Exercise 1

$$AB = \begin{pmatrix} 7 & 7 & 9 \\ 10 & 0 & 5 \\ 6 & 10 & 16 \end{pmatrix} \qquad BA = \begin{pmatrix} 13 & 15 & 6 \\ 6 & 12 & 4 \\ 7 & 9 & -2 \end{pmatrix}$$

Exercise 2

$$AB = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 7 & -11 & 0 & 7 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix} \qquad A^T = \begin{pmatrix} 0 & -3 \\ 1 & 1 \\ -2 & 4 \end{pmatrix} \qquad B^T A^T = \begin{pmatrix} 1 & -1 \\ 1 & -11 \\ 0 & 0 \\ -2 & 10 \end{pmatrix}$$

$$AC = \begin{pmatrix} -2 & -2 & -2 & 1 \\ -5 & 7 & -2 & -5 \end{pmatrix} \qquad AB + AC = \begin{pmatrix} -1 & -1 & -2 & -1 \\ 2 & -4 & -2 & 2 \end{pmatrix}$$

Exercise 3

a) **True.** Since elements of A and B are in $\mathbb{R}$, we have

$$A + B = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,n} + b_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} + b_{m,1} & \cdots & a_{n,n} + b_{n,n} \end{pmatrix} = \begin{pmatrix} b_{1,1} + a_{1,1} & \cdots & b_{1,n} + a_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} + a_{n,1} & \cdots & b_{n,n} + a_{n,n} \end{pmatrix} = B + A$$

b) **False.** We interpret A as a column vector of row vectors $(a_1, \dots a_n)$ and B as a row vector of column vectors $(b_1, \dots, b_n)$ and thus have

$$AB = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} = \begin{pmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_n \\ \vdots & \ddots & \vdots \\ a_n \cdot b_1 & \cdots & a_n \cdot b_n \end{pmatrix} \neq$$

$$BA = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_1 \cdot a_1 & \cdots & b_1 \cdot a_n \\ \vdots & \ddots & \vdots \\ b_n \cdot a_1 & \cdots & b_n \cdot a_n \end{pmatrix}$$

c) **True.**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad AA^T = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{1,1} & \cdots & a_{n,1} \\ \vdots & \ddots & \vdots \\ a_{1,n} & \cdots & a_{n,n} \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \sum_{k=1}^{n} a_{k,1}^2 & \cdots & \sum_{k=1}^{n} a_{1,k} \cdot a_{n,k} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{n,k} \cdot a_{1,k} & \cdots & \sum_{k=1}^{n} a_{n,k}^2 \end{pmatrix}$$

Clearly, AA^T is symmetrical.

d) **False.**

$$AA^T = \begin{pmatrix} 7 & 15 \\ 10 & 22 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \qquad A^T A = \begin{pmatrix} 10 & 14 \\ 14 & 20 \end{pmatrix}$$

e) **False.**

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6 & 8 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 8 \end{pmatrix}$$

f) **True.**

$$A = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \qquad\qquad B = \begin{pmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$AB = \begin{pmatrix} a_1 b_1 & 0 & \dots & 0 \\ 0 & a_2 b_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n b_n \end{pmatrix} = BA$$

g) **False.** Since according to the script distributivity holds, we have

$$(A + B)^2 = (A + B)(A + B) = A(A + B) + B(A + B) = A^2 + AB + BA + B^2$$

which shows that the above statement is only equal to $A^2 + 2AB + B^2$ if $AB = BA$.

h) **True.** In accordance to the previous point, we need only show that $E_n \cdot A = E_n \cdot A$ which is trivially true.

i) **False.**

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad\qquad C = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$AB = AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{but} \quad B \neq C$$

Exercise 4

a)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

b)

$$aI = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad cI = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \qquad bJ = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \qquad dJ = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}$$

$$aI + bJ = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad cI + dJ = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \quad (aI + bJ)(cI + dJ) = \begin{pmatrix} ac - bd & ad + bc \\ -ad - bc & ac - bd \end{pmatrix}$$

c)   (i) Associativity of addition and multiplication. For arbitrary $x, y, z \in \mathcal{C}$ we have for addition

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} + \left(\begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} + \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix} \right)$$
$$= \begin{pmatrix} x_1 + y_1 + z_1 & x_2 + y_2 + z_2 \\ -x_2 - y_2 - z_2 & x_1 + y_1 + z - 1 \end{pmatrix}$$
$$= \left(\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} \right) + \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}$$

and similarly for multiplication

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \cdot \left(\begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} \cdot \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix} \right) = \left(\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \cdot \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} \right) \cdot \begin{pmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{pmatrix}$$

which follows trivially from the associativity of matrix multiplication. (Which itself follows trivially from the associativity of function composition.)

(ii) Existence of neutral element for addition and multiplication. For any matrix x

$$x + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = x$$

The additive neutral element is in $\mathcal{C}$ for $a = b = 0$.

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

The multiplicative neutral element is in $\mathcal{C}$ for $a = 1$ and $b = 0$.

(iii) Existence of inverses for addition and multiplication.

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} -x_1 & -x_2 \\ x_2 & -x_1 \end{pmatrix}$$

For an element in $\mathcal{C}$ with given $a, b \in \mathbb{R}$ the additive inverse is the element with $-a, -b$.

A 2x2 matrix is invertible if and only if its determinant is nonzero. In the given case the determinant is $x_1^2 - (x_2 \cdot -x_2) = x_1^2 + x_2^2$ which is zero only if x_1 and x_2 are both zero. Since that is equivalent to the additive inverse we can discard that case.

(iv) Commutativity of addition and multiplication.

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} + \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ -x_2 - y_2 & x_1 - y_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} x_1 y_1 - x_2 y_2 & x_1 y_2 + x_2 y_1 \\ -x_2 y_1 - x_1 y_2 & -x_2 y_2 + x_1 y_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$$

(v) Distributivity of multiplication over addition. Let $x, y, z \in \mathcal{C}$ be arbitrary.

$$(x(y + z))_{i,j} = \sum_{k=1}^{n} x_{i,k} \cdot (y_{k,j} + z_{k,j}) = \sum_{k=1}^{n} x_{i,k} \cdot y_{k,j} + x_{i,k} \cdot z_{k,j}$$

$$= \sum_{k=1}^{n} x_{i,k} \cdot y_{k,j} + \sum_{k=1}^{n} x_{i,k} \cdot z_{k,j} = (xy)_{i,j} + (xz)_{i,j}$$

Exercise 5   The given matrix is invertible for all $k \neq -1$. The inverses are

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k+1} & \frac{k}{k+1} \\ 0 & \frac{1}{k+1} & -\frac{1}{k+1} \end{pmatrix}$$

Exercise 7

a)

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 2 & 2 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 4z \\ y + z \\ 2x + 2y + 6z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

After solving the linear system of equations this is true for $y = \frac{x}{2}$, $z = -\frac{x}{2}$ with arbitrary $x \in \mathbb{R}$.

b)

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 2 & 2 & 6 \end{pmatrix} = \begin{pmatrix} x + 2z \\ 2x + y + 2z \\ 4x + y + 6z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$$

After solving the linear system of equations this is true for $y = -x$, $z = -\frac{x}{2}$ with arbitrary $x \in \mathbb{R}$.

c)

$$\begin{pmatrix} x \\ 0 \\ 2x \end{pmatrix} + \begin{pmatrix} 2y \\ y \\ 2y \end{pmatrix} + \begin{pmatrix} 4z \\ z \\ 6z \end{pmatrix} = \begin{pmatrix} x + 2y + 4z \\ y + z \\ 2x + 2x + 6z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

After solving the linear system of equations this is true for $y = \frac{x}{2} + \frac{7}{2}$, $z = -\frac{x}{2} - \frac{3}{2}$ with arbitrary $x \in \mathbb{R}$.

d)

$$\begin{pmatrix} x \\ 2x + y \\ 4x + y \end{pmatrix} + \begin{pmatrix} 2z \\ 2z \\ 6z \end{pmatrix} = \begin{pmatrix} x + 2z \\ 2x + y + 2z \\ 4x + y + 6z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}$$

After solving the linear syste mof equations this is true for $y = 1 - x$, $z = -\frac{x}{2} - \frac{1}{2}$ with arbitrary $x \in \mathbb{R}$.