

# Counting labeled graphs

## Labeled graphs

Let the number of vertices in a graph be  $n$ . We have to compute the number  $G_n$  of labeled graphs with  $n$  vertices (labeled means that the vertices are marked with the numbers from 1 to  $n$ ). The edges of the graphs are considered undirected, and loops and multiple edges are forbidden.

We consider the set of all possible edges of the graph. For each edge  $(i, j)$  we can assume that  $i < j$  (because the graph is undirected, and there are no loops). Therefore the set of all edges has the cardinality  $\binom{n}{2}$ , i.e.  $\frac{n(n-1)}{2}$ .

Since any labeled graph is uniquely determined by its edges, the number of labeled graphs with  $n$  vertices is equal to:

$$G_n = 2^{\frac{n(n-1)}{2}}$$

## Connected labeled graphs

Here, we additionally impose the restriction that the graph has to be connected.

Let's denote the required number of connected graphs with  $n$  vertices as  $C_n$ .

We will first discuss how many **disconnected** graphs exists. Then the number of connected graphs will be  $G_n$  minus the number of disconnected graphs. Even more, we will count the number of **disconnected, rooted graphs**. A rooted graph is a graph, where we emphasize one vertex by labeling it as root. Obviously we have  $n$  possibilities to root a graph with  $n$  labeled vertices, therefore we will need to divide the number of disconnected rooted graphs by  $n$  at the end to get the number of disconnected graphs.

The root vertex will appear in a connected component of size  $1, \dots, n-1$ . There are  $k \binom{n}{k} C_k G_{n-k}$  graphs such that the root vertex is in a connected component with  $k$  vertices (there are  $\binom{n}{k}$  ways to choose  $k$  vertices for the component, these are connected in one of  $C_k$  ways, the root vertex can be any of the  $k$  vertices, and the remainder  $n-k$  vertices can be connected/disconnected in any way, which gives a factor of  $G_{n-k}$ ). Therefore the number of disconnected graphs with  $n$  vertices is:

$$\frac{1}{n} \sum_{k=1}^{n-1} k \binom{n}{k} C_k G_{n-k}$$

And finally the number of connected graphs is:

$$C_n = G_n - \frac{1}{n} \sum_{k=1}^{n-1} k \binom{n}{k} C_k G_{n-k}$$

## Labeled graphs with $k$ connected components

Based on the formula from the previous section, we will learn how to count the number of labeled graphs with  $n$  vertices and  $k$  connected components.

This number can be computed using dynamic programming. We will compute  $D[i][j]$  - the number of labeled graphs with  $i$  vertices and  $j$  components - for each  $i \leq n$  and  $j \leq k$ .

Let's discuss how to compute the next element  $D[n][k]$  if we already know the previous values. We use a common approach, we take the last vertex (index  $n$ ). This vertex belongs to some component. If the size of this component be  $s$ , then there are  $\binom{n-1}{s-1}$  ways to choose such a set of vertices, and  $C_s$  ways to connect them. After removing this component from the graph we have  $n - s$  remaining vertices with  $k - 1$  connected components. Therefore we obtain the following recurrence relation:

$$D[n][k] = \sum_{s=1}^n \binom{n-1}{s-1} C_s D[n-s][k-1]$$

Contributors:

[jakobkogler](#) (89.47%) [adamant-pwn](#) (10.53%)