Presolve Reductions for Individual Constraints and Variables

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Introduction

Presolve is a technique used in optimization to simplify the problem before it is solved by the main solver. Here is a brief overview of what Presolve does:

- It eliminates redundant constraints or variables, reducing the size of the problem after Presolve processing
- It tightens the bounds of integer programming, improving the bounds of the linear programming relaxation after Presolve processing
- It reduces the size of the coefficients in the problem, making the absolute values of the coefficients smaller after Presolve processing

Introduction

Based on the different criteria used for applying the reductions, the presolve algorithms can be categorized into three distinct classes:

- Reductions for Individual Constraints and Variables
- Reductions that Consider Multiple Constraints and Variables Simultaneously
- Reductions that Consider the Whole Problem

Notation

Given a matrix $A \in \mathbb{R}^{m \times n}$, vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $\ell \in (\mathbb{R} \cup \{-\infty\})^n$, $u \in (\mathbb{R} \cup \{\infty\})^n$, variables $x \in \mathbb{R}^n$ with $x_j \in \mathbb{Z}$ for $j \in I \subseteq N = \{1, \ldots, n\}$, and relations $\circ_i \in \{=, \leq, \geq\}$ for every row $i \in M = \{1, \ldots, m\}$ of A, then the optimization problem MIP $=$ $(M, N, I, A, b, c, \circ, \ell, u)$ defined as

$$\begin{aligned} \min \ & c^T x \\ \text{s.t.} \quad & Ax \circ b \\ & \ell \leq x \leq u \\ & x_j \in \mathbb{Z} \text{ for all } j \in I \end{aligned}$$

Notation

- The elements of the matrix A are denoted by a_{ij}, $i \in M, j \in N$
- We use the notation $A_{i\cdot}$ to identify the row vector given by the ith row of A. Similarly, $A_{\cdot j}$ is the column vector given by the jth column of A
- With $\text{supp}(A_{i\cdot}) = \{j \in N : a_{ij} \neq 0\}$ we denote the support of $A_{i\cdot}$, and $\text{supp}(A_{\cdot j}) = \{i \in M : a_{ij} \neq 0\}$ denotes the support of $A_{\cdot j}$
- For a subset $S \subseteq N$ we define A_{iS} to be the vector $A_{i\cdot}$ restricted to the indices in S. Similarly, x_S denotes the vector x restricted to S.
- $\inf\{A_{i\cdot}x\} = \sum_{j \in N, a_{ij} > 0} a_{ij} l_j + \sum_{j \in N, a_{ij} < 0} a_{ij} u_j$
- $\sup\{A_{i\cdot}x\} = \sum_{j \in N, a_{ij} > 0} a_{ij} u_j + \sum_{j \in N, a_{ij} < 0} a_{ij} l_j$

Model Cleanup

Remove $A_{i\cdot}x \leq b_i$

- $b_i \geq \psi$
- $\sup\{A_{i\cdot}x\} \leq b_i + \epsilon$
- If $\inf\{A_{i\cdot}x\} \geq b_i + \epsilon$, the problem is infeasible

Remove $A_{i\cdot}x = b_i$

- $\inf\{A_{i\cdot}x\} \geq b_i - \epsilon$ and $\sup\{A_{i\cdot}x\} \leq b_i + \epsilon$
- If $\inf\{A_{i\cdot}x\} > b_i + \epsilon$ or $\sup\{A_{i\cdot}x\} < b_i - \epsilon$, the problem is infeasible

Set $a_{ik} \neq 0$ to be 0 and $b_i = b_i - a_{ik} \cdot l_k$

- $|a_{ik}| < 10^{-3}$ and $|a_{ik}| \cdot (u_k - l_k) \cdot |supp(A_{i\cdot})| < 10^{-2} \cdot \epsilon$
- $|a_{ik}| \cdot (u_k - l_k) < 10^{-1} \cdot \epsilon$
- $|a_{ik}| < 10^{-10}$

Bound Strengthening

Consider an inequality constraint: $A_{iS}x_S + a_{ik}x_k \leq b_i$, where $S = supp(A_{i\cdot}) \setminus \{k\}$ and $a_{ik} \neq 0$.

Let $l_{iS} = \inf\{A_{iS}x_S\} \leq A_{iS}x_S$ and if $l_{iS} > -\infty$:

- $a_{ik} > 0$: $x_k \leq (b_i - A_{iS}x_S)/a_{ik} \leq (b_i - l_{iS})/a_{ik}$, then $u_k = \min\{u_k, (b_i - l_{iS})/a_{ik}\}$

- $a_{ik} < 0$: $x_k \geq (b_i - A_{iS}x_S)/a_{ik} \geq (b_i - l_{iS})/a_{ik}$, then $l_k = \max\{l_k, (b_i - l_{iS})/a_{ik}\}$

Coefficient Strengthening

Consider an inequality constraint: $A_{iS}x_S + a_{ik}x_k \leq b_i$, where $S = supp(A_{i\cdot}) \setminus \{k\}$ and $a_{ik} \neq 0$, but now assume that x_k is an integer variable.

Let $u_{iS} = \sup\{A_{iS}x_S\} \geq A_{iS}x_S$ and if $u_{iS} < \infty$:

- $a_{ik} > 0$: If $a_{ik} \geq d = b_i - u_{iS} - a_{ik}(u_k - 1) > 0$, then replace the original constraint by $A_{iS}x_S + (a_{ik} - d)x_k \leq b_i - du_k$ to obtain an equivalent model with a tighter LP relaxation

Coefficient Strengthening

Equivalence

- For $x_k = u_k$, $A_{iS}x_S + (a_{ik} - d)u_k \leq b_i - du_k \Rightarrow A_{iS}x_S + a_{ik}u_k \leq b_i$
- For $x_k \leq u_k - 1$,
  $A_{iS}x_S + a_{ik}(u_k - 1) \leq u_{iS} + a_{ik}(u_k - 1) = b_i - d \leq b_i$ and
  $A_{iS}x_S + (a_{ik} - d)(u_k - 1) \leq u_{iS} + (a_{ik} - d)(u_k - 1) = b_i - du_k$

Tighter LP relaxation

- For $x_k = u_k - 1$
- $A_{iS}x_S + a_{ik}(u_k - 1) \leq b_i \Rightarrow A_{iS}x_S + b_i - u_{iS} - d \leq b_i \Rightarrow A_{iS}x_S \leq u_{iS} + d$
- $A_{iS}x_S + (a_{ik} - d)(u_k - 1) \leq b_i - du_k \Rightarrow$
  $A_{iS}x_S + b_i - u_{iS} - d - du_k + d \leq b_i - du_k \Rightarrow A_{iS}x_S \leq u_{iS}$

Simple Probing

Consider an equality constraint: $A_{i\cdot}x = b_i$ with $x_k \in \{0,1\}$, $a_{ik} \neq 0$, and

- $|a_{ik}| = \sup\{A_{i\cdot}x\} - b_i$
- $|a_{ik}| = b_i - \inf\{A_{i\cdot}x\}$
- Because replacing x_j with $x_j' = u_j - x_j$ does not affect these two conditions, we can assume that $a_{ij} \geq 0$ for all j

The first and second conditions imply

- $\sum_{j \in supp(A_{i\cdot})\setminus\{k\}} a_{ij}u_j = b_i$, then $x_k = 0 \Rightarrow x_j = u_j$ for all $j \in supp(A_{i\cdot})\setminus\{k\}$
- $\sum_{j \in supp(A_{i\cdot})\setminus\{k\}} a_{ij}l_j + a_{ik} = b_i$, then $x_k = 1 \Rightarrow x_j = l_j$ for all $j \in supp(A_{i\cdot})\setminus\{k\}$
- $x_j = u_j - (u_j - l_j)x_k$ for all $j \in supp(A_{i\cdot})\setminus\{k\}$

Removal of Fixed Variables

Variables x_j with bounds $l_j = u_j$ can be removed from the problem by subtracting $A_{\cdot j} l_j$ from the right-hand side b and by accumulating the contributions $c_j l_j$ to the objective function in a constant $c_0 \in R$.

Rounding Bounds of Integer Variables

If x_j is an integer variable with fractional lower or upper bound, this bound can be replaced by $\lceil l_j \rceil$ or $\lfloor u_j \rfloor$, respectively.

Strengthen Semi-continuous and Semi-integer Bounds

Let a semi-continuous variable x_j with $x_j = 0 \vee (l_j \leq x_j \leq u_j)$ be given. If we can prove $x_j \geq l'_j > 0$, then we can convert x_j into a regular continuous variable with $\max\{l_j, l'_j\} \leq x_j \leq u_j$. Conversely, if we can prove $x_j < l_j$ the variable must be zero and we can fix $x_j = 0$. Finally, if $l_j = 0$ we can discard the "semi" property of the variable and interpret the variable as a regular continuous variable.

Substitute-Implied Free Variables

Let l_j and u_j denote the explicit bounds of variable x_j, and $\bar{l}_j$ and $\overline{u}_j$ be the tightest implied bounds that can be determined using bound strengthening.If $[\bar{l}_j, \overline{u}_j] \subseteq [l_j, u_j]$, x_j is called an implied free variable: we could remove the bounds of the variable without modifying the feasible space.