

# Stars and bars

Stars and bars is a mathematical technique for solving certain combinatorial problems. It occurs whenever you want to count the number of ways to group identical objects.

## Theorem

The number of ways to put  $n$  identical objects into  $k$  labeled boxes is

$$\binom{n+k-1}{n}.$$

The proof involves turning the objects into stars and separating the boxes using bars (therefore the name). E.g. we can represent with  $\star|\star\star||\star\star$  the following situation: in the first box is one object, in the second box are two objects, the third one is empty and in the last box are two objects. This is one way of dividing 5 objects into 4 boxes.

It should be pretty obvious, that every partition can be represented using  $n$  stars and  $k-1$  bars and every stars and bars permutation using  $n$  stars and  $k-1$  bars represents one partition. Therefore the number of ways to divide  $n$  identical objects into  $k$  labeled boxes is the same number as there are permutations of  $n$  stars and  $k-1$  bars. The [Binomial Coefficient](#) gives us the desired formula.

## Number of non-negative integer sums

This problem is a direct application of the theorem.

You want to count the number of solution of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

with  $x_i \geq 0$ .

Again we can represent a solution using stars and bars. E.g. the solution  $1 + 3 + 0 = 4$  for  $n = 4, k = 3$  can be represented using  $\star|\star\star\star|$ .

It is easy to see, that this is exactly the stars and bars theorem. Therefore the solution is  $\binom{n+k-1}{n}$ .

## Number of positive integer sums

A second theorem provides a nice interpretation for positive integers. Consider solutions to

$$x_1 + x_2 + \cdots + x_k = n$$

with  $x_i \geq 1$ .

We can consider  $n$  stars, but this time we can put at most *one bar* between stars, since two bars between stars would represent  $x_i = 0$ , i.e. an empty box. There are  $n - 1$  gaps between stars to place  $k - 1$  bars, so the solution is  $\binom{n-1}{k-1}$ .

## Number of lower-bound integer sums

This can easily be extended to integer sums with different lower bounds. I.e. we want to count the number of solutions for the equation

$$x_1 + x_2 + \cdots + x_k = n$$

with  $x_i \geq a_i$ .

After substituting  $x'_i := x_i - a_i$  we receive the modified equation

$$(x'_1 + a_1) + (x'_2 + a_2) + \cdots + (x'_k + a_k) = n$$

$$\Leftrightarrow x'_1 + x'_2 + \cdots + x'_k = n - a_1 - a_2 - \cdots - a_k$$

with  $x'_i \geq 0$ . So we have reduced the problem to the simpler case with  $x'_i \geq 0$  and again can apply the stars and bars theorem.

## Number of upper-bound integer sums

With some help of the [Inclusion-Exclusion Principle](#), you can also restrict the integers with upper bounds. See the [Number of upper-bound integer sums](#) section in the corresponding article.

## Practice Problems

- [Codeforces - Array](#)
- [Codeforces - Kyoya and Coloured Balls](#)
- [Codeforces - Colorful Bricks](#)

- [Codeforces - Two Arrays](#)
- [Codeforces - One-Dimensional Puzzle](#)

Contributors:

[jakobkogler](#) (70.73%)   [jxu](#) (13.41%)   [Electron1997](#) (6.1%)   [adamant-pwn](#) (6.1%)   [fork52](#) (3.66%)