

# Real Analysis: Homework #1

Due on Mar 4, 2024 at 10:00am

*Professor Yakun Xi Tuesday*

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## Problem 1

Complete the proof of Proposition 1.2.

*Proof.* • choose  $N_0 \in \mathbb{N}$  such that  $\frac{1}{N_0} < \frac{1}{b-a}$ ,

$$(a, b) = \cup_{n=N_0}^{\infty} \left( a, b - \frac{1}{n} \right].$$

- choose  $N_0 \in \mathbb{N}$  such that  $\frac{1}{N_0} < \frac{1}{b-a}$ ,

$$(a, b) = \cup_{n=N_0}^{\infty} \left[ a + \frac{1}{n}, b \right),$$

•

$$(a, b) = (a, \infty) \cap \left( \cap_{n=1}^{\infty} \left( b - \frac{1}{n}, \infty \right) \right)^c,$$

•

$$(a, b) = (-\infty, b) \cap \left( \cap_{n=1}^{\infty} \left( -\infty, a + \frac{1}{n} \right) \right)^c,$$

•

$$(a, b) = [b, \infty)^c \cap \left( \cup_{n=1}^{\infty} \left[ a + \frac{1}{n}, \infty \right) \right),$$

•

$$(a, b) = (-\infty, a]^c \cap \left( \cup_{n=1}^{\infty} \left( -\infty, b - \frac{1}{n} \right] \right).$$

By the definition of generated  $\sigma$ -algebra, it means  $(a, b) \in \mathcal{M}(\mathcal{E}_i)$ ,  $\forall i = 3, 4, 5, 6, 7, 8$ . By Lemma 1.1,  $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_i)$ . Then we complete the proof of Proposition 1.2.  $\square$

## Problem 2

Let  $\mathcal{M}$  be an infinite  $\sigma$ -algebra

1.  $\mathcal{M}$  contains an infinite sequence of disjoint sets.
2.  $\text{card}(\mathcal{M}) \geq \aleph$ .

*Proof.* (1).  $\mathcal{M}$  be an infinite  $\sigma$ -algebra means that there exists at least  $\aleph_0$  distinct non-empty sets  $\{\mathcal{S}_i\}_{i=1}^{\infty} \subset \mathcal{M}$ . Unfortunately,  $\{\mathcal{S}_i\}_{i=1}^{\infty}$  isn't a sequence of disjoint sets in general.

So, we should check the following lemma:

**Lemma 1.**  $\forall n \in \mathbb{N}$ , if  $\{\mathcal{S}_i\}_{i=1}^n \subset \mathcal{M}$  are distinct non-empty sets, there exists a sequence of disjoint non-empty sets  $\{\mathcal{T}_i\}_{i=1}^n$  such that  $\mathcal{T}_i \in \mathcal{M}$ , and  $\cup_{i=1}^n \mathcal{T}_i = \cup_{i=1}^n \mathcal{S}_i$ .

*Proof.* For  $n = 1$ , just set  $\mathcal{T}_1 = \mathcal{S}_1$ . Assume the lemma holds for  $n = k$ , consider  $\{\mathcal{S}_i\}_{i=1}^{k+1}$ . By the assumption, there exists a sequence of disjoint non-empty sets  $\{\tilde{\mathcal{S}}_i\}_{i=1}^k \subset \mathcal{M}$  such that  $\cup_{i=1}^k \mathcal{S}_i = \cup_{i=1}^k \tilde{\mathcal{S}}_i$ . For  $\mathcal{S}_{k+1}$ , there are three distinct cases:

- $\exists i \leq k, \mathcal{S}_{k+1} \subset \tilde{\mathcal{S}}_i$ .
- $\forall i \leq k, \mathcal{S}_{k+1} \cap \tilde{\mathcal{S}}_i = \emptyset$ .
- Other cases.

For case 1, we choose  $\mathcal{T}_j = \tilde{\mathcal{S}}_j$  for  $j \leq k$  and  $j \neq i$ ,  $\mathcal{T}_i = \tilde{\mathcal{S}}_i \setminus \mathcal{S}_{k+1}$ ,  $\mathcal{T}_{k+1} = \mathcal{S}_{k+1}$ . For case 2, we choose  $\mathcal{T}_i = \tilde{\mathcal{S}}_i$  for  $i \leq k$ ,  $\mathcal{T}_{k+1} = \mathcal{S}_{k+1}$ . For case 3, we choose  $\mathcal{T}_i = \tilde{\mathcal{S}}_i$  for  $i \leq k$ ,  $\mathcal{T}_{k+1} = \mathcal{S}_{k+1} \setminus \left(\bigcup_{i=1}^n \tilde{\mathcal{S}}_i\right)$ . Then the sequence  $\{\mathcal{T}_i\}_{i=1}^n$  satisfies the condition in Lemma 1. By induction, we complete the proof.  $\square$

The lemma means that  $\forall n, \exists$  a sequence of distinct non-empty disjoint sets  $\{\mathcal{T}_i\}_{i=1}^n$  such that  $\mathcal{T}_i \in \mathcal{M}$ . Then,  $\{\mathcal{T}_i\}_{i=1}^\infty$  is just the sequence of disjoint sets.

(2) Choose the sequence  $\{\mathcal{T}_i\}_{i=1}^\infty$  in (1), as  $\{\mathcal{T}_i\}$  are non-empty disjoint sets,  $\forall \mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{N}$  and  $\mathcal{S}_1 \neq \mathcal{S}_2$ ,

$$(\bigcup_{i \in \mathcal{S}_1} \mathcal{T}_i) \neq (\bigcup_{j \in \mathcal{S}_2} \mathcal{T}_j).$$

And by the definition of  $\sigma$ -algebra,  $\forall \mathcal{S} \subset \mathbb{N}$ ,  $\bigcup_{i \in \mathcal{S}} \mathcal{T}_i \in \mathcal{M}$ . It means that:

$$\text{card}(\mathcal{M}) \geq \text{card}(2^{\mathbb{N}}) = \aleph.$$

$\square$

### Problem 3

An algebra  $\mathcal{A}$  is a  $\sigma$ -algebra iff  $\mathcal{A}$  is closed under countable increasing unions.

*Proof.* " $\Rightarrow$ ": Just by the definition of  $\sigma$ -algebra.

" $\Leftarrow$ ":  $\mathcal{A}$  is an algebra, so we just need to check its countable unions.

Choose a series of non-empty sets  $\{\mathcal{S}_i\}_{i=1}^\infty \subset \mathcal{M}$ , mark  $\mathcal{F}_i := \bigcup_{j=1}^i \mathcal{S}_j$ , it's clear that  $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ . And by the definition of algebra,  $\mathcal{F}_i \in \mathcal{A}$ . As  $\mathcal{A}$  is closed under countable increasing unions,

$$\bigcup_{i=1}^\infty \mathcal{S}_i = \bigcup_{i=1}^\infty \mathcal{F}_i \in \mathcal{A}.$$

So  $\mathcal{A}$  is closed under countable unions, i.e.  $\mathcal{A}$  is a  $\sigma$ -algebra.  $\square$

### Problem 4

If  $\mathcal{M}$  is the  $\sigma$ -algebra generated by  $\mathcal{E}$ , then  $\mathcal{M}$  is the union of the  $\sigma$ -algebras generated by  $\mathcal{F}$  as  $\mathcal{F}$  ranges over all countable subsets of  $\mathcal{E}$ .

*Proof.* We complete this proof by three steps. Mark

$$\tilde{\mathcal{M}} := \bigcup_{\mathcal{F} \subset \mathcal{E}, |\mathcal{F}| = \aleph_0} \mathcal{M}(\mathcal{F}).$$

First, we show  $\tilde{\mathcal{M}}$  is a  $\sigma$ -algebra. For the countable unions, we choose  $\{\mathcal{S}_i\}_{i=1}^\infty \subset \tilde{\mathcal{M}}$ . WLOG, assume  $\mathcal{S}_i \in \mathcal{M}(\mathcal{F}_i)$ , where  $\mathcal{F}_i \subset \mathcal{E}$  and  $\text{card}(\mathcal{F}_i) = \aleph_0$ . Mark  $\tilde{\mathcal{F}} = \bigcup_{j=1}^\infty \mathcal{F}_j$ , as  $\mathcal{F}_i \subset \tilde{\mathcal{F}}$ ,  $\mathcal{S}_i \in \mathcal{M}(\mathcal{F}_i) \subset \mathcal{M}(\tilde{\mathcal{F}})$ , i.e.

$$\bigcup_{i=1}^\infty \mathcal{S}_i \in \mathcal{M}(\tilde{\mathcal{F}}).$$

$\text{card}(\mathcal{F}_i) = \aleph_0$  means  $\text{card}(\tilde{\mathcal{F}}) = \aleph_0$ , i.e.  $\mathcal{M}(\tilde{\mathcal{F}}) \subset \tilde{\mathcal{M}}$ , so  $\bigcup_{i=1}^\infty \mathcal{S}_i \in \tilde{\mathcal{M}}$ .

On the other hand,  $\tilde{\mathcal{M}}$  is closed under complement as  $\forall \mathcal{F} \subset \mathcal{E}$ ,  $\mathcal{M}(\mathcal{F})$  is closed under complement. So  $\tilde{\mathcal{M}}$  is a  $\sigma$ -algebra.

Second, we show that  $\tilde{\mathcal{M}} \subset \mathcal{M}$ . As  $\forall \mathcal{F} \subset \mathcal{E}$ ,  $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}$ , it's clear that

$$\tilde{\mathcal{M}} = \bigcup_{\mathcal{F} \subset \mathcal{E}, |\mathcal{F}| = \aleph_0} \mathcal{M}(\mathcal{F}) \subset \mathcal{M}.$$

Finally, we show  $\mathcal{M} \subset \tilde{\mathcal{M}}$ . As  $\tilde{\mathcal{M}}$  be a  $\sigma$ -algebra, if  $\mathcal{E} \subset \tilde{\mathcal{M}}$ , we can see  $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subset \tilde{\mathcal{M}}$ .

$\forall \mathcal{S} \in \mathcal{E}$ ,  $\exists \mathcal{F} \subset \mathcal{E}$ ,  $|\mathcal{F}| = \aleph_0$ , s.t.  $\mathcal{S} \in \mathcal{F}$ , i.e.

$$\mathcal{S} \in \mathcal{F} \subset \mathcal{M}(\mathcal{F}) \subset \tilde{\mathcal{M}}.$$

So  $\mathcal{E} \subset \tilde{\mathcal{M}}$ , it means  $\mathcal{M} \subset \tilde{\mathcal{M}}$ .  $\square$

## Problem 5

If  $\mu_1, \dots, \mu_n$  are measures on  $(X, \mathcal{M})$  and  $a_1, \dots, a_n \in [0, \infty)$ , then  $\sum_1^n a_j \mu_j$  is a measure on  $(X, \mathcal{M})$ .

*Proof.* First, choose  $\mathcal{S} \in \mathcal{M}$ , as  $a_j \geq 0$  and  $\mu_j(\mathcal{S}) \geq 0$ ,  $\sum_1^n a_j \mu_j(\mathcal{S}) \geq 0$ , i.e.  $\sum_1^n a_j \mu_j$  is a map from  $\mathcal{M}$  to  $[0, \infty]$ .

Then, as  $\forall j$ ,  $\mu_j(\emptyset) = 0$ ,  $\sum_1^n a_j \mu_j(\emptyset) = 0$ .

Finally, choose a sequence of disjoint sets  $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ ,

$$\sum_1^n a_j \mu_j(\cup_{k=1}^\infty E_k) = \sum_{j=1}^n \sum_{k=1}^\infty a_j \mu_j(E_k). \quad (1)$$

We claim that

$$\sum_{j=1}^n \sum_{k=1}^\infty a_j \mu_j(E_k) = \sum_{k=1}^\infty \sum_{j=1}^n a_j \mu_j(E_k). \quad (2)$$

If LHS =  $\infty$ , i.e.  $\exists j_0 \leq n$ ,  $\sum_{k=1}^\infty a_{j_0} \mu_{j_0}(E_k) = \infty$ ,

$$\text{RHS} \geq \sum_{k=1}^\infty a_{j_0} \mu_{j_0}(E_k) = \infty.$$

If LHS <  $\infty$ , i.e.  $\forall j \leq n$ ,  $\sum_{k=1}^\infty a_j \mu_j(E_k)$  is convergent, as  $n < \infty$ ,  $\exists M > 0$  such that  $\forall j \leq n$ ,  $\sum_{k=1}^\infty a_j \mu_j(E_k) \leq M$ . By dominant convergent theorem(DCT), (2) is true. From (2), (1) and the first two steps,  $\sum_1^n a_j \mu_j$  is a measure.  $\square$