INTRODUCTION TO LANGUAGES AND FINITE AUTOMATA

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Given a task, two questions arise:

1. Can it be carried out using a computer?

2. How can the task be carried out?

Models of computation are used to help answer these questions.

Types of structures used in models of computation:

- Grammars

Types of structures used in models of computation:

- Grammars
  - Grammars are used to generate the words of a language and to determine whether a word is in a language.
  - Formal languages, which are generated by grammars, provide models for both natural languages, such as English, and for programming languages, such as Pascal, Fortran, Prolog, C, and Java.
  - American linguist Noam Chomsky in the 1950s.

- Finite-state machines

- Finite-state machines
  - Various types of finite-state machines are used in modeling.

- Turing machines

Finite-State Machines

- Many kinds of machines, including components in computers, can be modeled using a structure called a finite-state machine.
- All versions of finite-state machines include a finite set of states
  1. a designated starting state, an input alphabet,
  2. a transition function that assigns a next state to every state
  3. input pair
- Applications in computer science and data networking.
- Spell checking, grammar checking, indexing or searching large bodies of text etc.

Finite-State Machines with Output

- Finite-state machines that produce output.
- We will see how finite-state machines can be used to model a vending machine, a machine that delays input, a machine that adds integers, and a machine that determines whether a bit string contains a specified pattern.

Definition 1

- A finite-state machine $M = (S, I, O, f, g, s_0)$ consists of a finite set S of states, a finite input alphabet I, a finite output alphabet O, a transition function f that assigns to each state and input pair a new state, an output function g that assigns to each state and input pair an output, and an initial state s_0.

Representation

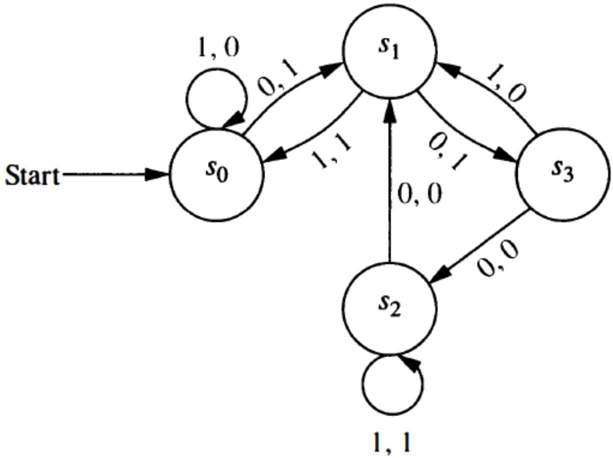
- Let $M = (S, I, O, f, g, s_0)$ be a finite-state machine. We can use a **state table** to represent the values of the transition function f and the output function g for all pairs of states and input.
- Another way to represent a finite-state machine is to use a **state diagram**, which is a directed graph with labeled edges. In this diagram, each state is represented by a circle. Arrows labeled with the input and output pair are shown for each transition.

Example 1

- The state table shown in Table 2 describes a finite-state machine with $S = \{s_0, s_1, s_2, s_3\}$, $I = \{0, 1\}$, and $O = \{0, 1\}$. The values of the transition function f are displayed in the first two columns, and the values of the output function g are displayed in the last two columns.

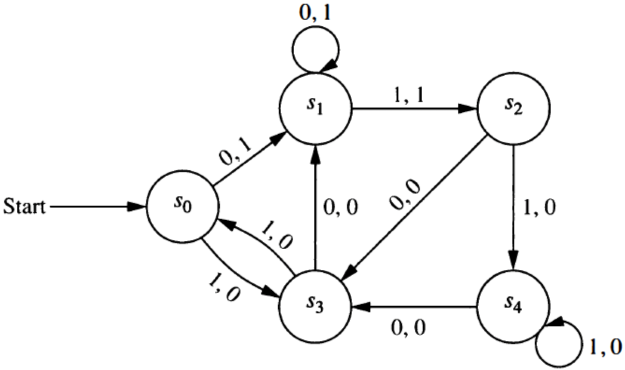
TABLE 2

	f		*g*	
	Input		***Input***	
State	**0**	**1**	**0**	**1**
s_0	s_1	s_0	1	0
s_1	s_3	s_0	1	1
s_2	s_1	s_2	0	1
s_3	s_2	s_1	0	0


Start
s_0
s_1
s_2
s_3
1, 0
0, 1
1, 1
0, 0
0, 1
1, 0
0, 0
1, 1

Example 2

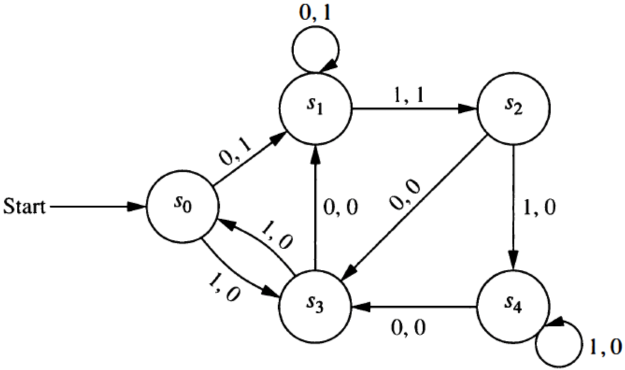
- Construct the state table for the finite-state machine with the state diagram shown in figure below.



	f		g	
	Input		Input	
State	**0**	**1**	**0**	**1**
s_0	s_1	s_3	1	0
s_1	s_1	s_2	1	1
s_2	s_3	s_4	0	0
s_3	s_1	s_0	0	0
s_4	s_3	s_4	0	0

Example 3

- Find the output string generated by the finite-state machine in if the input string is 101011.



Input	1	0	1	0	1	1	—
State	s_0	s_3	s_1	s_2	s_3	s_0	s_3
Output	0	0	1	0	0	0	—

Definition 2

- Let $M = (S, I, O, f, g, s_0)$ be a finite-state machine and $L \subseteq I^*$. We say that M recognizes(or accepts) L if an input string x belongs to L if and only if the last output bit produced by M when given x as input is a 1.

Example 4

- In a certain coding scheme, when three consecutive 1 s appear in a message, the receiver of the message knows that there has been a transmission error. Construct a finite-state machine that gives a 1 as its current output bit if and only if the last three bits received are all 1s.

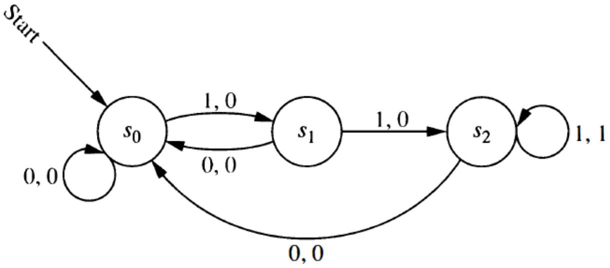
Example 4

- Three states are needed in this machine:
- The start state s_0 remembers that the previous input value, if it exists, was not a 1.
- The state s_1 remembers that the previous input was a 1, but the input before the previous input, if it exists, was not a 1.
- The state s_2 remembers that the previous two inputs were 1s.

Example 4 Ctd.

Input 1: 1011

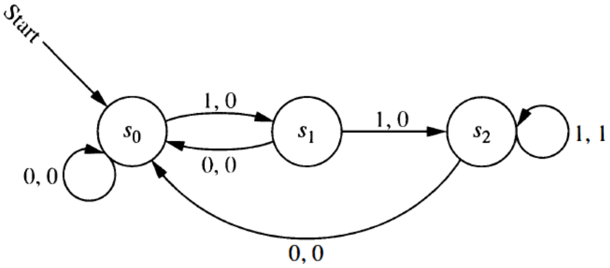
Input 2: 0100111



Example 4 Ctd.

Input 1: 1011

Input 2: 0100111



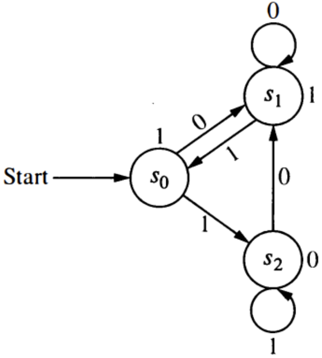
The final output bit of the finite-state machine we constructed here is 1 if and only if the input string ends with 111 .

Types of Finite-State Machines

- Mealy machines: outputs correspond to transitions between states.
- Moore machine: output is determined only by the state.

Example of Moore Machine

Construct the state table for the Moore machine with the state diagram shown here.



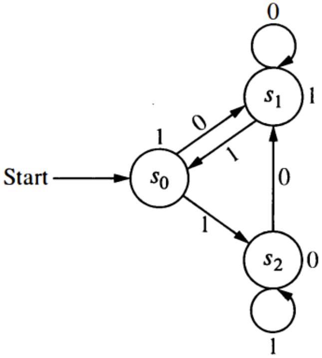
Example of Moore Machine

	f		
	Input		
State	**0**	**1**	g
s_0	s_1	s_2	1
s_1	s_1	s_0	1
s_2	s_1	s_2	0

Example of Moore Machine

Inputs: a) 0101

b) 111111



Strings

- Sequences of the form $a_1, a_2, ..., a_n$ or $a_1 a_2 ... a_n$ are called strings.
- The length of the string S is the number of terms in this string.
- The **empty string**, denoted by λ, is the string that has no terms. The empty string has length zero.

Vocabulary

- A vocabulary (or alphabet) V is a finite, nonempty set of elements called symbols.
- A word (or sentence) over V is a string of finite length of elements of V.
- The empty string or null string, denoted by λ, is the string containing no symbols.
- The set of all words over V is denoted by V^*. A language over V is a subset of V^*.

Definition: Concatenation

- Suppose that A and B are subsets of V^*, where V is a vocabulary. The concatenation of A and B, denoted by AB, is the set of all strings of the form xy, where x is a string in A and y is a string in B.

Example

- Let $A = \{0, 11\}$ and $B = \{1, 10, 110\}$. Find AB and BA.
- Let $A = \{1, 00\}$. Find A^n for $n = 0, 1, 2, 3$.

Definition: Kleene closure of A

- Suppose that A is a subset of V^*. Then the Kleene closure of A, denoted by A^*, is the set consisting of concatenations of arbitrarily many strings from A. That is, $A^* = \bigcup_{k=0}^{\infty} A^k$.

- Example: What are the Kleene closures of the sets $A = \{0\}$, $B = \{0, 1\}$, and $C = \{11\}$?

Solution

- The Kleene closure of A is the concatenation of the string 0 with itself an arbitrary finite number of times. Hence,
  $A^* = \{0^n : n = 0, 1, 2, 3...\}$

- The Kleene closure of B is the concatenation of an arbitrary number of strings, where each string is either 0 or 1. This is the set of all strings over the alphabet $V = \{0, 1\}$.

- The Kleene closure of C is the concatenation of the string 11 with itself an arbitrary number of times. Hence, C^* is the set of strings consisting of an even number of 1s.
  That is, $C^* = \{1^{2n} | n = 0, 1, 2, ...\}$.

Finite-State Automata

- A finite-state machine with no output is called finite-state automaton.

- A finite-state automaton $M = (S, I, f, s_0, F)$ consists of a finite set S of states, a finite input alphabet I, a transition function f that assigns a next state to every pair of state and input (so that $f : S \times I \to S$), an initial or start state s_0, and a subset F of S consisting of final (or accepting states).

- We can represent finite-state automata using either state tables or state diagrams. Final states are indicated in state diagrams by using **double circles**.

Exercise

- Construct the state diagram for the finite-state automaton $M = (S, I, f, s_0, F)$, where $S = \{s_0, s_1, s_2, s_3\}$, $I = \{0, 1\}$, $F = \{s_0, s_3\}$, and the transition function f is given in Table 1.

TABLE 1

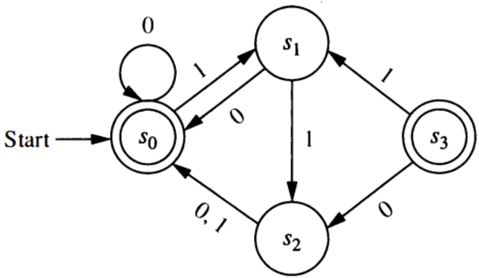
	f	
	Input	
State	**0**	**1**
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_0
s_3	s_2	s_1

Exercise

- Construct the state diagram for the finite-state automaton $M = (S, I, f, s_0, F)$, where $S = \{s_0, s_1, s_2, s_3\}$, $I = \{0, 1\}$, $F = \{s_0, s_3\}$, and the transition function f is given in Table 1.

TABLE 1

State	***f*** Input 0	***f*** Input 1
s_0	s_0	s_1
s_1	s_0	s_2
s_2	s_0	s_0
s_3	s_2	s_1


0
1
0
1
1
0
0, 1
Start
s_0
s_1
s_2
s_3

EXTENDING THE TRANSITION FUNCTION

- The transition function I of a finite-state machine $M = (S, I, f, s_0, F)$ can be extended so that it is defined for all pairs of states and strings.
  i.e. f can be extended to a function $f : S \times I^* \to S$.

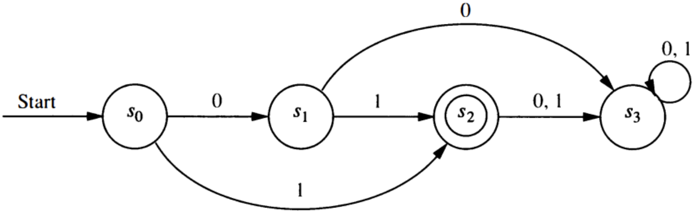
- Let $x = x_1 x_2 ... x_k$ be a string in I^*. Then $f(s_1, x)$ is the state obtained by using each successive symbol of x , from left to right, as input, starting with state s_1. From s_1 we go on to state $s_2 = f(s_1, x_1)$, then to state $s_3 = f(s_2, x_2)$, and so on, with $f(s_1, x) = f(s_k, x_k)$.

Language Recognition by Finite-State Machines

- A string x is said to be recognized or accepted by the machine $M = (S, I, f, s_0, F)$ if it takes the initial state s_0 to a final state. i.e. $f(s_0, x)$ is a state in F.

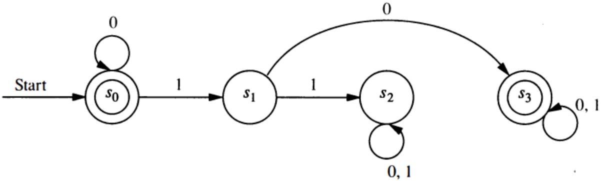
- Two finite-state automata are called equivalent if they recognize the same language.

Determine the languages recognized by the finite-state automaton



Ans: $L(M) = \{0, 01\}$

Determine the languages recognized by the finite-state automaton

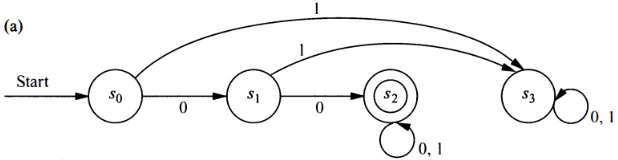


Ans: $L(M) = \{0^n, 0^n 10x : n = 0, 1, 2, ...and$x is any string$\}$

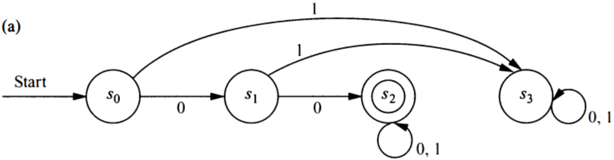
Deterministic Finite-State Automaton

Deterministic finite-state automaton is a finite-state machine in which for each pair of state and input value there is a unique next state given by the transition function.

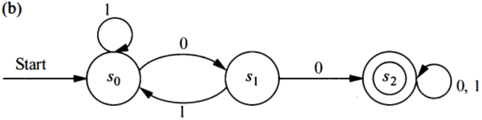
- Construct deterministic finite-state automata that recognize each of these languages.
  - a the set of bit strings that begin with two 0s
  - b the set of bit strings that contain two consecutive 0s
  - c the set of bit strings that do not contain two consecutive 0s
  - d the set of bit strings that end with two 0s
  - e the set of bit strings that contain at least two 0s


(a)
Start
s_0
s_1
s_2
s_3
0
0
1
1
0, 1
0, 1

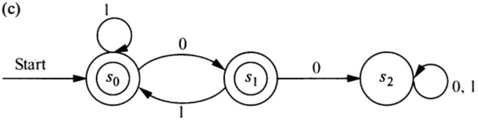
(a)



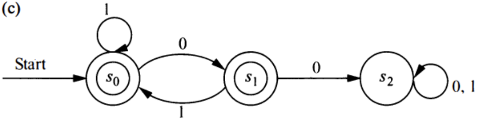
(b)



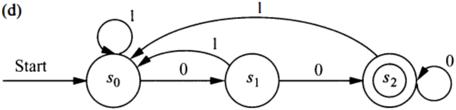
(c)



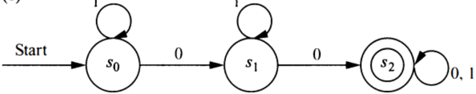
(c)



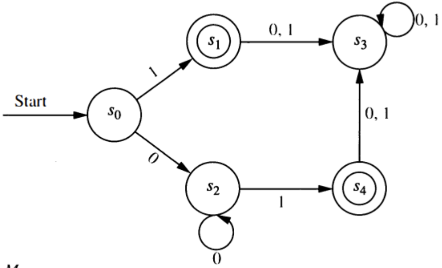
(d)



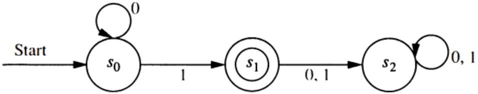
(e)



- Show that the two finite-state automata M_0 and M_1 in the figures are equivalent.



M_0



M_1

Solution: Let x be any string.

For M_0 to recognise x, x must reach one of the final stages, s_1 or s_4, from the starting stage s_0. Now, the only string that reaches s_1 from s_0 is 1.

The strings that will reach s_4 from s_0 should reach s_2 and then from s_2 to s_4.

The strings from s_0 to reach s_4 should start with 0, followed by a 1 or start with 0 followed by more additional 0s, which keep the machine in state s_2, followed by a 1 to reach the final stage, s_4.

All other strings takes the machine from s_0 to a state that is not final.

$\therefore$

$$L(M_0) = \{1, 0^n 1 : n = 1, 2, ...\} \tag{1}$$

Solution ctd: For a string x to be recognized by M_1, x must take the machine from s_0 to the final state, i.e. s_1.
If x is just 1, it reaches s_1 from s_0.
If x starts with a number of 0s, which leave the machine in state s_0, followed by a 1, which takes it to the final state s_1. A string of all 0s is not recognized because it leaves the machine in state s_0, which is not final. All strings that contain a start with 1 followed by a 0 are not recognized because it will reach s_2, which is not final. $\therefore$

$$L(M_1) = \{1, 0^n 1 : n = 1, 2, ...\} \tag{2}$$

From (1) and (2) we can conclude that the finite state machines M_0 and M_1 recognise the same language. Hence by the definition of equivalent machines, M_0 and M_1 are equivalent.

Nondeterministic Finite-State Automaton

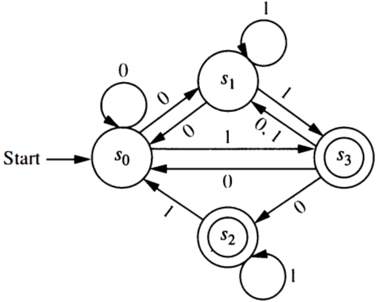
A nondeterministic finite-state automaton $M = (S, I, f, s_0, F)$ consists of a set S of states, an input alphabet I, a transition function f that assigns a set of states to each pair of state and input (so that $f : S \times I \to P(S)$), a starting state s_0, and a subset F of S consisting of the final states.

Example

Find the state diagram for the nondeterministic finite-state automaton with the state table shown below.

	f	
	Input	
State	0	1
s_0	s_0, s_1	s_3
s_1	s_0	s_1, s_3
s_2		s_0, s_2
s_3	s_0, s_1, s_2	s_1

Example



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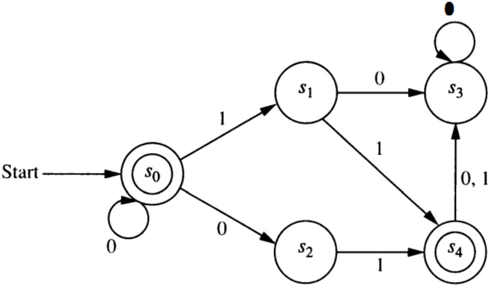
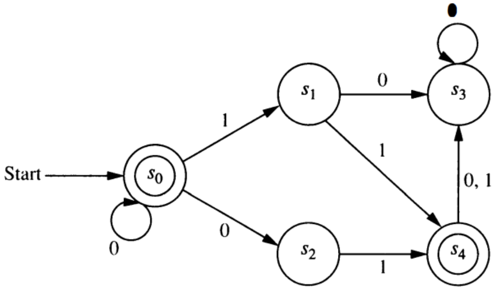


Example

	f	
	Input	
State	**0**	**1**
s_0	$s_0,\ s_2$	s_1
s_1	s_3	s_4
s_2		s_4
s_3	s_3	
s_4	s_3	s_3

Example

Find the language recognized by the nondeterministic finite-state automaton, M_2 .

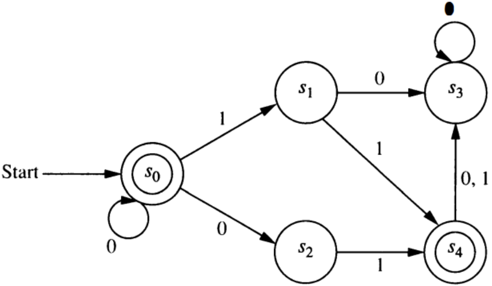


Theorem

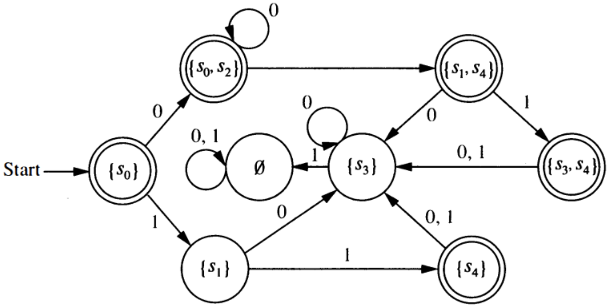
If the language L is recognized by a nondeterministic finite-state automaton M_0, then L is also recognized by a deterministic finite-state automaton M_1.

Example

Find a deterministic finite-state automaton that recognizes the same language as the nondeterministic finite-state automaton given below.

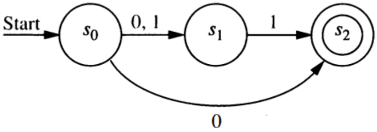


Solution



Example

Find a deterministic finite-state automaton that recognizes the same language as the nondeterministic finite-state automaton given below.



Example

Find a deterministic finite-state automaton that recognizes the same language as the nondeterministic finite-state automaton given below.