T&R Team of Algorithm Design

College of Computer Science and Engineering, CQU

Algorithm Analysis & Design

Introduction to Algorithm

Chapter 7: Quick Sort

Outline

- **6.1 Basic Quick Sort**
- **6.2 Improving Quick Sort with Medians**

7.1 Basic Quick Sort

QUICK SORT

- **We have seen two O(*n* log *n*) sorting algorithms:**
  - **Merge sort which is faster but requires more memory**
  - **Heap sort which allows in-place sorting**
- **We will now look at a recursive algorithm which may be done *almost* in place and usually faster than heap sort**
  - **Use an object in the array (a pivot) to divide the two**
  - **Average case: O(*n* log *n*) time and O(log *n*) memory**
  - **Worst case: O(n^2) time and O(*n*) memory**

QUICK SORT

- Merge sort splits the array sub-lists and sorts them
- The larger problem is split into two sub-problems based on location in the array
- Consider the following alternative:
  - Chose an object in the array and partition the remaining objects into two groups relative to the chosen entry

QUICK SORT

- For example, given an unsorted array:

2	8	7	1	3	5	6	4

- We can select the last entry, 4, and sort the remaining entries into two groups, those less than 4 and those greater than 4:

2	1	3	4	7	5	6	8

- Note that 4 is now in the correct location once the list is sorted
  - Proceed by applying the algorithm to the first 3 and last 4 entries

A Simple Implementation – PARTITION

- **PARTITION** (A, p, r)

  $x \leftarrow A[r]$

  $i \leftarrow p-1$

  FOR $j \leftarrow p$ **TO** $r-1$

  IF $A[j] \leq x$

  THEN $i \leftarrow i + 1$

  exchange $A[i] \leftrightarrow A[j]$

  exchange $A[i+1] \leftrightarrow A[r]$

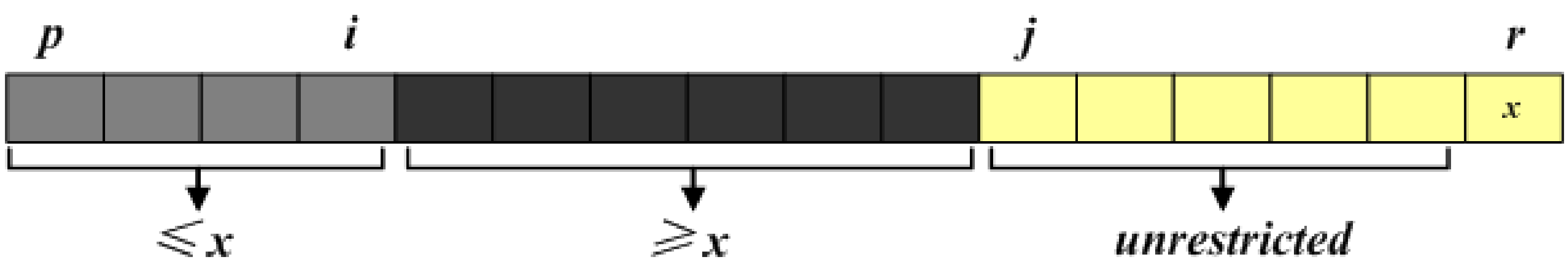
  RETURN $i+1$

A Simple Implementation – PARTITION

PARTITION

```
PARTITION(A, p, r)        //A[p..r]
  x ← A[r]                //the rightmost element as pivot
  i ← p-1
for j ← p to r-1
      do if A[j] ≤ x
              then i ← i+1
                   exchange A[i]↔A[j]
exchange A[i+1]↔A[r]
return i+1
```

Running time = $O(n)$ for n elements



A Simple Implementation – PARTITION

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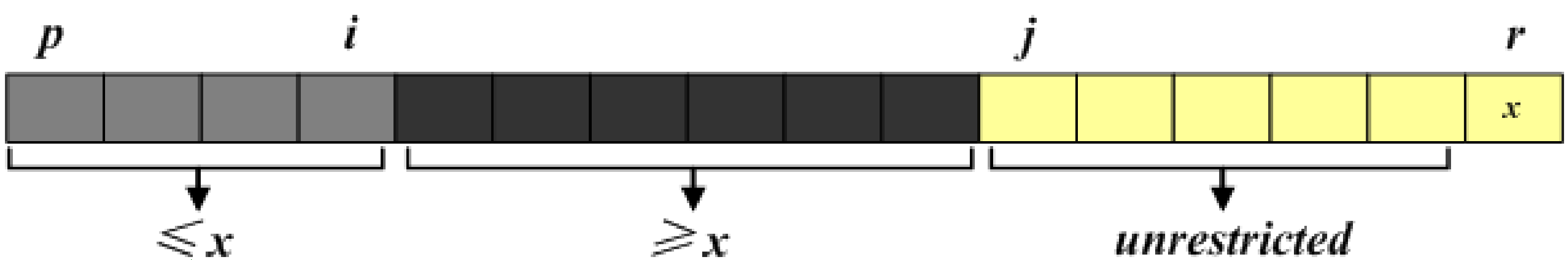

A[p, … , i]: known to be ≤ pivot

A Simple Implementation – PARTITION

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PARTITION(A, p, r)          //A[p..r]
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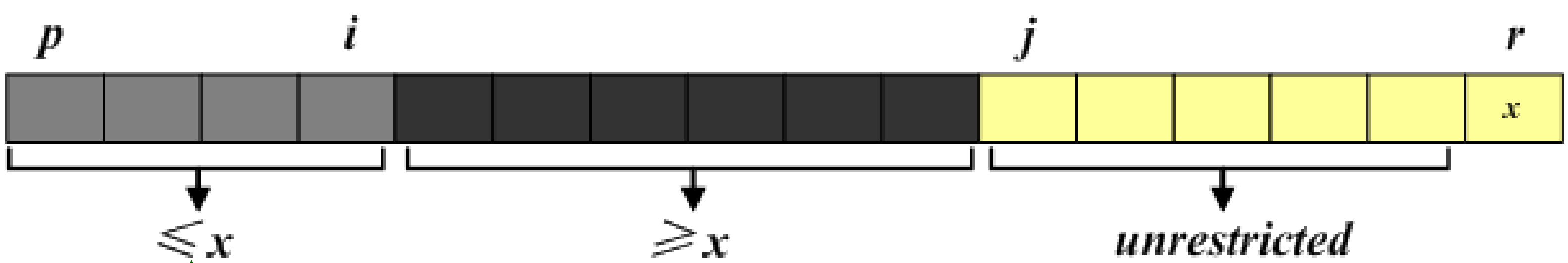

A[p, … , i]: known to be ≤ pivot

A[i+1, … , j-1]: known to be > pivot

A Simple Implementation – PARTITION

PARTITION

```
PARTITION(A, p, r)          //A[p..r]
  x ← A[r]                  //the rightmost element as pivot
  i ← p-1
for j ← p to r-1
      do if A[j] ≤ x
            then i ← i+1
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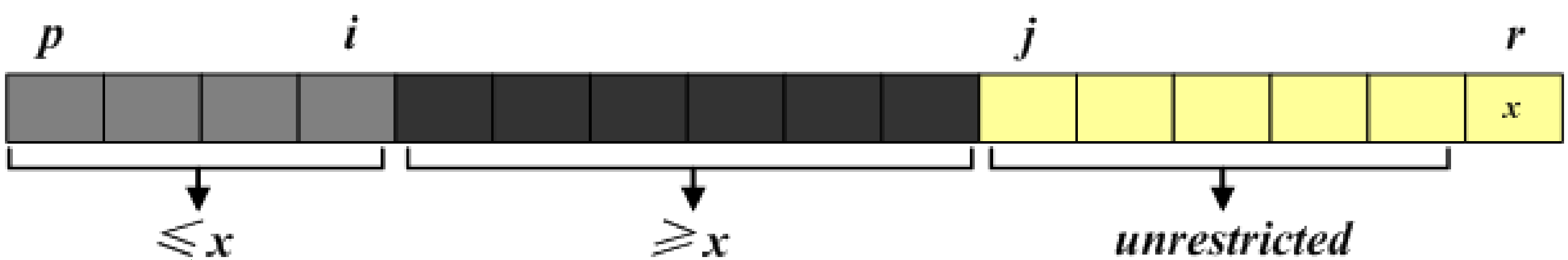

A[p, … , i]: known to be ≤ pivot

A[i+1, … , j-1]: known to be > pivot

A[j … r-1]: not yet examined

A Simple Implementation – PARTITION

PARTITION

```
PARTITION(A, p, r)         //A[p..r]
   x ← A[r]                //the rightmost element as pivot
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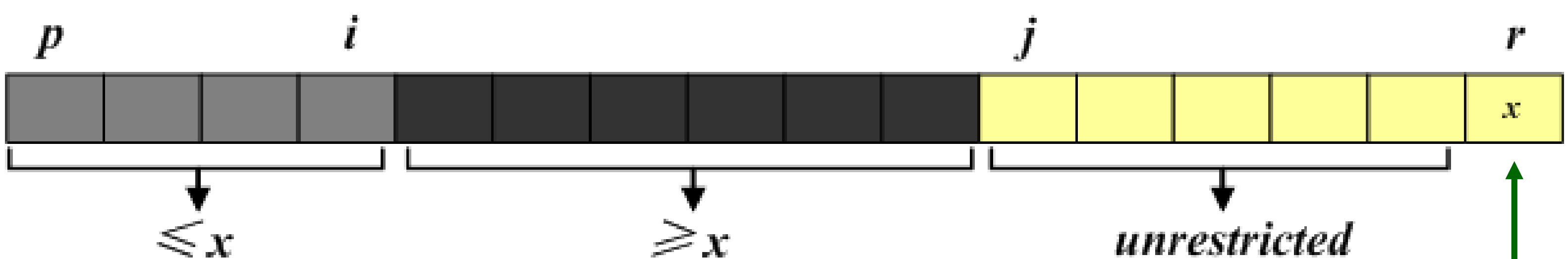
Running time = $O(n)$ for n elements



A[p, … , i]: known to be ≤ pivot

A[j … r-1]: not yet examined

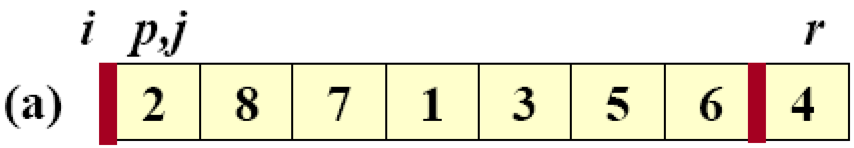
A[i+1, … , j-1]: known to be > pivot

A[r]: pivot

A Simple Implementation – PARTITION

- **The operation of Partition on the sample array. Lightly shaded array elements are all with values no greater than *x* (the pivot). Heavily shaded array elements are all with values greater than *x*.**

A Simple Implementation – PARTITION



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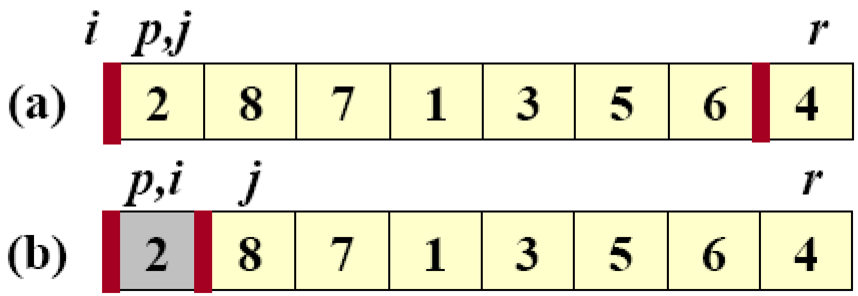

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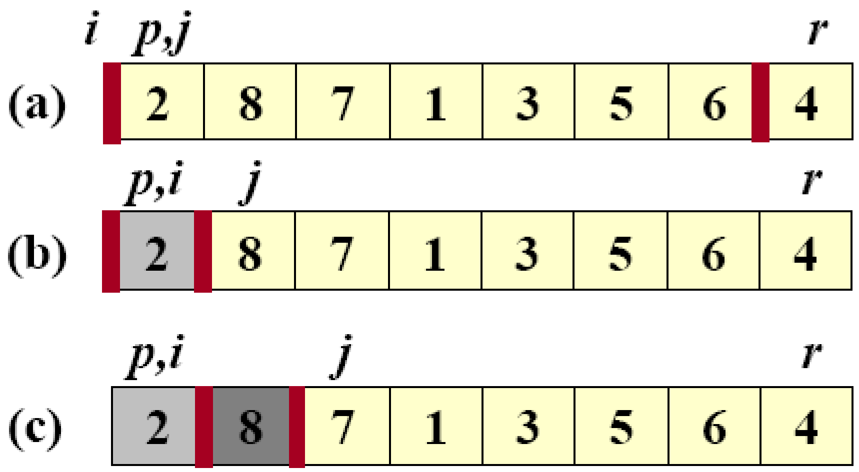

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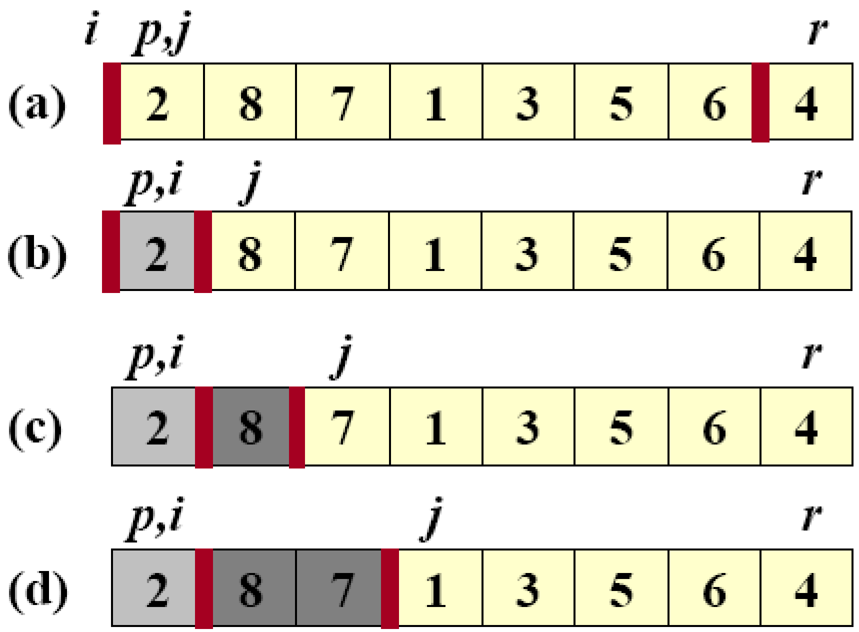

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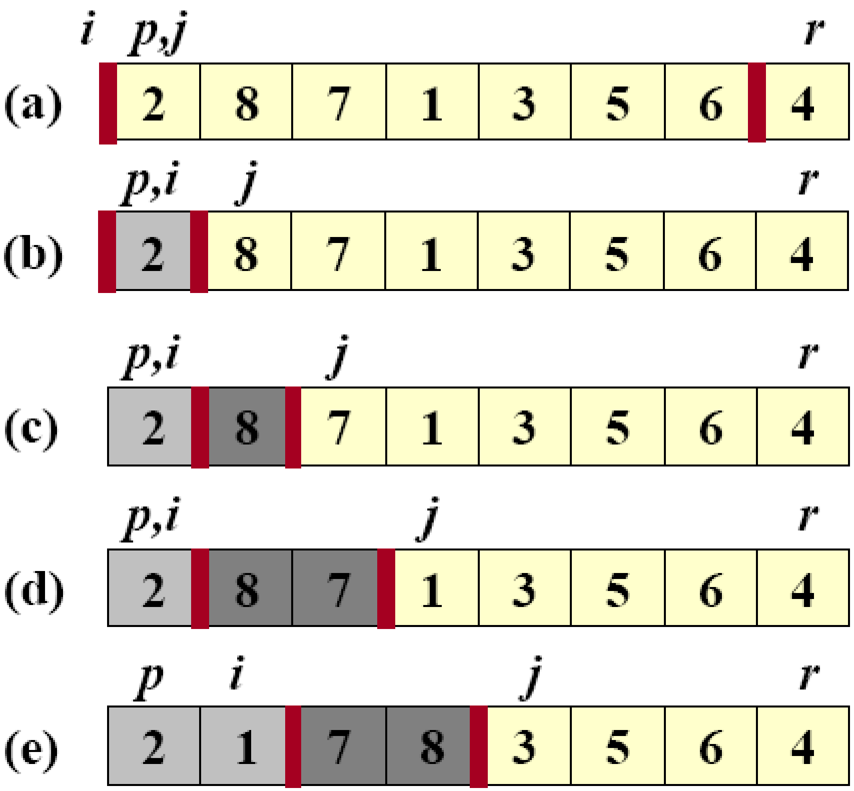

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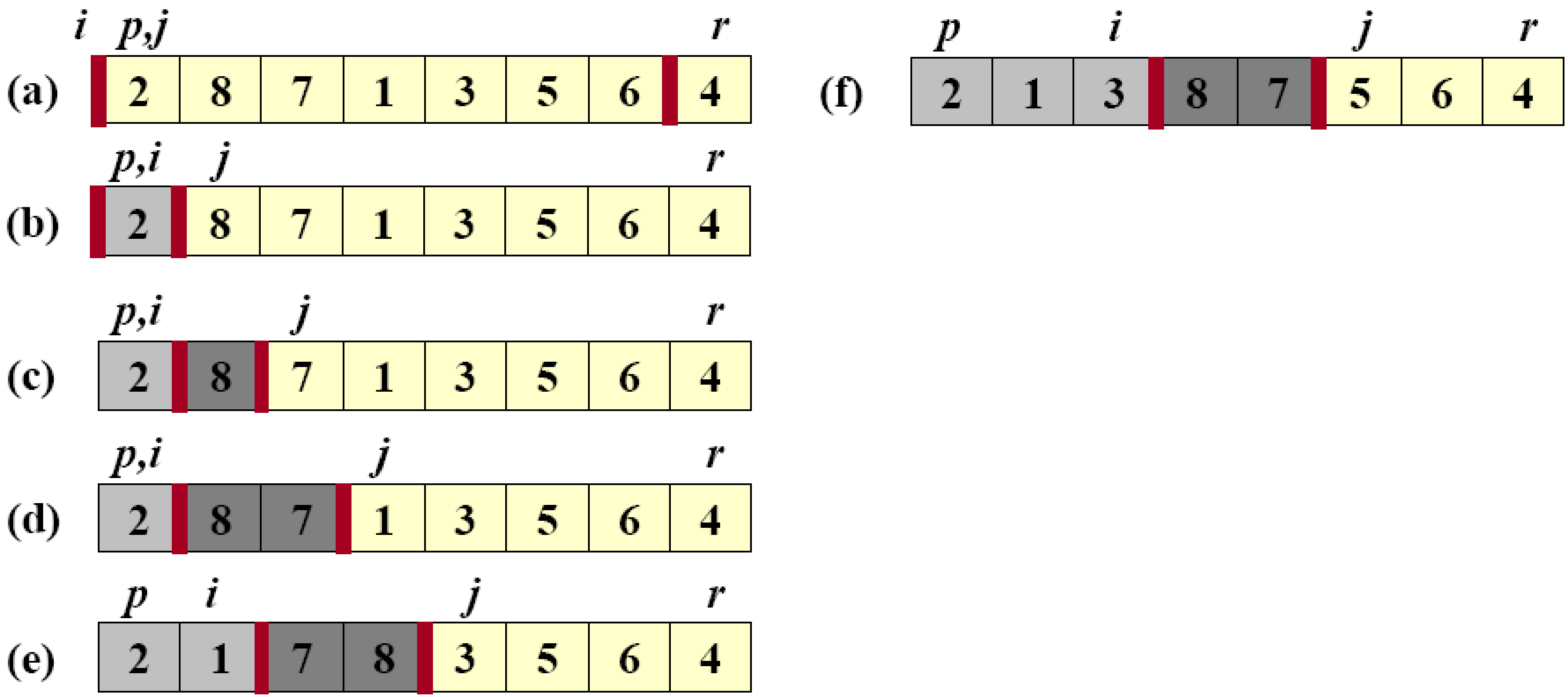

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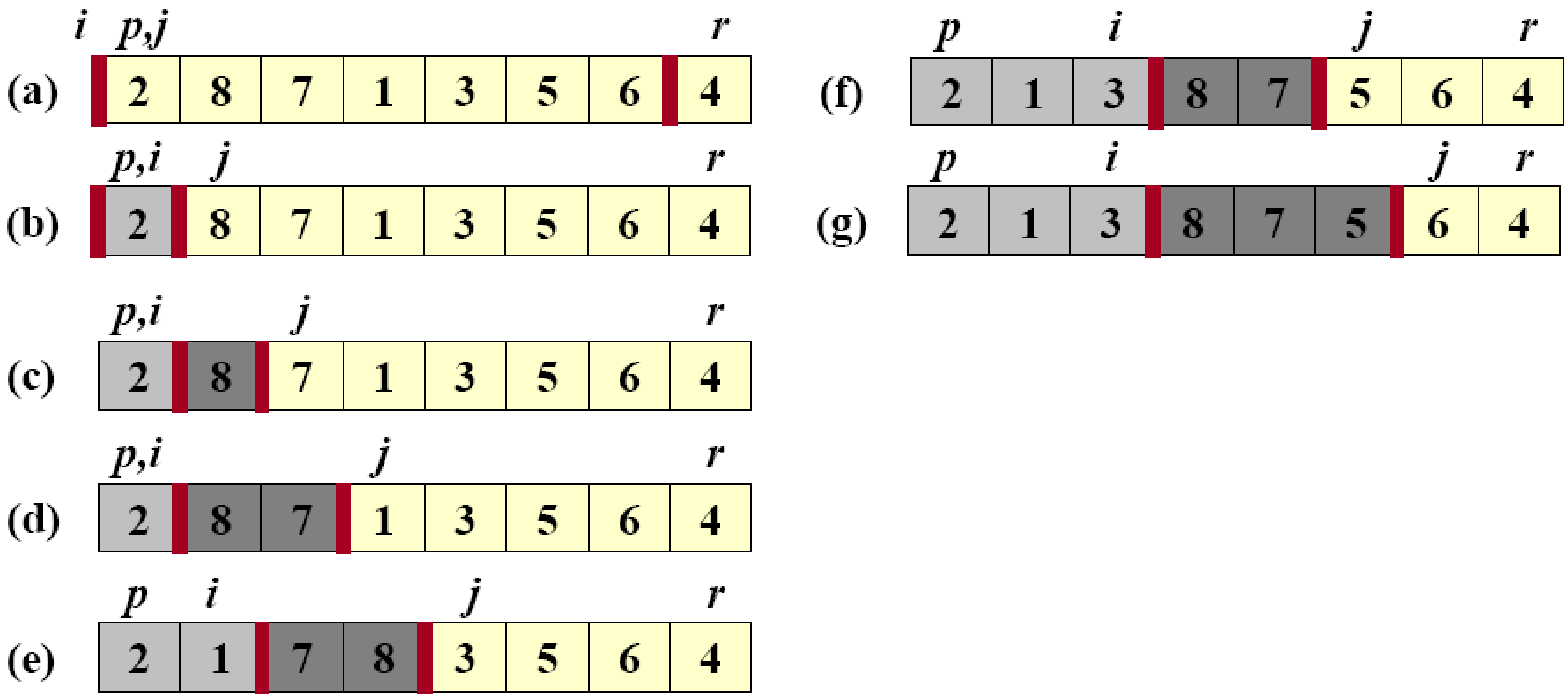

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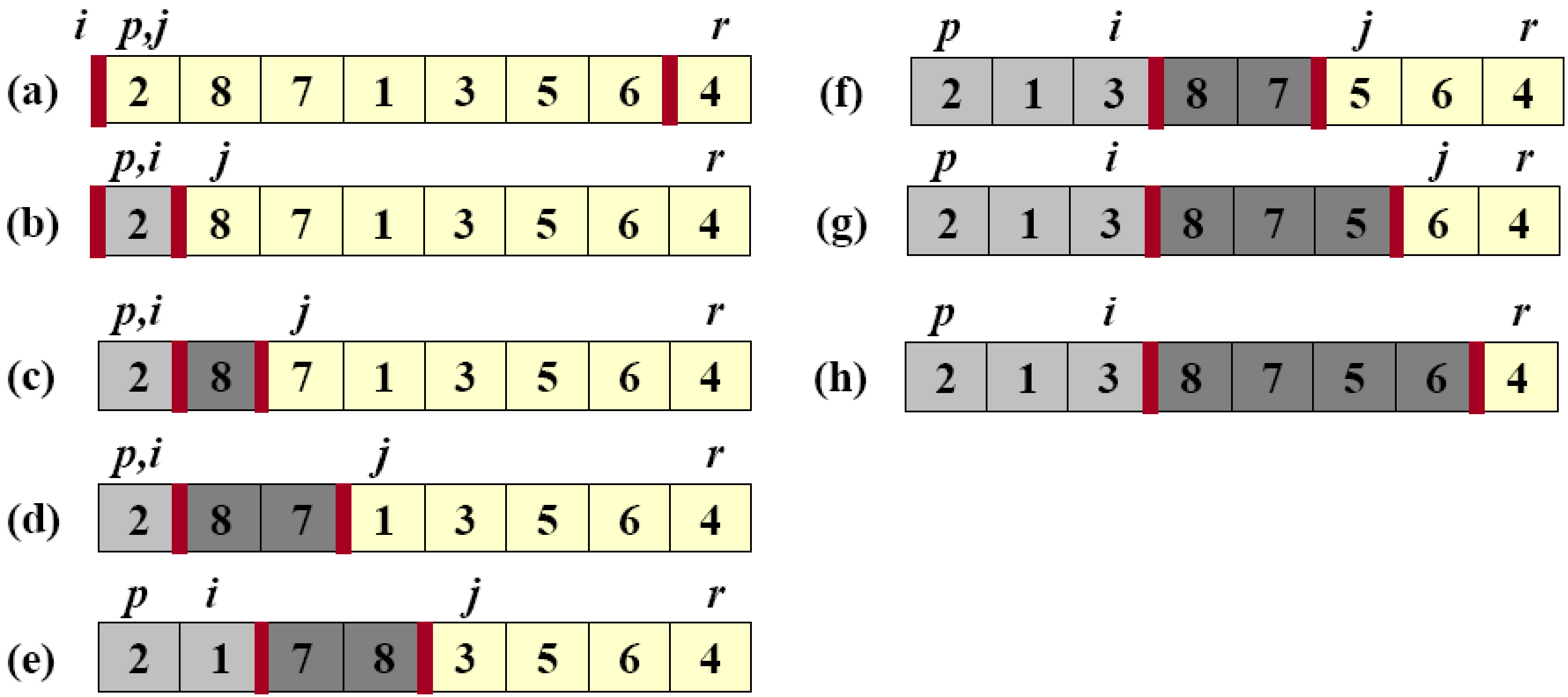

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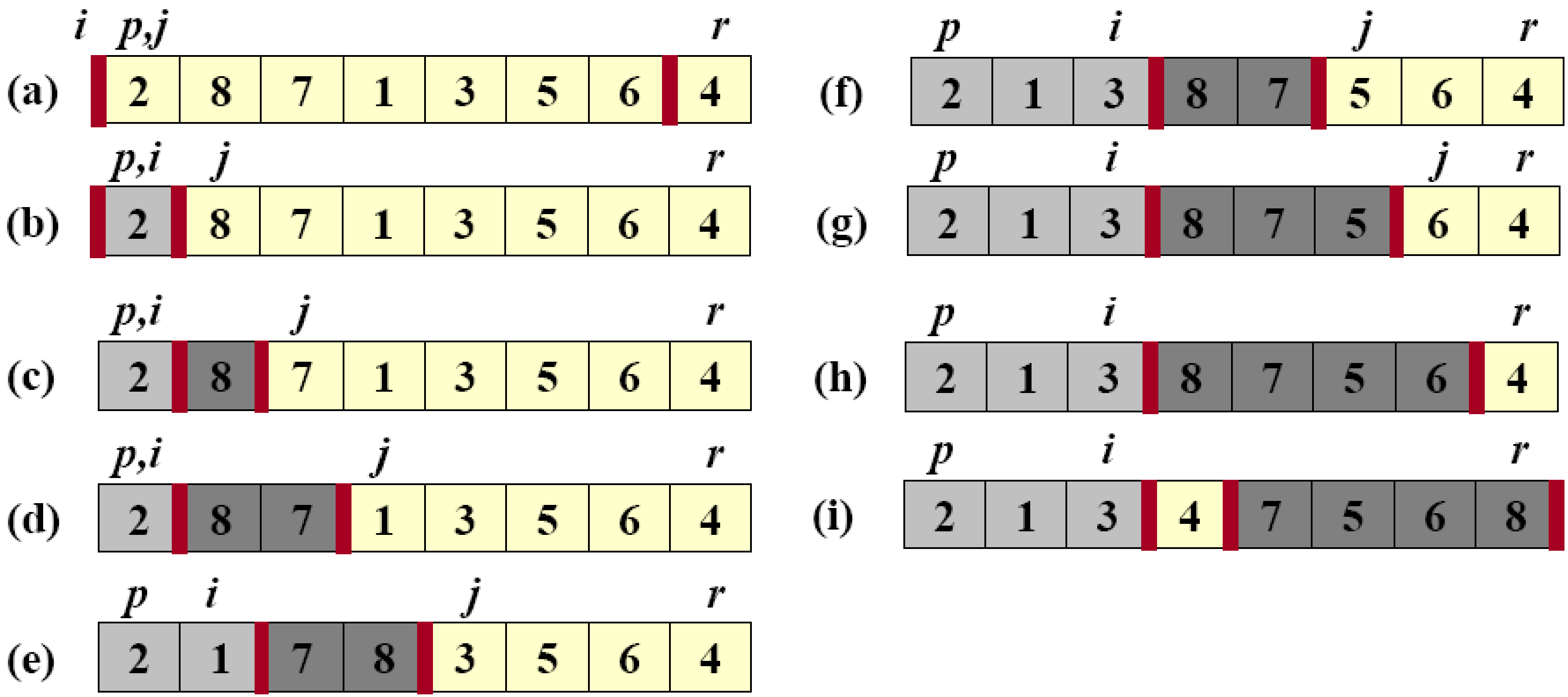
A Simple Implementation – PARTITION



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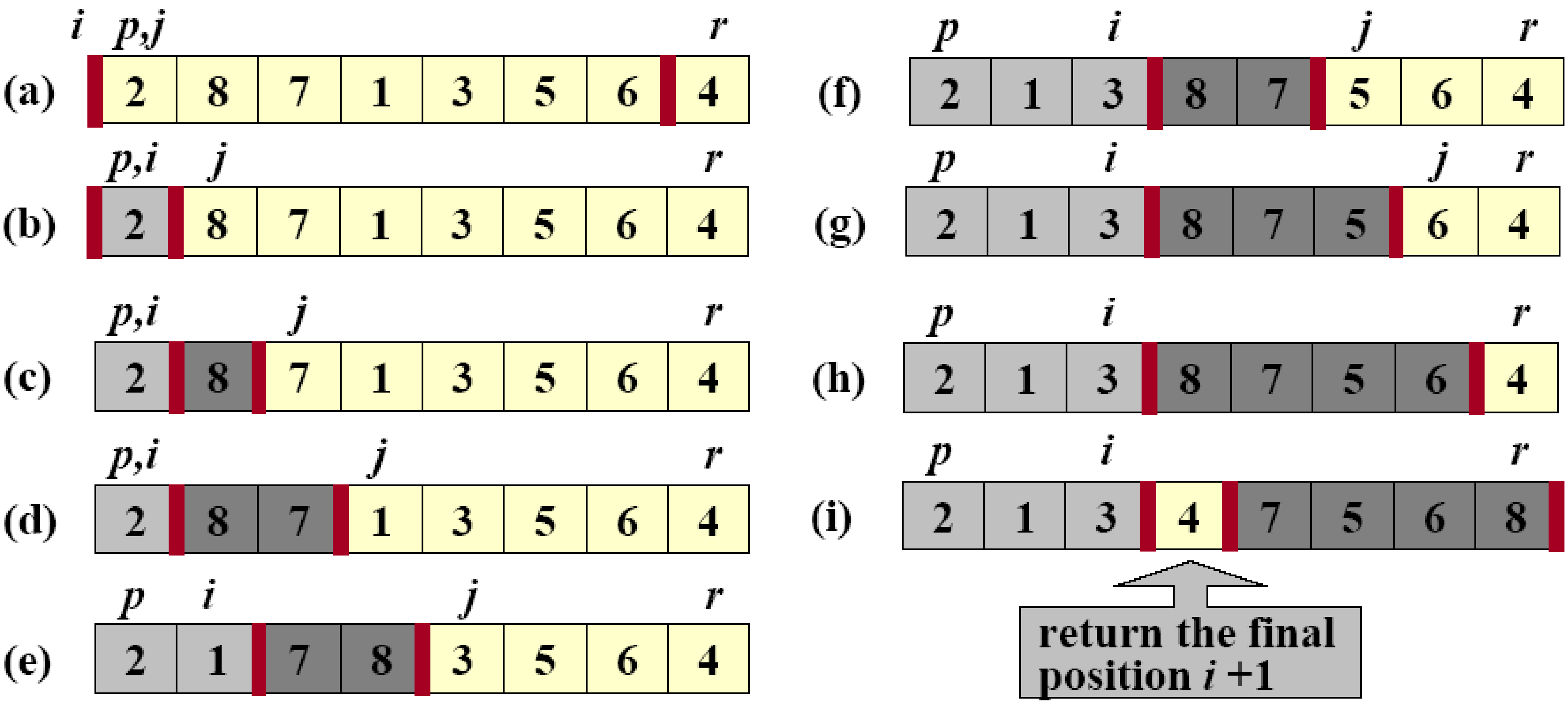
A Simple Implementation – PARTITION



- The operation of Partition on the sample array. Lightly shaded array elements are all with values no greater than x (the pivot). Heavily shaded array elements are all with values greater than x.

A Simple Implementation – QUICKSORT

- **QUICKSORT (A, p, r)**
  - **IF p < r**
  - **THEN q ← PARTITION (A, p, r)**
    - **QUICKSORT (A, p, q–1)**
    - **QUICKSORT (A, q+1, r)**

- **Initial call: QUICKSORT(A, 1, n)**

Run-time Analysis

- **In the best case, the list will be split into two approximately equal sub-lists, and thus, the run time could be very similar to that of merge sort: $\Theta(n \log n)$**

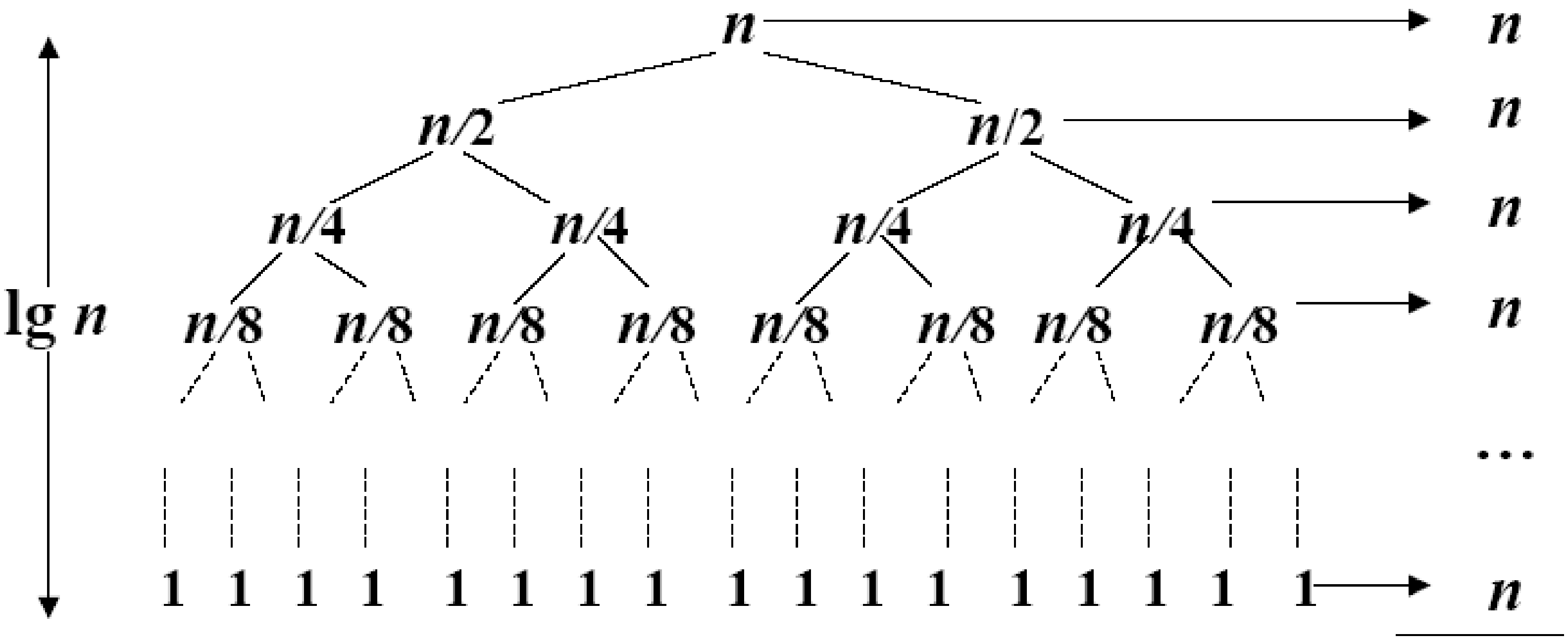
Recursive Tree of the Best Case

- **A recursion tree for quick sort in which the partition always balances the two sides of the partition equally. The resulting running time is $\Theta(n \log n)$**
- **The question is: WHAT happens if we don't get that lucky?**

Recursive Tree of the Best Case

- A recursion tree for quick sort in which the partition always balances the two sides of the partition equally. The resulting running time is $\Theta(n \log n)$

- The question is: WHAT happens if we don't get that **lucky**?



Worst-case Scenario

- Suppose we choose the smallest element as our pivot and we try ordering a sorted list:

80	38	95	84	66	10	79	**2**	26	87	96	12	43	81	3

- Using 2, we partition the original list into

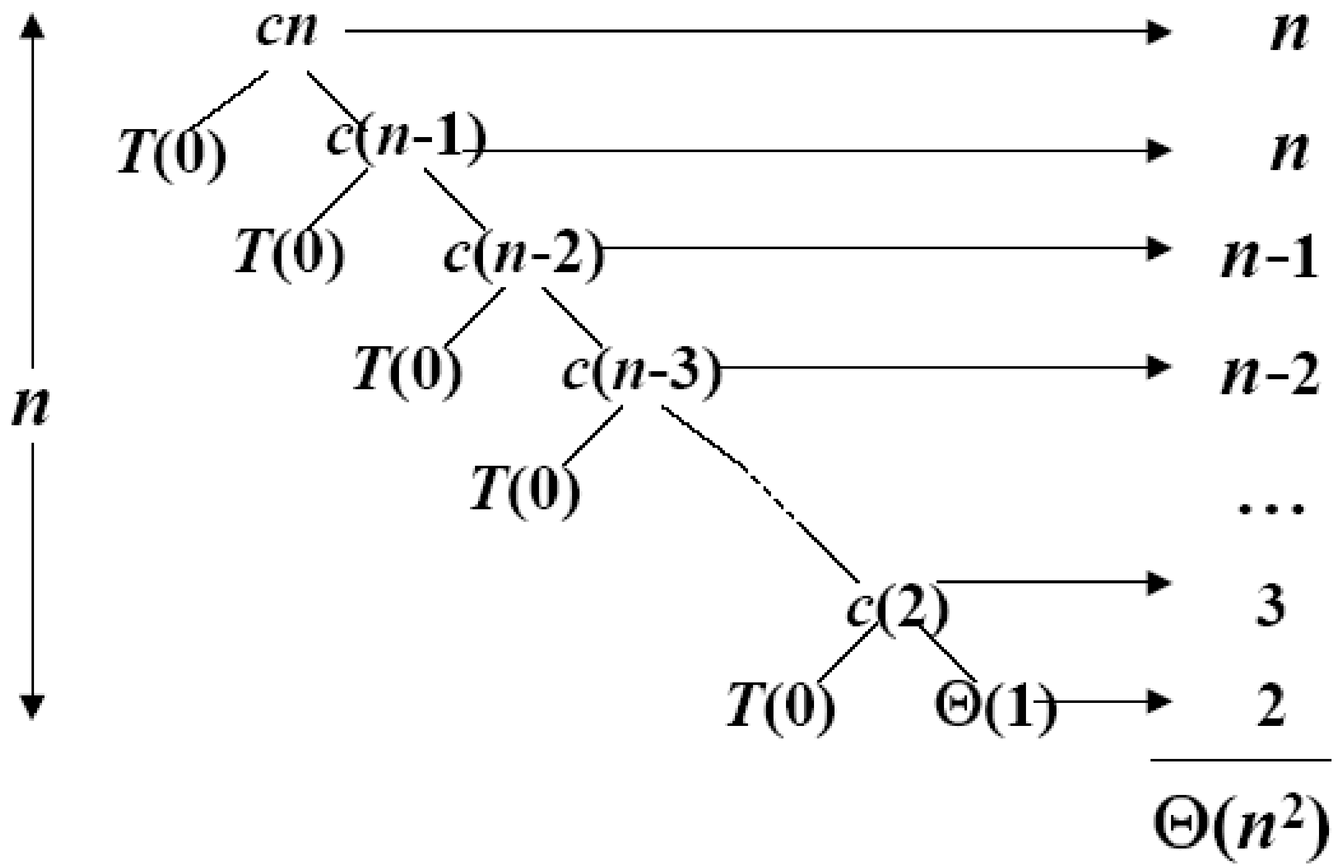
2	80	38	95	84	66	10	79	26	87	96	12	43	81	3

- We still have to sort a list of size n – 1

- The run time is $T(n) = T(n - 1) + \Theta(n) = \Theta(n^2)$
  - Thus, the run time drops from $\Theta(n \log n)$ to $\Theta(n^2)$

Recursive Tree of the Worst Case

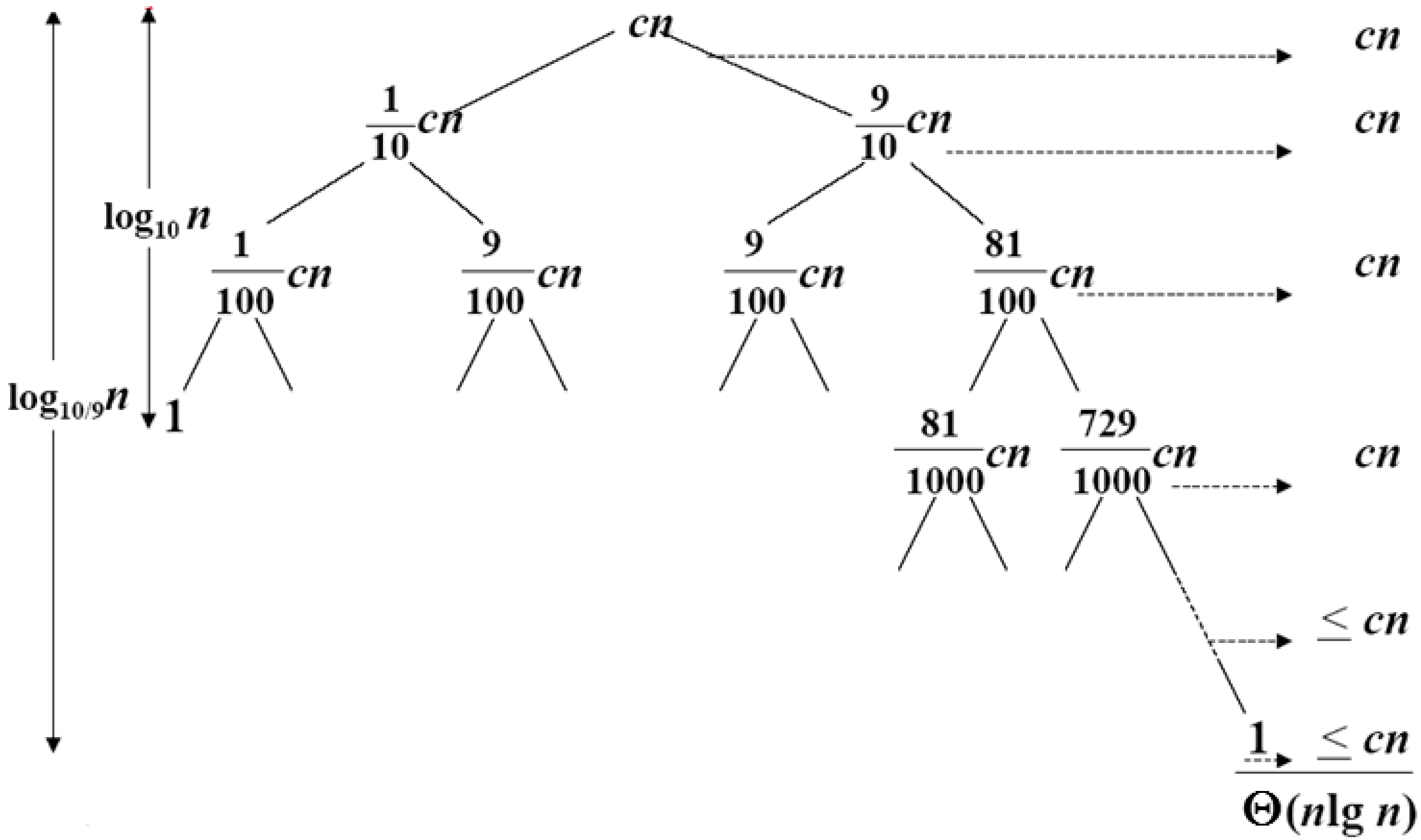
- **A recursion tree for quick sort in which the partition always puts only a single element on one side of the partition. The resulting running time is $\Theta(n^2)$**



Recursive Tree of the Balanced Case

- **What if the split is always 1:9?**
  - **$T(n) = T(9n/10) + T(n/10) + \Theta(n)$**
  - **What is the solution to this recurrence?**

Recursive Tree of the Balanced Case



- **A recursion tree for quick sort in which partition always produces a 9-to-1 split, yielding a running time of $\Theta(n \log n)$**

7.2 Improving Quick Sort with Medians

Alternate Strategy

- **Our goal is to choose the median element in the list as our pivot:**

80	38	95	84	66	10	79	2	26	87	96	12	43	81	3

- **Unfortunately, it's DIFFICULT to find**

- **Alternate strategy: take the median of a subset of entries**

  - **For example, take the median of the first, middle, and last entries**

Choose the Median-of-Three

- **It is difficult to find the median so consider another strategy:**
  - **Choose the median of the first, middle, and last entries in the list**

80	38	95	84	99	10	79	44	26	87	96	12	43	81	3

- **This will usually give a much better approximation of the actual median**

Choose the Median-of-Three

- **Sorting the elements based on 44 results in two sub-lists, each of which must be sorted (again, using quicksort)**

- **We select the 26 to partition the first sub-list:**

38	10	26	12	43	3	44	80	95	84	99	79	87	96	81

- **and 81 to partition the second sub-list:**

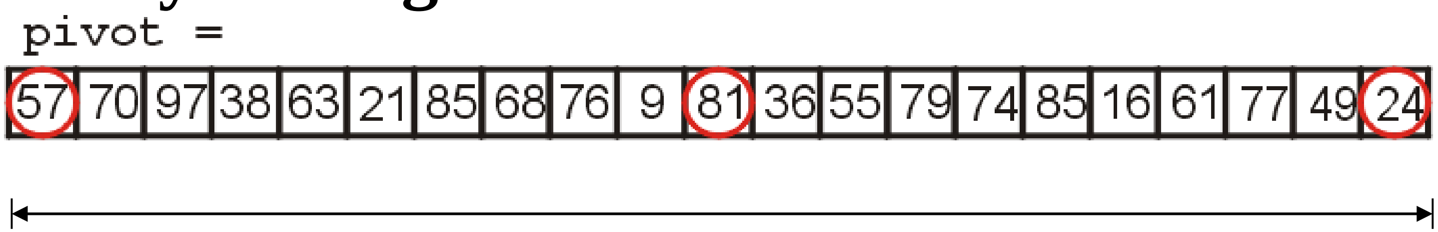
38	10	26	12	43	3	44	80	95	84	99	79	87	96	81

Choose the Median-of-Three

- If we choose a random pivot, this will, on average, divide a set of *n* items into two sets of size $\frac{n}{4}$ and $\frac{3n}{4}$.
  - 90 % of the time the width will have a ratio of 1:19 or better.
- Choosing the median-of-three will, on average, divide the *n* items into two sets of size $\frac{5n}{16}$ and $\frac{11n}{16}$.
  - Median-of-three helps speed the algorithm
  - 90 % of the time the width will have a ratio of 1:6.388 or better.
- Further, we can apply insertion sort to sorting the small sub-arrays.

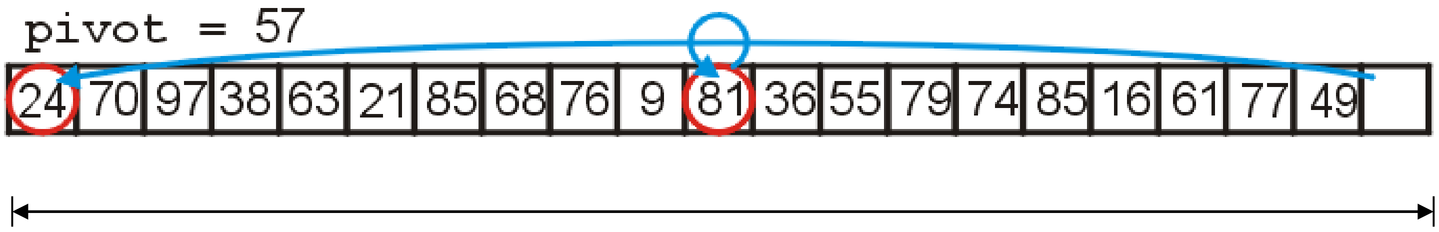
Improved Quick Sort Example

- First, we examine the first, middle, and last entries of the full list
- The span below will indicate which list we are currently sorting



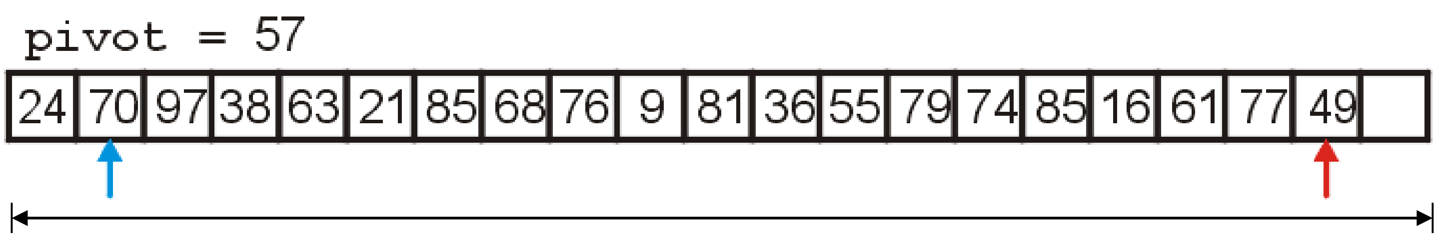
Improved Quick Sort Example

- **We select 57 to be our pivot**
- **We move 24 into the first location**



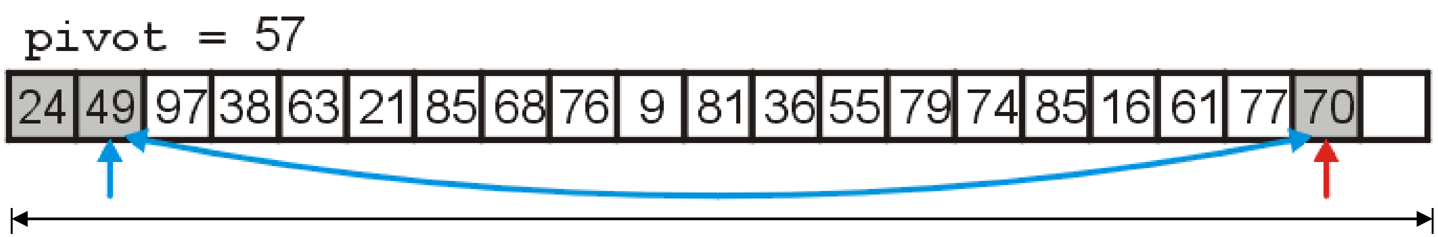
Improved Quick Sort Example

- Starting at the 2nd and 2nd-last locations:
- we search forward till we find 70 > 57
- we search backward till we find 49 < 57



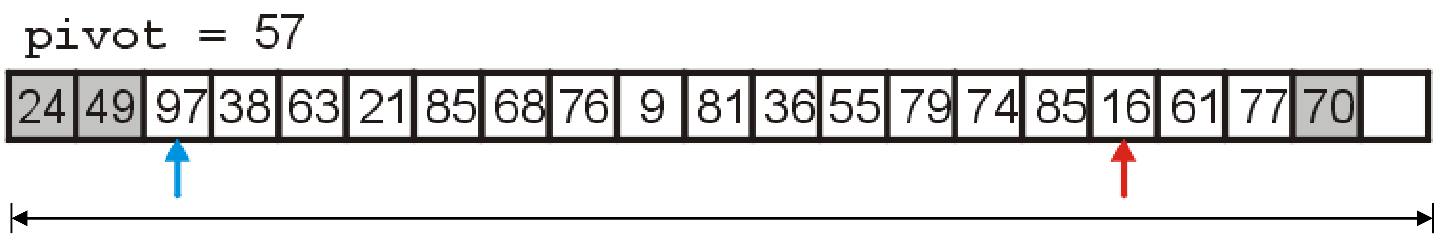
Improved Quick Sort Example

- **We swap 70 and 49, placing them in order with respect to each other**



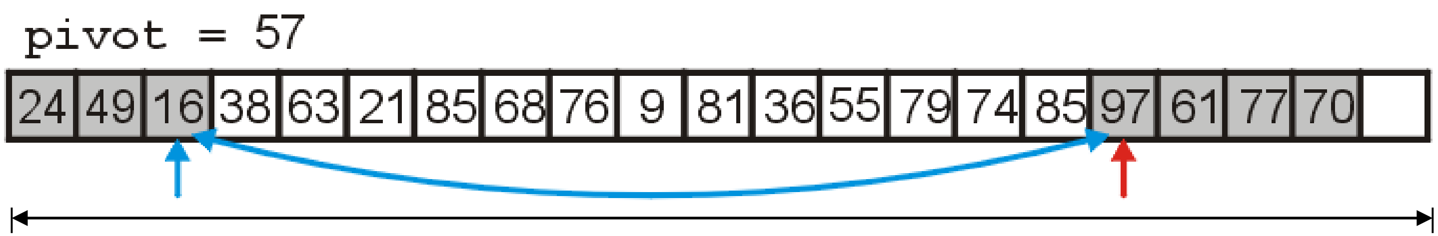
Improved Quick Sort Example

- **We search forward until we find** **97 > 57**
- **We search backward until we find** **16 < 57**



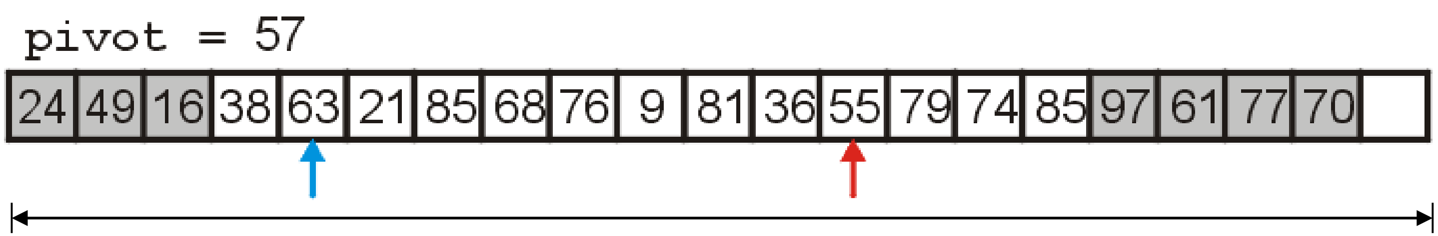
Improved Quick Sort Example

- **We swap 16 and 97 which are now in order with respect to each other**



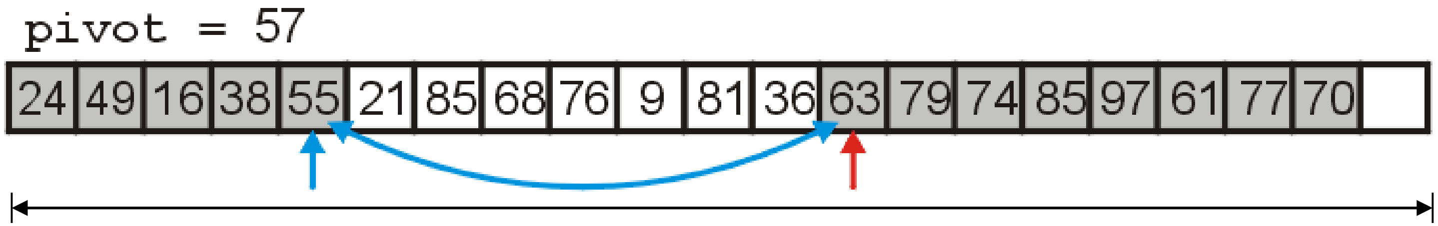
Improved Quick Sort Example

- We search forward till we find **63 > 57**
- We search backward till we find **55 < 57**



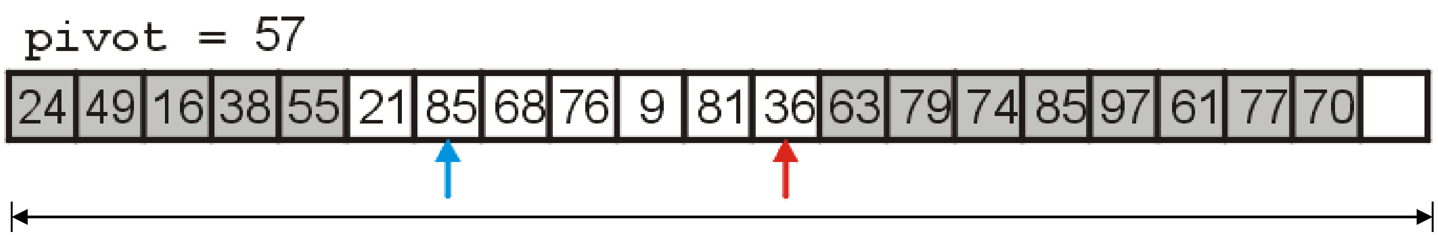
Improved Quick Sort Example

- **We swap 63 and 55**



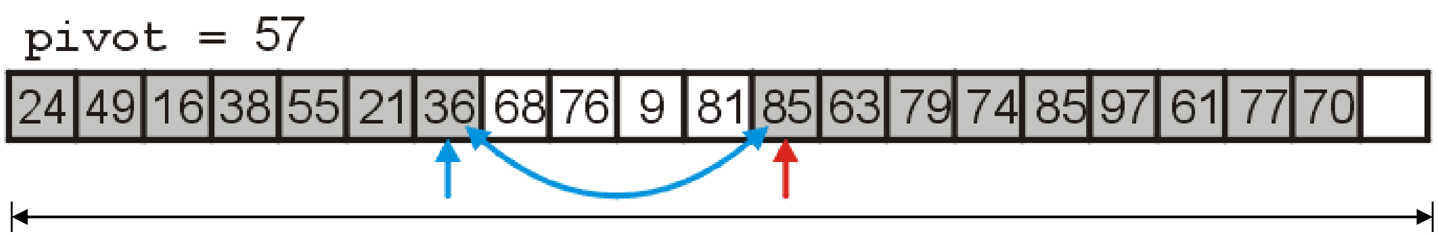
Improved Quick Sort Example

- **We search forward till we find** **85 > 57**
- **We search backward till we find** **36 < 57**



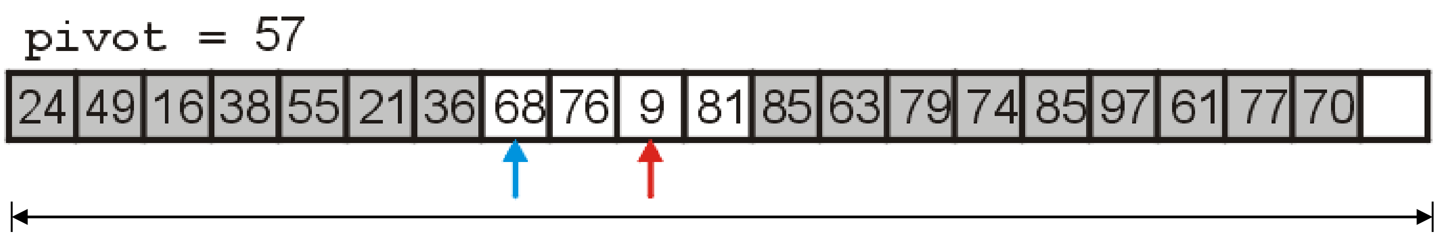
Improved Quick Sort Example

- **We swap 85 and 36, placing them in order with respect to each other**



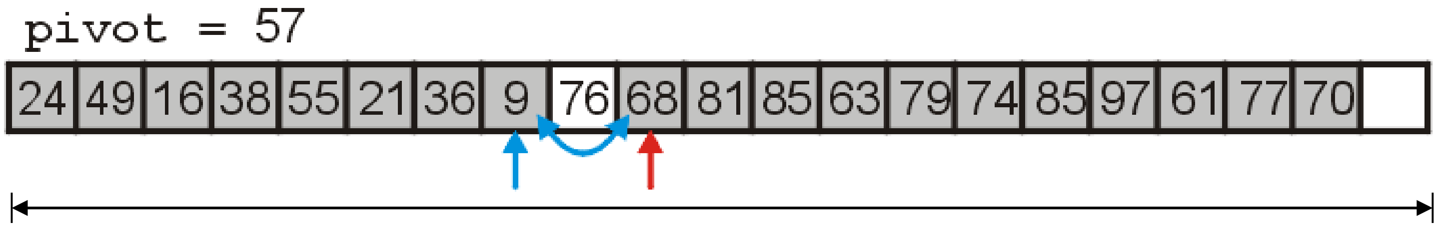
Improved Quick Sort Example

- We search forward until we find **68 > 57**
- We search backward until we find **9 < 57**



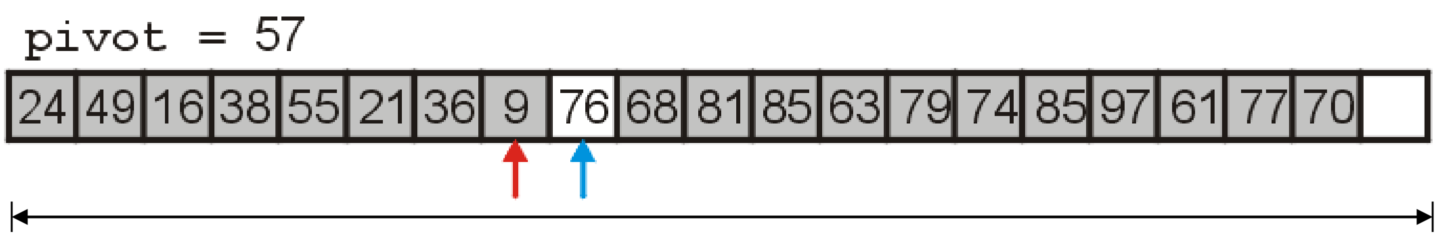
Improved Quick Sort Example

- **We swap 68 and 9**



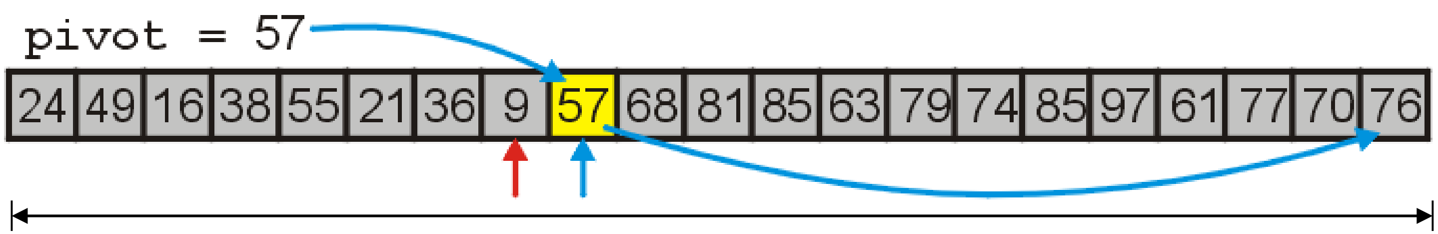
Improved Quick Sort Example

- We search forward until we find $76 > 57$
- We search backward until we find $9 < 57$
  - The indices are out of order, so we stop



Improved Quick Sort Example

- We move the larger indexed item to the vacancy at the end of the array
- We fill the empty location with the pivot, 57
- The pivot is now in the correct location



Improved Quick Sort Example

- **We will now recursively call quick sort on the first half of the list**
- **When we are finished, all entries < 57 will be sorted**

24	49	16	38	55	21	36	9	57	68	81	85	63	79	74	85	97	61	77	70	76	

Improved Quick Sort Example

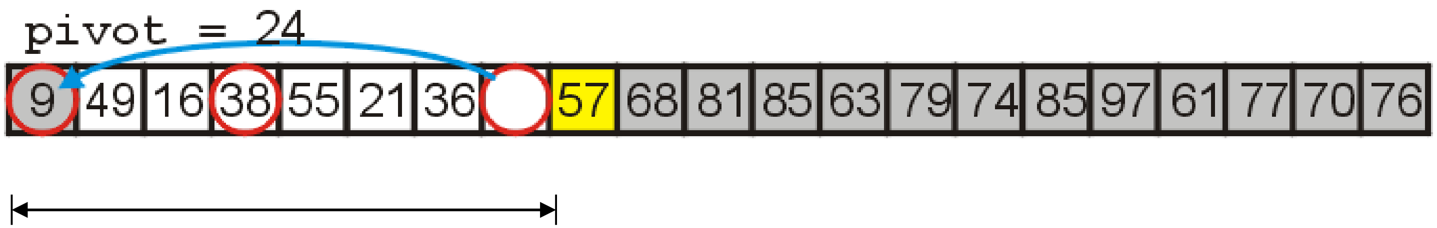
- **We examine the first, middle, and last elements of this sub list**

pivot =

24	49	16	38	55	21	36	9	57	68	81	85	63	79	74	85	97	61	77	70	76

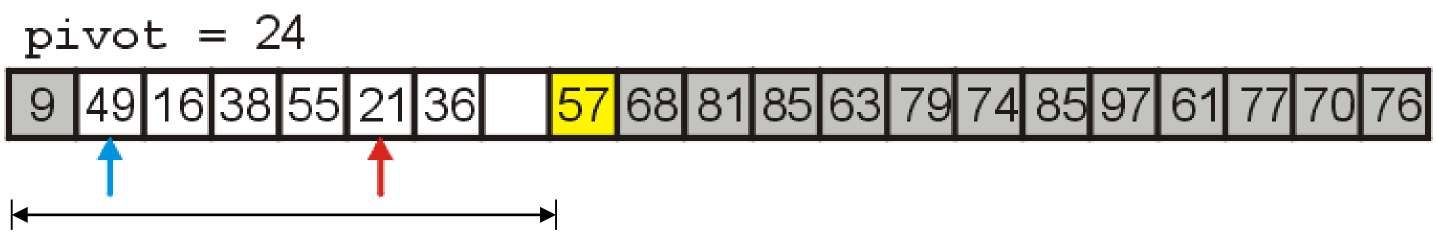
Improved Quick Sort Example

- We choose 24 to be our pivot
- We move 9 into the first location in this sub-list



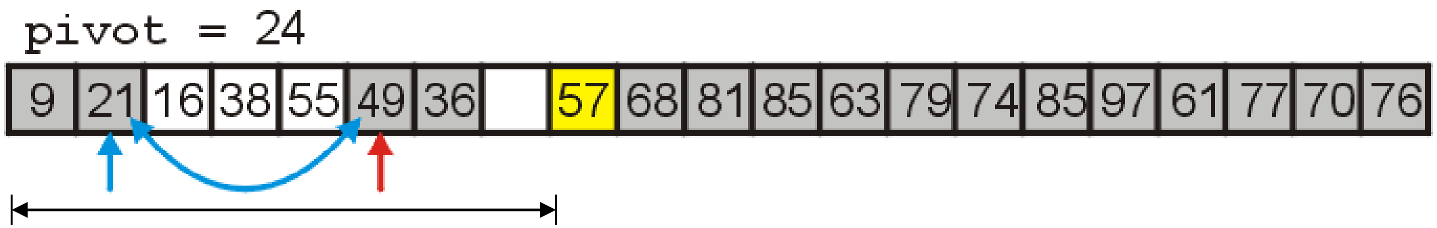
Improved Quick Sort Example

- **We search forward till we find** **49 > 24**
- **We search backward till we find** **21 < 24**



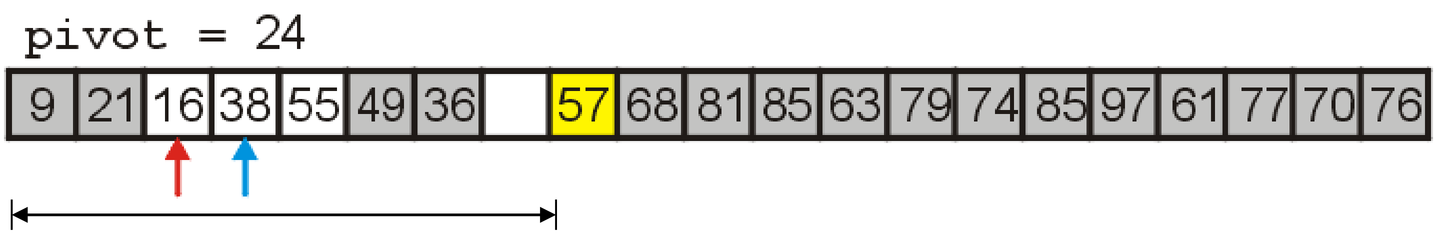
Improved Quick Sort Example

- **We swap 49 and 21, placing them in order with respect to each other**



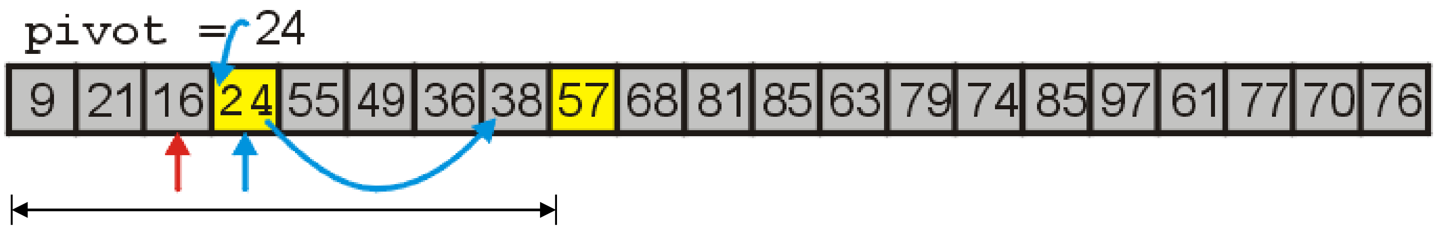
Improved Quick Sort Example

- **We search forward till we find** **38 > 24**
- **We search backward till we find** **16 < 24**
- **The indices are reversed, so we stop**



Improved Quick Sort Example

- We move 38 to the vacant location and move the pivot 24 into the previous location of 38
- 24 is now in the correct location



Improved Quick Sort Example

- **We will now recursively call quick sort on the left and right halves of those entries which are < 57**

9	21	16	24	55	49	36	38	57	68	81	85	63	79	74	85	97	61	77	70	76

Improved Quick Sort Example

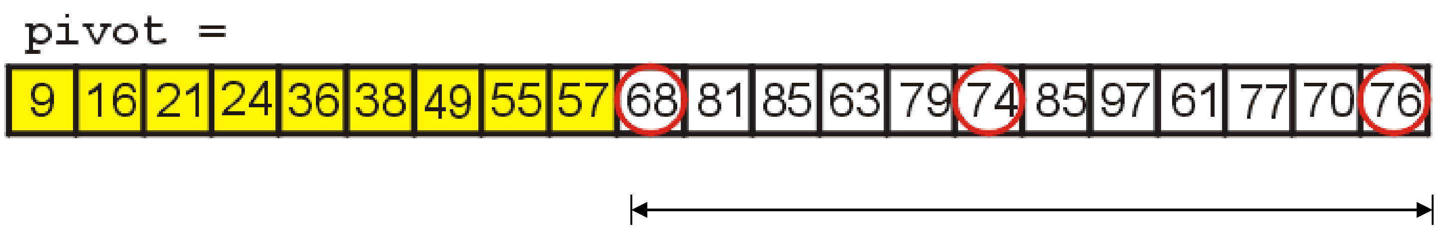
- **The first partition has three entries, so we sort it using insertion sort**

Improved Quick Sort Example

- **The second partition also has only four entries, so again, we use insertion sort**

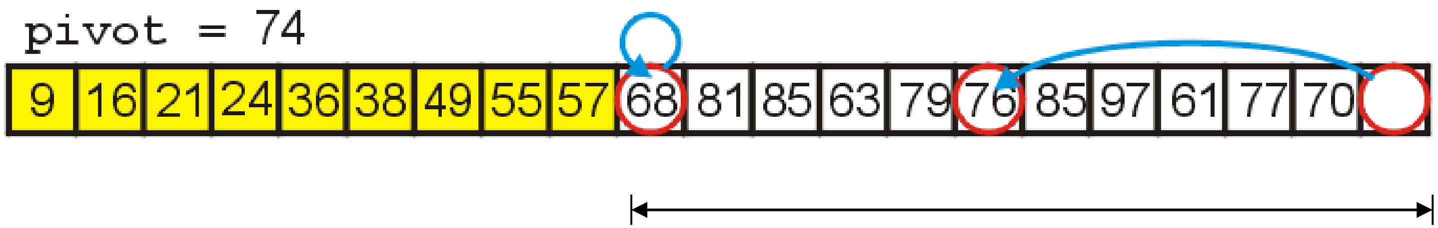
Improved Quick Sort Example

- First we examine the first, middle, and last entries of the sub-list



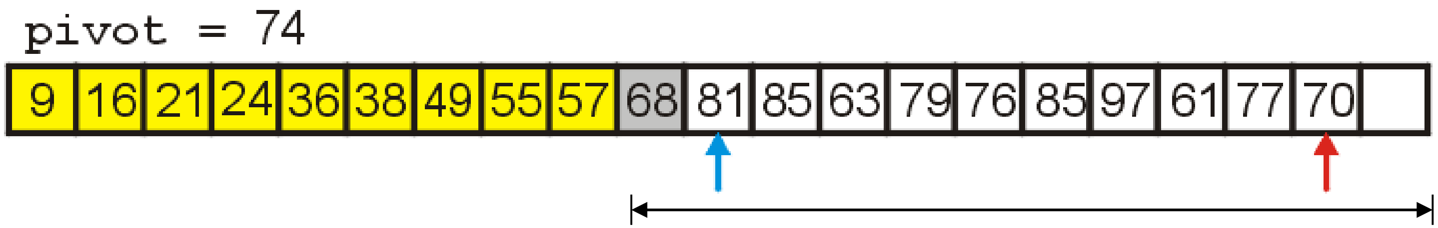
Improved Quick Sort Example

- We choose 74 to be our pivot
- We move 76 to the vacancy left by 74



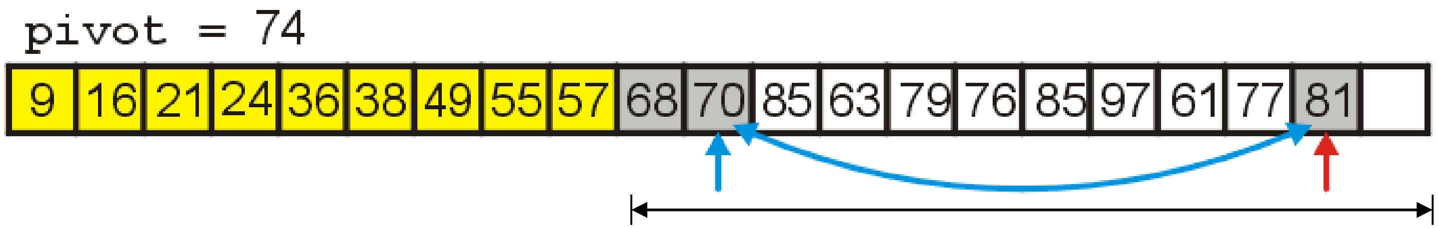
Improved Quick Sort Example

- We search forward till we find 81 > 74
- We search backward till we find 70 < 74



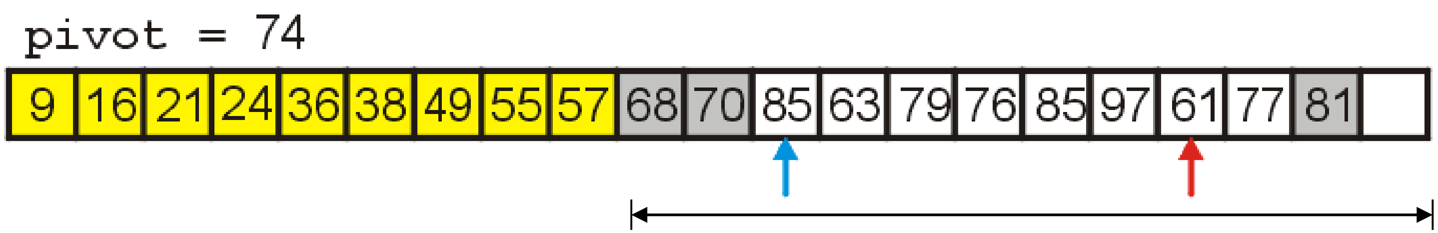
Improved Quick Sort Example

- We swap 70 and 84 placing them in order



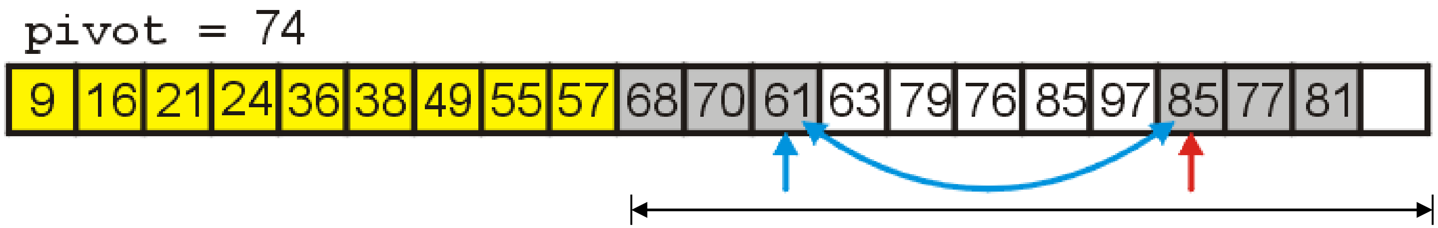
Improved Quick Sort Example

- **We search forward till we find** **85 > 74**
- **We search backward till we find** **61 < 74**



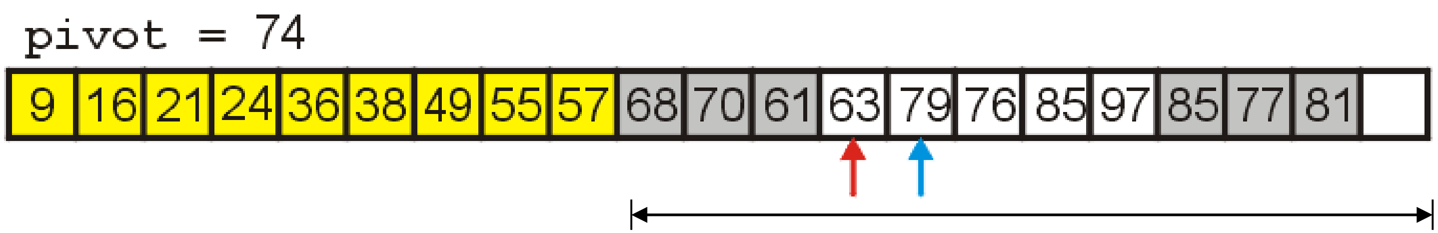
Improved Quick Sort Example

- **We swap 85 and 61 placing them in order**



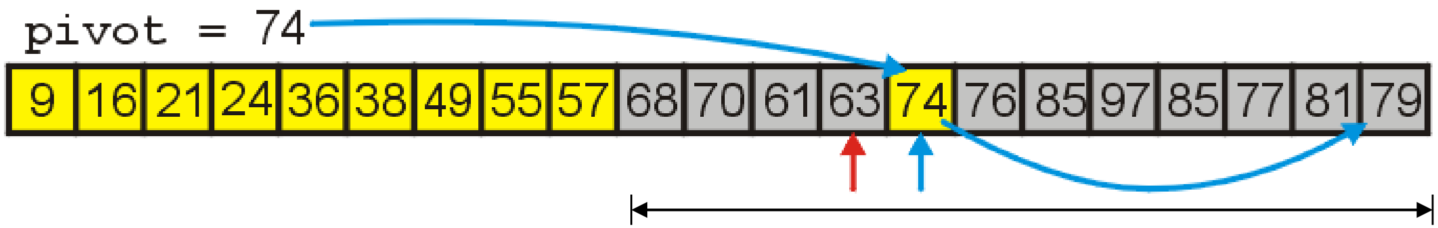
Improved Quick Sort Example

- **We search forward till we find** **79 > 74**
- **We search backward till we find** **63 < 74**
- **The indices are reversed, so we stop**



Improved Quick Sort Example

- We move 79 to the vacant location and move the pivot 74 into previous location of 79
- 74 is now in the correct location



Improved Quick Sort Example

- We sort the left sub-list first
- It has 4 elements, so we simply use insertion sort

9	16	21	24	36	38	49	55	57	68	70	61	63	74	76	85	97	85	77	81	79

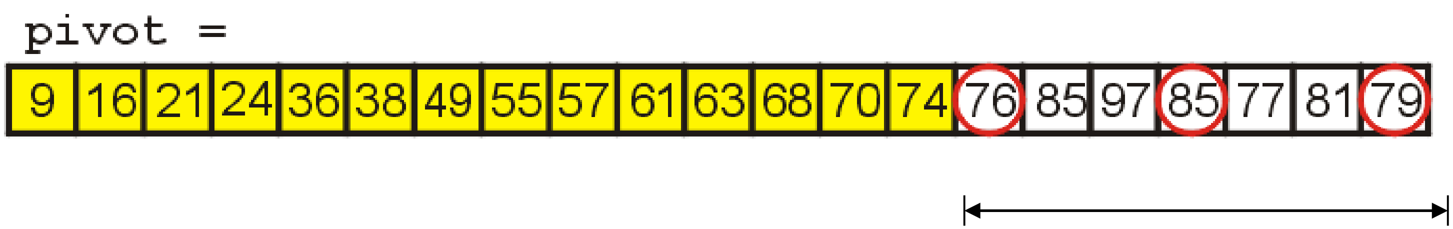
Improved Quick Sort Example

- **Having sorted the four elements, we focus on the remaining sub-list of seven entries**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	85	97	85	77	81	79

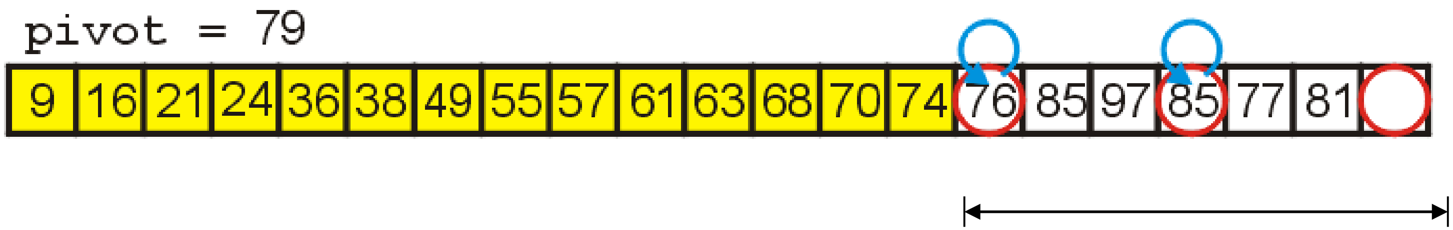
Improved Quick Sort Example

- **To sort the next sub-list, we examine the first, middle, and last entries**



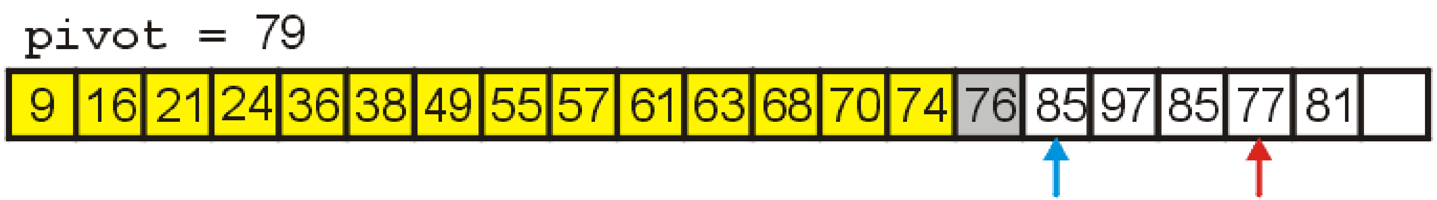
Improved Quick Sort Example

- We select 79 as our pivot and move:
- 76 into the lowest position
- 85 into the highest position



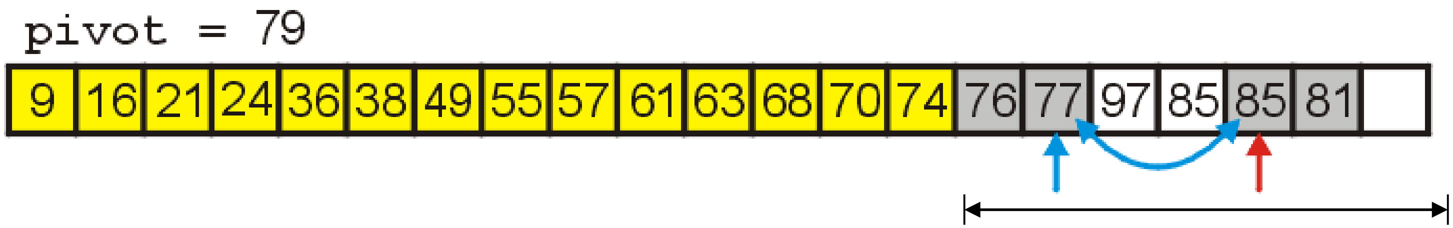
Improved Quick Sort Example

- We search forward till we find **85 > 79**
- We search backward till we find **77 < 79**



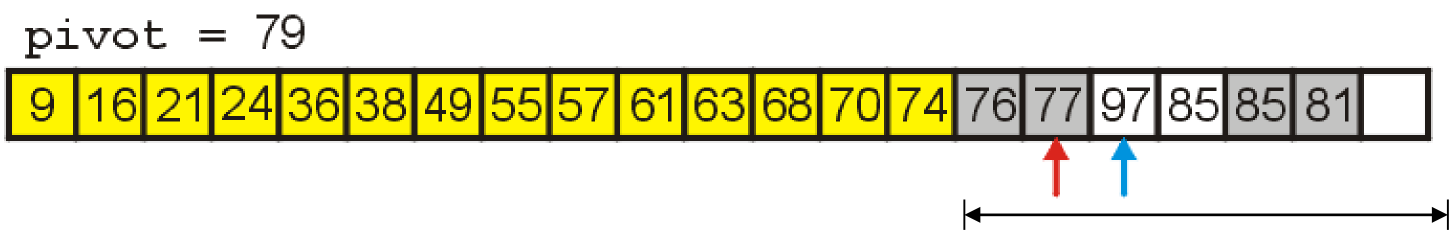
Improved Quick Sort Example

- We swap **85** and **77**, placing them in order



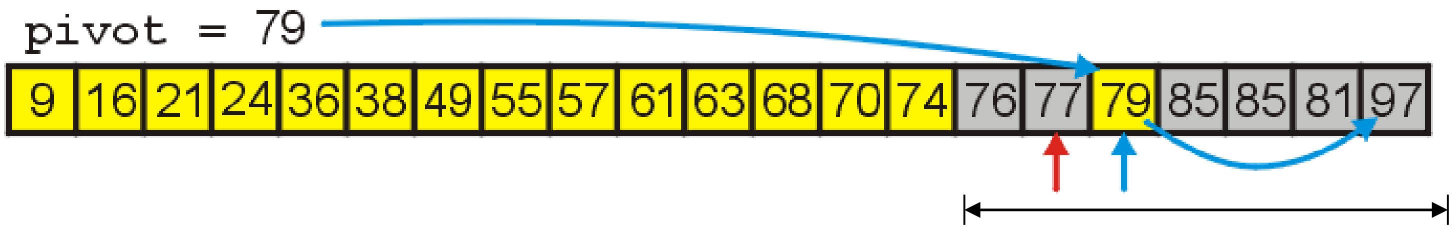
Improved Quick Sort Example

- We search forward till we find **97 > 79**
- We search backward till we find **77 < 79**
- The indices are reversed, so we stop



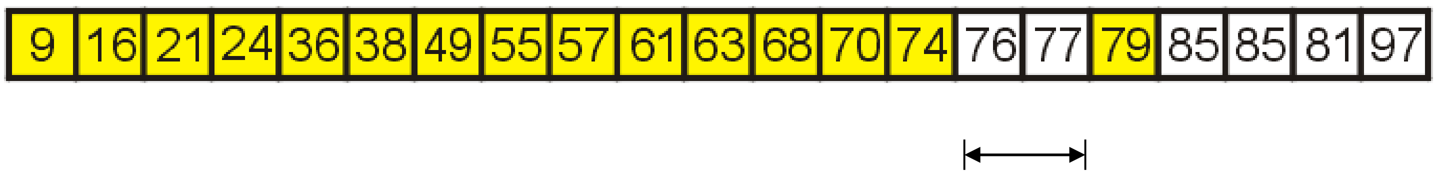
Improved Quick Sort Example

- Finally, we move 97 to the vacant location and copy 79 into the appropriate location
- 79 is now in the correct location



Improved Quick Sort Example

- This splits the sub-list into two sub-lists of size 2 and 4
- We use insertion sort for the first sub-list



Improved Quick Sort Example

- **We are left with one sub-list with four entries, so again, we use insertion sort**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	77	79	85	85	81	97

Improved Quick Sort Example

- **Sorting the last sub-list, we arrive at an ordered list**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	77	79	81	85	85	97

The Memory Requirement

- The additional memory required is O(log n)
- Each recursive function call places its local variables, parameters, etc., on a stack
  - The depth of the recursion tree is O(log n)
  - Unfortunately, if the run time is O(n^2), the memory use is O(n)

Run Time Summery

- To summarize all three O(n log n) algorithms

	Average Run Time	Worst-case Run Time	Average Memory	Worst-case Memory
Heap Sort	O(n log n)		O(1)	
Merge Sort	O(n log n)		O(n)	
Quick Sort	O(n log n)	O(n^2)	O(log n)	O(n)

算法分析课程组

重庆大学计算机学院

End of Section.