

Loopless Gray Code Enumeration and the Tower of Bucharest

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Abstract

We give new algorithms for generating all n -tuples over an alphabet of m letters, changing only one letter at a time (Gray codes). These algorithms are based on the connection with variations of the Towers of Hanoi game. Our algorithms are loopless, in the sense that the next change can be determined in a constant number of steps, and they can be implemented in hardware. We also give another family of loopless algorithms that is based on the idea of working ahead and saving the work in a buffer.

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1 Introduction: the binary reflected Gray code and the Towers of Hanoi

1.1 The Gray code

The Gray code, or more precisely, the reflected binary Gray code G_n , orders the 2^n binary strings of length n in such a way that successive strings differ in a single bit. It is defined inductively as follows, see Figure 1 for an example. The Gray code $G_1 = 0, 1$, and if $G_n = C_1, C_2, \dots, C_{2^n}$ is the Gray code for the bit strings of length n , then

$$G_{n+1} = 0C_1, 0C_2, \dots, 0C_{2^n}, 1C_{2^n}, 1C_{2^n-1}, \dots, 1C_2, 1C_1. \quad (1)$$

In other words, we prefix each word of G_n with 0, and this is followed by the reverse of G_n with 1 prefixed to each word.

000000	001011	010111	110100	101110	0000	0111	0222	1112	1001	2120	2220
000001	001001	010110	111100	101111	0001	0112	1222	1111	1000	2110	2221
000011	001000	010010	111101	101101	0002	0102	1221	1110	2000	2111	2222
000010	011000	010011	111111	101100	0012	0101	1220	1120	2001	2112	
000110	011001	010001	111110	100100	0011	0100	1210	1121	2002	2102	
000111	011011	010000	111010	100101	0010	0200	1211	1122	2012	2101	
000101	011010	110000	111011	100111	0020	0201	1212	1022	2011	2100	
000100	011110	110001	111001	100110	0021	0202	1202	1021	2010	2200	
001100	011111	110011	111000	100010	0022	0212	1201	1020	2020	2201	
001101	011101	110010	101000	100011	0122	0211	1200	1010	2021	2202	
001111	011100	110110	101001	100001	0121	0210	1100	1011	2022	2212	
001110	010100	110111	101011	100000	0120	0220	1101	1012	2122	2211	
001010	010101	110101	101010		0110	0221	1102	1002	2121	2210	

Figure 1 The binary Gray code G_6 for 6-tuples and the ternary Gray code for 4-tuples.

1.2 Loopless algorithms

The Gray code has an advantage over alternative algorithms for enumerating the binary strings, for example in lexicographic order: one can change a binary string $a_n a_{n-1} \dots a_1$ to the successor in the sequence by a single update of the form $a_i := 1 - a_i$ in constant time. However, we also have to *compute* the position i of the bit which has to be updated. A straightforward implementation of the recursive definition (1) leads to an algorithm with an optimal overall runtime of $O(2^n)$, i.e., constant average time per enumerated bit string.

A stricter requirement is to compute each successor string in constant *worst-case* time. Such an algorithm is called a *loopless* generation algorithm. Loopless enumeration algorithms for various combinatorial structures were pioneered by Ehrlich [3], and different loopless algorithms for Gray codes are known, see Bitner, Ehrlich, and Reingold [1] and Knuth [6, Algorithms 7.2.1.1.L and 7.2.1.1.H]. These algorithms achieve constant running time by maintaining additional pointers.

1.3 The Tower of Hanoi

The Tower of Hanoi is the standard textbook example for illustrating the principle of recursive algorithms. It has n disks D_1, D_2, \dots, D_n of increasing radii and three pegs P_0, P_1, P_2 , see Fig. 2. The goal is to move all disks from the peg P_0 , where they initially rest, to another peg, subject to the following rules:

1. Only one disk may be moved at a time: the topmost disk from one peg can be moved on top of the disks of another peg
2. A disk can never lie on top of a smaller disk.

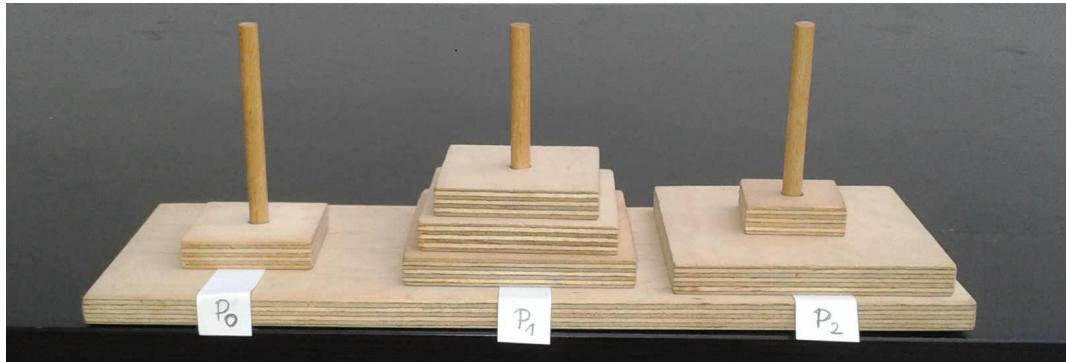


Figure 2 The Towers of Hanoi with $n = 6$ (square) disks. When running the algorithm HANOI from Section 1.5, the configuration in this picture occurs together with the bit string 110011. (There is no easy relation between the positions of the disks and this bit string.) The next disk to move is D_1 ; it moves clockwise to peg P_0 , and the last bit is complemented. The successor in the Gray code is the string 110010. After that, D_1 pauses for one step, while disk D_3 moves, again clockwise, from P_1 to P_2 , and the third bit from the right is complemented, leading to the string 110110.

For moving a tower of height n , one has to move disk D_n at some point. But before moving disk D_n from peg A to B , one has to move the disks D_1, \dots, D_{n-1} , which lie on top of D_n , out of the way, onto the third peg. After moving D_n to B , these disks have to be moved from the third peg to B . This reduces the problem for a tower of height n to two towers of height $n - 1$, leading to the following recursive procedure.

```

move_tower( $k, A, B$ ): (Move the  $k$  smallest disks  $D_1 \dots D_k$  from peg  $A$  to peg  $B$ )
  if  $k \leq 0$ : return
  auxiliary :=  $3 - A - B$ ; (auxiliary is the third peg, different from  $A$  and  $B$ .)
  move_tower( $k - 1, A, auxiliary$ )
  move disk  $D_k$  from  $A$  to  $B$ 
  move_tower( $k - 1, auxiliary, B$ )

```

1.4 Connections between the Towers of Hanoi and Gray codes

The *delta sequence* of the Gray code is the sequence $1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, \dots$ of bit positions that are updated. (In contrast to the usual convention, we number the bits starting from 1.) This sequence has an obvious recursive structure which results from (1). It also describes the number of changed bits when incrementing from i to $i + 1$ in binary counting. Moreover, it is easy to observe that the same sequence also describes the disks that are moved by the recursive algorithm *move_tower* above. It has thus been noted that the Gray code G_n can be used to solve the Tower of Hanoi puzzle, cf. Scorer, Grundy, and Smith [8] or Gardner [4]. In the other direction, the Tower of Hanoi puzzle can be used to generate the Gray code G_n , see Buneman and Levy [2].

Several loopless ways to compute the next move for the Towers of Hanoi are known, and they lead directly to loopless algorithms for the Gray code. We describe one such algorithm.

1.5 Loopless Tower of Hanoi and binary Gray code

From the recursive algorithm *move_tower*, it is not hard to derive the following fact.

► **Proposition 1.** *If the tower should be moved from P_0 to P_1 and n is odd, or if the tower should be moved from P_0 to P_2 and n is even, the moves of the odd-numbered disks always proceed in forward (“clockwise”) circular direction: $P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow P_0$, and the even-numbered disks always proceed in the opposite circular direction: $P_0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$. ◀*

In the other case, when the assumption does not hold, the directions are simply swapped. Since we are interested not in moving the tower to a specific target peg, but in generating the Gray code, we stick with the proposition as stated.

Algorithm HANOI. Loopless algorithm for the binary Gray code.

Initialize: Put all disks on P_0 .

loop:

 Move D_1 clockwise.

 Let D_k be the smaller of the topmost disks on the two pegs that don't carry D_1 .

 If there is no such disk, terminate.

 Move D_k clockwise if k is odd; otherwise, move it counterclockwise.

To obtain the Gray code, we simply set $a_k := 1 - a_k$ whenever we move the disk D_k [2]. See Fig. 2 for a snapshot of the procedure.

We would not need the clockwise/counterclockwise rule for D_k : Since we must not put D_k on top of D_1 , there is anyway no choice of where to move it [2]. We have chosen the above formulation since it is better suited for generalization.

1.6 Overview

In the remainder of this paper, we will generalize these connections to Gray codes for larger radices (alphabet sizes). Section 2 is devoted to ternary Gray codes and their connections to the so-called *Towers of Bucharest*. After defining Gray codes with general radices in Section 3, we extend the ternary algorithm from Section 2 to arbitrary odd radices m in Section 4, and even to mixed (odd) radices (Section 6). In Section 5, we generalize the binary Gray code algorithm of Section 1.5 to arbitrary even m . Finally, in Section 8, we develop loopless algorithms bases on an entirely different idea of “working ahead” that is related to converting amortized running-time bounds to worst-case bounds. In the appendix, we give prototype code for simulating all our algorithms in PYTHON.

2 Ternary Gray codes and the Towers of Bucharest

A ternary Gray code enumerates the 3^n n -tuples (a_n, \dots, a_1) with $a_i \in \{0, 1, 2\}$. Successive tuples differ in one entry, and in this entry they differ by ± 1 .

The following simple variation of the Towers of Hanoi will yield a ternary Gray code ($m = 3$): *We disallow the direct movement of a disk between pegs P_0 and P_2 :* a disk can only be moved to an adjacent peg. We call this the Towers of Bucharest.¹

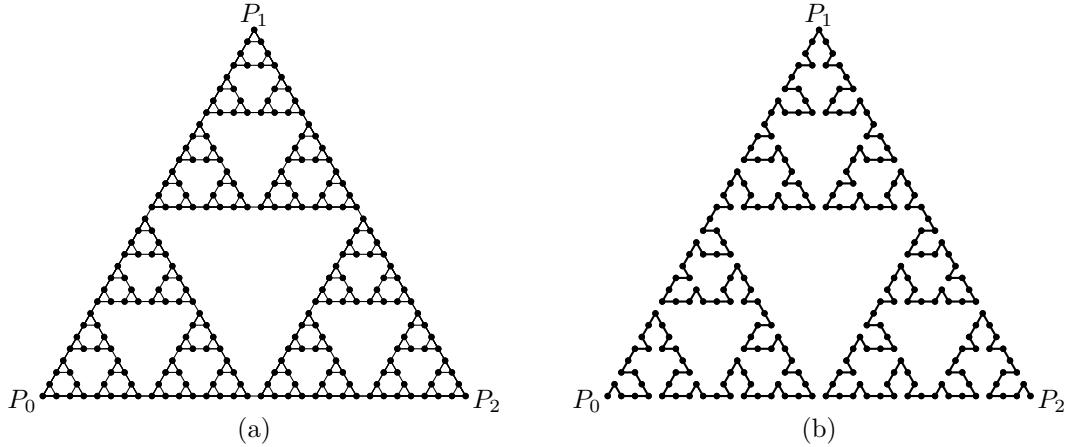


Figure 3 The state graphs of (a) the Tower of Hanoi and (b) the Tower of Bucharest with 5 disks

Figure 3 shows the state space of the Towers of Bucharest in comparison with the Towers of Hanoi. In accordance with this figure, we can make the following easy observations:

- **Proposition 2.** 1. *In the towers of Hanoi, there are three possible moves from any position, except when the whole tower is on one peg: In these cases, there are only two possible moves.*
- 2. *In the towers of Bucharest, there are two possible moves from any position, except when the whole tower is on peg P_0 or P_2 : In those cases, there is only one possible move.*

¹ The custom of naming variations of the Tower of Hanoi game after different cities, instead of using ordinary names such as “three-in-a-row” [7], has a long tradition. The name “Towers of Bucharest” has been suggested by Günter M. Ziegler. Several legends rank themselves around these towers.

A little count from Transylvania had conquered the whole country and had become a powerful Lord. In order to celebrate his glory, he built a magnificent palace in the capital city Bucharest, and he suppressed his people as best he could. He also had a dog named Heisenberg. In a nearby monastery, the monks had golden disks of different sizes on three pegs, and they had played the Towers of Hanoi game for centuries. It was already foreseeable that the game was drawing towards its conclusion. According to an ancient prophecy, the palace of the ruler of the country would crumble and his rule would come to an end when the game would be finished. When the count, who called himself king by this time, heard this story, he did not like it. *First* of all, he had the monks beheaded and told them to do some useful job instead. *Secondly*, he removed the pegs with the discs and took them to his palace. He made sure that they were placed very far away from each other: The first peg was put in the South wing, the second peg in the North wing, and the third peg again in the South wing. One may wonder why he did not place them in some more logical arrangement like South-South-North or South-North-North, or perhaps North-middle-South or South-middle-North. The reader will soon understand that this placement was a clever decision when she or he learns what else he did. The North wing could only be reached from the South wing through the middle wing, or by going out on the street and reentering on the other side, but I don't think it is very wise to go into the street carrying a heavy golden disk.

Anyway, his *third* action was his most wicked and smartest move: It occurred to him that *he was powerful and he was the ruler, and he therefore had the power to change the rules*. He decreed that the discs can only be moved between the first and the second peg or between the second and the third peg. Direct moves between the first and third peg were henceforth forbidden. This would delay the moves of

- Proof.** 1. The disk D_1 can be moved to any of the other pegs (two possible moves). In addition, the smaller of the topmost disks on the other pegs (if those pegs aren't both empty) can be moved to the other peg which is not occupied by D_1 .
2. If the disk D_1 is in the middle, it can be moved to any of the other pegs, but no other move is possible. If the disk D_1 is on P_0 or P_2 , it has only one possible move, and the smaller of the topmost disks on the other pegs (if those pegs aren't empty) also has one possible move, similarly as above. \blacktriangleleft

Both games have the same set of 3^n states, corresponding to the possible ways of assigning each disk to one of the pegs P_0, P_1, P_2 . The nodes in the corners marked P_0, P_1, P_2 represent the states where all disks are on one peg. The graph of the Towers of Hanoi in Figure 3a approaches the Sierpiński gasket. The optimal path of length $2^n - 1$ is the straight path from P_0 to P_2 . (The directions of the edges in this drawing of the state graph are not directly related to the pegs that are involved in the exchange, and the relation between a state and its position on the drawing is not straightforward.) By contrast, we see that the graph of the Towers of Bucharest in Figure 3b is a single path through all nodes.

Let us see why this is true. By Proposition 2, this graph has maximum degree 2, and it follows that it must consist of a path between P_0 and P_2 (the only degree-1 nodes), plus a number of disjoint cycles. However, it is known that the path has length $3^n - 1$ and does therefore indeed go through all nodes. Since we will prove a more general statement later (Theorem 3), we only sketch the argument here: Solving the problem recursively in an analogous way to the procedure *move_tower*, we reduce the problem of moving a tower of n disks from P_0 to P_2 (or vice versa) to three problem instances with $n - 1$ disks, plus two movements of disk D_n , and the resulting recursion establishes that $3^n - 1$ moves are required.

The states of the Towers of Bucharest correspond in a natural way to the ternary n -tuples: The digit $a_i \in \{0, 1, 2\}$ gives the position of disk D_i . It follows now easily that the solution of the Towers of Bucharest yields a ternary Gray code: Since we can move only one disk at a time, it means that we change only one digit at a time, and by the special rules of the Towers of Bucharest, we change it by ± 1 . (This connection has apparently not been made before.) In fact, the algorithm produces the ternary reflected Gray code, which we are about to define below in Section 3; see also Theorem 3.

the disks, since they always had to be carried all the way from the South wing to the North wing and back. As shown in this article, the consequences of the new rule in delaying the game are even more spectacular. These measures were definitely overcautious, in particular since nobody was there to move the discs any more, and moreover, the pegs with the golden discs were carefully guarded. Nevertheless, he was worried that his wife and children would wander around in the palace and play with the disks, thereby setting the prophecy into motion again, like in that movie, "Jumanji" with Robin Williams. He was not sure how the guarding officers would behave in a conflict between the loyalty to their orders and the authority of his family members. You may draw your own conclusions, but in my opinion, this count, or king if you wish, was pretty paranoid. In the end, it served him nothing. He was swept away by the revolution. What became of the golden disks? Nobody knows. It is sometimes claimed that they were hidden in a subterranean cave, and hobby archaeologists are still looking for them occasionally. But probably they have found their way to the black market. Today, tourists that visit the palace are led to a stump on the floor in the North wing, which is supposedly the remainder of one of the pegs. The South wing is closed for restoration.

Another story, even more unbelievable but no less bloody than the first one, puts the Towers of Bucharest in the context of the legendary caliph Harun-al-Rashid. One night, the caliph was again wandering through the streets of Baghdad, as usual dressed like an ordinary businessman, in order to assure himself that the people were still loving his reign, admiring his wisdom, and praising his justice. He noticed a crowd of lookers-on who were gathered around a man and a woman sitting on the ground side by side, silently and solemnly executing the moves of the Towers of Bucharest. One of them would pick up a disk and set it on an adjacent peg. By the rules of the game, there was always one of them for whom the two involved pegs were easy to reach, and this was the one who carried out the move. Only on the infrequent occasions when one of the larger and heavier disks had to be moved, they helped each other. The man wore a

Moreover, since there are only two possible moves, one just has to always choose the move which does not undo the previous move, and this leads to a very easy loopless Gray code enumeration algorithm.

It is remarkable that ternary Gray codes can be generated on the same hardware as binary Gray codes (Fig. 2). In the context of generating the ternary Gray code, the Gray code string can be directly read off the disks. For example, the configuration in Fig. 2 represents the string 211102. It is D_1 's turn to move, and the disk D_1 will make two steps to the left, generating the strings 211101 and 211100, and pauses there for one step, while disk D_3 moves to the right, leading to the string 211200, etc.

3 Gray codes with general radices

An m -ary Gray code enumerates the n -tuples (a_n, \dots, a_1) with $0 \leq a_i < m$, changing a single digit at a time by ± 1 . The m -ary reflected Gray code can be recursively described as follows: Let C_1, C_2, \dots, C_{m^n} be the Gray code for the strings of length n . Then the strings of length $n + 1$ are generated in the order

$$\begin{aligned} C_{10}, C_{11}, C_{12}, \dots, C_{1(m-2)}, C_{1(m-1)}, \quad & C_{2(m-1)}, C_{2(m-2)}, \dots, C_{22}, C_{21}, C_{20}, \\ C_{30}, C_{31}, C_{32}, \dots, C_{3(m-2)}, C_{3(m-1)}, \quad & C_{4(m-1)}, C_{4(m-2)}, \dots, C_{42}, C_{41}, C_{40}, \quad (2) \\ C_{50}, C_{51}, C_{52}, \dots, C_{5(m-2)}, C_{5(m-1)}, \quad & \dots \end{aligned}$$

Each digit alternates between an upward sweep from 0 to $m - 1$ and a return sweep from $m - 1$ to 0.

4 Generating the m -ary Gray code with odd m

For odd m , the ternary algorithm can be generalized. We need m pegs P_0, \dots, P_{m-1} . The leftmost peg P_0 and the rightmost peg P_{m-1} play a special role. The other pegs are called the *intermediate pegs*.

Algorithm ODD. Generation of the m -ary Gray code for odd m .

Initialize: Put all disks on P_0 .

loop:

Move D_1 for $m - 1$ steps, from P_0 to P_{m-1} or vice versa.

Let D_k be the smallest of the topmost disks on the $m - 1$ pegs that don't carry D_1 .

If there is no such disk, terminate.

Move D_k by one step:

If D_k is on P_0 or P_{m-1} , there is only one possible direction where to go.

Otherwise, the disk D_k continues in the same direction as in its last move.

cowboy hat, and the woman was in her bikini. After all, it might have been the Towers of Hanoi that they played. Some witnesses have later reported that they had seen a disc jumping between the first and the third peg, but this has never been conclusively confirmed.

May that as it be, something unexpected happened. As the khaliph was engrossed in watching the spectacle and drew a bit closer, a small door in the wall beside him opened, which he had not noticed before. It gave onto a small garden. The moon had risen over the rooftops, and her light gave a sort-of surreal atmosphere to the whole scene. In the middle of the garden, at the corner of a fountain, a woman sang, accompanying herself on the lute. She had a beautiful voice, a bit like Mariah Carey or Adele. The calif would have listened longer, but he was quickly escorted into a house, where a maid-servant took charge of him and handed him a black gown. "Hurry up, you are late. We were waiting only for you!" The gown covered his whole stature and hid his face, and he entered a room that was barely lit by an open fire. Seven other men in black gowns were already sitting on small stools in a circle around the fire. One stool was free, and he sat down. Beside the fire, there was a small ivory model of the Towers of Bucharest, with the four largest of the $n = 6$ disks on the final peg. Disc 1 was on the middle peg, and disc 2 was on the first peg. The khaliph, having watched the game just before, understood immediately what that meant. Nowbody said a word. The tension rose. After six minutes, a lady entered and addressed them. She was the singer from the fountain. "Gentlemen. You have sworn to come to my rescue when I would be in need. Now the time has come to fulfill your oath. You see seven discs of different sizes. He who will draw the smallest disk will bring me the head of the detestable caliph Harun-al-Rashid (ca. 763–809). Should he fail to fulfill this task, the other eight will kill him, and we will come together and draw again." With these words, she dropped the discs into a chalice. In silence, each man picked a disk. The khaliph was last to draw. As he opened his hand, sure enough, he found the smallest disc, disc number 1. He rose and said: "Fair lady, I will fulfill your order as I have promised. But pray tell me: by which deeds or words has the kalif enraged you so much that you wish him to

Figure 4 shows an example with $m = 5$. The game with 5 pegs is called the Tower of Klagenfurt, after the birthplace of the senior author.²



Figure 4 The Towers of Klagenfurt. This configuration represents the string 321411 over the radix $m = 5$. The next step of the Gray code moves the smallest disk D_1 onto peg P_0 , changing the string to 321410. After that, disk D_2 moves from P_1 to P_2 and the next string is 321420. In the background, the two-headed Lindworm monster.

In this procedure, the movement of D_1 is “externally given”, whereas the movement of the other disks, whenever D_1 is at rest, is somehow “determined by the algorithm”. It is not obvious that the algorithm does not put a larger disk on top of D_1 .

► **Theorem 3.** *Algorithm ODD generates the m -ary reflected Gray code defined in (2).*

Proof. It is clear from the algorithm that the last digit, which is controlled by the movement of D_1 , changes in accordance with (2). We still have to show that when we discard the last digit and observe only the movement of the disks D_2, \dots, D_n , the algorithm produces the Gray code for the strings of length $n - 1$. This is proved by induction.

By the rules of the algorithm, whenever D_1 rests, the disk that moves is D_2 , unless D_2 is covered by D_1 . Let us now observe the motion pattern of D_1 and D_2 that results from this rule. We start with D_1 on top of D_2 , say, on peg P_0 , with D_1 about to start its sweep. Whenever D_1 pauses for one step, D_2 will make a step towards P_{m-1} . After D_2 reaches P_{m-1} , it turns out that, because m is odd, D_1 will make its next sweep from P_0 to P_{m-1} , resting on top of D_2 . Now, since D_2 is covered, it will be one of the other disks D_3, D_4, \dots that will move. Then the same routine repeats in the other direction.

If we now ignore D_1 and look only at the motions of the other disks, the following pattern emerges: D_2 makes $m - 1$ steps from one end to the other, and then the smallest disk that is not covered by D_2 makes its move, according to the rules. This is precisely the same procedure as Algorithm ODD, with D_2 taking the role of the “externally controlled” disk D_1 , and we have assumed by induction that this algorithm correctly produces the Gray code for the strings of length $n - 1$, and it does not put a larger disk on top of D_2 . Since the larger disks are moved only when D_2 lies under D_1 , it follows that a larger disk cannot be moved on top of D_1 either. ◀

² When Klagenfurt was founded, it was surrounded by a swamp. The swamp was inhabited by a dinosaur, the so-called *Lindworm*. The Lindworm would regularly come to the city and eat citizens. Occasionally, she would devour one of the towers of the city. The coat of arms of Klagenfurt shows the Lindworm dragon in front of the only remaining tower, see Figure 4. (Initially, there were five towers.) Over the centuries, the swamp has been drained, and the Lindworm is practically extinct.

One can actually apply one induction step of the proof in the opposite direction, introducing an additional “control disk” D_0 which does not have a digit associated with it. Its only role is to alternately cover P_0 and P_{m-1} and exclude these pegs from the selection of the disk D_k that should be moved. The algorithm becomes simpler because it does not have to treat D_1 separately from the other disks. (See Appendix A.3, where this idea is applied to the algorithm of Section 6 below).

5 Generating the m -ary Gray code with even m

For even m , we generalize Algorithm HANOI, which solves the case $m = 2$. We use $m + 1$ pegs P_0, \dots, P_m , which we arrange in a cyclic clockwise order. We stipulate that disks D_i with odd i move only clockwise, and disks with even i move only counterclockwise.

Algorithm EVEN. Generation of the m -ary Gray code for even m .

Initialize: Put all disks on P_0 .

loop:

Move D_1 for $m - 1$ steps, in clockwise direction.

Let D_k be the smallest of the topmost disks on the m pegs that don’t carry D_1 .

If there is no such disk, terminate.

Move D_k by one step, in the direction determined by the parity of k .

The Gray code is determined by changing the digit a_i whenever disk D_i is moved. The digit a_i runs through the sequence $0, 1, 2, \dots, m - 2, m - 1, m - 2, \dots, 2, 1, 0, 1, 2, \dots$. Thus it changes always by ± 1 , but we have to remember whether it is on the increasing or the decreasing part of the cycle. The position of disk D_i is no longer directly correlated with the digit a_i ; thus the digits a_i have to be maintained separately, in addition to the disks on the pegs. It is far from straightforward to relate the disk configuration to the Gray code.

For example, when carrying out the algorithm for $m = 4$, the configuration in Figure 4 appears when the string is 211030. Disk D_1 has just made three steps and is going to rest for one step. The next step moves D_3 clockwise, since 3 is odd, and the string is changed to 211130. After that, D_1 resumes its clockwise motion, and the string changes into 211131.

► **Theorem 4.** *Algorithm EVEN generates the m -ary reflected Gray code defined in (2).*

Proof. This follows along the same lines as Theorem 3. When we look at the pattern of motion of D_1 and D_2 , we observe again that D_2 makes $m - 1$ steps until it is covered by D_1 , see Fig. 5: After the first move of D_2 , the clockwise cyclic distance from D_1 to D_2 is 1, and with each move of D_2 , this distance increases by 1. Thus, after $m - 1$ moves, the distance becomes $m - 1$, and D_1 will land on top of D_2 with its next sweep. ◀

Algorithms ODD and EVEN do not generate a shortest sequence of moves to the target configuration except when $m = 3$ or $m = 2$. We could not come up with some set of natural constraints under which our algorithms give a shortest solution.

6 The Towers of Bucharest++

In Algorithm ODD, the intermediate pegs P_1, \dots, P_{m-2} will always be available for selecting the smallest disk D_k to be moved. Thus, one can coalesce these pegs into one peg, keeping only the two extreme pegs separate. With three pegs, we can use the same hardware as the tower of Bucharest, but we have to record the value of the digits, since they are no longer expressed by the position. A simple method is to provide the disks with *marks* that indicate

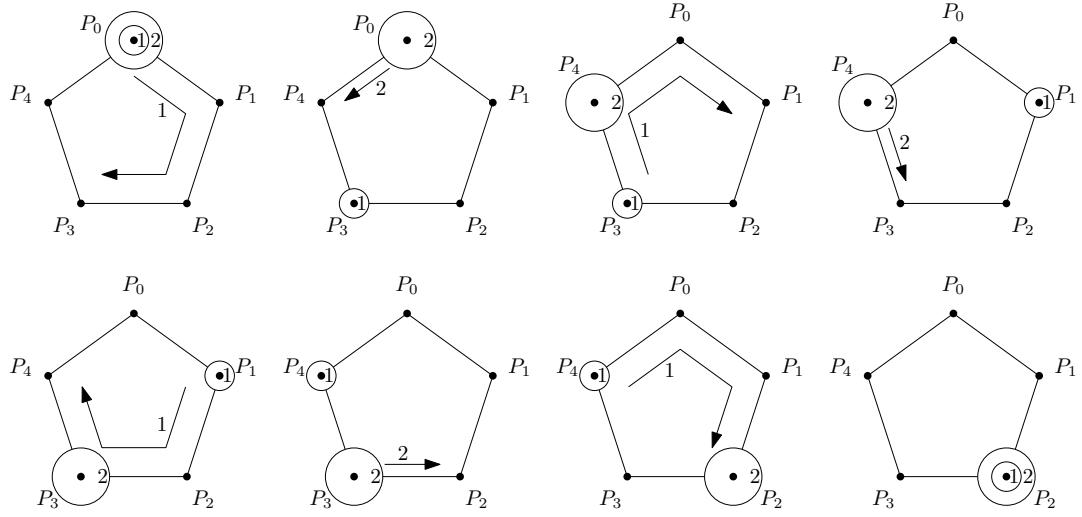


Figure 5 One period of movement of the two smallest disks D_1 and D_2 when Algorithm EVEN generates all tuples over an alphabet of size $m = 4$ using $m + 1 = 5$ pegs.

the value as well as the direction of movement, which we have to remember anyway. Each disk cycles through $2m - 2$ values, potentially augmented with direction information:

$$0, 1\uparrow, 2\uparrow, \dots, (m-2)\uparrow, m-1, (m-2)\downarrow, \dots 2\downarrow, 1\downarrow, 0, 1\uparrow, \dots \quad (3)$$

It makes sense to encode this information like a dial with $2m - 2$ equally spaced directions, as shown in Fig. 6a. A disk whose mark shows 0 is always on the left peg P_0 . A disk whose mark shows $m - 1$ is always on the right peg P_2 . Otherwise, it is on the middle peg P_1 . When we say we *turn a disk*, this means that we turn it clockwise to the next dial position, and if necessary, move it to the appropriate peg.

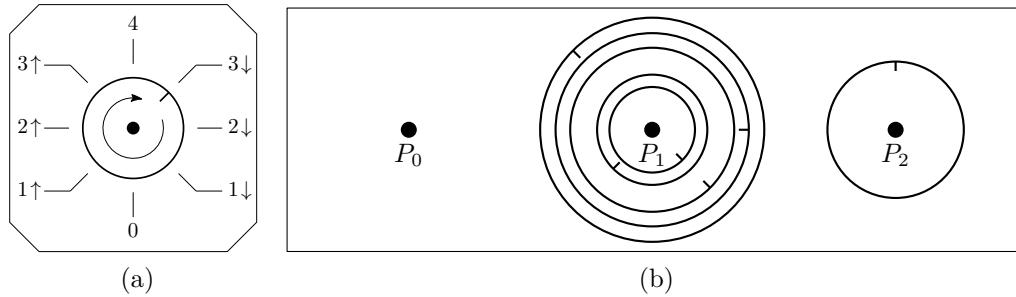


Figure 6 (a) The upgraded disk of the Towers of Bucharest++ and the meaning of its positions, for $m = 5$. (b) The situation of Figure 4, compressed to 3 pegs.

Algorithm ODD-COMPRESSED. Generation of the m -ary Gray code for odd m .

Initialize: Put all disks on P_0 , and turn them to show 0.

loop:

Turn disk D_1 $m - 1$ times until it arrives at one of the extreme pegs P_0 or P_2 .

Let D_k be the smaller of the topmost disks on the two pegs not covered by D_1 .

If there is no such disk, terminate.

Turn D_k once.

The digits a_i can be read off from the dial positions. Correctness follows by comparison with Algorithm ODD, checking that the transition between successive states is preserved when merging the intermediate pegs into one peg. \blacktriangleleft

This algorithm can now even be generalized to mixed-radix Gray codes for the n -tuples (a_n, \dots, a_1) with $0 \leq a_i < m_i$, for some sequence of radixes $m_i \geq 2$, provided that all m_i are odd.

7 Simulation

All our algorithms can be easily simulated in software on a digital computer.³ A stack will do for each peg. If there are k pegs, the algorithm takes $O(k)$ time to compute the next move and accordingly produce the next element of the Gray code sequence. If m is constant, then $k = m$ or $k = m + 1$ in Algorithms ODD and EVEN, and these algorithms can pass as loopless algorithms. If k is large, Algorithm ODD can be replaced by ODD-COMPRESSED, which has only 3 pegs, independent of m .

To make a truly loopless algorithm out of Algorithm EVEN, at the expense of an increased overhead, we can use the following easy fact, which follows directly from the algorithm statement.

► **Lemma 5.** *In the algorithms EVEN, ODD, and ODD-COMPRESSED, when a disk D_k is moved, all smaller disks D_1, \dots, D_{k-1} are on the same peg.* \blacktriangleleft

To get a loopless implementation, the set of disks on a peg has to be maintained as a sequence of maximal intervals of successive integers, instead of storing them as a stack in the usual way. Then, whenever D_1 is at rest, the disk D_k to be moved can be determined in constant time as the smallest missing disk on the peg containing D_1 .

8 Working ahead

While we are at the topic of Gray codes, we might as well mention another approach for loopless generation of Gray codes, which results from a generally applicable technique for converting amortized bounds into worst-case bounds. We start from the observation that was already mentioned in connection with the delta-sequence in Section 1.4:

► **Proposition 6.** *Consider the enumeration of the n -tuples (b_n, \dots, b_1) with $0 \leq b_i < m_i$ in lexicographic order. If, between two successive tuples of the sequence, the j rightmost digits are changed, then, at the corresponding transition in the Gray code, the j -th digit from the right is changed.* \blacktriangleleft

We can thus find the position j that has to be changed in the Gray code by lexicographically “incrementing” n -tuples (b_n, \dots, b_1) in a straightforward way:

³ Nowadays, most households will more readily have access to a computer than to towers of Hanoi.

Algorithm DELTA. Generation of the delta-sequence for the Gray code.

Initialize: $(b_n, \dots, b_2, b_1) := (0, 0, \dots, 0, 0)$

$Q :=$ an empty list

loop:

$j := 1$

while $b_j = m_j - 1$:

$b_j := 0$

$j := j + 1$

if $j = n + 1$: TERMINATE

$b_j := b_j + 1$

$Q.append(j)$

The delta sequence is stored in Q . It is known that the *average* number of loop iterations for producing an entry of Q is less than 2. We use this fact to coordinate the *production* of entries Q by Algorithm DELTA with their *consumption* in the Gray code generation, turning Q into a buffer of bounded capacity. This leads to the following loopless algorithm:

Algorithm WORK-AHEAD. Generation of the Gray code.

procedure STEP:

if $b_j = m_j - 1$:

$b_j := 0$

$j := j + 1$

else:

if Q is not filled to capacity:

$b_j := b_j + 1$

$Q.append(j)$

$j := 1$

$(a_n, \dots, a_2, a_1) := (0, \dots, 0, 0)$

$(d_n, \dots, d_2, d_1) := (1, \dots, 1, 1)$

$(b_{n+1}, b_n, \dots, b_2, b_1) := (0, 0, \dots, 0, 0); m_{n+1} := 2$

$Q :=$ queue of capacity $B := \lceil \frac{n}{2} \rceil$, initially empty

$j := 1$

loop:

visit the n -tuple (a_n, \dots, a_2, a_1)

STEP

STEP

remove j from Q

if $j = n + 1$: TERMINATE

$a_j := a_j + d_j$

if $a_j = 0$ or $a_j = m_j - 1$: $d_j := -d_j$

The procedure STEP on the left side encompasses one loop iteration of Algorithm DELTA. By programmer's license, we have moved the initialization $j := 1$ of the loop variable to the end of the previous loop. We have also moved the termination test $j = n + 1$ to the side of the consumer. Accordingly, we had to extend the n -tuple b into an $(n + 1)$ -tuple, setting m_{n+1} arbitrarily to 2. When Q is full, nothing is done in the procedure STEP, and the repeated call of STEP will try to insert the same value into Q . Thus, apart from the termination test, a repeated execution of STEP will faithfully carry out Algorithm DELTA.

The Gray code algorithm on the right couples two production STEPs with one consumption step, which takes out an entry j of Q and carries out the update $a_j := a_j \pm 1$. Every digit a_j must cycle up and down through its values in the sequence (3), and thus, we have to remember the direction $d_j = \pm 1$ in which it moves, as in Algorithm ODD.

To show that the algorithm is correct, we have to ensure that the queue Q is never empty when the algorithm retrieves an element from it. This is proved below in Lemma 7.

The clean way to terminate the algorithm would be to stop inserting elements into Q as soon as $j = n + 1$ is *produced* in STEP, as in Algorithm DELTA. Instead, termination is triggered when the value $j = n + 1$ is *removed* from Q . Due to this delayed termination test, a few more iterations of STEP can be carried out, but they cause no harm.

For the *binary* Gray code ($m_i = 2$ for all $i = 1, \dots, n$), the algorithm can be simplified. With a slightly larger buffer Q of size $B' := \max\{\lceil \frac{n+1}{2} \rceil, 2\}$, the test whether Q is filled to

capacity can be omitted, see Lemma 9 below. The reason is that the average number of production STEPs per item approaches 2 in the limit, and accordingly, the queue automatically does not grow beyond the minimum necessary size. The directions d_i are of course also superfluous in the binary case.

The idea of “working ahead” is opposite to the approach of delaying work as long as possible that underlies many “lazy” data structures and also lazy evaluation in some functional programming languages. In a similar vein, Guibas, McCreight, Plass, and Janet R. Roberts [5] have obtained worst-case bounds of $O(\log k)$ for updating a sorted linear list at distance k from the beginning.⁴ Their algorithm works ahead to hedge against sudden bursts of activity. Our setting is much simpler, because we do not depend on the update requests of a “user” and we can plan everything in advance.

At a different level of complexity, the idea of working ahead occurs in an algorithm of Wettstein [9, Section 6]. This trick, credited to Emo Welzl, is used to achieve *polynomial delay* between successive solutions when enumerating non-crossing perfect matching of a planar point set, despite having to build up a network with exponential space in a preprocessing phase.

8.1 An alternative STEP procedure

As an alternative to the organization of Algorithm WORK-AHEAD, we can incorporate the termination test into the STEP procedure:

```
procedure STEP':
  if  $j = n + 1$ : TERMINATE
  if  $b_j = m_j - 1$ :
     $b_j := 0$ 
     $j := j + 1$ 
  else:
    if  $Q$  is not filled to capacity:
       $b_j := b_j + 1$ 
       $Q.append(j)$ 
       $j := 1$ 
```

With this modified procedure STEP', the termination test in the main part of Algorithm WORK-AHEAD can of course be omitted. We also need not extend the arrays b and m to $n + 1$ elements.

The algorithm still works correctly because there are no unused entries in the queue when STEP' signals termination. Let us prove this:

The termination signal is sent instead of producing the value $j = \bar{\rho}(k) = n + 1$ for $k = m_0m_1\dots m_{n-1}$. Generating this signal takes $n + 1$ iterations of STEP. In this time, no new values are added to the queue. Let us assume that the production of $\bar{\rho}(k)$ was started during iteration k_0 , and the buffer was filled with $B_0 \leq B$ entries at that time. The first of these entries is consumed at the end of iteration k_0 , and all B_0 entries of the buffer have been used up at the beginning of iteration $k_0 + B_0$. By this time, at most $2B_0 \leq 2B \leq n + 1$ iterations of STEP were carried out and contributed to the production of the termination signal. It follows that when STEP discovers that $j = n + 1$, no unused entries are in the stack, and it is safe to terminate the program.

⁴ We thank Don Knuth for leading us to this reference.

It is important not to “speed up” the program by moving the termination test into the **if**-branch after the statement $j := j + 1$. Also, we must use exactly the prescribed buffer size for Q . Therefore, this variation is incompatible with the simplification for the binary case mentioned above.

8.2 Correctness proofs for the work-ahead algorithms

We define the ruler function ρ and the modified ruler function $\bar{\rho}$ with respect to a sequence of radices m_1, \dots, m_n as follows:

$$\rho(k) := \max\{ i : 0 \leq i \leq n, m_1 m_2 \dots m_i \text{ divides } k \}, \quad \bar{\rho}(k) := \rho(k) + 1$$

Then the k -th value that is entered into Q is $\bar{\rho}(k)$, and for computing this value, Algorithm DELTA needs $\bar{\rho}(k)$ iterations, and accordingly, Algorithm WORK-AHEAD needs $\bar{\rho}(k)$ STEPs.

► **Lemma 7.** *In Algorithm WORK-AHEAD, the buffer Q never becomes empty.*

Proof. We number the iterations of the main loop as $1, 2, \dots, m_1 m_2 \dots m_n$. In the last iteration, the algorithm terminates.

Let us show that the queue Q is not empty in iteration k . We distinguish two cases.

- (i) Up to and including iteration k , two repetitions of STEP were always completed.
- (ii) Some repetitions of STEP had no effect because the buffer Q was full.

In case (i), production of all values $\rho(i)$ for $i = 1, \dots, k$ requires

$$S(k) := \sum_{i=1}^k \bar{\rho}(i)$$

calls to STEP. To show that these calls are completed by the time when $\bar{\rho}(k)$ is needed, we have to show

$$S(k) \leq 2k. \tag{4}$$

In case (ii), let k_0 be the last iteration when an execution of STEP was “skipped”. This means that the queue Q was filled to capacity B just before removing the value $j = \bar{\rho}(k_0)$, and it contained the values $\bar{\rho}(k_0), \bar{\rho}(k_0 + 1), \dots, \bar{\rho}(k_0 + B - 1)$. Since then, STEP was called $2(k - k_0)$ times, and $\bar{\rho}(k)$ is ready when it is needed, provided that

$$1 + \sum_{i=k_0+B+1}^k \bar{\rho}(i) \leq 2(k - k_0)$$

whenever $k \geq k_0 + B$. The left-hand side of this inequality is the number of necessary STEPs for computing the values up to $\bar{\rho}(k)$. Computing $\bar{\rho}(k_0 + B)$ takes just one more STEP, since the STEP that would have stored this value in Q was abandoned in iteration k_0 . Setting $k' = k_0 + B$, we can express the inequality equivalently as

$$S(k) - S(k') \leq 2(k - k' + B) - 1 \text{ for } k' \leq k \tag{5}$$

We can write an explicit formula for $S(k)$:

$$S(k) = k + \left\lfloor \frac{k}{m_1} \right\rfloor + \left\lfloor \frac{k}{m_1 m_2} \right\rfloor + \dots + \left\lfloor \frac{k}{m_1 m_2 \dots m_n} \right\rfloor$$

Since all $m_i \geq 2$, we get $S(k) \leq k + k/2 + k/4 + k/8 + \cdots + k/2^n < 2k$, proving (4). For the other bound (5), we use the relation

$$\left\lfloor \frac{k}{x} \right\rfloor - \left\lfloor \frac{k'}{x} \right\rfloor < \frac{k - k'}{x} + 1$$

to get

$$S(k) - S(k') < (k - k') + (k - k') \cdot (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}) + n < 2(k - k') + n$$

Since the left-hand side is an integer, we obtain $S(k) - S(k') \leq 2(k - k') + n - 1$ and this implies (5) since the buffer size $B := \lceil \frac{n}{2} \rceil$ satisfies $2B \geq n$. \blacktriangleleft

In Algorithm WORK-AHEAD, the STEPs should generate entries $\bar{\rho}(1), \bar{\rho}(2), \dots$ of Q up to $\bar{\rho}(N)$, where $N := m_1 m_2 \dots m_n$.

The following lemma shows that production of the STEPs may overrun their target by at most one. Since the algorithm has already made provisions to generate $\bar{\rho}(N) = n + 1$ by extending the arrays b and m to size $n + 1$ instead of n , this one extra entry does not cause any harm.

► Lemma 8. *In Algorithm WORK-AHEAD, the last entry that is added to Q is $\bar{\rho}(N)$ or $\bar{\rho}(N + 1)$.*

Proof. The production of $\bar{\rho}(N) = n + 1$ takes $n + 1 \geq 2B$ STEPs. It follows that the buffer Q is empty when $\bar{\rho}(N) = n + 1$ is inserted, regardless of whether the production of $\bar{\rho}(N)$ is started in the first or second STEP of an iteration.

If the production of $\bar{\rho}(N) = n + 1$ is completed in the second STEP of an iteration, it is thus immediately consumed, which leads to termination. If $\bar{\rho}(N)$ is completed in the first STEP of an iteration, the second STEP will produce the value $\bar{\rho}(N + 1) = 1$, but then the algorithm will terminate as well. \blacktriangleleft

Finally, we prove the simplification of the algorithm for the binary case.

► Lemma 9. *In the binary version of Algorithm WORK-AHEAD, i.e., when $m_i = 2$ for all $i = 1, \dots, n$, the buffer Q automatically never gets more than $B' := \max\{\lceil \frac{n+1}{2} \rceil, 2\}$ entries, even if the test in STEP whether the buffer is full is omitted.*

Proof. Let us assume for contradiction that the buffer becomes overfull in iteration k , $1 \leq k \leq 2^n$. This means that, before $j = \bar{\rho}(k)$ is removed from Q , the $2k$ STEP operations have produced more than $k - 1 + B'$ values. But this is impossible, since, as we will show, the production of the first $k_1 = k + B'$ values would have taken

$$S(k_1) = k_1 + \left\lfloor \frac{k_1}{2} \right\rfloor + \left\lfloor \frac{k_1}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{k_1}{2^n} \right\rfloor > 2k$$

STEPS. To show the last inequality, we first consider the case $k_1 < 2^n$. We apply the inequality $\lfloor x \rfloor > x - 1$ and obtain $S(k_1) > 2k_1 - k_1/2^n - n$, and since $k_1/2^n < 1$ and $S(k_1)$ is an integer, we get

$$S(k_1) \geq 2k_1 - n = 2k + 2B' - n > 2k.$$

Let us now see at what time $\bar{\rho}(k_1)$ for $k_1 \geq 2^n$ is entered into Q . When $k_1 = 2^n$, no round-off takes place in the formula for $S(k_1)$, and we have $S(2^n) = 2 \cdot 2^n - 1$. This shows that the production of $\bar{\rho}(2^n)$ is completed in the first STEP of iteration 2^n . In the second STEP of this iteration, $\bar{\rho}(2^n + 1) = 1$ is added to Q . Thus, when $\bar{\rho}(2^n)$ is about to be retrieved, the buffer contains $2 \leq B'$ elements. Then the algorithm terminates, and no more elements are produced. \blacktriangleleft

9 Concluding Remarks

By our approach of modeling the Gray code in terms of a motion-planning game, we were able get a mixed-radix Gray code only when all radices m_i are odd. It remains to find a model that would work for different even radices or even for radices of mixed parity.

Another motion-planning game which is related to the binary Gray code is the *Chinese rings puzzle*, see Scorer, Grundy, and Smith [8], Gardner [4], or Knuth [6, pp. 285–6]. In this puzzle, there are at most two possible moves in every state, like in the towers of Bucharest, and by simulating the Chinese rings directly, one can obtain another loopless algorithm for the binary Gray code. However, this algorithm does not seem to extend to other radices.

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A Appendix: PYTHON simulations of the algorithms

Here we list prototype implementations in PYTHON. They should run equally with Python 2.7 and Python 3. The pegs, the string a , and the array of `directions` are kept as global variables.

A.1 Basic procedures

The procedure visit prints the string and the contents of the pegs.

```
def visit():
    print ("\".join(str(x) for x in reversed(a[1:]))) + " "+
          ".join("P{}:".format(k)+",".join(map(str,p))
                 for k,p in enumerate(peg)))
```



```
def find_smallest_disk(exclude=None):
    list_d_k = [(p[-1],k) for k,p in enumerate(peg) if k!=exclude and p]
    if list_d_k:
        _,k = min(list_d_k) # smallest disk not covered by D1
        return k
    return None
```

A.2 Algorithm ODD, Section 4

```
def initialize_m_ary_odd(m,n):
    # n-tuple of entries from the set {0,1,...,m-1}
    global pegs, a, direction
    pegs = tuple([] for k in range(m))
    for i in range(n,0,-1):
        pegs[0].append(i)
    a = (n+1)*[0] # a[0] and direction[0] is wasted
    direction = (n+1)*[+1]

def Gray_code_m_ary_odd(m):
    visit()
    while True:
        for _ in range(m-1): # repeat m-1 times:
            move_disk(m,peg=a[1])
            visit()
            k = find_smallest_disk(exclude=a[1]) # smallest disk not covered by D1
            if k==None: return
            move_disk(m,peg=k)
            visit()

    def move_disk(m,peg): # move topmost disk on peg
        disk = pegs[peg].pop()
        peg += direction[disk]
        a[disk] += direction[disk]
        if peg==m-1: direction[disk] = -1
        elif peg==0: direction[disk] = +1
        pegs[peg].append(disk)

    # run the program for a test:
    m=3
    initialize_m_ary_odd(m,6)
    Gray_code_m_ary_odd(m)
```

A.3 Algorithm ODD-COMPRESSED, Section 6

This implementation uses the idea of a “control disk” D_0 mentioned at the end of section 4. For uniformiy, we attach a direction also to the dial positions 0 and $m - 1$, namely the direction in with the next move will proceed (in contrast to the convention (3) used in Section 6).

```

def initialize_m_ary_odd_compressed(m,n):
    # n-tuple of entries from the set {0,1,...,m-1}
    global pegs, a
    pegs = tuple([] for k in range(3))
    for i in range(n,0,-1):
        pegs[0].append((i,0,+1))
    a = (n+1)*[0] # a[0] is wasted

def Gray_code_m_ary_odd_compressed(m):
    visit()
    while True:
        for control_disk in (2,0):
            k = find_smallest_disk(exclude=control_disk)
            if k==None: return # All disks are on the same peg.
            move_disk_compressed(m,peg=k)
            visit()

def move_disk_compressed(m,peg):
    # retrieve topmost disk on peg:
    disk,value,direction = pegs[peg].pop()
    if value in (0,m-1) or value+direction in (0,m-1):
        peg += direction
        value += direction
        a[disk] += direction
        if value in (0,m-1):
            direction = -direction
        pegs[peg].append((disk,value,direction))

# run the program for a test:
m=5
initialize_m_ary_odd_compressed(m,5)
Gray_code_m_ary_odd_compressed(m)

```

A.4 Algorithm EVEN, Section 5

```

def initialize_m_ary_even(m,n):
    initialize_m_ary_odd(m+1,n) # use m+1 pegs

def Gray_code_m_ary_even(m):
    peg_disk1 = 0 # position of disk D1
    visit()
    while True:
        for _ in range(m-1): # repeat m-1 times:
            turn_disk(m,peg=peg_disk1)
            peg_disk1 = (peg_disk1+1) % (m+1)
            visit()
        k = find_smallest_disk(exclude=peg_disk1)
        # smallest disk not covered by D1
        if k==None: return
        turn_disk(m,peg=k)
        visit()

def turn_disk(m,peg): # move the topmost disk on peg
    disk = pegs[peg].pop()
    if disk%2==1:
        peg = (peg + 1) % (m+1)
    else:
        peg = (peg - 1) % (m+1)
    pegs[peg].append(disk)
    a[disk] += direction[disk]
    if a[disk]==m-1: direction[disk] = -1
    elif a[disk]==0: direction[disk] = +1

# run the program for a test:
m=4
initialize_m_ary_even(m,6)
Gray_code_m_ary_even(m)

```

A.5 Truly loopless implementation of Algorithm EVEN, Section 7

A peg is a list of pairs (a, b) with $a \leq b$, denoting a maximal subset $a, a+1, \dots, b$ of consecutive disks (an interval). The pairs are sorted, with the smallest disks at the end (at the “top”).

```

def initialize_m_ary_even_intervals(m,n):
    global pegs,position
    initialize_m_ary_even(m,n)
    pegs[0][:]=[1,n]
    position=[0]*(n+1)

```

```

def Gray_code_m_ary_even_intervals(m,n):
    visit()
    while True:
        for _ in range(m-1): # repeat m-1 times:
            turn_disk_intervals(m,peg=position[1])
            visit()
        d = find_smallest_missing_disk(position[1])
        # smallest disk not covered by D1
        if d>n: return
        turn_disk_intervals(m,peg=position[d])
        visit()

def turn_disk_intervals(m,peg): # move topmost disk on peg
    disk,d2 = pegs[peg] [-1]
    # remove disk from peg:
    if disk==d2:
        pegs[peg].pop()
    else:
        pegs[peg] [-1]=(disk+1,d2)
    if disk%2==1:
        peg = (peg + 1) % (m+1) # turn the disk "clockwise"
    else:
        peg = (peg - 1) % (m+1) # turn the disk "counterclockwise"
    position[disk]=peg
    # add disk to peg:
    if pegs[peg]:
        d1,d2 = pegs[peg] [-1]
        if disk<d1-1:
            pegs[peg].append((disk,disk))
        else:
            pegs[peg] [-1]=(disk,d2)
    else:
        pegs[peg].append((disk,disk))
    a[disk] += direction[disk]
    if a[disk]==m-1: direction[disk] = -1
    elif a[disk]==0: direction[disk] = +1

def find_smallest_missing_disk(peg):
    (_,d2)=pegs[peg] [-1]
    return d2+1

# run the program for a test:
m=4
initialize_m_ary_even_intervals(m,6)
Gray_code_m_ary_even_intervals(m,6)

```

A.6 Algorithm WORK-AHEAD, Section 8

```

def STEP():
    global j, b,m,B,Q
    if b[j]==m[j]-1:
        b[j]=0
        j += 1
    else:
        if len(Q)<B:
            b[j] += 1
            Q.append(j)
            j = 1

def initialize_work_ahead(n):
    global a,b,direction,B,Q,j
    a = (n+1)*[0] # a[0], b[0], and direction[0] is wasted
    direction = (n+1)*[+1]
    b = (n+2)*[0]
    from collections import deque
    B = (n+1)//2
    Q = deque()
    j = 1

def Gray_code_work_ahead(n):
    while True:
        VISIT()
        STEP()
        STEP()
        j = Q.popleft()
        if j==n+1: break
        a[j] += direction[j]
        if a[j] in (0,m[j]-1): direction[j] *= -1

def VISIT():
    print ("".join(str(x) for x in reversed(a[1:])))

# run the program for a test:
n=4
m=[0]+[2,4,5,2]+[2] # initial 0 and final 2 are artificial
initialize_work_ahead(n)
Gray_code_work_ahead(n)

```

A.7 Algorithm WORK-AHEAD for the binary Gray code, Section 8

```

def STEP_binary():
    global j, b,B,Q
    if b[j]==1:
        b[j]=0
        j += 1
    else:
        if len(Q)>=B:
            print ("error")
            exit(1)
        b[j]=1
        Q.append(j)
        j = 1

def initialize_work_ahead_binary(n):
    global a,b,B,Q,j
    a = (n+1)*[0] # a[0], b[0], and direction[0] is wasted
    b = (n+2)*[0]
    from collections import deque
    B = max(2,(n+2)//2)
    Q = deque()
    j = 1

def Gray_code_work_ahead_binary(n):
    while True:
        VISIT()
        STEP_binary()
        STEP_binary()
        j = Q.popleft()
        if j==n+1: break
        a[j] = 1-a[j]

# run the program for a test:
n=4
initialize_work_ahead_binary(n)
Gray_code_work_ahead_binary(n)

```

A.8 Algorithm WORK-AHEAD with the modification of Section 8.1

```

def STEP_x():
    global j, b,m,B,Q,n
    if j==n+1: raise StopIteration
    if b[j]==m[j]-1:
        b[j]=0
        j += 1
    else:
        if len(Q)<B:
            b[j] += 1
            Q.append(j)
            j = 1

def initialize_work_ahead_x():
    global n,a,b,direction,B,Q,j
    a = (n+1)*[0] # a[0], b[0], and direction[0] is wasted
    direction = (n+1)*[+1]
    b = (n+1)*[0]
    from collections import deque
    B = (n+1)//2
    Q = deque()
    j = 1

def Gray_code_work_ahead_x():
    try:
        while True:
            VISIT()
            STEP_x()
            STEP_x()
            j = Q.popleft()
            a[j] += direction[j]
            if a[j] in (0,m[j]-1): direction[j] *= -1
    except StopIteration:
        return

# run the program for a test:
n=4
m=[0]+[2,4,5,2] # initial entry 0 is redundant
initialize_work_ahead_x()
Gray_code_work_ahead_x()

```

A.9 General mixed-radix Gray code generation according to the recursive definition of Section 3

In order to have a reference implementation for comparing the results, we give a program straight from the definition (2) of Section 3, extended to arbitrary mixed radices (m_1, \dots, m_n) . When this program is run with the data specified below, it should produce the same Gray codes as all the previous example programs combined (apart from the additional state of the pegs that is reported by these programs). The outputs coincide precisely after stripping everything after the first blank on each line. The source files of this arXiv preprint include scripts that will extract the program code from the L^AT_EX file of this appendix (`extract-code-from-appendix.py`) and compare the outputs to check the results (`check-output.sh`) after running the examples,

```

def mixed_Gray_code(ms):
    "a generator for the mixed-radix Gray code"
    if ms:
        m = ms[0]
        G1 = mixed_Gray_code(ms[1:])
        while True:
            g = next(G1)
            for lastdigit in range(0,m):
                yield g+(lastdigit,)
            g = next(G1)
            for lastdigit in reversed(range(0,m)):
                yield g+(lastdigit,)
    else:
        yield ()
for result in (
    mixed_Gray_code([3]*6),
    mixed_Gray_code([5]*5),
    mixed_Gray_code([4]*6),
    mixed_Gray_code([4]*6),
    mixed_Gray_code([2,4,5,2]),
    mixed_Gray_code([2]*4),
    mixed_Gray_code([2,4,5,2]),
):
    print ("#####")
    for g in result:
        print ("".join(str(x) for x in g))

```