

Mathematical Experiments

数学规划

— 多目标规划



重庆大学数学与统计学院

Example1: 快乐学习问题

- (1) 学习成绩。教学内容的掌握程度，考试分数 f_1 ;
- (2) 其他能力。组织、社交、体育和文艺等， f_2 ;
- (3) 快乐每一天。快乐指数 f_3 。



决策变量和约束：

考虑最大工作时间 T ，学习时间为 t_1 ，用来发展课外活动的时间为 t_2 ，其中 $t_1 + t_2 \leq T$ 。

决策变量和目标函数：

学习成绩关于学习时间 t_1 是函数。如 $f_1 = f_1(t_1) = \arctan(t_1)$ 。

其他能力关于时间 t_2 是增函数。如 $f_2 = f_2(t_2) = a + b t_2$ 。

快乐指数与 t_1 和 t_2 都有关，如 $f_3 = k_1/t_1 + k_2 t_2$ 。

多目标规划问题：

$$\begin{aligned} \max \quad & \begin{pmatrix} f_1(t_1) \\ f_2(t_2) \\ f_3(t_1, t_2) \end{pmatrix} \\ \text{s.t.} \quad & \\ & t_1 + t_2 \leq T \\ & t_1, t_2 \geq 0 \end{aligned}$$

目标往往难以同时达到，因为他们有一些是相互矛盾的。

特点：目标函数是一个关于决策变量的向量值函数，也称为向量优化问题。



Example2: 证券投资投资组合问题

Portfolio Optimization

Markowitz Shares the 1990 Nobel Prize

1952年美
国经济学家
Markowitz用概
率统计的方法,
建立了完整的
组合投资理论,
于1990年获得
诺贝尔经济学
奖。



Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences
in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS
AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize
in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called,
Capital Asset Pricing Model (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary

Financial markets serve a key purpose in a modern market economy by allocating productive resources
among various areas of production. It is to a large extent through financial markets that saving in
different sectors of the economy is transferred to firms for investments in buildings and machines.
Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread
and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry
Markowitz who developed a theory for households' and firms' allocation of financial assets under
uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally
invested in assets which differ in regard to their expected return and risk, and thereby also how risks can
be reduced.





问题描述: 8种投资的年收益率的历史数据如下表, 投资者应如何分配他的投资资金, 是使得收益尽可能大, 风险尽可能小。

分析: 设投资的期限是一年, 设投资总数为1个单位, 用于第 i 项投资的资金比例为 x_i , $x=(x_1, x_2, \dots, x_n)$ 称为投资组合向量, 即决策变量。问题的约束条件为

$$x_1 + x_2 + \dots + x_n = 1, x_i \geq 0$$



项目 年份	股票1	股票2	股票3	股票4	股票5	股票6	股票7	股票8
1973	.075	-.058	-.148	-.185	-.302	.023	-.149	.677
1974	.084	.020	-.265	-.284	-.338	.002	-.232	.722
1975	.061	.056	.371	.385	.318	.123	.354	-.260
1976	.052	.175	.236	.266	.280	.156	.025	-.040
1977	.055	.002	-.074	-.026	.093	.030	.181	.200
1978	.077	-.018	.064	.093	.146	.012	.326	.295
1979	.109	-.022	.184	.256	.307	.023	.048	.212
1980	.127	-.053	.323	.337	.367	.031	.226	.296
1981	.156	.003	-.051	-.037	-.010	.073	-.023	-.312
1982	.117	.465	.215	.187	.213	.311	-.019	.084
1983	.092	-.015	.224	.235	.217	.080	.237	-.128
1984	.103	.159	.061	.030	-.097	.150	.074	-.175
1985	.080	.366	.316	.326	.333	.213	.562	.006
1986	.063	.309	.186	.161	.086	.156	.694	.216
1987	.061	-.075	.052	.023	-.041	.023	.246	.244
1988	.071	.086	.165	.179	.165	.076	.283	-.139
1989	.087	.212	.316	.292	.204	.142	.105	-.023
1990	.080	.054	-.032	-.062	-.170	.083	-.234	-.078
1991	.057	.193	.304	.342	.594	.161	.121	-.042
1992	.036	.079	.076	.090	.174	.076	-.122	-.064
1993	.031	.217	.100	.113	.162	.110	.326	.146
1994	.045	-.111	.012	-.001	-.032	-.035	.078	-.010



目标函数

根据Markowitz的理论，证券投资组合的平均收益可以写成

$$R(x) = \sum_{i=1}^8 x_i \bar{r}_i$$

其中 \bar{r}_i 为第*i*种投资的历史平均收益率。

而投资组合的风险可以使用收益的波动程度加以度量,比如使用如下的定义

$$Q(x) = \frac{1}{2} x^T Q x$$

其中Q为8种证券的协方差矩阵。



双目标规划模型：

$$\max \begin{pmatrix} R(x) \\ -Q(x) \end{pmatrix}$$

$$s.t. \quad x_1 + x_2 + \dots + x_8 = 1,$$

$$x_i \geq 0, \quad i=1, 2, \dots, 8$$



模型1: 控制风险最大化收益

$$\begin{array}{ll}\max & R(x) \\ s.t. & Q(x) \leq \sigma \\ & x_1 + x_2 + \cdots + x_8 = 1, \\ & x_i \geq 0\end{array}$$

控制值 σ 如何确定?

模型2: 控制赢利最小化风险

$$\begin{array}{ll}\min & Q(x) \\ s.t. & R(x) \geq \beta \\ & x_1 + x_2 + \cdots + x_8 = 1, \\ & x_i \geq 0\end{array}$$

控制值 β 如何确定?



模型3: 收益和风险加权平均 ($0 \leq \rho \leq 1$)

$$\max \quad (1 - \rho)R(x) - \rho Q(x),$$

$$s.t. \quad x_1 + x_2 + \cdots + x_n = 1$$

$$x_i \geq 0 \quad i = 1, 2, \cdots, n$$

权系数 ρ 如何确定?

上述三种方法都能将多目标规划问题转化成为单目标规划问题。



A的数据略

```
r=mean(A);
```

```
Q=corrcoef(A);
```

```
rho=0:0.05:1;%取不同的权重
```

```
n=length(rho);x0=zeros(8,1);
```

```
X=zeros(8,n);Fval=zeros(8,1);
```

```
Aeq=ones(1,8);beq=1;Lb=zeros(8,1);
```

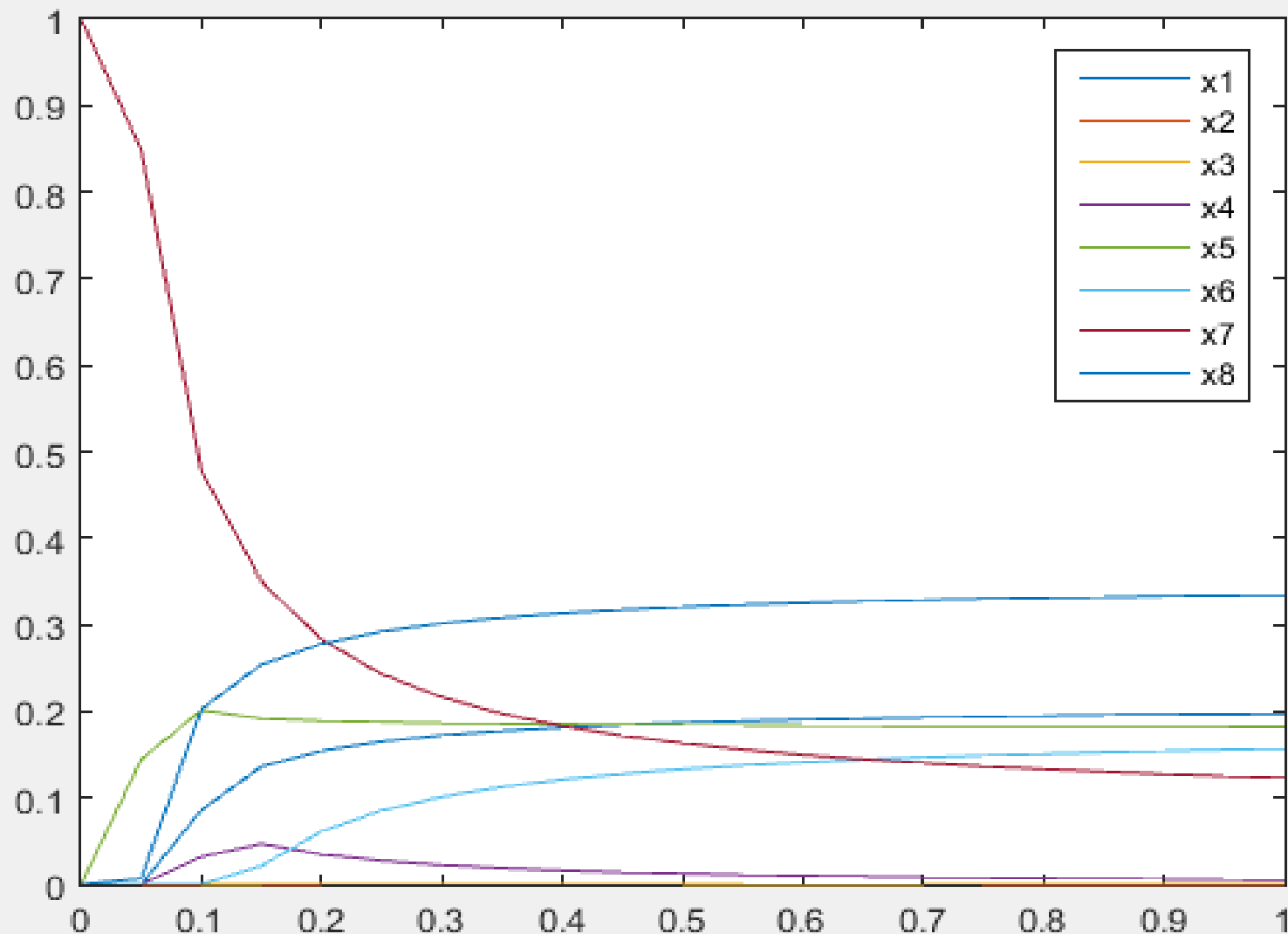
```
for i=1:n
```

```
    H=rho(i)*Q;
```

```
    c=-(1-rho(i))*r;    [X(:,i),Fval(i)]=quadprog(H,c,[],[],Aeq,beq,Lb,[],x0);
```

```
end
```

```
plot(rho,X)
```

结果分析：投资组合随着权系数变化。权系数越大，表示越重视风险。当取值为1时，等价于风险最小化的单目标问题；当取值为0时，等价于收益极大化的单目标问题。决策者可以根据自己的风险偏好程度选择投资组合。

Thanks



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