T&R Team of Algorithm Design
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Algorithm Analysis & Design

Introduction to Algorithm

Chapter 7: Quick Sort

Outline

- **6.1 Basic Quick Sort**
- **6.2 Improving Quick Sort with Medians**
- **6.3 Quick Search**

7.1 Basic Quick Sort

QUICK SORT

- We have seen two O(n log n) sorting algorithms:
  - Merge sort which is faster but requires more memory
  - Heap sort which allows in-place sorting
- We will now look at a recursive algorithm which may be done *almost* in place and usually faster than heap sort
  - Use an object in the array (a pivot) to divide the two
  - Average case: O(n log n) time and O(log n) memory
  - Worst case: O(n^2) time and O(n) memory

QUICK SORT

- Merge sort splits the array sub-lists and sorts them
- The larger problem is split into two sub-problems based on location in the array
- Consider the following alternative:
  - Chose an object in the array and partition the remaining objects into two groups relative to the chosen entry

QUICK SORT

- For example, given an unsorted array:

2	8	7	1	3	5	6	4

- We can select the last entry, 4, and sort the remaining entries into two groups, those less than 4 and those greater than 4:

2	1	3	4	7	5	6	8

- Note that 4 is now in the correct location once the list is sorted
  - Proceed by applying the algorithm to the first 3 and last 4 entries

A Simple Implementation – PARTITION

- **PARTITION** (A, p, r)

  $x \leftarrow A[r]$

  $i \leftarrow p$-1

  FOR $j \leftarrow p$ **TO** r-1

      IF $A[j] \leq x$

          THEN $i \leftarrow i + 1$

              exchange $A[i] \leftrightarrow A[j]$

  exchange $A[i+1] \leftrightarrow A[r]$

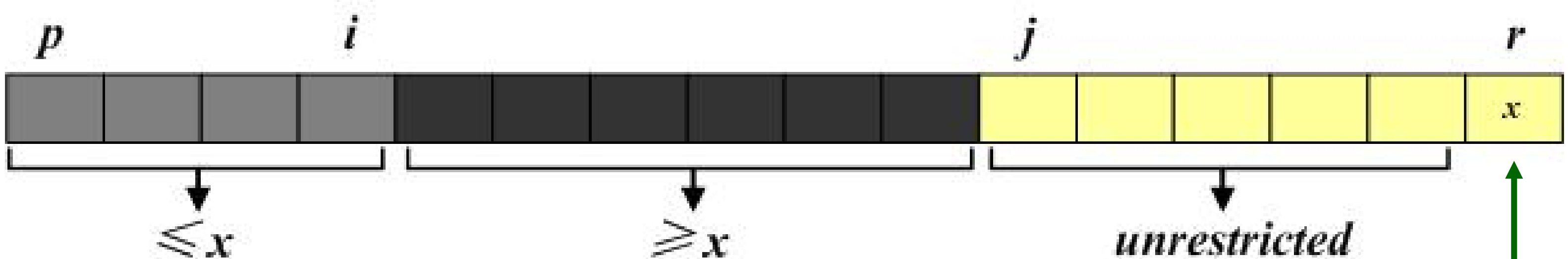
  RETURN $i+1$

A Simple Implementation – PARTITION

PARTITION

```
PARTITION(A, p, r)          //A[p..r]
    x ← A[r]                //the rightmost element as pivot
    i ← p-1
  for j ← p to r-1
        do if A[j] ≤ x
                then i ← i+1
                     exchange A[i]↔A[j]
  exchange A[i+1]↔A[r]
  return i+1
```

Running time = O(n) for n elements



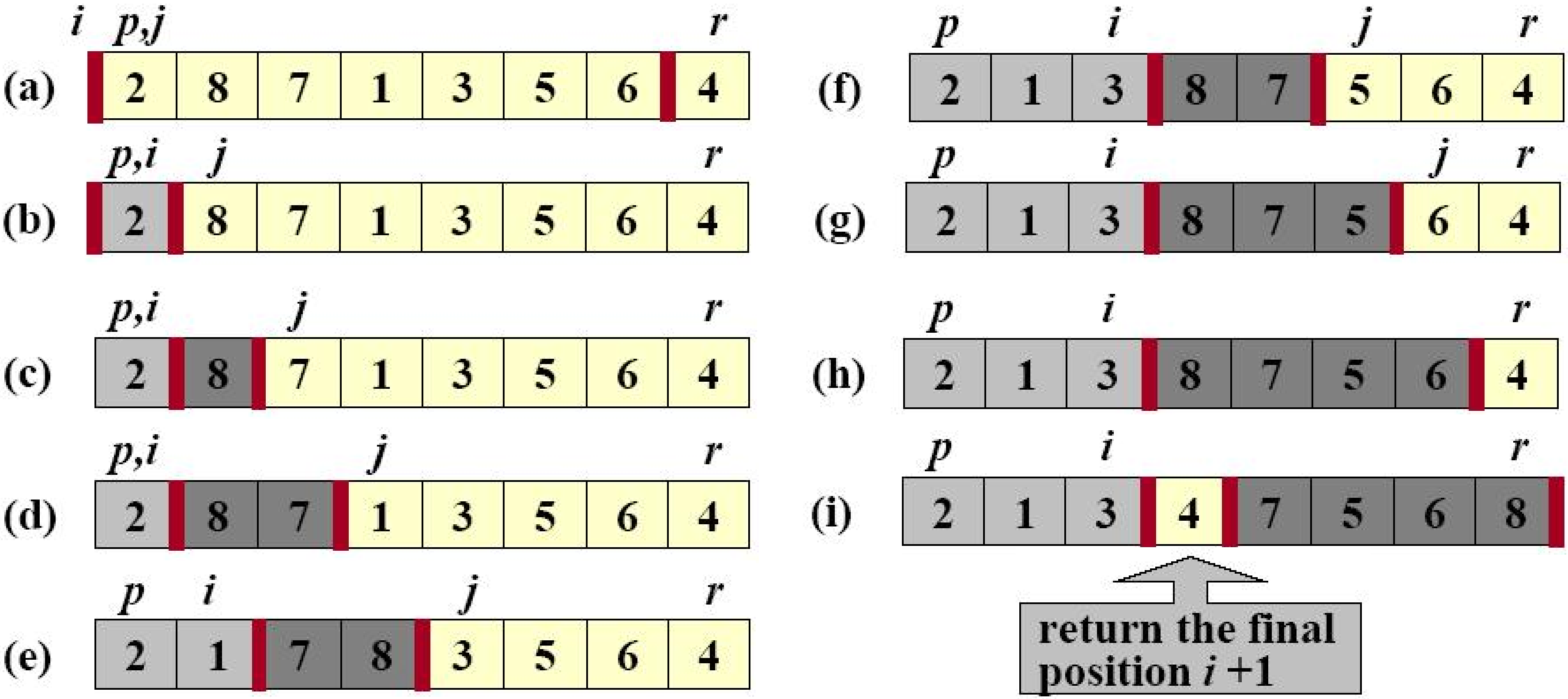
A[p, … , i]: known to be ≤ pivot

A[j … r-1]: not yet examined

A[i+1, … , j-1]: known to be > pivot

A[r]: pivot

A Simple Implementation – PARTITION



- **The operation of Partition on the sample array. Lightly shaded array elements are all with values no greater than *x* (the pivot). Heavily shaded array elements are all with values greater than *x*.**

A Simple Implementation – QUICKSORT

- **QUICKSORT (A, p, r)**

  IF p < r

  THEN q ← PARTITION (A, p, r)

  QUICKSORT (A, p, q–1)

  QUICKSORT (A, q+1, r)

- **Initial call: QUICKSORT(A, 1, n)**

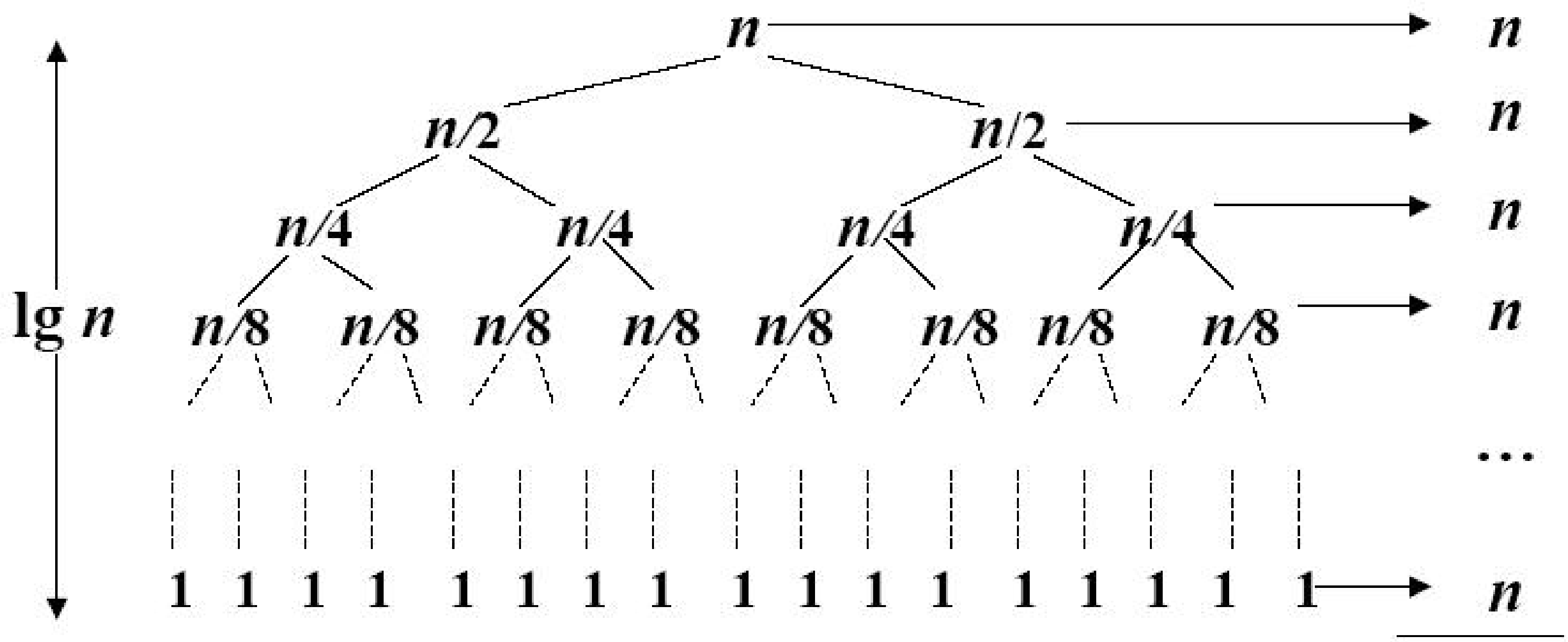
Run-time Analysis

- In the best case, the list will be split into two approximately equal sub-lists, and thus, the run time could be very similar to that of merge sort: $\Theta(n \log n)$

Recursive Tree of the Best Case

- **A recursion tree for quick sort in which the partition always balances the two sides of the partition equally. The resulting running time is $\Theta(n \log n)$**

- **The question is: WHAT happens if we don't get that lucky?**



Worst-case Scenario

- Suppose we choose the smallest element as our pivot and we try ordering a sorted list:

80	38	95	84	66	10	79	2	26	87	96	12	43	81	3

- Using 2, we partition the original list into

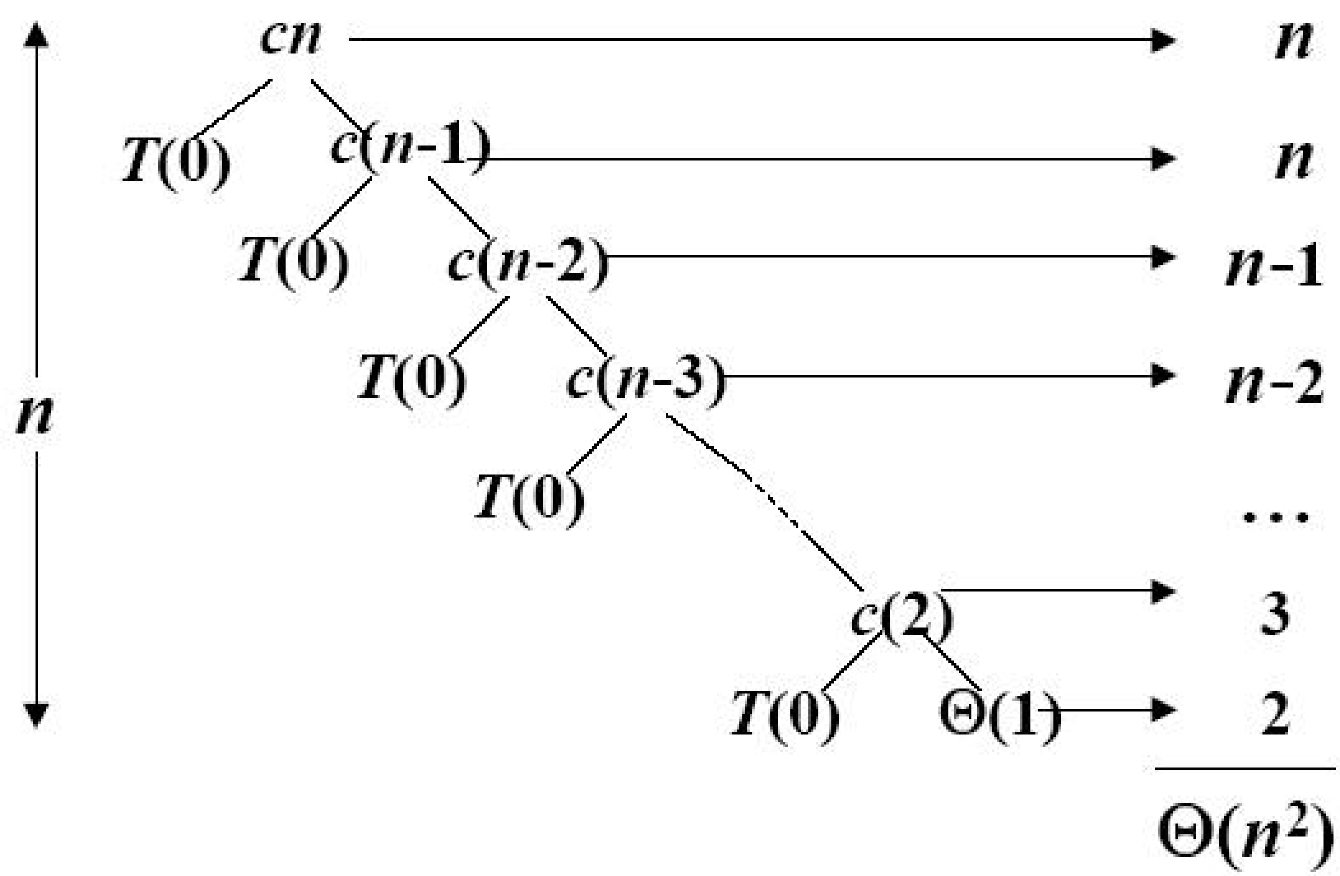
2	80	38	95	84	66	10	79	26	87	96	12	43	81	3

- We still have to sort a list of size n – 1

- The run time is $T(n) = T(n-1) + \Theta(n) = \Theta(n^2)$
  - Thus, the run time drops from $\Theta(n \log n)$ to $\Theta(n^2)$

Recursive Tree of the Worst Case

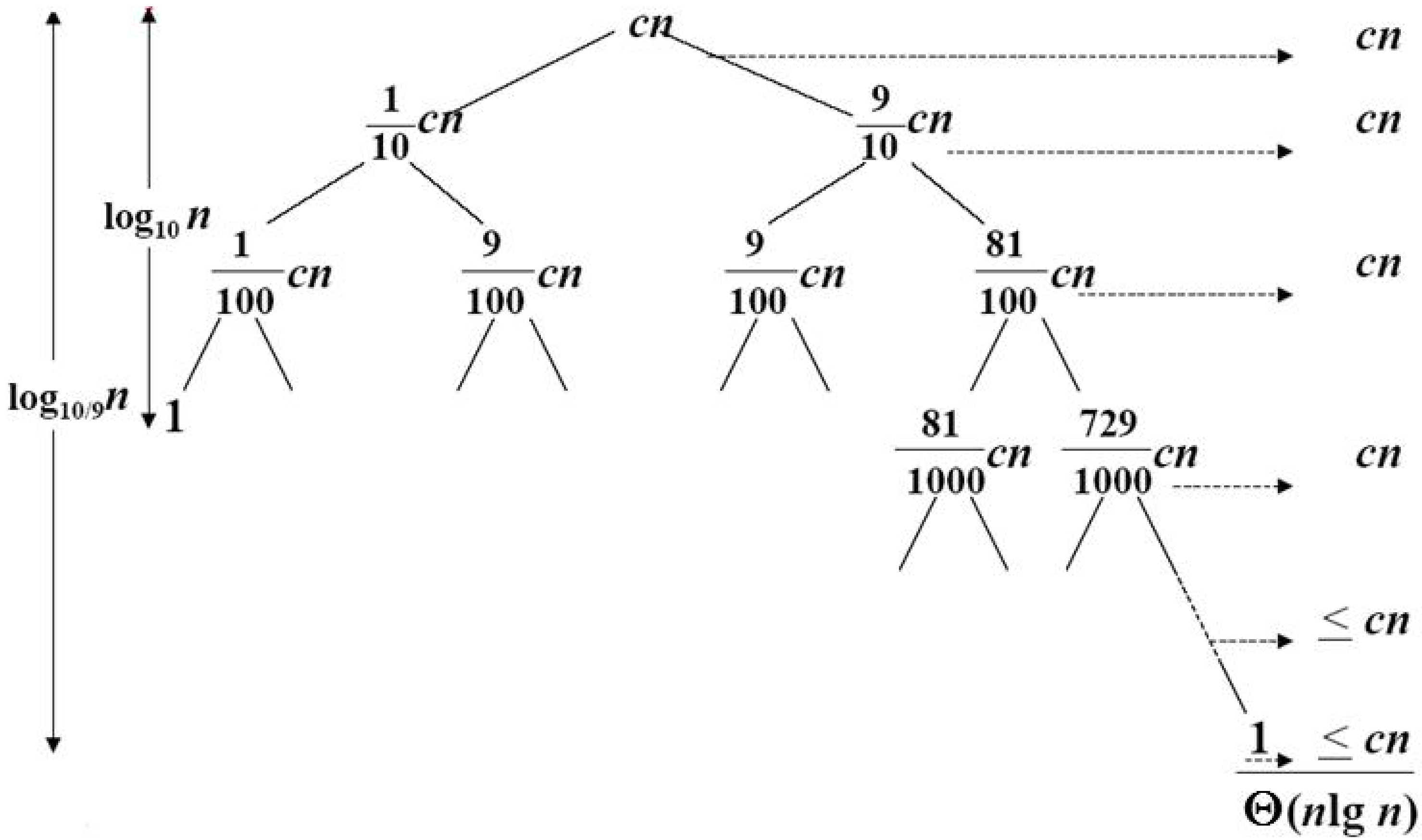
- A recursion tree for quick sort in which the partition always puts only a single element on one side of the partition. The resulting running time is $\Theta(n^2)$



Recursive Tree of the Balanced Case

- **What if the split is always 1:9?**
  - $T(n) = T(9n/10) + T(n/10) + \Theta(n)$
  - **What is the solution to this recurrence?**

Recursive Tree of the Balanced Case



- **A recursion tree for quick sort in which partition always produces a 9-to-1 split, yielding a running time of $\Theta(n \log n)$**

Average-case Senario

- **A recursion tree for quick sort in which partition always produces a 9-to-1 split, yielding a running time of $\Theta(n \log n)$**

If we choose a random pivot, this will, on average, divide a set of *n* items into two sets of size $\frac{n}{4}$ and $\frac{3n}{4}$.

90 % of the time the width will have a ratio of 1:19 or better.

7.2 Improving Quick Sort with Medians

Alternate Strategy

- **Our goal is to choose the median element in the list as our pivot:**

80	38	95	84	66	10	79	2	26	87	96	12	43	81	3

- **Unfortunately, it's DIFFICULT to find**
- **Alternate strategy: take the median of a subset of entries**
  - **For example, take the median of the first, middle, and last entries**

Choose the Median-of-Three

- **It is difficult to find the median so consider another strategy:**
  - **Choose the median of the first, middle, and last entries in the list**

80	38	95	84	99	10	79	44	26	87	96	12	43	81	3

- **This will usually give a much better approximation of the actual median**

Choose the Median-of-Three

- **Sorting the elements based on 44 results in two sub-lists, each of which must be sorted (again, using quicksort)**

- **We select the 26 to partition the first sub-list:**

38	10	26	12	43	3	44	80	95	84	99	79	87	96	81

- **and 81 to partition the second sub-list:**

38	10	26	12	43	3	44	80	95	84	99	79	87	96	81

Choose the Median-of-Three

- If we choose a random pivot, this will, on average, divide a set of *n* items into two sets of size $\frac{n}{4}$ and $\frac{3n}{4}$.
  - 90 % of the time the width will have a ratio of 1:19 or better.
- Choosing the median-of-three will, on average, divide the *n* items into two sets of size $\frac{5n}{16}$ and $\frac{11n}{16}$.
  - Median-of-three helps speed the algorithm
  - 90 % of the time the width will have a ratio of 1:6.388 or better.
- Further, we can apply insertion sort to sorting the small sub-arrays.

Improved Quick Sort Example

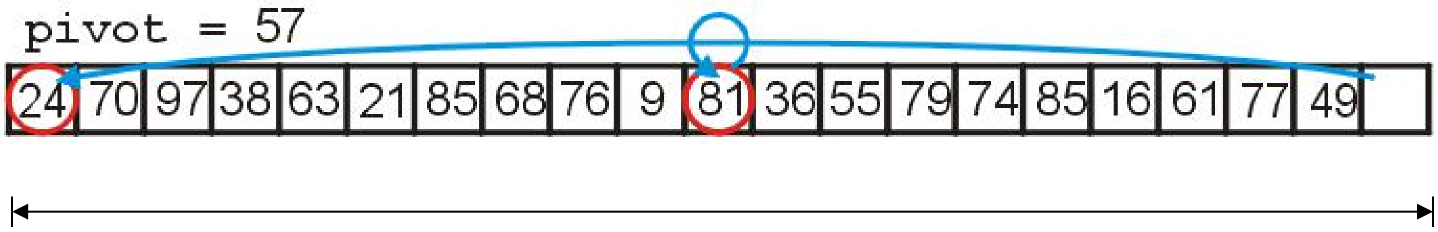
- **First, we examine the first, middle, and last entries of the full list**
- **The span below will indicate which list we are currently sorting**

pivot =

57	70	97	38	63	21	85	68	76	9	81	36	55	79	74	85	16	61	77	49	24

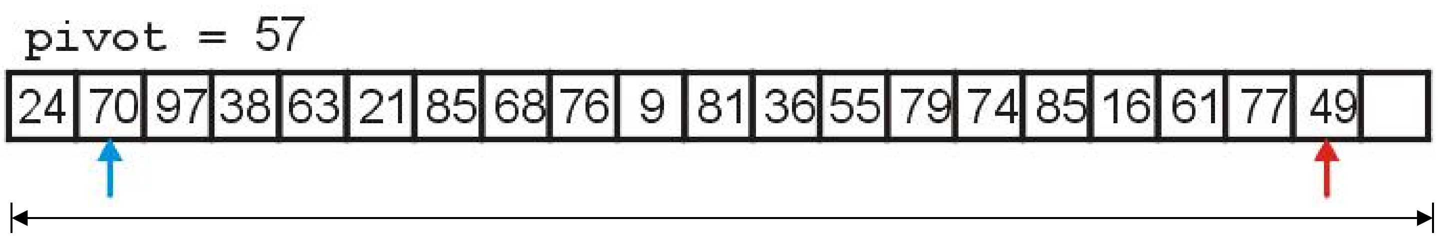
Improved Quick Sort Example

- We select 57 to be our pivot
- We move 24 into the first location



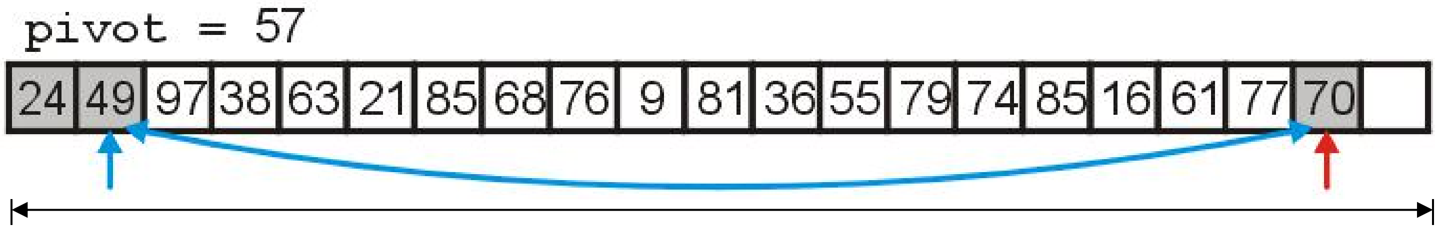
Improved Quick Sort Example

- **Starting at the 2nd and 2nd-last locations:**
- **we search forward till we find** **$70 > 57$**
- **we search backward till we find** **$49 < 57$**



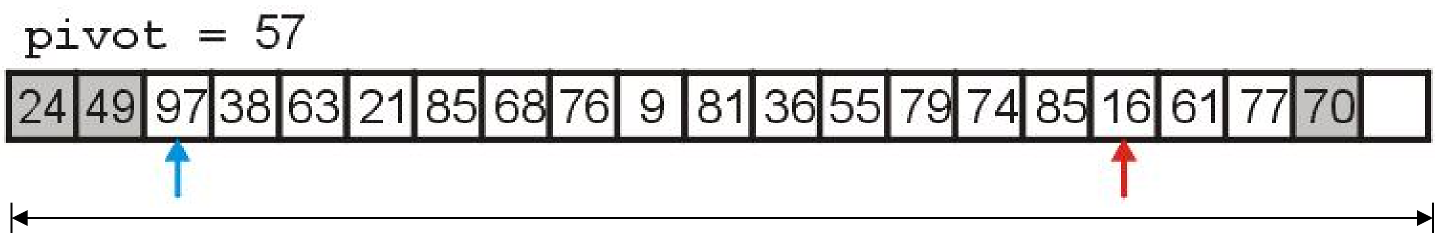
Improved Quick Sort Example

- We swap 70 and 49, placing them in order with respect to each other



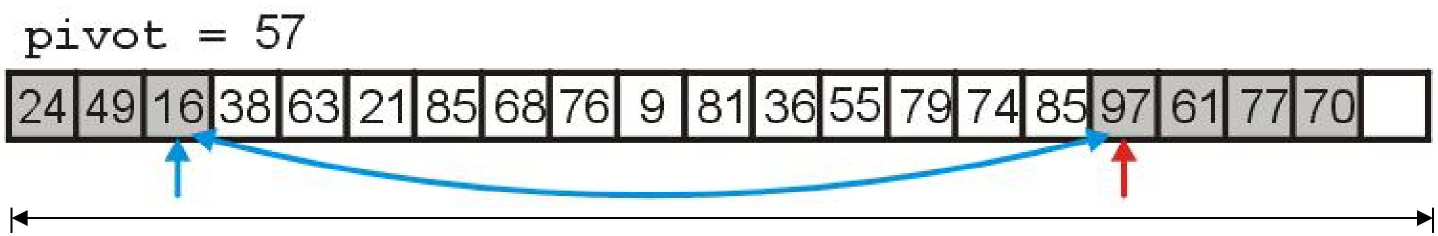
Improved Quick Sort Example

- **We search forward until we find** **97 > 57**
- **We search backward until we find** **16 < 57**



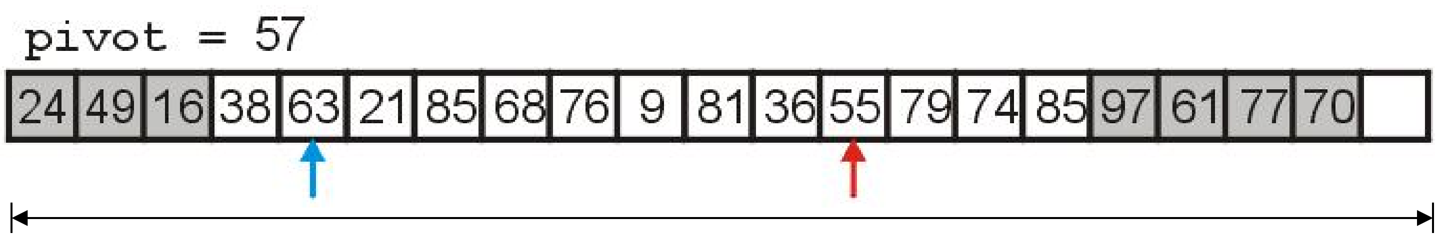
Improved Quick Sort Example

- We swap 16 and 97 which are now in order with respect to each other



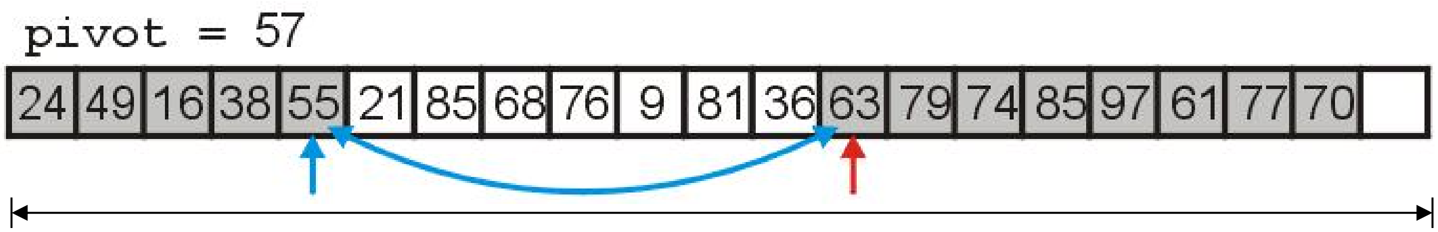
Improved Quick Sort Example

- We search forward till we find $63 > 57$
- We search backward till we find $55 < 57$



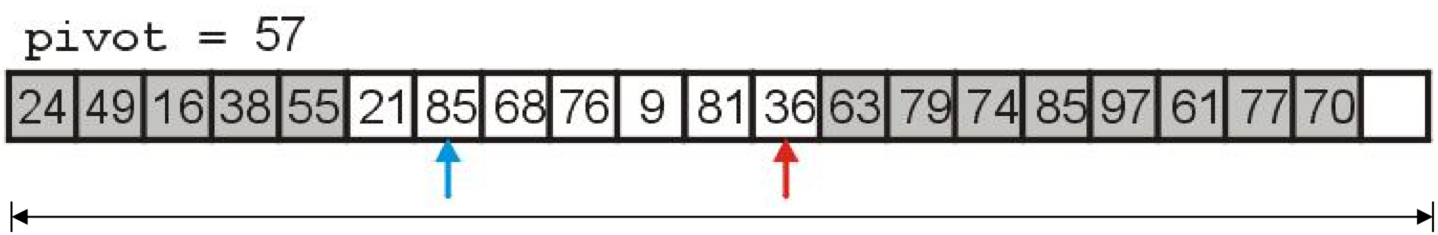
Improved Quick Sort Example

- We swap **63** and **55**



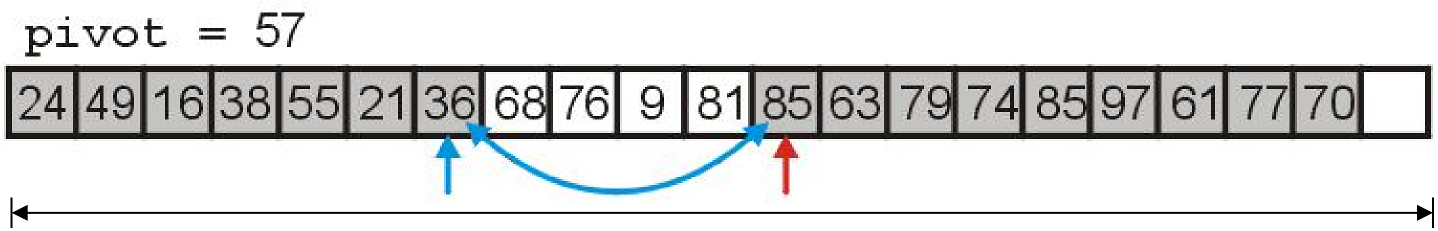
Improved Quick Sort Example

- **We search forward till we find** **$85 > 57$**
- **We search backward till we find** **$36 < 57$**



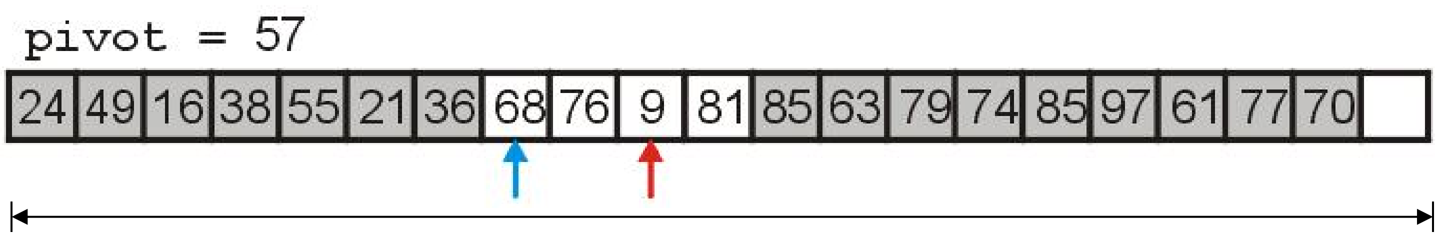
Improved Quick Sort Example

- We swap **85** and **36**, placing them in order with respect to each other



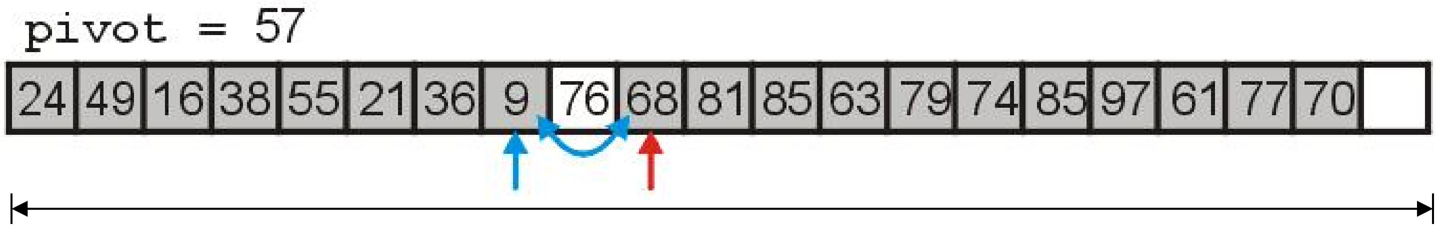
Improved Quick Sort Example

- **We search forward until we find** $68 > 57$
- **We search backward until we find** $9 < 57$



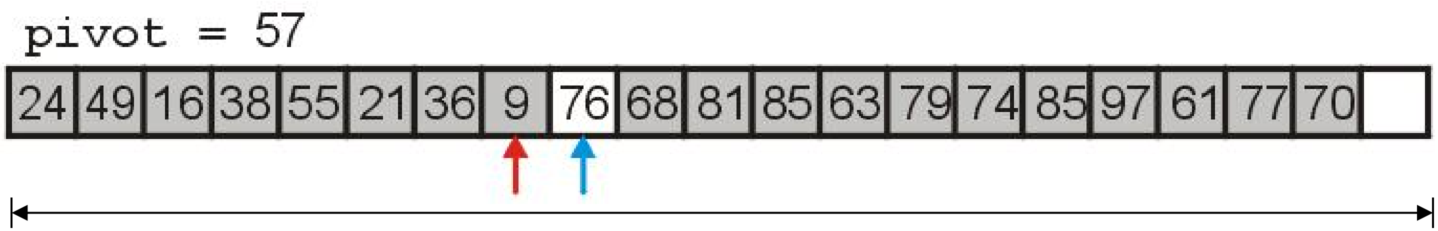
Improved Quick Sort Example

- We swap **68** and **9**



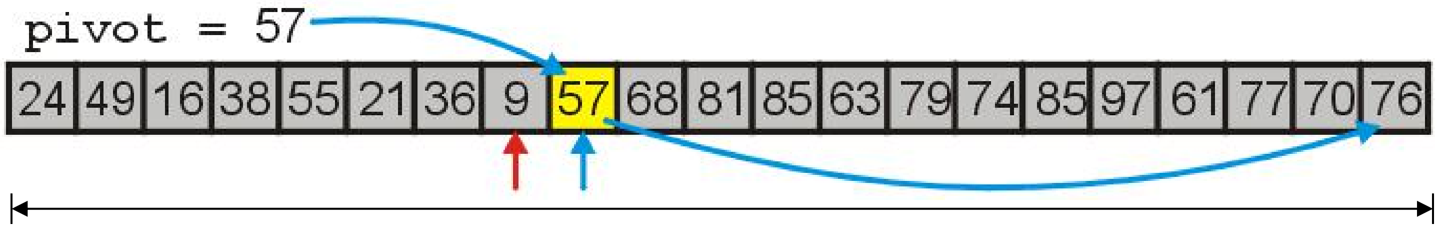
Improved Quick Sort Example

- We search forward until we find $76 > 57$
- We search backward until we find $9 < 57$
  - The indices are out of order, so we stop



Improved Quick Sort Example

- **We move the larger indexed item to the vacancy at the end of the array**
- **We fill the empty location with the pivot, 57**
- **The pivot is now in the correct location**



Improved Quick Sort Example

- **We will now recursively call quick sort on the first half of the list**
- **When we are finished, all entries < 57 will be sorted**

24	49	16	38	55	21	36	9	57	68	81	85	63	79	74	85	97	61	77	70	76

Improved Quick Sort Example

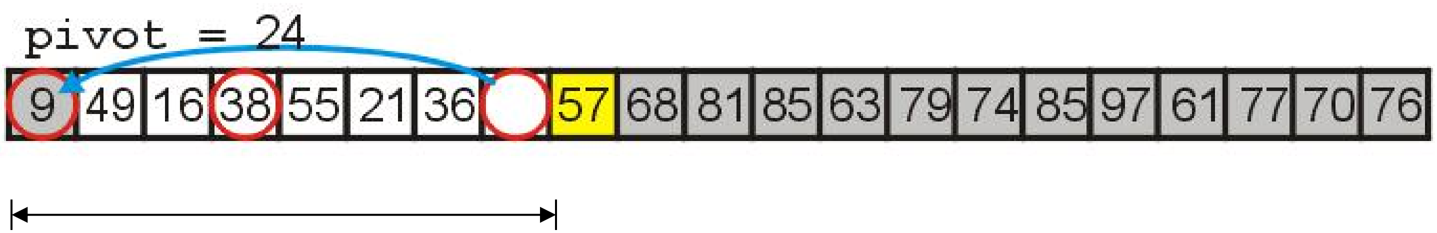
- **We examine the first, middle, and last elements of this sub list**

pivot =

24	49	16	38	55	21	36	9	57	68	81	85	63	79	74	85	97	61	77	70	76

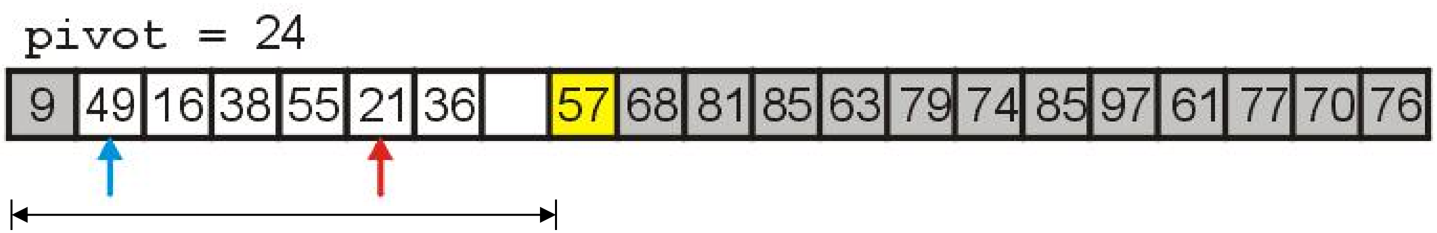
Improved Quick Sort Example

- We choose 24 to be our pivot
- We move 9 into the first location in this sub-list



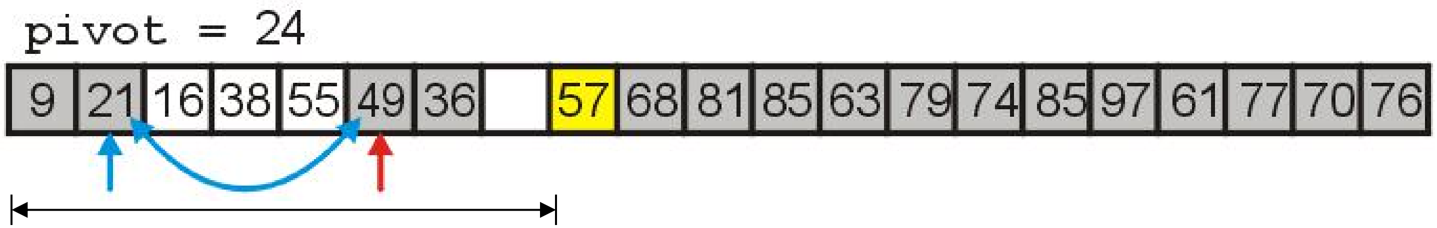
Improved Quick Sort Example

- **We search forward till we find** **49 > 24**
- **We search backward till we find** **21 < 24**



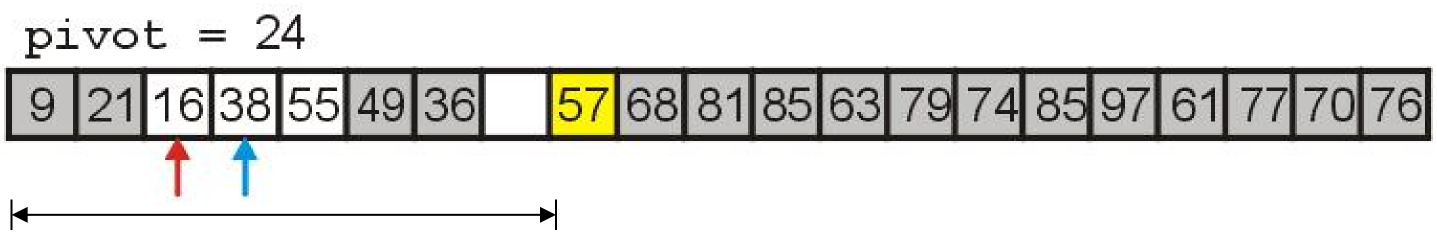
Improved Quick Sort Example

- We swap **49** and **21**, placing them in order with respect to each other



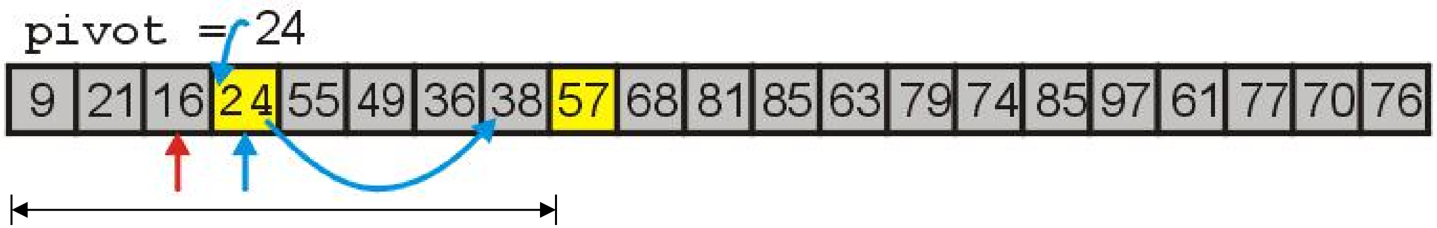
Improved Quick Sort Example

- We search forward till we find **38 > 24**
- We search backward till we find **16 < 24**
- The indices are reversed, so we stop



Improved Quick Sort Example

- We move **38** to the vacant location and move the pivot **24** into the previous location of **38**
- **24** is now in the correct location



Improved Quick Sort Example

- **We will now recursively call quick sort on the left and right halves of those entries which are < 57**

9	21	16	24	55	49	36	38	57	68	81	85	63	79	74	85	97	61	77	70	76

Improved Quick Sort Example

- The first partition has three entries, so we sort it using insertion sort

Improved Quick Sort Example

- **The second partition also has only four entries, so again, we use insertion sort**

9	16	21	24	36	38	49	55	57	68	81	85	63	79	74	85	97	61	77	70	76

Improved Quick Sort Example

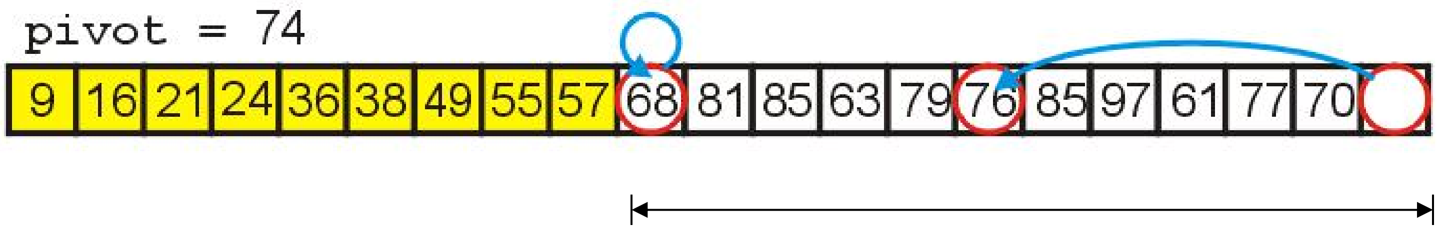
- **First we examine the first, middle, and last entries of the sub-list**

`pivot =`

9	16	21	24	36	38	49	55	57	68	81	85	63	79	74	85	97	61	77	70	76

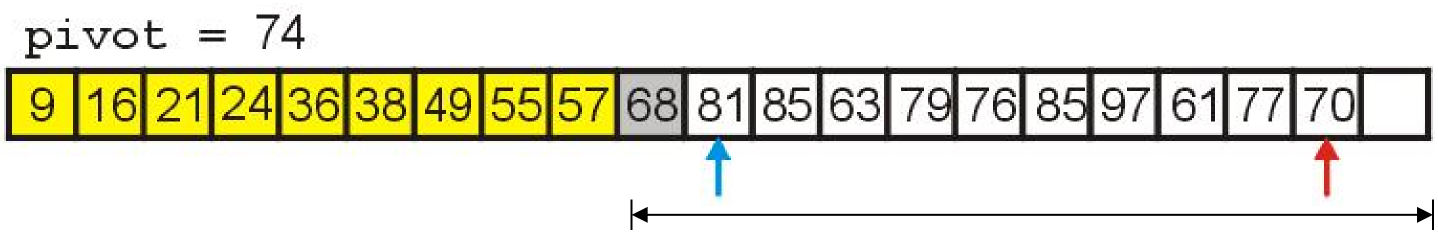
Improved Quick Sort Example

- We choose 74 to be our pivot
- We move 76 to the vacancy left by 74



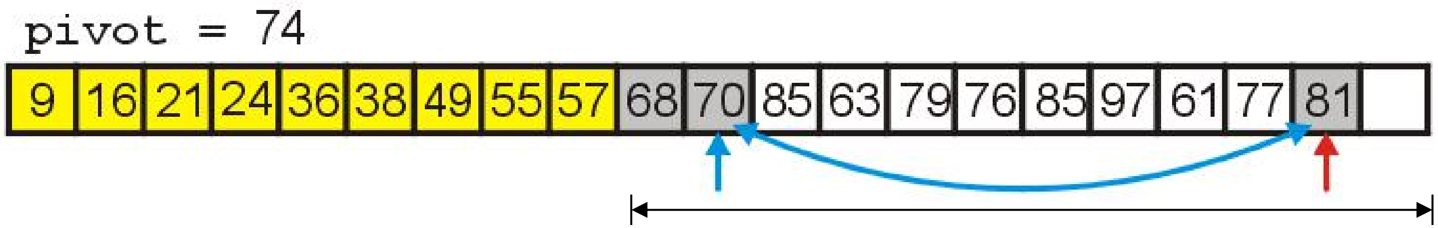
Improved Quick Sort Example

- **We search forward till we find** **$81 > 74$**
- **We search backward till we find** **$70 < 74$**



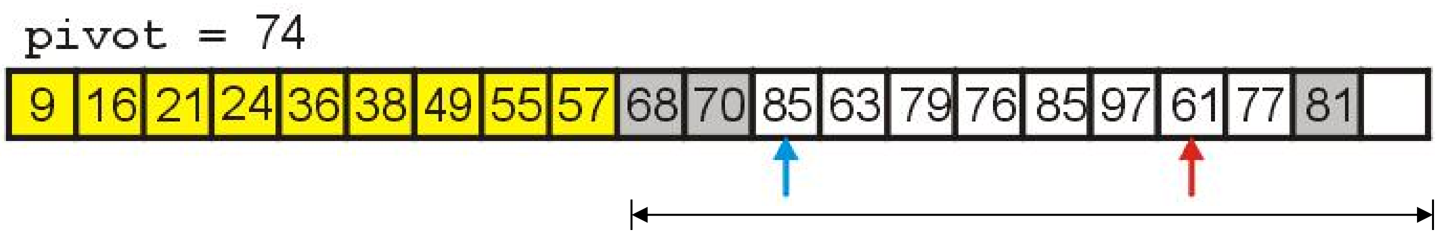
Improved Quick Sort Example

- We swap 70 and 84 placing them in order



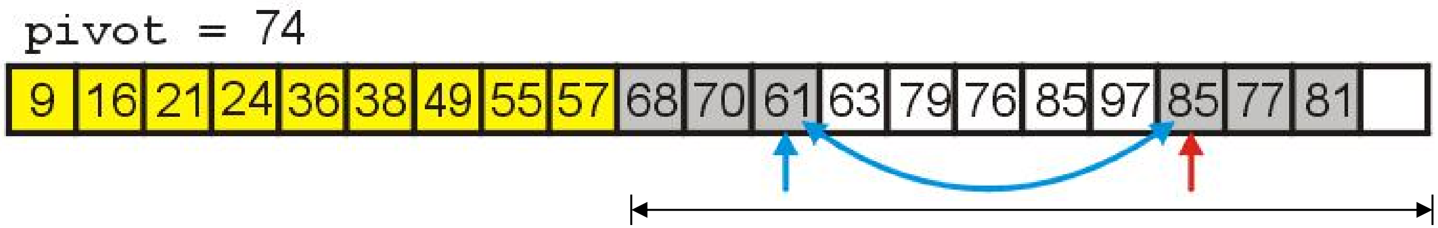
Improved Quick Sort Example

- We search forward till we find **85 > 74**
- We search backward till we find **61 < 74**



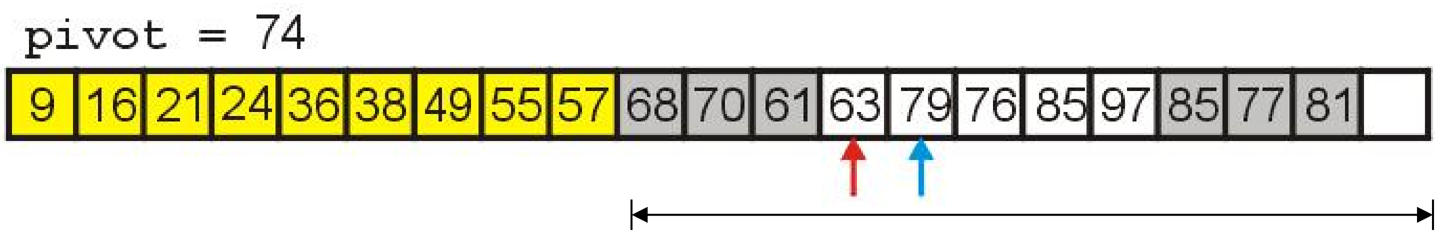
Improved Quick Sort Example

- **We swap 85 and 61 placing them in order**



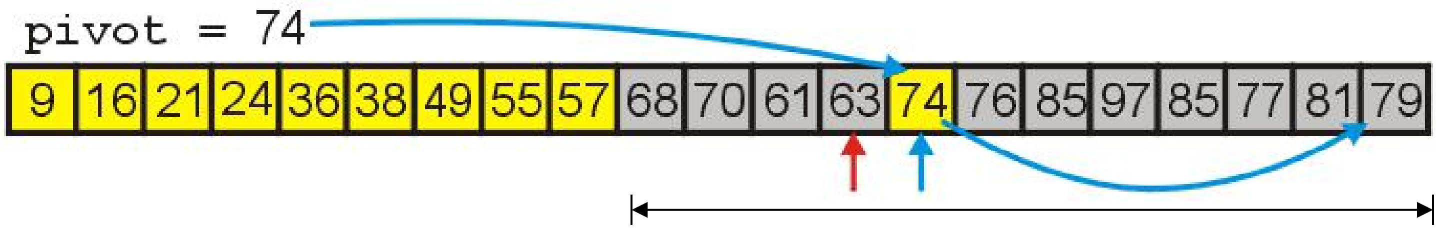
Improved Quick Sort Example

- We search forward till we find 79 > 74
- We search backward till we find 63 < 74
- The indices are reversed, so we stop



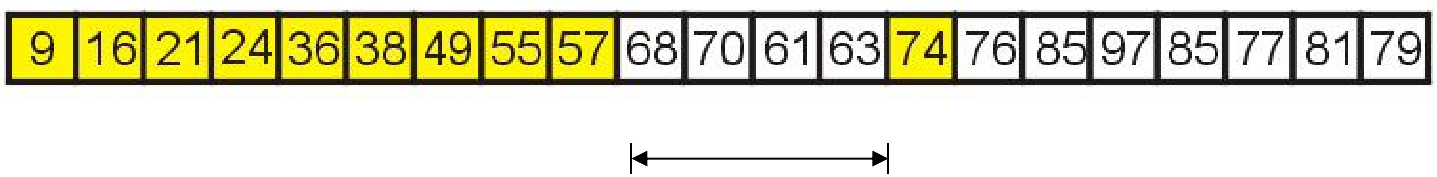
Improved Quick Sort Example

- We move 79 to the vacant location and move the pivot 74 into previous location of 79
- 74 is now in the correct location



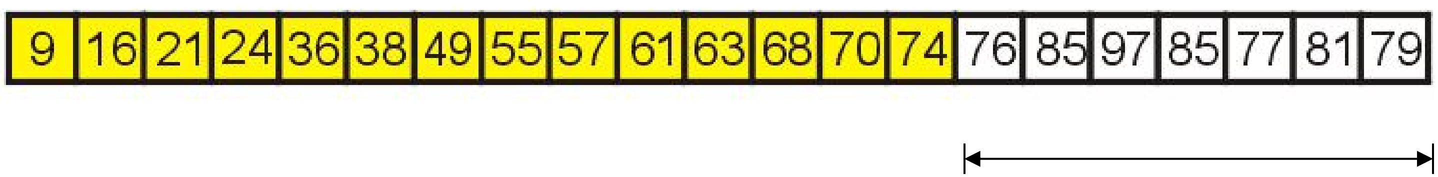
Improved Quick Sort Example

- We sort the left sub-list first
- It has 4 elements, so we simply use insertion sort



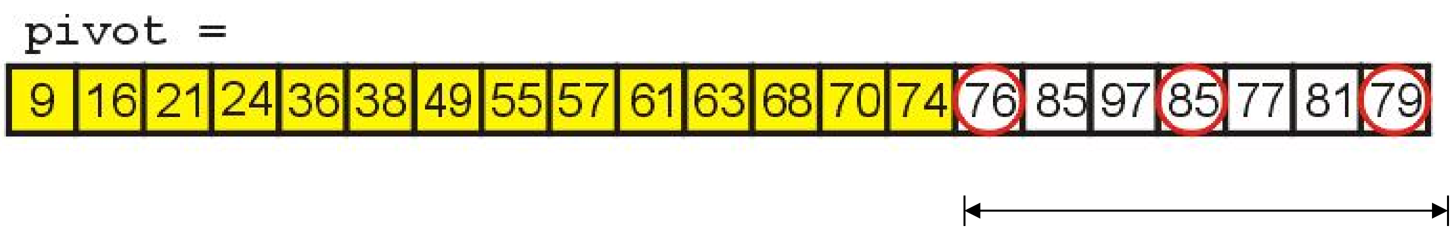
Improved Quick Sort Example

- **Having sorted the four elements, we focus on the remaining sub-list of seven entries**



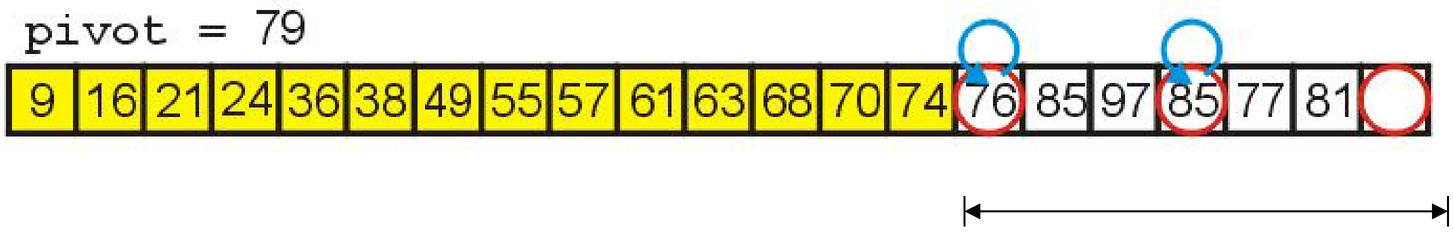
Improved Quick Sort Example

- To sort the next sub-list, we examine the first, middle, and last entries



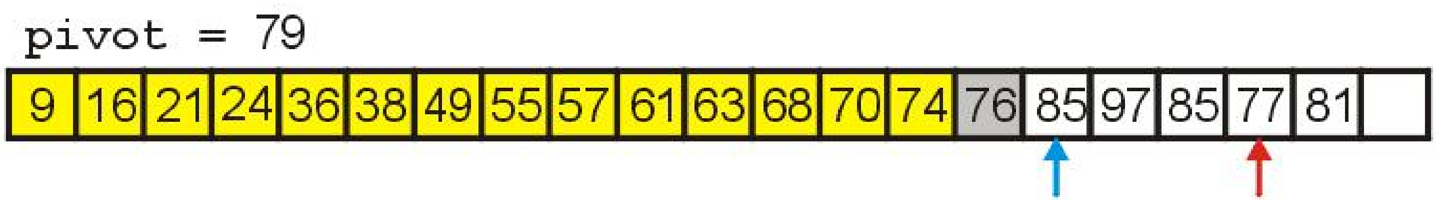
Improved Quick Sort Example

- We select 79 as our pivot and move:
- 76 into the lowest position
- 85 into the highest position



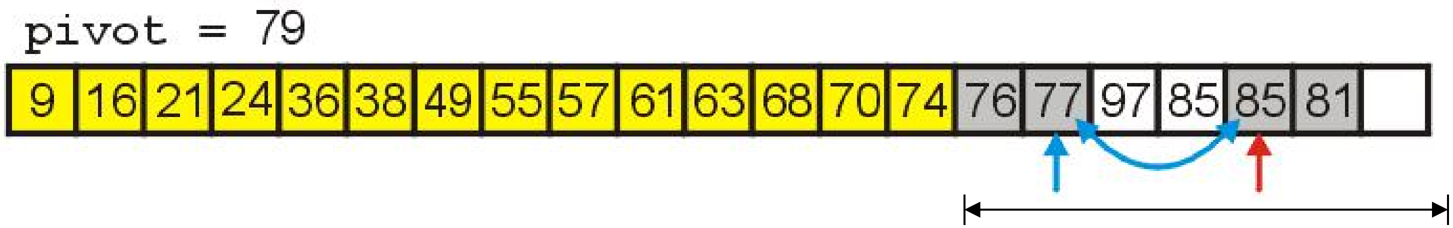
Improved Quick Sort Example

- **We search forward till we find** **85 > 79**
- **We search backward till we find** **77 < 79**



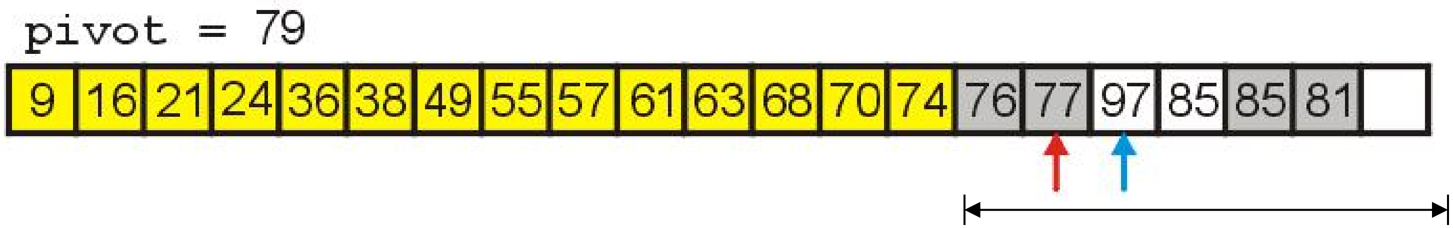
Improved Quick Sort Example

- We swap **85** and **77**, placing them in order



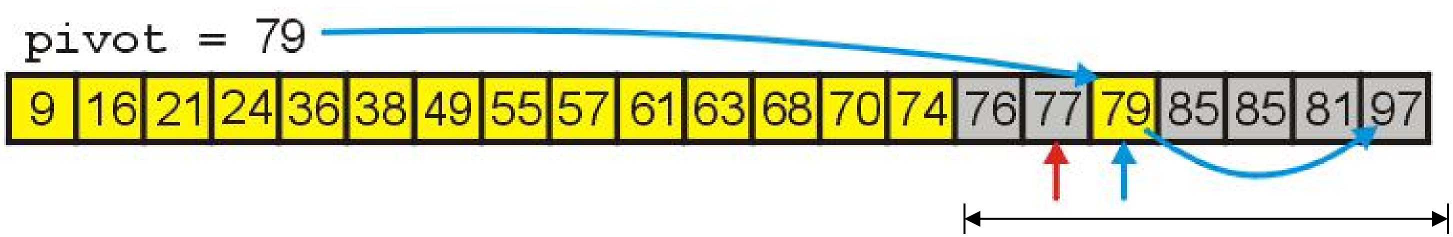
Improved Quick Sort Example

- We search forward till we find 97 > 79
- We search backward till we find 77 < 79
- The indices are reversed, so we stop



Improved Quick Sort Example

- Finally, we move 97 to the vacant location and copy 79 into the appropriate location
- 79 is now in the correct location



Improved Quick Sort Example

- **This splits the sub-list into two sub-lists of size 2 and 4**
- **We use insertion sort for the first sub-list**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	77	79	85	85	81	97

Improved Quick Sort Example

- **We are left with one sub-list with four entries, so again, we use insertion sort**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	77	79	85	85	81	97

Improved Quick Sort Example

- **Sorting the last sub-list, we arrive at an ordered list**

9	16	21	24	36	38	49	55	57	61	63	68	70	74	76	77	79	81	85	85	97

Searching k-th smallest element

Input: A[1…n] unsorted array
Output: index of the k-th smallest element
($1 \leqq k \leqq n$)

Algorithm 1:
Sorting A
Time complexity: O(nlog n)

Algorithm 2:
Sequential selection
Time complexity: O(kn)

Algorithm 3:
Quick Search

Quick Search

```
QuickSearch(A, p, r, k)
    if(p = r) return p        //p=r=k
    t ← Partition(A, p, r)
    if(t = k) return t

    if( k < t)
        QuickSearch(A, p, t-1, k)
                // find k-th smallest element in A[p..t-1]
    else
        QuickSearch(A, t+1, r, k)
                // find k-th smallest element in A[t+1..r]
```

Quick Search

```
QuickSearch(A, p, r, k)
    if(p = r) return p        //p=r=k
    t ← Partition(A, p, r)
    if(t = k) return l

    if( k < l)
        QuickSearch(A, p, t-1, k)
                // find k-th smallest element in A[p..t-1]
    else
        QuickSearch(A, t+1, r, k)
                // find k-th smallest element in A[t+1..r]
```

Time complexity

Worst case: $O(n^2)$

Best case: O(1)

Average case: O(n) ?

- **If we choose a random pivot, this will, on average, divide a set of n items into two sets of size $\frac{n}{4}$ and $\frac{3n}{4}$.**
  - **90 % of the time the width will have a ratio of 1:19 or better.**

The Memory Requirement

- **The additional memory required is O(log n)**
- **Each recursive function call places its local variables, parameters, etc., on a stack**
  - **The depth of the recursion tree is O(log n)**
  - **Unfortunately, if the run time is O(n^2), the memory use is O(n)**

Run Time Summery

- To summarize all three $O(n \log n)$ algorithms

	Average Run Time	Worst-case Run Time	Average Memory	Worst-case Memory
Heap Sort	$O(n \log n)$		$O(1)$	
Merge Sort	$O(n \log n)$		$O(n)$	
Quick Sort	$O(n \log n)$	$O(n^2)$	$O(\log n)$	$O(n)$

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End of Section.