University of Amsterdam

Assignment 1: Black-Scholes and Binomial Tree

Computational Finance

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1 Introduction

A derivative is a type of financial product whose value is dependant on the value of a more basic underlying asset. In this report we will mainly focus on the derivatives called options. An example of an option is a call option which gives the owner the right but not the obligation to buy a stock at the maturity time at strike price K. In the case of a call option an example of a more basic underlying asset can be a stock of a company. The way these options and other derivatives can be used are to hedge risk, to speculate and to lock in an arbitrage profit.

In the option market there are two parties that agree to a deal. The party that agrees to buy the option and therefore takes a long position and the party that has agreed to sell the option and therefore takes a short position. To sell the option both parties need to agree on the price of a certain option. Market makers use different pricing models to price options of which we will study two in this report called the binomial tree model and the Black-Scholes model.

2 Methods

2.1 Continuous compounding

First of all we will cover one important assumption in both the binomial tree model and the Black-Scholes model, which is that when money is invested in a bank account or in the money market than it will yield an interest risk-free rate r. The interest over the money is compounded at a certain frequency and is defined as the compounding type, which for example can be annual compounding or monthly compounding. For both models a continuous compounding is assumed, which means that the account will receive compounding interest over an infinite number of periods per year. This compounding interest will be added into the account. The formula for compounding types can be seen in eq. (1).

$$B = C\left(1 + \frac{r}{n}\right)^{n\Delta t} \tag{1}$$

Where C is the money invested in the money market, n the amount of periods during Δt and Δt the amount of time we want to calculate the compounding interest over. B is the amount of money at the end of the compounding interest.

Taking the limit $n \to \infty$, so we will get the formula of continuous compounding in eq. (4):

$$B = \lim_{n\to\infty} C\left(1 + \frac{r}{n}\right)^{n\Delta t} \tag{2}$$

$$\text{and } \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n} = e^r \tag{3}$$

$$\implies B = Ce^{r\Delta t} \tag{4}$$

2.2 Coupon bond

As an example of continuous-compounding in action, consider a financial instrument called the coupon bond, which is a debt obligation with coupons attached that represent semiannual interest payments. The fair value of the bond depends on the free-market interest rate as it determines the time value of money. So, the present value of the future cash flows, assuming the free-market interest rate applies, is given by eq. (6)

$$\text{PV} = Ce^{-r} + Ce^{-2r} + \dots Ce^{-nr} + Pe^{-nr} \tag{5}$$

$$\implies \text{PV} = Ce^{-r}\left(\frac{1 - e^{-nr}}{1 - e^{-r}}\right) + Pe^{-nr} \tag{6}$$

where C is the coupon rate, r is the effective interest rate, and n is the number of installments. In the present example of coupon bonds with quarterly compounded interest, $C = \$300$, $r = 0.015 \times 0.25 = 0.00375$, $P = \$50.000$, $n = 8$. So, using (6), we get (to 5 decimal places) PV = $\$50.882,20359$.

2.3 Forward contracts

Another important derivative is a forward contact, which is a contract between two parties to buy or sell a certain asset S at specific time in the future T for a specific price K. The long position is obliged to buy the asset from the party that is in the short position. The long-position party can borrow an amount S_0 at an interest rate r, purchase the underlying asset for S_0, and at maturity fulfill the contract by selling to the short-position party at price F, and repay the loan amount with interest $S_0 e^{rT}$.

Now, an arbitrage strategy would exist for the long position if $F > S_0 e^{rT}$, as they can make a riskless profit of $F - S_0 e^{rT} > 0$, which contradicts the no-arbitrage principle. Therefore,

$$F \leq S_0 e^{rT} \tag{7}$$

Alternatively, an arbitrage opportunity can also emerge when $F < S_0 e^{rT}$. The underlying asset's owner can sell it for S_0, invest the revenue earned at interest rate r, and entering the forward contract to buy the asset at the price F. The proceeds from the interest would amount to $S_0 e^{rT}$, and can be used to pay the forward price F, resulting in a riskless profit of $S_0 e^{rT} - F > 0$. Therefore according the no-arbitrage principle the forward price of a contract at time zero is $S_0 e^{rT} = F$.

2.4 Payoff diagrams of multiple financial instruments

We will draw different payoff diagrams to get a better idea how the different financial instruments that are used later in the report behave under the increasing of the underlying asset value S at maturity time T. First we will begin with the payoff diagrams of stock S and an investment in the money market of Ke^{-rT}, which can be seen in fig. 1.

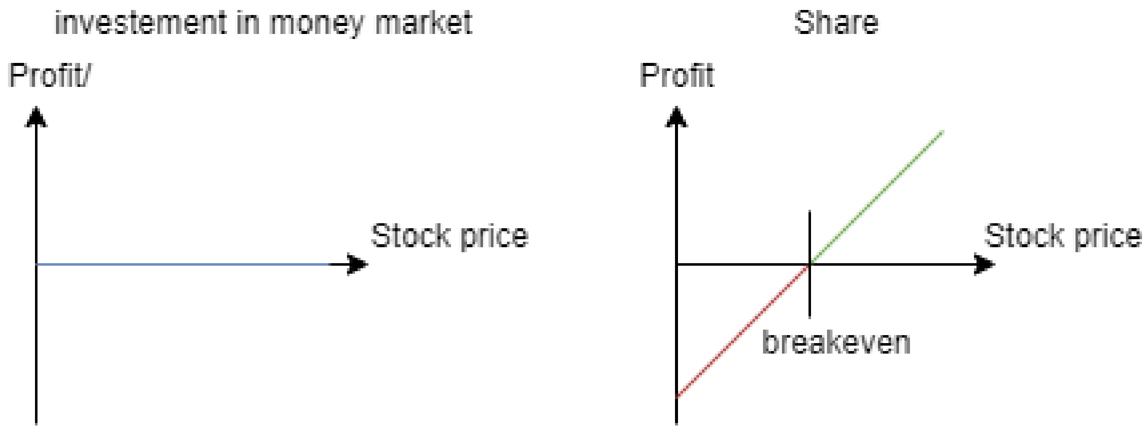

Figure 1: Payoff diagram of an investment in the money market and of stock S at maturity time T.

What is seen is that the investment in the money market is independent of the stock price and that the stock has a profit or less that is linearly dependent on the stocks price. Now we will show the payoff diagrams of a long call option and of a long put option, which can be seen in fig. 2.

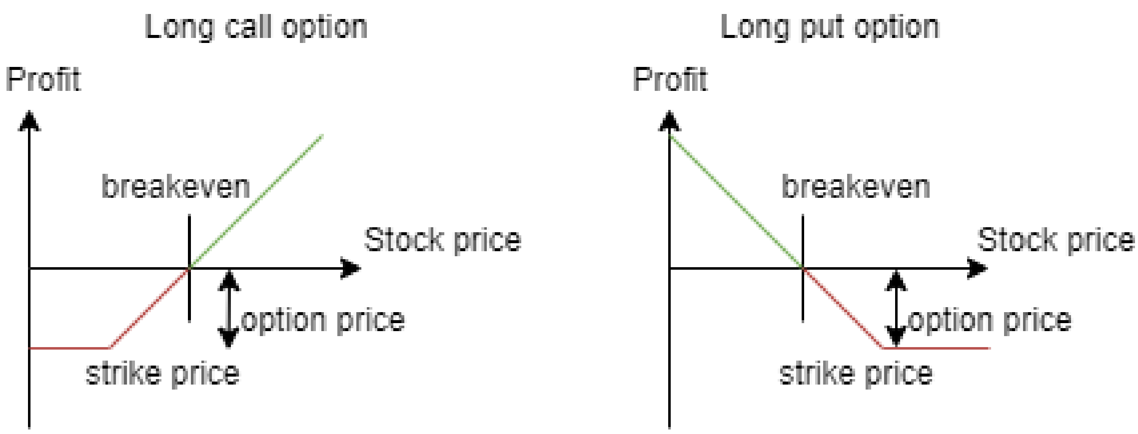

Figure 2: Payoff diagram of an European long call option and an European long put option.

Finally we can combine the payoff diagrams: 1) a European long call option and an investment in the money market of Ke^{-rT} and 2) A European long put option and one share of the stock. Both portfolios can be seen in fig. 3.

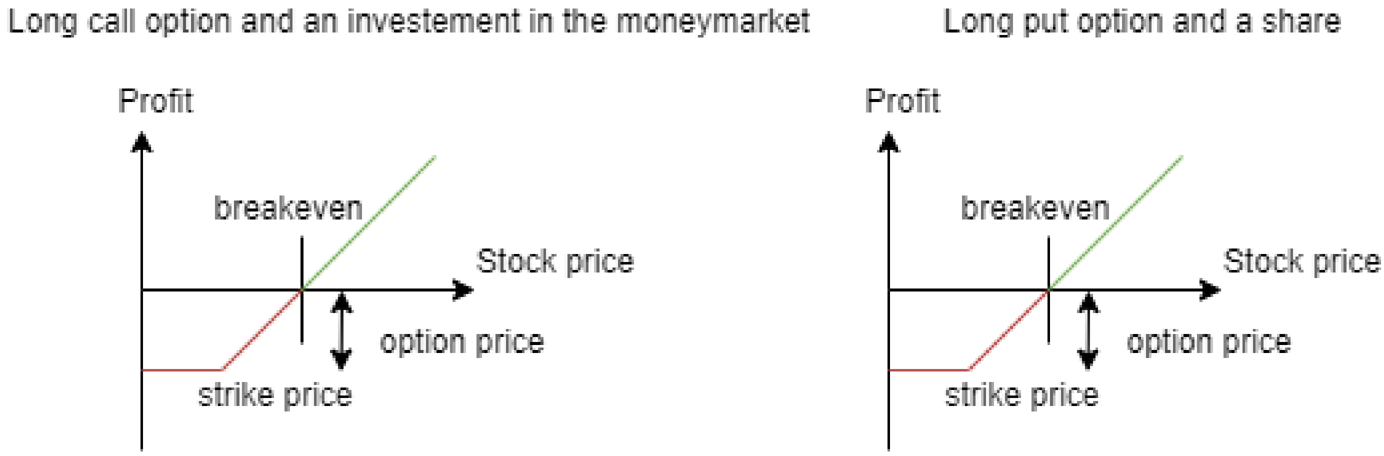

Figure 3: Payoff diagram of an European long call option an investment in the money market and an long put option and a share of the underlying share of the options. On the x-axis is the price of the stock. X is the strike price.

For the call option and the investment in the money market of Ke^{-rT} it is seen that the total value of the portfolio remains constant until the underlying stock price of the long call option is larger than the strike price of the call option. Than the call option is worth the difference between the strike price and the stock price at maturity. The reason the total worth of the portfolio first is at a constant loss is because the call option also has a cost. The long put option can be used to sell the stock for the same price which keeps total worth of the portfolio constant, until the stock price is larger than the strike price of the put option. It would not make sense to use the put option than to sell the stock because you would sell it for less. Then the long put option is worth nothing anymore, but the stock is getting worth more. The reason the total worth of the portfolio first is at a constant loss is because the put option also has a cost. Both investing strategies result in the same payoff diagram.

2.5 Put-call parity

The four financial products that combine to the payoff diagrams of fig. 3 come together in the put-call parity, which is a principle that defines the relationship between an European call an put option. If one side of the put-call parity equation, which can be seen in eq. (8), is greater than it is possible to sell the more expensive side and buy the cheaper side to make risk-free profit. It is risk-free, because at maturity time T it does not matter if the stock goes up or down for the total value of you portfolio. If for example the total value of the right side of the equation is larger than the left side than we would short the put option and the stock and we would long the call option and the investment in the money market. In the beginning when we do this we would receive the money of the difference between the shorted part of the equation and the longed part of the other side of the equation. This creates an arbitrage opportunity since it does not matter if the stock goes up or down because you can pay of the shorted part with the longed part and you can keep the difference between the shorted and longed part of the equation. Therefore the put-call parity must hold, because of the no-arbitrage principle.

$$C_t + Ke^{-rT} = P_t + S_t \tag{8}$$

2.6 Binomial Tree Method for option pricing

As earlier said in the introduction one of the ways to price an option is with the binomial tree model. For our pricing models we will use an option on a non-dividend paying stock. If we divide the time until maturity time T into N sub intervals of length Δt. Than the stock price at the (i,j) node can be seen in eq. (9)

$$S_{i,j} = S_0 u^j d^{i-j} \tag{9}$$

Where we will refer to the j^{th} node at time $i * \Delta t$ as node (i, j). In eq. (9) S_0 is known, u^j can be calculated with eq. (10) and d^{i-j} can be calculated with eq. (11). In both equation 10 and 11 the volatility σ and Δt are known.

$$e^{\sigma\sqrt{\Delta t}} \tag{10}$$

$$e^{-\sigma\sqrt{\Delta t}} \tag{11}$$

We can use the stock prices of the last nodes of the last sub interval to calculate the option prices at the last nodes with eq. (12) for call options.

$$f = max\{0, S - K\} \tag{12}$$

For put options the price of the last nodes can be calculated with eq. (13).

$$f = max\{0, K - S\} \tag{13}$$

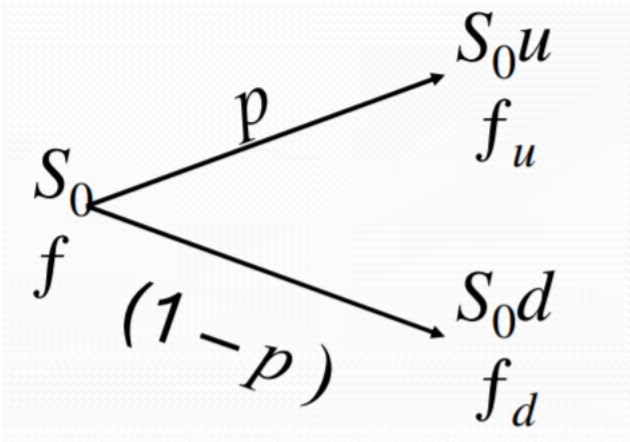

Figure 4: Illustration of how a binomial tree looks and how you could calculate the option price at $t = 0$ [1].

With a backward induction scheme we can calculate the option prices of the other nodes with the option prices of the last nodes. You have the values of the option at time $(i+1)\Delta t$ for very j, which would mean that in fig. 4 you would know the option prices of the last nodes f_u and f_d. Then there are risk neutral probabilities p and $(1-p)$ to move from the $(i+j)$ node to the $(i+1, i+j)$ node and the $(i+1, j)$ node respectively at time $i\Delta t$. Combined with a risk neutral valuation this leads to eq. (14) to calculate the option price at node (i, j).

$$f_{i,j} = e^{r\Delta t}(pf_{i+j,j+1} + (1-p)f_{i+1,j}) \tag{14}$$

In this formula r and Δt are known. $f_{i+j,j+1}$ and $f_{i+1,j}$ are known because of the backward induction scheme and p can be calculated with eq. (15).

$$p = \frac{e^{r\Delta t} - d}{u - d} \tag{15}$$

With eq. (14) we can calculate the option prices of all the nodes and therefore also the node where $t = 0$. The price of the option at $t = 0$ is the price that the option should be sold for.

Until now we talked about an option where the only time that it can be exercised is at the maturity time. These type of options are called European options. In this report we will also study another type of options called American options. These types of options can be exercised at any moment between $t = 0$ and the maturity time of the option. In theory the American option should be more valuable, because it can be exercised at any moment. The difference in the calculation of the nodes is that the nodes with an European option can be calculated with eq. (14) and the nodes of the American call option with eq. (16).

$$f_{i,j} = max\{e^{r\Delta t}(pf_{i+j,j+1} + (1-p)f_{(i+1),j}), S - K\} \tag{16}$$

The the nodes of the American put option can be calculated with eq. (17).

$$f_{i,j} = max\{e^{r\Delta t}(pf_{i+j,j+1} + (1-p)f_{i+1,j}), K - S\} \tag{17}$$

For both the American call and put option the option price can be determined with the option price of the node at $t = 0$

2.7 Analytical solution of the Black-Scholes Formula for option pricing

The solution

$$V^{put}(S,t) = V^{call}(S,t) - S + Ee^{-r(T-t)}$$

Can be written in a similar form as the solution for a call option:

$$V^{ep}(S,t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1)$$

In this report we want to compare the option value at $t = 0$ of the binomial tree method to the analytical option price of the Black-Scholes model.

Proposition 2.1. *The analytical option value can be calculated with eq.* (18)*.*

$$V(t) = S_0 N(d_1) - Ke^{-rT} N d_2 \tag{18}$$

$$d_1 = \frac{log \frac{S_t}{k} + (r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}} \tag{19}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{20}$$

With the put-call parity and the analytical Black-Scholes formula in eq. (18) we can also make a formula for the European put option, which can be seen in .

2.8 Delta-Hedging for the binomial tree method

With a method called hedging we can make a portfolio of assets and cash so that it copies the value of the option. With the right hedge parameter Δ, which stand for how many stocks are needed to hedge the portfolio, we can make the portfolio so that the final value of the portfolio is independent of the the movement of the stock and therefore the portfolio would be risk less. For the binomial tree method the Δ at $t = 0$ can be calculated with eq. (21).

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \tag{21}$$

2.9 Delta-Hedging using the Black-Scholes Model

Similar to the binomial-tree method, we can set up a risk-free portfolio with continuous hedging using the Black-Scholes model. This time we determine the Δ parameter under the framework of the Black-Scholes model.

Starting with eq. (18), the definition of the Δ parameter in this continuous case becomes.

$$\Delta(t) = \frac{\partial V(t)}{\partial S(t)} \tag{22}$$

$$\implies \Delta(t) = N(d_1) + S\frac{\partial N(d_1)}{\partial S} - Ke^{-r\tau}\frac{\partial N(d_2)}{\partial S} \tag{23}$$

Consider the standard normal CDF terms.

$$\frac{\partial N(x)}{x} = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \Phi(x) \tag{24}$$

Substituting eq. (24) in eq. (23), we get

$$\Delta(t) = \Phi(d_1) - S\Phi(d_1)\frac{\partial d_1}{\partial S} + Ke^{-r\tau}\Phi(d_2)\frac{\partial d_2}{\partial S} \tag{25}$$

Consider eq. (19) and eq. (20). Differentiating with respect to S, we get

$$\frac{\partial d_1}{\partial S} = \frac{1}{\sigma\sqrt{\tau}}\frac{1}{S} = \frac{\partial d_2}{\partial S} \tag{26}$$

$\Phi(d_2)$ can also be simplified further.

$$\begin{aligned}
\Phi(d_2) &= \Phi(d_1 - \sigma\sqrt{\tau}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d_1 - \sigma\sqrt{\tau})^2}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(d_1^2 + \sigma^2\tau - d_1\sigma\sqrt{\tau})\right) \\
&= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{d_1^2}{2}\right) \exp\left(-\frac{\sigma^2\tau}{2}\right) \exp\left(d_1\sigma\sqrt{\tau}\right) \\
&= \Phi(d_1) \exp\left(\frac{\sigma^2\tau}{2}\right) \exp\left(d_1\sigma\sqrt{\tau}\right) \\
&= \Phi(d_1) \exp\left(\frac{\sigma^2\tau}{2}\right) \exp\left(\log\left(\frac{S}{K}\right) + \tau\left(r + \frac{\sigma^2}{2}\right)\right) \\
\implies N(d_2) &= N(d_1)\frac{S}{K}e^{r\tau}
\end{aligned}$$

$$\implies \Delta(t) = N(d_1) - S\Phi(d_1)\frac{\partial d_1}{\partial S} + S\Phi(d_1)\frac{\partial d_2}{\partial S} \tag{27}$$

Substituting eq. (26) into eq. (27), we finally get

$$\Delta(t) = N(d_1). \tag{28}$$

As an example of hedging a portfolio using eq. (28), consider a portfolio with Δ shares long at spot price $S(t=0) = S_0$ that is hedged using a short position in a call option with strike price K at maturity date T. The investor can value the premium of this option using eq. (18). This portfolio is financed using a bond that earns the risk-free interest rate. Thus, at any given time, the value of the portfolio is given by

$$\Pi(t) = \Delta(t)S(t) + B(t) - V(t) \tag{29}$$

Where B is the value of the bond. The bond value is chosen to be just enough to finance the portfolio at $t = 0$ so that $\Pi(0) = 0$.

While the Black-Scholes formula describes a continuously valued option, practically investors can only adjust their portfolios at discrete time-instances. Let M be the adjustment frequency, so that the portfolio is adjusted at time intervals of size $\delta t = T/M$. For the purpose of simulation we let geometric Brownian motion (GBM) be the process that governs the stock-price in a risk-neutral market, i.e. the stochastic process is given by

$$dS(t) = rSdt + \sigma S_t dz, \tag{30}$$

where $dz = \varepsilon\sqrt{dt}, \varepsilon\Phi(0,1)$. This stochastic process is discretised with $\delta_t = T/M$ and numerically simulated using Euler's method. So, the portfolio is updated at discrete intervals.

$$\delta\Delta = \Delta(t+\delta t) - \Delta(t) \tag{31}$$

$$B(t+\delta t) = B(t)e^{r\delta t} - \delta\Delta \cdot S(t+\delta t) \tag{32}$$

The value of the portfolio at $t + \delta t$ is then recalculated using eq. (29).

3 Results

Throughout this section, we will describe the various simulations that we have conducted using the theory outlined in section 2.

Our starting point was a simulation of the price of a European call option for a non-dividend-paying underlying stock, with a maturity date of one year by using the the binomial tree method described in section 2.6. In our simulation we have used the following parameters:

- $S_0 = 100$ denoting the initial stock price
- $K = 99$ denoting the strike price
- $r = 0.06$, denoting the risk-free rate of 6%
- $\sigma = 0.2$, denoting the volatility rate
- $N = 50$, denoting the number of steps for the Binomial Tree

This resulted in a price approximation of 11.546 for the European call option and 4.781 for the European put option.

As discussed in section 2.7, we know that an analytical solution for the option price can be obtained by using the Black Scholes Formula. Therefore, we continued by applying the same parameters to the Black Scholes Formula and comparing the result obtained with the Binomial Tree Method. We obtained a result of 11.544 using the Black Scholes Formula for the price of the call option, which indicates that the Binomial Tree Method yielded a fairly accurate result. Then, in order to investigate the difference between the methods, we calculated the price of a call option with the same parameters, but varying the volatility rate σ in equal intervals between 0.01 and 1, representing volatility rates between 0% and 100%. The result is illustrated in the following figure:

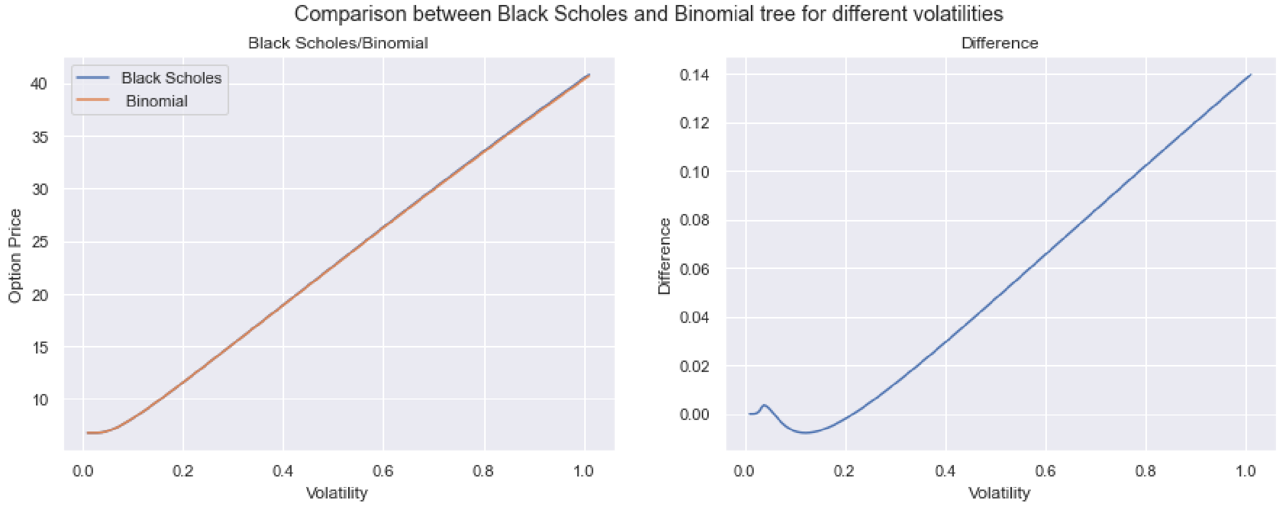

Figure 5: Comparison of pricing an option using the Binomial Tree method and the Black-Scholes Equation

As can be seen from fig. 5, the left-hand-side shows the call option price using both methods, while the right-hand-side shows the difference between the two prices. For very low volatility rates, the result between the methods is largely similar. However, as we start to increase the volatility rate, the differences start growing in linear trend, which reaches a difference of 0.14 between the methods at 100% volatility. Due note that this difference actually grows in proportion to the option price, so as a percentage of the actual value the difference stays quite similar.

We then continued by investigating what other parameters might have a big influence on the result. We decided to simulate the prices using a fixed volatility rate of 20%, and instead varying the steps used in the Binomial Tree method - As this would allow us to see in how many steps does the binomial method converges reasonably close to the result from the Black Scholes method. We performed the binomial method using equally distributed, varying steps sizes for N from 1 to 100. The result is illustrated in the following figure:

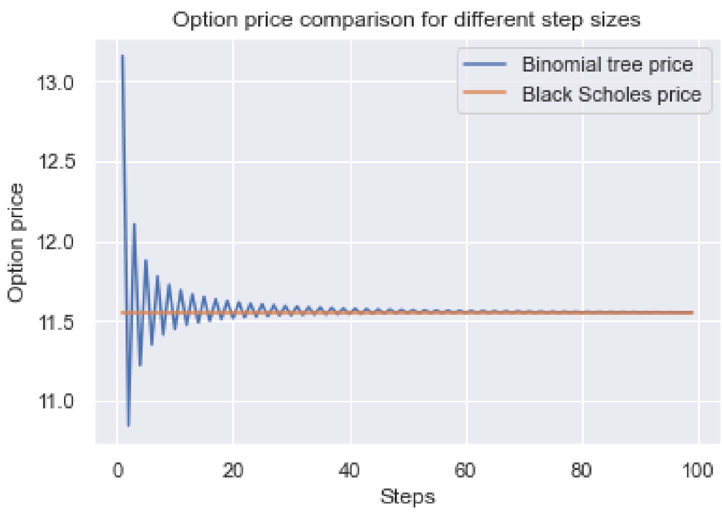

Figure 6: Price approximations using the Binomial Tree method for various step sizes.

Furthermore, the following figure illustrates the convergence rate of the Binomial method:

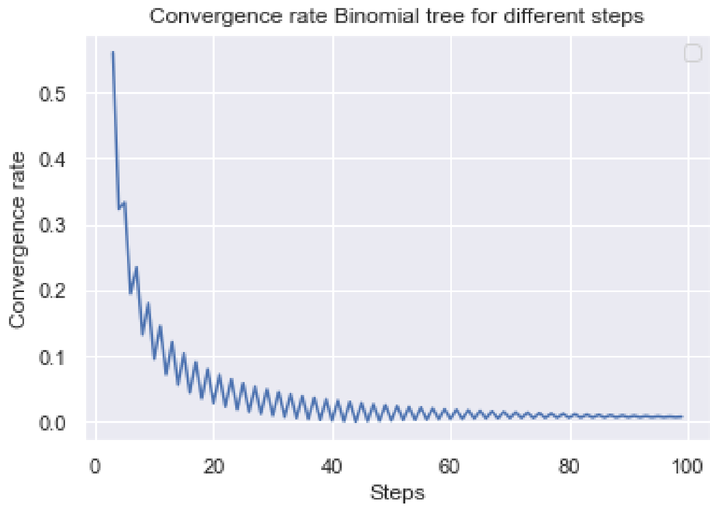

Figure 7: Convergence rate

As can be seen, the Binomial method converges linearly towards the result of the Black Scholes formula, and reaches a sufficiently close result after around 50 steps.

Furthermore, the complexity of the Binomial method is $O(n^2)$ - So while the Binomial method converges linearly, the time complexity increases quadratically - Thus, we would like to keep N as small as possible, while keeping differences between the Binomial and Black Scholes results as low as possible. We believe that a step size of $N = 50$ meet both conditions, but if further accuracy is required from the Binomial method, we see diminishing returns after $N = 100$.

Additionally, in order to verify that the convergence rate of the Binomial method is indeed linear, we introduce the following definition:

Definition 3.1. Order of convergence: A sequence (x_k) converges to L with order $q \geq 1$ if

$$\lim_{k \to \infty} \frac{|x_{k+1} - L|}{|x_k - L|^q} < M \tag{33}$$

where M is some positive constant.
Furthermore, for linear convergence, we require that $q = 1$ and $M < 1$.

Where in our case x_k denotes the approximated price given by the binomial method at step k, and L denotes the price given by the Black Scholes formula. The next figure verifies our assumption that the Binomial Method converges with an order of 1 towards the Black Scholes price since the limit is strictly below 1.

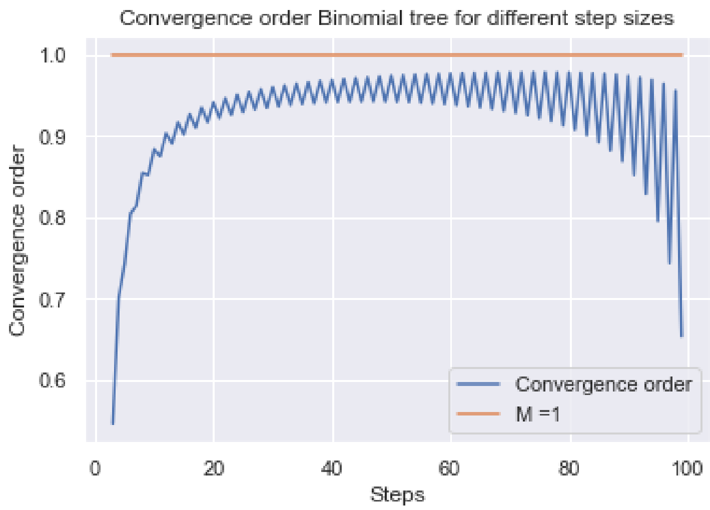

Figure 8: Convergence order

We then continued by calculating the hedge parameter Δ at $t = 0$ for both methods. Δ denotes the change in the stock price compared to the option price and can vary between 0 and 1 for a call option. As discussed in the methods section, Δ is calculated using eq. (21) for the Binomial Method while it is calculated by eq. (28) for the Black Scholes method. For the same initial parameters, we obtained $\Delta = 0.6725$ and $\Delta = 0.6737$ for the Binomial and Black Scholes methods respectively. In order to make a meaningful comparison, we repeated the simulation for various rates of volatility; we started with a rate of 1% up until 300%. The result is illustrated in the following figure:

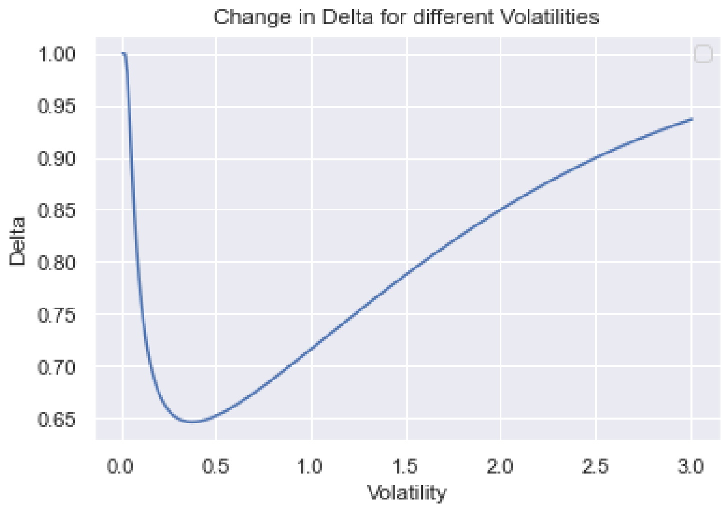

Figure 9: Δ calculated at $t = 0$ using the Black Scholes/Binomial methods for various volatilities

As can be seen, the result given by both methods is almost identical. The differences fluctuated slightly but did need exceed $1e-3$. Furthermore, we can observe the the value of Δ displays a *U-shape* smile, where the value starts high, then quickly decreases, only to start increasing again. Therefore, whenever the volatility rate is between roughly 20% to 50%, the value of Δ is lowest. Up until this point, we were strictly considering European type options for our simulations. As opposed to European options, were the option can only be exercised at the maturity date, American type options can be exercised at any point in time between their purchase and the maturity date. Therefore, we repeated some of the simulations for American options, which required a small change in the algorithm. We calculated the option price for an American option on a non-dividend-paying stock with a maturity of one year with the binomial tree method. The parameters that were used are identical to the ones used for the European option. Namely;

- $S_0 = 100$ denoting the initial stock price
- $K = 99$ denoting the strike price
- $r = 0.06$, denoting the risk-free rate of 6%
- $\sigma = 0.2$, denoting the volatility rate
- $N = 50$, denoting the number of steps for the Binomial Tree

This resulted in an option price of 11.546 for the American call option and an option price of 5.347 for the American put option.

In section 4 the price of an American put and call option is compared for various rates of volatility:

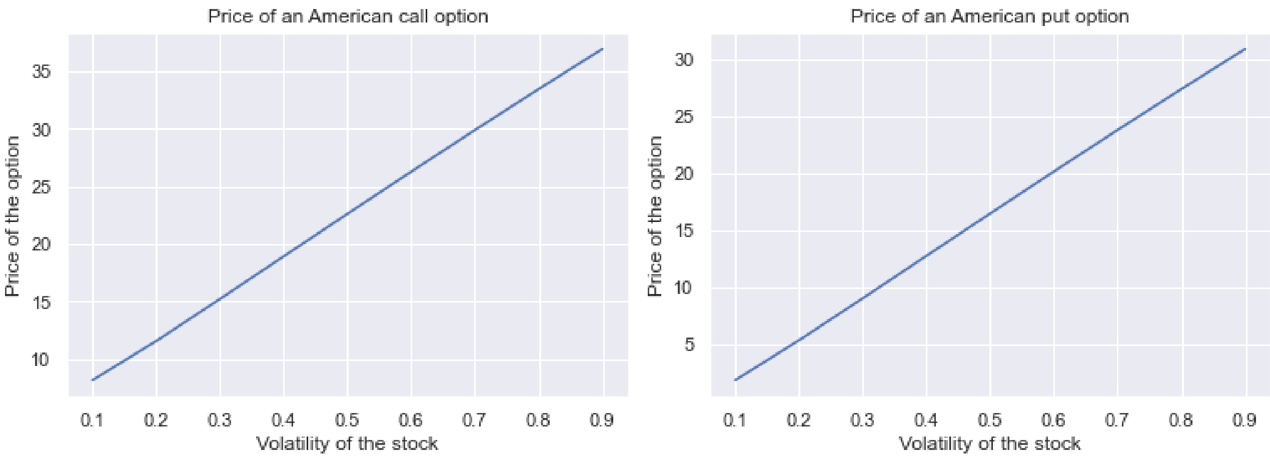

Figure 10: The price of American call and put options under different volatility's of the underlying non-dividend-paying stock determined with the binomial tree method

3.1 Delta-Hedging Simulations

We performed the delta-hedging simulations using the procedure described in section 2.9 and the parameter values listed in section 3.

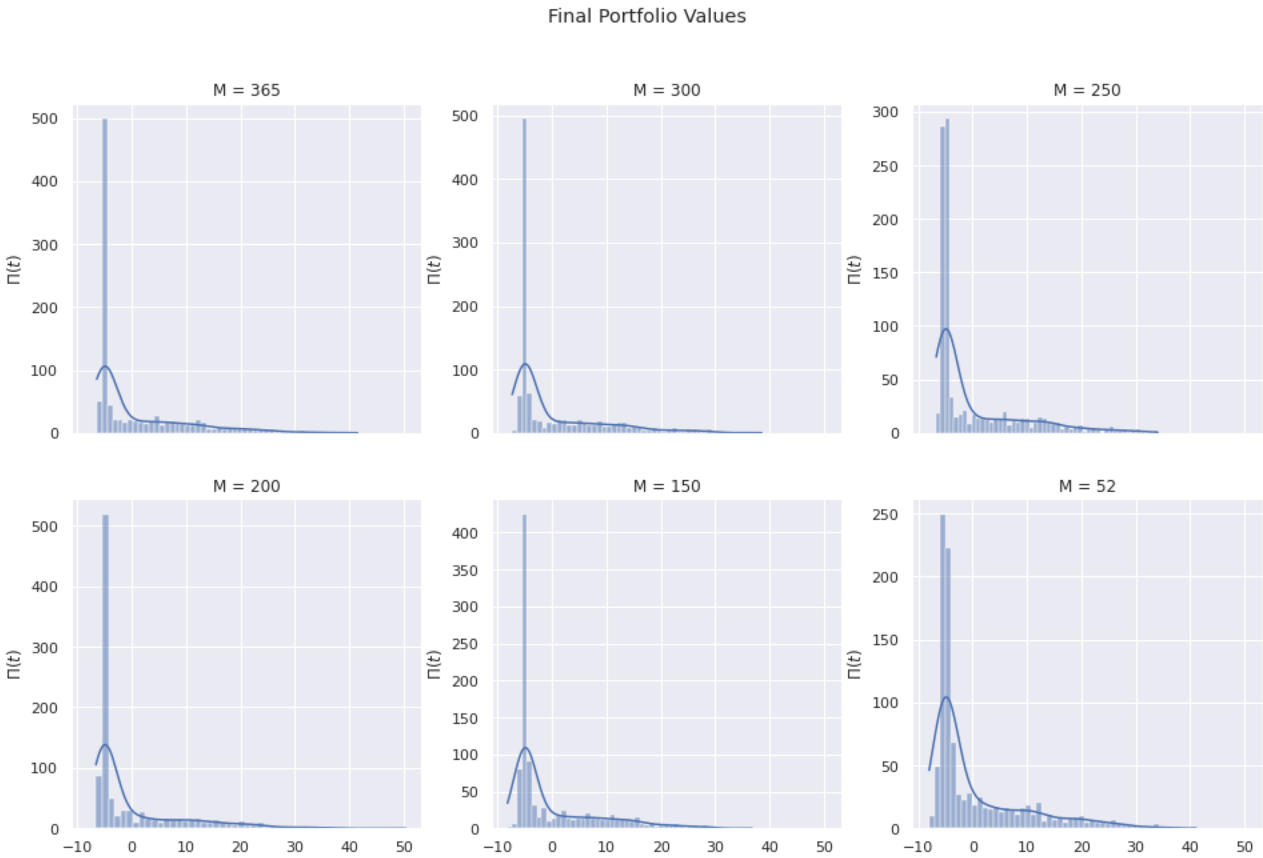

Figure 11: Histograms of $\Pi(T)$ for different hedging frequencies along with kernel density estimations, 1000 trials per M-value.

The simulation in fig. 11 used the same volatility for simulating the stock price and calculating the Δ parameter. We also tried to probe what happens when the volatility used for Δ calculation differs from the volatility used in the stock-price simulation.

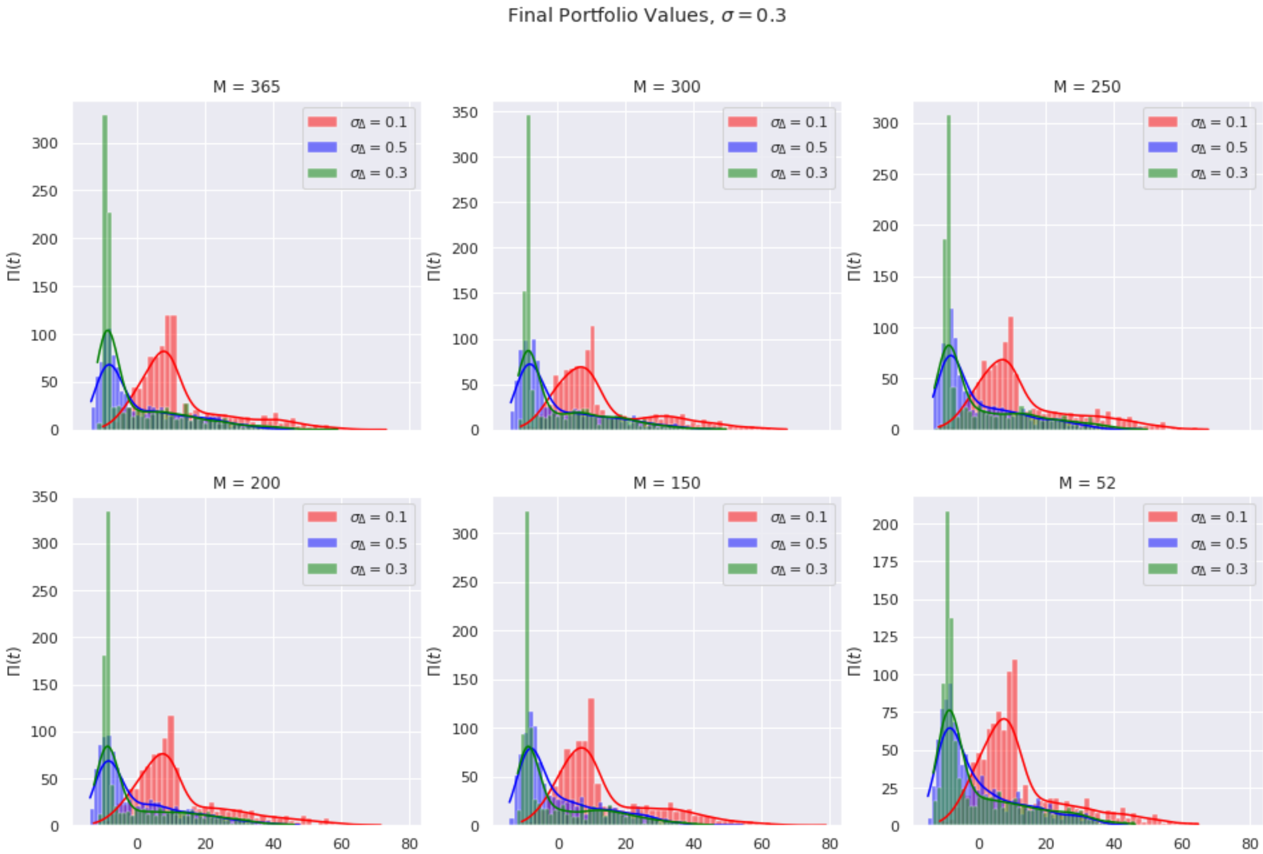

Figure 12: Hedging Simulations when there is a mismatch between the σ values used for the $\Delta(t)$ calculation and that used for the GBM process of the asset price. σ = 0.3 for the share price and σ_Δ is the corresponding value used for the Δ estimation.

4 Discussion

In this paper, we started by introducing some of the underlying theory and background information required for *Option pricing*. We then followed by a number of simulations, exploring how different parameters affect the prices of options and compared the approximated result obtained with the *Binomial Tree Method* to the analytical solution obtained by the *Black Scholes Formula*. When comparing *European-type* options, we have found that for N = 50 steps, the binomial method converges quite closely to the analytical solution. We tried the same thing for various time steps, to find the rate and order of convergence to be equal to 1, while the time complexity of the binomial method is $O(n^2)$. We then explored how this might be affected for different volatility rates, that higher volatilities lead to higher call prices, but that the difference between the binomial method and the black scholes grows in proportion to the option price and stays relatively low for sufficiently large step sizes. However, we did observe that the volatility rate had a significant effect on the value of the hedging parameter, Δ at t = 0. The result showed that the value begins quite high, then drops significantly, and starts growing up again. Hence, the lowest values for Δ were found in the 10% to 50% volatility rates.

We then compared the option price of the *American-type* options to the European options and noticed that the price of the call option does not differ for both type of options, but the price of the put option does differ. This finding is earlier explained in the book of Hull, where it is stated that American and European calls on non-dividend paying stocks should have the same value. However, American put options on non-dividend paying stocks should have an equal price or a larger price compared to European put options [1]. The reason for this is that With a put option you get the option to sell a stock at the strike price K of the put option. If you decide to exercise this option at t=0 then you will receive the strike price K for the stock and can invest it at the risk free rate. This means that at t=1 you will have Ke^{rT}. If you do not exercise the put option at t = 0, but at the maturity time of the option then you would simply receive K. This is why an American put option is worth more at t=0 compared to an European put option.

With an American call option you can not sell a stock but you can buy a stock, so you do not have the option to invest this money. If you would exercise an American call option on a non-dividend-paying stock before expiration than this would discard the time value inherent in the option. Therefore an American and European call option are worth the same.

In a linear relationship is seen between the volatility and the option price for both American options. The more volatile the underlying stock is the more expensive the option will be.

For the delta-hedging simulations, in fig. 11 we expected the portfolio to become “more riskless” for higher hedging frequencies, as the larger hedging frequencies are closer to the continuum approximation of the Black-Scholes model. This would have reflected in the simulation output as a smaller variance in the final portfolio value. But we do not observe meaningful variance reduction in fig. 11. This could be because either our sample size of 1000 was too small, or there is a numerical bug in our simulation source code.

References

[1] J. Hull, Options, Futures, and Other Derivatives. Pearson, 2015.