# Real Analysis: Homework #1

Due on Mar 4, 2024 at 10:00am

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# Problem 1

Complete the proof of Proposition 1.2.

*Proof.*
- choose $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \frac{1}{b-a}$,
$$(a,b) = \cup_{n=N_0}^{\infty} \left(a, b - \frac{1}{n}\right].$$
- choose $N_0 \in \mathbb{N}$ such that $\frac{1}{N_0} < \frac{1}{b-a}$,
$$(a,b) = \cup_{n=N_0}^{\infty} \left[a + \frac{1}{n}, b\right),$$
- $$(a,b) = (a,\infty) \cap \left(\cap_{n=1}^{\infty} \left(b - \frac{1}{n}, \infty\right)\right)^c,$$
- $$(a,b) = (-\infty,b) \cap \left(\cap_{n=1}^{\infty} \left(-\infty, a + \frac{1}{n}\right)\right)^c,$$
- $$(a,b) = [b,\infty)^c \cap \left(\cup_{n=1}^{\infty} \left[a + \frac{1}{n}, \infty\right)\right),$$
- $$(a,b) = (-\infty,a]^c \cap \left(\cup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n}\right]\right).$$

By the definition of generated $\sigma$-algebra, it means $(a,b) \in \mathcal{M}(\mathcal{E}_i)$, $\forall i = 3,4,5,6,7,8$. By Lemma 1.1, $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_i)$. Then we complete the proof of Proposition 1.2. □

# Problem 2

Let $\mathcal{M}$ be an infinite $\sigma$-algebra

1. $\mathcal{M}$ contains an infinite sequence of disjoint sets.
2. $\operatorname{card}(\mathcal{M}) \geq \aleph$.

*Proof.* (1). $\mathcal{M}$ be an infinite $\sigma$-algebra means that there exists at least $\aleph_0$ distinct non-empty sets $\{\mathcal{S}_i\}_{i=1}^{\infty} \subset \mathcal{M}$. Unfortunately, $\{\mathcal{S}_i\}_{i=1}^{\infty}$ isn't a sequence of disjoint sets in general.
So, we should check the following lemma:

**Lemma 1.** *$\forall n \in \mathbb{N}$, if $\{\mathcal{S}_i\}_{i=1}^{n} \subset \mathcal{M}$ are distinct non-empty sets, there exists a sequence of disjoint non-empty sets $\{\mathcal{T}_i\}_{i=1}^{n}$ such that $\mathcal{T}_i \in \mathcal{M}$, and $\cup_{i=1}^{n}\mathcal{T}_i = \cup_{i=1}^{n}\mathcal{S}_i$.*

*Proof.* For $n=1$, just set $\mathcal{T}_1 = \mathcal{S}_1$. Assume the lemma holds for $n=k$, consider $\{\mathcal{S}_i\}_{i=1}^{k+1}$. By the assumption, there exists a sequence of disjoint non-empty sets $\{\tilde{\mathcal{S}}_i\}_{i=1}^{k} \subset \mathcal{M}$ such that $\cup_{i=1}^{k}\mathcal{S}_i = \cup_{i=1}^{k}\tilde{\mathcal{S}}_i$. For $\mathcal{S}_{k+1}$, there are three distinct cases:

- $\exists i \leq k$, $\mathcal{S}_{k+1} \subset \tilde{\mathcal{S}}_i$.
- $\forall i \leq k$, $\mathcal{S}_{k+1} \cap \tilde{\mathcal{S}}_i = \emptyset$.
- Other cases.

For case 1, we choose $\mathcal{T}_j = \tilde{\mathcal{S}}_j$ for $j \le k$ and $j \ne i$, $\mathcal{T}_i = \tilde{\mathcal{S}}_i \setminus \mathcal{S}_{k+1}$, $\mathcal{T}_{k+1} = \mathcal{S}_{k+1}$. For case 2, we choose $\mathcal{T}_i = \tilde{\mathcal{S}}_i$ for $i \le k$, $\mathcal{T}_{k+1} = \mathcal{S}_{k+1}$. For case 3, we choose $\mathcal{T}_i = \tilde{\mathcal{S}}_i$ for $i \le k$, $\mathcal{T}_{k+1} = \mathcal{S}_{k+1} \setminus \left(\cup_{i=1}^n \tilde{\mathcal{S}}_i\right)$. Then the sequence $\{\mathcal{T}_i\}_{i=1}^n$ satisfies the condition in Lemma 1. By induction, we complete the proof. □

The lemma means that $\forall n$, $\exists$ a sequence of distinct non-empty disjoint sets $\{\mathcal{T}_i\}_{i=1}^n$ such that $\mathcal{T}_i \in \mathcal{M}$. Then, $\{\mathcal{T}_i\}_{i=1}^\infty$ is just the sequence of disjoint sets.
(2) Choose the sequence $\{\mathcal{T}_i\}_{i=1}^\infty$ in (1), as $\{\mathcal{T}_i\}$ are non-empty disjoint sets, $\forall \mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{N}$ and $\mathcal{S}_1 \ne \mathcal{S}_2$,

$$\left(\cup_{i\in\mathcal{S}_1}\mathcal{T}_i\right) \ne \left(\cup_{j\in\mathcal{S}_2}\mathcal{T}_j\right).$$

And by the definition of $\sigma$-algebra, $\forall \mathcal{S} \subset \mathbb{N}$, $\cup_{i\in\mathcal{S}}\mathcal{T}_i \in \mathcal{M}$. It means that:

$$\text{card}(\mathcal{M}) \ge \text{card}\left(2^{\mathbb{N}}\right) = \aleph.$$

□

# Problem 3

An algebra $\mathcal{A}$ is a $\sigma$-algebra iff $\mathcal{A}$ is closed under countable increasing unions.

*Proof.* " $\Rightarrow$ ": Just by the definition of $\sigma$-algebra.
" $\Leftarrow$ ": $\mathcal{A}$ is an algebra, so we just need to check its countable unions.
Choose a series of non-empty sets $\{\mathcal{S}_i\}_{i=1}^\infty \subset \mathcal{M}$, mark $\mathcal{F}_i := \cup_{j=1}^i \mathcal{S}_j$, it's clear that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$. And by the definition of algebra, $\mathcal{F}_i \in \mathcal{A}$. As $\mathcal{A}$ is closed under countable increasing unions,

$$\cup_{i=1}^\infty \mathcal{S}_i = \cup_{i=1}^\infty \mathcal{F}_i \in \mathcal{A}.$$

So $\mathcal{A}$ is closed under countable unions, i.e. $\mathcal{A}$ is a $\sigma$-algebra. □

# Problem 4

If $\mathcal{M}$ is the $\sigma$-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the $\sigma$-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

*Proof.* We complete this proof by three steps. Mark

$$\tilde{\mathcal{M}} := \cup_{\mathcal{F}\subset\mathcal{E},|\mathcal{F}|=\aleph_0}\mathcal{M}(\mathcal{F}).$$

First, we show $\tilde{\mathcal{M}}$ is a $\sigma$-algebra. For the countable unions, we choose $\{\mathcal{S}_i\}_{i=1}^\infty \subset \tilde{\mathcal{M}}$. WLOG, assume $\mathcal{S}_i \in \mathcal{M}(\mathcal{F}_i)$, where $\mathcal{F}_i \subset \mathcal{E}$ and $\text{card}(\mathcal{F}_i) = \aleph_0$. Mark $\tilde{\mathcal{F}} = \cup_{j=1}^\infty \mathcal{F}_j$, as $\mathcal{F}_i \subset \tilde{\mathcal{F}}$, $\mathcal{S}_i \in \mathcal{M}(\mathcal{F}_i) \subset \mathcal{M}(\tilde{\mathcal{F}})$, i.e.

$$\cup_{i=1}^\infty \mathcal{S}_i \in \mathcal{M}(\tilde{\mathcal{F}}).$$

$\text{card}(\mathcal{F}_i) = \aleph_0$ means $\text{card}(\tilde{\mathcal{F}}) = \aleph_0$, i.e. $\mathcal{M}(\tilde{\mathcal{F}}) \subset \tilde{\mathcal{M}}$, so $\cup_{i=1}^n \mathcal{S}_i \subset \tilde{\mathcal{M}}$.
On the other hand, $\tilde{\mathcal{M}}$ is closed under complement as $\forall \mathcal{F} \subset \mathcal{E}$, $\mathcal{M}(\mathcal{F})$ is closed under complement. So $\tilde{\mathcal{M}}$ is a $\sigma$-algebra.
Second, we show that $\tilde{\mathcal{M}} \subset \mathcal{M}$. As $\forall \mathcal{F} \subset \mathcal{E}$, $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}$, it's clear that

$$\tilde{\mathcal{M}} = \cup_{\mathcal{F}\subset\mathcal{E},|\mathcal{F}|=\aleph_0}\mathcal{M}(\mathcal{F}) \subset \mathcal{M}.$$

Finally, we show $\mathcal{M} \subset \tilde{\mathcal{M}}$. As $\tilde{\mathcal{M}}$ be a $\sigma$-algebra, if $\mathcal{E} \subset \tilde{\mathcal{M}}$, we can see $\mathcal{M} = \mathcal{M}(\mathcal{E}) \subset \tilde{\mathcal{M}}$.
$\forall \mathcal{S} \in \mathcal{E}$, $\exists \mathcal{F} \subset \mathcal{E}, |\mathcal{F}| = \aleph_0$, s.t. $\mathcal{S} \in \mathcal{F}$, i.e.

$$\mathcal{S} \in \mathcal{F} \subset \mathcal{M}(\mathcal{F}) \subset \tilde{\mathcal{M}}.$$

So $\mathcal{E} \subset \tilde{\mathcal{M}}$, it means $\mathcal{M} \subset \tilde{\mathcal{M}}$. □

# Problem 5

If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

*Proof.* First, choose $\mathcal{S} \in \mathcal{M}$, as $a_j \geq 0$ and $\mu_j(\mathcal{S}) \geq 0$, $\sum_1^n a_j \mu_j(\mathcal{S}) \geq 0$, i.e. $\sum_1^n a_j \mu_j$ is a map from $\mathcal{M}$ to $[0, \infty]$.
Then, as $\forall j$, $\mu_j(\emptyset) = 0$, $\sum_1^n a_j \mu_j(\emptyset) = 0$.
Finally, choose a sequence of disjoint sets $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$,

$$\sum_1^n a_j \mu_j(\cup_{k=1}^{\infty} E_k) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} a_j \mu_j(E_k). \tag{1}$$

We claim that

$$\sum_{j=1}^{n} \sum_{k=1}^{\infty} a_j \mu_j(E_k) = \sum_{k=1}^{\infty} \sum_{j=1}^{n} a_j \mu_j(E_k). \tag{2}$$

If LHS $= \infty$, i.e. $\exists j_0 \leq n$, $\sum_{k=1}^{\infty} a_{j_0} \mu_{j_0}(E_k) = \infty$,

$$\text{RHS} \geq \sum_{k=1}^{\infty} a_{j_0} \mu_{j_0}(E_k) = \infty.$$

If LHS $< \infty$, i.e. $\forall j \leq n$, $\sum_{k=1}^{\infty} a_j \mu_j(E_k)$ is convergent, as $n < \infty$, $\exists M > 0$ such that $\forall j \leq n$, $\sum_{k=1}^{n} a_j \mu_j(E_k) \leq M$. By dominant convergent theorem(DCT), (2) is true. From (2), (1) and the first two steps, $\sum_1^n a_j \mu_j$ is a measure. $\square$