

**Exercise 1**

a)

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

b)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{pmatrix}$$

c)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

d)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & 3a_{12} & a_{13} \\ a_{21} & 3a_{22} & a_{23} \\ a_{31} & 3a_{32} & a_{33} \end{pmatrix}$$

e)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{21} + a_{31} & 2a_{22} + a_{32} & 2a_{23} + a_{33} \end{pmatrix}$$

f)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} + 2a_{13} & a_{13} \\ a_{21} & a_{22} + 2a_{23} & a_{23} \\ a_{31} & a_{32} + 2a_{33} & a_{33} \end{pmatrix}$$

g)

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{pmatrix}$$

h)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{pmatrix}$$

**Exercise 2**

$$\begin{aligned}
R_{2,1}(-3) \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ -3 & 0 & -6 & -12 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} &= \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} \\
R_{4,2}(-2) \cdot R_{3,2} \left( -\frac{2}{3} \right) \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} &= \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 4 & 17 \end{pmatrix} \\
R_{4,3}(-2) \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 4 & 17 \end{pmatrix} &= \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}
\end{aligned}$$

$$R_{4,3}(-2) \cdot R_{4,2}(-2) \cdot R_{3,2} \left( -\frac{2}{3} \right) \cdot R_{2,1}(-3) \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ -3 & 0 & -6 & -12 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} = \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} -1 & 1 & -2 & -3 \\ -3 & 0 & -6 & -12 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} &= R_{4,3}(2) \cdot R_{4,2}(2) \cdot R_{3,2} \left( \frac{2}{3} \right) \cdot R_{2,1}(3) \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix} \\
\begin{pmatrix} -1 & 1 & -2 & -3 \\ -3 & 0 & -6 & -12 \\ 0 & -2 & 2 & 3 \\ 0 & -6 & 4 & 11 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 2 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & -2 & -3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 7 \end{pmatrix}
\end{aligned}$$

**Exercise 3**

a)

$$R_{4,3}(-3) \cdot R_{3,1}(-1) \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

b)

$$A\vec{x} = \vec{0} \rightarrow LU(\vec{x}) = \vec{0} \rightarrow L(U\vec{x}) = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and now the previous system becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with  $\vec{y} = \vec{0}$ . Now we can restate the intermediate system to

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with  $\vec{x} = \vec{0}$ .

c)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 6 \\ 3 \end{pmatrix}$$

Let

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and now the previous system becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 6 \\ 3 \end{pmatrix}$$

with  $y_1 = 5$ ,  $y_2 = -4$ ,  $y_3 = 1$ ,  $y_4 = 0$ . Now we can restate the intermediate system to

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

with  $x_4 = 0$ ,  $x_3 = -1$ ,  $x_2 = 1$  and  $x_1 = 5$ .

**Exercise 4** The inverse of the given matrix is

$$\begin{array}{ccc} \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \end{array}$$

$$\begin{array}{ccc} \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \end{array}$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

$$\begin{array}{ccc} \begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \boxed{1} \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \boxed{1} \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \end{array}$$

$$\begin{array}{ccc} \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \boxed{1} \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} & \rightarrow & \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \boxed{1} & \boxed{-1} & 0 & 1 \end{array} \end{array}$$

$$A^{-1} = R_{4,2}(-1) \cdot R_{4,1}(1) \cdot R_{3,4}(1) \cdot R_{2,4}(1) \cdot R_{1,2}(1)$$

**Exercise 5**

- The (two) columns are linearly independent thus the rank is 2.
- The zero matrix has rank zero.
- The columns are linearly dependent, the largest set of linearly independent columns has one element, thus the rank is 1.
- g) See c).

**Exercise 6**

a) Using only elementary transformations,

$$P_1^{-1} = T_{1,3} \cdot T_{2,3}, \quad \text{thus} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) The given matrix is equivalent to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \text{whose inverse is} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \text{which is equivalent to} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

c)

$$\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

d)

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

e)

$$\begin{aligned} (P \cdot P^T)_{i,j} &= \sum_{k=1}^n P_{i,k} \cdot P_{k,j}^T \\ &= \sum_{k=1}^n P_{i,k} \cdot P_{j,k}. \end{aligned}$$

Consider that  $P_{i,k}$  and  $P_{j,k}$  are both in the same column. Only one element in a column may be nonzero. Thus the only way for their product to be nonzero is if  $i = j$ , or

$$\sum_{k=1}^n P_{i,k} \cdot P_{j,k} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This is the definition of the identity matrix.