Notes for Project on Spectral Methods for DEs

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A) Collocation Methods for Second-order BVPs

Consider the second-order boundary value problem (BVP):

$$\begin{aligned}
&\mathcal{L}[u](x) := -\varepsilon u''(x) + p(x)u'(x) + q(x)u(x) = f(x), \quad x \in I := (-1,1), \\
&u(-1) = c_-, \quad u(1) = c_+,
\end{aligned} \tag{1}$$

where $p(x), q(x), f(x)$ is a given continuous function on I, and $c_{\pm 1}$ are given constants.

A$_1$. Collocation Scheme: Find $u_N \in \mathbb{P}_N$ such that

$$\begin{aligned}
&\mathcal{L}[u_N](x_i) = -\varepsilon u_N''(x_i) + p(x_i)u_N'(x_i) + q(x_i)u_N(x_i) = f(x_i), \quad 1 \le i \le N-1, \\
&u_N(x_0) = c_-, \quad u_N(x_N) = c_+,
\end{aligned} \tag{2}$$

where $\{x_i\}_{i=0}^N$ (with $x_0 = -1$ and $x_N = 1$) are the Legendre-Gauss-Lobatto (LGL) points.

A$_2$. Basis and Linear System: We choose the Lagrange basis polynomials $\{h_j\}_{j=0}^N$ on $\{x_j\}_{j=0}^N$ as basis functions, that is,

$$\mathbb{P}_N = \mathrm{span}\{h_0, h_1, \cdots, h_N\}.$$

Denote $v_j = u_N(x_j)$. Then we can write the numerical solution as

$$u_N(x) = \sum_{j=0}^N v_j h_j(x) = \underbrace{c_- h_0(x) + c_+ h_N(x)}_{\text{known}} + \sum_{j=1}^{N-1} v_j h_j(x). \tag{3}$$

Substituting it into the scheme (2) yields

$$\sum_{j=1}^{N-1} v_j \, \mathcal{L}[h_j](x_i) = f(x_i) - c_- \, \mathcal{L}[h_0](x_i) - c_+ \, \mathcal{L}[h_N](x_i), \quad 1 \le i \le N-1. \tag{4}$$

Introduce the matrix $\boldsymbol{A} \in \mathbb{R}^{(N-1)^2}$ with the entries

$$A_{ij} := \mathcal{L}[h_j](x_i) = -\varepsilon h_j''(x_i) + p(x_i)h_j'(x_i) + q(x_i)h_j(x_i), \quad 1 \le i, j \le N-1,$$

and introduce the vectors of unknowns and source terms

$$\boldsymbol{v} = (v_1, \cdots, v_{N-1})^t; \quad \boldsymbol{b} = (b_1, \cdots, b_{N-1})^t,$$

with

$$b_i = f(x_i) - c_- \mathcal{L}[h_0](x_i) - c_+ \mathcal{L}[h_N](x_i). \tag{5}$$

Then we can formulate the scheme (2) as

$$\boldsymbol{A}\,\boldsymbol{v} = \boldsymbol{b}. \tag{6}$$

A$_3$. Differentiation Matrix: It is seen that one has to evaluate $\{d_{ij}^{(m)} := h_j^{(m)}(x_i)\}$ for $0 \le i, j \le N$, that is, the mth-order differentiation matrix: $\boldsymbol{D}^{(m)} \in \mathbb{R}^{(N+1)^2}$. From the general theory, we know

$$\boldsymbol{D}^{(m)} = \boldsymbol{D} \cdots \boldsymbol{D} = \boldsymbol{D}^m, \tag{7}$$

which is the product of m first-order differentiation matrices $\boldsymbol{D} = \boldsymbol{D}^{(m)}$ with $m = 1$. Then we can express the matrix $\boldsymbol{A}$ in (6) as

$$\boldsymbol{A} = -\varepsilon\,\boldsymbol{D}_{\text{in}}^{(2)} + \boldsymbol{P}\boldsymbol{D}_{\text{in}} + \boldsymbol{Q}, \tag{8}$$

where $\boldsymbol{D}_{\text{in}}^{(m)}$ denotes the sub-matrix of $\boldsymbol{D}^{(m)}$ by deleting its first and last rows and columns, and $\boldsymbol{P}, \boldsymbol{Q}$ are diagonal matrices given by

$$\boldsymbol{P} = \text{diag}(p(x_1), \cdots, p(x_{N-1})), \quad \boldsymbol{Q} = \text{diag}(q(x_1), \cdots, q(x_{N-1})). \tag{9}$$

The elements of the vector $\boldsymbol{b}$ in (5) involve the first and last columns of the differentiation matrices, that is,

$$b_i = f(x_i) - c_- \big(-\varepsilon d_{i0}^{(2)} + p(x_i) d_{i0}^{(1)}\big) - c_+ \big(-\varepsilon d_{iN}^{(2)} + p(x_i) d_{iN}^{(1)}\big), \tag{10}$$

for $1 \le i \le N-1$.

In summary, we carry out the **Collocation Algorithm** as follows.

(i) **Initialization:**

Compute the Legendre-Gauss-Lobatto (LGL) points $\{x_j\}_{j=0}^N$ via: `legslb.m`

Compute the first-order LGL differentiation matrix $\boldsymbol{D}$ via: `legslbdiff.m` and obtain $\boldsymbol{D}^{(m)}$ via (7)

(ii) **Form and solve the linear system:**

Form the matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$ in (6) via (8)-(10), and solve it by a suitable linear solver.

(ii) **Output the results.**

B) **Pseudo-spectral Methods**

We multiply the first equation of (2) by $\phi(x_i)\omega_i$ with $\phi \in \mathbb{P}_N^0$, and then sum for $0 \le i \le N$:

$$\sum_{i=0}^{N} \mathcal{L}[u_N](x_i)\phi(x_i)\omega_i = \sum_{i=0}^{N} f(x_i)\phi(x_i)\omega_i,$$

so we can write it in terms of discrete inner products:

$$\langle \mathcal{L}[u_N], \phi\rangle_N = \langle f, \phi\rangle_N, \quad \forall \phi \in \mathbb{P}_N^0. \tag{11}$$

Using the exactness of Gauss-Lobattor quadrature and integration by parts, we get the pseudospectral scheme: Find $u_N \in \mathbb{P}_N$ (satisfying the boundary conditions) such that

$$a(u_N, \phi) := \varepsilon\langle u_N', \phi'\rangle_N + \langle p u_N', \phi\rangle_N + \langle q u_N, \phi\rangle_N = \langle f, \phi\rangle_N, \quad \forall \phi \in \mathbb{P}_N^0. \tag{12}$$

Similarly, we expand u_N as in (3) and insert into the above with $\phi = h_i(x)$ for $1 \le i \le N-1$ (which belongs to $\mathbb{P}_N^0$):

$$\sum_{j=1}^{N-1} v_j\, a(h_j, h_i) = \langle f, h_i\rangle_N - c_-\, a(h_0, h_i) - c_+\, a(h_N, h_i) := \tilde{b}_i,$$

Denote the coefficient matrix by $\tilde{\boldsymbol{A}}$ with the entries given by

$$\begin{aligned}
\tilde{A}_{ij} &= a(h_j, h_i) = \varepsilon\langle h_j', h_i'\rangle_N + \langle p h_j', h_i\rangle_N + \langle q h_j, h_i\rangle_N \\
&= \varepsilon \sum_{\ell=0}^{N} h_j'(x_\ell) h_i'(x_\ell)\omega_\ell + \sum_{\ell=0}^{N} p(x_\ell) h_j'(x_\ell) h_i(x_\ell)\omega_\ell + \sum_{\ell=0}^{N} q(x_\ell) h_j(x_\ell) h_i(x_\ell)\omega_\ell \\
&= \varepsilon \sum_{\ell=0}^{N} d_{\ell j} d_{\ell i}\omega_\ell + \sum_{\ell=0}^{N} p(x_\ell) d_{\ell j}\delta_{\ell i}\,\omega_\ell + \sum_{\ell=0}^{N} q(x_\ell)\delta_{\ell j}\,\delta_{\ell i}\,\omega_\ell \\
&= \varepsilon(\widetilde{\boldsymbol{D}}^t \boldsymbol{W}\widetilde{\boldsymbol{D}})_{ij} + p(x_i)\omega_i d_{ij} + q(x_i)\omega_i\delta_{ij}, \quad 1 \le i, j \le N-1,
\end{aligned} \tag{13}$$

where $\widetilde{\boldsymbol{D}}$ of size $(N+1)\times(N-1)$ is the sub-matrix of the first-order differentiation matrix $\boldsymbol{D}$ by deleting its first and last columns, and $\boldsymbol{W} = \mathrm{diag}(\omega_0, \cdots, \omega_N)$. As a result, we have

$$\tilde{\boldsymbol{A}} = \varepsilon\widetilde{\boldsymbol{D}}^t \boldsymbol{W}\widetilde{\boldsymbol{D}} + \boldsymbol{P}\boldsymbol{W}_{\text{in}}\boldsymbol{D}_{\text{in}} + \boldsymbol{Q}\boldsymbol{W}_{\text{in}}, \tag{14}$$

where $\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{D}_{\text{in}}$ are the same as in (8)-(9) and $\boldsymbol{W}_{\text{in}} = \mathrm{diag}(\omega_1, \cdots, \omega_{N-1})$.

Similarly, we can compute the elements of the vector $\tilde{\boldsymbol{b}}$. By (13) with $j = 0, N$, we find

$$a(h_0, h_i) = \varepsilon \sum_{\ell=0}^{N} d_{\ell 0} d_{\ell i}\omega_\ell + p(x_i)\omega_i d_{i0}, \quad a(h_N, h_i) = \varepsilon \sum_{\ell=0}^{N} d_{\ell N} d_{\ell i}\omega_\ell + p(x_i)\omega_i d_{iN}.$$

Therefore, we have

$$\tilde{\boldsymbol{b}} = \boldsymbol{W}_{\text{in}}\boldsymbol{f} - c_{-}\left(\varepsilon\widetilde{\boldsymbol{D}}^{t}\boldsymbol{W}\boldsymbol{d}_0 + \boldsymbol{P}\boldsymbol{W}_{\text{in}}\boldsymbol{d}_{\text{in},0}\right) - c_{+}\left(\varepsilon\widetilde{\boldsymbol{D}}^{t}\boldsymbol{W}\boldsymbol{d}_N + \boldsymbol{P}\boldsymbol{W}_{\text{in}}\boldsymbol{d}_{\text{in},N}\right), \tag{15}$$

where $\boldsymbol{d}_1, \boldsymbol{d}_N$ are respectively the first and last columns of $\boldsymbol{D}$, and $\boldsymbol{d}_{\text{in},0}, \boldsymbol{d}_{\text{in},N}$ denote the column vectors by deleting the first and last elements of $\boldsymbol{d}_1, \boldsymbol{d}_N$, respectively.