

**Exercise 1**

a) We have

| $r$ | $x$ | $y$ | $q$ |
|-----|-----|-----|-----|
| 135 | 1   | 0   |     |
| 54  | 0   | 1   | 2   |
| 27  | 1   | -2  | 2   |
| 0   | -2  | 5   |     |

and thus  $\gcd(135, 54) = 27$ . Since  $27 \mid 0$ , there is a solution.

From the last row we know that  $125 \cdot -2 + 54 \cdot 5 = 0$ , thus  $(-2, 5) \in L$ . We can now describe  $L$  as  $L = \{(-2k, 5k) \mid k \in \mathbb{Z}\}$ .

b) We have (note that  $x$  and  $y$  are reversed)

| $r$ | $y$ | $x$ | $q$ |
|-----|-----|-----|-----|
| 105 | 1   | 0   |     |
| 99  | 0   | 1   | 1   |
| 6   | 1   | -1  | 16  |
| 3   | -16 | 17  | 2   |
| 0   | 33  | -35 |     |

and thus  $\gcd(105, 99) = 3$ . Since  $3 \mid 12$ , there is a solution.

From the second to last row we know

$$\begin{aligned} 3 &= (17 \cdot 99) + (-16 \cdot 105) \quad \text{and, after multiplying by 4} \\ 12 &= (68 \cdot 99) + (-64 \cdot 105). \end{aligned}$$

Thus we have that  $(68, -64) \in L$  and further  $L = \{(68 - 35k, -64 + 33k) \mid k \in \mathbb{Z}\}$ .

c) We have

| $q$ | $x$ | $y$ | $r$ |
|-----|-----|-----|-----|
| 38  | 1   | 0   |     |
| 19  | 0   | 1   | 2   |
| 0   | 1   | -2  |     |

and thus  $\gcd(38, 19) = \gcd(19, -38) = 19$ . Since  $19 \nmid 5$  this equation does not have a solution.

**Exercise 2** We are looking for solutions to

$$35x + 45y = 1000$$

where  $x$  is the number of linear Algebra books and  $y$  is the number of Analysis books. We have (note that  $x$  and  $y$  are reversed)

| $r$ | $y$ | $x$ | $q$ |
|-----|-----|-----|-----|
| 45  | 1   | 0   |     |
| 35  | 0   | 1   | 1   |
| 10  | 1   | -1  | 3   |
| 5   | -3  | 4   | 2   |
| 0   | 7   | -9  |     |

and thus  $\gcd(45, 35) = 5$ . Since  $5 \mid 1000$ , there is a solution.

From the second to last row we know

$$5 = (4 \cdot 35) + (-3 \cdot 45) \quad \text{and, after multiplying by 200}$$

$$1000 = (800 \cdot 35) + (-600 \cdot 45)$$

and thus  $(800, -600) \in L$ , allowing us to state  $L = \{(800 \cdot -9k, -600 \cdot 7k) \mid k \in \mathbb{Z}\}$ . For  $86 \leq k \leq 88$  neither of the values in the pairs  $\in L$  are negative. Thus we can either buy

| Lineare Algebra | Analysis |
|-----------------|----------|
| 26              | 2        |
| 17              | 9        |
| 8               | 16       |

books.

If the total available money were 1001 then we would have  $5 \nmid 1001$ , thus we would not be able to spend all of our budget.

**Exercise 3** Idk.

**Exercise 4** Interpreting the polynomials as being in  $\mathbb{Z}_5$ .

$$\begin{array}{cccccc}
 x^5 & x^4 & x^3 & x^2 & x^1 & x^0 \\
 3 & 1 & 4 & 1 & 0 & 4 \\
 3 & 3 & 4 & 2 & & 
 \end{array}
 : 
 \begin{array}{cccccc}
 x^3 & x^2 & x^1 & x^0 & & \\
 2 & 2 & 1 & 3 & & 
 \end{array}
 = 
 \begin{array}{cccccc}
 x^2 & x^1 & x^0 & & & \\
 4 & 4 & 1 & & & 
 \end{array}$$


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$$\begin{array}{cccccc}
 & 3 & 0 & 4 & 0 & 4 \\
 & 3 & 3 & 4 & 2 & \\
 & 2 & 0 & 3 & 4 & \\
 & 2 & 2 & 1 & 3 & \\
 & 3 & 2 & 1 & & 
 \end{array}$$

Interpreting the polynomials as being in  $\mathbb{Q}$ .

$$\begin{array}{cccccc}
 x^5 & x^4 & x^3 & x^2 & x^1 & x^0 \\
 3 & 1 & 4 & 1 & 5 & 9 \\
 3 & 10.5 & 1.5 & 12 & & 
 \end{array}
 : 
 \begin{array}{cccccc}
 x^3 & x^2 & x^1 & x^0 & & \\
 2 & 7 & 1 & 8 & & 
 \end{array}
 = 
 \begin{array}{cccccc}
 x^2 & x^1 & x^0 & & & \\
 1.5 & -4.75 & 17.875 & & & 
 \end{array}$$


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$$\begin{array}{cccccc}
 -9.5 & 3.5 & -11 & 5 & 9 & \\
 -9.5 & -33.25 & -4.75 & -38 & & \\
 35.75 & -6.25 & 43 & 9 & & \\
 35.75 & 125.12 & 17.975 & 143 & & \\
 -132.37 & 25.125 & -134 & & & 
 \end{array}$$

**Exercise 5**

a) Consider that a general table for the GCD is

| $r$   | $u$   | $v$   | $q$   |
|-------|-------|-------|-------|
| $P_1$ | 1     | 0     |       |
| $P_2$ | 0     | 1     | $q_1$ |
| $r_1$ | 1     | $v_1$ | $q_2$ |
| $r_2$ | $u_2$ | $v_2$ | $q_3$ |
| $r_3$ | $u_3$ | $v_3$ | $q_4$ |

We begin by calculating  $q_1$  and  $r_1$ .

$$\begin{array}{cccccc}
 x^5 & x^4 & x^3 & x^2 & x^1 & x^0 \\
 1 & 6 & 9 & -6 & -22 & -12 \\
 1 & 1 & -4 & -2 & 4 & \\
 \hline
 & 5 & 13 & -4 & -26 & -12 \\
 & 5 & 5 & -20 & -10 & 20 \\
 \hline
 & 8 & 16 & -16 & -32 & 
 \end{array}
 \quad : \quad
 \begin{array}{cccccc}
 x^4 & x^3 & x^2 & x^1 & x^0 & \\
 1 & 1 & -4 & -2 & 4 & \\
 1 & 1 & -4 & -2 & 4 & \\
 \hline
 & -1 & -2 & 2 & 4 & \\
 & -1 & -2 & 2 & 4 & \\
 \hline
 & 0 & 0 & 0 & 0 & 
 \end{array}
 \quad = \quad
 \begin{array}{cccccc}
 x^1 & x^0 & & & & \\
 1 & 5 & & & & 
 \end{array}$$

Thus  $q_1 = x + 5$ ,  $r_1 = 8x^3 + 16x^2 - 16x - 32$  and  $v_1 = 0 - q_1 = -x - 5$ .

We continue by calculating  $q_2$  and  $r_2$ .

$$\begin{array}{cccccc}
 x^4 & x^3 & x^2 & x^1 & x^0 & \\
 1 & 1 & -4 & -2 & 4 & \\
 1 & 1 & -4 & -2 & 4 & \\
 \hline
 & -1 & -2 & 2 & 4 & \\
 & -1 & -2 & 2 & 4 & \\
 \hline
 & 0 & 0 & 0 & 0 & 
 \end{array}
 \quad : \quad
 \begin{array}{cccccc}
 x^3 & x^2 & x^1 & x^0 & & \\
 8 & 16 & -16 & -32 & & \\
 8 & 16 & & & & \\
 \hline
 & -16 & -32 & & & \\
 & -16 & -32 & & & \\
 \hline
 & 0 & 0 & 0 & 0 & 
 \end{array}
 \quad = \quad
 \begin{array}{cccccc}
 x^1 & x^0 & & & & \\
 \frac{1}{8} & -\frac{1}{8} & & & & 
 \end{array}$$

Thus  $q_2 = \frac{1}{8}x - \frac{1}{8}$  and  $r_2 = 0$ . We have  $\gcd(P_1, P_2) = r_1 = 8x^3 + 16x^2 - 16x - 32$ .

b) If  $\frac{a}{b}$  is a root of  $\gcd(P_1, P_2)$  then  $a$  must be a divisor of 32 and  $b$  must be a divisor of 8. The candidates are thus

$$\pm 1 \quad \boxed{-2} \quad \pm 4 \quad \pm 8 \quad \pm 16 \quad \pm 32 \quad \pm \frac{1}{2} \quad \pm \frac{1}{4} \quad \pm \frac{1}{8}$$

where boxed numbers are actual roots. We can thus factor out  $x + 2$  by division.

$$\begin{array}{cccccc}
 x^3 & x^2 & x^1 & x^0 & & \\
 8 & 16 & -16 & -32 & & \\
 8 & 16 & & & & \\
 \hline
 & -16 & -32 & & & \\
 & -16 & -32 & & & \\
 \hline
 & 0 & 0 & 0 & 0 & 
 \end{array}
 \quad : \quad
 \begin{array}{cccccc}
 x^1 & x^0 & & & & \\
 1 & 2 & & & & \\
 1 & 2 & & & & \\
 \hline
 & -16 & -32 & & & \\
 & -16 & -32 & & & \\
 \hline
 & 0 & 0 & 0 & 0 & 
 \end{array}
 \quad = \quad
 \begin{array}{cccccc}
 x^2 & x^1 & x^0 & & & \\
 8 & 0 & -16 & & & 
 \end{array}$$

We can now solve  $8x^2 - 16 = 0$  through

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm \frac{\sqrt{-4 \cdot 8 \cdot -16}}{2 \cdot 8} = \pm \frac{\sqrt{512}}{16} = \pm \frac{16\sqrt{2}}{16} = \pm \sqrt{2}.$$

The roots are thus  $-2, -\sqrt{2}$  and  $\sqrt{2}$ .

**Exercise 6** We are looking for the rational roots of

$$P(x) = 18x^6 - 51x^5 - 7x^4 + 106x^3 - 62x^2 - 8x + 8$$

Using the fact that, if  $\frac{a}{b}$  is a root of a polynomial then  $a \mid a_0$  and  $b \mid a_n$ , we get

$$\pm 1 \quad \boxed{2} \quad \pm 4 \quad \pm 8 \quad \boxed{\frac{1}{2}} \quad \boxed{-\frac{1}{3}} \quad \pm \frac{1}{6} \quad \pm \frac{1}{9} \quad \pm \frac{1}{18} \quad \boxed{\frac{2}{3}} \quad \pm \frac{2}{9} \quad \pm \frac{2}{18} \quad \pm \frac{4}{3} \quad \pm \frac{4}{9} \quad \pm \frac{8}{3} \quad \pm \frac{8}{9}$$

as potential roots. Boxed numbers are actual roots.

We can thus factor out

$$\left(x + \frac{1}{3}\right) \left(x - \frac{1}{2}\right) \left(x - \frac{2}{3}\right) (x - 2) = x^4 - \frac{17}{6}x^3 + \frac{29}{18}x^2 + \frac{2}{9}x - \frac{2}{9}$$

by division.

|       |       |       |       |       |       |       |    |       |                 |                 |               |                |   |       |       |       |
|-------|-------|-------|-------|-------|-------|-------|----|-------|-----------------|-----------------|---------------|----------------|---|-------|-------|-------|
| $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ | $x^1$ | $x^0$ |    | $x^4$ | $x^3$           | $x^2$           | $x^1$         | $x^0$          |   | $x^2$ | $x^1$ | $x^0$ |
| 18    | -51   | -7    | 106   | -62   | -8    | 8     | :  | 1     | $-\frac{17}{6}$ | $\frac{29}{18}$ | $\frac{2}{9}$ | $-\frac{2}{9}$ | = | 18    | 0     | -36   |
| 18    | -51   | 29    | 4     | -4    |       |       |    |       |                 |                 |               |                |   |       |       |       |
|       |       |       |       | -36   | 102   | -58   | -8 | 8     |                 |                 |               |                |   |       |       |       |
|       |       |       |       | -36   | 102   | -58   | -8 | 8     |                 |                 |               |                |   |       |       |       |

We can now solve  $18x^2 - 36 = 0$  through

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \pm \frac{\sqrt{-4 \cdot 18 \cdot -36}}{2 \cdot 18} = \pm \frac{\sqrt{2592}}{36} = \pm \frac{36\sqrt{2}}{36} = \pm\sqrt{2},$$

yielding no rational roots. The rational roots are thus these obtained previously.

**Exercise 7** We have

$$p(x) = x^7 - 6x^6 + 10x^5 - 6x^4 + 9x^3$$

$$p'(x) = 7x^6 - 36x^5 + 50x^4 - 24x^3 + 27x^2$$

and we are looking for a square-free factorisation of  $p$ . Calculating the GCD of  $p$  and  $p'$  we first divide  $p$  by  $p'$

|       |                 |                  |                   |                  |                   |   |       |       |       |       |       |   |               |                 |
|-------|-----------------|------------------|-------------------|------------------|-------------------|---|-------|-------|-------|-------|-------|---|---------------|-----------------|
| $x^7$ | $x^6$           | $x^5$            | $x^4$             | $x^3$            | $x^2$             |   | $x^6$ | $x^5$ | $x^4$ | $x^3$ | $x^2$ |   | $x^1$         | $x^0$           |
| 1     | -6              | 10               | -6                | 9                | 0                 | : | 7     | -36   | 50    | -24   | 27    | = | $\frac{1}{7}$ | $-\frac{6}{49}$ |
| 1     | $-\frac{36}{7}$ | $\frac{50}{7}$   | $-\frac{24}{7}$   | $\frac{27}{7}$   |                   |   |       |       |       |       |       |   |               |                 |
|       | $-\frac{6}{7}$  | $\frac{20}{7}$   | $-\frac{17}{7}$   | $\frac{36}{7}$   | 0                 |   |       |       |       |       |       |   |               |                 |
|       | $-\frac{6}{7}$  | $\frac{216}{49}$ | $-\frac{300}{49}$ | $\frac{144}{49}$ | $-\frac{162}{49}$ |   |       |       |       |       |       |   |               |                 |
|       |                 | $-\frac{76}{49}$ | $\frac{174}{49}$  | $\frac{108}{49}$ | $\frac{162}{49}$  |   |       |       |       |       |       |   |               |                 |

and get  $r_1 = -\frac{76x^5}{49} + \frac{174x^4}{49} + \frac{108x^3}{49} + \frac{162x^2}{49}$  and  $q_1 = \frac{x}{7} - \frac{6}{49}$ . We can simplify  $r_1 = -38x^5 + 87x^4 + 54x^3 + 81x^2$ .

Now we divide  $p'$  by  $r_1$

|       |                   |                         |                         |                          |   |       |       |       |       |   |                 |                    |
|-------|-------------------|-------------------------|-------------------------|--------------------------|---|-------|-------|-------|-------|---|-----------------|--------------------|
| $x^6$ | $x^5$             | $x^4$                   | $x^3$                   | $x^2$                    |   | $x^5$ | $x^4$ | $x^3$ | $x^2$ |   | $x^1$           | $x^0$              |
| 7     | -36               | 50                      | -24                     | 27                       | : | -38   | 87    | 54    | 81    | = | $-\frac{7}{38}$ | $\frac{759}{1444}$ |
| 7     | $\frac{126}{19}$  | $\frac{175}{19}$        | $\frac{84}{19}$         | $-\frac{189}{38}$        |   |       |       |       |       |   |                 |                    |
|       | $-\frac{810}{19}$ | $\frac{775}{19}$        | $-\frac{540}{19}$       | $\frac{1215}{38}$        |   |       |       |       |       |   |                 |                    |
|       | $-\frac{810}{19}$ | $\frac{66033}{1444}$    | $\frac{20493}{722}$     | $\frac{61479}{1444}$     |   |       |       |       |       |   |                 |                    |
|       |                   | $\frac{20531x^4}{1444}$ | $-\frac{13524x^3}{361}$ | $-\frac{22491x^2}{1444}$ |   |       |       |       |       |   |                 |                    |

and get  $r_2 = \frac{20531x^4}{1444} - \frac{13524x^3}{361} - \frac{22491x^2}{1444}$ . We then divide  $r_1$  by  $r_2$  and get  $r_3 = \frac{11696400x^2}{175561} - \frac{3898800x^3}{175561}$ . We then divide  $r_2$  by  $r_3$  and get  $r_4 = 0$ .

Thus  $r_3$ , which can be simplified to  $x^3 - 3x^2$ , is the GCD we are looking for.

We now divide  $p$  by this result which yields  $x^4 - 3x^3 + x^2 - 3x$  with no remainder.



**Exercise 8** To show that  $p \mid \binom{p}{k}$  note that

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

$$p! = \binom{p}{k} (k!(p-k)!).$$

Since the left hand side of the equation is clearly divisible by  $p$ , the right hand side must also be divisible by it. The expression  $k!(p-k)!$  is not divisible by  $p$  since it is a product of numbers smaller than  $p$  and  $p$  is prime. Thus the binomial coefficient must be the part which is divisible by  $p$ .

Now consider that, by the binomial theorem

$$(x+y)^p = \sum_{k=0}^p \binom{p}{k} x^{p-k} y^k$$

$$(x+y)^p = \binom{p}{0} x^p y^0 + \binom{p}{1} x^{p-1} y^1 + \cdots + \binom{p}{p-1} x^1 y^{p-1} + \binom{p}{p} x^0 y^p$$

$$(x+y)^p = x^p + \binom{p}{1} x^{p-1} y^1 + \cdots + \binom{p}{p-1} x^1 y^{p-1} + y^p$$

$$(x+y)^p \equiv_6 x^p + y^p$$