T&R Team of Algorithm Design
College of Computer Science and Engineering, CQU

Algorithm Analysis & Design

Introduction to Algorithm

Chapter 25:

All-Pairs Shortest Paths

Outlines

- **All-pairs shortest paths**
- **First solution: using Dijkstra’s algorithm**
- **Second solution: dynamical programming**

  1. matrix manipulation

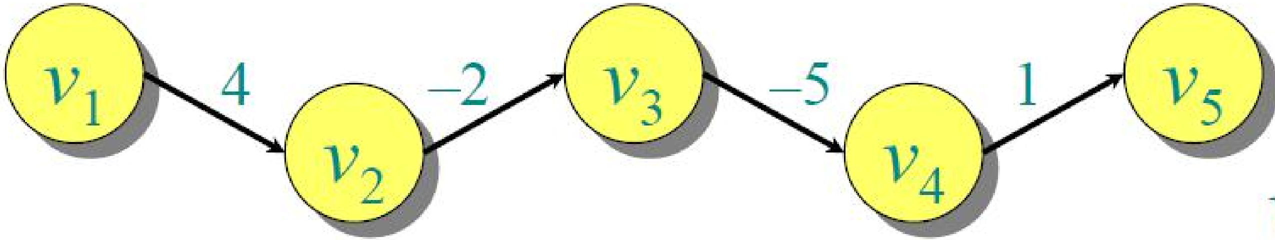
  2. Floyd-warshall algorithm

Paths in Graphs

Consider a digraph $G = (V, E)$ with edge-weight function $w : E \to \mathbb{R}$. The ***weight*** of path $p = v_1 \to v_2 \to \cdots \to v_k$ is defined to be

$$w(p) = \sum_{i=1}^{k-1} w(v_i, v_{i+1}).$$

Example:



$w(p) = -2$

Shortest Paths

A ***shortest path*** from u to v is a path of minimum weight from u to v. The ***shortest-path weight*** from u to v is defined as

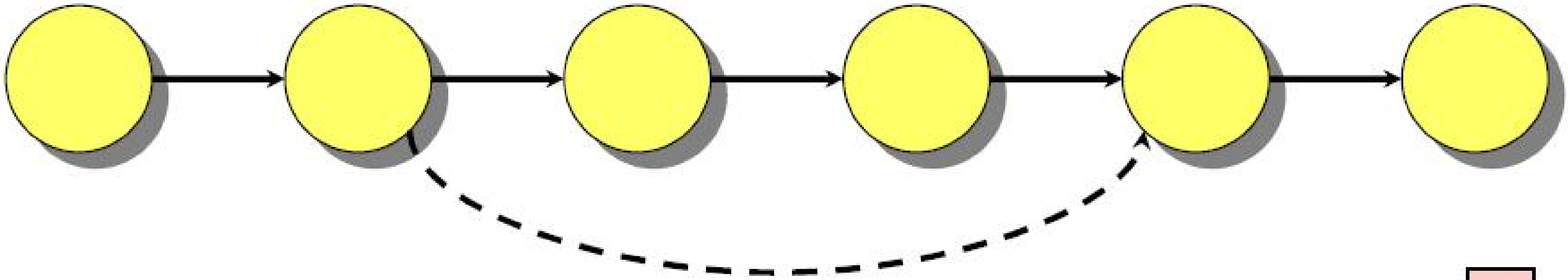
$$\delta(u, v) = \min\{w(p) : p \text{ is a path from } u \text{ to } v\}.$$

Note: $\delta(u, v) = \infty$ if no path from u to v exists.

Optimal Sub-Structure

Theorem. A subpath of a shortest path is a shortest path.

Proof. Cut and paste:



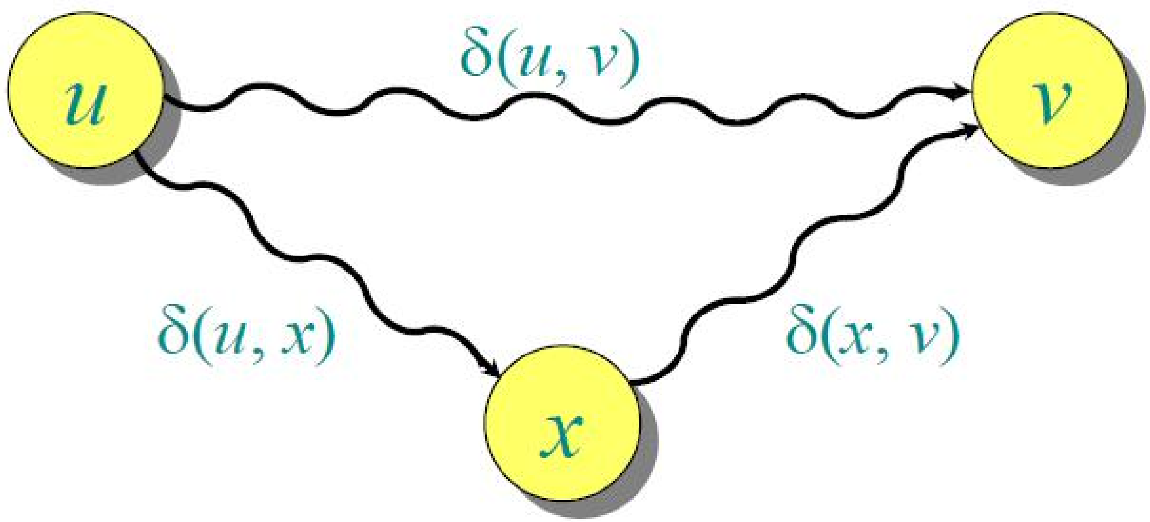
□

Triangle Inequality

Theorem. For all $u, v, x \in V$, we have

$$\delta(u, v) \leq \delta(u, x) + \delta(x, v).$$

Proof.

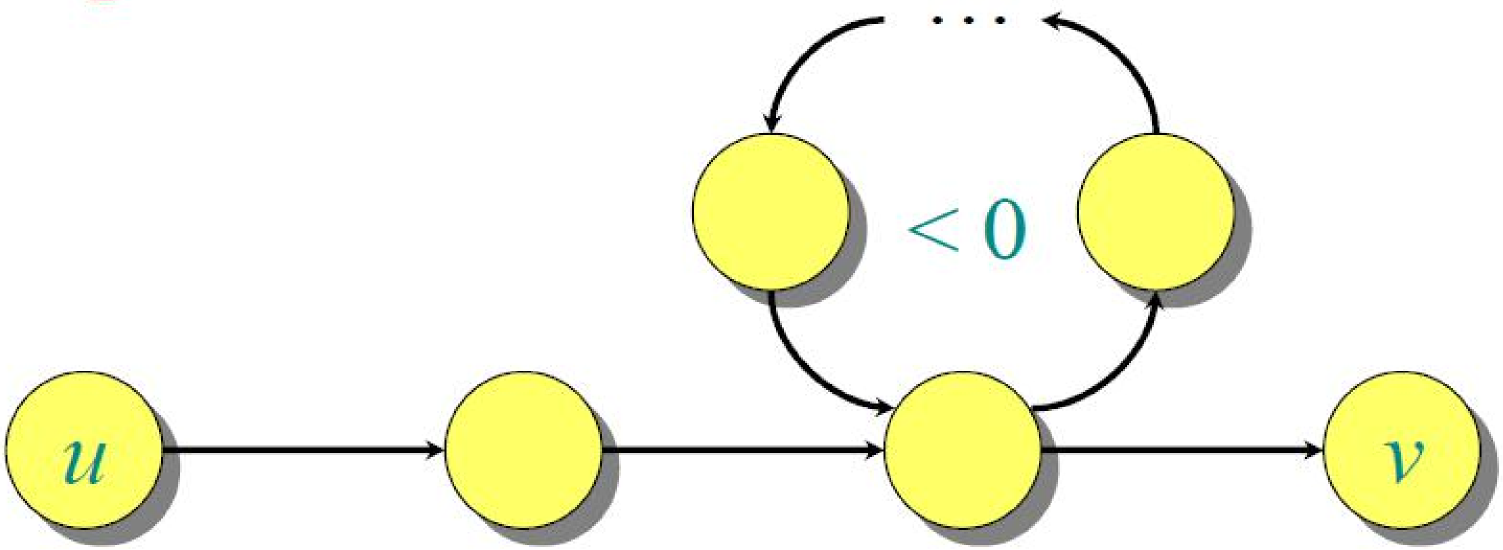


$\square$

Well-Definedness of SP

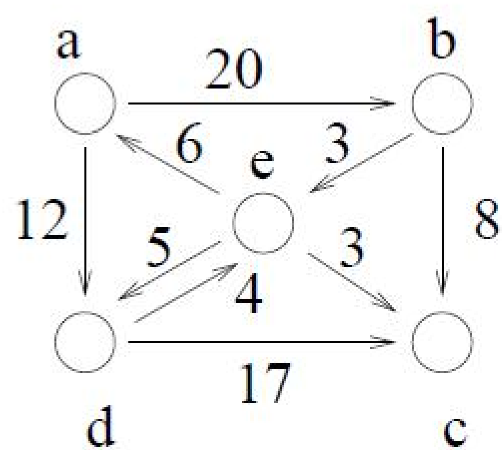
If a graph G contains a negative-weight cycle, then some shortest paths may not exist.

Example:

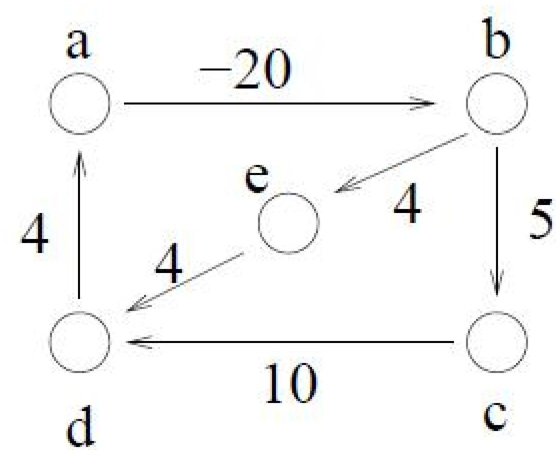


All-Pairs Shortest Paths

Given a weighted digraph $G = (V, E)$ with weight function $w : E \rightarrow \mathbf{R}$, ($R$ is the set of real numbers), determine the length of the shortest path (i.e., distance) between all pairs of vertices in G.



without negative cost cycle



with negative cost cycle

Dijkstra’s Algorithm

Solution 1: Dijkstra's Algorithm

If there are no negative cost edges apply Dijkstra's algorithm to each vertex (as the source) of the digraph.

- Recall that D's algorithm runs in $\Theta((n+e)\log n)$. This gives a

$$\Theta(n(n+e)\log n) = \Theta(n^2 \log n + ne \log n)$$

time algorithm, where $n = |V|$ and $e = |E|$.

Dynamical Programming

To make DP work:

(1) How do we decompose the all-pairs shortest paths problem into subproblems?

(2) How do we express the optimal solution of a subproblem in terms of optimal solutions to some subsubproblems?

(3) How do we use the recursive relation from (2) to compute the optimal solution in a bottom-up fashion?

(4) How do we construct all the shortest paths?

Matrix multiplication

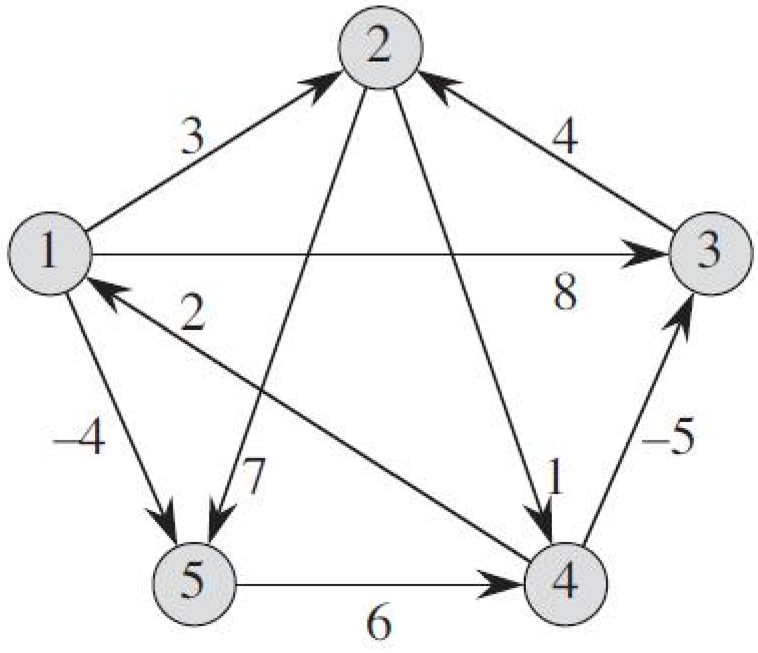
To simplify the notation, we assume that $V = \{1, 2, \ldots, n\}$.

Assume that the graph is represented by an $n \times n$ matrix with the weights of the edges:

$$w_{ij} = \begin{cases} 0 & \text{if } i = j, \\ w(i,j) & \text{if } i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{if } i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

Output Format: an $n \times n$ matrix $D = [d_{ij}]$ where d_{ij} is the length of the shortest path from vertex i to j.

Example



Without negative circle

Input
$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Output
$$\begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

How to decompose the problem

- Subproblems with smaller sizes should be easier to solve.

- An optimal solution to a subproblem should be expressed in terms of the optimal solutions to subproblems with smaller sizes.

These are guidelines ONLY.

Step 1: Decompose in a Natural Way

- Define $d_{ij}^{(m)}$ to be the length of the shortest path from i to j that contains at most m edges.
  Let $D^{(m)}$ be the $n \times n$ matrix $[d_{ij}^{(m)}]$.

- $d_{ij}^{(n-1)}$ is the true distance from i to j (see next page for a proof this conclusion).

- Subproblems: compute $D^{(m)}$ for $m = 1, \cdots, n-1$.

Question: Which $D^{(m)}$ is easiest to compute?

$d_{ij}^{(n-1)}$ = True Distance from i to j

Proof: We prove that any shortest path P from i to j contains at most $n-1$ edges.

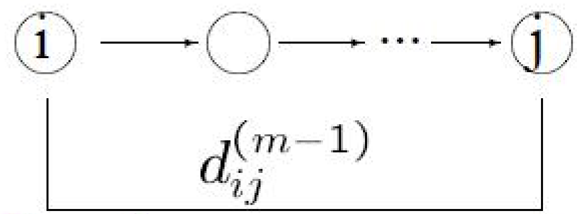
First note that since all cycles have positive weight, a shortest path can have no cycles (if there were a cycle, we could remove it and lower the length of the path).

A path without cycles can have length at most $n-1$ (since a longer path must contain some vertex twice, that is, contain a cycle).

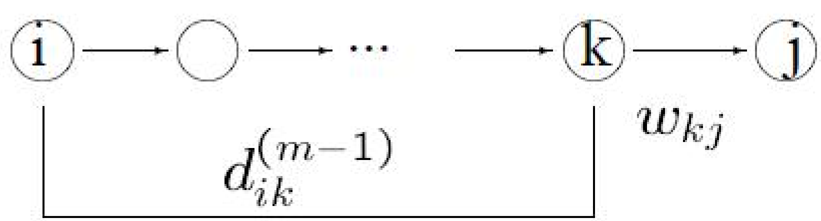
Step 2： Recursive Formula

Consider a shortest path from i to j of length $d_{ij}^{(m)}$.

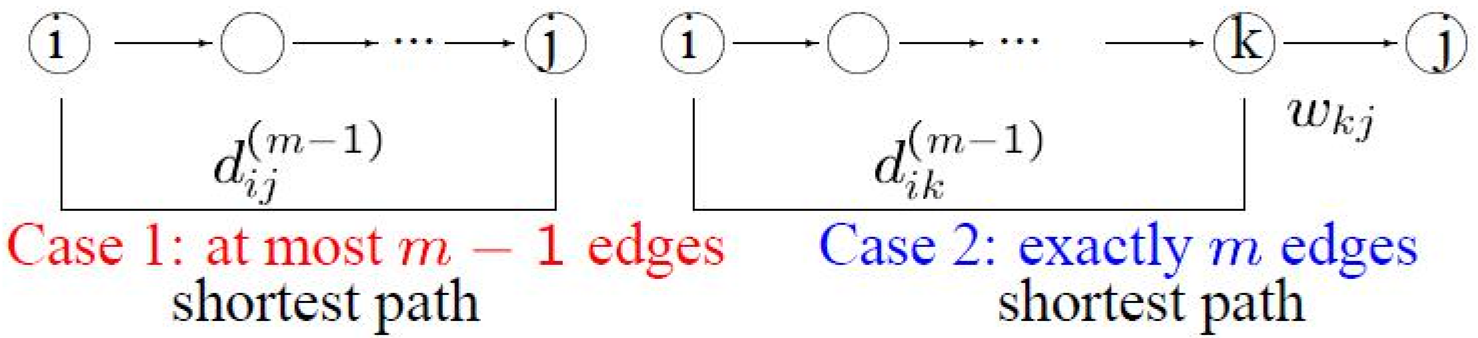
Case 1: It has at most $m-1$ edges.
Then $d_{ij}^{(m)} = d_{ij}^{(m-1)} = d_{ij}^{(m-1)} + w_{jj}$.



Case 2: It has m edges. Let k be the vertex before j on a shortest path.
Then $d_{ij}^{(m)} = d_{ik}^{(m-1)} + w_{kj}$.



Step 2: Recursive Formula



Combining the two cases,

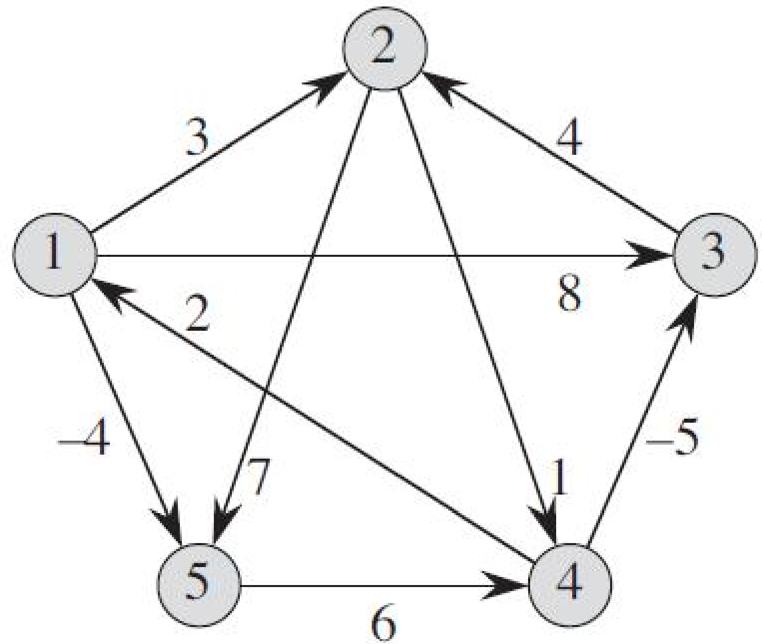
$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

Step 3: Bottom-Up Computation

- Bottom: $D^{(1)} = \left[w_{ij}\right]$, the weight matrix.
- Compute $D^{(m)}$ from $D^{(m-1)}$, for $m = 2, ..., n-1$, using

$$d_{ij}^{(m)} = \min_{1 \le k \le n} \left\{ d_{ik}^{(m-1)} + w_{kj} \right\}.$$

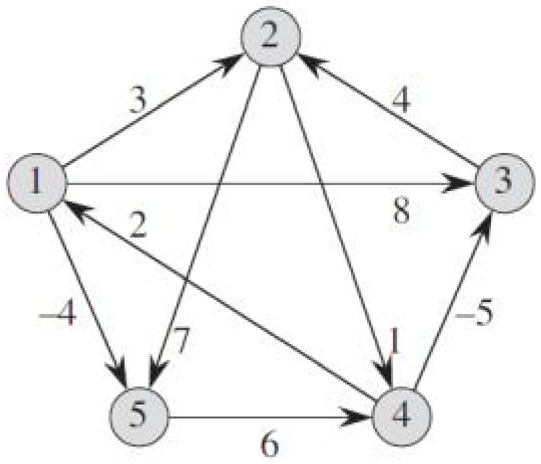
Example



$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

weight matrix

Example



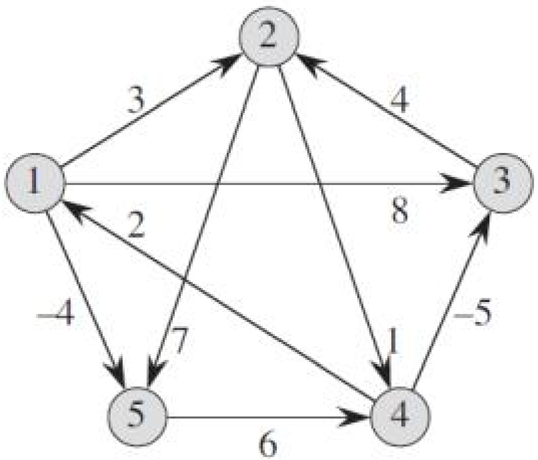
$$D^{(1)} \times D^{(1)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \text{x} \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(2)} = \min_{1 \leq k \leq 5}\{d_{ik}^{(1)} + d_{kj}^{(1)}\}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

Example

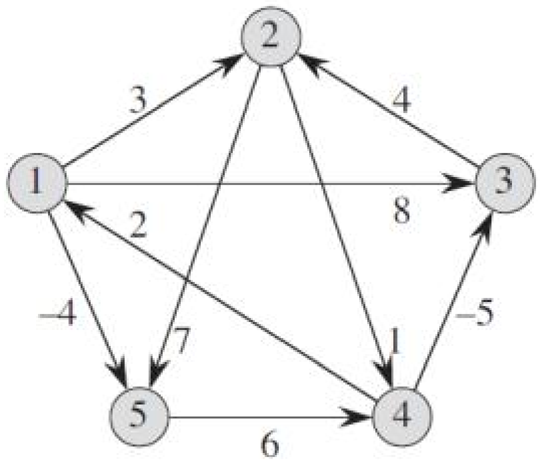


$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix} \quad \text{x} \quad D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(3)} = \min_{1 \le k \le 5}\{d_{ik}^{(2)} + d_{kj}^{(1)}\}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Example



$$D^{(3)} \qquad\qquad\qquad\qquad D^{(1)}$$

$$\begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(4)} = \min_{1 \le k \le 5}\{d_{ik}^{(3)} + d_{kj}^{(1)}\}$$

$$D^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

The shortest distances between any pair of vertices

Algorithm

for $m = 1$ to $n - 1$
    for $i = 1$ to n
        for $j = 1$ to n
        $\{$
            $min = \infty;$
            for $k = 1$ to n
            $\{$
                $new = d_{ik}^{(m-1)} + w_{kj};$
                if $(new < min)\ min = new;$
            $\}$
            $d_{ij}^{(m)} = min;$
        $\}$

Comments

- Algorithm uses $\Theta(n^3)$ space; how can this be reduced down to $\Theta(n^2)$?
- How can we extract the actual shortest paths from the solution?
- Running time $O(n^4)$, much worse than the solution using Dijkstra's algorithm. Can we improve this?

Improvement: Repeated Squaring

$$D^{(n-1)} = D^{i}, \text{ for all } i \geq n.$$

In particular, this implies that $D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n-1)}$.

We can calculate $D^{\left(2^{\lceil \log_2 n \rceil}\right)}$ using "repeated squaring" to find

$$D^{(2)}, D^{(4)}, D^{(8)}, \ldots, D^{\left(2^{\lceil \log_2 n \rceil}\right)}$$

Improvement: Repeated Squaring

- Bottom: $D^{(1)} = \left[w_{ij}\right]$, the weight matrix.

- For $s \geq 1$ compute $D^{(2s)}$ using

$$d_{ij}^{(2s)} = \min_{1 \leq k \leq n} \left\{ d_{ik}^{(s)} + d_{kj}^{(s)} \right\}.$$

Given this relation we can calculate $D^{(2^i)}$ from $D^{(2^{i-1})}$ in $O(n^3)$ time. We can therefore calculate all of

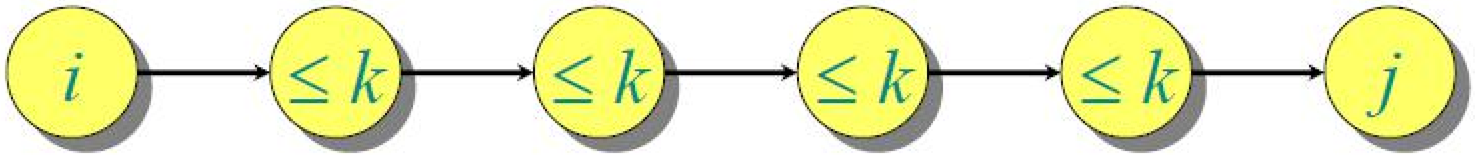
$$D^{(2)}, D^{(4)}, D^{(8)}, \ldots, D^{\left(2^{\lceil \log_2 n \rceil}\right)} = D^{(n)}$$

in $O(n^3 \log n)$ time, improving our running time.

Floyd-Warshall Algorithm

Definition: The vertices $v_2, v_3, ..., v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, ..., v_{l-1}, v_l \rangle$.

- Let $d_{ij}^{(k)}$ be the length of the shortest path from i to j such that *all* intermediate vertices on the path (if any) are in set $\{1, 2, \ldots, k\}$.



$d_{ij}^{(0)}$ is set to be w_{ij}, i.e., no intermediate vertex.
Let $D^{(k)}$ be the $n \times n$ matrix $[d_{ij}^{(k)}]$.

Floyd-Warshall Algorithm

Definition: The vertices $v_2, v_3, ..., v_{l-1}$ are called the *intermediate vertices* of the path $p = \langle v_1, v_2, ..., v_{l-1}, v_l \rangle$.

- Claim: $d_{ij}^{(n)}$ is the distance from i to j. So our aim is to compute $D^{(n)}$.
- Subproblems: compute $D^{(k)}$ for $k = 0, 1, \cdots, n$.

The Structure of Shortest Paths

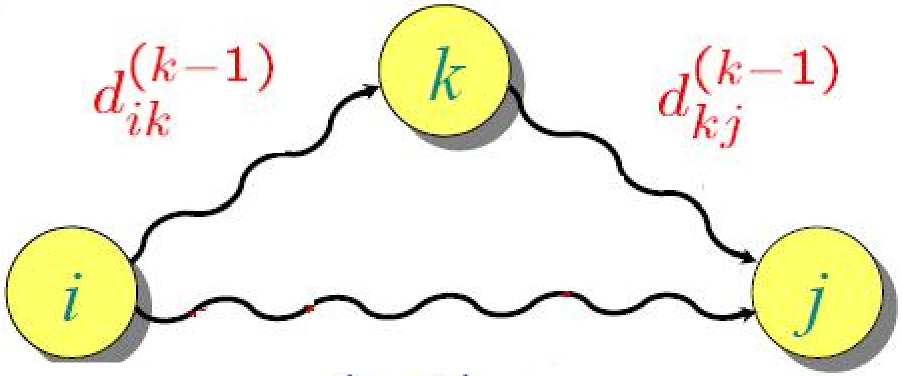
Observation 1: A shortest path does not contain the same vertex twice.

Non-negative circle!

Step 2: The Structure of Shortest Paths

Observation 2: For a shortest path from i to j such that any intermediate vertices on the path are chosen from the set $\{1, 2, \ldots, k\}$, there are two possibilities:

k is a vertex on the path.



k is not a vertex on the path, $d_{ij}^{(k-1)}$

$$d_{ij}^{(k)} = \min\left\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right\}.$$

Step 3： Bottom-Up Computation

- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, ..., n$.

Step 3： Bottom-Up Computation

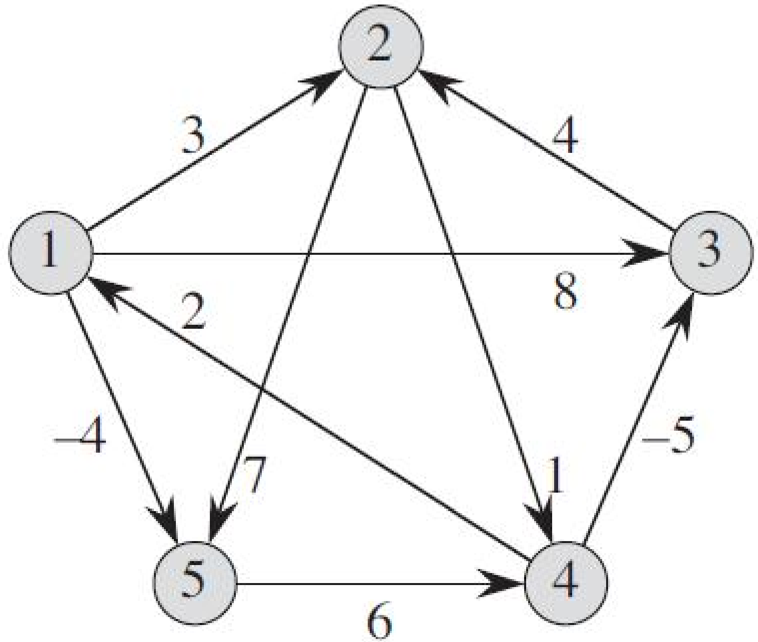
- Bottom: $D^{(0)} = [w_{ij}]$, the weight matrix.

- Compute $D^{(k)}$ from $D^{(k-1)}$ using

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

for $k = 1, ..., n$.

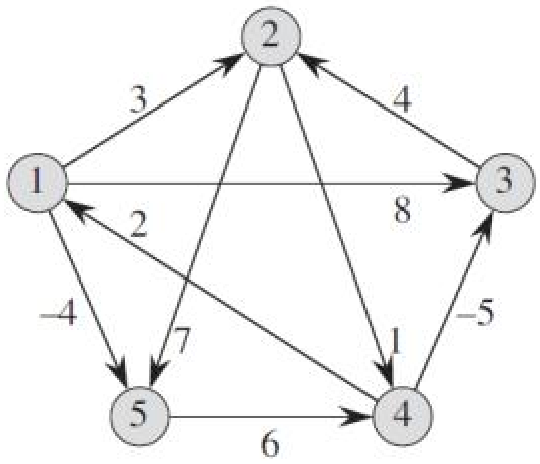
Example



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

weight matrix

Example



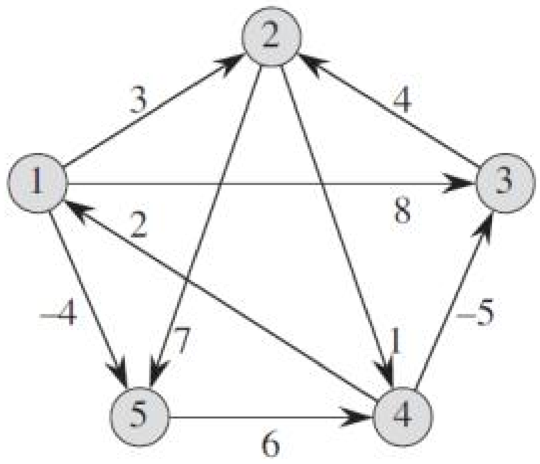
$$D^{(0)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(1)} = min\{d_{ij}^{(0)},\ d_{i1}^{(0)} + d_{1j}^{(0)}\}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



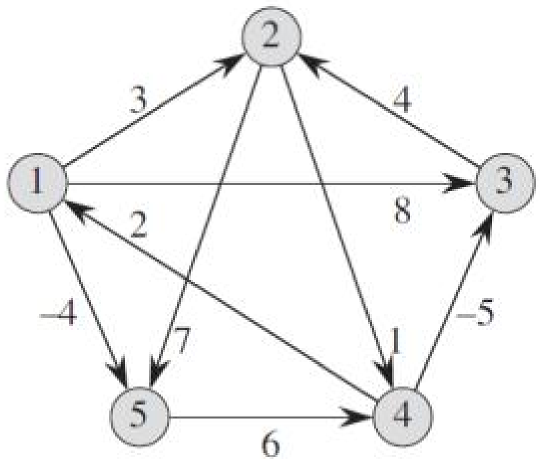
$$D^{(1)}$$

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(2)} = min\{d_{ij}^{(1)},\ d_{i2}^{(1)} + d_{2j}^{(1)}\}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



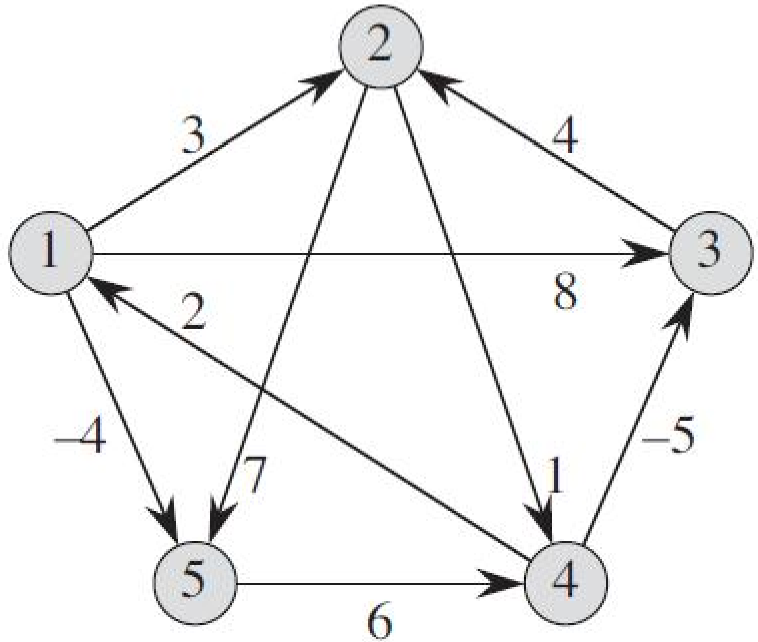
$$D^{(2)}$$

$$\begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$d_{ij}^{(3)} = min\{d_{ij}^{(2)},\ d_{i3}^{(2)} + d_{3j}^{(2)}\}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Example



$$d_{ij}^{(5)} = min\{d_{ij}^{(4)},\ d_{i5}^{(4)} + d_{5j}^{(4)}\}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

The shortest distances between any pair of vertices

Algorithm

Floyd-Warshall(w, n)

$\{$ for $i = 1$ to n do ⠀⠀⠀⠀ initialize

  for $j = 1$ to n do

  $\{$ $d[i,j] = w[i,j];$

   $pred[i,j] = nil;$

  $\}$

 for $k = 1$ to n do ⠀⠀⠀⠀ dynamic programming

  for $i = 1$ to n do

   for $j = 1$ to n do

    if $(d[i,k] + d[k,j] < d[i,j])$

     $\{d[i,j] = d[i,k] + d[k,j];$

     $pred[i,j] = k;\}$

 return $d[1..n, 1..n];$

$\}$

Comments

- The algorithm's running time is clearly $\Theta(n^3)$.
- The predecessor pointer $\texttt{pred[i, j]}$ can be used to extract the final path (see later).
- Problem: the algorithm uses $\Theta(n^3)$ space.
  It is possible to reduce this down to $\Theta(n^2)$ space by keeping only one matrix instead of n.

Extracting The Shortest Paths

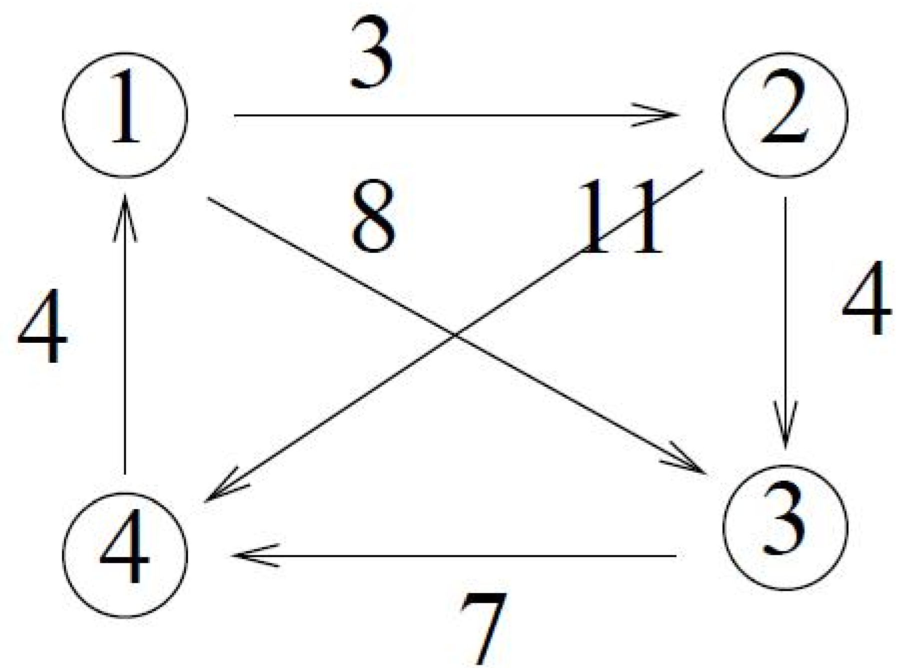
To find the shortest path from i to j, we consult $pred[i, j]$. If it is nil, then the shortest path is just the edge (i, j). Otherwise, we recursively compute the shortest path from i to $pred[i, j]$ and the shortest path from $pred[i, j]$ to j.

Exercises

- **25.1-1**
- **24.1-7**
- **25.2-1**
- **25.2-4**

Short Test in Class

Give $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ with matrix multiplication algorithm, or $D^{(0)}$, $D^{(1)}$, $D^{(2)}$ by Floyd-Warshell algorithm.



算法分析课程组

重庆大学计算机学院

End of Section.