

**Exercise 1**

- a) We first determine a base of  $U$  by noting that  $x = -y - z$  solves  $x + y + z = 0$  for arbitrary  $y$  and  $z$  leading to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

which shows that  $(-1, 1, 0)$  and  $(-1, 0, 1)$  form a basis for  $U$  (since any element of  $U$  can be written as a linear combination of them).

Consider

$$\begin{aligned} \dim(R^3) &= \dim(U) + \dim(R^3/U) \\ 3 &= 2 + \dim(R^3/U) \\ \dim(R^3/U) &= 1 \end{aligned}$$

thus we are looking for one more element in  $R^3$  such that it forms a basis of  $R^3$  alongside our existing vectors.

- b) We first determine a base of  $U$ . By solving the linear system  $x + y + z = 0$  and  $x + 2y + 3z = 0$ .

$$\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 & 2 & 0 \end{array}$$

Thus

$$\begin{array}{ll} y + 2z = 0 & x + y + z = 0 \\ y = -2z & x - 2z + z = 0 \\ & x = z \end{array}$$

leading to

$$\begin{pmatrix} x \\ y \\ -2x \end{pmatrix} = x \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

which makes  $(1, -1, -2)$  a basis of  $U$ .

**Exercise 2**

- a) To show that the given vectors form an orthogonal system it is necessary to show that they are pairwise orthogonal. (That for any vectors  $\vec{v}$  and  $\vec{u}$  we have  $\vec{v} \cdot \vec{u} = 0$ ) This is the case here.
- b) To show that the given vectors form a basis we show that they are linearly independent (skipped,  $x_1, \dots, x_4 = 0$ ). Since we have four independent vectors of a four-dimensional vector space they form a basis.
- c) To determine the coordinates of  $\vec{a}$  in relation to the given basis we solve

$$\begin{array}{cccc|c} 0 & -10 & 26 & 9 & 6 \\ 0 & -5 & 47 & -12 & 37 \\ 1 & 6 & 66 & 4 & 78 \\ 2 & -3 & -33 & -2 & -39 \end{array}$$

and get  $\lambda_1 = 0$ ,  $\lambda_2 = 2$ ,  $\lambda_3 = 1$ ,  $\lambda_4 = 0$ .

### Exercise 3

a) (i)

$$\begin{aligned} x_1^2 + 2x_2^2 &\geq 0 \\ x_1^2 + 2x_2^2 = 0 &\Leftrightarrow \vec{x} = \vec{0} \end{aligned}$$

(ii)

$$x_1y_1 + 2x_2y_2 = y_1x_1 + 2y_2x_2$$

(iii)

$$\begin{aligned} (\lambda x_1 + \phi y_1)z_1 + 2(\lambda x_2 + \phi y_2)z_2 &= \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2) \\ \lambda x_1z_1 + \phi y_1z_1 + 2(\lambda x_2z_2 + \phi y_2z_2) &= \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2) \\ \lambda x_1z_1 + \phi y_1z_1 + 2\lambda x_2z_2 + 2\phi y_2z_2 &= \lambda(x_1z_1 + 2x_2z_2) + \phi(y_1z_1 + 2y_2z_2) \end{aligned}$$

b) (i) Same as regular scalar product. (No, contradiction.)

(ii)

$$x_1y_2 + x_2y_1 = y_1x_2 + y_2x_1$$

(iii)

$$\begin{aligned} (\lambda x_1 + \phi y_1)z_2 + (\lambda x_2 + \phi y_2)z_1 &= \lambda(x_1z_2 + x_2z_1) + \phi(y_1z_2 + y_2z_1) \\ \lambda x_1z_2 + \phi y_1z_2 + \lambda x_2z_1 + \phi y_2z_1 &= \lambda(x_1z_2 + x_2z_1) + \phi(y_1z_2 + y_2z_1) \end{aligned}$$

c) Not a scalar product.

(i) Same as regular scalar product.

(ii)

$$\begin{aligned} x_1y_1 + x_2y_1 &= y_1x_1 + y_2x_1 \\ x_2y_1 &= y_2x_1 \end{aligned}$$

Consider  $\vec{x} = (1, 2)$  and  $\vec{y} = (3, 4)$ , we now have  $6 = 4$ .

d) Not a scalar product.

(i)

$$x_1 + x_2 + x_1 + x_2 \geq 0$$

Consider  $\vec{x} = (-1, 0)$  we now have  $-1 + 0 - 1 + 0 \geq 0$ .

e) Not a scalar product, definition requires  $V \times V \rightarrow \mathbb{R}$  but  $R_3 \neq R_2$ .

f) Can be restated to

$$\left( \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \right) \mapsto \begin{pmatrix} \langle (x_1, x_2), (y_1, y_2) \rangle \\ \langle (x_1, x_3), (y_1, y_3) \rangle \end{pmatrix}$$

and is thus a scalar product. (No, contradiction.)

#### Exercise 4

a) To show that a set of vectors form an orthogonal basis we show that they are a basis (they are linearly independent, done?) and that they are pairwise orthogonal (trivial).

b) Since the system

$$\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 6 \\ 1 & -1 & 0 & 3 \end{array}$$

does not have a solution, the given vector is not in  $U$ . (Cannot be constructed from linear combination of vectors in  $U$ .)

#### Exercise 5

$i = 1$

$$w_1 = u$$

$i = 2$

$$w_2 = v - (\text{proj}_{w_1}(v))$$

with

$$\text{proj}_{w_1}(v) = \frac{v \cdot w_1}{w_1 \cdot w_1} \cdot w_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

thus

$$w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$$i = 3$$

$$w_3 = w - (\text{proj}_{w_1}(w) + \text{proj}_{w_2}(w))$$

with

$$\text{proj}_{w_1}(w) = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\text{proj}_{w_2}(w) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$$

thus

$$w_3 = \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

### Exercise 6

(i)

$$\begin{aligned} x_1^2 + x_1x_2 + x_2x_1 + 2x_2^2 - x_1x_3 - x_3x_1 + 3x_3^2 &\geq 0 \\ x_1^2 + 2x_1x_2 + 2x_2^2 - 2x_1x_3 + 3x_3^2 &\geq 0 \\ x_1^2 + 2x_2^2 + 3x_3^2 &\geq -2x_1x_2 + 2x_1x_3 \end{aligned}$$

(ii)

$$\begin{aligned} x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 - x_1y_3 - x_3y_1 + 3x_3y_3 &= y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 - y_1x_3 - y_3x_1 + 3y_3x_3 \\ &= 0 \end{aligned}$$

(iii)

$$\begin{aligned} &(\lambda x_1 + \phi y_1)z_1 + (\lambda x_1 + \phi y_1)z_2 + (\lambda x_2 + \phi y_2)z_1 + 2(\lambda x_2 + \phi y_2)z_2 - (\lambda x_1 + \phi y_1)z_3 - (\lambda x_3 + \phi y_3)z_1 + 3(\lambda x_3 + \phi y_3)z_3 \\ &= \lambda(x_1z_1 + x_1z_2 + x_2z_1 + 2x_2z_2 - x_1z_3 - x_3z_1 + 3x_3z_3) + \phi(y_1z_1 + y_1z_2 + y_2z_1 + 2y_2z_2 - y_1z_3 - y_3z_1 + 3y_3z_3) \end{aligned}$$

**Exercise 7** The angle can be calculated with

$$\cos(\alpha) = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} = \frac{1}{\sqrt{(\sqrt{2} + 1)^2 - 2\sqrt{2} + 1 \cdot 1}}$$

$$\alpha = 60 \text{ deg}$$

**Exercise 8** Gram-Schmidt auf Standardbasis von  $\mathbb{R}^3$  anwenden.