Vertex Cover is NP-complete

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A vertex cover of a graph is a set of vertices that includes at leat one endpoint of every edge of the graph[1]. The VERTEX COVER problem is: Given a graph G and an integer k, does G have a vertex cover of size $\leq k$?

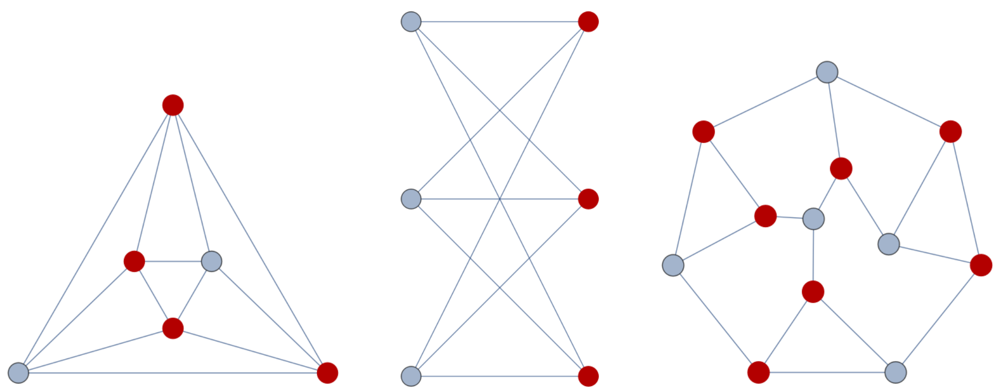

Figure 1: Various graphs with a highlighted minimum vertex cover.

We have chosen this problem for its rich representation in the literature and its reasonably straightforward proof of **NP**-completeness. We will first show that another decision problem, INDEPENDENT SET, is **NP**-complete by giving a reduction from 3-SAT, and then demonstrate a simple reduction from it to VERTEX COVER. We realize that this is not (exactly) what was required but believe that it is within the spirit of the exercise.

An independent set is a set of vertices in a graph, no two of which are adjacent[2]. The INDEPENDENT SET problem is: Given a graph G and an integer k, does G have an independet set of size $\geq k$?

Lemma INDEPENDENT SET is **NP**-complete.[3]

We can verify that a given set $S \subset V$ is independent in a graph $G(V, E)$ by checking for all $v_1, v_2 \in S$ that $(v_1, v_2) \notin V$. This can be done in polytime. We can check that $|S| \geq k$ in constant time. Thus INDEPENDENT SET is in **NP**.

We will now reduce from 3-SAT. Let F be an arbitrary instance of 3-SAT with clauses $C_1, \ldots C_k$ in the variables $x_1, \ldots x_n$. We construct the graph G such that it has a triangle (three interconnected nodes) for each clause C_i. Nodes of these triangles are labeled with the terms of its clause (either x_i or $\overline{x_i}$). We then add edges between every pair of nodes $(x_i, \overline{x_i})$.

If F is satisfiable there is a true term in each triangle. We pick that node from each triangle, leading to an independent set of size k. These nodes are independent because we only picked one node

[1] `https://en.wikipedia.org/wiki/Vertex_cover`

[2] `https://en.wikipedia.org/wiki/Independent_set_(graph_theory)`

[3] See `https://www.cs.cmu.edu/~15451-f17/lectures/lec23-np.pdf`, p.5 and `http://www.cs.cornell.edu/courses/cs482/2005su/handouts/NPComplete`, p.2.

from each triangle, and the only edges between triangles go between nodes with labels x_i and $\overline{x_i}$ (which can't both be satisfied). So if and only if there is a satisifiable assignment, we will find an independent set of size k.

Theorem VERTEX COVER is **NP**-complete.[4]

We can verify that a given set S covers a graph $G(V, E)$ by checking that for all $(u, v) \in E$ we have either $u \in S$ or $v \in S$ and $|S| \leq k$. This is possible in polytime. Thus VERTEX COVER is in **NP**.

We will reduce from INDEPENDENT SET. Consider that if C is a vertex cover in a graph $G(V, E)$ then $V \backslash S$ is an independent set – there cannot be an edge between any two vertices in $V \backslash S$ because otherwise C would not cover all edges. In the same spirit, if S is an independent set then $V \backslash S$ is a vertex cover.

This allows us to transform a given instance (G, k) of INDEPENDENT SET to an instance $(G, |V| - k)$ for VERTEX COVER. This works because we have just established that "is there an independent set of size $\geq k$" is equivalent to "is there a vertex cover of size $\leq |V| - k$".

[4] See `https://www.cs.cmu.edu/~15451-f17/lectures/lec23-np.pdf`, p.5 (again) and `http://cs.williams.edu/~shikha/teaching/spring20/cs256/lectures/Lecture22.pdf`, p.17 (goes in the other direction, but still useful as reference).