Logic Lab Exercise

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Section 1 contains an analysis of the evaluation tree for `testOut2a(...)`. The requested screenshots (Figures 1 and 2) as well as a complete listing of the RISCAL file can be found in section 2.

1 Analysis of the evaluation

We have $a = [2, 0, 1, 3]$, $b = [2, 1, 1, 0]$, $n = 4$ and Index $= \mathbb{Z}[-1, 5]$.

$$\begin{aligned}\text{array}(b, n-1) \wedge (\exists\, p : \text{Index}.\ & p \geq 0 \wedge p < n \wedge \\ & (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < n) \Rightarrow (a[p] \leq a[i])) \wedge \\ & (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < p) \Rightarrow (b[i] = a[i])) \wedge \\ & (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < n) \Rightarrow (b[i] = a[i+1]))\end{aligned}$$

Let us first consider the outermost conjunction. The left-hand side, $\text{array}(b, n-1)$, is true since b is an array where all elements at and beyond the index $n - 1 = 3$ are zero. We are left with the existential quantifier on the right-hand side. To evaluate the expression $\exists\, p : \text{Index}.\ F$ we will iteratively apply all possible $p \in \text{Index}$ to F.

It will be useful to analyze the trees of subformulas of $\exists$ and $\forall$ in a bottom-up manner — starting from leaves (or other existential/universal quantifiers, which will themselves be analyzed in this manner) and going toward the root of the subtree.

Thus we have for $p = -1$ (subformula 0 of $\exists$) a leaf of $p \geq 0$ which is false. (In the interest of avoiding redundancy, formulas known to be true or false will be replaced with $\top$ or $\bot$ respectively, instead of being specified manyfold. It will be clear from context what is being replaced.) Going upwards we see that this leads to

- $\bot \wedge (p < n)$ being false, leading to
- $\bot \wedge (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < n) \Rightarrow (a[p] \leq a[i]))$ being false, leading to
- $\bot \wedge (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < p) \Rightarrow (b[i] = a[i]))$ being false, leading to
- $\bot(\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < n) \Rightarrow (b[i] = a[i+1]))$ being false,

which finally allows us to conclude that $p = -1$ is not the value we seek.

Thus we continue with $p = 0$ (subformula 1 of $\exists$), where, on our way down the tree, we encounter an universal quantifier $\forall\, i : \text{Index}.\ F$. We will iterate over all $i \in \text{Index}$ to evaluate the expression and analyze it in a bottom-up manner, as mentioned previously. The leaves make up the expression $((i \geq 0) \wedge (i < n)) \Rightarrow (a[p] \leq a[i])$. Since we only care about the right-hand side of an implication when the left-hand side is true we will discard iterations where the left-hand side is false.

Therefore let $i = 1$. Our leaves on the left-hand side are $i \geq 0$ and $i < n$, both of which are true. We then have $\top \wedge \top$, leading to $\top \Rightarrow (a[p] \leq a[i])$. Recall that $a = [2, 0, 1, 3]$, thus $2 \nleq 0$. We now have $\top \Rightarrow \bot$, which is false.

Thus the universal quantifier is false, which leads to

- $\bot \wedge (p \geq 0 \wedge p < n)$ being false, leading to
- $\bot \wedge (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < p) \Rightarrow (b[i] = a[i]))$ being false, leading to
- $\bot \wedge (\forall\, i : \text{Index}.\ (i \geq 0 \wedge i < n) \Rightarrow (b[i] = a[i+1]))$ being false,

which finally allows us to conclude that $p = 0$ is not the value we seek.

We continue with $p = 1$ (subformula 2 of $\exists$). Again we encounter an universal quantifier $\forall\, i$: Index., the leaves of which make up the expression $((i \geq 0) \wedge (i < n)) \Rightarrow (b[i] = a[i+1])$. Let $i = 2$, we now have $i \geq 0$ and $i < n$ leading to $\top \Rightarrow (b[i] = a[i+1])$. Recall that $b = [2, 1, 1, 0]$, thus $1 \neq 3$ leading to $\top \Rightarrow \bot$ which is false.

Recall that this universal quantifier is part of a series of conjunctions, thus $p = 1$ is not the value we seek.

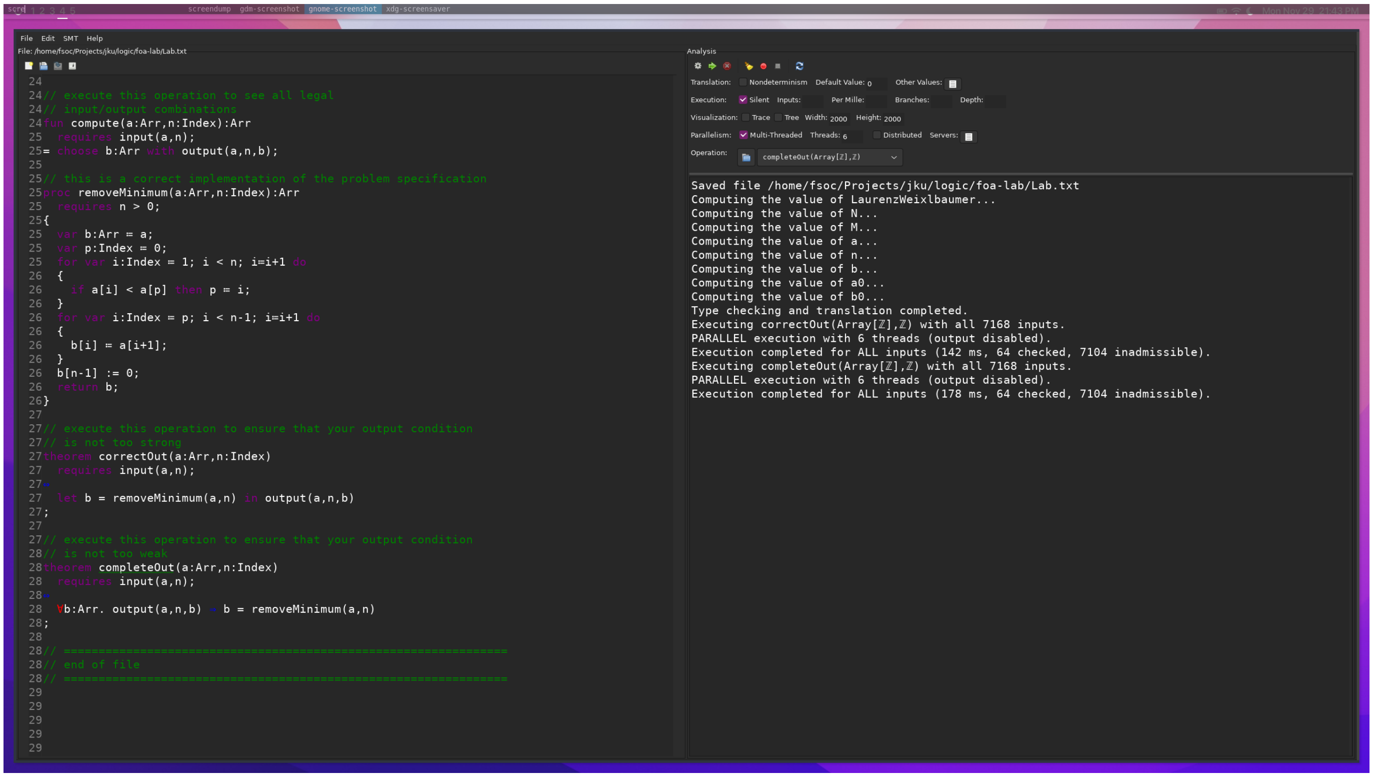
We continue with $p = 2$ (subformula 3 of $\exists$). Again we encounter an universal quantifier $\forall\, i$: Index., the leaves of which make up the expression $((i \geq 0) \wedge (i < n)) \Rightarrow (a[p] \leq a[i])$. Let $i = 1$, we now have $i \geq 0$ and $i < n$ leading to $\top \Rightarrow (a[p] \leq a[i])$. Since $1 \nleq 0$ we can conclude that $p = 2$ is not the value we seek.

We continue with $p = 3$ (subformula 4 of $\exists$). Again we encounter an universal quantifier $\forall\, i$: Index., the leaves of which make up the expression $((i \geq 0) \wedge (i < n)) \Rightarrow (a[p] \leq a[i])$. Let $i = 0$, we now have $\top \Rightarrow (a[p] \leq a[i])$. Since $3 \nleq 2$ we can conclude that $p = 3$ is not the value we seek.

We continue with $p_1 = 4$ and $p_2 = 5$ (subformulas 5 and 6 of $\exists$). One of our leaves for both of them is $p < n$. Since $n = 4$ this is false for both p_1 and p_2. Going upwards we see that this conjunction is part of the series of conjunctions our $\exists$ is made up of, so we can conclude that neither $p = 4$ nor $p = 5$ is the value we seek.

Thus there is no p that satisfies our existential quantifier which overall leads to our formula yielding false for the given inputs.

2 Appendix



File: /home/fsoc/Projects/jku/logic/foa-lab/Lab.txt

```

// execute this operation to see all legal
// input/output combinations
fun compute(a:Arr,n:Index):Arr
  requires input(a,n);
= choose b:Arr with output(a,n,b);

// this is a correct implementation of the problem specification
proc removeMinimum(a:Arr,n:Index):Arr
  requires n > 0;
{
  var b:Arr ≔ a;
  var p:Index ≔ 0;
  for var i:Index ≔ 1; i < n; i≔i+1 do
  {
    if a[i] < a[p] then p ≔ i;
  }
  for var i:Index ≔ p; i < n-1; i≔i+1 do
  {
    b[i] ≔ a[i+1];
  }
  b[n-1] := 0;
  return b;
}

// execute this operation to ensure that your output condition
// is not too strong
theorem correctOut(a:Arr,n:Index)
  requires input(a,n);
⇔
  let b = removeMinimum(a,n) in output(a,n,b)
;

// execute this operation to ensure that your output condition
// is not too weak
theorem completeOut(a:Arr,n:Index)
  requires input(a,n);
⇔
  ∀b:Arr. output(a,n,b) ⇒ b = removeMinimum(a,n)
;

// ==============================================================
// end of file
// ==============================================================
```

Translation: □ Nondeterminism Default Value: 0 Other Values:
Execution: ☑ Silent Inputs: Per Mille: Branches: Depth:
Visualization: □ Trace □ Tree Width: 2000 Height: 2000
Parallelism: ☑ Multi-Threaded Threads: 6 □ Distributed Servers:
Operation: completeOut(Array[ℤ],ℤ)

```
Saved file /home/fsoc/Projects/jku/logic/foa-lab/Lab.txt
Computing the value of LaurenzWeixlbaumer...
Computing the value of N...
Computing the value of M...
Computing the value of a...
Computing the value of n...
Computing the value of b...
Computing the value of a0...
Computing the value of b0...
Type checking and translation completed.
Executing correctOut(Array[ℤ],ℤ) with all 7168 inputs.
PARALLEL execution with 6 threads (output disabled).
Execution completed for ALL inputs (142 ms, 64 checked, 7104 inadmissible).
Executing completeOut(Array[ℤ],ℤ) with all 7168 inputs.
PARALLEL execution with 6 threads (output disabled).
Execution completed for ALL inputs (178 ms, 64 checked, 7104 inadmissible).
```

Figure 1: Screenshot of the RISCAL terminal output when executing `correctOut(...)` and `completeOut(...)`.

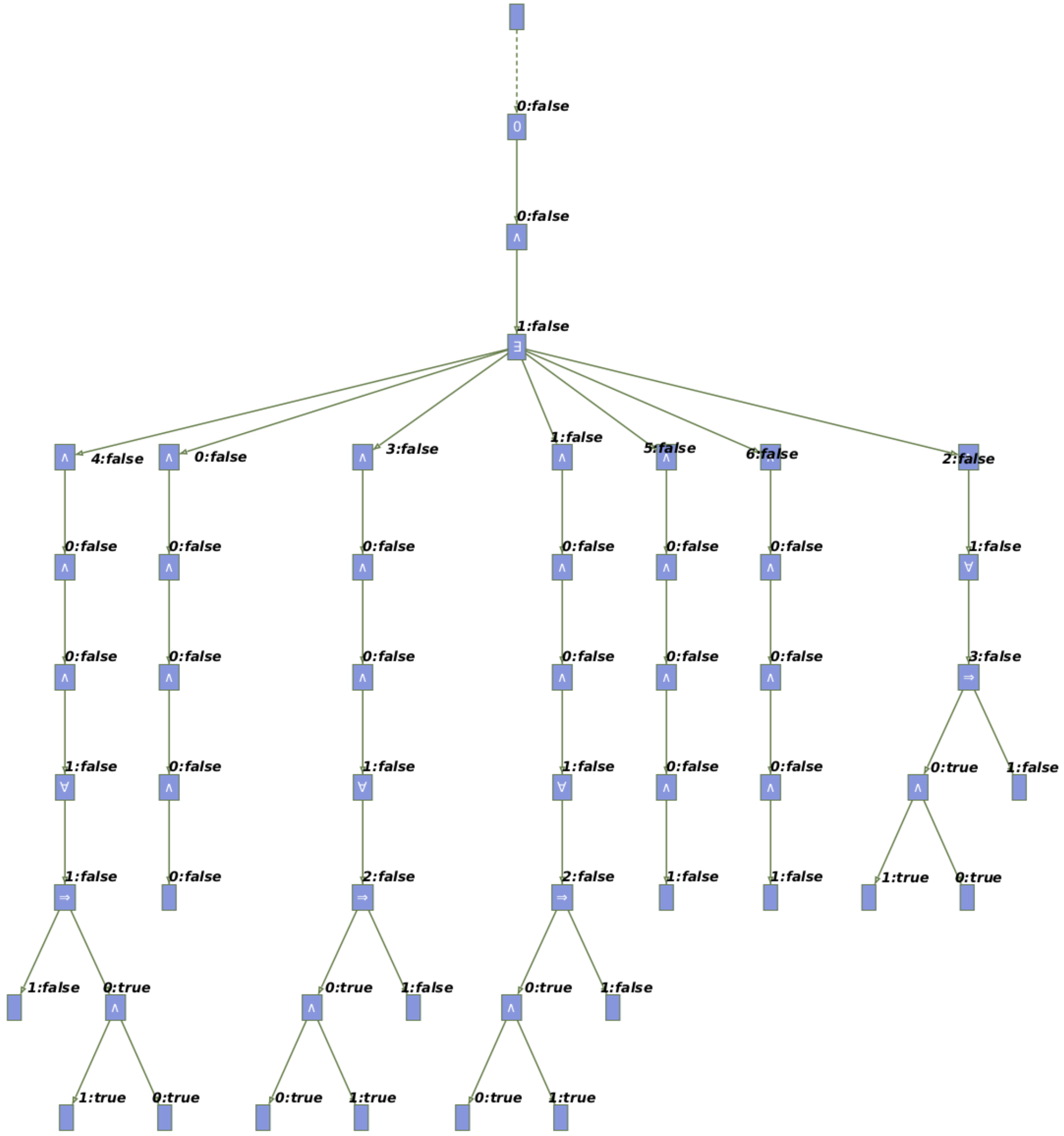


Figure 2: Cropped screenshot of the visualisation tree for `testOutput2a` (which tests the `output` predicate) when given an illegal input.

```
/* ==================================================================
   Course Logic, Lab Exercise
   "Syntax, Semantics, and Pragmatics of First−Order Logic"
   ==================================================================

  PROBLEM:

  Let Elem be the set of all integers in the range 0..M (for some
  constant M) which we call "elements". Let Arr be the set of all
  "arrays" of length N (for some constant N) of "element" values.
  Let Index be the set of all integers in the range −1..N (which
  includes all valid array indices 0..N−1 but also −1 and N).

  We introduce the predicate "a has n elements" to state that the
  last N−n slots a[n ],.., a[N−1] of array "a" are filled with value 0
  (thus array "a" may have arbitrary elements only in the first "n"
  slots a [0],.., a[n−1]).

  We introduce the predicate "a has no duplicates in the first
  n positions" to state that there are not any two different
  array indices less than "n" such that "a" holds the same
  element at those indices.

  Now consider the following problem: given an array "a" with n>0
  elements that has no duplicates in the first "n" positions, remove
  from "a" the smallest element; this results in an array that has
  n−1 elements. For example, for N=5 and M=3, the legal inputs
  a =[2,0,1,3,0] and n=4 result in the output b =[2,1,3,0,0].
  Since "a" has no duplicates in the first "n" positions,
  "b" is uniquely determined.

  In more detail, this problem can be described as follows:

    Input: a ∈ Arr, n ∈ Index where
      n is greater than 0
      a has n elements
      a has no duplicates in the first n positions
    Output: b ∈ Arr where
      b has n−1 elements
      there exists some position p ∈ Index such that
        p is greater equal 0 and less than n
        a[p] is less than equal a [0],..., a[n−1]
        b holds at positions 0,.., p−1 the same values as a
        b holds at positions p ,.., n−2 those values that
           a holds at positions p+1,..,n−1
```

YOUR TASKS:

0. Replace the name of the constant "WolfgangSchreiner" given below by your own name.

1. Complete the definition of predicate "array(a,n)" below to state "a has n elements" (as described above).

   Complete the definition of predicate "nodup(a,n)" below to state "a has no duplicates in the first n positions" (as described above).

   Validate your definitions by executing operations "array(..)" and "nodup(...)" with execution option "Silent" *not* selected. This prints the truth values of the predicates for *all* possible inputs. Check that the truth values are as expected.

2. Complete the definition of predicate "input(a,n)" below to state the input condition of the problem (as described above).

   Validate your input condition by executing the operations "testIn1()" and "testIn2()". These check the input condition for one legal input and for one illegal input (the first should run without error, the second with an error message).

3. Complete the definition of predicate "output(a,n,b)" below to state the output condition of the problem (as described above).

   Validate your output condition by executing the operations "testOut1()" and "testOut2()". These check the output condition for one legal output and for one illegal output (the first should run without error, the second with an error message).

4. Further validate your input/output conditions by executing "compute(..)" with execution option "Silent" *not* selected and translation option "Nondeterminism" selected. This prints all input/output pairs allowed by your specification. Check that these pairs conform to your expectations.

5. Finally verify your specification by executing the operations "correctOut()" and "completeOut()".

   – If "correctOut()" fails, then your output condition is too strong (or your input condition is too weak).

   – If "completeOut()" fails, then your output condition is too weak.

If these checks do not fail , your specification is correct (unless your input condition is too strong, which can be not ruled out by automatic verification).

DELIVERABLES:

The result of this assignment consists of a document that includes the following elements:

1. A complete listing of this file (with your formulas nicely formatted and indented).

2. A screenshot of the software displaying in the right terminal

   – the text ”Computing the value of WolfgangSchreiner...” (where the name ”WolfgangSchreiner” is replaced by your name).

   – the execution of operations correctOut(..) and completeOut(..) (with execution option ”Silent” selected).

   For this clear the terminal by pressing the ”Clear Output” button abeled by a broom, modify and save the file (which results in the ”Computing...” output) and then run the two operations.

3. A screenshot of the visualization window for operation testOut2a() which displays the evaluation of the formula ”output()” on an ∗ illegal ∗ output (yielding the truth value ”false”). Make sure

   – you have inlined the definition of ”output” into this operation (i.e., do not call ”output” but insert its definition)

   – you have switched on the visualization option ”Tree” (and set ”Width” to a suitably large value e.g. 800).

4. A systematic and detailed written analysis of the evaluation tree for testOut2a() where you describe in detail how the evaluation of each subformula yields the truth value ”false” of the overall formula.

   Please note that the label ”n:t” at each arrow denotes the number ”n” of the subformula to which the arrow points and its truth value ”t”; numbering starts with 0, so the first subformula is the one with label n=0 (subformulas may be shuffled by the automatic layout of the graph).

```
   Important: hover with the mouse pointer over each node to see
   the values of the variables on which each subformula depends and
   the truth value of the subformula. Use this information
   to explain the overall result.

   Take a look at the RISCAL manual, Section 2.9 "Visualizing
   Evaluation Trees", for more details on how to interpret the tree.

You will be asked in your presentation to illustrate points 3 and 4
(a laptop will be provided that runs RISCAL with your specification).

*/

// ====================================================================
// perform here your changes
// ====================================================================

// replace the name of this constant by your name
val LaurenzWeixlbaumer = 1 ;

val N = 5;
val M = 3;

type Elem = ℕ[M];
type Index = ℤ[−1,N];
type Arr = Array[N,Elem];

pred array(a:Arr,n:Index)
  requires n ≥ 0;
⇔
  // formulate here "a has n elements"
  ∀i:Index. ((i ≥ n ∧ i < N) ⇒ (a[i] = 0))
;

pred nodup(a:Arr,n:Index)
  requires n ≥ 0;
⇔
  // formulate here "a has no duplicates in the first n positions"
  ∀i:Index. ((i ≥ 0 ∧ i < n) ⇒ (
    ∀j:Index. (j ≥ 0 ∧ j < i) ⇒ (a[j] ≠ a[i]))
  )
;

pred input(a:Arr,n:Index) ⇔
  // formulate here the input condition
  n > 0 ∧ array(a, n) ∧ nodup(a, n)
;

pred output(a:Arr,n:Index,b:Arr)
  requires n > 0;
⇔
  // formulate here the output condition
  array(b, n − 1) ∧
  (∃p:Index. p ≥ 0 ∧ p < n ∧ (
    ∀i:Index. (i ≥ 0 ∧ i < n) ⇒ (a[p] ≤ a[i])
  ) ∧ (
    ∀i:Index. (i ≥ 0 ∧ i < p) ⇒ (b[i] = a[i])
  ) ∧ (
    ∀i:Index. (i ≥ p ∧ i < n − 1) ⇒ (b[i] = a[i + 1])
  ))
;

// ========================================================
// no more changes needed below (except for testOut2a())
// ========================================================

// legal inputs a,n and output b
val a = Array[N,Elem](0)
  with [0]=2 with [1]=0 with [2]=1 with [3]=3;
val n = 4 ;
val b = Array[N,Elem](0)
  with [0]=2 with [1]=1 with [2]=3;

// illegal input a0 and illegal output b0
val a0 = Array[N,Elem](0)
  with [0]=2 with [1]=3 with [2]=1 with [3]=3;
val b0 = Array[N,Elem](0)
  with [0]=2 with [1]=1 with [2]=1;

// execute this operation to ensure that your input condition
// holds/does not hold for the sample inputs
theorem testIn1() ⇔ input(a,n);
theorem testIn2() ⇔ let a = a0 in input(a,n);

// execute this operation to ensure that your output condition
// holds for the sample inputs and the correct output
theorem testOut1() ⇔ output(a,n,b);

// execute this operation to ensure that your output condition
// does *not* hold for the sample inputs and a *wrong* output
theorem testOut2() ⇔ let b = b0 in output(a,n,b);
```

```
// the visualization gets nicer if you inline the postcondition
// formula and then run this operation
theorem testOut2a() ⇔
  let b = b0 in
  // inline here your postcondition formula
  array(b, n − 1) ∧
  (∃p:Index. p ≥ 0 ∧ p < n ∧ (
    ∀i:Index. (i ≥ 0 ∧ i < n) ⇒ (a[p] ≤ a[i])
  ) ∧ (
    ∀i:Index. (i ≥ 0 ∧ i < p) ⇒ (b[i] = a[i])
  ) ∧ (
    ∀i:Index. (i ≥ p ∧ i < n − 1) ⇒ (b[i] = a[i + 1])
  ))
;

// execute this operation to see all legal
// input/output combinations
fun compute(a:Arr,n:Index):Arr
  requires input(a,n);
= choose b:Arr with output(a,n,b);

// this is a correct implementation of the problem specification
proc removeMinimum(a:Arr,n:Index):Arr
  requires n > 0;
{
  var b:Arr := a;
  var p:Index := 0;
  for var i:Index := 1; i < n; i:=i+1 do
  {
    if a[i] < a[p] then p := i;
  }
  for var i:Index := p; i < n−1; i:=i+1 do
  {
    b[i] := a[i+1];
  }
  b[n−1] := 0;
  return b;
}

// execute this operation to ensure that your output condition
// is not too strong
theorem correctOut(a:Arr,n:Index)
  requires input(a,n);
⇔
  let b = removeMinimum(a,n) in output(a,n,b)
```

```
;

// execute this operation to ensure that your output condition
// is not too weak
theorem completeOut(a:Arr,n:Index)
  requires input(a,n);
⇔
  ∀b:Arr. output(a,n,b) ⇒ b = removeMinimum(a,n)
;

// =========================================================================
// end of file
// =========================================================================
```