

BLOKEA 5 - Integralak - Ariketak

Integral trigonometrikoak

Integral trigonometrikoak ebazterako orduan gehien erabiltzen diren berdintza trigonometrikoak honakoak dira:

Formula Trigonometrikoak:

$$\sin^2 a + \cos^2 a = 1 \quad (1)$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} \quad (2)$$

$$\sin^2 a = \frac{1 - \cos 2a}{2} \quad (3)$$

$$\cos(2a) = \cos^2 a - \sin^2 a \quad (4)$$

$$\sin(2a) = 2 \cdot \sin a \cdot \cos a \quad (5)$$

$$\cos(a \pm b) = \cos a \cdot \cos b \mp \sin a \cdot \sin b \quad (6)$$

$$\sin(a \pm b) = \sin a \cdot \cos b \pm \sin b \cdot \cos a \quad (7)$$

$$\cos a \cdot \cos b = \frac{\cos(a + b) + \cos(a - b)}{2} \quad (8)$$

$$\sin a \cdot \sin b = \frac{\cos(a - b) - \cos(a + b)}{2} \quad (9)$$

$$\sin a \cdot \cos b = \frac{\sin(a + b) + \sin(a - b)}{2} \quad (10)$$

1. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \sin 6x \cdot \sin 2x \, dx$$

Ebazpena:

Ondorengo berdintza erabiliz:

$$\sin a \cdot \sin b = \frac{\cos(a - b) - \cos(a + b)}{2}$$

$$\begin{aligned} \int \sin 6x \cdot \sin 2x \, dx &= \int \frac{\cos 4x - \cos 8x}{2} \, dx = \frac{1}{2} \int \cos 4x \, dx - \frac{1}{2} \int \cos 8x \, dx = \\ &= \frac{1}{2} \frac{\sin 4x}{4} - \frac{1}{2} \frac{\sin 8x}{8} + C = \frac{1}{8} \sin 4x + \frac{1}{16} \sin 8x + C \end{aligned}$$

2. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \cos^6 x \, dx$$

Ebazpena:

Honako berdintza hau erabiliz:

$$\cos^2 a = \frac{1 + \cos 2a}{2}$$

$$\begin{aligned} \int \cos^6 x \, dx &= \int (\cos^2 x)^3 \, dx = \int \left(\frac{1 + \cos 2x}{2} \right)^3 \, dx \\ &= \int \frac{1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x}{8} \, dx = \\ &= \frac{1}{8} \int dx + \frac{3}{8} \int \cos 2x \, dx + \frac{3}{8} \int \cos^2 2x \, dx + \frac{1}{8} \int \cos^3 2x \, dx = \\ &= \frac{x}{8} + \frac{3}{8} \frac{\sin 2x}{2} + C_3 + \frac{3}{8} \underbrace{\int \cos^2 2x \, dx}_{(A)} + \frac{1}{8} \underbrace{\int \cos^3 2x \, dx}_{(B)} \end{aligned}$$

(A) $\int \cos^2 2x \, dx$ kalkulatzeko berriro ere $\cos^2 a = \frac{1 + \cos 2a}{2}$ formula aplikatuz:

$$\int \cos^2 2x \, dx = \int \frac{1 + \cos 4x}{2} \, dx = \frac{x}{2} + \frac{\sin 4x}{8} + C_1$$

(B) $\int \cos^3 2x \, dx$ integrala kalkulatzeko $\sin^2 a + \cos^2 a = 1$ berdintza erabiliz:

$$\begin{aligned} \int \cos^3 2x \, dx &= \int \cos 2x \cos^2 2x \, dx = \int \cos 2x (1 - \sin^2 2x) \, dx \\ &= \int \cos 2x \, dx - \int \cos 2x \sin^2 2x \, dx = \frac{\sin 2x}{2} - \frac{1}{6} \sin^3 2x + C_2 \end{aligned}$$

Ondorioz, integral hauek hasierakoan ordezkatzuz:

$$\begin{aligned} \int \cos^6 x \, dx &= \frac{x}{8} + \frac{3}{8} \frac{\sin 2x}{2} + \frac{3}{8} \left(\frac{x}{2} + \frac{\sin 4x}{8} \right) + \frac{1}{8} \left(\frac{\sin 2x}{2} - \frac{1}{6} \sin^3 2x \right) + C = \\ &= \frac{x}{8} + \frac{3}{16} \sin 2x + \frac{3}{16} x + \frac{3}{16} \sin 4x + \frac{1}{16} \sin 2x - \frac{1}{48} \sin^3 2x + C = \\ &= \frac{5x}{16} + \frac{1}{4} \sin 2x + \frac{3}{64} \sin 4x - \frac{1}{48} \sin^3 2x + C \end{aligned}$$

Aldagai aldaketa orokorra:

Integral trigonometrikoak ebazteko era oso zabaldu bat aldagai aldaketa orokorra da.

Aldagai aldaketa orokorra honako hau da:

$$\tan\left(\frac{x}{2}\right) = t \Rightarrow x = 2 \arctan(t) \Rightarrow dx = \frac{2dt}{1+t^2}$$

Berdintza honetatik ondorengoa berdintzak lor daitezke:

$$\sin(x) = \frac{2t}{t^2+1}; \cos(x) = \frac{1-t^2}{t^2+1}; \tan x = \frac{2t}{1-t^2}$$

Aldagai aldaketa honi esker integral trigonometrikoak integral arrazionalak bilakatzen dira.

1. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \frac{dx}{4 - 2 \cos(x)}$$

Ebazpena:

Aldagai aldaketa orokorra erabiliz:

$$\cos(x) = \frac{1 - t^2}{t^2 + 1}, \quad dx = \frac{2dt}{1 + t^2}$$

Integralari aplikatuz:

$$\int \frac{dx}{4 - 2 \cos(x)} = \int \frac{2dt}{1 + t^2} \frac{1}{4 - 2 \cdot \frac{1 - t^2}{t^2 + 1}} = \int \frac{2dt}{1 + t^2} \frac{1}{\frac{4(t^2 + 1) - 2(1 - t^2)}{t^2 + 1}} =$$

$$\int \frac{2dt}{4t^2 + 4 - 2 + 2t^2} = \int \frac{2dt}{6t^2 + 2} = \int \frac{dt}{3t^2 + 1} = \int \frac{dt}{(\sqrt{3}t)^2 + 1} = \frac{1}{\sqrt{3}} \int \frac{\sqrt{3}}{(\sqrt{3}t)^2 + 1} =$$

$$\frac{1}{\sqrt{3}} \arctan(\sqrt{3}t) + C$$

Bukatzeko, aldagai aldaketa desegiten:

$$\int \frac{dx}{4 - 2 \cos(x)} = \frac{1}{\sqrt{3}} \arctan\left(\sqrt{3} \tan\left(\frac{x}{2}\right)\right) + C$$

2. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \frac{\tan x}{1 - \cos(x)} \cdot dx$$

Ebazpena:

Aldagai aldaketa orokorra erabiliz:

$$\tan x = \frac{2t}{1-t^2}, \cos(x) = \frac{1-t^2}{t^2+1}, dx = \frac{2dt}{1+t^2}$$

Integralari aplikatuz:

$$\begin{aligned} \int \frac{\tan x}{1 - \cos(x)} \cdot dx &= \int \frac{2t}{1-t^2} \frac{1}{1 - \frac{1-t^2}{t^2+1}} \cdot \frac{2dt}{1+t^2} = \int \frac{4t}{(1-t^2) \frac{[t^2+1 - (1-t^2)](1+t^2)}{t^2+1}} dt = \\ &= \int \frac{4t}{(1-t^2)(t^2+1-1+t^2)} dt = \int \frac{4t}{(1-t^2)(2t^2)} dt = 2 \int \frac{1}{(1-t^2)t} dt = \end{aligned}$$

Integral arrazionala lortu dugu, integral hau ebatziz:

$$\begin{aligned} \frac{1}{(1-t^2)t} &= \frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{1-t} + \frac{C}{1+t} \Rightarrow \\ \frac{1}{(1-t^2)t} &= \frac{A(1-t)(1+t) + Bt(1+t) + Ct(1-t)}{t(1-t)(1+t)} \Rightarrow \\ 1 &= A(1-t)(1+t) + Bt(1+t) + Ct(1-t) \end{aligned}$$

$$t = 0 \Rightarrow 1 = A$$

$$t = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$t = -1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$$

Ondorioz,

$$\begin{aligned} 2 \int \frac{1}{(1-t^2)t} dt &= 2 \int \frac{dt}{t} + 2 \int \frac{1/2}{1-t} dt + 2 \int \frac{-1/2}{1+t} dt = \\ 2 \int \frac{dt}{t} + \int \frac{dt}{1-t} - \int \frac{dt}{1+t} &= 2 \ln|t| - \ln|1-t| - \ln|1+t| + C \end{aligned}$$

Bukatzeko, aldagai aldaketa desegiten:

$$\int \frac{\tan x}{1 - \cos(x)} \cdot dx = 2 \ln \left| \tan \left(\frac{x}{2} \right) \right| - \ln \left| 1 - \tan \left(\frac{x}{2} \right) \right| - \ln \left| 1 + \tan \left(\frac{x}{2} \right) \right| + C$$