Lifting Minimal Cover Inequalities for Knapsack Problem

Exploit combinatorial structure of Integer Programming to derive facet inequalities.

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Minimal Cover Inequalities

Consider the $0,1$ knapsack set

$$K := \left\{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$$

where $b > 0$ and $b > a_j > 0$ for $j \in N := \{1, \cdots, n\}$. conv$(K)$ has dimension n.

A **cover** is a subset $C \subset N$ such that $\sum_{j \in C} a_j > b$. For any cover C, the **cover inequality** associated with C is

$$\sum_{j \in C} x_j \leq |C| - 1,$$

and it is **valid** for conv(K). A cover inequality is **minimal** if $\sum_{j \in C \setminus \{k\}} a_j \leq b$ for all $k \in C$.

Minimal cover inequalities could reformulate the question.

Proposition 1

The set

$$K^C := \left\{ x \in \{0,1\}^n : \sum_{i \in C} x_i \leq |C| - 1 \textit{ for every minimal cover } C \textit{ for } K \right\}$$

coincide with the knapsack set $K := \left\{ x \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b \right\}$.

However, we could construct examples one of the formulation is better than other:

$$K_{K^C \text{ better}} := \{x \in \{0,1\}^3 : 3x_1 + 3x_2 + 3x_3 \leq 5\},$$

and vice versa:

$$K_{K \text{ better}} := \{x \in \{0,1\}^3 : x_1 + x_2 + x_3 \leq 1\},$$

or neither is better, i.e. their linear relaxations can not be compared.

Every polyhedron admits a **minimal representation** in which none of its constraints are redundant. A **face** of a polyhedron P is a set of the form

$$F := P \cap \{x \in \mathbb{R}^n : cx = \delta\}$$

where $cx \le \delta$ is a valid inequality for P. Inclusionwise maximal **proper faces** of P are called **facets**.

Theorem 1

Let $P := \{x \in \mathbb{R}^n : Ax \le b\}$ be a nonempty polyhedron, and let f be the number of its facets. ***(i)*** *For each facet F of P, there exists $j \in I^<$ such that the inequality $a^j x \le b_j$ defines F.* ***(ii)*** *If $a^i x = b_i, i \in I^=, a^i x \le b_i, i \in I^<$ is a minimal representation for P, then $|I^<| = f$ and the facets of P are $F_i := \{x \in P : a^i x = b_i\}$, for $i \in I^<$.* ***(iii)*** *A face F of P is a facet if and only if F is nonempty and $\dim(F) = \dim(P) - 1$.*

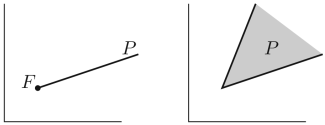

Figure: Facets of dimension 0,1

Facet-defining inequality could bring no extra information:

Proposition 2

The inequality $x_j \leq 1$ *defines a facet of* K *if and only if*

$$a_{j_*} + a_j \leq b,$$

where $a_{j_*} = \max_{j \in N - \{j\}} a_i$.

Proposition 3

Let C *be a cover for* K*. The cover inequality associated with* C *is facet-defining for* $P_C := \mathrm{conv}(K) \cap \{x \in \mathbb{R}^n : x_j = 0, j \in N \setminus C\}$ *if and only if* C *is a minimal cover.*

Given a minimal cover C, how can one compute coefficients $\alpha_j, j \in N \setminus C$, so that the inequality

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq \mid C \mid -1$$

is facet-defining for $\mathrm{conv}(K)$?

Separation Problem: a Cutting Plane Scheme

Given a vector $\bar{x} \in [0,1]^n$, find a cover inequality for K that is violated by $\bar{x}$, or show that none exists.

Notice a cover inequality relative to C is violated by $\bar{x} \iff \sum_{j \in C} (1 - \bar{x}_j) < 1$.

Deciding whether a violated cover inequality exists reduces to solving the problem

$$\eta = \min_{C \subseteq K} \{ \sum_{j \in C} (1 - \bar{x}_j) : C \text{ is a cover for } K \}.$$

$$\implies \quad \eta = \min_{\mathbf{z}} \sum_{j=1}^{n} (1 - \bar{x}_j) z_j$$

$$\text{s.t.} \quad \sum_{j=1}^{n} a_j z_j \geq b + 1.$$

$$z \in \{0,1\}^n.$$

In general, it is NP-hard solvable. Heuristics method uses relaxed optimal solution of $\eta < 1$ to form cover $C := \{ j \in N : z_j^* > 0 \}$. However, it doesn't guarantee cutting off the fractional point $\bar{x}$.

Lifting

Proposition 4

Consider a set $S \subseteq \{0,1\}^n$ *such that* $S \cap \{x : x_n = 1\} \neq \varnothing$, *and let* $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ *be a valid inequality for* $S \cap \{x : x_n = 0\}$. *Then*

$$\alpha_n := \beta - \max\left\{ \sum_{i=1}^{n-1} \alpha_i x_i : x \in S, x_n = 1 \right\}$$

is the largest coefficient such that $\sum_{i=1}^{n-1} \alpha_i x_i + \alpha_n x_n \leq \beta$ *is valid for* S.

Furthermore, if $\sum_{i=1}^{n-1} \alpha_i x_i < \beta$ *defines a* d*-dimensional face of* conv$(S) \cap \{x_n = 0\}$, *then* $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ *defines a face of* conv(S) *of dimensional at least* $d+1$.

Sequential Lifting

Consider a set $S := \{x \in \{0,1\}^n : Ax \le b\}$ of dimension n, where A is a nonnegative matrix. Proposition 4 suggests the following way of lifting a facet-defining inequality $\sum_{j \in C} \alpha_j x_j \le \beta$ of $\mathrm{conv}(S) \cap \{x : x_j = 0, j \in N \setminus C\}$:

1. Choose an ordering $j_1, \cdots, j_l$ of the indices in $N \setminus C$. Let $C_0 = C$ and $C_h = C_{h-1} \cup \{j_h\}$ for $h = 1, \cdots, l$.

2. For $h = 1$ up to $h = l$, compute

$$\alpha_{j_h} := \beta - \max \left\{ \sum_{j \in C_{h-1}} \alpha_j x_j : x \in S, x_j = 0, j \in N \setminus C_h, x_{j_h} = 1 \right\}.$$

By Proposition 4 the inequality $\sum_{j=1}^{n} \alpha_j x_j \le \beta$ obtained this way is facet-defining for $\mathrm{conv}(S)$. $A \ge 0$ and $\dim(S) = n$ guarantee '$\max$' is feasible.

Example 2

Consider the knapsack set

$$8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 + 6x_6 + 6x_7 \leq 22, x_j \in \{0,1\} \text{ for } j = 1, \cdots, 7.$$

The index set $C := \{1,2,3,4\}$ is a minimal cover. The corresponding minimal cover inequality is $x_1 + x_2 + x_3 + x_4 \leq 3$. We perform sequential lifting according to the order $5, 6, 7$:

$$\alpha_5 = 3 - \max\{\sum_{i=1}^{4} x_i : 8x_1 + 7x_2 + 6x_3 + 4x_4 \leq 22 - 6, x_i \in \{0,1\}\},$$

$$\alpha_5 = 1, \alpha_6 = 3 - \max\{\sum_{i=1}^{5} x_i : 8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 \leq 22 - 6, x_i \in \{0,1\}\}.$$

It follows $\alpha_6 = \alpha_7 = 0$. By symmetry, the inequalities $\sum_{i=1}^{4} x_i + x_j \leq 3$, for $j \in \{5,6,7\}$, are all facet defining. However, not all possible facet-defining lifted inequalities can be obtained sequentially. For example, see $\sum_{i=1}^{4} x_i + 0.5 \sum_{j=5}^{7} x_j \leq 3$, a valid facet inequality whose coefficients are fractional.

Lifting Minimal Cover Inequalities

Theorem 2

Let C be a minimal cover for K, and let

$$\sum_{j\in C} x_j + \sum_{j\in N\setminus C} \alpha_j x_j \le |C|-1$$

be a lifting of the cover inequality associated with C. Up to permuting the indices, assume that $C=\{1,\cdots,t\}$ and $a_1 \ge a_2 \ge \cdots \ge a_t$. Let $\mu_0 := 0$ and $\mu_h := \sum_{l=1}^{h} a_l$ for $h=1,\cdots,t$. Let $\lambda := \mu_t - b > 0$. If it defines a facet of $\text{conv}(K)$, then the following hold for every $j \in N\setminus C$.

(i) If, for some h, $\mu_h \le a_j \le \mu_{h+1} - \lambda$, then $\alpha_j = h$.

(ii) If, for some h, $\mu_{h+1} - \lambda < a_j < \mu_{h+1}$, then $h \le \alpha_j \le h+1$.

Furthermore, for every $j \in N\setminus C$, if $\mu_{h+1} - \lambda < a_j < \mu_{h+1}$, then there exists a facet-defining inequality of the form such that $\alpha_j = h+1$.

Remark of Theorem 2

For every $j \in N \setminus C$, let $h(j)$ be the index such that $\mu_{h(j)} \leq a_j \leq \mu_{h(j)+1}$. The inequality $\sum_{j \in C} x_j + \sum_{j \in N \setminus C} h(j) x_j \leq |C| - 1$ is a lifting of the minimal cover inequality associated with C. Furthermore, if $a_j \leq \mu_{h(j)+1} - \lambda$ for all $j \in N \setminus C$, then the above is the unique facet-defining lifting.

Example 3

We consider the knapsack set

$$K := \{x \in \{0,1\}^5 : 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5 \leq 5\}.$$

The set $C := \{3,4,5\}$ is a minimal cover. We would like to lift the inequality $x_3 + x_4 + x_5 \leq 2$ into a facet of $\text{conv}(K)$. We have $\mu_0 = 0, \mu_1 = 3, \mu_2 = 5, \mu_3 = 6$, and $\lambda = 1$. Therefore $\alpha_1 = 2$ since $\mu_2 \leq a_1 \leq \mu_3 - \lambda$. Similarly $\alpha_2 = 1$ since $\mu_1 \leq a_2 \leq \mu_2 - \lambda$. By Theorem 2, the inequality $2x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$ defines a facet of $\text{conv}(K)$. Furthermore, by remark, this is the unique facet-defining lifting.

Superadditive Lifting Functions

The **lifting funciton** of the inequality $\sum_{j \in C} \alpha_j x_j \leq \beta$ is defined by

$$f(z) := \beta - \max \sum_{i \in C} \alpha_i x_i,$$

$$\sum_{i \in C} a^i x_i \leq b - z, x_i \in \{0,1\} \text{ for } i \in C.$$

By Proposition 4, if $A \geq 0$ and $\dim(S) = n$, for all $j \in N \setminus C$, the coefficients of any lifting must satisfy $\alpha_j \leq f(a^j)$.

A function $g : U \to \mathbb{R}$ is **superadditive** if $g(u+v) \geq g(u) + g(v)$ for all $u, v \in U$ such that $u + v \in U$.

Theorem 3

Let $g : [0,b] \to \mathbb{R}$ be a superadditive function such that $g \leq f$. Then $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} g(a^j) x_j \leq \beta$ is a valid inequality for S. In particular, if f is superadditive, then the inequality $\sum_{j \in C} \alpha_j x_j + \sum_{j \in N \setminus C} f(a^j) x_j \leq \beta$ is the unique maximal lifting of $\sum_{j \in C} \alpha_j x_j \leq \beta$.

Sequence Independent Lifting for Minimal Cover Inequalities

We consider a knapsack set K, i.e. the matrix A is a nonnegative vector. Theorem 2 shows that

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq \mu_1 - \lambda \\ h & \text{if } \mu_h - \lambda < z \leq \mu_{h+1} - \lambda, \text{ for } h = 1, \cdots, t-1. \end{cases}$$

The function f is not superadditive in general. Consider the function g:

$$g(z) := \begin{cases} 0 & \text{if } z = 0 \\ h & \text{if } \mu_h - \lambda < z \leq \mu_{h+1} - \lambda, \text{ for } h = 1, \cdots, t-1 \\ h - \frac{\mu_h - \lambda + \rho_h - z}{\rho_1} & \text{if } \mu_h - \lambda < z \leq \mu_h - \lambda + \rho_h, \text{ for } h = 1, \cdots, t-1 \end{cases}$$

where $\rho_h = \max\{0, a_{h+1} - (a_1 - \lambda)\}$ for $h = 0, \cdots, r-1$. Note that $g \leq f$. It can be shown that the function g is superadditive. Hence by Theorem 3 the inequality

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} g(a^j) x_j \leq |C| - 1.$$

is a lifting of the minimal cover inequality.

Example 4

Consider the knapsack set from Example 2

$$8x_1+7x_2+6x_3+4x_4+6x_5+6x_6+6x_7 \leq 22, x_j \in \{0,1\} \text{ for } j=1,\cdots,7.$$

This time we lift the minimal cover $C := \{1,2,3,4\}$ using the supperadditive function g and the lifted minimal cover inequality is

$$x_1+x_2+x_3+x_4+0.5x_5+x_6+0.5x_7 \leq 3.$$

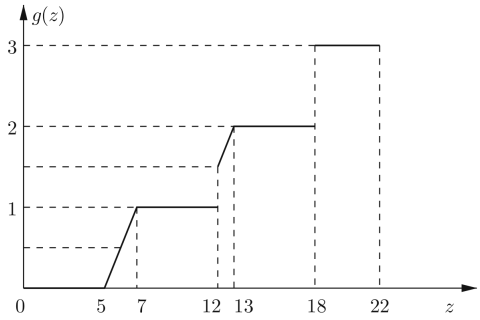

Figure: Sequence independent lifting function g