1 Gas Continuity Equation

Mass continuity on gray (gas) control volume (Vg) surround by boundary Ag

$$\frac{\partial}{\partial t}\int_{V_g} \rho \mathrm{d}V + \int_{A_g} \rho(\hat{u}\cdot\hat{n})\mathrm{d}A = 0$$

Using second-order approximations of the integrals and a first-order time derivative

$$\frac{\rho V_g - \rho^0 V_g^0}{\Delta t} + \sum_{f,g} \rho(\hat{u}\cdot \hat{A_{f,g}}) = 0$$

Translate this to the full cell

$$V_c \frac{\rho\alpha_g - \rho^0\alpha_g^0}{\Delta t} + \sum_f \rho(\hat{u}\cdot\hat{A}_f)\alpha_f = \dot{m}'' A_b$$

Divide by cell volume, V_c, to give the equation in OpenFOAM form that we want to solve

$$\boxed{\frac{\rho\alpha_g - \rho^0\alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f \rho(\hat{u}\cdot\hat{A}_f)\alpha_f = \frac{\dot{m}'' A_b}{V_c}} \tag{1}$$

2 Gas Momentum Equation

Conservation of momentum on gray (gas) control volume (Vg) surround by boundary Ag

$$\frac{\partial}{\partial t}\int_{V_g} \rho\hat{u} \mathrm{d}V + \int_{A_g} \hat{u}(\rho\hat{u}\cdot\hat{n})\mathrm{d}A + \hat{F}_p = S(\hat{u})$$

Using second-order approximations of the integrals and a first-order time derivative

$$\frac{\rho\hat{u}V_g - \rho^0\hat{u}^0 V_g^0}{\Delta t} + \sum_{f,g} \hat{u}(\rho\hat{u}\cdot \hat{A_{f,g}}) + \hat{F}_p = S(\hat{u})$$

Translate this to the full cell

$$V_c \frac{\rho\hat{u}\alpha_g - \rho^0\hat{u}^0\alpha_g^0}{\Delta t} + \sum_f \hat{u}(\rho\hat{u}\cdot\hat{A}_f)\alpha_f + \hat{F}_p\alpha_f = S(\hat{u}) + \hat{u}_b\dot{m}'' A_b$$

Divide by cell volume, V_c, to give the equation in OpenFOAM form that we want to solve

$$\boxed{\frac{\rho\hat{u}\alpha_g - \rho^0\hat{u}^0\alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f \hat{u}(\rho\hat{u}\cdot\hat{A}_f)\alpha_f + \frac{\hat{F}_p\alpha_f}{V_c} = \frac{S(\hat{u})}{V_c} + \frac{\hat{u}_b\dot{m}'' A_b}{V_c}} \tag{2}$$

3 Gas Energy Equation

Conservation of energy on the gray (gas) control volume surrounded by boundary Ag

$$\frac{\partial}{\partial t}\int_{V_g}\rho E \mathrm{d}V + \int_{A_g}(E+\frac{p}{\rho})(\rho\hat{u}\cdot\hat{n})\mathrm{d}A + \int_{A_g}(\hat{q}\cdot\hat{n})\mathrm{d}A = S(E)$$

The total energy of the fluid can be rewritten in terms of the specific enthalpy (h_s) such that

$$\begin{aligned}
K =& \frac{u^2+v^2+w^2}{2}\\
i =& h_s - \frac{p}{\rho}\\
E =& i + K\\
E =& h_s + K - \frac{p}{\rho}
\end{aligned}$$

Substituting into the energy equation then yields

$$\frac{\partial}{\partial t}\int_{V_g}\rho(h_s + K - \frac{p}{\rho})\mathrm{d}V + \int_{A_g}(h_s+K)(\rho\hat{u}\cdot\hat{n})\mathrm{d}A + \int_{A_g}(\hat{q}\cdot\hat{n})\mathrm{d}A = S(E)$$

Using second-order approximations of the integrals and a first-order time derivative

$$\begin{aligned}
&\frac{\rho h_s V_g - \rho^0 h_s^0 V_g^0}{\Delta t} + \sum_{f,g} h_s(\rho\hat{u}\cdot\hat{A}_{f,g})\\
+&\frac{\rho K V_g - \rho^0 K^0 V_g^0}{\Delta t} + \sum_{f,g} K(\rho\hat{u}\cdot\hat{A}_{f,g})\\
+&\frac{p V_g - p^0 V_g^0}{\Delta t} + \sum_{f,g}(\hat{q}\cdot\hat{A}_{f,g})\\
=&S(E)
\end{aligned}$$

Translate equation to full cell

$$\begin{aligned}
&V_c\frac{\rho h_s \alpha_g - \rho^0 h_s^0 \alpha_g^0}{\Delta t} + \sum_{f} h_s(\rho\hat{u}\cdot\hat{A}_{f})\alpha_f\\
+&V_c\frac{\rho K \alpha_g - \rho^0 K^0 \alpha_g^0}{\Delta t} + \sum_{f} K(\rho\hat{u}\cdot\hat{A}_{f})\alpha_f\\
+&V_c\frac{p \alpha_g - p^0 \alpha_g^0}{\Delta t} + \sum_{f}(\hat{q}\cdot\hat{A}_{f})\alpha_f\\
=&S(E) + K_b\dot{m}'' A_b + \hat{q}_b\cdot\hat{A}_b
\end{aligned}$$

Divide by cell volume, V_c, to give the equation in OpenFOAM form that we want to solve

$$\boxed{\begin{aligned}&\frac{\rho h_s \alpha_g - \rho^0 h_s^0 \alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f h_s(\rho \hat{u}\cdot\hat{A}_f)\alpha_f\\ +&\frac{\rho K \alpha_g - \rho^0 K^0 \alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f K(\rho \hat{u}\cdot\hat{A}_f)\alpha_f\\ +&\frac{p\alpha_g - p^0\alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f (\hat{q}\cdot\hat{A}_f)\alpha_f\\ =&\frac{S(E)}{V_c} + \frac{K_b \dot{m}'' A_b}{V_c} + \frac{\hat{q}_b\cdot\hat{A}_b}{V_c}\end{aligned}} \tag{3}$$

4 Gas Species Equation

Conservation of a given species mass fraction (Y_i) on gray (gas) control volume (Vg) surround by boundary Ag

$$\frac{\partial}{\partial t}\int_{V_g}\rho Y_i \mathrm{d}V + \int_{A_g} Y_i(\rho\hat{u}\cdot\hat{n})\mathrm{d}A + \int_{A_g}(\hat{G}_i\cdot\hat{n})\mathrm{d}A = S(Y_i)$$

Using second-order approximations of the integrals and a first-order time derivative

$$\frac{\rho Y_i V_g - \rho^0 Y_i^0 V_g^0}{\Delta t} + \sum_{f,g} Y_i(\rho\hat{u}\cdot\hat{A_{f,g}}) + \sum_{f,g}(\hat{G}_i\cdot\hat{A_{f,g}}) = S(Y_i)$$

Translate this to the full cell

$$V_c\frac{\rho Y_i\alpha_g - \rho^0 Y_i^0\alpha_g^0}{\Delta t} + \sum_f Y_i(\rho\hat{u}\cdot\hat{A}_f)\alpha_f + \sum_{f,g}(\hat{G}_i\cdot\hat{A}_f)\alpha_f = S(Y_i) + Y_{i,b}\dot{m}''A_b$$

Divide by cell volume, V_c, to give the equation in OpenFOAM form that we want to solve

$$\boxed{\begin{aligned}&\frac{\rho Y_i\alpha_g - \rho^0 Y_i^0\alpha_g^0}{\Delta t} + \frac{1}{V_c}\sum_f Y_i(\rho\hat{u}\cdot\hat{A}_f)\alpha_f\\ +&\frac{1}{V_c}\sum_{f,g}(\hat{G}_i\cdot\hat{A}_f)\alpha_f = \frac{S(Y_i)}{V_c} + \frac{Y_{i,b}\dot{m}''A_b}{V_c}\end{aligned}} \tag{4}$$