

Binary relations  $R$  operating on a set  $A$  are said to be asymmetric, antisymmetric or irreflexive according to the following definitions for arbitrary but fixed  $R$  and  $A$ .

$$\text{asymmetric}(R, A) \iff \forall x, y \in A : R(x, y) \Rightarrow \neg R(y, x) \quad (1)$$

$$\text{antisymmetric}(R, A) \iff \forall x, y \in A : (R(x, y) \wedge R(y, x)) \Rightarrow (x = y) \quad (2)$$

$$\text{irreflexive}(R, A) \iff \forall x \in A : \neg R(x, x) \quad (3)$$

We show that, again for arbitrary but fixed  $R$  and  $A$ , the implication

$$\text{asymmetric}(R, A) \Rightarrow (\text{antisymmetric}(R, A) \wedge \text{irreflexive}(R, A))$$

holds by assuming that  $\text{asymmetric}(R, A)$  is true and showing that  $\text{antisymmetric}(R, A)$  and  $\text{irreflexive}(R, A)$  then hold.

It is elementary that, for arbitrary but fixed  $x, y \in A$  the statement  $R(x, y) \wedge R(y, x)$  in (2) is a contradiction.  $R(x, y)$  implies that  $\neg R(y, x)$ , therefore  $R(x, y) \wedge R(y, x)$  can never be true for fixed  $x, y$  under the assumption of (1). Since the left-hand side of the implication in (2) is always false the statement will always be true. We have thus shown that  $\text{antisymmetric}(R, A)$  holds.

If  $R(x, x)$  were true for arbitrary but fixed  $x \in A$  it would lead to  $R(x, x) \Rightarrow \neg R(x, x)$  being false. Since that is a contradiction (we know that (1) holds for our  $R$  and  $A$ ) we conclude that  $\neg R(x, x)$ . We have thus shown that  $\text{irreflexive}(R, A)$  holds.