

## Time Hierarchy Theorem

Informally, the theorem states that given more time, a Turing machine can solve more problems. (In other words, more languages can be decided with more time.) One way to state it formally is that if  $f$  and  $g$  are time-honest functions with  $f(n) \log f(n) = O(g(n))$ , then

$$\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n)).$$

As suggested, we have chosen to present a proof of a weaker version, showing that  $\text{DTIME}(n)$  is a strict subset of  $\text{DTIME}(n^{1.5})$ . The following are notes based on the proof presented in *Computational Complexity: A Modern Approach* by S. Arora and B. Barak, available on p. 66 of the public draft.

Let  $M_x$  be the TM represented by a string  $x$  with outputs  $\{0, 1\}$  and let  $U$  be an Universal TM (i.e. an „interpreter“ of TMs, which is itself a TM). Now consider a TM  $D$  which, given as input  $x$ , runs  $U$  for  $|x|^{1.4}$  steps, simulating the execution of  $M_x$  on  $x$ . If an output  $M_x(x)$  is obtained in time it outputs  $1 - M_x(x)$ , the opposite. Otherwise it outputs zero.

Clearly,  $D$  finishes execution within  $n^{1.4}$  steps since it only lets  $U$  run for as many steps and doesn't do anything else that scales with  $n$ . Thus the language decided by  $D$  is in  $\text{DTIME}(n^{1.5})$ . The following will show that this language is not in  $\text{DTIME}(n)$ , thus providing a proof of the initial statement.

For the sake of contradiction assume that there exists a TM  $M$  which decides the same language as  $D$  but runs in time  $kn$  for input sizes  $n$  where  $k$  is a constant factor independent of  $n$ . Then for all possible inputs  $x$  we have  $M(x) = D(x)$ : The machines are functionally equivalent but  $M$  runs in linear time.

The time needed to simulate  $M$  on  $U$  is at most  $ck|x| \log |x|$  for some constant  $c$  which is independent of  $x$ . (This makes use of the general statement that the runtime of an Universal TM is bounded by  $cT \log T$  where the simulated machine halts within  $T$  steps.) Naturally, at some point  $n^{1.4}$  will grow larger than  $ckn \log n$ . Let  $n_0$  be the smallest  $n$  such that this is the case and let  $x$  (representing the TM  $M_x$ ) have a length of  $n_0$ . Then  $D(x)$  will obtain the output of  $M(x)$  within  $|x|^{1.4}$  steps (since it stops at that point) but by definition we have

$$D(x) = 1 - M(x) \neq M(x),$$

and have arrived at a contradiction.

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