\Rightarrow (D^2 + 8D + 17)y = 4e^t + 3 \cdots (iii)$

A.E. $\Rightarrow$

$m^2 + 8m + 17 = 0$

$\therefore m = -4 \pm i$

$\therefore y_c = e^{-4t}[A\cos t + B\sin t]$

$\therefore y_p = \dfrac{1}{D^2 + 8D + 17} 4e^t + \dfrac{1}{D^2 + 8D + 17}(3)$

$= \dfrac{1}{1+8+17} 4e^t + \dfrac{1}{17} \cdot 3$

$= \dfrac{4}{26} e^t + \dfrac{3}{17} = \dfrac{2}{13} e^t + \dfrac{3}{17}$

$\therefore y = e^{-4t}[A\cos t + B\sin t] + \dfrac{2}{13} e^t + \dfrac{3}{17}$

$$\frac{dy}{dt} = e^{-4t}[-A\sin t + B\cos t] - 4e^{-4t}[A\cos t + B\sin t] + \frac{2}{13}e^{t}$$

from (ii),

$$-2x + Dy + 3y = e^{t} - 1$$

$$\Rightarrow 2x = Dy + 3y - e^{t} + 1$$

$$= e^{-4t}[B\cos t - A\sin t] - 4e^{-4t}[A\cos t + B\sin t] + \frac{2}{13}e^{t}$$

$$+ 3e^{-4t}[A\cos t + B\sin t] + \frac{2}{13}e^{t} + \frac{3}{17} - e^{t} + 1$$

$$= [B\cos t - A\sin t - 4A\cos t - 4B\sin t + 3A\cos t + 3B\sin t]\, e^{-4t}$$

$$- \frac{9}{13} + \frac{20}{17}$$

$$= [(-A - 4B + 3B)\sin t + (B - 4A + 3A)\cos t]\, e^{-4t} - \frac{9}{13} + \frac{20}{17}$$

$$= [(B-A)\cos t - (A+B)\sin t]\, e^{-4t} - \frac{9}{13} + \frac{20}{17}$$

7]

$$\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t \quad \Rightarrow \quad Dx + (D-2)y = 2\cos t - 7\sin t \cdots (i)$$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t \qquad (D+2)x - Dy = 4\cos t - 3\sin t \cdots (ii)$$

Operating (i) by D and operating (ii) by $(D-2)$

$$D^2x + D(D-2)y = D(2\cos t - 7\sin t)$$

$$(D+2)(D-2)x - D(D-2)y = (D-2)(4\cos t - 3\sin t)$$

$$= -2\sin t - 7\cos t - 4\sin t - 3\cos t - 8\cos t + 6\sin t$$

$$(D^2 + D^2 - 4)x$$

$$(2D^2 - 4)x = -18\cos t$$

$$\therefore (D^2 - 2)x = -9\cos t$$

A.E. $\Rightarrow$ $m^2 - 2 = 0$

$m = \pm 2$

$\therefore x_c = c_1 e^{2t} + c_2 e^{-2t}$

$\therefore x_p = \frac{1}{D^2 - 2}(-9\cos t)$

$= \frac{1}{-1-3}(-9\cos t)$

$= \frac{9}{4}\cos t$

$$\therefore x = c_1 e^{2t} + c_2 e^{-2t} + \frac{9}{4}\cos t$$

$$\therefore \frac{dx}{dt} = 2c_1 e^{2t} - 2c_2 e^{-2t} - \frac{9}{4}\sin t$$

From (ii)

$$Dx+2x-Dy = 4\cos t - 3\sin t$$

$$Dy = Dx+2x - 4\cos t + 3\sin t$$

$$= 2c_1e^{2t} - 2c_2e^{-2t} - \frac{2}{4}\sin t + 2c_1e^{2t} + 2c_2e^{-2t} + \frac{2}{2}\cos t$$
$$- 4\cos t + 3\sin t$$

$$Dy = 4c_1e^{2t} + \frac{3}{4}\sin t + \frac{1}{2}\cos t$$

$$\therefore y = 2c_1e^{2t} - \frac{3}{4}\cos t + \frac{1}{2}\sin t$$

Ans

From Zill's Book

Exercise. 4.9

7|

$$\frac{d^2x}{dt^2} = 4y + e^t \quad \Rightarrow \quad D^2x - 4y = e^t \quad \cdots (i)$$

$$-4x + D^2y = -e^t \quad \cdots (ii)$$

$$\frac{d^2y}{dt^2} = 4x - e^t$$

Operating (i) by D^2 and multiplying (ii) by 4,

$$D^4x - 4D^2y = D^2(e^t)$$
$$-16x + 4D^2y = -4e^t$$

$$(D^4-16)x = e^t - 4e^t$$

$$\therefore (D^4-16)x = -3e^t$$

$\therefore AE \Rightarrow$

$$m^4-16=0$$

$$(m^2+4)(m^2-4)=0$$

$$\therefore m = \pm 2, \pm 2i$$

$$\therefore x_c = c_1 e^{2t} + c_2 e^{-2t} + A\cos 2t + B\sin 2t$$

$$\therefore x_p = \frac{1}{D^4-16}(-3e^t)$$

$$= \frac{1}{1-16}(-3e^t)$$

$$= \frac{3}{15}e^t$$

$$= \frac{1}{5}e^t$$

$$\therefore x = c_1 e^{2t} + c_2 e^{-2t} + A\cos 2t + B\sin 2t + \frac{1}{5}e^t$$

$$\therefore \frac{dx}{dt} = 2c_1 e^{2t} - 2c_2 e^{-2t} - 2A\sin 2t + 2B\cos 2t + \frac{1}{5}e^t$$

$$\therefore \frac{d^2x}{dt^2} = 4c_1 e^{2t} + 4c_2 e^{-2t} - 4A\cos 2t - 4B\sin 2t + \frac{1}{5}e^t$$

From (i),

$$D^2x - 4y = e^t$$

$$\therefore 4y = D^2x - e^t$$

$$= 4c_1 e^{2t} + 4c_2 e^{-2t} - 4A\cos 2t - 4B\sin 2t + \frac{1}{5}e^t - e^t$$

$$\therefore y = c_1 e^{2t} + c_2 e^{-2t} - A\cos 2t - B\sin 2t - \frac{1}{5}e^t$$

9|

$$Dx + D^2y = e^{3t} \quad \cdots (i)$$

$$(D+1)x + (D-1)y = 4e^{3t} \quad \cdots (ii)$$

Operating (i) by $(D-1)$ and Operating (ii) by D^2,

$$D(D-1)x + D^2(D-1)y = (D-1)e^{3t}$$

$$D^2(D+1)x + D^2(D-1)y = D^2(4e^{3t})$$

$$(-) \quad (-) \quad (-)$$

$$(D^2-D-D^3-D^2)x = 3e^{3t} - e^{3t} - 36e^{3t}$$

$$\therefore (D^3+D)x = 34e^{3t}$$

A.E. $\Rightarrow$

$$m^3+m=0$$

$$m(m^2+1)=0$$

$$\therefore m = 0, \pm i$$

$$\therefore x_c = c_1 + A\cos t + B\sin t$$

$$\therefore x_p = \frac{1}{D^3+D} 34e^{3t}$$

$$= \frac{34}{27+3} e^{3t}$$

$$= \frac{17}{15} e^{3t}$$

$$\therefore x = c_1 + A\cos t + B\sin t + \frac{17}{15} e^{3t} \quad \text{A}$$

$$\therefore \frac{dx}{dt} = -A\sin t + B\cos t + \frac{17}{5} e^{3t}$$

From (i),

$$D^2y = e^{3t} - Dx$$

$$= e^{3t} + A\sin t - B\cos t - \frac{17}{5} e^{3t}$$

$$= A\sin t - B\cos t - \frac{12}{5} e^{3t}$$

$$\therefore Dy = -A\cos t - B\sin t - \frac{12}{15}e^{3t} + C_2$$

$$\therefore y = -A\sin t + B\cos t - \frac{4}{15}e^{3t} + C_2 t + C_3$$

$$\cancel{\frac{dy}{dt} = -A\cos t - B\sin t - \frac{4}{5}e^{3t}} + C_2$$

from (i)

$$Dx + x + Dy - y = 4e^{3t}$$

$$-A\sin t + B\cos t + \frac{17}{5}e^{3t} + C_1 + A\cos t + B\sin t + \frac{17}{15}e^{3t}$$

$$-A\cos t - B\sin t - \frac{4}{5}e^{3t} + C_2 + A\sin t - B\cos t + \frac{4}{15}e^{3t}$$

$$-C_2 t - C_3 - 4e^{3t} = 0$$

$$\cancel{(-A+B-B+A)\sin t} + \cancel{(B+A-A-B)\cos t}$$

$$C_1 + C_2 - C_2 t - C_3 = 0$$

$$\cancel{C_1 + (1-t)C_2 - C_3 = 0}$$

$$C_1 + C_2 = C_2 t + C_3$$

$$C_3 = 0$$

$$\therefore y = B\cos t - A\sin t - \frac{4}{15}e^{3t} + C_1 + \cancel{C_2}$$

$$\cancel{\frac{dy}{dt} = -B\sin t - A\cos t - \frac{4}{5}e^{3t}}$$

$$\cancel{\frac{d^2y}{dt^2} = -B\cos t + A\sin t - \frac{12}{5}e^{3t}}$$

~~From (i),~~

~~$Dx + D^2y = e^{3t}$~~

~~$\Rightarrow -A\sin t + B\cos t + \frac{17}{5}e^{3t}$~~ $+ A\sin t - B\cos t - \frac{12}{5}e^{3t} - e^{3t} = 0$

~~10)~~

~~$D^2x - D^2y = e^{3t}$~~

~~$(D+1)$~~

10)

$$D^2x - Dy = t \quad \text{--- (i)}$$

$$(D+3)x + (D+3)y = 2 \quad \text{--- (ii)}$$

Operating (i) by $(D+3)$ and operating (ii) by D,

$$D^2(D+3)x - D(D+3)y = (D+3)t$$

$$D(D+3)x + D(D+3)y = D\cdot 2$$

$$(D^3 + 3D^2 + D^2 + 3D)x = 1 + 3t + 0$$

$$\therefore \; (D^3 + 4D^2 + 3D)x = 3t + 1$$

$$A.E \Rightarrow \quad m^3 + 4m^2 + 3m = 0$$

$$m(m^2 + 4m + 3) = 0$$

$$m = 0, -1, -3$$

$\therefore x_c = c_1 + c_2 e^{-t} + c_3 e^{-3t}$

$\therefore x_p = \dfrac{1}{D^3 + 4D^2 + 3D}(3t+1)$

$= \dfrac{1}{3D}\left(1 + \dfrac{D^2}{3} + \dfrac{4D}{3}\right)^{-1}(3t+1)$

$= \dfrac{1}{3D}\left(1 - \dfrac{D^2}{3} - \dfrac{4D}{3} + \dots\right)(3t+1)$

$= \dfrac{1}{3D}\left(3t + 1 - \dfrac{12}{3}\right)$

$= \dfrac{1}{3}\cdot\left(\dfrac{3}{2}t^2 - 3t\right)$

$= \dfrac{1}{2}t^2 - t$

$\therefore x = c_1 + c_2 e^{-t} + c_3 e^{-3t} + \dfrac{1}{2}t^2 - t$

$\therefore Dx = -c_2 e^{-t} - 3c_3 e^{-3t} + t - 1$

$\therefore D^2x = c_2 e^{-t} + 9c_3 e^{-3t} + 1$

From (1),

$Dy = D^2x - t$

$= c_2 e^{-t} + 9c_3 e^{-3t} + 1 - t$

$\therefore y = -c_2 e^{-t} + 3c_3 e^{-3t} + t - \dfrac{1}{2}t^2 + c_4$

from (ii)

$$Dx + 3x + Dy + 3y = 2$$

$$-c_2e^{-t} - 3c_3e^{-3t} + t - 1 + 3c_1 + 3c_2e^{-t} + 3c_3e^{-3t} + \frac{3}{2}t^2 - 3t$$

$$+ c_2e^{-t} + 9c_3e^{-3t} - t + 1 - 3c_2e^{t} + 9c_3e^{-3t} + 3t - \frac{3}{2}t^2$$

$$+ 3c_4 = 2$$

$$3c_1 + 9c_3e^{-3t} + 9c_3e^{-3t} + 3c_4 = 2$$

$$c_4 = \frac{2 - 3c_1 - 18c_3e^{-3t}}{3}$$

$$= \frac{2}{3} - c_1 - 6c_3e^{-3t}$$

$$\therefore y = -c_2e^{-t} + 3c_3e^{-3t} + t - \frac{1}{2}t^2 + \frac{2}{3} - c_1 - 6c_3e^{-3t}$$

$$= \frac{2}{3} - c_1 - c_2e^{-t} - 3c_3e^{-3t} + t - \frac{1}{2}t^2$$

Ans

13)

$$2\frac{dx}{dt} - 5x + \frac{dy}{dt} = e^t \quad \Rightarrow \quad (2D-5)x + Dy = e^t$$

$$\frac{dx}{dt} - x + \frac{dy}{dt} = 5e^t \qquad (D-1)x + Dy = 5e^t$$

$$(-) \qquad (-) \qquad (-)$$

$$(2D-5-D+1)x = e^t - 5e^t$$

$$(D-4)x = -4e^t$$

A.E $\Rightarrow$

$$m-4=0$$

$$m=4$$

$$\therefore x_c = c_1 e^{4t}$$

$$\therefore x_p = \frac{1}{D-4}(-4e^t)$$

$$= \frac{-4}{1-4}e^t$$

$$= \frac{4}{3}e^t$$

$$\therefore x = c_1 e^{4t} + \frac{4}{3}e^t$$

$$\therefore \frac{dx}{dt} = 4c_1 e^{4t} + \frac{4}{3}e^t$$

From (ii),

$$Dx - x + Dy = 5e^t$$

$$Dy = 5e^t - Dx + x$$

$$= 5e^t - 4c_1e^{4t} - \frac{4}{3}e^t + c_1e^{4t} + \frac{4}{3}e^t$$

$$= 5e^t - 3c_1e^{4t}$$

$\therefore y = 5e^{t} - \frac{3}{4} c_1 e^{4t} + c_2$

Ans

16|

$D^2 x - 2(D^2+D)y = \sin t \quad \cdots (i)$

$x + Dy = 0 \quad \cdots (ii)$

~~Open~~

multiplying (i) by 1 and operating (ii) by D^2

$D^2 x - 2(D^2+D)y = \sin t$

$D^2 x + D^3 y = 0$

$(-) \quad (-) \quad (-)$

$(-2D^2 - 2D - D^3)y = \sin t$

$\Rightarrow (D^3 + 2D^2 + 2D)y = -\sin t$

$\therefore$ A.E. $\Rightarrow m^3 + 2m^2 + 2m = 0$

$m(m^2 + 2m + 2) = 0$

$\therefore m = 0, -1 \pm i$

$y_p = \frac{1}{D^3 + 2D^2 + 2D}(-\sin t)$

$= \frac{1}{-D - 2 + 2D}(-\sin t)$

$= \frac{1}{D-2}(-\sin t)$

$= \frac{D+2}{D^2-4}(-\sin t)$

$= \frac{D+2}{-1-4}(-\sin t)$

$\frac{1}{5}(\cos t + 2\sin t)$

$\therefore y_c = c_1 + e^{-t}[A\cos t + B\sin t]$

$y = c_1 + e^{-t}[A\cos t + B\sin t] + \frac{1}{5}\cos t + \frac{2}{5}\sin t$

$\Rightarrow Dy = e^{-t}[-A\sin t + B\cos t] - e^{-t}[A\cos t + B\sin t]$

$- \frac{1}{5}\sin t + \frac{2}{5}\cos t$

From (ii),

$$x + Dy = 0$$

$$\Rightarrow x = -Dy$$

$$= e^{t}[A\cos t + B\sin t] - e^{t}[B\cos t - A\sin t] + \frac{1}{5}\sin t - \frac{2}{5}\cos t$$

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$$Dx + z = e^{t} \quad \cdots (i)$$

$$(D-1)x + Dy + Dz = 0 \quad \cdots (ii)$$

$$x + 2y + Dz = e^{t} \quad \cdots (iii)$$

From (i),

$$Dx + z = e^{t}$$

$$\Rightarrow z = e^{t} - Dx$$

$$\Rightarrow Dz = e^{t} - D^2x$$

(ii) $\Rightarrow$ $(D-1)x + Dy + e^{t} - D^2x = 0$ $\Rightarrow$ $(-D^2+D-1)x + Dy = -e^{t} \quad \cdots (iv)$

(iii) $\Rightarrow$ $x + 2y + e^{t} - D^2x = e^{t}$ $\quad$ $(-D^2+1)x + 2y = 0 \quad \cdots (v)$

multiplying (iv) by 2 and operating (v) by D,

$$2(-D^2+D-1)x + 2Dy = -2e^{t}$$

$$D(-D^2+1)x + 2Dy = 0$$

$$(-) \qquad (-) \qquad (-)$$

$$(-2D^2+2D-2+D^3-D)x = -2e^{t}$$

$\therefore (D^3-2D^2+D-2)x = -2e^t$

A.E. $\Rightarrow$

$$m^3-2m^2+m-2=0$$

$$\therefore m = 2, \pm i$$

$$\therefore x_c = c_1e^{2t} + A\cos t + B\sin t$$

$$\therefore x_p = \frac{1}{D^3-2D^2+D-2}(-2e^t)$$

$$= \frac{1}{1-2+1-2}(-2e^t)$$

$$= e^t$$

$$\therefore x = c_1e^{2t} + A\cos t + B\sin t + e^t$$

$$\therefore Dx = 2c_1e^{2t} - A\sin t + B\cos t + e^t$$

from ①,

$$-D^2x + x + 2y = 0$$

$$\Rightarrow 2y = D^2x - x$$

$$= 4c_1e^{2t} - A\cos t - B\sin t - c_1e^{2t} + A\cos t - B\sin t + e^t - e^t$$

$$= 3c_1e^{2t} - 2A\cos t - 2B\sin t$$

$$\therefore y = \frac{3}{2}c_1e^{2t} - A\cos t - B\sin t$$

From (i),

$$z = e^t - Dx$$

$$= e^t - 2c_1e^{2t} + A\sin t - B\cos t - e^t$$

Ans

$$= -2c_1e^{2t} + A\sin t - B\cos t$$

Ans

HW ⇒ from Lecture - 16

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Example - 8/

$$y'' + (\cos x)y = 0 \quad \text{---(i)}$$

Let,
$y = \sum_{n=0}^{\infty} c_n x^n$ be the solution of (i)

$$\therefore y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting y'', y', y in (i)

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + (\cos x)\sum_{n=0}^{\infty} c_n x^n = 0$$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n \cdot \cos x = 0$

Let,

$k = n-2$	$k = n$
$n = k+2$	$n = k$
if, $n=2$, $k=0$	if, $n=0$, $k=0$

$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=0}^{\infty} c_k x^k \cos x = 0$

$\Rightarrow \sum_{k=0}^{\infty} \left[(k+2)(k+1) c_{k+2} + c_k \cos x\right] x^k = 0$

$\Rightarrow 2(2-1) c_2 x^0 + c_0 x^0 \cdot \cos 0 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} c_n x^n \cdot \cos x = 0$

$\Rightarrow 2c_2 + c_0 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} c_n x^n \cdot \cos x = 0$

Let,

$k = n-2$	$k = n$
$n = k+2$	$n = k$
if $n=3$, $k=1$	if, $n=1$, $k=1$

$\Rightarrow 2c_2 + c_0 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} c_k x^k \cdot \cos x = 0$

$= (2c_2 + c_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} + c_k \cos x\right] x^k = 0$

Equating coefficient of the light terms,

x^0: $2c_2 + c_0 = 0$

$$c_2 = -\frac{c_0}{2}$$

x^k: $(k+2)(k+1)c_{k+2} + c_k \cos x = 0$

$$c_{k+2} = -\frac{c_k \cos x}{(k+2)(k+1)} \quad ; k = 1, 2, 3, 4, 5, \ldots$$

$k = 1$:

$$c_3 = -\frac{c_1 \cos x}{3 \cdot 2} = -\frac{c_1}{6} \cos x$$

$k = 2$:

$$c_4 = -\frac{c_2 \cos x}{4 \cdot 3} = \frac{c_0}{2 \cdot 3 \cdot 4} \cos x = \frac{c_0}{24} \cos x$$

$k = 3$:

$$c_5 = -\frac{c_3 \cos x}{5 \cdot 4} = \frac{c_1}{120} \cos^2 x$$

$k = 4$:

$$c_6 = -\frac{c_4 \cos x}{6 \cdot 5} = -\frac{c_0}{720} \cos^2 x$$

$k = 5$:

$$c_7 = -\frac{c_5 \cos x}{7 \cdot 6} = -\frac{c_1}{5040} \cos^3 x$$

$\therefore$ Hence, the solution is,

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0 x^0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots$$

$$= c_0 + c_1 x - \frac{c_0}{2} x^2 - \frac{c_1}{6} \cos x \cdot x^3 + \frac{c_0}{24} \cos x \cdot x^4 + \frac{c_1}{120} x^5 \cdot \cos^2 x$$
$$- \frac{c_0}{720} x^6 \cos^2 x - \frac{c_1}{5040} x^7 \cos^3 x + \cdots$$

$$= \left(1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 \cos x - \frac{x^6}{720} \cos^2 x + \cdots\right) c_0$$
$$+ \left(x - \frac{x^3}{6} \cos x + \frac{x^5}{120} \cos^2 x - \frac{x^7}{5040} \cos^3 x + \cdots\right) c_2$$

Ans

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$$y'' - 2xy' + y = 0 \quad \cdots \text{(i)}$$

Let,

$y = \sum_{n=0}^{\infty} c_n x^n$ be the solution of (i)

$$\therefore y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\therefore y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substitute y'', y', y in ①

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2(2-1) c_2 x^0 + c_0 x^0 + \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=\cancel{0}1}^{\infty} c_n x^n = 0$$

Let,

$k = n-2$	$k = n$	$k = n$
$n = k+2$	$n = k$	$n = k$
if, $n = 3, k = 1$	if, $n = 1, k = 1$	if, $n = \cancel{0}1$
		$k = \cancel{0}1$

$$\Rightarrow (2c_2 + c_0) x^0 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$\Rightarrow (2c_2 + c_0) x^0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} - 2k c_k + c_k \right] x^k = 0$$

Equating coefficient of the light terms,

x^0: $2c_2 + c_0 = 0$

$\therefore c_2 = -\frac{c_0}{2}$

x^k:

$(k+2)(k+1) c_{k+2} - 2k c_k + c_k = 0$

$$\Rightarrow c_{k+2} = \frac{2k c_k - c_k}{(k+2)(k+1)} = \frac{(2k-1) c_k}{(k+2)(k+1)}$$

$; k = 1, 2, 3, 4, 5, \dots$

$\underline{k=1:}$,

$$c_3 = \frac{1\cdot c_1}{3\cdot 2} = \frac{c_1}{6}$$

$\underline{k=2:}$,

$$c_4 = \frac{3c_2}{4\cdot 3} = \frac{1}{4}c_2 = -\frac{1}{8}c_0$$

$\underline{k=3:}$,

$$c_5 = \frac{5c_3}{5\cdot 4} = \frac{1}{24}c_1$$

$\underline{k=4:}$,

$$c_6 = \frac{7c_4}{6\cdot 5} = -\frac{7}{240}c_0$$

$\underline{k=5:}$,

$$c_7 = \frac{9c_5}{7\cdot 6} = \frac{1}{112}c_1$$

Hence, the solution is

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7$$

$$= c_0 + c_1x - \frac{x^2}{2}c_0 + \frac{x^3}{6}c_1 - \frac{x^4}{8}c_0 + \frac{x^5}{24}c_1 - \frac{7x^6}{240}c_0$$

$$+ \frac{x^7}{112}c_1$$

$$= \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{7x^6}{240} - \cdots\right)c_0 + \left(x + \frac{x^3}{6} + \frac{x^5}{24} + \frac{x^7}{112} + \cdots\right)c_1$$

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$(x-1)y'' + y' = 0 \quad \cdots \text{(i)}$

Let,

$y = \sum_{n=0}^{\infty} c_n x^n$ be the solution of (i)

$\therefore y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$\therefore y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Substituting y'', y', y in (i),

$$(x-1)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^{n-1} = 0$$

$$\Rightarrow -2(2-1)c_2 x^0 + c_1 x^0 + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} - \sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=2}^{\infty} n c_n x^{n-1} = 0$$

Let,

$k = n-1$	$k = n-2$	$k = n-1$
$n = k+1$	$n = k+2$	$n = k+1$
if $n = 2$	if $n = 3$	if, $n = 2$
$k = 1$	$k = 1$	$k = 1$

$$\Rightarrow (c_1 - 2c_2)x^0 + \sum_{k=1}^{\infty} (k+1)(k) c_{k+1} x^k - \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} (k+1) c_{k+1} x^k = 0$$

$$= (c_1 - 2c_2)x^0 + \sum_{k=1}^{\infty}\left[k(k+1)c_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1}\right]x^k = 0$$

Equating coefficient of light terms,

x^0: $c_1 - 2c_2 = 0$

$\therefore c_2 = \frac{1}{2}c_1$

x^k:

$$k(k+1)c_{k+1} - (k+2)(k+1)c_{k+2} + (k+1)c_{k+1} = 0$$

$$k \cdot c_{k+1} - (k+2)c_{k+2} + c_{k+1} = 0$$

$$\Rightarrow c_{k+2} = \frac{k \cdot c_{k+1} + c_{k+1}}{k+2}$$

$$= \frac{(k+1)}{k+2} c_{k+1} \quad ; k = 1, 2, 3 \ldots$$

$\underline{k=1}$

$$c_3 = \frac{2}{3}c_2 = \frac{1}{3}c_1$$

$\underline{k=2}$,

$$c_4 = \frac{3}{4}c_3 = \frac{1}{4}c_1$$

$\underline{k=3}$,

$$c_5 = \frac{4}{5}c_4 = \frac{1}{5}c_1$$

$\underline{k=4}$,

$$c_6 = \frac{5}{6}c_5 = \frac{1}{6}c_1$$

$$\therefore y = \sum_{n=0}^{\infty} c_n x^n$$

$$= c_0x^0 + c_1x^1 + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6$$

$$= c_0 + c_1x + \frac{x^2}{2}c_1 + \frac{x^3}{3}c_1 + \frac{x^4}{4}c_1 + \frac{x^5}{5}c_1 + \frac{x^6}{6}c_1$$

$$= c_0 + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \cdots\right)c_1$$

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$$(x^2+2)y'' + 3xy' - y = 0 \quad \cdots (1)$$

Let,

$y = \sum_{n=0}^{\infty} c_n x^n$ be the solution of (1)

$$\Rightarrow y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$\Rightarrow y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

Substituting y'', y', y in (1)

$$(x^2+2)\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 3x\sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 3n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\Rightarrow 2\cdot 2(2-1) c_2 x^0 - c_0 x^0 + \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=3}^{\infty} 2n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 3n c_n x^n$$

$$- \sum_{n=1}^{\infty} c_n x^n = 0$$

Let,

$k = n$	$k = n-2$	$k = n$
$n = k$	$n = k+2$	$n = k$
if, $n = 2$	if, $n = 3$	if, $n = 1$
$k = 2$	$k = 1$	$k = 1$

$$\Rightarrow (4c_2 - c_0)x^0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=1}^{\infty} 2(k+2)(k+1)c_{k+2} x^k + \sum_{k=1}^{\infty} (3kc_k - c_k)x^k = 0$$

$$\Rightarrow (4c_2 - c_0)x^0 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=1}^{\infty} \left[2(k+2)(k+1)c_{k+2} + 3kc_k - c_k\right] x^k = 0$$

$$\Rightarrow (4c_2 - c_0)x^0 + \left[2(1+2)(1+1)c_3 + 3 \cdot 1 \cdot c_1 - c_1\right]x^1 + \sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=2}^{\infty} \left[2(k+2)(k+1)c_{k+2} + 3kc_k - c_k\right] x^k = 0$$

$$\Rightarrow (4c_2 - c_0)x^0 + (12c_3 + 2c_1)x + \sum_{k=2}^{\infty} \left[k(k+1)c_k + 2(k+2)(k+1)c_{k+2} + 3kc_k - c_k\right] x^k = 0$$

Equating coefficient of the light term,

x^0: $4c_2 - c_0 = 0$	x^1: $12c_3 + 2c_1 = 0$
$\therefore c_2 = \frac{1}{4}c_0$	$\therefore c_3 = -\frac{1}{6}c_1$

x^k:

$$k(k+1)c_k + 2(k+2)(k+1)c_{k+2} + 3kc_k - c_k = 0$$

$$c_{k+2} = \frac{c_k - k(k+1)c_k - 3kc_k}{2(k+2)(k+1)} = \frac{(1 - k(k+1) - 3k)}{2(k+2)(k+1)} c_k$$

$$= \frac{1 - k(k+1-3)}{2(k+2)(k+1)} c_k$$

$$= \frac{1 - k(k-2)}{2(k+2)(k+1)} c_k \quad ; k = 2, 3, 4, \ldots$$

$$\therefore c_{k+2} = \frac{1-k(k-2)}{2(k+2)(k+1)} c_k$$

$\underline{k=2}$,

$$c_4 = \frac{1-2(2-2)}{2(2+2)(2+1)} c_2 = \frac{1}{24} c_2 = \frac{1}{96} c_0$$

$\underline{k=3}$,

$$c_5 = \frac{1-3(3-2)}{2(3+2)(3+1)} c_3 = -\frac{1}{20} c_3 = \frac{1}{120} c_1$$

$\underline{k=4}$,

$$c_6 = \frac{1-4(4-2)}{2(4+2)(4+1)} c_4 = -\frac{7}{60} c_4 = -\frac{7}{5760} c_0$$

$\underline{k=5}$,

$$c_7 = \frac{1-5(5-2)}{2(5+2)(5+1)} c_5 = -\frac{1}{6} c_5 = -\frac{1}{720} c_1$$

Hence

Solution is,

$$y = \sum_{n=0}^{\alpha} c_n x^n = c_0 x^0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \dots$$

$$= c_0 + c_1 x + \frac{x^2}{4} c_0 - \frac{x^3}{6} c_1 + \frac{x^4}{96} c_0 + \frac{x^5}{120} c_1 - \frac{7x^6}{5760} c_0 - \frac{x^7}{720} c_1 - \dots$$

$$= \left(1 + \frac{x^2}{4} + \frac{x^4}{96} - \frac{7x^6}{5760} - \dots\right) c_0 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{720} \dots\right) c_1$$

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$$y'' + (\sin x)y = 0$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \sin x - \frac{x^6}{720} \sin^2 x + \cdots\right) c_0$$
$$+ \left(x - \frac{x^3}{6} \sin x + \frac{x^5}{120} \cos \sin^2 x - \frac{x^7}{5040} \sin^3 x + \cdots\right) c_1$$

H.W. ⇒ from Lecture - 18

Zill's Book - Exercise 6.3

17)

$$4xy'' + \frac{1}{2}y' + y = 0 \quad \cdots (i)$$

Let,

$y = \sum_{n=0}^{\infty} c_n x^{n+r}$ be the solution of (i)

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Substituting y'', y', y in (i)

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + \frac{1}{2}\sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 4(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} \frac{1}{2}(n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r-1} \left(4n+4r-4+\tfrac{1}{2}\right) + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\Rightarrow x^r \left[\sum_{n=0}^{\infty} \left(4n+4r-\tfrac{7}{2}\right)(n+r)\, c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right] = 0$$

$$\Rightarrow r\left(4r-\tfrac{7}{2}\right) c_0 x^{-1} + \sum_{n=1}^{\infty} \left(4n+4r-\tfrac{7}{2}\right)(n+r)\, c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

Let,

$k = n-1$	$k = n$
$n = k+1$	$n = k$
if, $n = 1$	if $n = 0$
$k = 0$	$k = 0$

$$\Rightarrow r\left(4r-\tfrac{7}{2}\right) c_0 x^{-1} + \sum_{k=0}^{\infty} \left(4k+4+4r-\tfrac{7}{2}\right)(k+r+1)\, c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\Rightarrow r\left(4r-\tfrac{7}{2}\right) c_0 x^{-1} + \sum_{k=0}^{\infty} \left[\left(4k+4r+\tfrac{1}{2}\right)(k+r+1)\, c_{k+1} + c_k\right] x^k = 0$$

Equating coefficient of the light terms,

x^{-1}:

$r\left(4r-\frac{7}{2}\right) c_0 = 0$

$r\left(4r-\frac{7}{2}\right) = 0$

$\therefore r = 0, \frac{7}{8}$

x^k:

$\left(4k+4r+\frac{1}{2}\right)(k+r+1)\, c_{k+1} + c_k = 0$

$$\therefore c_{k+1} = -\frac{1}{\left(4k+4r+\frac{1}{2}\right)(k+r+1)}\, c_k$$

$k = 0, 1, 2, \ldots$

$r = 0$,

$$C_{k+1} = \frac{-C_k}{(4k+\frac{1}{2})(k+1)}$$

$\underline{k=0}$,

$C_1 = -2C_0$

$\underline{k=1}$,

$$C_2 = -\frac{1}{9}C_1 = \frac{2}{9}C_0$$

$\underline{k=2}$,

$$C_3 = -\frac{2}{51}C_2 = -\frac{4}{459}C_0$$

$r = \frac{7}{8}$

$$C_{k+1} = \frac{-C_k}{(4k+\frac{7}{2}+\frac{1}{2})(k+\frac{7}{8}+1)} = \frac{-C_k}{(4k+4)(k+\frac{15}{8})}$$

$\underline{k=0}$,

$$C_1 = -\frac{2}{15}C_0$$

$\underline{k=1}$,

$$C_2 = -\frac{1}{23}C_1 = \frac{2}{345}C_0$$

$\underline{k=2}$,

$$C_3 = -\frac{2}{93}C_2 = -\frac{4}{32085}C_0$$

$\therefore$ When $r=0$

$$Y_1 = \sum_{n=0}^{\infty} c_n x^n$$

$$= C_0x^0 + C_1x^1 + C_2x^2 + C_3x^3 + \cdots$$

$$= C_0 - 2C_0x + \frac{2}{9}C_0x^2 - \frac{4}{459}C_0x^3 + \cdots$$

When, $r = \frac{7}{8}$

$$Y_2 = x^{7/8}\left[C_0x^0 + C_1x^1 + C_2x^2 + C_3x^3 + \cdots\right]$$

$$= x^{7/8}\left[C_0 - \frac{2}{15}C_0x + \frac{2}{345}C_0x^2 - \frac{4}{32085}C_0x^3 + \cdots\right]$$

$\therefore Y = C_1Y_1 + C_2Y_2$

$$= C_1\left[C_0 - 2C_0x + \frac{2}{9}C_0x^2 - \frac{4}{459}C_0x^3 + \cdots\right] + C_2\left[C_0 - \frac{2}{15}C_0x + \frac{2}{345}C_0x^2 - \frac{4}{32085}C_0x^3 \cdots\right]$$

$$= C_3\left[1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \cdots\right] + C_4\left[1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32085}x^3 + \cdots\right]$$