Exercise 1

a) We have

r	x	y	q
135	1	0	
54	0	1	2
27	1	-2	2
0	-2	5	

and thus $\gcd(135, 54) = 27$. Since $27 \mid 0$, there is a solution.

From the last row we know that $125 \cdot -2 + 54 \cdot 5 = 0$, thus $(-2, 5) \in L$. We can now describe L as $L = \{(-2k, 5k) \mid k \in \mathbb{Z}\}$.

b) We have (note that x and y are reversed)

r	y	x	q
105	1	0	
99	0	1	1
6	1	-1	16
3	-16	17	2
0	33	-35	

and thus $\gcd(105, 99) = 3$. Since $3 \mid 12$, there is a solution.

From the second to last row we know

$$\begin{aligned} 3 &= (17 \cdot 99) + (-16 \cdot 105) \quad \text{and, after multiplying by 4} \\ 12 &= (68 \cdot 99) + (-64 \cdot 105). \end{aligned}$$

Thus we have that $(68, -64) \in L$ and further $L = \{(68 - 35k, -64 + 33k) \mid k \in \mathbb{Z}\}$.

c) We have

q	x	y	r
38	1	0	
19	0	1	2
0	1	-2	

and thus $\gcd(38, 19) = \gcd(19, -38) = 19$. Since $19 \nmid 5$ this equation does not have a solution.

Exercise 2 We are looking for solutions to

$$35x + 45y = 1000$$

where x is the number of linear Algebra books and y is the number of Analysis books. We have (note that x and y are reversed)

r	y	x	q
45	1	0	
35	0	1	1
10	1	-1	3
5	-3	4	2
0	7	-9	

and thus $\gcd(45, 35) = 5$. Since $5 \mid 1000$, there is a solution.

From the second to last row we know

$$\begin{aligned} 5 &= (4 \cdot 35) + (-3 \cdot 45) \quad \text{and, after multiplying by 200} \\ 1000 &= (800 \cdot 35) + (-600 \cdot 45) \end{aligned}$$

and thus $(800, -600) \in L$, allowing us to state $L = \{(800 \cdot -9k, -600 \cdot 7k) \mid k \in \mathbb{Z}\}$. For $86 \leq k \leq 88$ neither of the values in the pairs $\in L$ are negative. Thus we can either buy

Lineare Algebra	Analysis
26	2
17	9
8	16

books.

If the total available money were 1001 then we would have $5 \nmid 1000$, thus we would not be able to spend all of our budget.

Exercise 3 Idk.

Exercise 4 Interpreting the polynomials as being in $\mathbb{Z}_5$.

x^5	x^4	x^3	x^2	x^1	x^0		x^3	x^2	x^1	x^0		x^2	x^1	x^0
3	1	4	1	0	4	:	2	2	1	3	=	4	4	1
3	3	4	2											
	3	0	4	0	4									
	3	3	4	2										
		2	0	3	4									
		2	2	1	3									
			3	2	1									

Interpreting the polynomials as being in $\mathbb{Q}$.

x^5	x^4	x^3	x^2	x^1	x^0		x^3	x^2	x^1	x^0		x^2	x^1	x^0
3	1	4	1	5	9	:	2	7	1	8	=	1.5	-4.75	17.875
3	10.5	1.5	12											
	-9.5	3.5	-11	5	9									
	-9.5	-33.25	-4.75	-38										
		35.75	-6.25	43	9									
		35.75	125.12	17.975	143									
			-132.37	25.125	-134									

Exercise 5

a) Consider that a general table for the GCD is

r	u	v	q
P_1	1	0	
P_2	0	1	q_1
r_1	1	v_1	q_2
r_2	u_2	v_2	q_3
r_3	u_3	v_3	q_4

We begin by calculating q_1 and r_1.

$$\begin{array}{ccccccccccccccc}
x^5 & x^4 & x^3 & x^2 & x^1 & x^0 & & x^4 & x^3 & x^2 & x^1 & x^0 & & x^1 & x^0 \\
1 & 6 & 9 & -6 & -22 & -12 & : & 1 & 1 & -4 & -2 & 4 & = & 1 & 5 \\
1 & 1 & -4 & -2 & 4 & & & & & & & & & & \\
\cline{1-5}
 & 5 & 13 & -4 & -26 & -12 & & & & & & & & & \\
 & 5 & 5 & -20 & -10 & 20 & & & & & & & & & \\
\cline{2-6}
 & & 8 & 16 & -16 & -32 & & & & & & & & &
\end{array}$$

Thus $q_1 = x + 5$, $r_1 = 8x^3 + 16x^2 - 16x - 32$ and $v_1 = 0 - q_1 = -x - 5$.

We continue by calculating q_2 and r_2.

$$\begin{array}{ccccccccccccc}
x^4 & x^3 & x^2 & x^1 & x^0 & & x^3 & x^2 & x^1 & x^0 & & x^1 & x^0 \\
1 & 1 & -4 & -2 & 4 & : & 8 & 16 & -16 & -32 & = & \frac{1}{8} & -\frac{1}{8} \\
1 & 2 & -2 & -4 & & & & & & & & & \\
\cline{1-4}
 & -1 & -2 & 2 & 4 & & & & & & & & \\
 & -1 & -2 & 2 & 4 & & & & & & & & \\
\cline{2-5}
 & 0 & 0 & 0 & 0 & & & & & & & &
\end{array}$$

Thus $q_2 = \frac{1}{8}x - \frac{1}{8}$ and $r_2 = 0$. We have $\gcd(P_1, P_2) = r_1 = 8x^3 + 16x^2 - 16x - 32$.

b) If $\frac{a}{b}$ is a root of $\gcd(P_1, P_2)$ then a must be a divisor of 32 and b must be a divisor of 8. The candidates are thus

$$\pm 1 \qquad \boxed{-2} \qquad \pm 4 \qquad \pm 8 \qquad \pm 16 \qquad \pm 32 \qquad \pm\frac{1}{2} \qquad \pm\frac{1}{4} \qquad \pm\frac{1}{8}$$

where boxed numbers are actual roots. We can thus factor out $x + 2$ by division.

$$\begin{array}{ccccccccccc}
x^3 & x^2 & x^1 & x^0 & & x^1 & x^0 & & x^2 & x^1 & x^0 \\
8 & 16 & -16 & -32 & : & 1 & 2 & = & 8 & 0 & -16 \\
8 & 16 & & & & & & & & & \\
\cline{1-2}
 & & -16 & -32 & & & & & & & \\
 & & -16 & -32 & & & & & & & \\
\cline{3-4}
\end{array}$$

We can now solve $8x^2 - 16 = 0$ through

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm\frac{\sqrt{-4 \cdot 8 \cdot -16}}{2 \cdot 8} = \pm\frac{\sqrt{512}}{16} = \pm\frac{16\sqrt{2}}{16} = \pm\sqrt{2}.$$

The roots are thus $-2, -\sqrt{2}$ and $\sqrt{2}$.

Exercise 6 We are looking for the rational roots of

$$P(x) = 18x^6 - 51x^5 - 7x^4 + 106x^3 - 62x^2 - 8x + 8$$

Using the fact that, if $\frac{a}{b}$ is a root of a polynom then $a \mid a_0$ and $b \mid a_n$, we get

$$\pm 1 \quad \boxed{2} \quad \pm 4 \quad \pm 8 \quad \boxed{\frac{1}{2}} \quad \boxed{-\frac{1}{3}} \quad \pm\frac{1}{6} \quad \pm\frac{1}{9} \quad \pm\frac{1}{18} \quad \boxed{\frac{2}{3}} \quad \pm\frac{2}{9} \quad \pm\frac{2}{18} \quad \pm\frac{4}{3} \quad \pm\frac{4}{9} \quad \pm\frac{8}{3} \quad \pm\frac{8}{9}$$

as potential roots. Boxed numbers are actual roots.

We can thus factor out

$$\left(x + \frac{1}{3}\right)\left(x - \frac{1}{2}\right)\left(x - \frac{2}{3}\right)(x - 2) = x^4 - \frac{17}{6}x^3 + \frac{29}{18}x^2 + \frac{2}{9}x - \frac{2}{9}$$

by division.

$$\begin{array}{ccccccccccccccccc}
x^6 & x^5 & x^4 & x^3 & x^2 & x^1 & x^0 & & x^4 & x^3 & x^2 & x^1 & x^0 & & x^2 & x^1 & x^0 \\
18 & -51 & -7 & 106 & -62 & -8 & 8 & : & 1 & -\frac{17}{6} & \frac{29}{18} & \frac{2}{9} & -\frac{2}{9} & = & 18 & 0 & -36 \\
18 & -51 & 29 & 4 & -4 & & & & & & & & & & & & \\
\cline{1-5}
 & & -36 & 102 & -58 & -8 & 8 & & & & & & & & & & \\
 & & -36 & 102 & -58 & -8 & 8 & & & & & & & & & & \\
\cline{3-7}
\end{array}$$

We can now solve $18x^2 - 36 = 0$ through

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm\frac{\sqrt{-4 \cdot 18 \cdot -36}}{2 \cdot 18} = \pm\frac{\sqrt{2592}}{36} = \pm\frac{36\sqrt{2}}{36} = \pm\sqrt{2},$$

yielding no rational roots. The rational roots are thus these obtained previously.

Exercise 7 We have

$$p(x) = x^7 - 6x^6 + 10x^5 - 6x^4 + 9x^3$$
$$p'(x) = 7x^6 - 36x^5 + 50x^4 - 24x^3 + 27x^2$$

and we are looking for a square-free factorisation of p. Calculating the GCD of p and p' we first divide p by p'

$$\begin{array}{cccccccccccccccc}
x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & & x^6 & x^5 & x^4 & x^3 & x^2 & & x^1 & x^0 \\
1 & \text{-6} & 10 & \text{-6} & 9 & 0 & : & 7 & \text{-36} & 50 & \text{-24} & 27 & = & \frac{1}{7} & -\frac{6}{49} \\
1 & -\frac{36}{7} & \frac{50}{7} & -\frac{24}{7} & \frac{27}{7} & & & & & & & & & & \\
\hline
 & -\frac{6}{7} & \frac{20}{7} & -\frac{17}{7} & \frac{36}{7} & 0 & & & & & & & & & \\
 & -\frac{6}{7} & \frac{216}{49} & -\frac{300}{49} & \frac{144}{49} & -\frac{162}{49} & & & & & & & & & \\
\hline
 & & -\frac{76}{49} & \frac{174}{49} & \frac{108}{49} & \frac{162}{49} & & & & & & & & &
\end{array}$$

and get $r_1 = -\frac{76x^5}{49} + \frac{174x^4}{49} + \frac{108x^3}{49} + \frac{162x^2}{49}$ and $q_1 = \frac{x}{7} - \frac{6}{49}$. We can simplify $r_1 = -38x^5 + 87x^4 + 54x^3 + 81x^2$.

Now we divide p' by r_1

$$\begin{array}{cccccccccccccc}
x^6 & x^5 & x^4 & x^3 & x^2 & & x^5 & x^4 & x^3 & x^2 & & x^1 & x^0 \\
7 & \text{-36} & 50 & \text{-24} & 27 & : & -38 & 87 & 54 & 81 & = & -\frac{7}{38} & \frac{759}{1444} \\
7 & \frac{126}{19} & \frac{175}{19} & \frac{84}{19} & -\frac{189}{38} & & & & & & & & \\
\hline
 & -\frac{810}{19} & \frac{775}{19} & -\frac{540}{19} & \frac{1215}{38} & & & & & & & & \\
 & -\frac{810}{19} & \frac{66033}{1444} & \frac{20493}{722} & \frac{61479}{1444} & & & & & & & & \\
\hline
 & & \frac{20531x^4}{1444} & -\frac{13524x^3}{361} & -\frac{22491x^2}{1444} & & & & & & & &
\end{array}$$

and get $r_2 = \frac{20531x^4}{1444} - \frac{13524x^3}{361} - \frac{22491x^2}{1444}$. We then divide r_1 by r_2 and get $r_3 = \frac{11696400x^2}{175561} - \frac{3898800x^3}{175561}$. We then divide r_2 by r_3 and get $r_4 = 0$.

Thus r_3, which can be simplified to $x^3 - 3x^2$, is the GCD we are looking for.

We now divide p by this result which yields $x^4 - 3x^3 + x^2 - 3x$ with no remainder.

Exercise 8 To show that $p \mid \binom{p}{k}$ note that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$
$$p! = \binom{p}{k}(k!(p-k)!).$$

Since the left hand side of the equation is clearly divisible by p, the right hand side must also be divisible by it. The expression $k!(p-k)!$ is not divisible by p since it is a product of numbers smaller than p and p is prime. Thus the binomial coefficient must be the part which is divisible by p.

Now consider that, by the binomial theorem

$$(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^{p-k} y^k$$
$$(x+y)^p = \binom{p}{0} x^p y^0 + \binom{p}{1} x^{p-1} y^1 + \cdots + \binom{p}{p-1} x^1 y^{p-1} + \binom{p}{p} x^0 y^p$$
$$(x+y)^p = x^p + \binom{p}{1} x^{p-1} y^1 + \cdots + \binom{p}{p-1} x^1 y^{p-1} + y^p$$
$$(x+y)^p \equiv_6 x^p + y^p$$