Discrete Differential Forms for Computational Modeling

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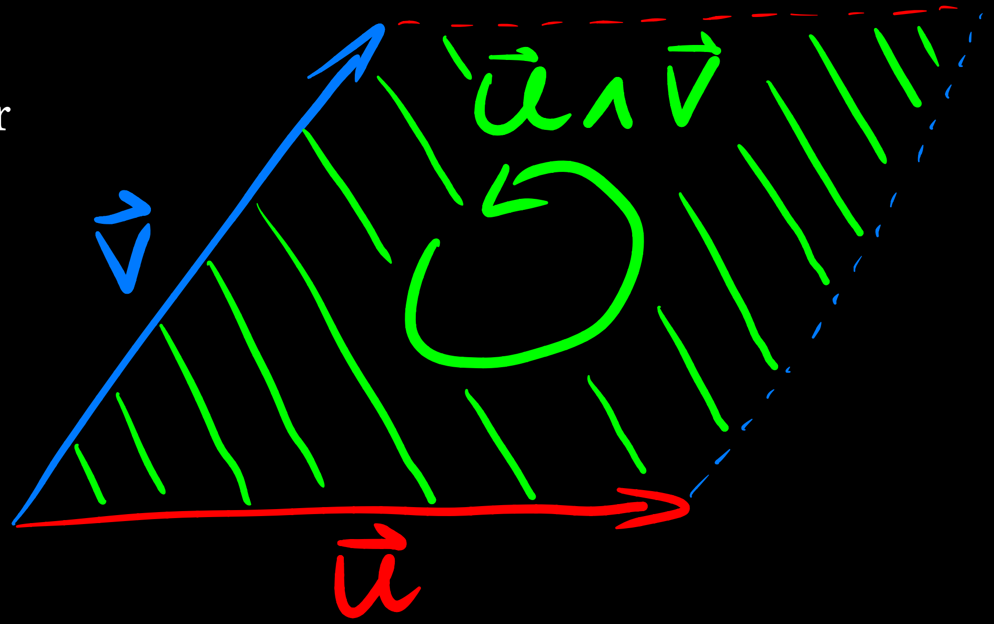
General idea of Differential Forms and Discrete Differential Forms

Idea of a Bivector

Take two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and form their **wedge** / **exterior** product

$$\mathbf{u} \wedge \mathbf{v}$$



The result is called a **bivector** or **2-vector**.

It represents an oriented plane segment.

Exterior product Wedge product

Given a basis of $\mathbb{R}^3$

$$\hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_3$$

This is the basis of the vectors. Using it we can build a basis for bivectors, by considering all the linear independent exterior product combinations of these basis vectors.

$$\hat{\mathbf{e}}_{12} = \hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2 \quad \hat{\mathbf{e}}_{13} = \hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_3 \quad \hat{\mathbf{e}}_{23} = \hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_3$$

The cross product induces this ordering of the bivector basis.

$$\hat{\mathbf{e}}_{23} \quad \hat{\mathbf{e}}_{31} \quad \hat{\mathbf{e}}_{12}$$

$$\hat{\mathbf{e}}_1 \wedge \hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_2 \wedge \hat{\mathbf{e}}_3, \quad \hat{\mathbf{e}}_3 \wedge \hat{\mathbf{e}}_1$$

$$\mathrm{d}f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \frac{\partial f}{\partial x_1}\,\mathrm{d}x^1 + \ldots + \frac{\partial f}{\partial x_n}\,\mathrm{d}x^n$$

differential form ω

Exterior Derivative

$$\mathrm{d}^k : \wedge^k(\Omega) \to \wedge^{k+1}(\Omega)$$

Stokes' Theorem

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$