

**Exercise 1**

a) Linear.

$$\begin{aligned}
 f(\lambda u + \mu v) &= \lambda f(u) + \mu f(v) \\
 f\left(\begin{pmatrix} \lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \\ \lambda u_3 + \mu v_3 \end{pmatrix}\right) &= \begin{pmatrix} \lambda(u_1 + 3u_2 + 4u_3) \\ \lambda u_3 \end{pmatrix} + \begin{pmatrix} \mu(v_1 + 3v_2 + 4v_3) \\ \mu v_3 \end{pmatrix} \\
 \begin{pmatrix} \lambda u_1 + \mu v_1 + 3(\lambda u_2 + \mu v_2) + 4(\lambda u_3 + \mu v_3) \\ \lambda u_3 + \mu v_3 \end{pmatrix} &= \begin{pmatrix} \lambda(u_1 + 3u_2 + 4u_3) + \mu(v_1 + 3v_2 + 4v_3) \\ \lambda u_3 + \mu v_3 \end{pmatrix}
 \end{aligned}$$

b) Not linear. The first equation produces  $u_1v_2$  and  $v_1u_2$  in the first row, which has no chance of happening in the second equation; they are not equal.

$$f(u+v) = f\left(\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}\right) = \begin{pmatrix} u_1+v_1 + 2(u_2+v_2) + (u_1+v_1)(u_2+v_2) + u_3+v_3 \\ u_1+v_1 + u_2+v_2 + u_3+v_3 \end{pmatrix}$$

$$\begin{aligned}
 f(u) + f(v) &= \begin{pmatrix} u_1 + 2u_2 + u_1u_2 + u_3 \\ u_1 + u_2 + u_3 \end{pmatrix} + \begin{pmatrix} v_1 + 2v_2 + v_1v_2 + v_3 \\ v_1 + v_2 + v_3 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 + 2u_2 + u_1u_2 + u_3 + v_1 + 2v_2 + v_1v_2 + v_3 \\ u_1 + u_2 + u_3 + v_1 + v_2 + v_3 \end{pmatrix}
 \end{aligned}$$

c) Not linear because  $\lambda f(u) + \mu f(v)$  will necessarily have  $\lambda + \mu$  in it. The statements are not equal.

$$f(\lambda u + \mu v) = f\left(\begin{pmatrix} \lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \\ \lambda u_3 + \mu v_3 \end{pmatrix}\right) = \lambda u_1 + \mu v_1 + \lambda u_2 + \mu v_2 + \lambda u_3 + \mu v_3 + 1$$

d) Not linear.

$$f(\lambda u + \mu v) = f\left(\begin{pmatrix} \lambda u_1 + \mu v_1 \\ \lambda u_2 + \mu v_2 \end{pmatrix}\right) = \begin{pmatrix} \lambda u_2 + \mu v_2 \\ \lambda u_1 + \mu v_1 \end{pmatrix} = \lambda \begin{pmatrix} u_2 \\ u_1 \end{pmatrix} + \mu \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} \neq \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mu \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

**Exercise 3**

a)

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

b)

**Exercise 4**

- a) The vectors in both B and C are linearly independent and thus both form a base of  $R^2$ .

$$\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array} \quad \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array} \quad \text{and} \quad \begin{array}{cc|c} 2 & 3 & 0 \\ 1 & -2 & 0 \end{array} \quad \begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -3.5 & 0 \end{array}$$

- b) Since

$$\begin{array}{cc|c} 2 & 3 & 1 \\ 1 & -2 & 1 \end{array} \quad \begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \end{array} \quad \text{with } x_1 = \frac{5}{7} \text{ and } x_2 = -\frac{1}{7}$$

$$\begin{array}{cc|c} 2 & 3 & -1 \\ 1 & -2 & 1 \end{array} \quad \begin{array}{cc|c} 2 & 3 & -1 \\ 0 & -\frac{7}{2} & \frac{3}{2} \end{array} \quad \text{with } x_1 = \frac{1}{7} \text{ and } x_2 = -\frac{3}{7}$$

we have

$$A_C^B = \begin{pmatrix} \frac{5}{7} & \frac{1}{7} \\ -\frac{1}{7} & -\frac{3}{7} \end{pmatrix}.$$

- c)

$$\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$

- d)

$$\begin{array}{cc|c} 1 & -1 & 2 \\ 1 & 1 & 1 \end{array} \quad \text{and} \quad \begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & -2 \end{array}$$

$$\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$