University of Amsterdam

Assignment 2: Monte Carlo Methods in Finance

Computational Finance

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1 Introduction

In our previous report, we explored how to price a vanilla option analytically using the *Black-Scholes* equation as well as by numerical simulation using the *Binomial tree method* and by simulating the asset price as a geometric Brownian motion (GBM) process. These methods work for European and American type options, which are not path dependant. In this report, we discuss *Monte Carlo methods* to price path dependant options, which include barrier options and Asian options. Path dependency in exotic options means that the value of the option depends not only on the price of the underlying asset at maturity, but also on the path which the underlying asset takes during the option's lifetime. There are two types of path dependant options. The first one is the soft path dependent option, e.g. the barrier option (also known as a binary or digital option), wherein if the price of the underlying asset goes above a specific price then this option becomes worthless. Therefore, the value of this option depends on a single price event of the underlying asset during its life. We will introduce this type of option in section 2, and present experimental results on pricing it using various methods in section 3. The second type of path dependent options the called *hard path dependant options*, e.g. Asian options. The payoff in these options is calculated by sampling some M asset price points uniformly from the asset price time-series up to the time of maturity of the option. A brief introduction is given in section 2.5.

The structure of this report is as follows: We begin by a theoretical introduction of the methods discussed throughout in section 2. Then, we will continue by discussing a number of experimentation comparing the various methods in section 3. For that purpose, we start by comparing the Monte Carlo pricing method to the binomial tree and Black-Scholes methods for vanilla options. Following that, a calculation of the hedge parameter Δ will be shown using the same methods and parameters for a European and digital call options, while also including additional methods such as the Pathwise and Likelihood Ratio methods to account for the discontinuous nature of the digital payoff function. Subsequently, we will study the Monte Carlo method for path dependant Asian type call options. As the Monte Carlo method relies heavily on performating a large number of simulations, we will introduce a variance reduction technique called control variates to reduce the number of simulations required and thus speed up the Monte Carlo method. Lastly, in section 4, we will discuss the results obtained by using the different methods throughout this report.

2 Methods

2.1 Monte Carlo method for option pricing

The *Black-Scholes* model describes the evolution of a stock price in the risk-neutral world through the stochastic differential equation (SDE)

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \tag{1}$$

where r is the risk-free interest rate, $S(t)$ is the asset price, and W is the standard Wiener process.

To solution of (1) is (2), which is used to calculate the stock price at maturity S_T.

$$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}N(0,1)\right) \tag{2}$$

In this equation S_0 is the stock price at $T = 0$, r is the risk free rate, σ is the volatility of the stock, T is the time until maturity of the option and $N(0,1)$ is a random variable from the normal distribution with a mean of 0 and standard deviation of 1. With S_t we can calculate the payoff of the European call option by using (3) or of the European put option by using (4).

$$f(S) = \max\{0, S_t - K\} \tag{3}$$

$$f(S) = \max\{0, K - S_t\} \tag{4}$$

In (3) and (4) K is the strike price. To get the fair price of the option the payoff of the option should be multiplied by the risk free rate to get (5) for an European option.

$$P = f(S)e^{-rT} \tag{5}$$

We should take the average of all the different option prices of the paths of the underlying asset that are used in the Monte Carlo method. To get the final (6) for the calculation of an European call option price with the Monte Carlo method and (7) for a European put option.

$$C = \exp(-rT)\frac{\sum_{n=1}^{N}\left(S_0\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z_n\right)-K\right)}{N} \tag{6}$$

$$P = \exp(-rT)\frac{\sum_{n=1}^{N}(K-S_0\exp\left(\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z_n\right)}{N} \tag{7}$$

, where $Z_n \sim N(0,1)\forall n \in [1\dots N]$.

An important result for the Monte Carlo method next to the actual option price is the standard error of this option price, which can be calculated with (8). In this equation N is the number of stock paths that were simulated.

$$\text{standard error} = \frac{\sigma_{\text{payoff of options}}}{\sqrt{N}} \tag{8}$$

Putting everything together, we used the following steps to value European options with the Monte Carlo method.

1. Simulate a path for the underlying stock price of the option in a risk neutral world.
2. Calculate the payoff of the European option at the maturity time
3. Calculate the option price by discounting the payoff of the option with the risk free rate e^{-rT}
4. Repeat steps 1,2 and 3 many times to get the sample mean of the European option price of these simulated paths.

And in psuedocode, the method described above is given by the following:

S_0 ←Price at time 0;
K ← Strike price;
σ ← Volatility rate;
T ← Time to maturity;
$S_T \leftarrow S_0\exp\left((r-\frac{\sigma^2}{2})T\right)$
N ← paths
$i \leftarrow 0$
while $i \leq N$ **do**
  $Z \leftarrow N(0,1)$ Generate a random normal number
  $S_T \leftarrow S_T\exp\left(\sigma\sqrt{T}Z\right)$ Simulate a path
  $P_i \leftarrow e^{-rT}\max(0, S_T - K)$ Calculate the option price
  $i \leftarrow i+1$
end while
$C \leftarrow \mathbb{E}[P_{i\in(0,N)}]$

Figure 1: Monte Carlo Algorithm for Simulating a European Option Price

2.2 Digital options

Consider a digital option which pays 1 euro if the stock price at expiry is higher than the strike and otherwise nothing.

Then, for a digital call option the payoff at maturity is:

$$C(T) = \begin{cases} 0 & \text{if } S(T) \leq K \\ 1 & \text{if } S(T) > K \end{cases} \tag{9}$$

And similarly for a put option, the payoff at maturity is:

$$P(T) = \begin{cases} 1 & \text{if } S_T \leq K \\ 0 & \text{if } S_T > K \end{cases} \tag{10}$$

From [1] we know that the expected discounted value of the digital call option is given by:

$$C(0) = e^{-rT}\Phi(-x) \tag{11}$$

where Φ denotes the Gaussian cumulative distribution function, and x is given by:

$$x = \frac{\ln K - \ln S_0 - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{12}$$

From (11), we can obtain the Δ of the digital option by differentiating it w.r.t. to the starting price to obtain:

$$\frac{dC}{dS_0} = \Delta = e^{-rT}\frac{1}{\sigma S_0\sqrt{T}}\Phi'(x)$$

Where $\Phi'(x)$ is easily obtained as the p.d.f. of the Gaussian distribution.

2.3 The hedge parameter Δ in Monte Carlo methods

In our previous report we studied the Δ of an option with the Binomial tree method and the Black-Scholes method. In this report we want to study the Δ with the Monte Carlo method to compare the obtained Δ to the two other methods. We will calculate the Δ with a forward finite difference approach which can be seen in (13). This forward finite difference approach is called the bump-and-revalue method and is based on the Euler method. The unbumped option value is $V(S)$ and the bumped option value is $V(S+\epsilon)$.

$$\Delta = \frac{V(S+\epsilon) - V(S)}{\epsilon} \tag{13}$$

In this report we will also research what happens when the calculation of $V(S)$ and $V(S+\epsilon)$ do have different seeds for the simulation of the random variable $N(0,1)$ or when they have the same seed.

2.4 The hedge parameter Δ for a digital option

When calculating greeks of the a digital option, there are a few considerations that we need to take into account due to the distribution of the digital option payoff.

When the shift-size is very small and with insufficient number of paths the value corresponding to the bumped-scenario might change drastically due to one path crossing the discontinuous boundary.

For this, we introduce the following methods:

2.4.1 The hedge parameter Δ with the Pathwise method

The pathwise method differentiates each simulated outcome with respect to the parameter of interest, and the estimator is calculated by interchanging the order of differentiation and integration.

For the digital option, we have the call price as:

$$C = \mathbb{E}\left[e^{-rT}1_{(S_T \geq K)}\right]$$

And so we simulate the following:

$$S_T = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right)$$

Where $Z \sim N(0,1)$
It then follows that

$$\Delta = \frac{dC}{dS_0} = \mathbb{E}\left[e^{-rT}1_{(S_T \geq K)}\frac{S_T}{S_0}\right]$$

So the estimator is calculated using

$$\hat{\Delta} = \frac{dC}{dS_0} = e^{-rT}1_{(S_T \geq K)}\frac{S_T}{S_0}$$

Since the calculation relies on interchanging the order of differentiation and integration, the payoff needs to be a smooth, continuous, and differentiable function almost everywhere, which is not the case for digital options. As such, we cannot immediately obtain an unbiased estimator. To get around that, we map values of $S_T - K$ to a normal cumulative distribution function, which is fully continuous and can be differentiated.

2.4.2 The hedge parameter Δ with the Likelihood ratio method

In contrast to the pathwise method, with the following method we differentiate a probability density with respect to the parameter of interest.By doing that, we obtain a delta based on a much smoother function - The likelihood function of a certain payoff. We introduce the following:

- S_0: Stock price at time 0
- $f(S_T)$: Payoff at time T
- $g(S_T, S_0)$: Probability density function of S_T

We know that the call price is given by $C(S_0) = \mathbb{E}[f(S_T)] = \int f(S_T)g(S_T, S_0)dS_T$ so that by differentiating C with respect to S_0 we obtain:

$$\Delta = \frac{dC}{dS_0} = \mathbb{E}\left[f\frac{\dot{g}}{g}\right]$$

And applying this to the digital options, we know that:

$$\ln S_T \approx \Phi\left[\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}\right]$$

$$g(x, S_0) = \frac{1}{xS_0\sqrt{T}}\Phi(\zeta(x, S_0))$$

$$\zeta(x, S_0) = \frac{\log\frac{x}{S_0} - (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

So we get

$$S_T = S_0 e^{(e - \frac{\sigma^2}{2})T + \sigma\sqrt{T}Z}$$

Where $Z \sim N(0,1)$ And finally, the value for delta is estimated by:

$$\Delta = \frac{dC}{dS_0} = \mathbb{E}\left[e^{-rT}1_{(S_T \geq K)}\frac{Z}{S_0\sigma\sqrt{T}}\right] \tag{14}$$

2.5 Path-Dependent Payoffs - Asian Options

The payoff of an Asian call option based on geometric averages is given by:

$$f(S, N, K) = \left(\tilde{A}_N - K\right)^+, \tag{15}$$

where

$$\tilde{A}_N = \left(\prod_{i=0}^{N} S\left(\frac{iT}{N}\right)\right)^{\frac{1}{N+1}} \tag{16}$$

Risk-neutral valuation of this option is then given by

$$\exp(-rT)\mathbb{E}\left[f(S, N, K)\right]. \tag{17}$$

It can be shown that the closed form of (17) is given by

$$C_g^A(S(0), T) = \exp\left((\tilde{r} - r)T\right)C(S(0), T) \tag{18}$$

$$\implies \quad = \exp\left(-rT\right)\left(S(0)e^{(\tilde{r}T)}\Phi(\tilde{d}_1) - K\Phi(\tilde{d}_1)\right), \tag{19}$$

where $C(S(0), T)$ is the value of the equivalent European option with the same parameters, and

$$\tilde{\sigma} = \sigma\sqrt{\frac{2N+1}{6(N+1)}}$$

$$\tilde{r} = \frac{\left[r - \frac{1}{2}\sigma^2\right] + \tilde{\sigma}^2}{2}$$

$$\tilde{d}_1 = \frac{\log\frac{S(0)}{K} + (\tilde{r} + 0.5\tilde{\sigma}^2)T}{\sqrt{T}\tilde{\sigma}}$$

$$\tilde{d}_2 = \frac{\log\frac{S(0)}{K} + (\tilde{r} - 0.5\tilde{\sigma}^2)T}{\sqrt{T}\tilde{\sigma}}$$

2.6 Variance reduction by control variates

For any random variable X, the simplest method of estimating $\mathbb{E}[X]$ is collecting n independent realisations and taking their sample mean

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Sample means are unbiased estimators of $\mathbb{E}[X]$ as

$$\mathbb{E}\left[\overline{X}_n\right] = \mathbb{E}[X].$$

From the central limit theorem, the variance of the sample mean is given by

$$\mathrm{Var}\left[\overline{X}_n\right] = \frac{\mathrm{Var}(X)}{n}. \tag{20}$$

So, this requires simulating X many times to reach some target variance. The variance of our estimate can be further reduced if we simulate a different random variable Y such that $\mathbb{E}[Y] = \mathbb{E}[X]$ and $\mathrm{Var}(Y) < \mathrm{Var}(X)$. The method of control variates constructs this new random variable Y by introducing a "control variate", which is another random C variable with known expectation value $\mathbb{E}[C]$ (and preferably known variance as well). Construct Y as

$$Y = X + b(C - \mathbb{E}[C]), \tag{21}$$

where b is a tunable constant. Clearly, $\mathbb{E}[Y] = \mathbb{E}[X]$, and

$$\mathrm{Var}(Y) = \mathrm{Var}(X) + b^2\mathrm{Var}(C) - 2b\mathrm{Cov}(X, C). \tag{22}$$

It is immediately apparent that positive correlation between X and C is necessary for variance reduction. Now,

$$\mathrm{Var}(Y) < \mathrm{Var}(X) \implies b < \frac{2\mathrm{Cov}(X, C)}{\mathrm{Var}(C)}. \tag{23}$$

So, (23) gives an upper bound on b for variance reduction. We can also minimize (22) with respect to b; differentiate with respect to b and equate to zero to get

$$b = \frac{\mathrm{Cov}(X, C)}{\mathrm{Var}(C)}. \tag{24}$$

In order to use the method of control variates, we must first estimate $\mathrm{Cov}(X, C)$ (and $\mathrm{Var}(C)$ if not known) in order to calculate b using (24). Once b is estimated, we can proceed to estimate $\mathbb{E}[X]$ using (21).

For pricing Asian options based on arithmetic means using Monte Carlo simulation, the corresponding Asian option based on geometric means with the same option parameters can be used as a control variate, as we have an analytic result for the expected value of the Asian option in (18). Since the option value has a strong positive correlation with the expected value of the payoff, and the arithmetic and geometric means of the same timeseries are also positively correlated, we can expect the geometric mean based Asian option to serve as a very effective control variate.

3 Results

3.1 Basic option valuation

First of all we want to price an European put option with the Monte Carlo method with the following parameters:

- $S_0 = 100$ denoting the initial stock price
- $K = 99$ denoting the strike price
- $r = 0.06$,   denoting the risk-free rate of 6%
- $\sigma = 0.2$,   denoting the volatility rate
- $T = 1$, denoting amount of time before maturity

To actually price the option we first need to do a convergence study, since the price of a option is different for every simulation with a Monte Carlo method, because of the stochastic nature of the method. We did this by by increasing the number of paths that the underlying stock takes. We set an arbitrary convergence criterion of a standard error of 0.05. Which resulted in 2.

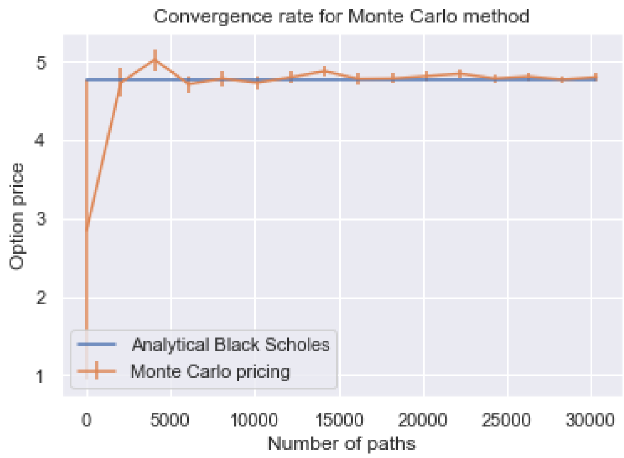

Figure 2: The convergence rate of the Monte Carlo method for the price of an European put option. The Monte Carlo method was converged to a standard error of less than 0.05 with 30311 paths

The arbitrary convergence criterion of a standard error of 0.05 was reached when the Monte Carlo method used 30311 paths. The European put price that was determined when the Monte Carlo method was converged was 4.80267 with a standard error of 0.048775. This standard error measures how much discrepancy there is likely to be in our sample mean compared to the population mean. The value of the European put option with the binomial tree method after 50 steps was 4.78112 and for the Black-Scholes method the value was 4.77897.

We agreed in the previous part that the Monte Carlo method converged when we used 30311 paths, so we will use this number of paths to test what happens when we vary the volatility of the underlying stock. In the left figure of 3 we can see the option price of the European put option under different volatility's calculated with the three different methods. In the right figure of 3 we can see the absolute difference between the analytical Black-Scholes method and the Binomial tree and between the analytical Black-Scholes method and the Monte Carlo method.

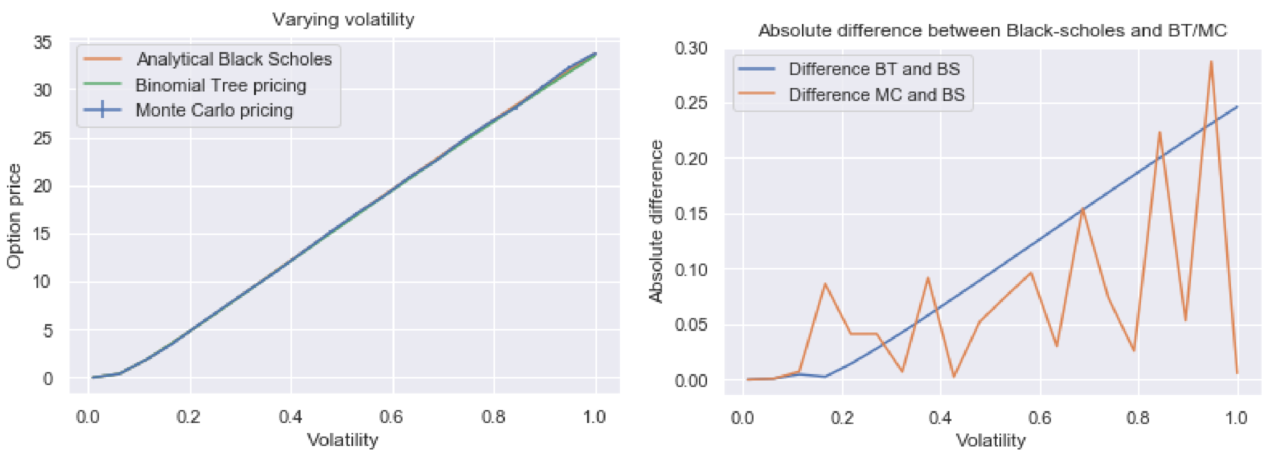

Figure 3: In the left Figure the price of an European put option is shown under different volatility's for the three method. In the right Figure the absolute difference of the binomial tree method and the Monte Carlo method is seen compared to the analytical Black-Scholes method under different volatilities.

We will also use the same number of paths to test what happens when we vary the strike price. In the left figure of 4 we can see the option price of the European put option under different strike prices calculated with the three different methods. In the right figure of 4 we can see the absolute difference between the analytical Black-Scholes method and the Binomial tree and between the analytical Black-Scholes method and the Monte Carlo method.

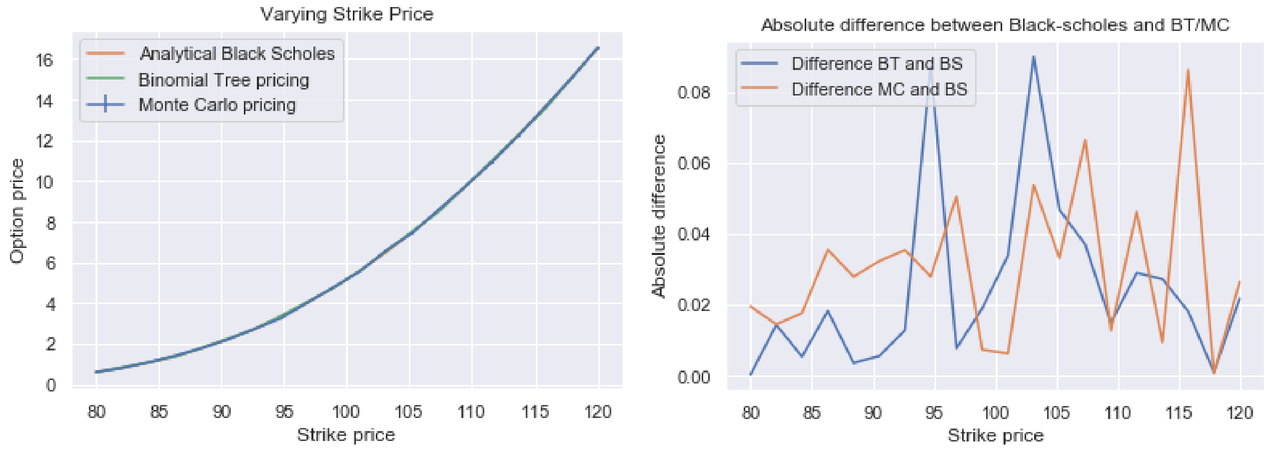

Figure 4: In the left Figure the price of an European put option is shown with different strike prices for the three method. In the right Figure the absolute difference of the binomial tree method and the Monte Carlo method is seen compared to the analytical Black-Scholes method with different strike prices.

3.1.1 Digital

In the following section we will be looking at the evaluation of a Digital call option price and the respective hedge parameter. We will begin with the Monte Carlo method in a similar way to the previous section, by using bumped and unbumped values to calculate the price and Δ by the finite difference method.

Starting with the analytical solution of the option, we apply the parameters on (11) to obtain the following values:

- Call value $C = 0.5639$
- Put value $P = 0.3778$
- Delta value $\Delta = 0.01821$

We then simulated the price of the digital option using the Monte Carlo method, with various bump values path lengths. The results were compared to the analytical solution of the option in the following graphs:

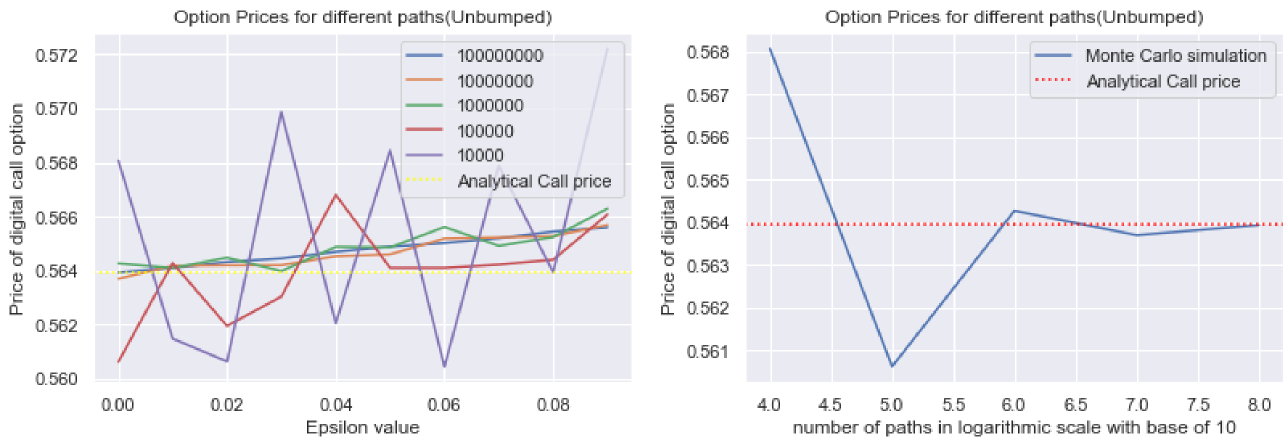

Figure 5: Digital option prices compared with the analytical solution for various path lengths. On the left: Small bumps. On the right: No bump

In the figures above, we can see that when the number of simulations were lower(below 10^6), the value fluctuated around, but was still relatively close to the actual price - both for the unbumped and bump values. Furthermore, by increasing the number of paths, we observed that the relative error decreased significantly.

This is also evident in the following tables, displaying the relative errors:

Relative price errors as % for bumped and umbumped/path length					
Bump value/ paths	10^4 paths	10^5 paths	10^6 paths	10^7 paths	10^8 paths
Unbumped	0.734191	0.588446	0.059011	0.042441	0.000614
0.01	0.434807	0.061182	0.028784	0.043246	0.029073
0.02	0.585106	0.351307	0.096252	0.046620	0.068128
0.03	1.051490	0.159257	0.008410	0.047488	0.091324
0.04	0.334607	0.508741	0.166893	0.105454	0.134796
0.05	0.800991	0.029452	0.163553	0.118313	0.168316
0.06	0.618506	0.029452	0.297153	0.221385	0.192360
0.07	0.700791	0.051162	0.173907	0.229084	0.223591
0.08	0.000608	0.082892	0.231188	0.237417	0.267281
0.09	1.468989	0.380152	0.418227	0.310813	0.294691
Mean Standard Error	0.004607	0.001460	0.000461	0.000146	0.000046

Table 1: Relative errors compared to the analytical solution of the option price

3.2 Estimation of the sensitivities

Afterwards we were interested in calculating the hedge parameter Δ for the Monte Carlo method with a forward finite difference method called the bump-and-revalue method. The same parameters were used as in 3.1, but now we calculated the delta for the European call option. We compared our calculated delta with the different epsilons that can be calculated with (13) to the analytical Black-Scholes delta. Next to varying the epsilon we also varied the number of paths that were used to calculate the delta with the Monte Carlo method. In 6 and 7 we show Δ for the different methods and we show the relative error between the Monte Carlo method and the Black-Scholes method and between the binomial tree method and the Black-Scholes method. For the Monte Carlo method we vary the number of paths and the epsilon. The other two methods are constant. To calculate both the binomial tree delta and the Black-Scholes just the stock price 100 was used without the epsilon. This was done, because the bump-and-revalue method is a forward finite difference method. In figure 6 the Δ and the relative error of the Δ are shown when the unbumped and bumped estimation of the option price have a different seed and in figure 7 the delta and the relative error are shown when the unbumped and bumped estimation of the option price have the same seed.

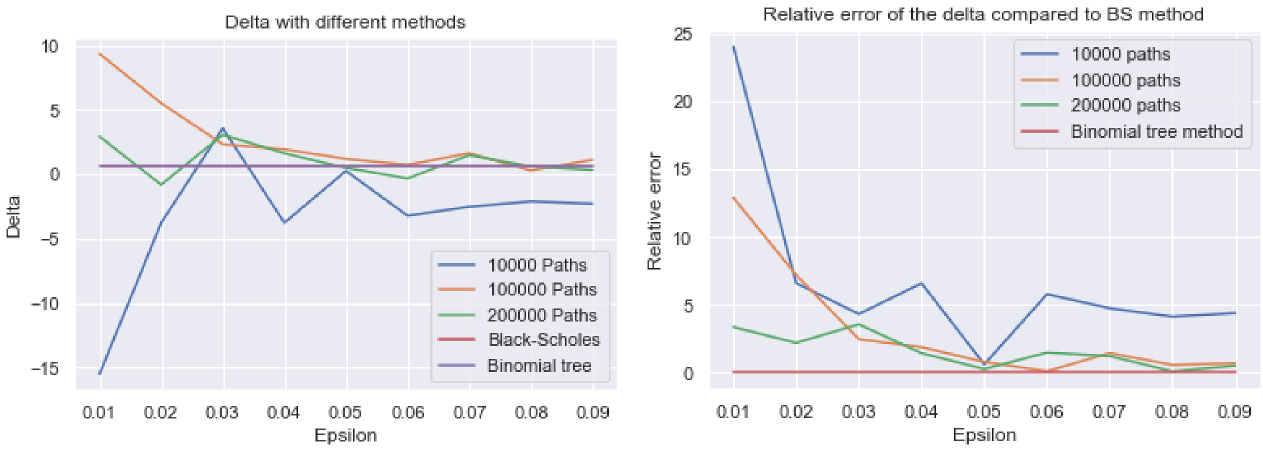

Figure 6: In the left figure the estimation of delta is shown with the Black-Scholes method, the binomial method and the Monte Carlo method with a different number of stock paths. In the right Figure the relative error of the Δ is shown between the analytical Black-Scholes and the binomial tree method and between the analytical Black-Scholes the Monte Carlo method with a different number of paths. In this Figure the unbumped and bumped estimation of the option price have a different seed.

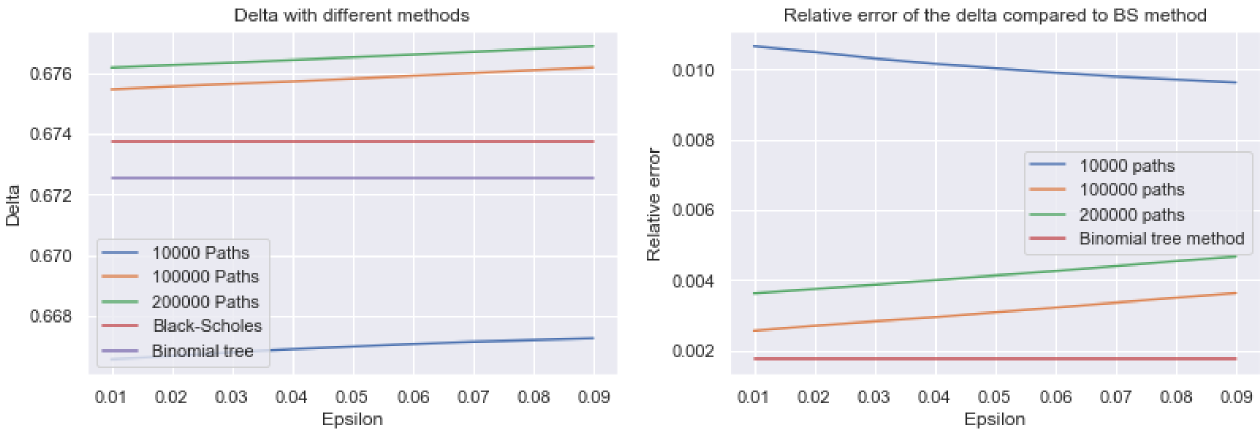

Figure 7: In the left figure the estimation of delta is shown with the Black-Scholes method, the binomial method and the Monte Carlo method with a different number of stock paths. In the right Figure the relative error of the Δ is shown between the analytical Black-Scholes and the binomial tree method and between the analytical Black-Scholes the Monte Carlo method with a different number of paths. In this Figure the unbumped and bumped estimation of the option price have the same seed.

In figures 6 and 7 it is shown that the approximation of delta with the Bump-and-revalue method has the smallest relative error from the analytical Black-Scholes method when 100000 stock paths were used,with an ϵ of 0.01 and when we use the same seed for the bumped and unbumped option price. In previous report we estimated Δ for the European call options with parameters named in section 3.1 to be 0.67256 with the binomial tree method and to be 0.67374 with the Black-Scholes method. Here we first estimated the option prices of the bumped and unbumped value with the Monte Carlo method and used the same seed for both the bumped and unbumped option price estimation. To use the Bump-and-revalue method 100 simulations with 100000 stock paths and with a ϵ of 0.01 to get a Δ of 0.67383 with a standard error of 0.00018.

3.2.1 Sensitivities of the Digital option

In this part, we followed the same methods used before, but applied it on the digital option to obtain the value of the hedge parameter. We started by calculating Δ using the same procedure of Monte Carlo Simulations as before, but included two additional methods as described in 2.4 - Namely, the pathwise method and likelihood ratio method. We then compared the result with the analytical solution by finding the delta values relative errors absolute errors, which are shown in the following tables respectively:

Delta Values					
Method	10^4 paths	10^5 paths	10^6 paths	10^7 paths	10^8 paths
MC	-0.138924	0.106209	0.008581	0.024206	0.018349
MC std. error	0.022275	0.009684	0.001265	0.000891	0.000083
Likelihood	0.018181	0.018162	0.018205	0.018220	0.018207
Pathwise	0.018411	0.018597	0.018240	0.018186	0.018183

Table 2: Delta Values for the Monte Carlo, Likelihood and Pathwise methods for various path lengths. Analytical value: Δ = 0.01821

Relative Delta errors as % per method/path length					
Method	10^4 paths	10^5 paths	10^6 paths	10^7 paths	10^8 paths
Monte Carlo	600.945998	304.760210	46.916151	18.287001	3.446286
Likelihood	0.141872	0.243542	0.010248	0.073694	0.003193
Pathwise	1.125195	2.143008	0.186602	0.113896	0.126350

Table 3: Relative errors compared to the analytical solution Delta

Absolute Delta errors as % per method/path length					
Method	10^4 paths	10^5 paths	10^6 paths	10^7 paths	10^8 paths
Monte Carlo	0.109410	0.055486	0.008542	0.003329	6.274435e-04
Likelihood	0.000026	0.000044	0.000002	0.000013	5.812986e-07
Pathwise	0.000205	0.000390	0.000034	0.000021	2.300366e-05

Table 4: Absolute errors to the analytical solution Delta

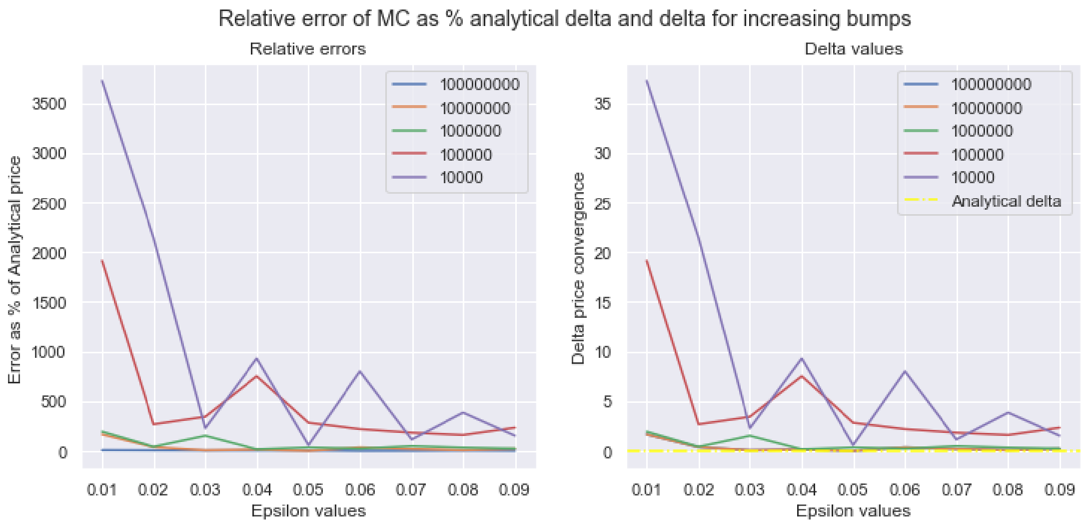

Figure 8: Relative errors and delta values for different epsilon values

As can be seen, the Monte Carlo simulation yields extremely large errors, which are exacerbated when the number of simulation is small. Only when we reach 10^8 simulation, the result is somewhat acceptable, albeit still significantly higher than what we would normally expect. This is duo to the payoff function of the digital option, which is not continuous and differentiable everywhere, as it includes a lot of jumps. When the shift-size is very small and with insufficient number of paths the value corresponding to the bumped-scenario might change drastically due to one path crossing the discontinuous boundary. In contrast, the pathwise and likelihood ratio methods both produce very good results with small relative errors, even for a fairly small number of simulations.

3.3 Variance Reduction

We start with Asian option with the same parameters as given in section 3.1 along with $M = 100$ points being used in the means for payoff calculations. We get $C_g^A(S(0), T) = 6.32288$ from (18). We then estimated C_g^A using Monte Carlo simulations. The stock price timeseries was simulated using the exact iteration method for geometric Brownian motion processes. The results are summarized in table 5; the most accurate estimate we have is $6.32769 \pm 0.00256$for 10^7 paths.

Number of Paths	Mean Option Value	Standard Error
10^2	5.10889	0.74744
10^3	6.55417	0.25824
10^4	6.48894	0.08172
10^5	6.48894	0.08172
10^6	6.32963	0.00810
10^7	6.32769	0.00256

Table 5: Results of Monte Carlo simulations of the GM based Asian option.

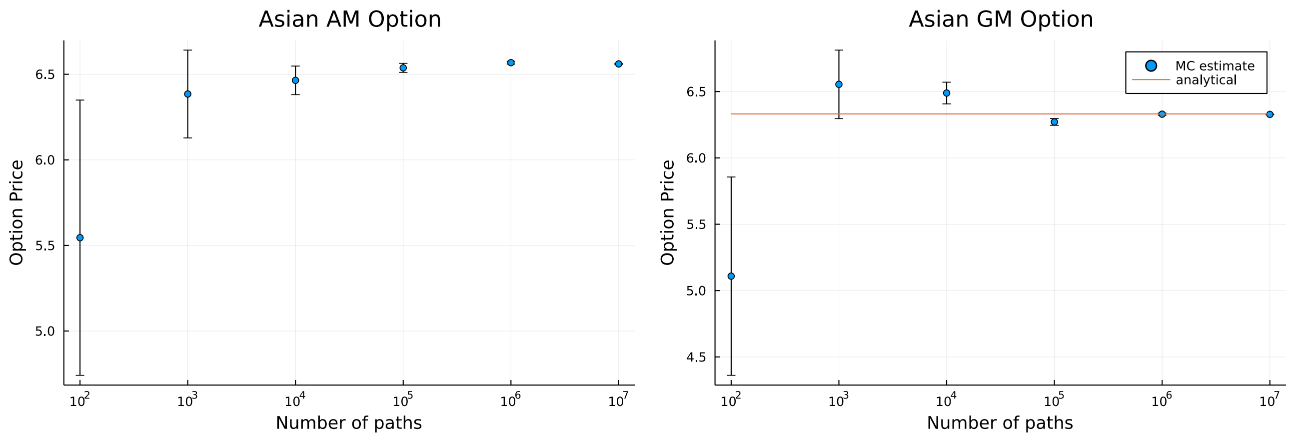

Figure (10) Monte Carlo simulations for geometric/arithmetic-mean based Asian options for different number of paths with standard error for error bars. Option parameters: $S\_0 = 100, K = 99, \sigma = 20\%, r = 6\%, M = 100$.

We can see in figure 10 that the simulation is converging to the analytical value. The same simulation for AM options is shown in figure 9a, although an analytical value is not known for this case. Nevertheless, it appears to be converging to some sensible value; we know from the AM-GM-HM inequality that AM $\geq$ GM, and hence the option value for an AM-based options should be greater than that of a GM-based options with the same parameters, which is satisfied in figures 9a and 10.

Number of Paths	Estimate without CV		Estimate with CV	
	Mean	Standard Error	Mean	Standard Error
10^2	5.54529	0.80417	6.56504	0.02207
10^3	6.38478	0.25645	6.57323	0.00769
10^4	6.46454	0.08349	6.56746	0.00230
10^5	6.53753	0.02636	6.56700	0.00073
10^6	6.56784	0.00838	6.56523	0.00022
10^7	6.56043	0.00265	6.56526	0.00007

Table 6: Monte Carlo simulation results for AM based Asian options with and without the GM Asian option control variate.

If we simulate the AM-based option with the GM-based option as the control variate, we get a much faster convergence, as seen in figure 10 and table 6.

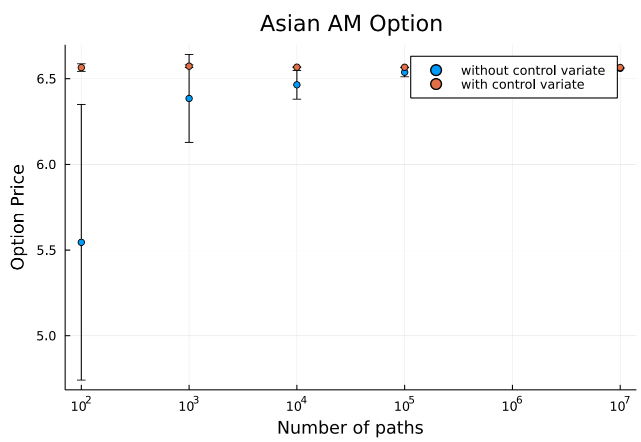

Figure 10: Monte Carlo simulation with and without the GM control variate for the same option parameters as in 10 and 9a.

Finally, we vary different option parameters and see how the method of control variates fares in those cases: we vary the strike K, number of points in the average M, and the volatility of the underlying σ. The results can be seen in figures 11 and 13.

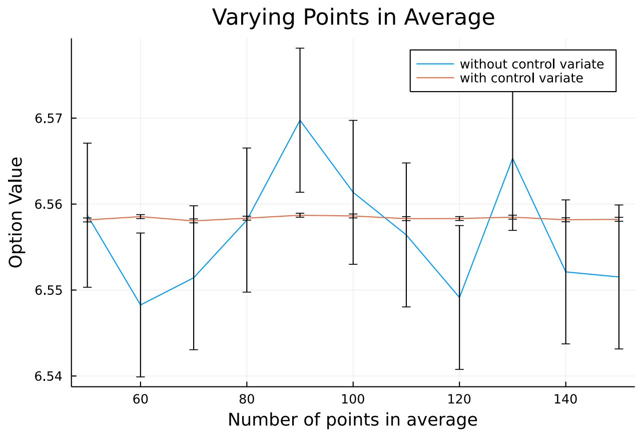

Figure 11: Monte Carlo simulations with and without control variates for different strike prices. $K \in \{50, 60, \ldots 150\}$

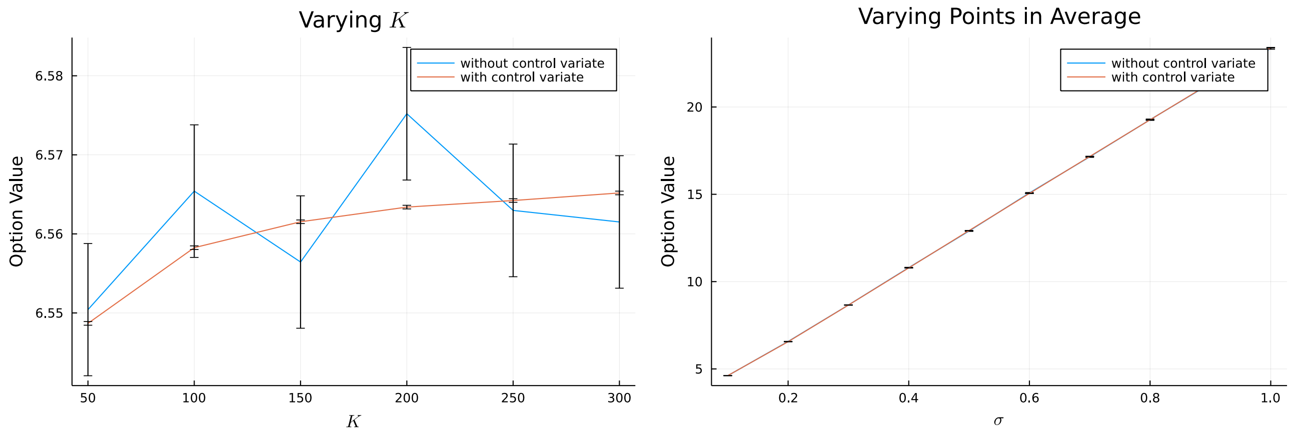

Figure (13) On the right: Monte Carlo simulations with and without control variates for different volatilities of the underlying asset: $\sigma \in \{0.1, 0.2, \ldots 1.0\}$. On the left: Monte Carlo simulations with and without control variates for different sample counts in mean calculation. $M \in \{50, 60, \ldots 150\}$, remaining option parameters equal to those in figure 10.

4 Discussion

In this report, we started with introducing the Monte Carlo method for the pricing of derivatives and to Δ hedge. The Monte Carlo method is especially very useful for path dependant options, like the Asian option and the barrier option. Afterwards we explained how two different type of options called the digital option and the Asian option work. For the digital option we also studied a different way to calculate Δ, called the likelihood ratio method and the pathwise method. This was needed because a digital option has a discontinuous boundary for the payoff of the option. Since one important drawback of the Monte Carlo method is that a lot of stock paths need to be simulated to give an accurate estimation of the option price, the last part we studied is variance reduction by control variates.

In part 3.1 we showed that it took 30311 steps to convergence the Monte Carlo to a standard error of less than 0.05. This amount of steps were used to estimate the option price of the European put option under different volatility's as can be seen in figure 3. For all three methods, it is seen that the option price increases under larger volatility's. The absolute difference between the binomial tree and the Black-Scholes method grows when the volatility increases, while the absolute difference between the Monte Carlo method and the Black-Scholes method seems to be more random under the different volatility's.

Afterwards, we did a comparison between the Binomial tree method, the Black-Scholes method and the Monte Carlo method to price the European put option with different strike prices. In figure 4 the three methods all show that the when the strike price of the European put option increases than the option increases with it. The absolute difference between the analytical Black-Scholes method and the Monte Carlo method and between the analytical Black-Scholes Binomial tree method both do not show a pattern.

In our last report we discussed the estimation of the hedge parameter Δ with the Binomial tree method and the Black-Scholes method. In this report we also studied the comparison between those methods with the

Monte Carlo method for the estimation of Δ. What was seen when we compared the results in figures 6 and 7 is that it is really important to use the same seed for the bumped and unbumped valuation of the option price for the Monte Carlo method, as otherwise the relative error between the analytical and Monte Carlo methods is a significantly larger. In figure 7 it was shown that a combination of 10^5 stock paths and an ϵ of 0.01 gave the smallest relative error compared to the analytical Δ. These parameters were used together with a seeded bumped option price and unbumped option price to get the estimate for Δ with the Monte Carlo method in section 3.2. We continued by applying the same method in pricing a digital call option - a type of option with a binary payoff that depends whether the spot price exceeds the strike price. We started with the Monte Carlo method using the same parameters as before, but varied the number of simulations to $10^4, 10^5, 10^6, 10^7$ and 10^8. We discovered that for the Monte Carlo method, the relative errors were very large, and in particular for the Δ valuation as was seen in figure 8. This is due to the dis-continuous nature of the payoff distribution, as the option price function is not differentiable. We then repeated the simulation using the Pathwise and Likelihood Ratio methods, where we used a smoothing function to account for the aforementioned discontinuities of the payoff functions. Overall, the results shown in Table 3 were very good and converged to the analytical price even for relatively small amounts of simulations. Generally, we observed that all methods were improved by increasing the amounts of simulations.

Finally, we used the method of control variates to price a path-dependent options, namely Asian options based on arithmetic means. The control variate in consideration is the corresponding Asian option based on geometric means, as the closed form of its price's expectation value is known. We first estimated the β parameter to be used with the control variate by estimating the variance of the GM based option and the covariance between the GM and AM based options using a large number of samples, and used that β value to perform Monte Carlo runs with the control variate. Significant variance reduction was observed - for the default option parameters the standard error at 10^3 paths with the control variate was close to the standard errror 10^5 runs without control variates. We note that there is a strong positive correlation between the strike price K and the option value, while the convergence for different volatilities is roughly the same for both with and without the control variate.

References

[1] T. Worrall, "The black-scholes formula.," Lecture notes ECO-30004 OPTIONS AND FUTURES.