Exercise 1

a)

$$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\cdot\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{21}&a_{22}&a_{23}\\a_{11}&a_{12}&a_{13}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$$

b)

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}a_{12}&a_{11}&a_{13}\\a_{22}&a_{21}&a_{23}\\a_{32}&a_{31}&a_{33}\end{pmatrix}$$

c)

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}\cdot\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\2a_{31}&2a_{32}&2a_{33}\end{pmatrix}$$

d)

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&3&0\\0&0&1\end{pmatrix}=\begin{pmatrix}a_{11}&3a_{12}&a_{13}\\a_{21}&3a_{22}&a_{23}\\a_{31}&3a_{32}&a_{33}\end{pmatrix}$$

e)

$$\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\cdot\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\2a_{21}+a_{31}&2a_{212}+a_{32}&2a_{23}+a_{33}\end{pmatrix}$$

f)

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}=\begin{pmatrix}a_{11}&a_{12}+2a_{13}&a_{13}\\a_{21}&a_{22}+2a_{23}&a_{23}\\a_{31}&a_{32}+2a_{33}&a_{33}\end{pmatrix}$$

g)

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\cdot\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\\a_{11}&a_{12}&a_{13}\end{pmatrix}$$

h)

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\cdot\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}=\begin{pmatrix}a_{13}&a_{11}&a_{12}\\a_{23}&a_{21}&a_{22}\\a_{33}&a_{31}&a_{32}\end{pmatrix}$$

Exercise 2

$$R_{2,1}(-3)\cdot\begin{pmatrix}-1&1&-2&-3\\-3&0&-6&-12\\0&-2&2&3\\0&-6&4&11\end{pmatrix}=\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&-2&2&3\\0&-6&4&11\end{pmatrix}$$

$$R_{4,2}(-2)\cdot R_{3,2}\left(-\frac{2}{3}\right)\cdot\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&-2&2&3\\0&-6&4&11\end{pmatrix}=\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&4&17\end{pmatrix}$$

$$R_{4,3}(-2)\cdot\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&4&17\end{pmatrix}=\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&0&7\end{pmatrix}$$

$$R_{4,3}(-2)\cdot R_{4,2}(-2)\cdot R_{3,2}\left(-\frac{2}{3}\right)\cdot R_{2,1}(-3)\cdot\begin{pmatrix}-1&1&-2&-3\\-3&0&-6&-12\\0&-2&2&3\\0&-6&4&11\end{pmatrix}=\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&0&7\end{pmatrix}$$

$$\begin{pmatrix}-1&1&-2&-3\\-3&0&-6&-12\\0&-2&2&3\\0&-6&4&11\end{pmatrix}=R_{4,3}(2)\cdot R_{4,2}(2)\cdot R_{3,2}\left(\frac{2}{3}\right)\cdot R_{2,1}(3)\cdot\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&0&7\end{pmatrix}$$

$$\begin{pmatrix}-1&1&-2&-3\\-3&0&-6&-12\\0&-2&2&3\\0&-6&4&11\end{pmatrix}=\begin{pmatrix}1&0&0&0\\3&1&0&0\\0&\frac{2}{3}&1&0\\0&2&2&1\end{pmatrix}\cdot\begin{pmatrix}-1&1&-2&-3\\0&-3&0&-3\\0&0&2&5\\0&0&0&7\end{pmatrix}$$

Exercise 3

a)

$$R_{4,3}(-3)\cdot R_{3,1}(-1)\cdot\begin{pmatrix}1&0&0&2\\0&-3&1&2\\1&0&-1&1\\0&0&-3&0\end{pmatrix}=\begin{pmatrix}1&0&0&2\\0&-3&1&2\\0&0&-1&-1\\0&0&0&3\end{pmatrix}$$

$$\begin{pmatrix}1&0&0&2\\0&-3&1&2\\1&0&-1&1\\0&0&-3&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\1&0&1&0\\0&0&3&1\end{pmatrix}\cdot\begin{pmatrix}1&0&0&2\\0&-3&1&2\\0&0&-1&-1\\0&0&0&3\end{pmatrix}$$

b)

$$A\vec{x} = \vec{0} \quad \rightarrow \quad LU(\vec{x}) = \vec{0} \quad \rightarrow \quad L(U\vec{x}) = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Let

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and now the previous system becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\vec{y} = \vec{0}$. Now we can restate the intermediate system to

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\vec{x} = \vec{0}$.

c)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 6 \\ 3 \end{pmatrix}$$

Let

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

and now the previous system becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 6 \\ 3 \end{pmatrix}$$

with $y_1 = 5$, $y_2 = -4$, $y_3 = 1$, $y_4 = 0$. Now we can restate the intermediate system to

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \end{pmatrix}$$

with $x_4 = 0$, $x_3 = -1$, $x_2 = 1$ and $x_1 = 5$.

Exercise 4 The inverse of the given matrix is

$$\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \rightarrow$$

$$\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \end{array} \rightarrow$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}.$$

$$\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & \boxed{1} & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \boxed{1} \\ 1 & -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \rightarrow$$

$$\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \boxed{1} \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array} \rightarrow \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \boxed{1} & \boxed{-1} & 0 & 1 \end{array}$$

$$A^{-1} = R_{4,2}(-1) \cdot R_{4,1}(1) \cdot R_{3,4}(1) \cdot R_{2,4}(1) \cdot R_{1,2}(1)$$

Exercise 5

a) The (two) columns are linearly independent thus the rank is 2.

b) The zero matrix has rank zero.

c) The columns are linearly dependent, the largest set of linearly independent columns has one element, thus the rank is 1.

d) – g) See c).

Exercise 6

a) Using only elementary transformations,

$$P_1^{-1} = T_{1,3} \cdot T_{2,3}, \quad \text{thus} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

b) The given matrix is equivalent to the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \text{whose inverse is} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \text{which is equvalent to} \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

c)

$$\left.\begin{array}{cccccc|cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right.$$

d)

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

e)

$$\begin{aligned} (P \cdot P^T)_{i,j} &= \sum_{k=1}^{n} P_{i,k} \cdot P^T_{k,j} \\ &= \sum_{k=1}^{n} P_{i,k} \cdot P_{j,k}. \end{aligned}$$

Consider that $P_{i,k}$ and $P_{j,k}$ are both in the same column. Only one element in a column may be nonzero. Thus the only way for their product to be nonzero is if $i = j$, or

$$\sum_{k=1}^{n} P_{i,k} \cdot P_{j,k} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

This is the definition of the identity matrix.