Report 2

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0.1 Plant Biomass

0.1.1 Background

Let the dry weight of some plant at time t be denoted by x(t). And suppose this plant feeds off a fixed amount of some single substrate, or a nutrient medium, for which the amount remaining at time t is denoted by S(t).

0.1.2 Assumption

We make the following assumptions and then build the model on them.

- The growth rate of the plant is proportional to its dry weight as well as to the amount of nutrient available.
- No material is lost in the conversion of S into x.

0.1.3 Word Equation

$$\begin{Bmatrix} growth\ rate \\ of\ plant \end{Bmatrix} = \begin{Bmatrix} growth\ rate \\ influenced\ by \\ dry\ weight \\ as\ well\ as \\ nutrient\ availabie \end{Bmatrix}$$

0.1.4 Formulating The Differential Equation

Given that $S(t) = x_f - x(t)$, where $k > 0$ is a proportionality coefficients of S(t)x(t). By word equation, we obtain differentiable equation

$$\frac{dx}{dt} = kS(t)x(t), \tag{1}$$

replace S(t) with $x_f - x(t)$, the equation becomes

$$\frac{dx}{dt} = k(x_f - x(t))x(t). \tag{2}$$

0.1.5 Solving The Differential Equation

To solve (2), we first separate the equation, we obtain

$$\frac{1}{k(x_f - x(t))x(t)}dx = 1dt, \tag{3}$$

that is

$$\frac{1}{x_f - x(t)} + \frac{1}{x(t)} = k \cdot x_f \tag{4}$$

integrate on both sides, we get

$$lnx(t) - ln(x_f - x(t)) = k \cdot x_f \cdot t + c, \tag{5}$$

for c is arbitrary constant, which can be simplified to

$$ln\frac{x(t)}{x_f - x(t)} = k \cdot x_f \cdot t + c, \tag{6}$$

that is

$$ln(\frac{x_f}{x_f - x(t)} - 1) = k \cdot x_f \cdot t + k \cdot x_f \cdot c, \tag{7}$$

we take exp on both sides,

$$\frac{x_f}{x_f - x(t)} = 1 + exp(k \cdot x_f \cdot t + c), \tag{8}$$

$$x(t) = x_f - \frac{x_f}{1 + C \cdot exp(k \cdot x_f \cdot t)}, \tag{9}$$

where denote $exp(c)$ as C. Note that $x(0) = x_0$, so we take x=0 into (8), obtaining $C = \frac{x_0}{x_f - x_o}$. Finally, the solution is

$$x(t) = x_f - \frac{x_f}{1 + \frac{x_0}{x_f - x_0} \cdot exp(k \cdot x_f \cdot t)}, \tag{10}$$

if we take k=1, $x_0 = 5$, $x_f = 15$, the graph of the solution will be shown in the following figure. .

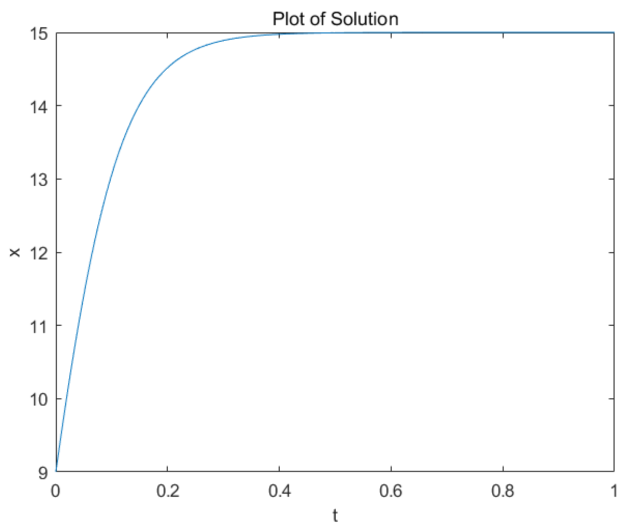

Figure 1: Shows the dry weight of some plant at time t

0.1.6 Limitation

The model we introduce above has a obvious limitation, that is using this model, the plant biomass of x_f will never be attained. To explain this, we need to imagine, according to our modeling, at a later stage of plant growth, when x(t) approaches x_f, the mass of remaining nutrients $S(t) = x_f - x(t)$ approaches 0. Since we saaume that the growth rate of the plant is proportional to S(t)x(t), the growth rate at this point also approaches 0. This implies that the closer the dry weight of the plant is to x_f , the closer the plant tends to not grow.

0.2 Modelling The Population of A Country

0.2.1 Background

Consider the population of a country. Assume constant per-capita birth and death rates, and that the population follows an exponential growth (or decay) process. There to be significant immigration and emigration of people into and out of the country.

0.2.2 Assumption

There are two possible assumptions and we can build the model on them respectively.

• The overall immigration and emigration rates are constant.

• All immigration and emigration occurs with a neighbouring country, such that the net movement from one country to the other is proportional to the population difference between the two countries and such that people move to the country with the larger population.

0.2.3 Word Equation

For the first assumption, we write the word equation

$$\left\{\begin{matrix} \textit{rate of} \\ \textit{change in} \\ \textit{population} \end{matrix}\right\} = \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{births} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{deaths} \end{matrix}\right\} + \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{immigration} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{emigration} \end{matrix}\right\}$$

For the second assumption, we have two word equations. For country that population flow in,

$$\left\{\begin{matrix} \textit{rate of} \\ \textit{change in} \\ \textit{population} \end{matrix}\right\} = \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{births} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{deaths} \end{matrix}\right\} + \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{immigration} \end{matrix}\right\}$$

For country that population flow out,

$$\left\{\begin{matrix} \textit{rate of} \\ \textit{change in} \\ \textit{population} \end{matrix}\right\} = \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{births} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{deaths} \end{matrix}\right\} - \left\{\begin{matrix} \textit{rate} \\ \textit{of} \\ \textit{emigration} \end{matrix}\right\}$$

0.2.4 Formulating the Differential Equation

For the first assumption

We define α as constant per-capita death rate, β as constant per-capita birth rate, i as rate of immigration, e as rate of emigration. x(t) represents the population changing over time t. Then the equation becomes

$$\frac{dx}{dt} = (\beta - \alpha)x + i - e, \tag{11}$$

For the second assumption

We define α_x as constant per-capita death rate of country that population flow in, define α_y as constant per-capita death rate of country that population flow out. We define β_x as constant per-capita birth rate of country that population flow in, define β_y as constant per-capita birth rate of country that population flow out. The proportionality coefficient to the population difference between two countries is represented by γ. x(t) represents the population changing over time t in the country that population flow in, y(t) represents the population changing over time t in the country that population flow out. Then the equation of country that population flow in is

$$\frac{dx}{dt} = (\beta_x - \alpha_x)x + \gamma(x - y), \tag{12}$$

the equation of country that population flow out is

$$\frac{dx}{dt} = (\beta_y - \alpha_y)x - \gamma(x - y), \tag{13}$$

0.3 Appendix

Code for plotting the solution by Matlab:

```
t = linspace(0, 1, 1000);
x = 15 - 15 ./ (1 + (3/2) * exp(15 * t));
plot(t, x);
title('Plot of Solution');
xlabel('t');
ylabel('x');
```