

**Exercise 1**

a) The given subset is linearly dependent.

$$\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 2 & 6 & 0 \\ 3 & 1 & 8 & 0 \end{array} \quad \begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 4 & 8 & 0 \end{array} \quad \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & \boxed{3} & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

b) The given subset is linearly dependent.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ t & t-1 & t+1 & 0 \\ t^2 & (t-1)^2 & (t+1)^2 & 0 \end{array} \quad \begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2t+1 & 2t+1 & 0 \end{array} \quad \begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

**Exercise 2**

a) First we show that

$$\forall \vec{u}, \vec{v} \in U \cap W : \vec{u} + \vec{v} \in U \cap W.$$

Since  $U$  is a subspace, for all elements  $\vec{u}_1, \vec{u}_2 \in U$  we know  $\vec{u}_1 + \vec{u}_2 \in U$ . The same holds for the vectors in  $W$ . We know that the vectors  $\vec{u}$  and  $\vec{v}$  are in  $U$  and  $W$ . Thus

$$\vec{u} + \vec{v} \in U \wedge \vec{u} + \vec{v} \in W \implies \vec{u} + \vec{v} \in U \cap W.$$

Second we show that

$$\forall \vec{v} \in U \cap W : \forall \lambda \in V : \lambda \cdot \vec{v} \in U \cap W$$

We again know for all  $\vec{u} \in U$  and  $\lambda \in V$  that we have  $\lambda \cdot \vec{u} \in U$  and analogously for  $W$ . Thus, since  $\vec{v}$  is in both  $U$  and  $W$  the same reasoning as above applies.

b) First we show that

$$\forall \vec{v}_1, \vec{v}_2 \in U + W : \vec{v}_1 + \vec{v}_2 \in U + W.$$

Since

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 + \vec{w}_1 \quad (\vec{u}_1 \in U, \vec{w}_1 \in W) \\ \vec{v}_2 &= \vec{u}_2 + \vec{w}_2 \quad (\vec{u}_2 \in U, \vec{w}_2 \in W) \end{aligned}$$

by definition of  $U + W$  we can restate the condition to

$$\begin{aligned} \vec{v}_1 + \vec{v}_2 &\in U + W \\ \vec{u}_1 + \vec{w}_1 + \vec{u}_2 + \vec{w}_2 &\in U + W \\ \vec{u}_1 + \vec{u}_2 + \vec{w}_1 + \vec{w}_2 &\in U + W \end{aligned}$$

which is true since  $\vec{u}_1 + \vec{u}_2 \in U$  and  $\vec{w}_1 + \vec{w}_2 \in W$ .

Second we show that

$$\forall \vec{v} \in U + W : \forall \lambda \in V : \lambda \cdot \vec{v} \in U + W$$

Since  $\vec{v} = \vec{u} + \vec{w}$  (for some vectors  $\vec{u}$  and  $\vec{w}$  in  $U$  and  $W$  respectively) we can restate the above condition to

$$\begin{aligned}\lambda(\vec{u} + \vec{w}) &\in U + W \\ \lambda \cdot \vec{u} + \lambda \cdot \vec{w} &\in U + W\end{aligned}$$

which is true since  $\lambda \cdot \vec{u} \in U$  and  $\lambda \cdot \vec{w} \in W$ . (We can decompose the multiplication with  $\lambda$  because  $V$  is a vector space.)

**Exercise 3** We show that  $W_1 \cup W_2$  is a subspace if and only if  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

First we show the „if“: Assume  $W_1 \subseteq W_2$ , then  $W_1 \cup W_2 = W_2$  which is a subspace. The reasoning for  $W_2 \subseteq W_1$  is very similar.

Second we show the „and only“: Let  $w_1 \in W_1$  such that  $w_1 \notin W_2$  and  $w_2 \in W_2$  with  $w_2 \notin W_1$ . We claim  $w_1 + w_2 \notin W_1 \cap W_2$  — thus that  $W_1 \cup W_2$  is not a subspace if neither is a subset of the other.

Proof by contradiction: Assume that  $w_1 + w_2 \in W_1$ . Since  $-w_1$  is in  $W_1$  (it is a subspace) we can construct the statement

$$\begin{aligned}(w_1 + w_2) - w_1 &\in W_1 \\ w_2 &\in W_1\end{aligned}$$

which is a contradiction. Same goes for the assumption that  $w_1 + w_2 \in W_2$ .

**Exercise 5**



There is no way to add the functions together to get  $f(x) = 0$ .

**Exercise 7** The base is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

and the dimension of the linear hull is 3.