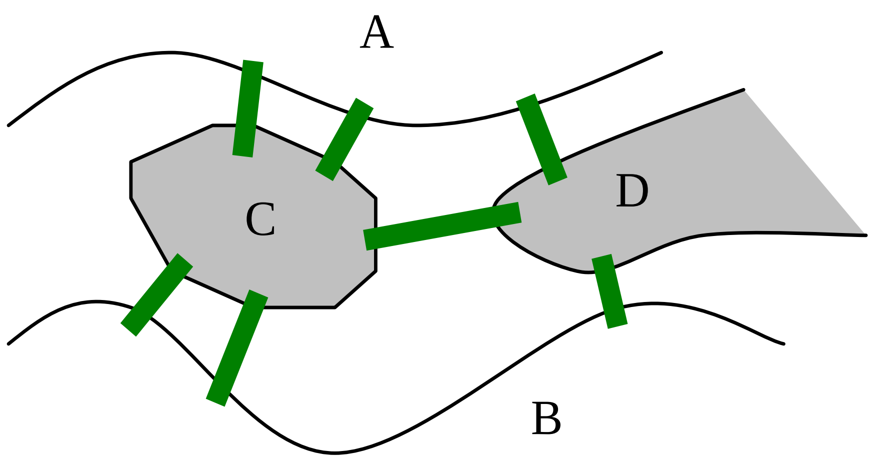
Euler and Hamilton Paths

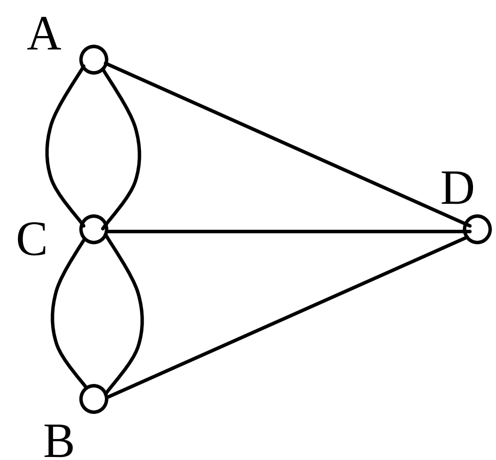
主要内容

- Euler paths and cycles
- Hamilton paths and cycles

Seven Bridges at Königsberg

- Abstraction
  - Vertices representing objects - areas
  - Edges representing the relationship between objects – connected by a bridge





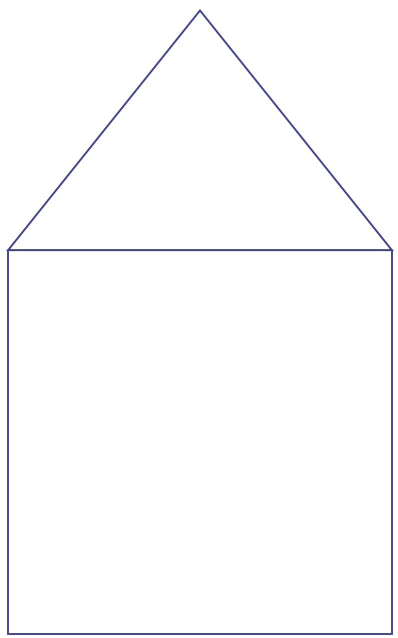
Euler and Hamilton Paths
-Motivation

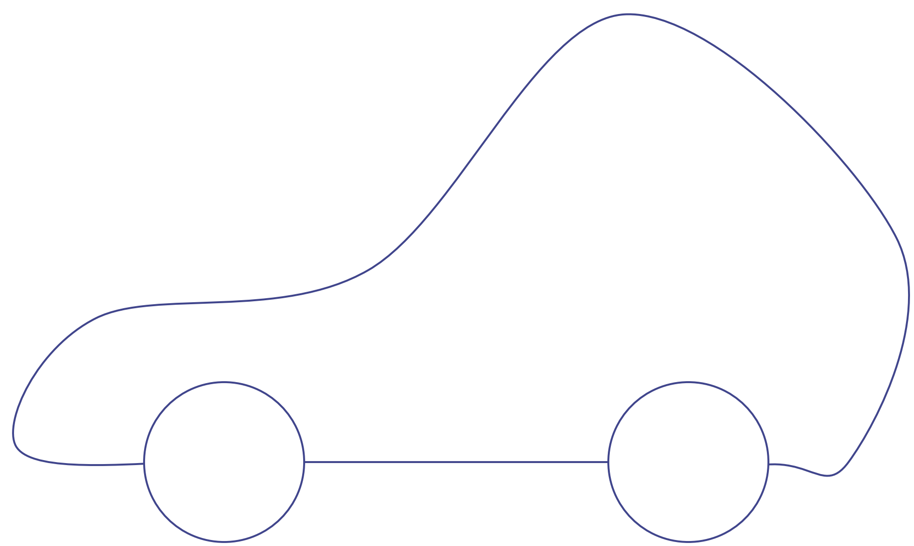
An pictorial way to motivate the graph theoretic concepts of Eulerian and Hamiltonian paths and circuits is with two puzzles:

- The pencil drawing problem(一笔画问题)
- The taxicab problem

Pencil Drawing Problem -Euler Paths

Which of the following pictures can be drawn on paper without ever lifting the pencil and without retracing over any segment?





欧拉路

欧拉路Euler path：a simple path including all edges
图中包含所有边的一条简单路；

欧拉回路Euler circuit ：欧拉路而且是回路；

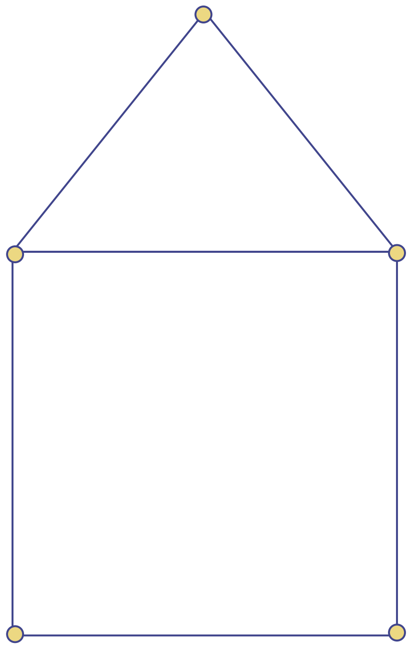
想想：欧拉开路是什么？

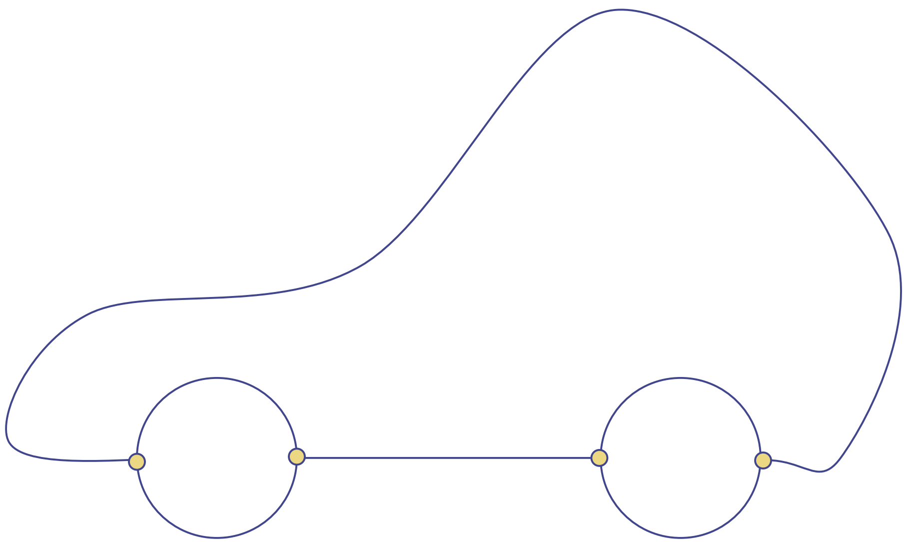
Note: The definition applies to both undirected
And digraphs of all types.

Euler Graph 欧拉图：存在欧拉回路的图。

Pencil Drawing Problem-Euler Paths

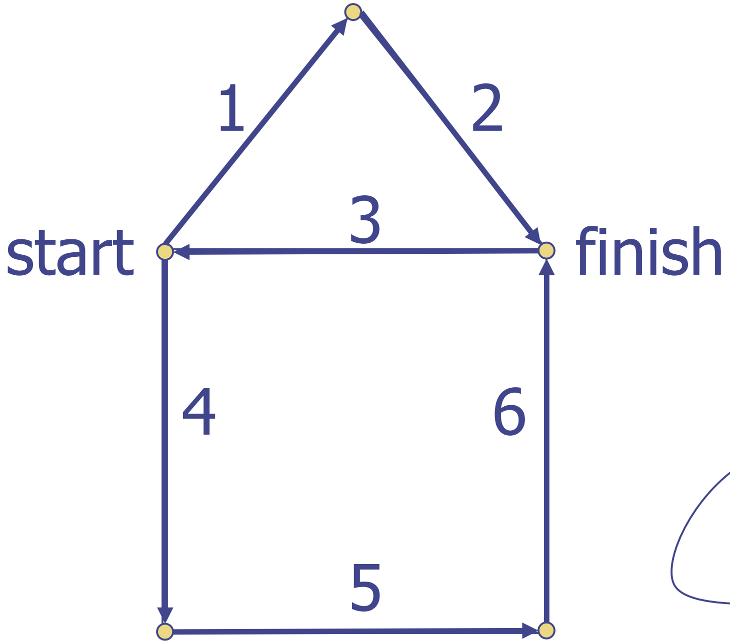
Graph Theoretically: Which of the following graphs has an Euler path?

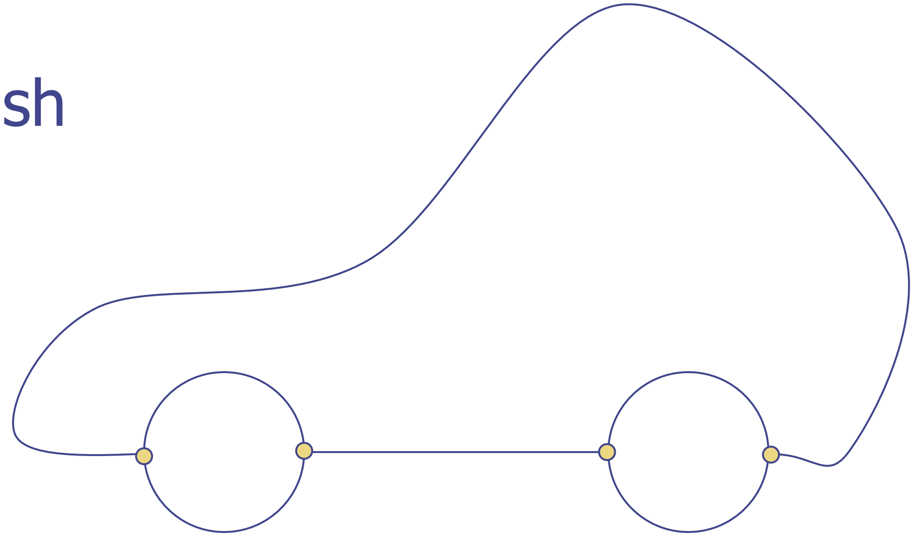




Pencil Drawing Problem -Euler Paths

Answer: the left, but not the right.





Euler Paths– 充分必要条件

欧拉回路存在的充分必要条件：（**Undirected graph)**

Theorem: A connected multi-graph *G* with at least two vertices has an Euler circuit if and only if every vertex has even degree. *(no isolated vertex)*

一个至少两个结点的连通多重图（或简单图）无向图为欧拉图的充要条件是每个节点的度都是偶数。

Digraph：

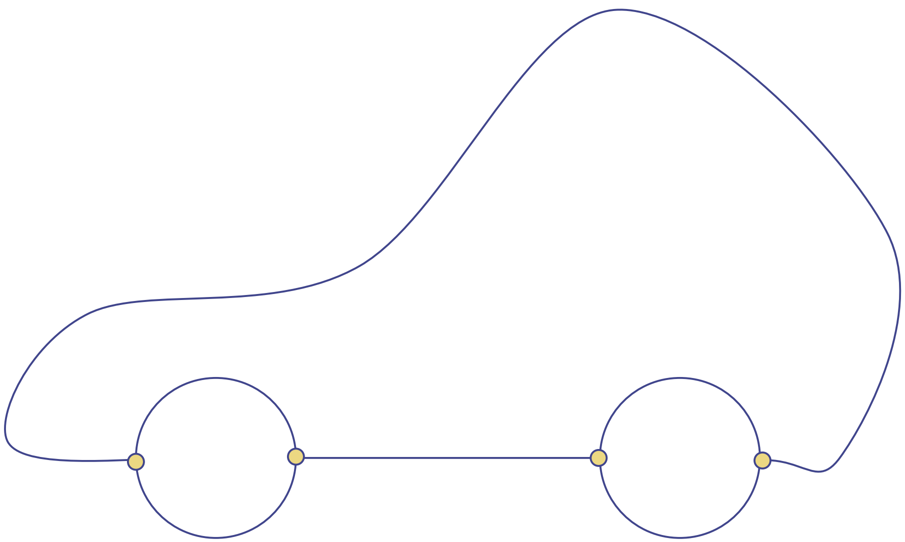
有向图的条件是：G be weakly connected and that every vertex has same in-degree as out-degree.

G是弱连通的而且每个结点的出度等于入度；

思考：定理中的条件“连通图”的作用。

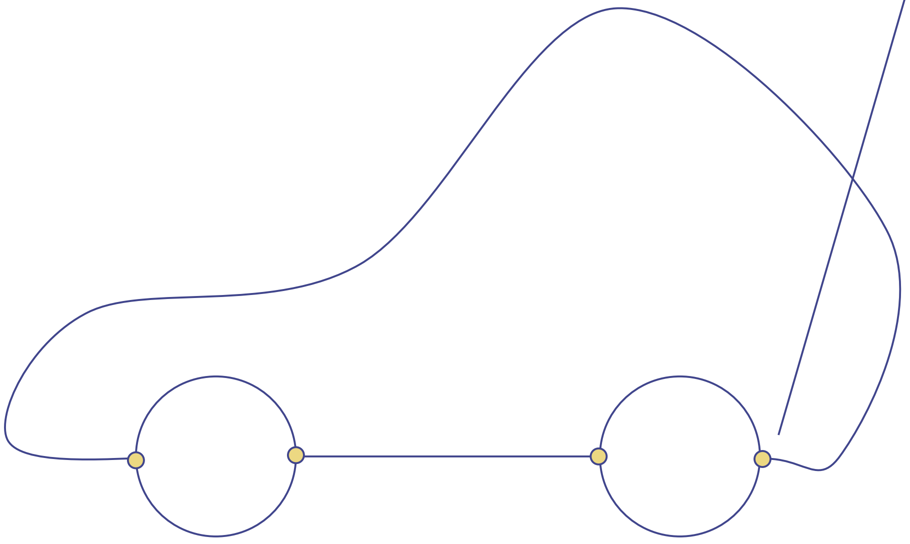
Finding Euler Circuits

Q: Why does the following graph have no Euler circuit?



Finding Euler Circuits

A: It contains a vertex of <u>odd degree</u>.



必要性证明思路

假定一个连通图存在一条欧拉回路。

欧拉回路结点起始于结点v_0;

每次进入到一个不是最终结点的点，必然要出去；
可能涉及到多次进出；这现象也包括结点v_0.

欧拉路经历所有边一次而且只一次，所以...

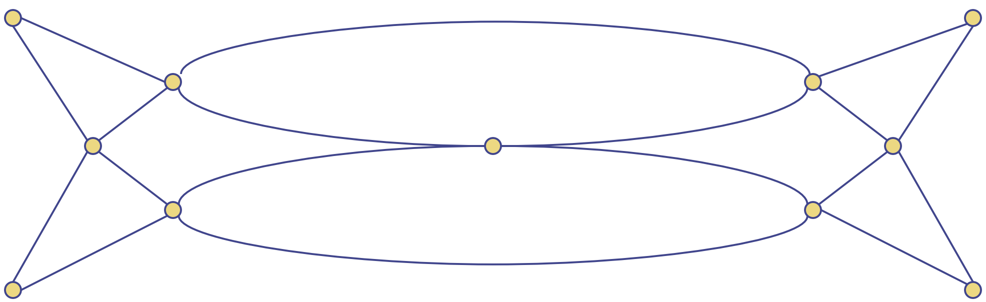
充分性证明思路

- 假定所有结点度均为偶数；
- 构造性证明：目标是构造一条欧拉回路。起点可以是任一个点v_0，随机地形成一条路，当到达某个点，关联这个点的边还没有走完时，继续。否则就暂时停下来。分析考察这条路。由于结点数和边数都是有限的，这个过程一定会经过有限次后停下来；
- 而且必然会是停于起点v_0;
- 思考：为什么？
- 那么这样走出来的回路就是欧拉回路吗？
- 还不一定！

Proof: “sufficient” direction

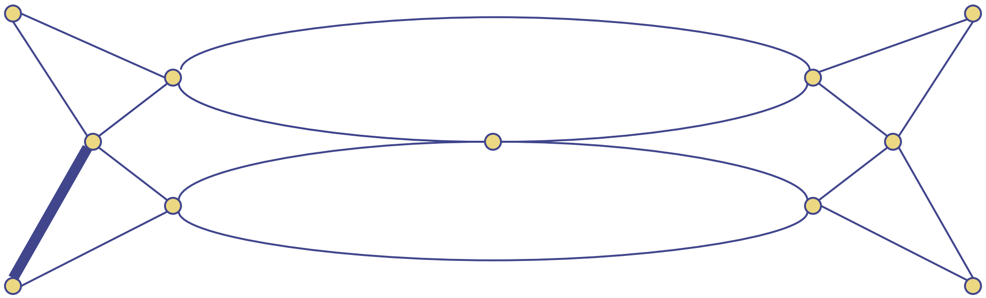
- 如果还不是欧拉回路,那么想象一下去掉刚刚的回路后剩下的子图，这子图跟刚刚的回路至少有一个共同的结点。
- 再使用上面的方法，在剩下的子图的每一个分支中重复使用上面的构造路的方法。
- 最后在将每条这样的简单回路沿着共同的结点“拼接起来”，形成一条欧拉回路。

Finding Euler Circuits 例子 Mohammed’s Scimitars 弯刀



Finding Euler Circuits Mohammed's Scimitars

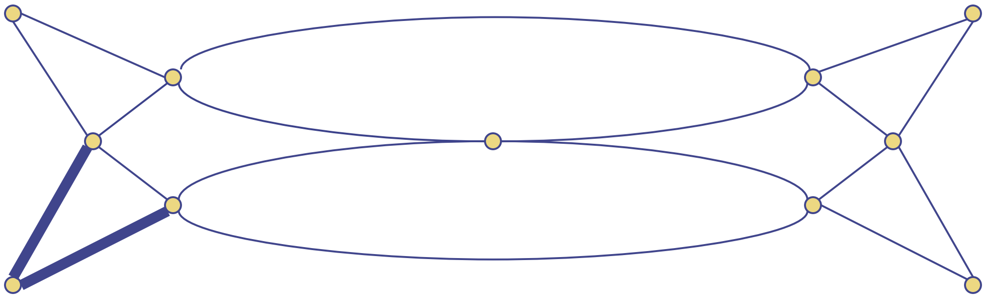
Found a cycle after starting from middle vertex.



Delete the cycle:

Finding Euler Circuits Mohammed's Scimitars

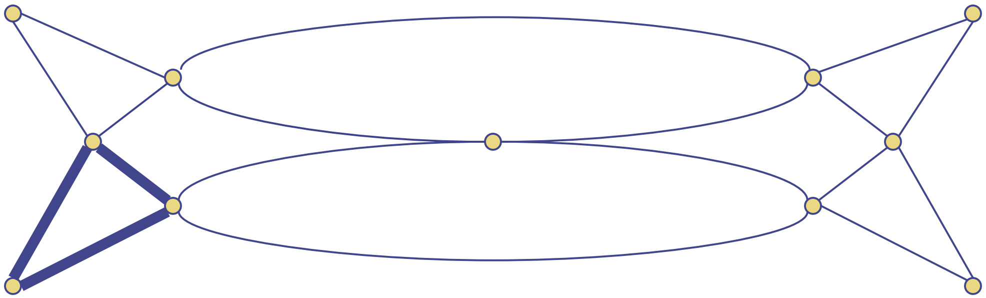
Found a cycle after starting from middle vertex.



Delete the cycle:

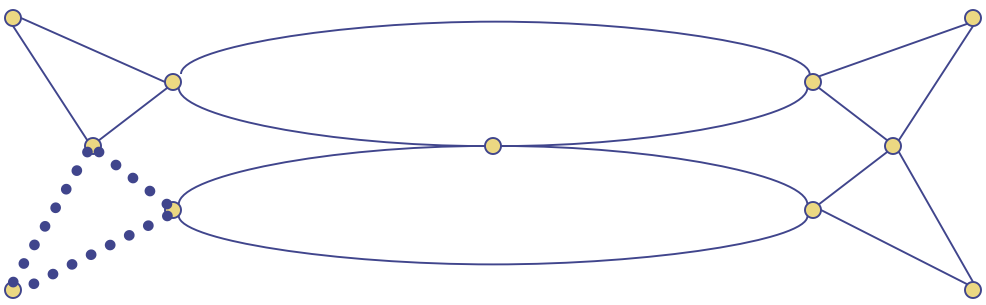
Finding Euler Circuits Mohammed's Scimitars

Found a cycle after starting from middle vertex.



Delete the cycle:

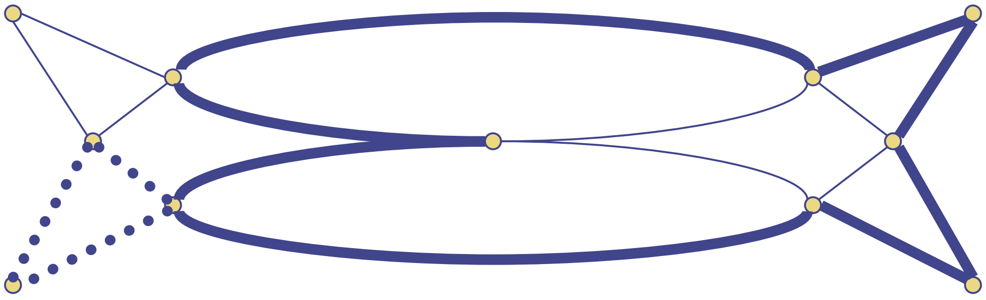
Finding Euler Circuits
Mohammed's Scimitars



Now try again (say from middle vertex):

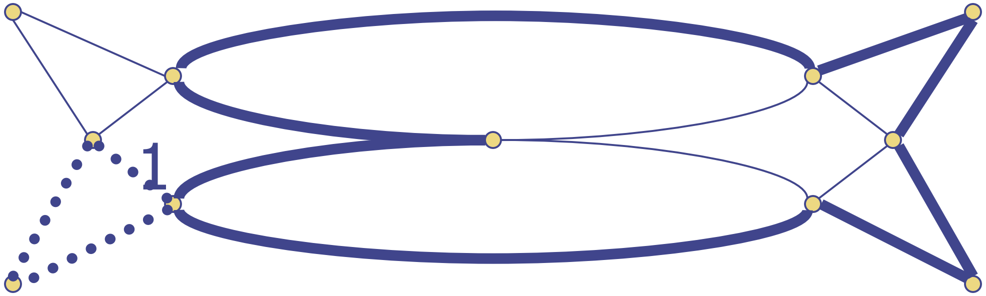
Finding Euler Circuits Mohammed's Scimitars

This time, found a cycle starting and ending at middle vertex:



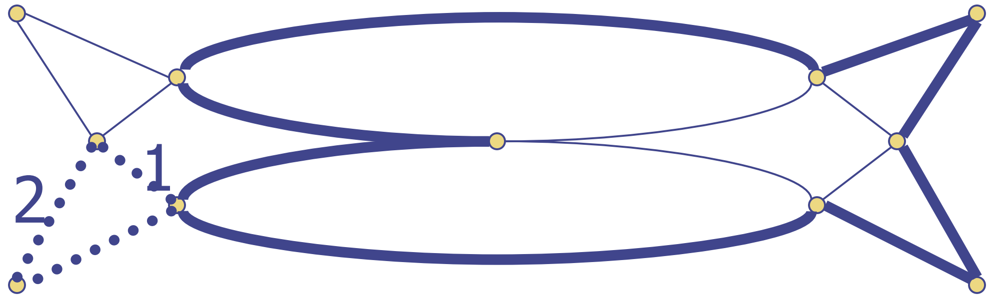
Amalgamate these cycles together from a point of intersection, and delete from graph:

Finding Euler Circuits Mohammed's Scimitars



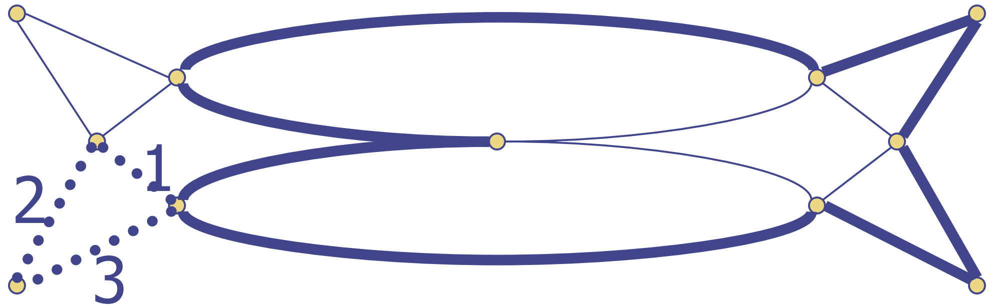
Find another cycle from middle vertex:

Finding Euler Circuits
Mohammed's Scimitars



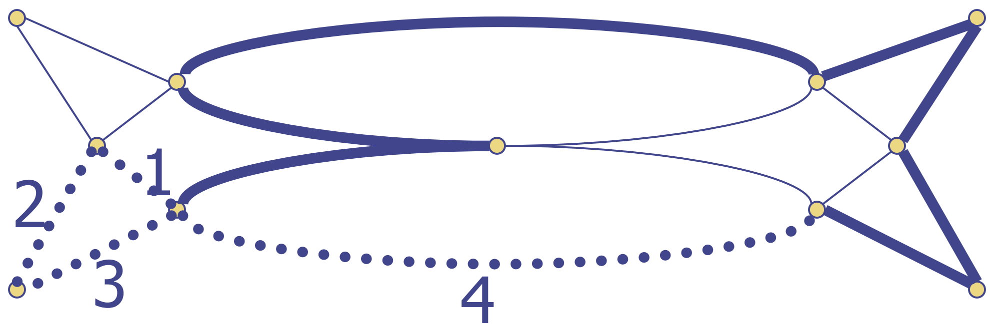
Find another cycle from middle vertex:

Finding Euler Circuits
Mohammed's Scimitars



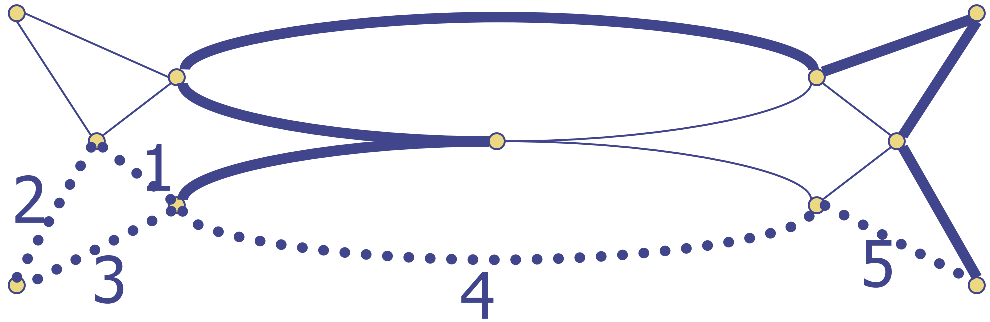
Find another cycle from middle vertex:

Finding Euler Circuits
Mohammed’s Scimitars



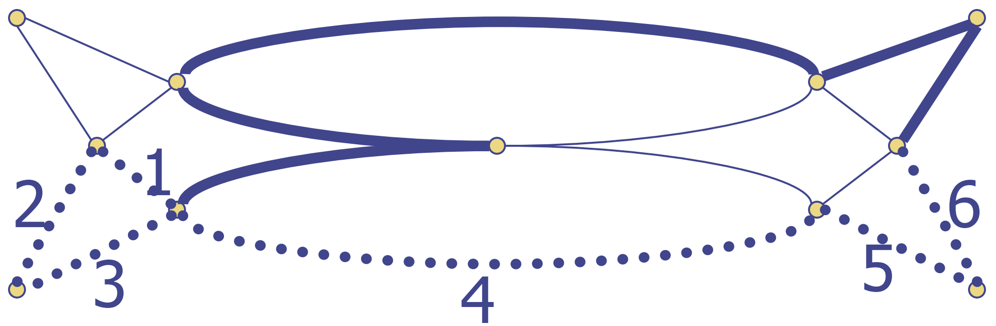
Find another cycle from middle vertex:

Finding Euler Circuits Mohammed's Scimitars



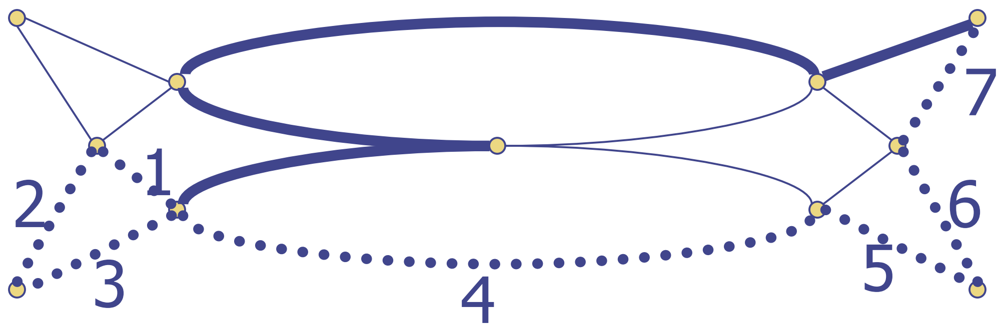
Find another cycle from middle vertex:

Finding Euler Circuits
Mohammed's Scimitars



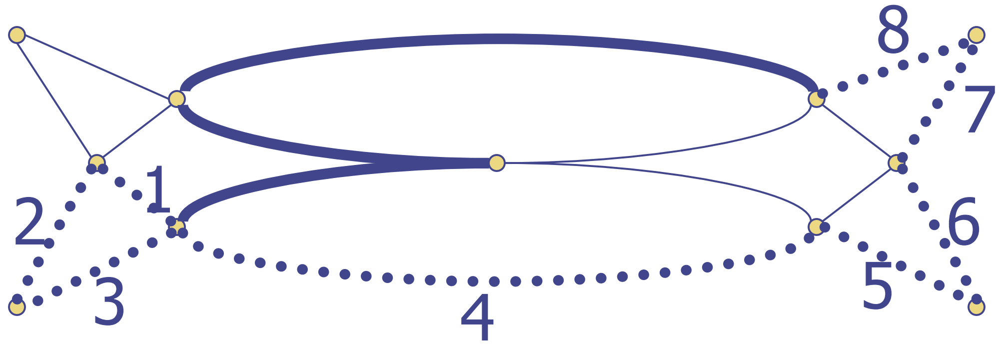
Find another cycle from middle vertex:

Finding Euler Circuits Mohammed's Scimitars



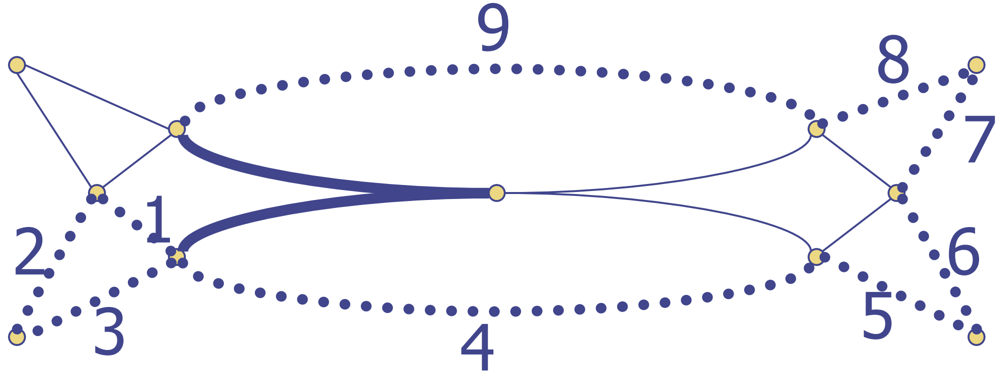
Find another cycle from middle vertex:

Finding Euler Circuits Mohammed's Scimitars



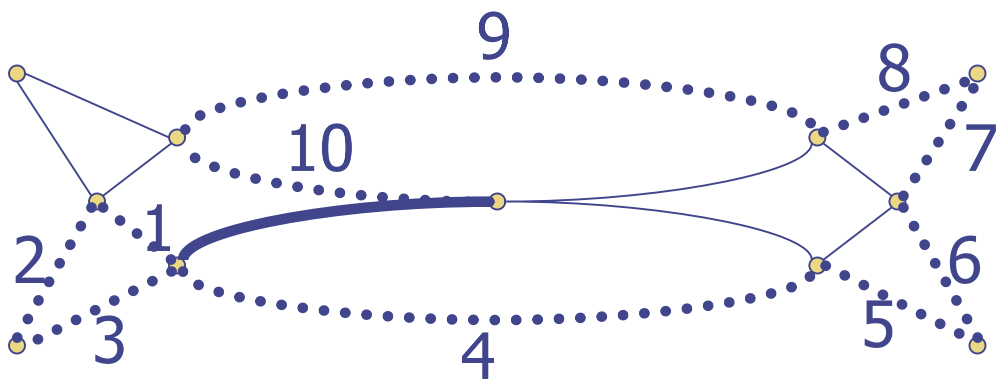
Find another cycle from middle vertex:

Finding Euler Circuits Mohammed's Scimitars



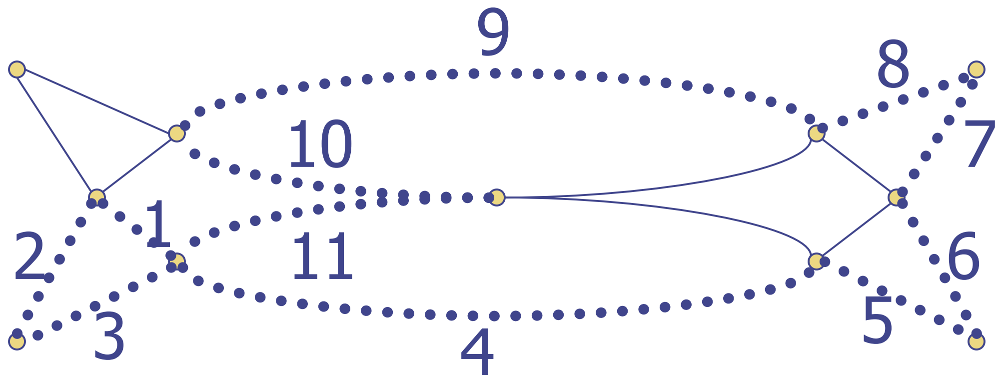
Find another cycle from middle vertex:

Finding Euler Circuits Mohammed's Scimitars



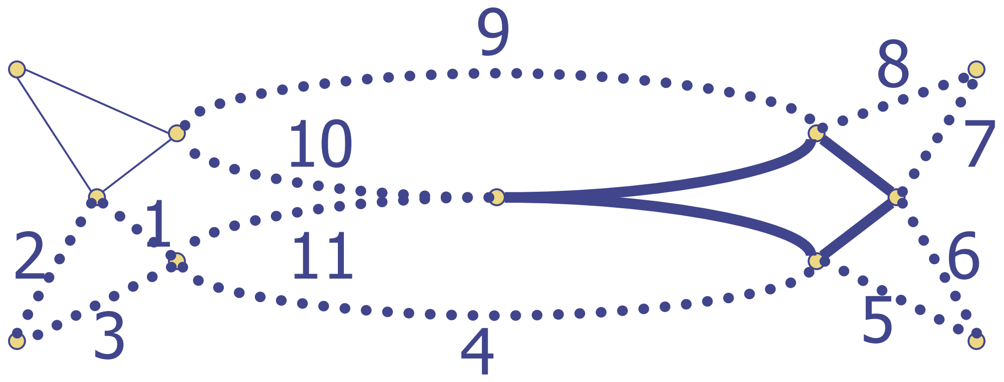
Find another cycle from middle vertex:

Finding Euler Circuits
Mohammed’s Scimitars



Find another cycle from middle vertex:

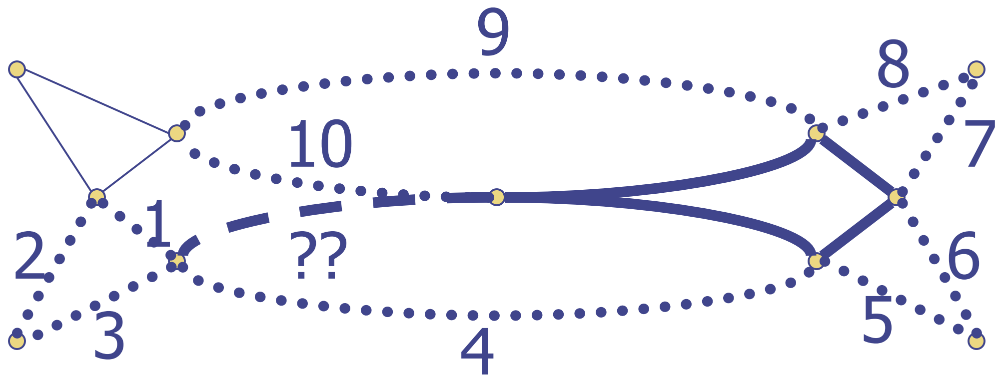
Finding Euler Circuits Mohammed's Scimitars



Amalgamate it to Euler cycle of deleted graph, and delete it. Need to insert cycle between former edges 10 & 11:

Finding Euler Circuits
Mohammed's Scimitars

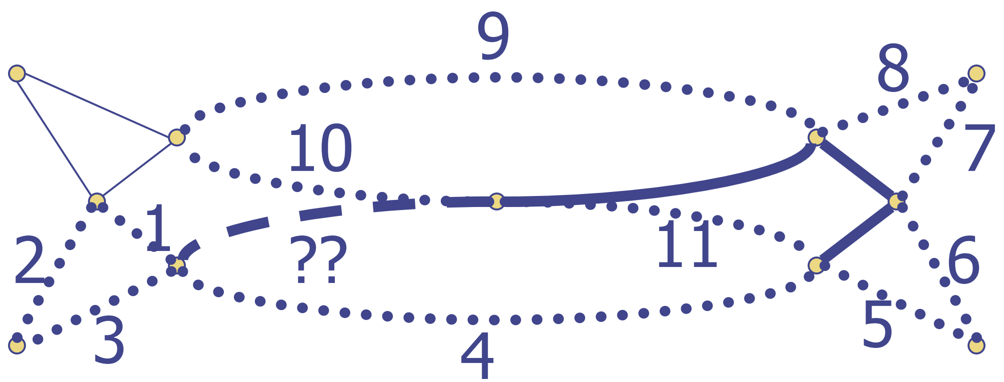
Finally, need to add the triangle.



Use same naïve approach looking for cycle in remaining component:

Finding Euler Circuits
Mohammed's Scimitars

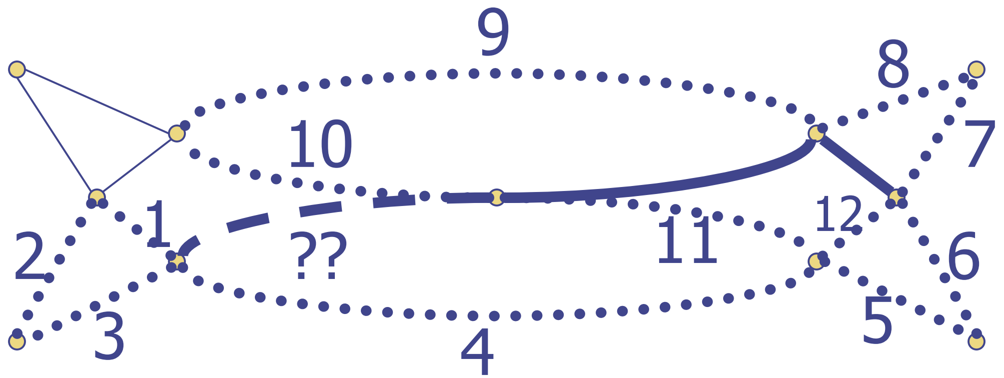
Finally, need to add the triangle.



Use same naïve approach looking for cycle in remaining component:

Finding Euler Circuits
Mohammed’s Scimitars

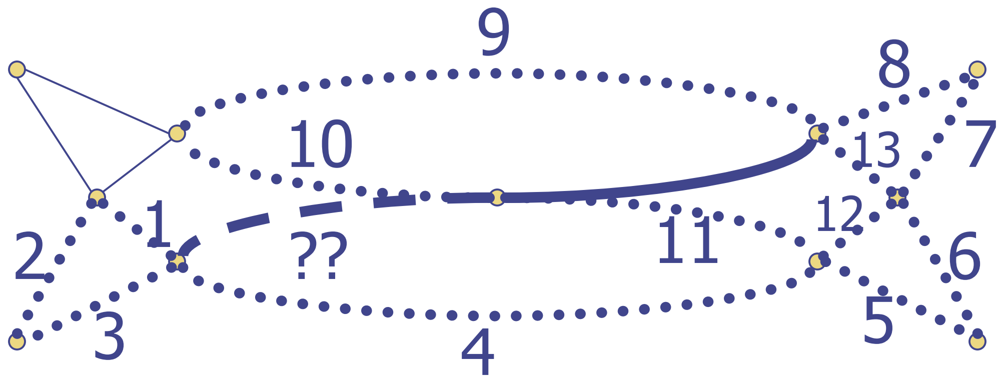
Finally, need to add the triangle.



Use same naïve approach looking for cycle in remaining component:

Finding Euler Circuits Mohammed's Scimitars

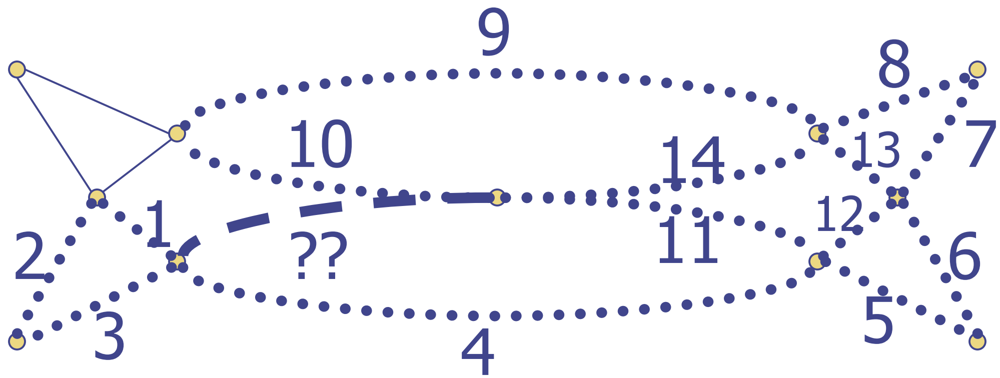
Finally, need to add the triangle.



Use same naïve approach looking for cycle in remaining component:

Finding Euler Circuits
Mohammed's Scimitars

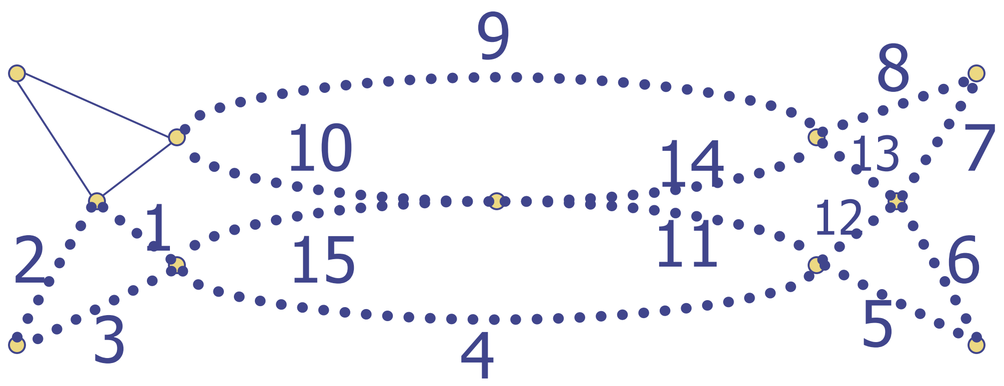
Finally, need to add the triangle.



Use same naïve approach looking for cycle in remaining component:

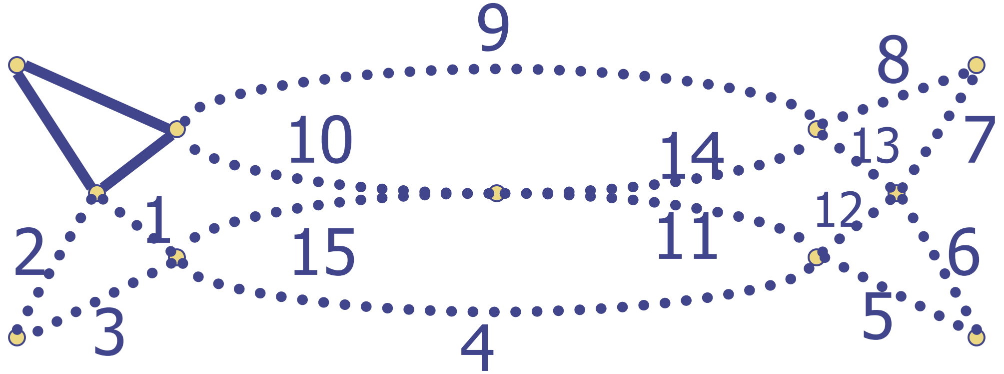
Finding Euler Circuits Mohammed's Scimitars

Finally, need to add the triangle.



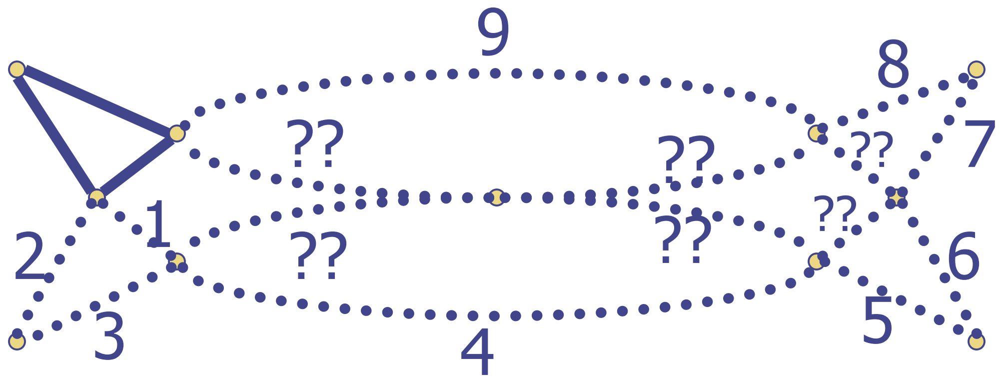
Use same naïve approach looking for cycle in remaining component:

Finding Euler Circuits Mohammed's Scimitars



Amalgamate the triangle cycle between edges formerly labeled 9 & 10:

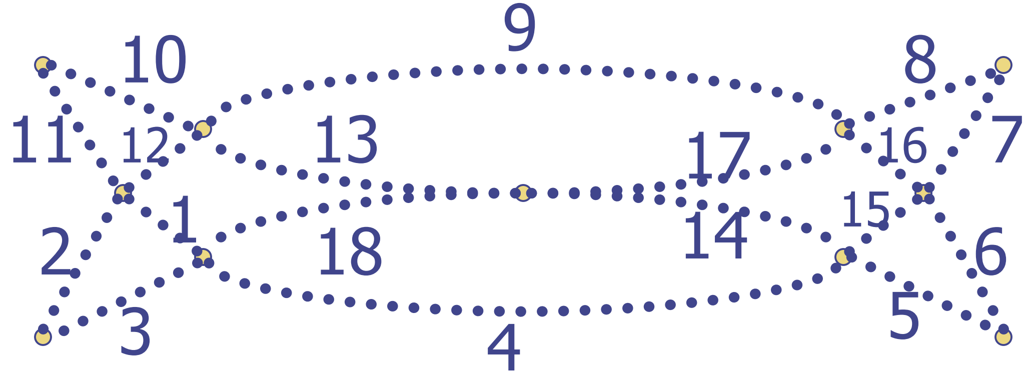
Finding Euler Circuits Mohammed's Scimitars



Amalgamate the triangle cycle between edges formerly labeled 9 & 10:

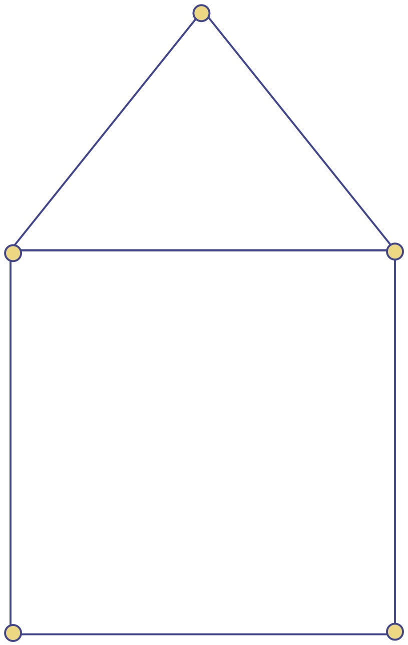
Finding Euler Circuits Mohammed's Scimitars

We found the Euler circuit!



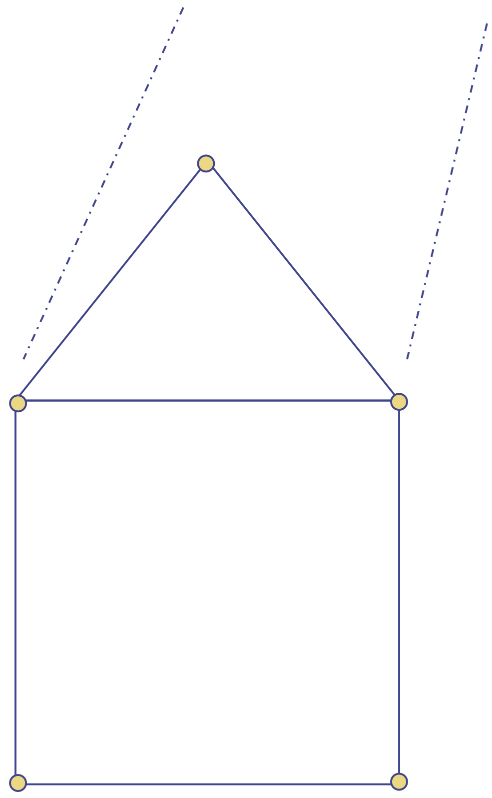
构造欧拉路 Finding Euler Paths

Q: Does the following have an Euler circuit?



Generalizing to Euler Paths

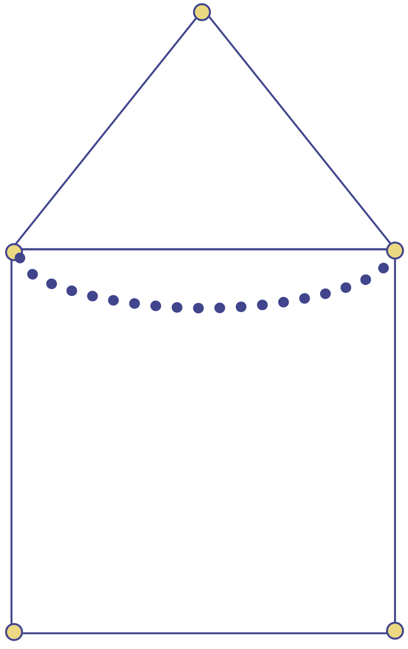
A: No, vertices of <u>odd degree</u>:



Q: How about an Euler path?

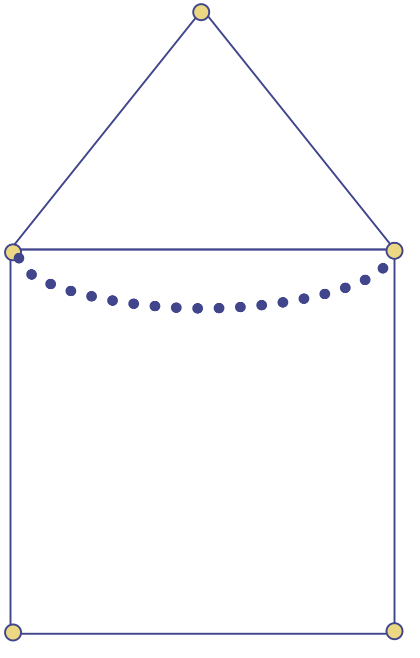
Generalizing to Euler Paths

A: YES! Because exactly 2 vertices of odd degree.



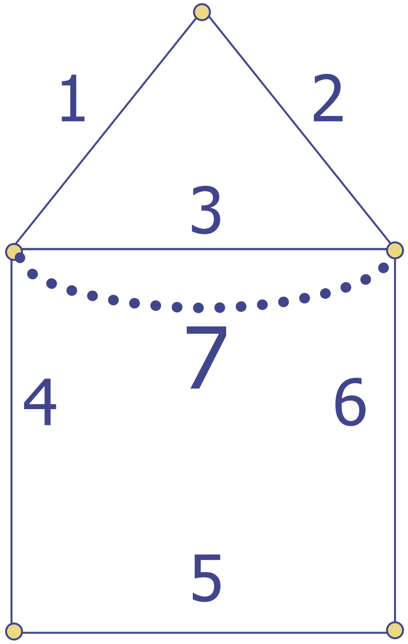
So can add a phantom edge between odd degree vertices。

Generalizing to Euler Paths



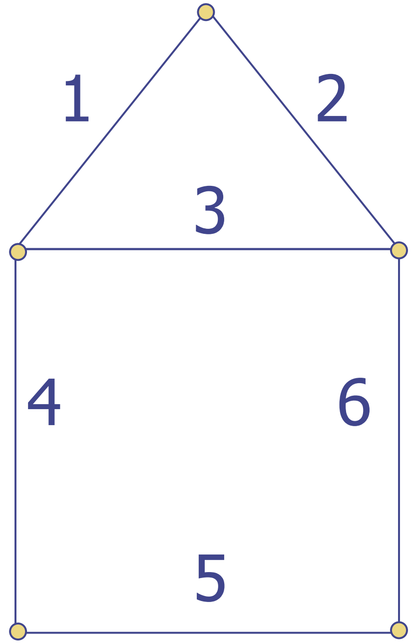
All degrees now even so find Euler cycle:

Generalizing to Euler Paths



Now remove phantom edge obtaining:

Generalizing to Euler Paths



Generalize…

Generalizing to Euler Paths

欧拉开路存在的充分必要条件：

定理：a connected multigraph has an Euler path，but not an Euler circuit iff it has exactly two vertices of odd degree.

一个无向连通图存在欧拉开路当且仅当恰有两个度为奇数的结点。

Back to the seven bridges problem: is it possible to start at some points in the town, travel across all the bridges and end up at another point in the town?

Impossible!

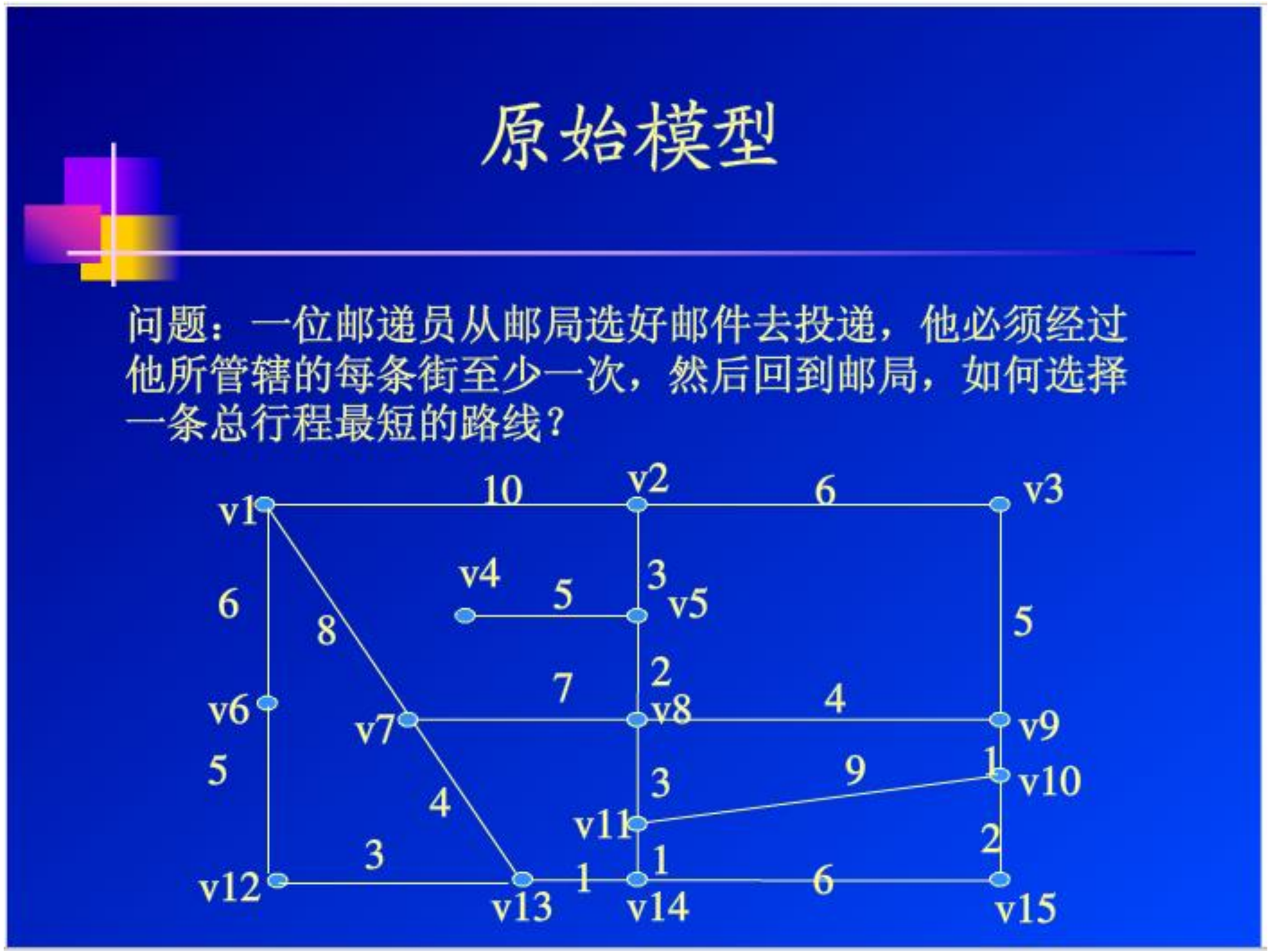
Why?

Applications of Euler Path

- Euler paths and circuits can be used to solve many practical problems.

- Some applications ask for a path or circuit that traverses each street in a neighborhood, each road in a transportation network.
- 应用举例：公路巡视检测员，需要检查某个区域的所有路段，希望不要走重复的路。

中国邮递员问题 the Chinese postman problem which posted by professor Guan Meigu



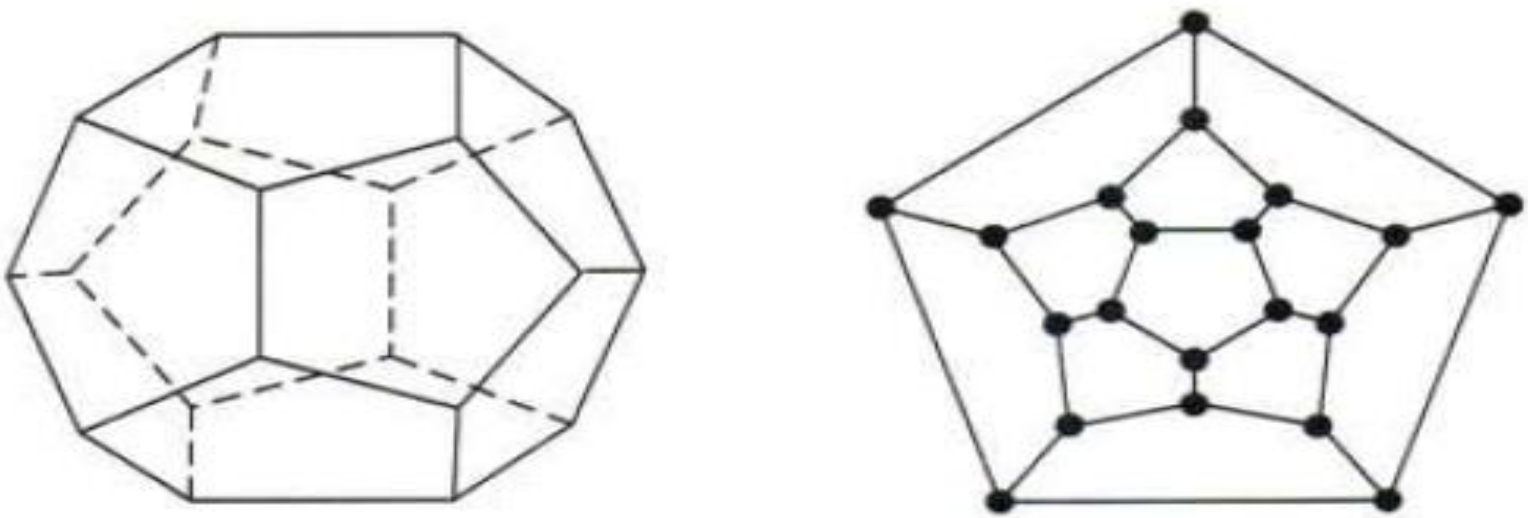
中国邮递员问题

- 中国邮递员问题可用图论语言叙述为：在一个具有非负权的赋权连通图G中，找出一条权最小的环游。这种环游称为最优环游。若G是欧拉图，则G的任意欧拉环游都是最优环游，从而任何寻找欧拉回路的算法都可以。若G不是欧拉图，则G的任意一个环游必定通过某些边不止一次。

- 这个问题由中国学者管梅谷在1960年首先提出，并给出了解法——“奇偶点图上作业法”，被国际上统称为“中国邮递员问题”。

All Around the World环游世界

The 12 regular pentagons as faces, with 20 vertices
20个顶点的12面体



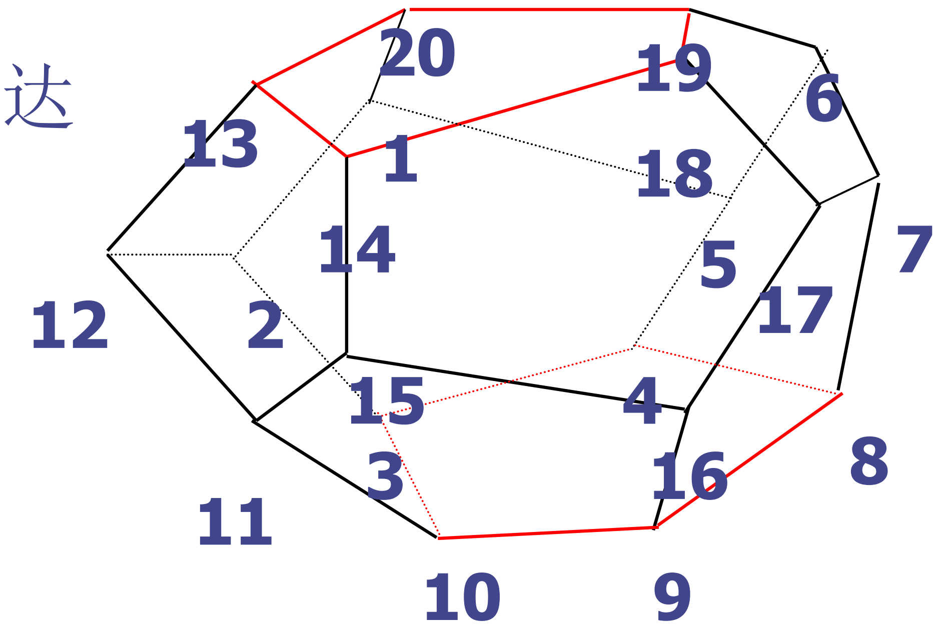
(a) (b)

Hamilton's "A Voyage Round the World" Puzzle.

All Around the World 环球旅游问题

- In 1859, an Irish mathematician, Hamilton, introduced the game called “All around the world”

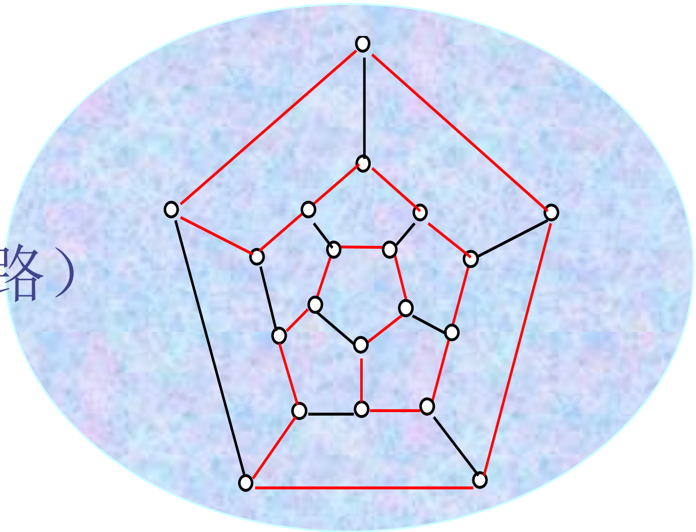
是否存在一条路线能到达每个顶点一次且只一次，然后回到起点？



Hamiltonian Path and Circuit
哈密尔顿路与哈密尔顿环

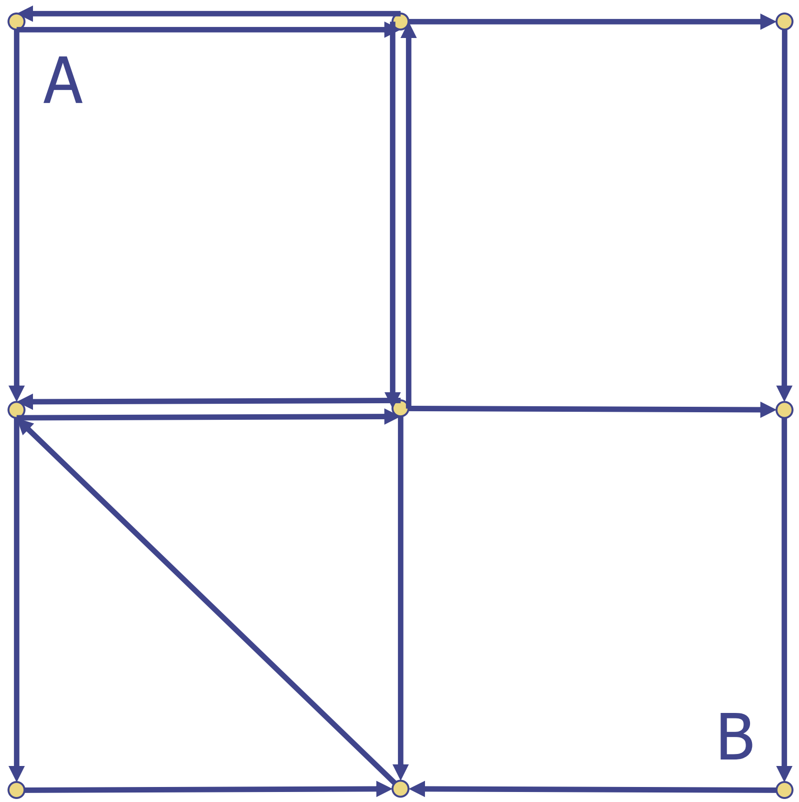
- In a graph G, a simple circuit is called a Hamiltonian circuit哈密尔顿环 if and only if it contains all the vertices in G exactly once. If G contains a Hamiltonian circuit, G itself is called a ***Hamiltonian graph*** 哈密尔顿图.

- A Hamiltonian path哈密尔顿路 is a simple path which contains all vertices exactly once. A **semi-Hamiltonian** graph contain a Hamiltonian path (开路) but not a Hamiltonian circuit.



Taxicab Problem-Hamilton Paths

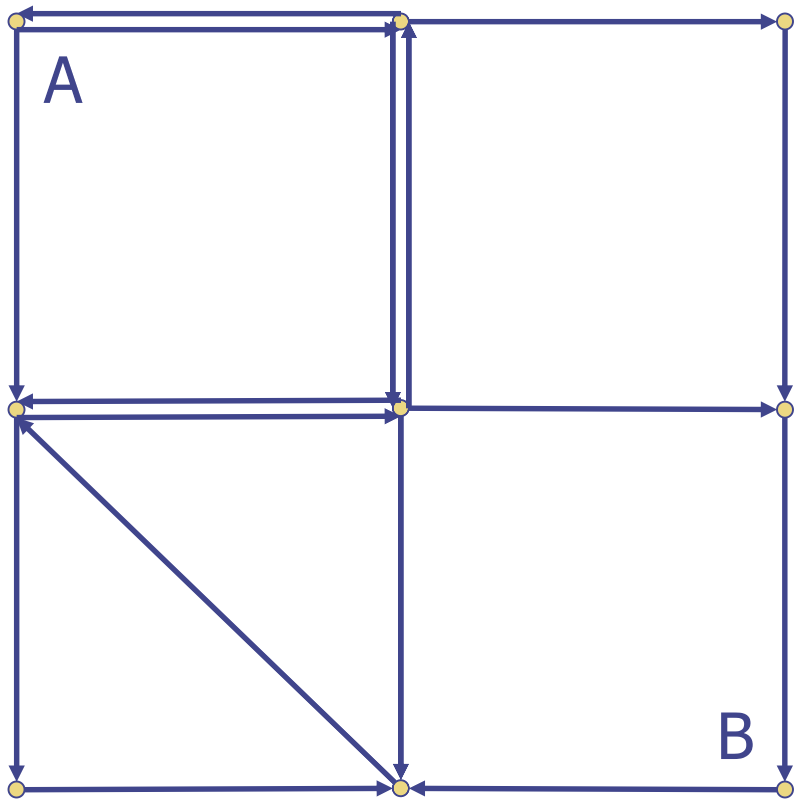
Can a taxicab driver milk his customers by visiting every intersection exactly once, when driving from point A to point B ?



Taxicab **Problem-Hamilton** Paths

Graph Theoretically: Is there a Hamilton path from A to B in the following graph?

(NO in this case)



Implications to CS

Finding Hamilton paths is very useful in CS.

EG: Visit every city (vertex) in a region using the least trips (edges) as possible.

EG: Encode all bit strings of a certain length as economically as possible so that only change one bit at a time. (Gray codes).

Hamilton Graph 哈密尔顿图

- **Question**: is there any simple way to know whether a graph has a Hamilton cycle or path, just like the necessary/sufficient conditions to find an Euler circuit?

- Unfortunately, so far there is no known way for Hamilton path! That is why "Finding Hamilton paths is **NP**-complete!"

- But even that, there are some sufficient conditions for the existence of Hamilton cycles, also some necessary conditions to show that some graphs have no Hamilton cycles.
- A basic fact is that the more edges a graph has, the more likely it has a Hamilton cycle.

Implications to CS 复杂度

Analyzing difficulty of Euler vs. Hamilton paths is a great CS case study.

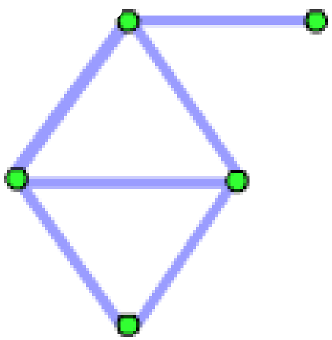
- Finding Euler paths can be done in $O(n)$ time

- Finding Hamilton paths is **NP**-complete!

Slight change in definition can result in dramatic algorithmic bifurcation!

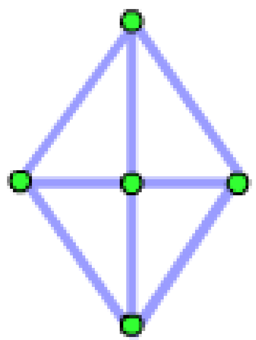
Hamilton Path/Cycle Examples

Example:Which of the following simple graphs have a Hamilton cycle/path or not?



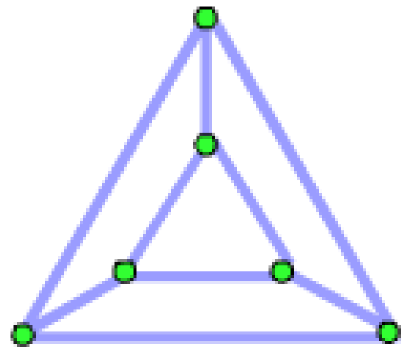
(a)

H-path



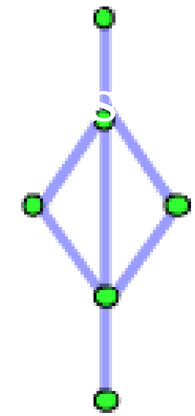
(b)

H-Cycle



(c)

H-Cycle



(d)

No

充分条件 for Hamiltonian Graph

- G 是不少于3个结点的简单图，那么：
- **Theorem 1（Ore定理）**：If for any pair of vertices *u*,*v* not adjacent in G:

  $\boldsymbol{deg(u)+deg(v)\geq n}$ (n is $|V_G|$)

  Then G is a Hamiltonian graph.
- **Theorem 2(Dirac定理)**: deg(v) ≥n/2 for any vertex v, then G is a Hamiltonian graph.
- **Theorem3** G=(V, E) is a simple graph, |v| = n, |E| = m, if $m \geq (n^2-3n+6)/2$ then G is a Hamiltonian graph.
- *We are not going to prove these here.*

Condition for Hamiltonian Graph

- 不少于3个结点的完全图有：

Theorem 4: K_n has a Hamilton circuit whenever n>2.

为什么？

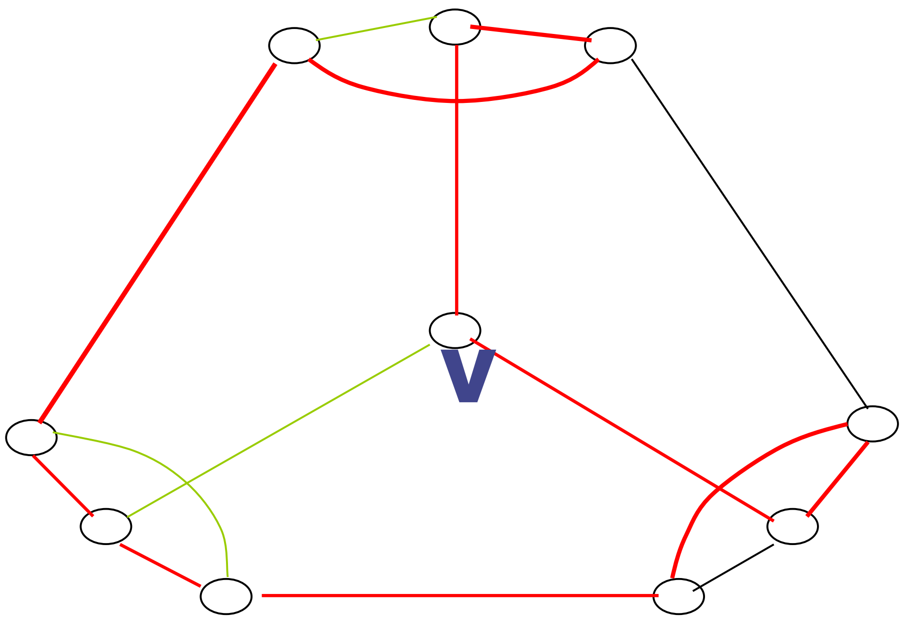
可以直接将n个结点布局成一个圈，由于K_n是完全图，任意的两个结点之间都有一条边。所以这一个圈的n个结点就可以形成一个圈 C_n, 这个圈本身是一个子图，就是K_n的一条Hamilton回路。

同学们可以自己画出来。

能否用上面的充分性条件判断K_n 是一个**Hamiltonian**图？

Hamilton Path/Cycle Examples

- G is Hamiltonian, but d(v) = 3 < 10/2



Note: it means the condition deg(v)>=n/2 is not necessary, but sufficient.

Another example: C_n (n>4)?

必要条件 for Hamiltonian Graph

Theorem 5: If a graph G=(V,E) 有一个哈密尔顿环，S 是V的真子集， 那么

$W(G-S) \leq |S|$.

注：W(G-S) 是子图 G-S of G 的连通分支数

Note: the subgraph G-S is the graph after removing all vertices and all edges incident to the vertices in S from G .

Proof: let C is a Hamilton cycle of G, obviously C includes all vertices of G.

Proof: let C is a Hamilton cycle of G, obviously C includes all vertices of G, and each vertex of S must be shown on the cycle C. So the number of connected components of (C-S) such that:

$$W(C-S) \le |S|$$

Because C-S is a subgraph G-S with same set of vertices and may include less edges than G-S. So

$W(G-S) \le W(C-S)$. We have the result:

$$W(G-S) \le |S|$$

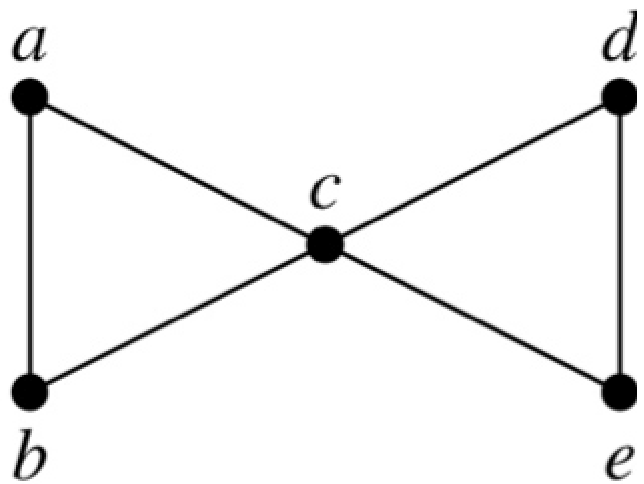
推论：如果**G**有一个割点，那么**G is non- Hamiltonian.**

Why?

Question： 必要条件可以用来判断图是否是H-图么？

Hamilton Path/Cycle Examples

- Look at this following graph, is it Hamiltonian? Why?



H

Hamilton Path/Cycle Examples 应用举例

- There are 10 guests in a party. Some are strangers to each other. Some are friends. We want to arrange the seats of the guests so that every guest sits between his two friends. How can we do that?

- Solution: we draw a graph. Each vertex represents a guest. An edge *a-b means a and b are friends*. *A good* seat assignment corresponds to a Hamiltonian path.

Applications using Hamilton Path

- The famous Traveling Salesman Problem（**TSP问题**）: ask for **the shortest route** a traveling salesman should take to visit a set of cities. This problem reduces to find a **Hamilton circuit** in a complete graph such that the total weight of its edges is as small as possible.

注：这个TSP问题与中国邮递员问题类似，但不一样。

- Gray Codes(格雷码): read the textbook to know what the gray-code is.

Exercises

6.5节

T1， T10， T16, T24