Real Analysis: Homework #2

Due on Mar 12, 2024 at 10:00am

Professor Yakun Xi Tuesday

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Problem 1

Complete the proof of Theorem 1.9.

Proof. First, we show that $\bar{\mu}$ be a complete measure on $\bar{\mathcal{M}}$. We divide this proof in the following three steps. First, $\forall \mathcal{S} \in \bar{\mathcal{M}}$, $\mathcal{S}$ can be divided to $\mathcal{S} = \mathcal{E} \cup \mathcal{F}$, where $\mathcal{E} \in \mathcal{M}$, $\mathcal{F} \subset N$ for $\mu(N) = 0$. By the definition of $\bar{\mu}$, $\bar{\mu}(\mathcal{S}) = \mu(\mathcal{E}) \geq 0$. So $\bar{\mu} : \bar{\mathcal{M}} \to [0, \infty]$.
Second, we show the additive. Choose $\{\mathcal{S}_i\}_{i=1}^{\infty} \subset \bar{\mathcal{M}}$ with disjoint sets $\mathcal{S}_i$, i.e. $\mathcal{S}_i = \mathcal{E}_i \cup \mathcal{F}_i$ and $\mathcal{E}_i \cap \mathcal{F}_i = \emptyset$, $\{\mathcal{E}_i\}$ disjoint, $\mathcal{F}_i \subset N_i$ with $\mu(N_i) = 0$. Then:

$$\bar{\mu}\left(\cup_{i=1}^{\infty} \mathcal{S}_i\right) = \bar{\mu}\left(\left(\cup_{i=1}^{\infty} \mathcal{E}_i\right) \cup \left(\cup_{i=1}^{\infty} \mathcal{F}_i\right)\right).$$

As $\mathcal{F}_i \subset N_i$ with $\mu(N_i) = 0$, $\cup_{i=1}^{\infty} \mathcal{F}_i \subset \cup_{i=1}^{\infty} N_i$ with $\mu(\cup_{i=1}^{\infty} N_i) = 0$. So:

$$\bar{\mu}(\cup_{i=1}^{\infty} \mathcal{S}_i) = \mu(\cup_{i=1}^{\infty} \mathcal{E}_i) = \sum_{i=1}^{\infty} \mu(\mathcal{E}_i) = \sum_{i=1}^{\infty} \bar{\mu}(\mathcal{E}_i \cup \mathcal{F}_i) = \sum_{i=1}^{\infty} \bar{\mu}(\mathcal{S}_i).$$

Third, by the definition, $\bar{\mu}(\emptyset) = \mu(\emptyset) = 0$. So $\bar{\mu}$ is a measure. And for N s.t. $\bar{\mu}(N) = 0$, i.e. $\exists \tilde{N}$ s.t. $N \subset \tilde{N}$, $\mu(\tilde{N}) = 0$, we have: $\forall \mathcal{F} \subset N$, $\mathcal{F} \subset \tilde{N}$, i.e. $\bar{\mu}(\mathcal{F}) = \bar{\mu}(\emptyset \cup \mathcal{F}) = 0$. So $\bar{\mu}$ is complete.
Then we show the uniqueness. Assume $\tilde{\mu}$ is a complete measure on $\bar{\mathcal{M}}$, then:

$$\tilde{\mu}(\mathcal{E}) \leq \tilde{\mu}(\mathcal{E} \cup \mathcal{F}) \leq \tilde{\mu}(\mathcal{E}) + \tilde{\mu}(\mathcal{F}).$$

As $\forall \mathcal{F} \subset N$ s.t. $\mu(N) = 0$, $\tilde{\mu}(\mathcal{F}) = 0$, we can see $\tilde{\mu}(\mathcal{E} \cup \mathcal{F}) = \tilde{\mu}(\mathcal{E})$, it means $\tilde{\mu} = \bar{\mu}$. □

Problem 2

A finitely additive measure μ is a measure iff it is continuous from below. If $\mu(X) < \infty$, μ is a measure iff it is continuous from above.

Proof. If μ is a measure, Theorem 1.8 shows the continuity from below, and the continuity from above in the case $\mu(X) < \infty$. So it suffices to show the opposite direction.
If μ be finitely additive and continuous from below, choose disjoint sets $\{\mathcal{E}_i\}_{i=1}^{\infty}$, mark $\mathcal{F}_i := \cup_{j=1}^{i} \mathcal{E}_j$, then

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots.$$

μ is continuous from below, i.e.

$$\mu(\cup_{i=1}^{\infty} \mathcal{E}_i) = \mu(\cup_{i=1}^{\infty} \mathcal{F}_i) = \lim_{n\to\infty} \mu(\mathcal{F}_n).$$

By the finitely additivity,

$$\lim_{n\to\infty} \mu(\mathcal{F}_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(\mathcal{E}_i) = \sum_{i=1}^{\infty} \mu(\mathcal{E}_i).$$

It means μ is additive.
If μ be finitely additive and continuous from above, $\mu(X) < \infty$, choose disjoint sets $\{\mathcal{E}_i\}_{i=1}^{\infty}$, $\mathcal{F}_i := \cup_{j=1}^{i} \mathcal{E}_i$, then:

$$\mathcal{F}_1^c \supset \mathcal{F}_2^c \supset \cdots \supset \mathcal{F}_n^c \supset \cdots,$$

and $\mu(\mathcal{F}_1^c) \leq \mu(X) < \infty$. Then:

$$\mu(\cap_{i=1}^{\infty} \mathcal{E}_i^c) = \mu(\cap_{i=1}^{\infty} \mathcal{F}_i^c) = \lim_{n\to\infty} \mu(\mathcal{F}_n^c) = \mu(X) - \lim_{n\to\infty} \mu(\mathcal{F}_n) = \mu(X) - \sum_{n=1}^{\infty} \mu(\mathcal{E}_n).$$

The second step is derived by the continuity from above, and the final step from the finitely additivity.
So:

$$\sum_{n=1}^{\infty} \mu(\mathcal{E}_n) = \mu(X) - \mu(\cap_{i=1}^{\infty} \mathcal{F}_i^c) = \mu((\cap_{i=1}^{\infty} \mathcal{F}_i^c)^c) = \mu(\cup_{i=1}^{\infty} \mathcal{F}_i) = \mu(\cup_{i=1}^{\infty} \mathcal{E}_i).$$

It means μ is additive. $\square$

Problem 3

Every σ-finite measure is semifinite.

Proof. $\forall \mathcal{E} \subset \mathcal{M}$ and $\mu(\mathcal{E}) = \infty$, we should construct $\mathcal{F} \subset \mathcal{E}$ s.t. $0 < \mu(\mathcal{F}) < \infty$.
As μ be σ-finite, $\exists \{\mathcal{E}_i\}_{i=1}^{\infty} \subset \mathcal{M}$ s.t. $\cup_{i=1}^{\infty} \mathcal{E}_i = X$, $\mu(\mathcal{E}_i) < \infty$, then $\mathcal{E} = \cup_{i=1}^{\infty}(\mathcal{E}_i \cap \mathcal{E})$.
Mark $\mathcal{F}_i := \mathcal{E}_i \cap \mathcal{E}$, as $\mu(\mathcal{E}_i) < \infty$, $\forall i$, $\mu(\mathcal{F}_i) < \infty$. If $\forall i$, $\mu(\mathcal{F}_i) = 0$, as $\mathcal{E} = \cup_{i=1}^{\infty} \mathcal{F}_i$, $\mu(\mathcal{E}) = 0$, contradict!
So $\exists j \in \mathbb{N}$ s.t. $\mu(\mathcal{F}_j) > 0$, $\mathcal{F}_j \subset \mathcal{E}$ and $0 < \mu(\mathcal{F}_j) < \infty$. $\square$

Problem 4

If μ is a semifinite measure and $\mu(E) = \infty$, for any $C > 0$ there exists $F \subset E$ with $C < \mu(F) < \infty$.

Proof. Proof by contradiction. Assume $\exists C_0 > 0$ s.t. $\forall F \subset E$ with $\mu(F) < \infty$, $\mu(F) < C_0$. Mark

$$\mathcal{F} := \{F | F \subset E, \mu(F) < C_0\},$$

$\mathcal{F}$ is an ordered set with relation $\subset$. By Zorn's Lemma, there exists a maximum element $\bar{F}$ in $\mathcal{F}$, and $\mu(\bar{F}) < C_0$.
As $\mu(E) = \infty$ and $\mu(\bar{F}) < C_0$, $\mu(E \setminus \bar{F}) = \infty$. μ be semifinite means $\exists \mathcal{S} \subset E \setminus \bar{F}$, $\mu(\mathcal{S}) < \infty$. So $\mu(\bar{F} \cup \mathcal{S}) < C_0 + \mu(\mathcal{S}) < \infty$.
On the other hand, by assumption, $\mu(\bar{F} \cup \mathcal{S}) < \infty$ means $\mu(\bar{F} \cup \mathcal{S}) < C_0$, i.e. $\bar{F} \cup \mathcal{S} \in \mathcal{F}$. However, $\bar{F}$ is the maximum element in $\mathcal{F}$, this leads to a contradiction, Q.E.D. $\square$

Problem 5

If μ^* is an outer measure on X and $\{A_j\}_1^{\infty}$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_{j=1}^{\infty} A_j)) = \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. By Caratheodory Theorem, μ^*-measurable sets form a σ-algebra. Mark $B_n := \cup_{j=1}^{n} A_j$, it suffices to show:

$$\forall n \in \mathbb{N},\ \mu^*(E \cap B_n) = \sum_{j=1}^{n} \mu^*(E \cap A_j). \tag{1}$$

Proof by induction. For $n = 1$, $B_1 = A_1$, so (1) holds. Assume (1) holds for $n = k$, consider $n = k + 1$.
By the definition of μ^*-measurable,

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_{n+1}) + \mu^*(E \cap B_{n+1}^c) \\ &= \mu^*(E \cap B_{n+1}) + \mu^*(E \cap B_n^c \cap A_{n+1}^c) \\ &= \mu^*(E \cap B_{n+1}) + \mu^*(E \cap B_n^c) - \mu^*(E \cap B_n^c \cap A_{n+1}) \\ &= \mu^*(E \cap B_{n+1}) + \mu^*(E \cap B_n^c) - \mu^*(E \cap A_{n+1}), \end{aligned} \tag{2}$$

the final step from the fact that $\{A_i\}$ disjoint. By (2) and the assumption,

$$\begin{aligned}\mu^*(E\cap B_{n+1}) &= \mu^*(E) - \mu^*(E\cap B_n^c) + \mu^*(E\cap A_{n+1}) \\ &= \mu^*(E\cap B_n) + \mu^*(E\cap A_{n+1}) \\ &= \sum_{j=1}^{n+1} \mu^*(E\cap A_j).\end{aligned} \tag{3}$$

So, by induction, (1) holds for all $n\in\mathbb{N}$.
If $\forall n\in\mathbb{N}$, $\mu^*(E\cap B_n)<\infty$, the proof is completed just by setting $n\to\infty$ in (1). If $\exists\ N_0\in\mathbb{N}$ s.t. $\mu^*(E\cap B_{N_0})=\infty$, as $B_{N_0}\subset\cup_{i=1}^\infty A_i$, $\mu^*(E\cap(\cup_{i=1}^\infty A_i))\geq\mu^*(E\cap B_{N_0})=\infty$, and $\sum_{j=1}^\infty\mu^*(E\cap A_j)\geq\sum_{j=1}^{N_0}(E\cap A_j)=\infty$. Now we complete the proof. □

Problem 6

Let $\mathcal{A}\subset\mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

1. For any $E\subset X$ and $\epsilon>0$ there exists $A\in\mathcal{A}_\sigma$ with $E\subset A$ and $\mu^*(A)\leq\mu^*(E)+\epsilon$.
2. If $\mu^*(E)<\infty$, then E is μ^*-measurable iff there exists $B\in\mathcal{A}_{\sigma\delta}$ with $E\subset B$ and $\mu^*(B\setminus E)=0$.
3. If μ_0 is σ-finite, the restriction $\mu^*(E)<\infty$ in (b) is superfluous.

Proof. (1) If $\mu^*(E)=\infty$, just set $A=\cup_{\mathcal{S}\in\mathcal{A}}\mathcal{S}$. So we only need to check the case $\mu^*(E)<\infty$. By the definition of outer measure derived by premeasure,

$$\mu^*(E)=\inf_{(\cup_{i=1}^\infty A_i)\supset E}\sum\mu_0(A_i),$$

where $\{A_i\}$ are disjoint. It means that $\forall\epsilon>0$, $\exists\{A_i\}\subset\mathcal{A}$, s.t. $E\subset\cup_{i=1}^\infty A_i$ and $\mu^*(E)+\epsilon\geq\sum_{i=1}^\infty\mu_0(A_i)$. Set $A:=\cup_{i=1}^\infty A_i$, we can see $A\in\mathcal{A}_\sigma$, and $\mu^*(A)=\sum_{i=1}^\infty\mu_0(A_i)$. It completes the proof. □
(2) Before the proof, we introduce the following three lemmas first.

Lemma 1. *$\mathcal{A}_{\sigma\delta}$ is a σ-algebra generated by elements in $\mathcal{A}$.*

Proof. By the definition of $\mathcal{A}_{\sigma\delta}$, we can see $\mathcal{A}\subset\mathcal{A}_{\sigma\delta}$ and $\mathcal{A}_{\sigma\delta}$ be closed under countable union. For complements, choose $A\in\mathcal{A}_{\sigma\delta}$, $\exists\{A_{ij}\}\subset\mathcal{A}$, s.t.

$$A=\cap_{i=1}^\infty\cup_{j=1}^\infty A_{ij}.$$

Then

$$A^c=\cup_{i=1}^\infty\cap_{j=1}^\infty A_{ij}^c=\cap_{j=1}^\infty\cup_{i=1}^\infty A_{ij}^c\in\mathcal{A}_{\sigma\delta}.$$

So $\mathcal{A}_{\sigma\delta}$ is a σ-algebra. On the other hand, for a σ-algebra $\tilde{\mathcal{A}}\supset\mathcal{A}$, σ-algebra is closed under countable unions and intersections, so $\mathcal{A}_{\sigma\delta}\subset\tilde{\mathcal{A}}$, i.e. $\mathcal{A}_{\sigma\delta}=\mathcal{M}(\mathcal{A})$. □

Lemma 2. *$\forall\mathcal{S}\in\mathcal{A}_{\sigma\delta}$, $\mathcal{S}$ is a μ^*-measurable set.*

Proof. By Caratheodory's Theorem, the μ^*-measurable sets form a σ-algebra. On the other hand, by Proposition 1.13, $\forall\ A\in\mathcal{A}$, A is μ^*-measurable. Choose $\mathcal{T}:=\{\mathcal{S}:\mathcal{S}\text{ is }\mu^*\text{-measurable}\}$, by Lemma 1, $\mathcal{A}_{\sigma\delta}\subset\mathcal{T}$. Q.E.D. □

Lemma 3. *$\mu^*(A)=0$ means A is μ^*-measurable.*

Proof. It suffices to show that $\forall \mathcal{S} \in \mathcal{P}(X)$, $\mu^*(\mathcal{S}) = \mu^*(S \cap A^c)$. $\forall$ disjoint sets $\{\mathcal{A}_i\} \subset \mathcal{A}$ satisfies $\cup_{i=1}^{\infty} \mathcal{A}_i \supset \mathcal{S}$, choose $\tilde{\mathcal{A}}_i := \mathcal{A}_i \setminus A$, it means that $\cup_{i=1}^{\infty} \tilde{\mathcal{A}}_i \supset \mathcal{S} \cap A^c$, and $\sum_{i=1}^{\infty} \mu_0(\mathcal{A}_i) = \sum_{i=1}^{\infty} \mu_0(\tilde{\mathcal{A}}_i)$. It completes the proof. ☐

Now we continue the proof.
" $\Rightarrow$ ": As $\mu^*(E) < \infty$, by (1), we can choose a sequence of sets $A_n \in \mathcal{A}_\sigma$, s.t.

- $E \subset A_n$.
- $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$.

Choose $B := \cap_{n=1}^{\infty} A_n$, it's clear that $B \in \mathcal{A}_{\sigma\delta}$, $E \subset B$. WLOG, we assume the sequence $\{A_n\}$ is decreasing. Then:

$$0 \leq \mu^*(B \setminus E) = \mu^*(\cap_{n=1}^{\infty} A_n \setminus E) = \lim_{n\to\infty} \mu^*(A_n \setminus E) \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

It completes the proof.
" $\Leftarrow$ ": Choose $F \in \mathcal{P}(X)$. By the monotony of μ^*,

$$\mu^*(F \cap E) + \mu^*(F \cap E^c) \leq \mu^*(F \cap B) + \mu^*(F \cap E^c).$$

By Lemma 2, B is μ^*-measurable, i.e. $\mu^*(F) = \mu^*(F \cap B) + \mu^*(F \cap B^c)$. So it suffices to show $\mu^*(F \cap E^c) = \mu^*(F \cap B^c)$. We have:

$$\begin{aligned}\mu^*(F \cap B^c) &= \mu^*(F \cap E^c \cap (B \setminus E)^c) \\ &= \mu^*(F \cap E^c) + \mu^*(F \cap E^c \cap (B \setminus E)) \\ &= \mu^*(F \cap E^c).\end{aligned}$$

The first step is from $B = E \cup B \setminus E$, The second from the fact that $\mu^*(B \setminus E) = 0$ and Lemma 3, and the third from the monotony of μ^*.
This completes the proof. ☐
(3) μ_0 be σ-finite means $\exists \{A_i\} \subset \mathcal{A}$ s.t.

- $X = \cup_{i=1}^{\infty} A_i$.
- $\mu_0(A_i) < \infty$.

" $\Rightarrow$ ": E is μ^*-measurable means $A_i \cap E$ is μ^*-measurable. By (2), $\exists B_i \in \mathcal{A}_{\sigma\delta}$ with $X_i \cap E \subset B_i$ and $\mu^*(B_i \setminus (X_i \cap E)) = 0$. Mark $B := \cup_{i=1}^{\infty} B_i$, it's clear that $E \subset \cup_{i=1}^{\infty} B_i = B$ and

$$\mu^*(B \setminus E) \leq \sum_{i=1}^{\infty} \mu^*(B_i \setminus (X_i \cap E)) = 0.$$

" $\Leftarrow$ ": $E \subset B$ means $A_i \cap E \subset A_i \cap B$, then

$$\mu^*((A_i \cap B) \setminus (A_i \cap E)) \leq \mu^*(B \cap E) = 0.$$

On the other hand, for $B \in \mathcal{A}_{\sigma\delta}$, $A_i \cap B \in \mathcal{A}_{\sigma\delta}$. By (2), $X_i \cap E$ is μ^*-measurable. By Caratheodory's Theorem, $\cup_{i=1}^{\infty}(X_i \cap E) = E$ is μ^*-measurable. ☐

Problem 7

Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the **inner measure** of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

Proof. " $\Rightarrow$ ": If E is μ^*-measurable and μ_0 be finite, by Problem 6(3), $\exists B \in \mathcal{A}_{\sigma\delta}$ s.t. $\mu^*(B \setminus E) = 0$.
By Lemma 2, B is μ^*-measurable, i.e. $\mu_0(X) = \mu^*(B) + \mu^*(B^c)$. On the other hand, we have:

- $\mu^*(E) \leq \mu^*(B) = \mu^*(E \cup (B \setminus E)) \leq \mu^*(E) + \mu^*(B \setminus E) = \mu^*(E)$.
- $\mu^*(B^c) = \mu^*(E^c \cap (B \setminus E)^c) = \mu^*(E^c) - \mu^*(E^c \cap (B \setminus E)) = \mu^*(E^c)$, this equality is derived from Lemma 3.

So $\mu_0(X) = \mu^*(E) + \mu^*(E^c) = \mu^*(E) + \mu_*(E)$. $\square$

" $\Leftarrow$ ": Since μ^* be finite, $\mu^*(E) < \infty$. Then by Problem 6(1), $\exists \ \{A_n\} \subset \mathcal{A}_\sigma$ s.t.

- $E \subset A_n$.
- $\mu^*(A_n) \leq \mu^*(E) + \frac{1}{n}$.

Mark $B := \cap_{n=1}^{\infty} A_n \in \mathcal{A}_{\sigma\delta}$, by Lemma 2, B is μ^*-measurable and $E^c \supset B^c$, so:

$$\mu^*(E^c) = \mu^*(E^c \cap B) + \mu^*(B^c).$$

It means:

$$\begin{aligned}
\mu^*(B \setminus E) &= \mu_0(X) - \mu^*(E) - \mu^*(B^c) \\
&\leq \frac{1}{n} + \mu_0(X) - \mu^*(A_n) - \mu^*(B^c) \\
&\leq \frac{1}{n} + \mu_0(X) - \mu^*(A_n \cup B^c) \\
&= \frac{1}{n}.
\end{aligned}$$

Choose $n \to \infty$, $\mu^*(B \setminus E) = 0$. By Problem 6(3), E is μ^*-measurable. $\square$