Lab Assignments Computational Finance

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Submission guidelines
These assignments can be done in groups of three students. Reports with a clear description of the assignment, the methods, the results and discussion should be submitted before the deadlines. You are free to choose the programming language/environment in which you would like to write your computer programs. If you have questions about the assignments you can ask them during the lab-sessions.

Grading scheme

- Each of the three assignments carries equal weight of 20% and the exam is worth 40%;
- The score of the exam should be 5 points (on the scale of 1 to 10) and higher for passing the course.

Assignment 1	Assignment 2	Assignment 3	Exam
20%	20%	20%	40%

Assignment 3: The finite difference Method

Part I

Background of PDE approach (30 points)

The Black-Scholes Partial Differential Equation (BS-PDE) derived for a plain vanilla option has the following form:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV \tag{1}$$

The equation can be transformed into a PDE with constant coefficients, by introducing $X = \ln(S)$. Since the equation is commonly solved backward in time, it is for further numerical treatment convenient to introduce the following transformation:

$$\frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$$

A. Show that the transformed BS-PDE has the following form:

$$\frac{\partial V}{\partial \tau} = \left(r - \frac{1}{2}\sigma^2\right)\frac{\partial V}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial X^2} - rV \tag{2}$$

The explicit FTCS (Forward Time Centred in Space) for the transformed BS-PDE (2) can be derived by means of Taylor Expansion Techniques. Recalling that

$$V(X + \Delta X, \tau) = V(X,\tau)\Delta X \frac{\partial V(X,\tau)}{\partial X} + \frac{1}{2!}\Delta X^2 \frac{\partial^2 V(X,\tau)}{\partial X^2} + \dots \tag{3}$$

B. Show that the discrete Finite-Difference (FD) approximation of the FTCS scheme equals:

$$\begin{aligned} V_i^{n+1} =& V_i^n + \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{2\Delta X}\left(V_{i+1}^n - V_{i-1}^n\right) + \frac{1}{2}\sigma^2 \frac{\Delta \tau}{\Delta X^2}\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n\right) \\ & - r\Delta\tau V_i^n, \end{aligned} \tag{4}$$

where the superscript n denotes the time point and the subscript denotes the spatial index of the log value X.

C. Now, show that the Crank-Nicolson scheme for equation (2) is given by:

$$\begin{aligned} V_i^{n+1} =& V_i^n + \left(r - \frac{1}{2}\sigma^2\right)\frac{\Delta\tau}{4\Delta X}\left(V_{i+1}^n - V_{i-1}^n + V_{i+1}^{n+1} - V_{i-1}^{n+1}\right) \\ & + \frac{1}{4}\sigma^2\frac{\Delta\tau}{\Delta X^2}\left(V_{i+1}^n - 2V_i^n + V_{i-1}^n + V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}\right) - \frac{r\Delta\tau}{2}\left(V_i^n + V_i^{n+1}\right), \end{aligned} \tag{5}$$

and show by using Taylor series expansion that this scheme is second order in space.

Part II

FD-Schemes for European call (50 points)

The FD-mesh for the derived schemes is drawn below. It should be noted that the cell-width is uniform over the whole computational domain. The infinite extent of $X = \ln(S)$ in the continuous problem is approximated by the truncated interval $[-M_1, M_2]$, where M_1 and M_2 are sufficiently large chosen positive constants so that the boundary conditions at the two ends of the infinite interval can be applied with sufficient accuracy. Approximated option values are computed at grid points $(i\Delta X, n\Delta\tau), i = 1, 2, \ldots, N$ and $n = 0, 1, \ldots$ The option values along the boundaries $i = 0$ and $i = N + 1$ are prescribed by boundary conditions of the option model, while initial values along $n = 0$ are given by the terminal payoff function. The FTCS and the

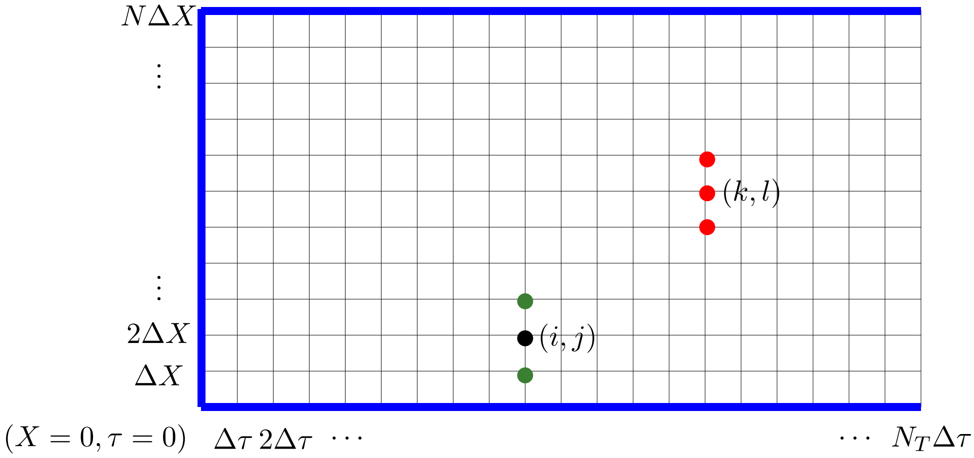

Figure 1: Schematic overview of discretized grid. The blue boundaries denote the known boundary or initial values and the green dots show the points needed for FD approximation of the first derivative at point (i, j) and the red points are the needed points for the appoximation of the second derivative at point (k, l).

Crank-Nicolson scheme can be represented in matrix vector notation as:

$$B\vec{V}^{n+1} = A\vec{V}^{n}, \tag{6}$$

where A and B is a sparse matrix (mainly zero values) with the three non-zero diagonals represented by the vectors:

$$B = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \vec{b}_{-1} & \vec{b}_0 & \vec{b}_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \text{ and } A = \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \vec{a}_{-1} & \vec{a}_0 & \vec{a}_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

A. Compute the vectors $\vec{a}_{-1}, \vec{a}_0, \vec{a}_1, \vec{b}_{-1}, \vec{b}_0, \vec{b}_1 \in \mathbb{R}^{n\times 1}$. for the FTCS scheme and the Crank-Nicolson scheme. Show which boundary conditions you use and how you incorporate them!

B. Solve the above tri-diagonal system by writing a computer program in Python, Matlab or any other programming language of your choice. Do this for both the FTCS-scheme and the Crank-Nicolson scheme. Use the following market parameters for a European call:

$$
\begin{array}{lllll}
r = 4\%, & \text{vol} = 30\%, & S_0 = 100, & K = 110, & T = 1 \text{ year (in the money)} \\
r = 4\%, & \text{vol} = 30\%, & S_0 = 110, & K = 110, & T = 1 \text{ year (at the money)} \\
r = 4\%, & \text{vol} = 30\%, & S_0 = 120, & K = 110, & T = 1 \text{ year (out of the money)}
\end{array}
$$

and additionally present your results in graphs.

Check whether your computer program is in line with your expectations from the theoretical analysis in Part I C. (second order convergence in number of grid points in space).

Determine your own optimal mesh size . Plot the delta as function of the underlying S, i.e.

$$\Delta = \frac{\partial V}{\partial S}. \tag{7}$$

Note (!) that the Greeks are in the original variable S and not in X. Explain the graphs.

Part III

Option price using COS method (20 points)

An asset price S_t (e.g. Stock or FX), can be modeled using Geometric Brownian motion:

$$dSt = rS_t dt + \sigma S_t dW_t^Q \tag{8}$$

where W_t^Q a Wiener process, r the interest rate and σ the volatility. For a European option with strike K, apply the following transformation:

$$x = \log(\frac{S_0}{K}) \text{ and } y = \log(\frac{S_t}{K}). \tag{9}$$

Then the value $V(x,t)$ of a European option must satisfy:

$$V(x,t) = e^{-rt} \int_{\mathbb{R}} g(y) f(y|x) dy, \tag{10}$$

where $g(y)$ is the payoff function and $f(y|x)$ the conditional density function.

A. Show that we can approximate $V(x,t)$ by a finite sum of Fourier cosine coefficients F_n and G_n of the conditional density function and payoff function respectively as:

$$e^{-rt} \frac{b-a}{2} \sum_{k=0}^{N} F_k(x) G_k$$

B. Assume the Fourier coefficients for European call option to be given by :

$$G_k = \frac{2}{b-a} \int_0^b K(e^y - 1) \cos\left(k\pi \frac{y-a}{b-a}\right) dy = \frac{2}{b-a} K(\chi_k(0,b) - \psi_k(0,b)),$$

where $\chi_k(a,b)$ and $\psi_k(a,b)$ are given as:

$$\begin{aligned}
\chi_k(c,d) :=& \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2} \left[\cos\left(k\pi \frac{d-a}{b-a}\right) e^d - \cos\left(k\pi \frac{c-a}{b-a}\right) e^c \right. \\
&\left. + \frac{k\pi}{b-a} \sin\left(k\pi \frac{d-a}{b-a}\right) e^d - \frac{k\pi}{b-a} \sin\left(k\pi \frac{c-a}{b-a}\right) e^c \right], \\
\psi_k(c,d) :=& \begin{cases} \left[\sin\left(k\pi \frac{d-a}{b-a}\right) - \sin\left(k\pi \frac{c-a}{b-a}\right)\right] \frac{b-a}{k\pi} & \text{if } k \neq 0; \\ d - c & \text{if } k = 0. \end{cases}
\end{aligned}$$

Furthermore, the Characteristic function of $y - x$ equals :

$$\phi_{\text{GBM}}(u) = e^{iu(r - \frac{1}{2}\sigma^2)t - (\frac{1}{2}\sigma^2 t u^2)};$$

(N.B. the term $\frac{S_0}{K}$ should be in $F_k(x)$.) Now price a one year ($T = 1$) call option with parameters from part IIIB. Use up to $N = 64$ Fourier cosine coefficients and define a and b as:

$$\begin{aligned}
a &= \log \frac{S_0}{K} + rT - 12\sqrt{\sigma^2 T} \\
b &= \log \frac{S_0}{K} + rT + 12\sqrt{\sigma^2 T};
\end{aligned}$$

Compare your results with the earlier derived exact Black Scholes pricing formula and comment on the error and speed of convergence (by increasing the Fourier cosine coefficients to $96, 128, 160, 192$