Planar Graphs 平面图

Def: ***Planar graphs*** are graphs that can be drawn in the plane without edges having to cross. 能不交叉地画在一个平面上的图。否则就叫非平面图。

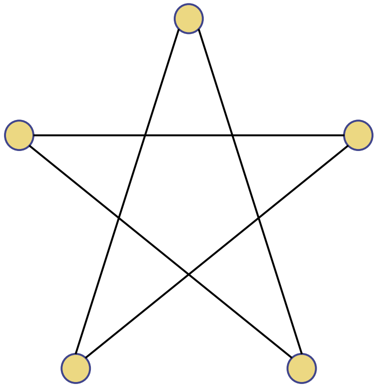
Understanding planar graph is important:

- Any graph representation of maps/ topographical information(地形图) is planar.
  - graph algorithms often specialized to planar graphs (e.g. traveling salesperson)
- Circuits usually represented by planar graphs

Planar Graphs
-Common Misunderstanding

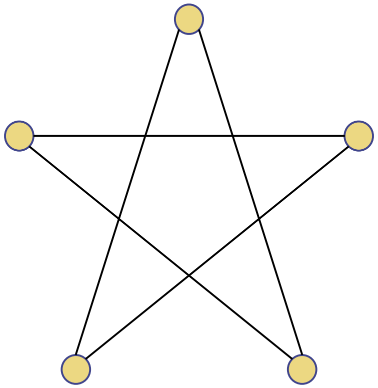
Just because a graph is drawn with edges crossing doesn't mean its not planar.

Q: Why can't we conclude that the following is non-planar?



Planar Graphs -Common Misunderstanding

A: Because it is isomorphic to a graph which *is* planar:



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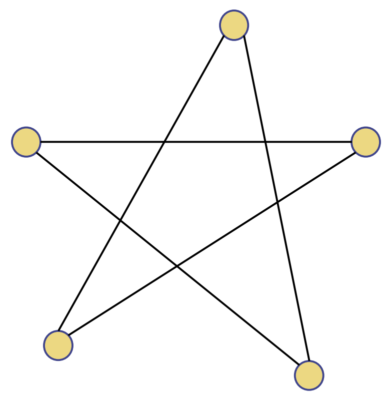

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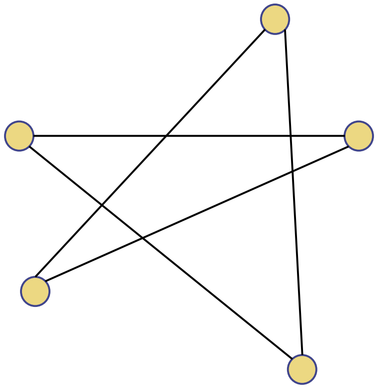

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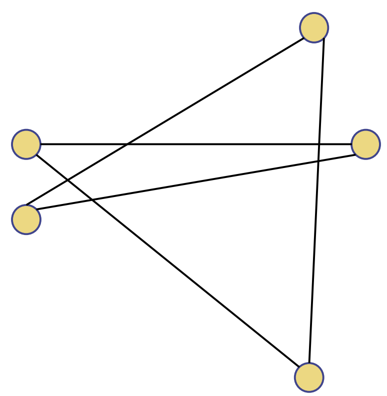

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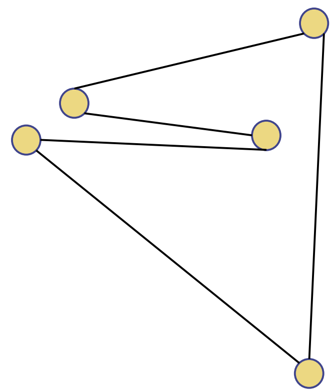

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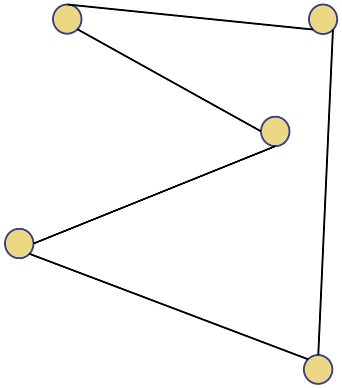

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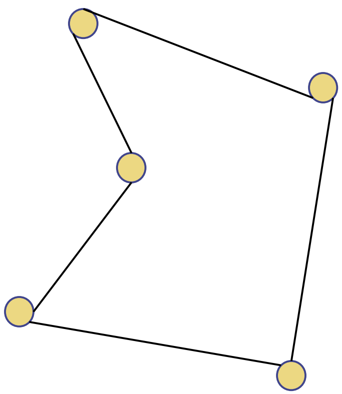

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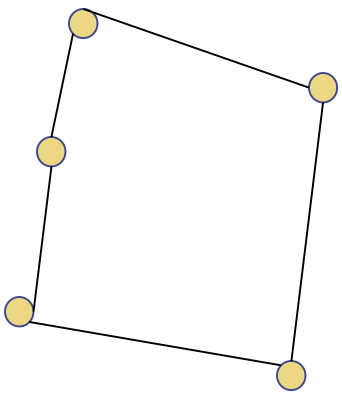

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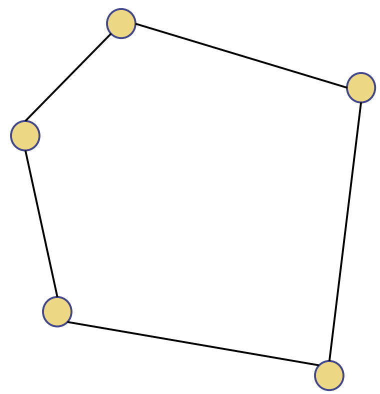


Do Edges Intersect?
下面图是平面图吗

- Planar graphs can sometimes be drawn as non-planar graphs. It is still a planar graph, because they are isomorphic.
- 以非平面图的方式画出来的图，仍然有可能是平面图。

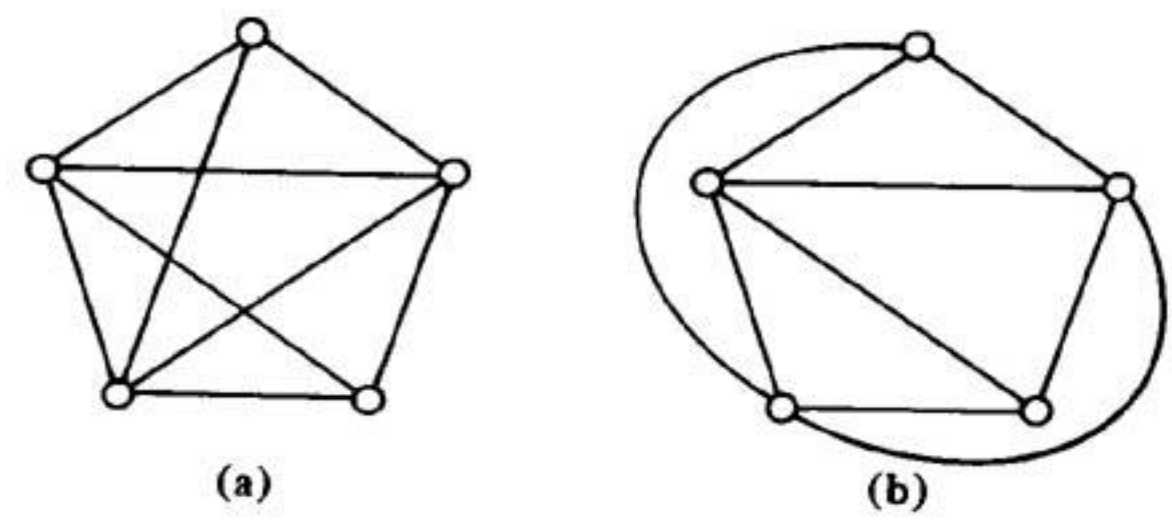
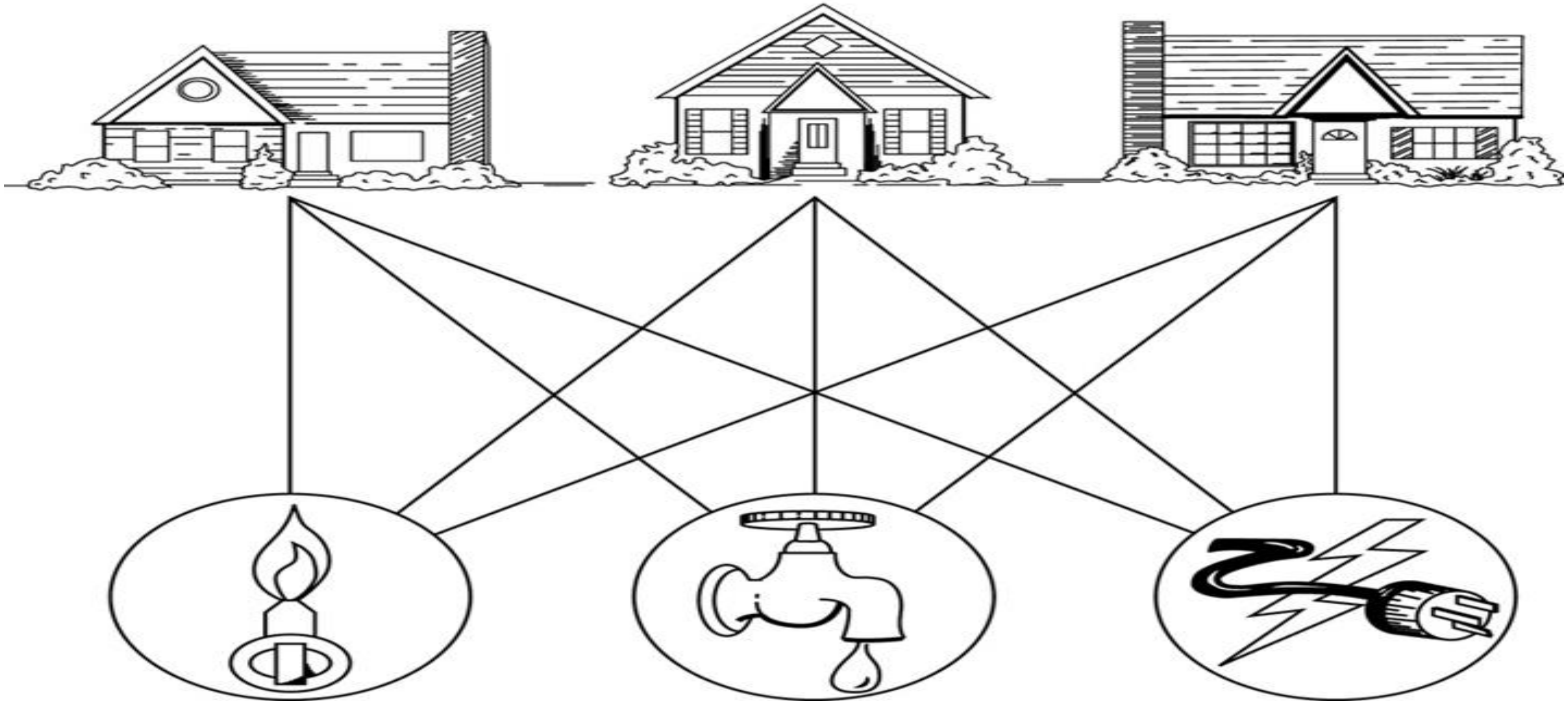


Figure 9.3

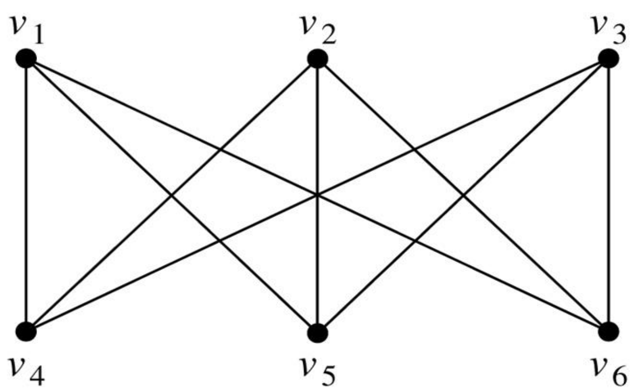
Three Houses / Three Utilities

- Q. Suppose we have three houses and three utilities. Is it possible to connect each utility to each of three houses without any lines crossing?
- Planar or Non-Planar ?
- This is also known as K(3,3) bipartite graph

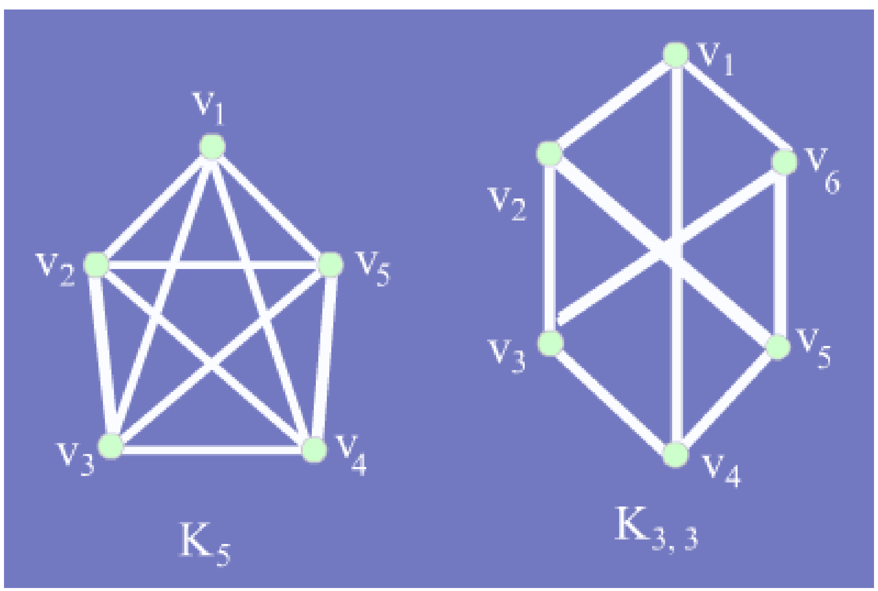


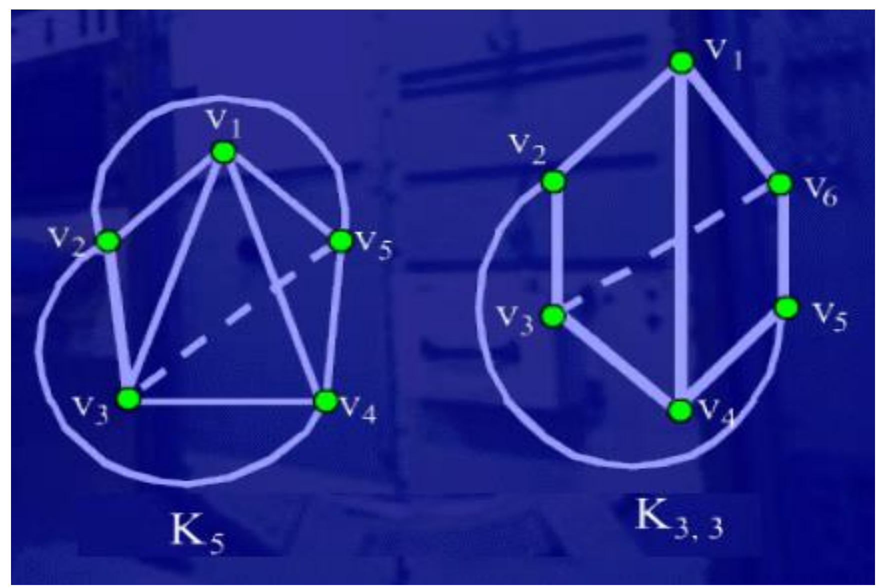
Two Examples of Non-planar

两个典型的非平面图的例子



The graph $K_{3,3}$ above is actually same with the $K_{3,3}$ below, it is just different way to draw on the plane.





K_5, $K_{3,3}$ are non-planar

平面图判断

从上面六边形里有3条对角线$K_{3,3}$是非平面图的判断方法中，总结出一种判断非平面图的方法：

如果一个图里面有一个子图（圈）C_n, n>=6. 而且对于这个子图，存在至少3条及以上的类似于$K_{3,3}$交错的对角线， 那么这个图一定是非平面图。

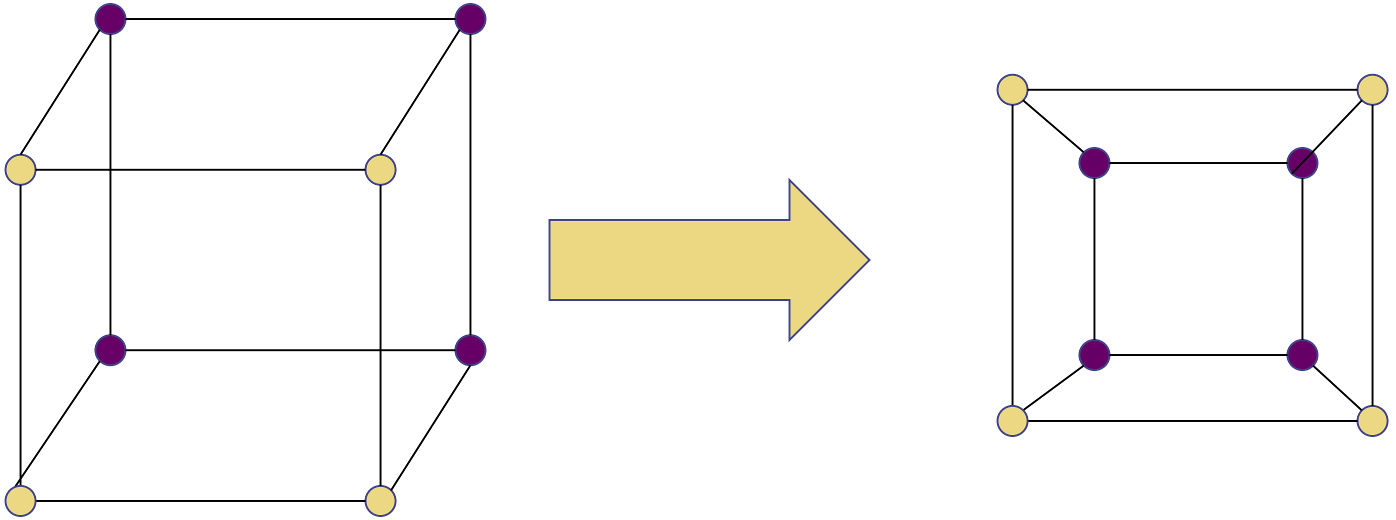
大家思考，为什么？

Proving Planarity 平面性

To prove that a graph is planar amounts to redrawing the edges in a way that no edges will cross. It may need to move vertices around and the edges may have to be drawn in a very indirect fashion.

Proving Planarity 3-Cube

E.G. show that the 3-cube is planar:



Disproving Planarity

有一些方法来判断一个图是或者不是平面图，可以根据具体情况选择任意一种。

Disproving Planarity

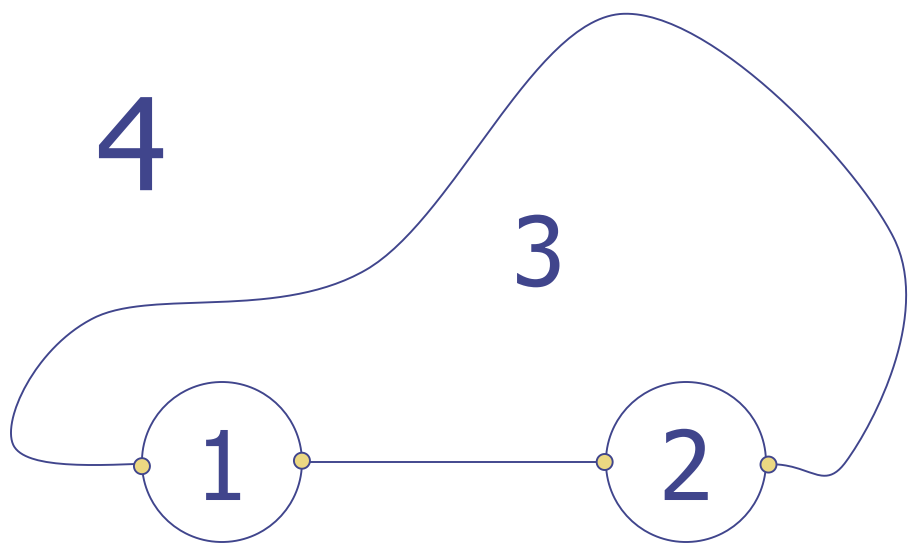
The idea tries to find some invariant quantities（不变量) possessed by graphs which are constrained to certain values, for planar graphs. Then to show that a graph is non-planar, compute the quantities and show that they do not satisfy the constraints on planar graphs.

一种想法是试图去寻找平面图的某些或某种不变的量（不变性），然后通过计算确定某些图不满足，从而否定一个图是平面图。

Regions 区域(面）

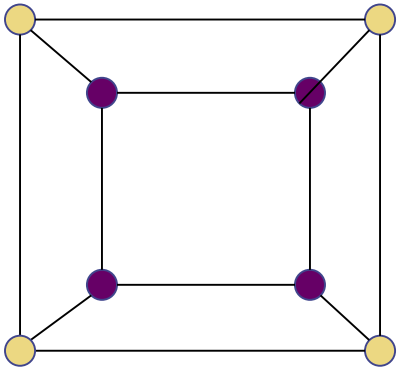
平面图的区域数：当一个图能不交叉地画于一个平面时，由它的所有边将平面分割成不同的部分（块），由一些边围成的封闭的或者是一个无限的，跟其它部分不重叠的区域（或者称为面），这些区域的个数，对于一个给定的图，只要是不交叉地画出来，无论画法如何，这个数是确定的不变的。

EG: the car graph has 4 regions:



Regions 区域

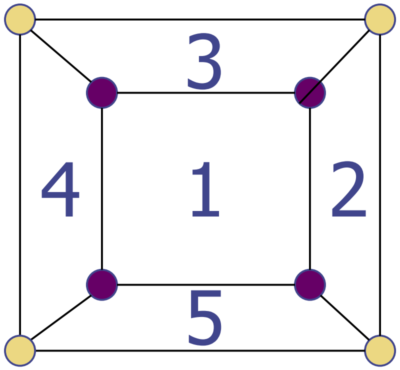
Q: How many regions does the 3-cube have?



Regions

A: 6 regions

6



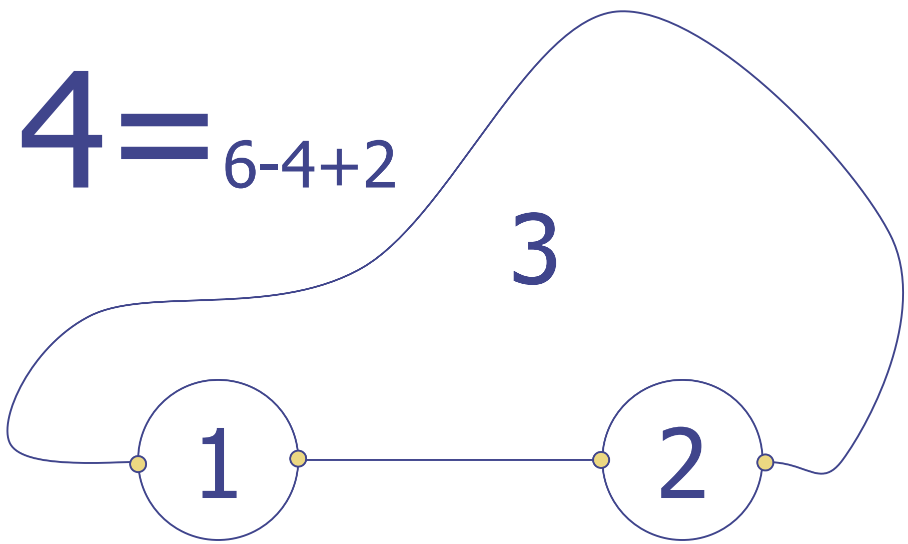
欧拉公式

Theorem: 一个**连通的**平面图所围成的区域数是一个与画法无关的不变量，这个区域数与结点数、边数的关系满足下面的公式（欧拉公式）

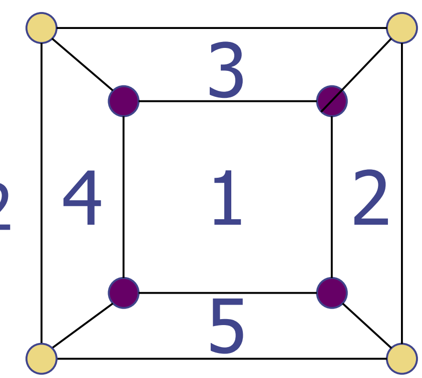
$$r = |E| - |V| + 2 \quad (\textbf{Euler Formula})$$

EG: Verify formula for car and 3-cube:

$4 = 6-4+2$



$6 = 12-8+2$



Euler Characteristic

The formula is proved by showing that the quantity $\chi = r - |E| + |V|$ must equal 2 for planar graphs. χ is called the ***Euler characteristic*** *欧拉特征值*.

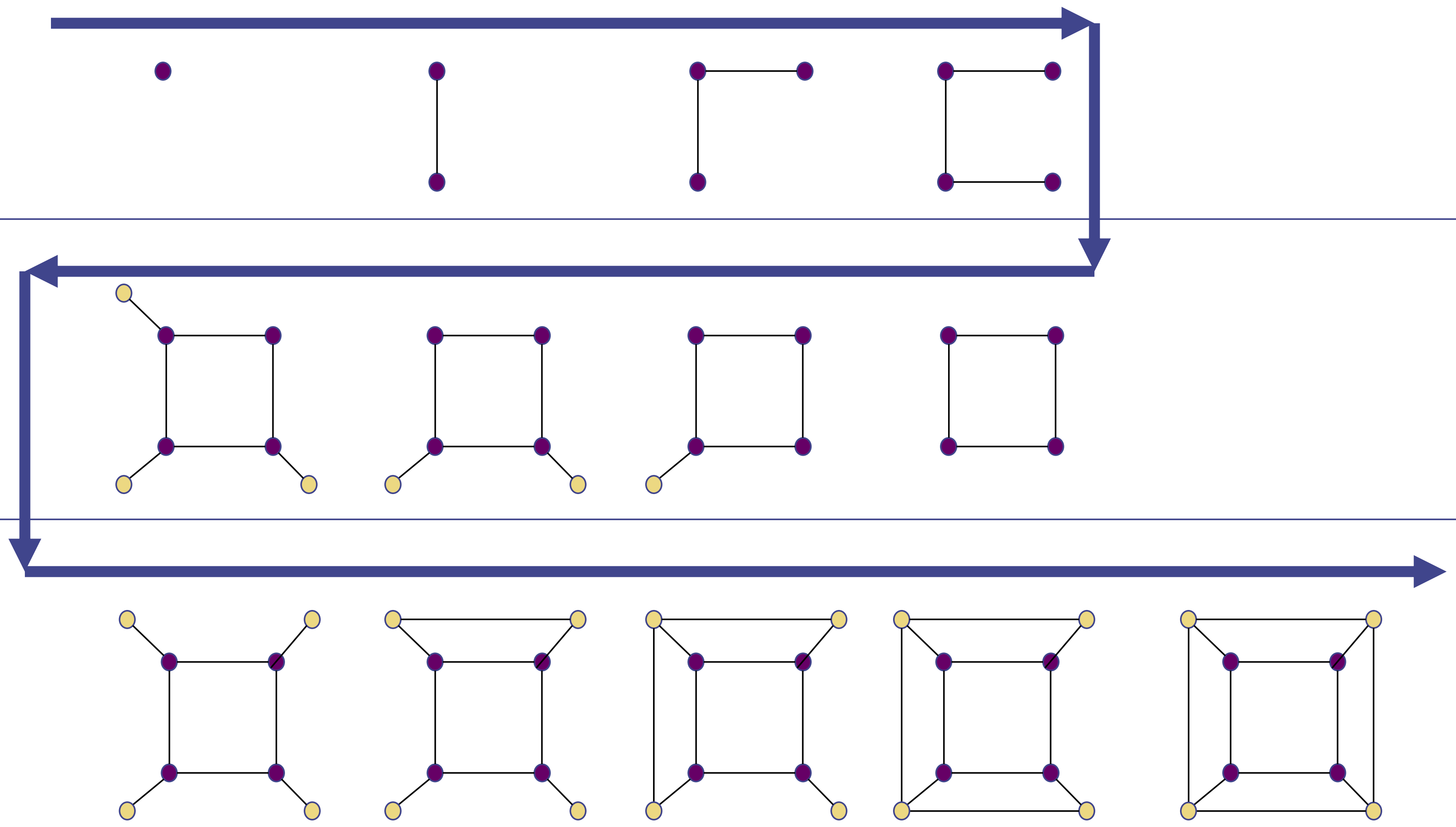
The idea is that any connected planar graph can be built up

From a vertex through a sequence of vertex and edge

additions. 一个连通的平面图可以从一个结点开始，再通过逐个加入点和不交叉地加入边的思路，画出来，然后总结分析其中边数、点数、面数的变化

For example, build 3- Cube as follows:

Euler Characteristic



Euler Characteristic

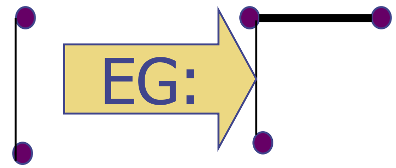
Thus to prove that χ is always 2 for planar graphs, one calculate χ for the trivial vertex graph:

$$\chi = 1-0+1 = 2$$

and then checks that each possible move does not change χ .

Euler Characteristic

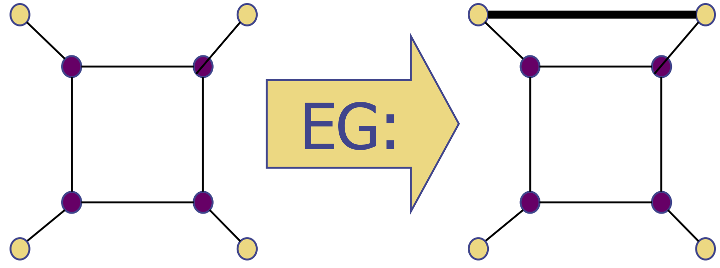
Check that moves don't change c :



1) Adding **a degree 1 vertex**:

r is unchanged. $|E|$ increases by 1. $|V|$ increases by 1. c = c + (0-1+1)

2) Adding **an edge between pre-existing vertices**:



r increases by 1. $|E|$ increases by 1. $|V|$ unchanged. c += (1-1+0)

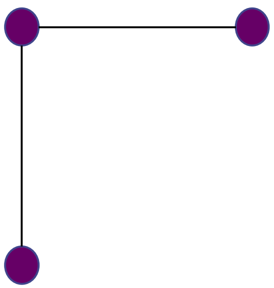
Animated Invariance of Euler Characteristic

$\lvert V\rvert$	$\lvert E\rvert$	r	$\chi = r - \lvert E\rvert + \lvert V\rvert$
1	0	1	2

Animated Invariance of Euler Characteristic

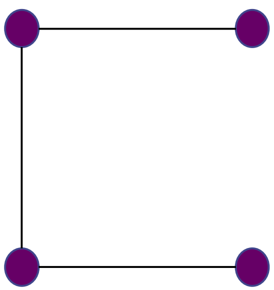
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
2	1	1	2

Animated Invariance of Euler Characteristic



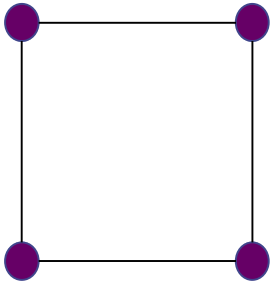
$\|V\|$	$\|E\|$	r	$\chi =$ $r - \|E\| + \|V\|$
3	2	1	2

Animated Invariance of Euler Characteristic



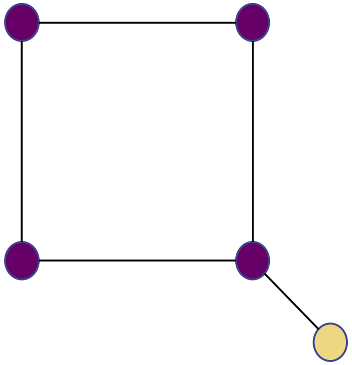
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
4	3	1	2

Animated Invariance of Euler Characteristic



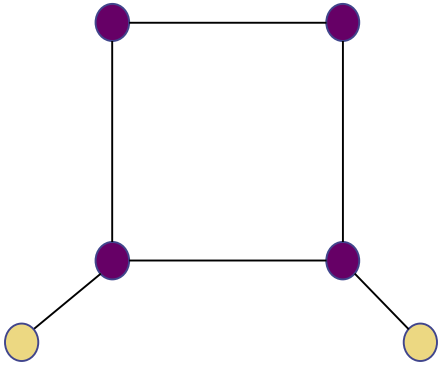
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
4	4	2	2

Animated Invariance of Euler Characteristic



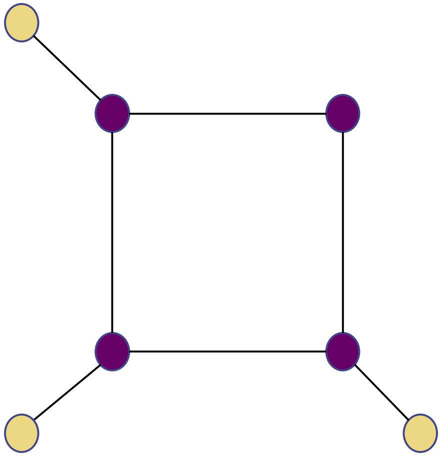
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
5	5	2	2

Animated Invariance of Euler Characteristic



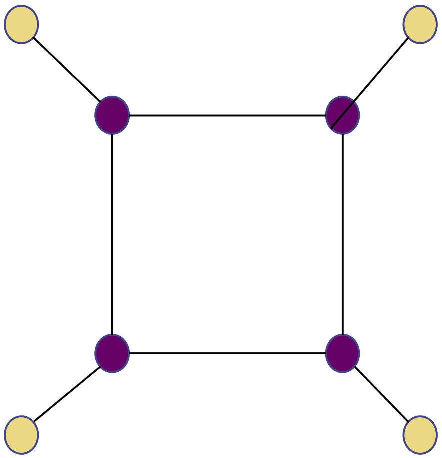
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
6	6	2	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
7	7	2	2

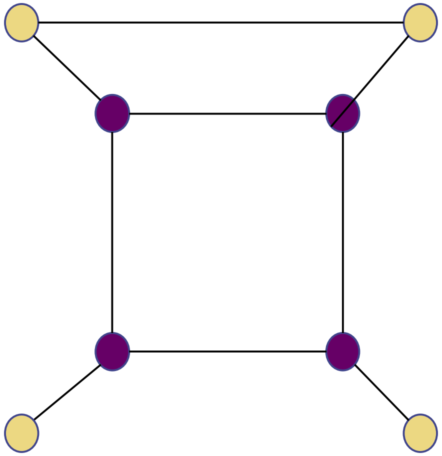
Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	8	2	2

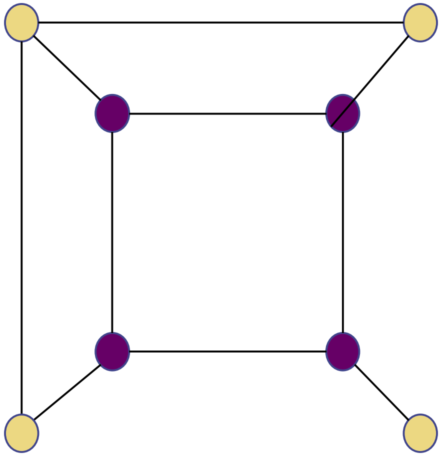
依次加入度为一的点，以及在现有点间加入边。观察点数、边数、面数的变化

Animated Invariance of Euler Characteristic



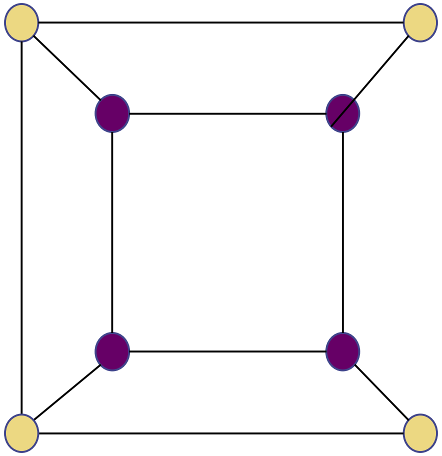
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	9	3	2

Animated Invariance of Euler Characteristic



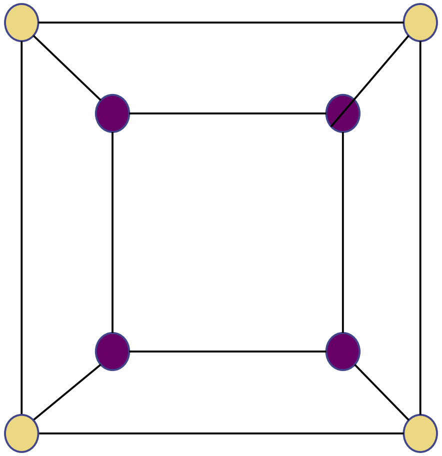
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	10	4	2

Animated Invariance of Euler Characteristic



$\lvert V \rvert$	$\lvert E \rvert$	r	$\chi = r - \lvert E \rvert + \lvert V \rvert$
8	11	5	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
8	12	6	2

平面图的必要条件

- **推论 1**: 如果图G是一个连通的简单平面图，那么当结点数 $v \geq 3$时，有不等式 **$e \leq 3v-6$**.
- ***证明思路:***
- Concept "the degree of a region": the number of edges on the boundary of this region. N: total degree of all regions.
- (0): 特殊情况，没有围城有限区域时，只有一个无限面。此时 $e = v-1$， $e \leq 3v-6$ 是成立的。
- (1):For simple graph (no loop, no multi-edge), the degree of each region is at least 3. N >= 3r (r is the number of regions)
- (2) because each edge occurs on the boundary of a region exactly twice. N<= 2e
- (3) USING Euler formula. r=e-v+2

K_5 is non-planar 非平面图

- n=5
- e= n * (n – 1) / 2 = 10
- Using necessary conditions of planar graphs:
- e <= 3n – 6
- 10 <= 3(5) – 6
- 10 <= 9 ???
- By contradiction, K_5 must be non-planar

平面图的必要条件

- **推论 2**: 一个**简单连通的**平面图，至少有一个结点的度不大于5 **($deg(v_i) \le 5$)**.
- ***证明思路:*** v denotes the number of vertices.
  - When v=1,2 or ≥3
- If *G* has one or two vertices, the result is true. If *G* has at least three vertices, by Corollary 1
- we know that $e \le 3v - 6$, so $2e \le 6v - 12$. If the degree of every vertex were at least six, then…

- because $2e =$ 总度数 (by the handshaking theorem), we would have $2e \ge 6v$. But this contradicts the inequality $2e \le 6v - 12$. It follows that there must be a vertex with degree no greater than 5.

平面图的必要条件

- 推论3：若连通的简单平面图有e条边，v个结点，v>2, 并且没有长度为3的回路，则e<=2v-4

- 请同学们自己分析证明

$K_{3,3}$ is Non-Planar 非平面图

- Proof by contradiction of theorems 反证法思路

- Since graph is bipartite, no edge connects two edges within same subset of vertices.
- The total degrees of all regions N >= 4r must be true, since graph contains no simple triangle regions of 3 edges, where r is the number of distinct regions. 每个区域至少由4条边围成
- N <= 2e must be true, since no edge can be used more than twice in forming a region

没有哪条边会对N贡献2次以上，顶多贡献2.

(con’t) Proof of $K_{3,3}$

- For K(3,3) v=6, e= 9, r= ??
- 4r <= N <= 2e
- 4r <= (2e = 2 * 9 = 18)
- r <= 4.5

- Using 欧拉定理, v – e + r = 2
- 6 – 9 + r = 2
- r = 5

- Proof by contradiction:
- r cannot be both equal to 5 and less than 4.5
- Therefore, K(3,3) is a non-planar graph

Complete Graphs

- Denoted by K_n
- All vertices are connected to all vertices
- $e = n * (n - 1) / 2$

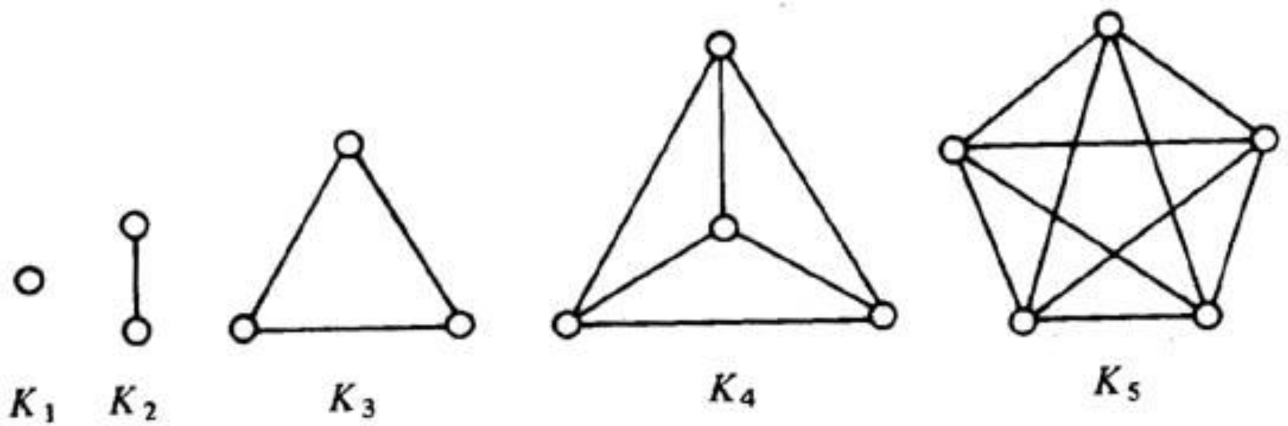


Figure 2.2

Question about K_n

- For the complete graph K_n with $n>5$, is it planar? Why?

- For $n>5$, the K_5 is a subgraph of K_n, so…

How about $K_{n,n}$?

思考问题

- 欧拉定理加入了条件---图是连通图。那么如果一个图不是连通图，如何？

  假定有t个连通分支, 引导学生自己得出一个结论或者公式。

  如果性质1中的连通条件去掉，能有什么类似结论吗？

- 如果一个图的有某个子图是非平面图，那么有什么结论？

- 如果一个图有一个子图为平面图，又如何？

Subdivisions of graph G

- **Elementary Subdivision** – a graph obtained from a graph G, by inserting vertices of degree two into any edge
- (H is a valid subdivision of G, while F is not)

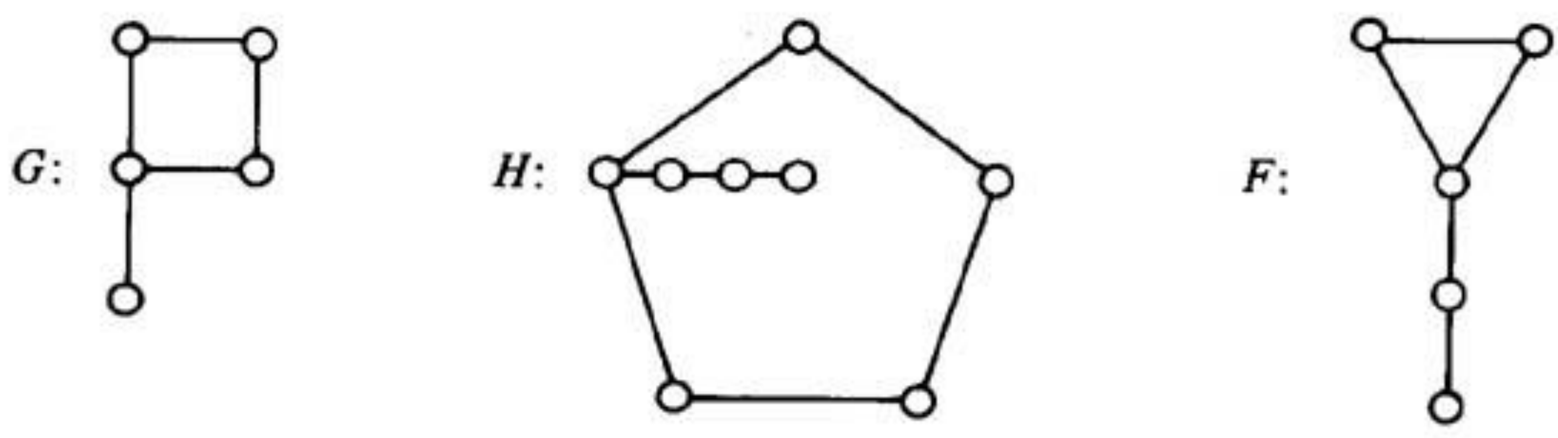


Figure 9.5

Kuratowski Reduction Theorem

- Homeomorphic: G1=(V1,E1), G2=(V2,E2) are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.

- 中文翻译:增减度为2的结点变换意义下同构（教材:同胚）:

通过一系列的删除或者添加度为2的结点使得图发生变化，变到另一个图, 但平面性不变。

Kuratowski Reduction Theorem

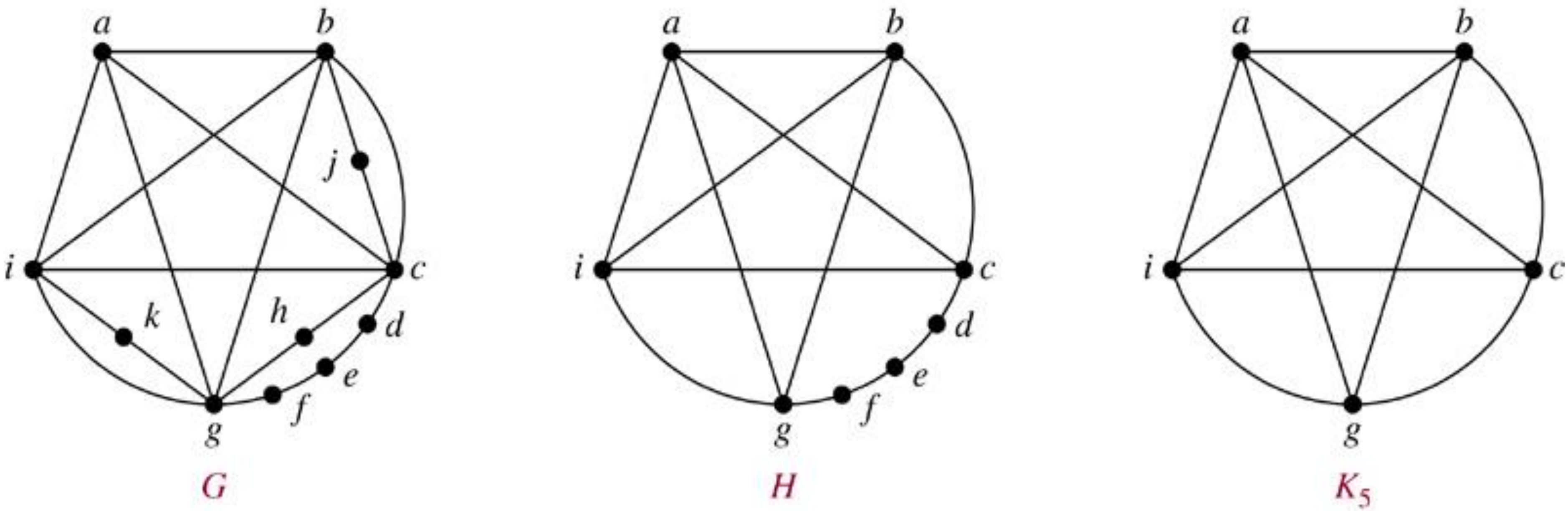
- **Kuratowski定理:** 一个图为非平面图的充分必要条件是它包含有与$K_{3,3}$ or K_5在增减度为2的结点变换意义下同构的子图。

- *The proof of it is very complicated, will not shown here.*
- *Kuratowski's theorem– in principle always works, though in practice can be quite unwieldy.)*

- *一个图为平面图的充分必要条件又是什么？*

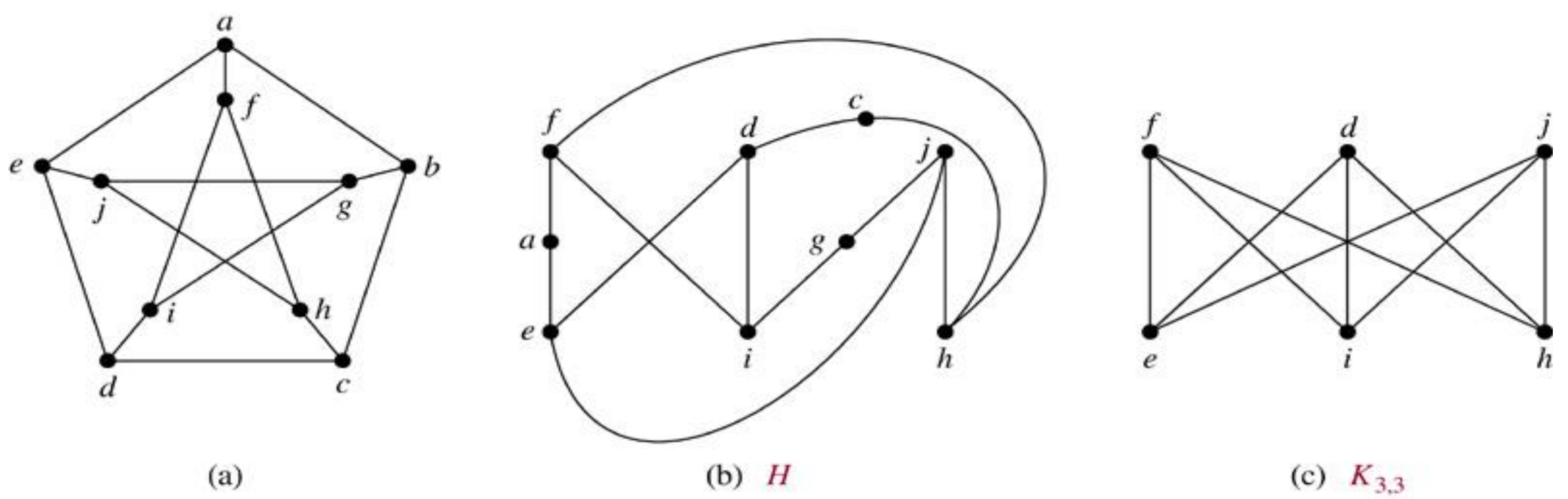
Examples using **Kuratowski THM**

- Determine whether the graph G is planar



Examples using **Kuratowski THM**

- Determine whether the Petersen graph (a) is planar
- Solution: to obtain H by removing vertex b and the three edges have b as a endpoint

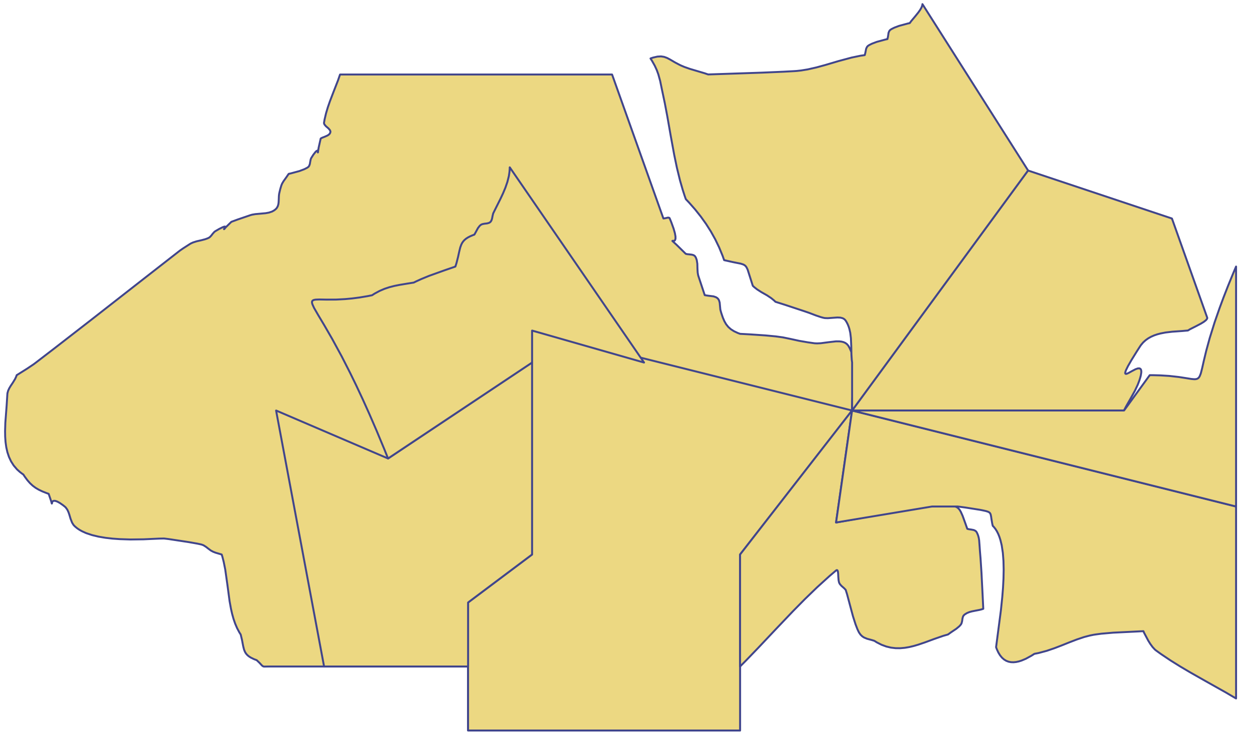


Planar Graph Exercises

- 6.7节 T3, T4，T7， T8

图着色

Consider a fictional continent分析下面这个虚构的陆地图：要把不同地区区分（分割开来）

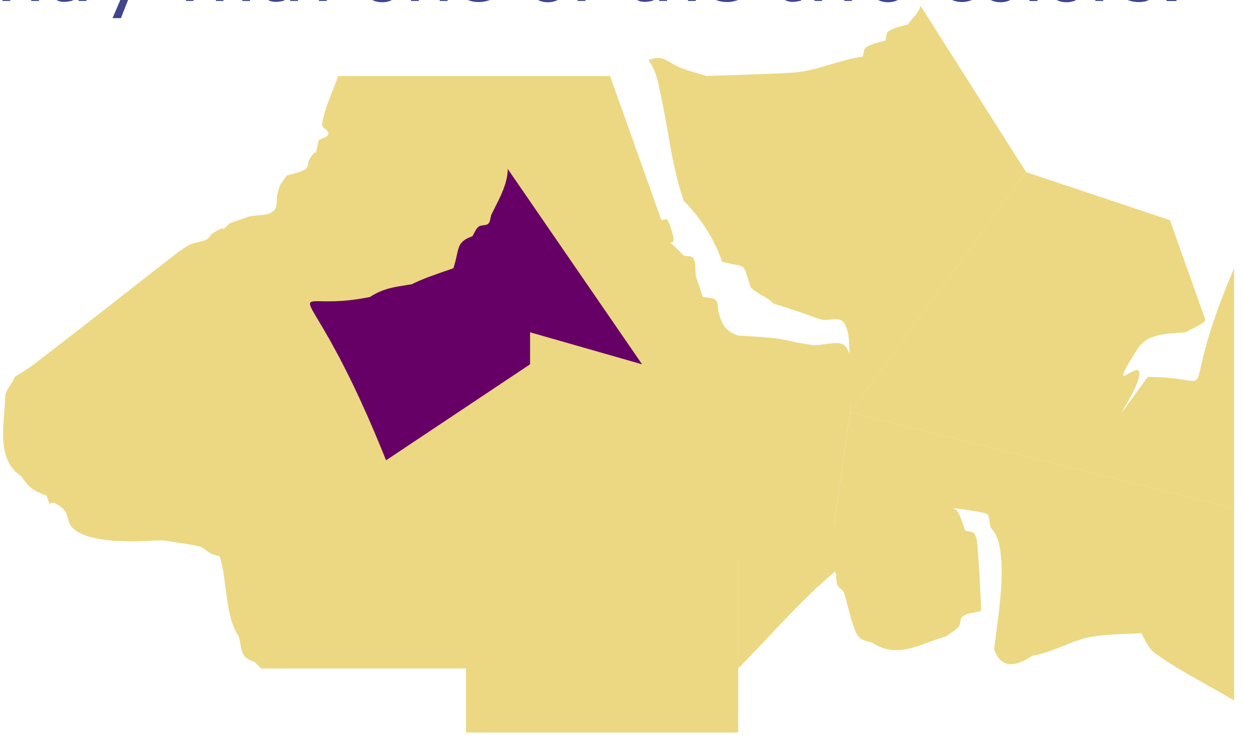


Map Coloring

- Suppose removed all borders but still wanted to see all the countries. 如果用颜色来区分的话， 1 color insufficient.

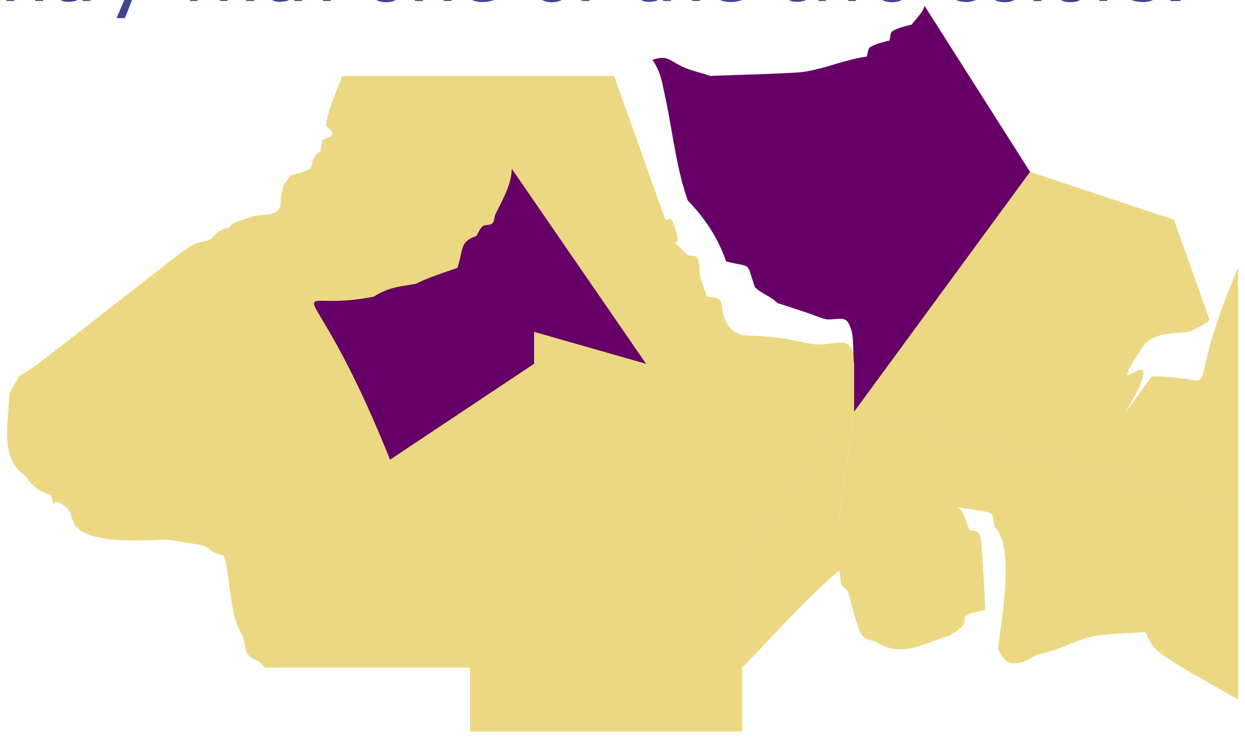
Map Coloring

So add another color. Try to fill in every country with one of the two colors.



Map Coloring

- So add another color. Try to fill in every country with one of the two colors.



Map Coloring

So add another color. Try to fill in every country with one of the two colors.

Map Coloring

So add another color. Try to fill in every country with one of the two colors.

Map Coloring

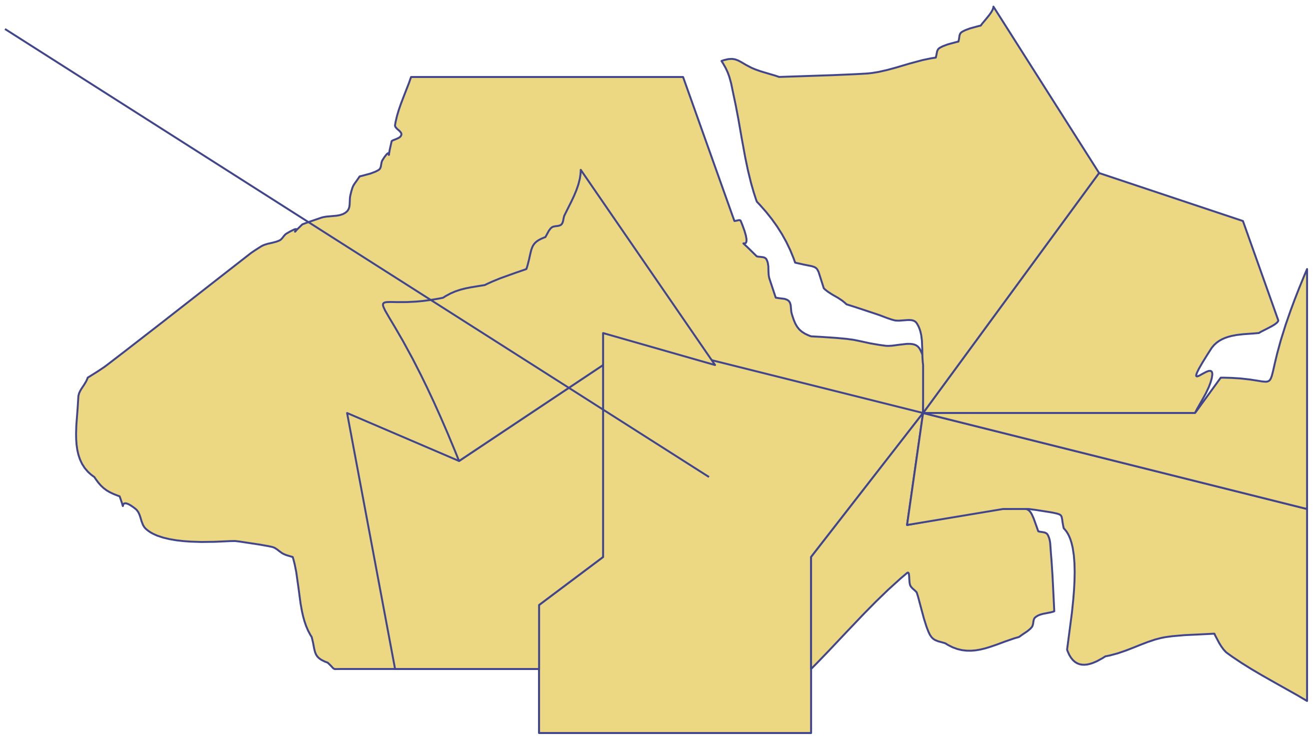
PROBLEM: Two adjacent countries forced to have same color. Border unseen.

Map Coloring

So add another color:

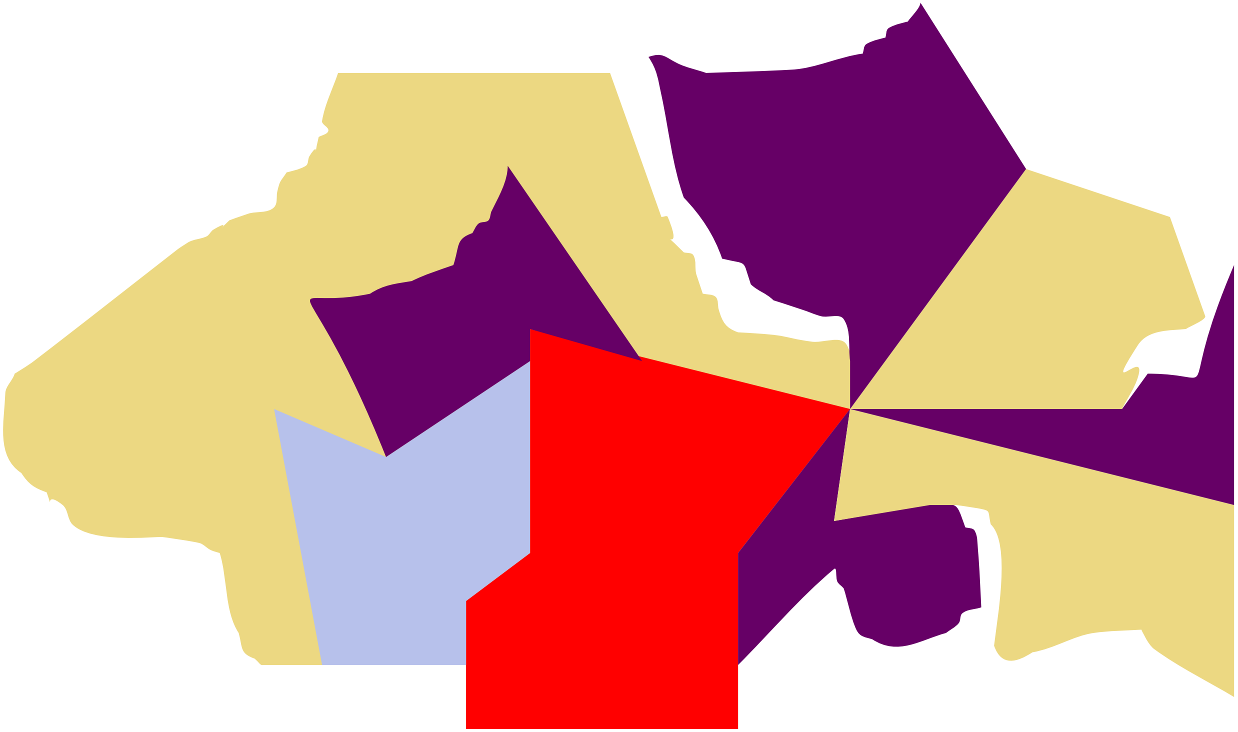
Map Coloring

Insufficient. Need 4 colors because of <u>this</u> country.
4种不同颜色才能区分开来



Map Coloring

- With 4 colors, could do it.



4-Color Theorem—四色定理

Theorem：任何平面图的区域都可以用4种颜色足够将所有区域分割开来，使得有共享边界的区域之间颜色不一样。

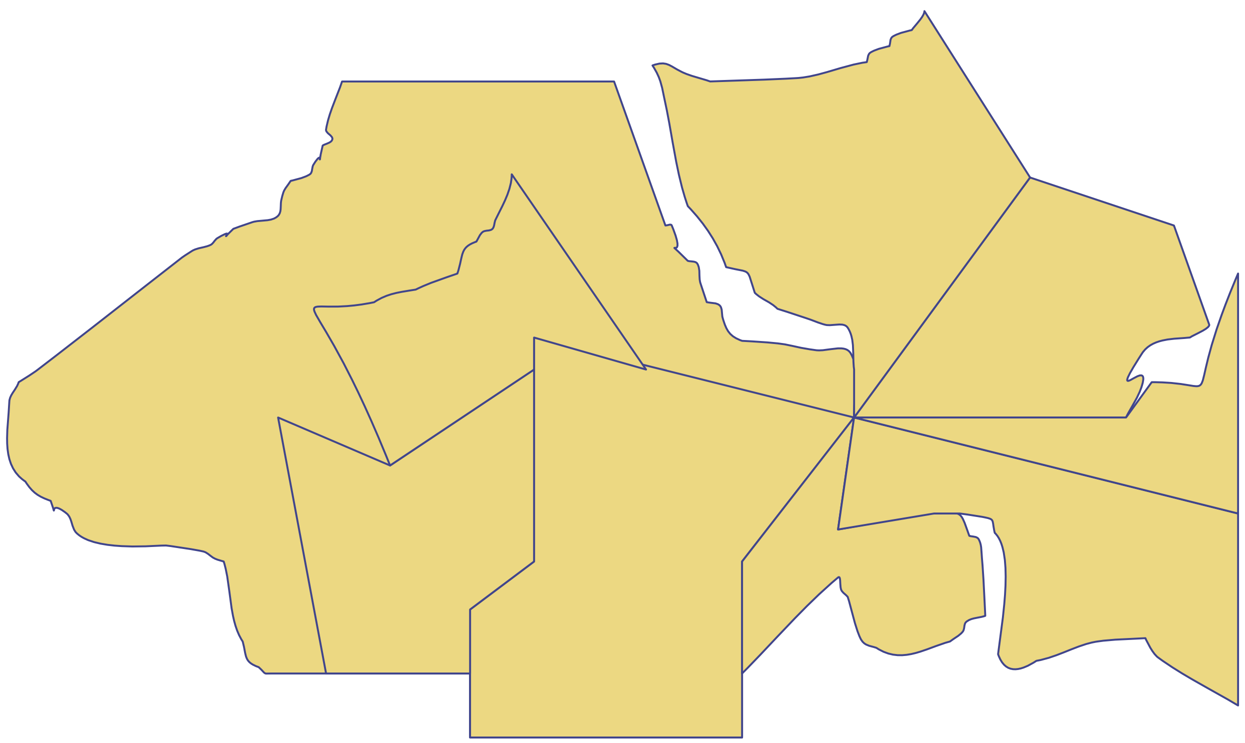
（也就是说最多是4色图）

Proof by Haaken and Appel used exhaustive computer search. (四色猜想)

It took more than 100 years to get the correct prove.

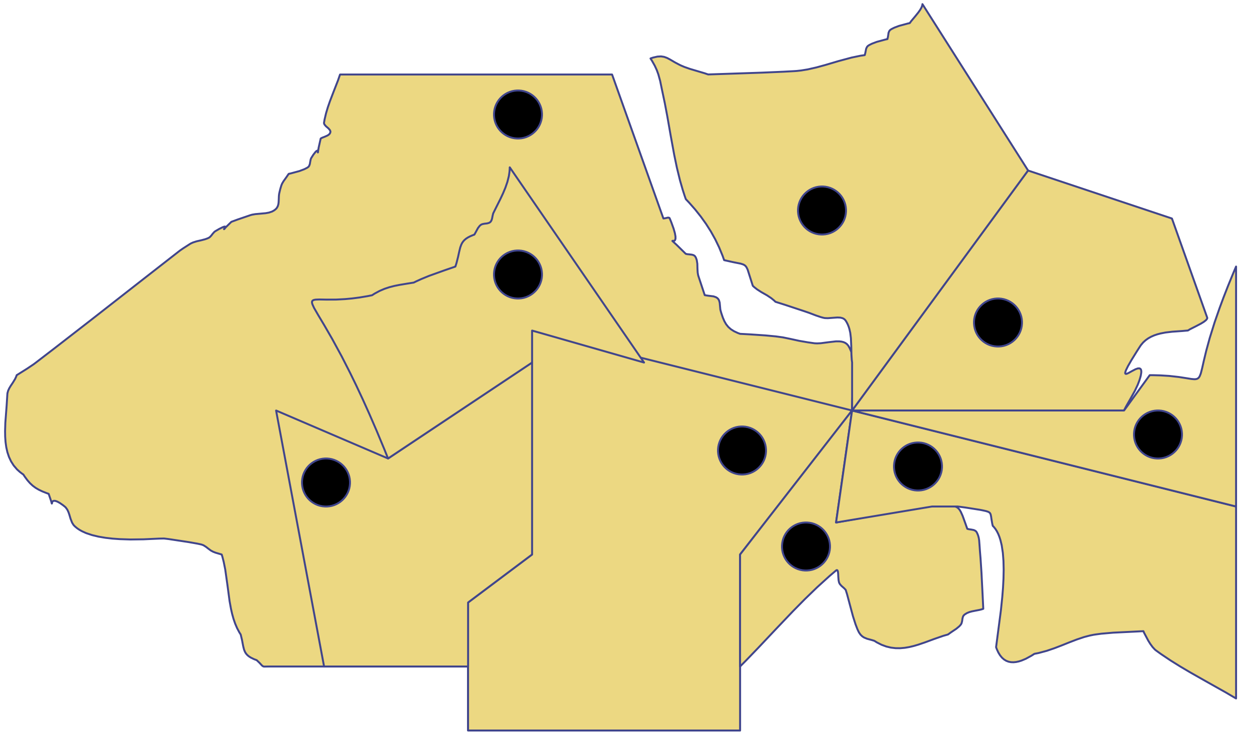
从地图着色到图着色的建模

对地图着色的问题建模转化成图着色的问题



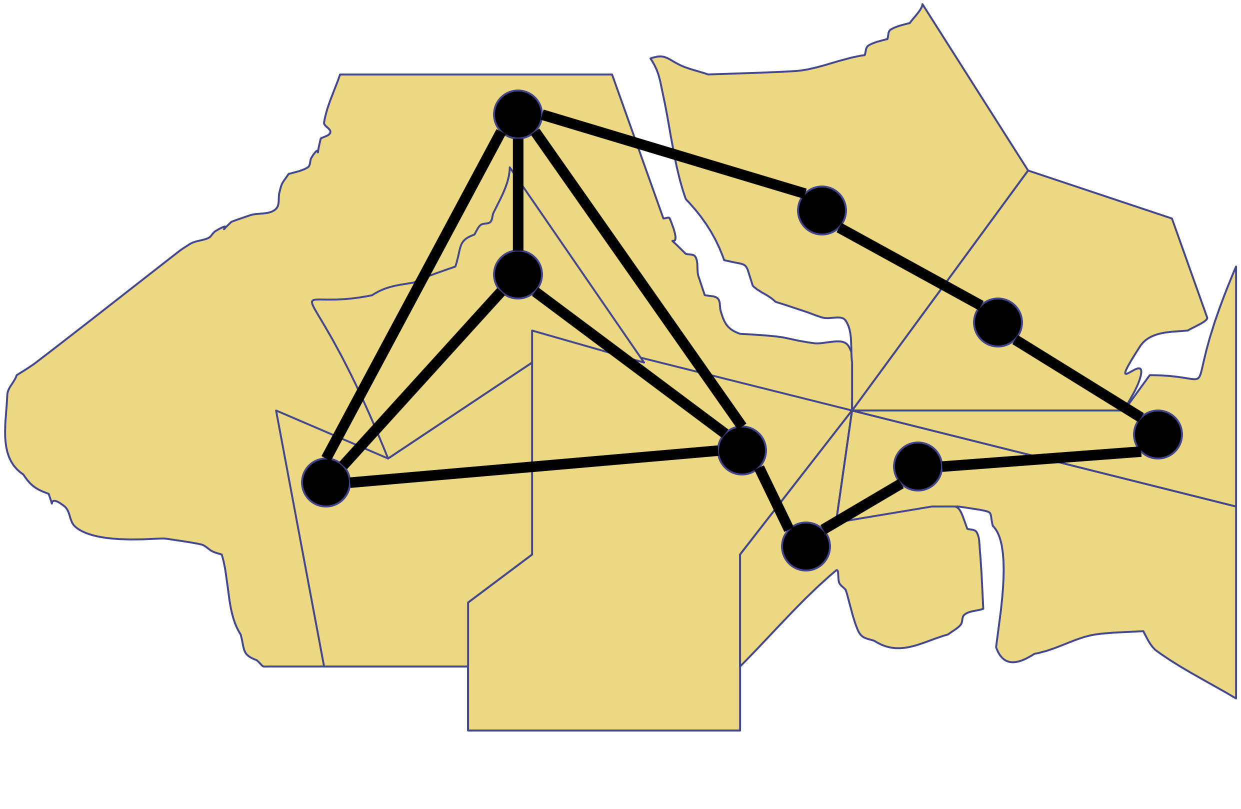
From Map Coloring to Graph Coloring

For each region introduce a vertex:



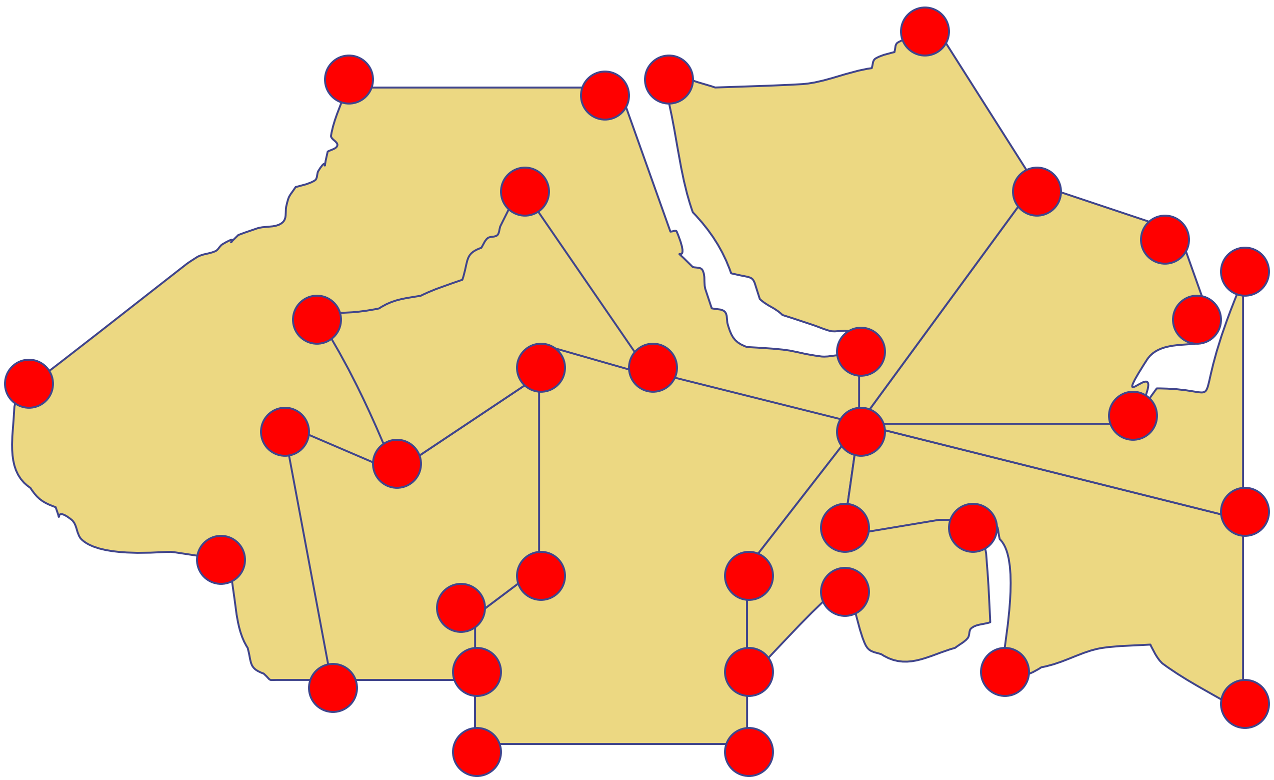
From Map Coloring to Graph Coloring

For each pair of regions with a positive-length common border introduce an edge:



From Maps to Graphs to Dual Graphs (对偶图)

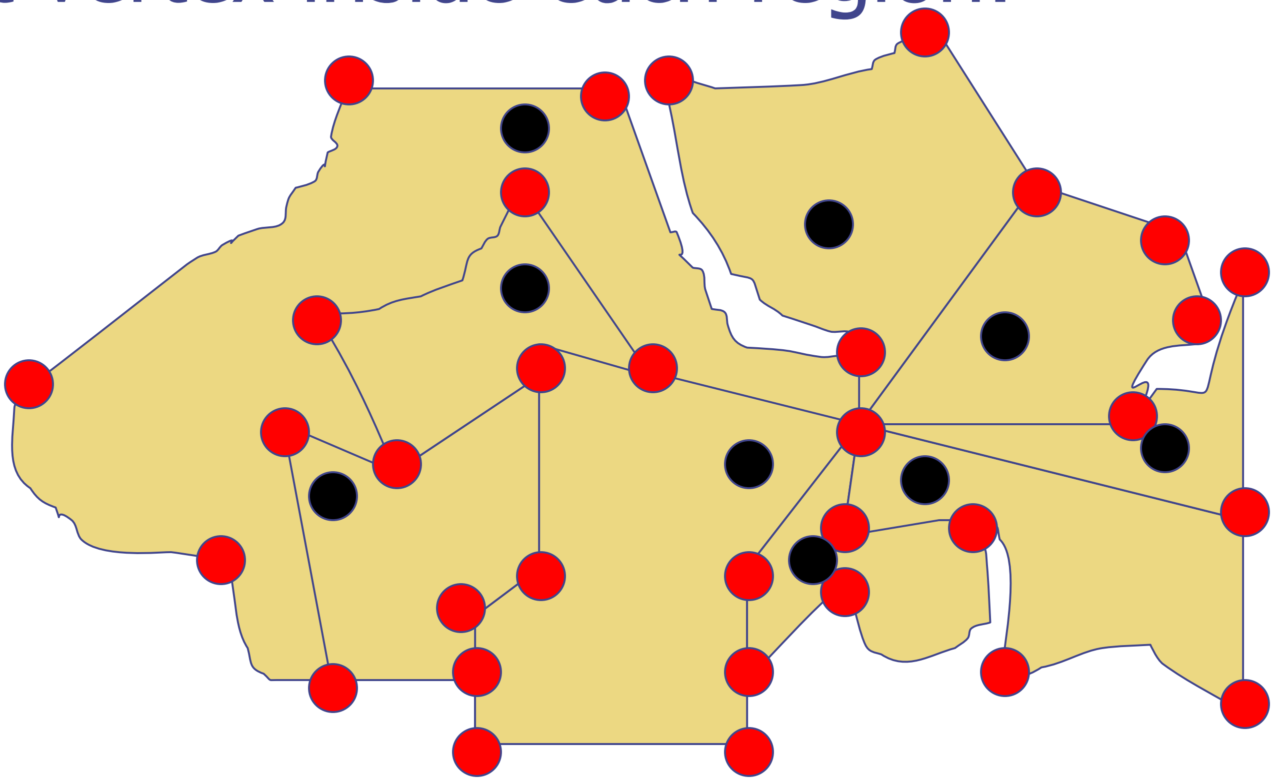
think of original map as a graph, and we are looking at *dual graph*:



From Maps to Graphs to Dual Graphs (对偶图)

Dual Graphs :

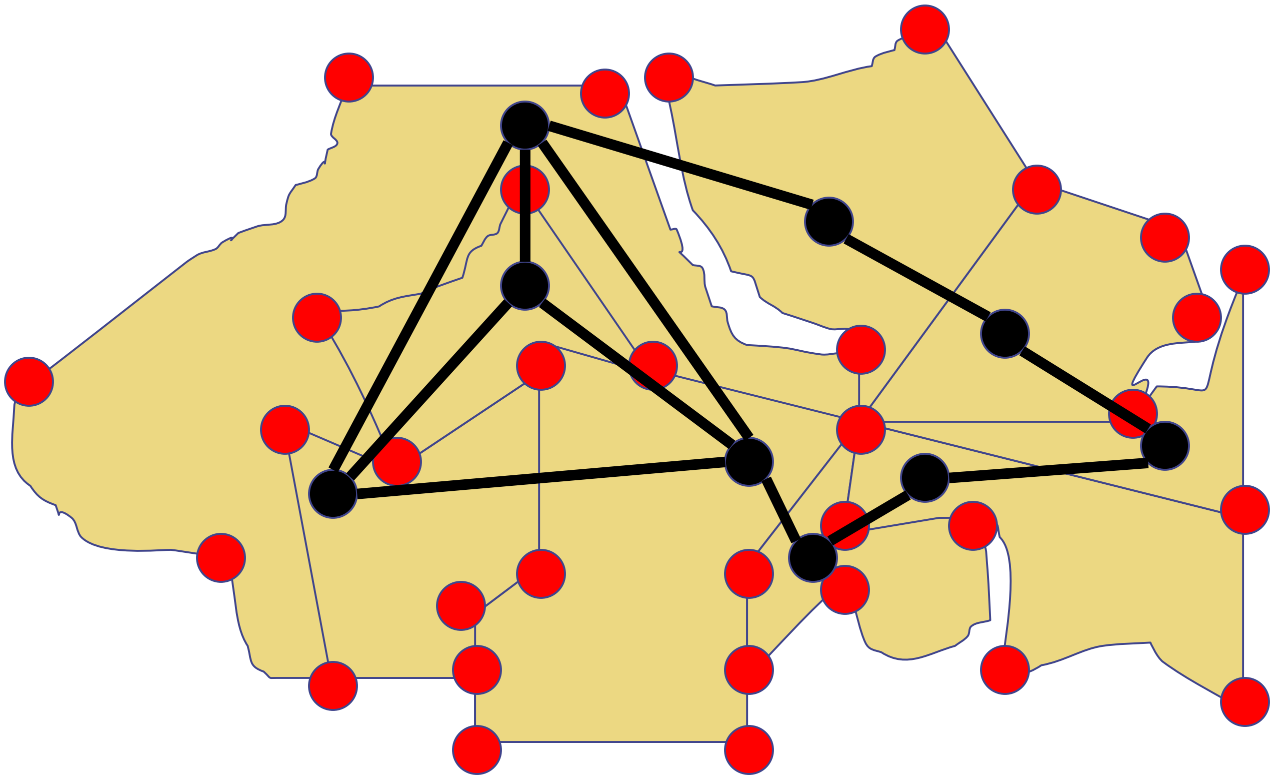
1) Put vertex inside each region:



From Maps to Graphs to Dual Graphs (对偶图)

Dual Graphs :

2) Connect vertices across common edges:



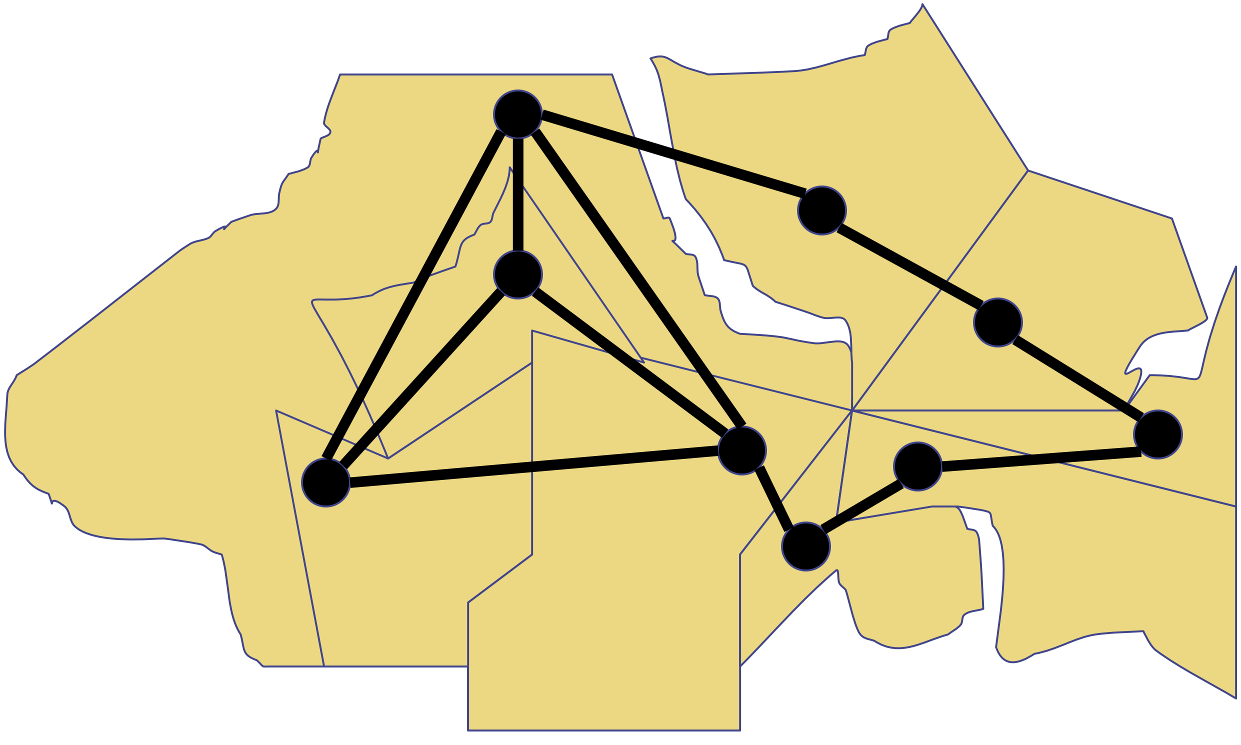
Def. of Dual Graph对偶图定义

DEF:一个平面图 $G = (V, E, R)$ [Vertices, Edges, Regions]的对偶图G^定义为如下的图:

- Vertices of G^: $V(G\hat{\ })= R$
- Edges of G^: $E(G\hat{\ })$ = set of edges of the form $\{F_1,F_2\}$ where 区域F_1 and *区域*F_2 share a common edge.

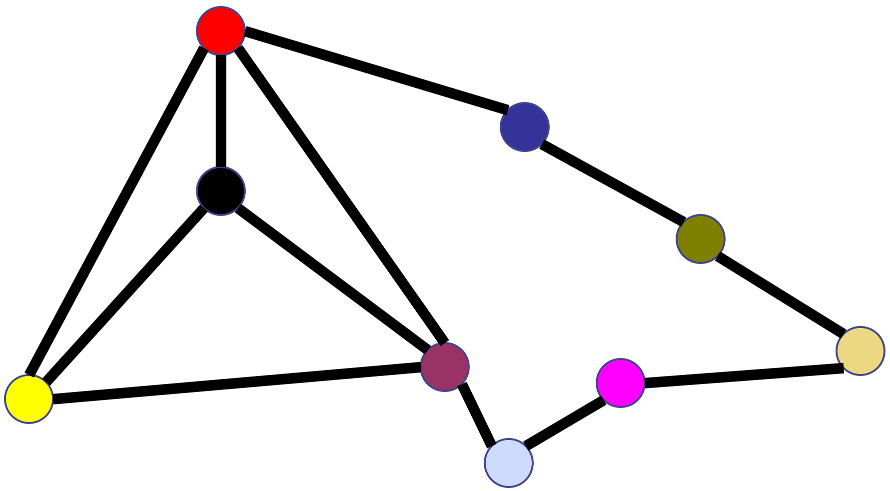
From Maps to Graphs to Dual Graphs

So take dual graph:



地图着色到图着色

Coloring regions is equivalent to coloring vertices of dual graph.



Definition of Colorable

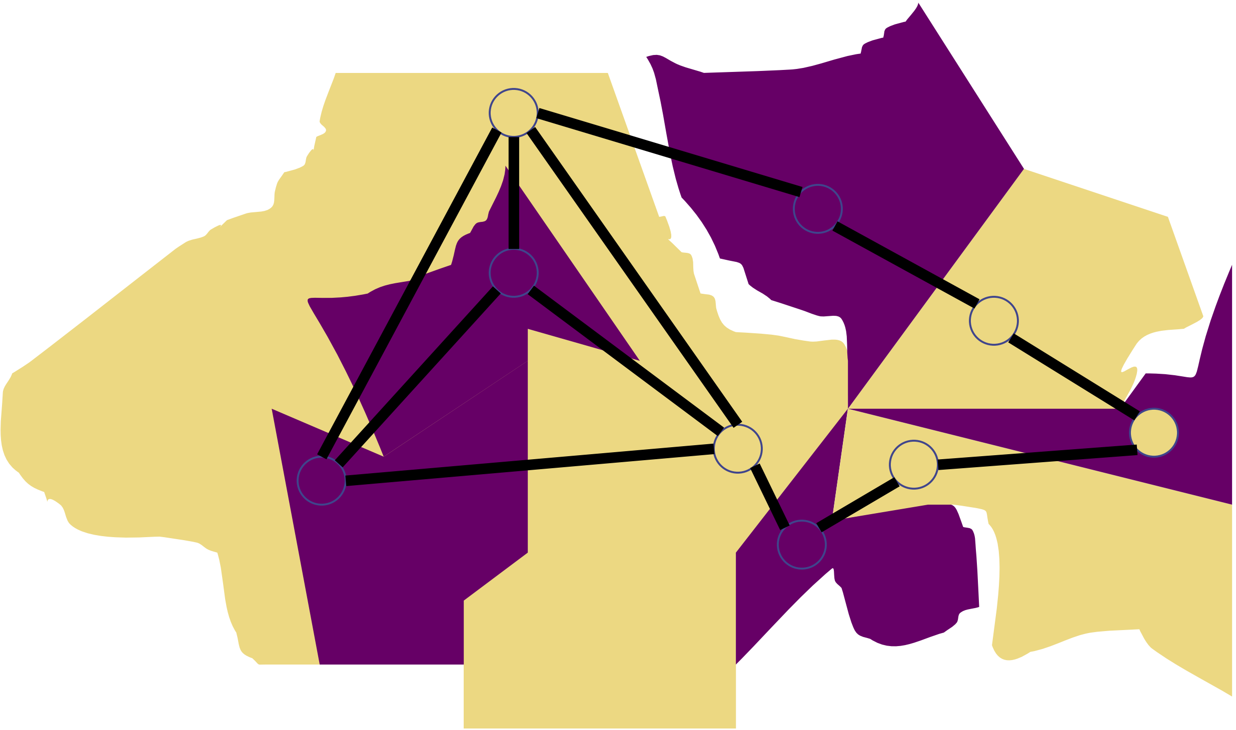
DEF: Let *n* be a positive number. 一个简单图称为n-色图或者说可n-色图，如果能用不超过n种颜色标记所有结点，使得任意邻接的结点都有不同的颜色。*(n种颜色不一定要用完)*

The ***chromatic number*** 颜色数 is smallest number *n* for which it is *n* -colorable.

EG: A graph is bipartite if and only if it is 2-colorable.
一个图为偶图当且仅当图是2-色图。
Think about why?

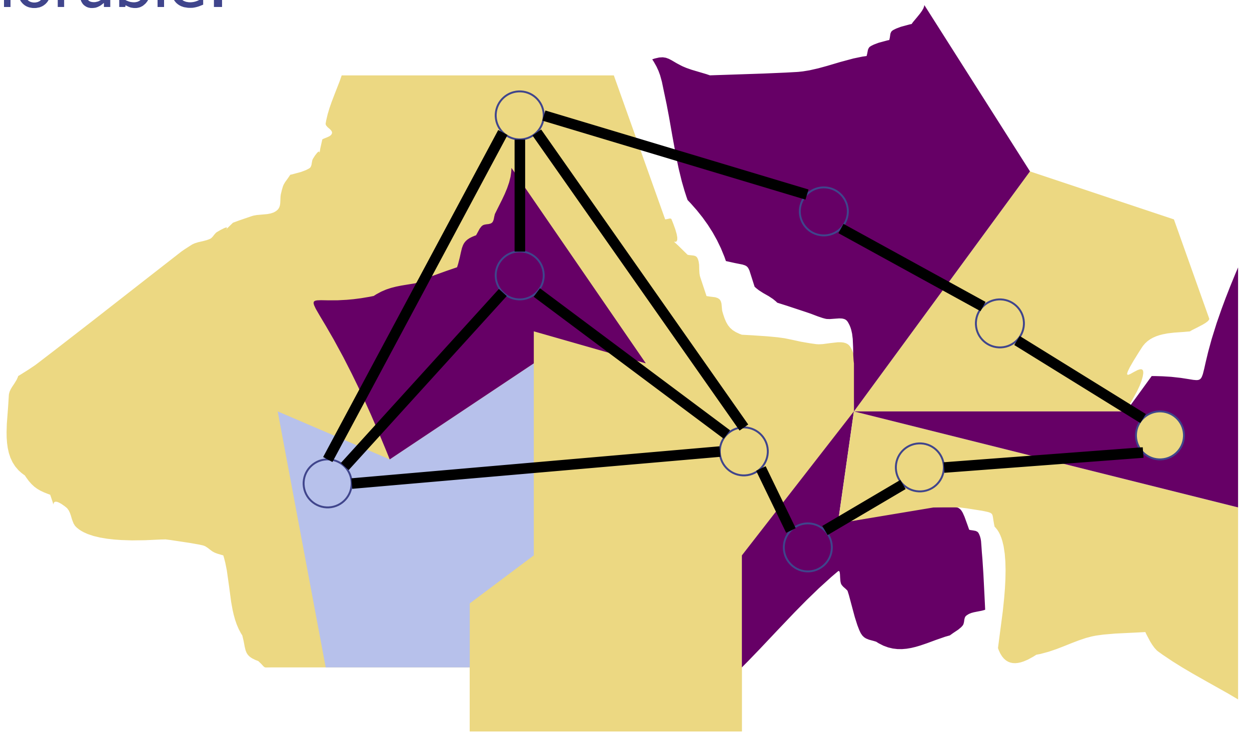
From Map Coloring to Graph Coloring

This map is not 2-colorable, so dual graph not 2-colorable:



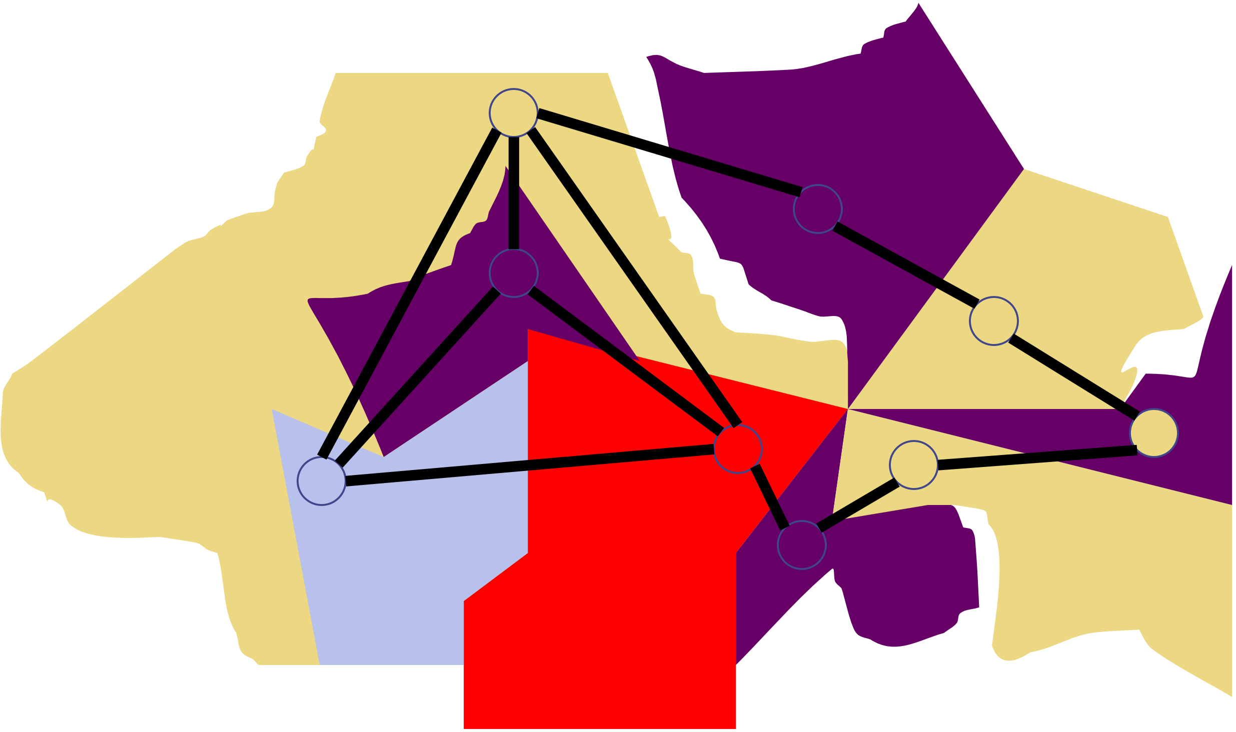
From Map Coloring to Graph Coloring

The following map is not 3-colorable, so graph not 3-colorable:



From Map Coloring to Graph Coloring

Graph is 4-colorable, so map is as well:



4-Color Theorem

四色定理：　任何平面图的颜色数不大于 4。

Note: it had been more than 100 years before a correct proof was given.

注：目前有的四色定理的证明是依赖计算机的，脱离计算机的人工证明还只有**五色定理。**

五色定理：也就是任何平面图的颜色数≤5

思考：　K_n的颜色数是多少？

颜色数是n. 因为任何两个点都是邻接的，所以颜色数不能少于n. 否则就会有两个点颜色相同，然而是邻接的。

反过来，如果一个简单图的颜色数是n，那么必然是完全图。

颜色数

- 性质：图G的颜色数不小于任何一个子图的颜色数。

- 求颜色数是一个难题，至今没有一个已知的好算法。

- 定理：假设G是一个简单图，其所有结点的最大度数是D_{max}. 那么G的颜色数≤ $D_{max}+1$

图着色与规划

EG: Suppose we want to schedule some final exams for CS courses with following course numbers:

1007, 3137, 3157, 3203, 3261, 4115, 4118, 4156

Suppose also that there are **no common students** in the following pairs of courses because of prerequisites:

1007-3137

1007-3157, 3137-3157

1007-3203

1007-3261, 3137-3261, 3203-3261

1007-4115, 3137-4115, 3203-4115, 3261-4115

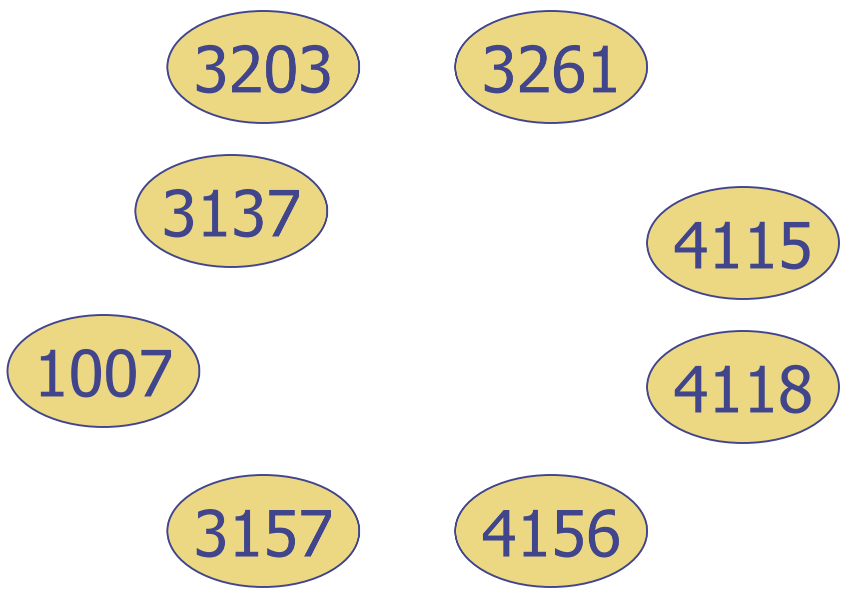
1007-4118, 3137-4118

1007-4156, 3137-4156, 3157-4156

How many exam slots are necessary to schedule exams?

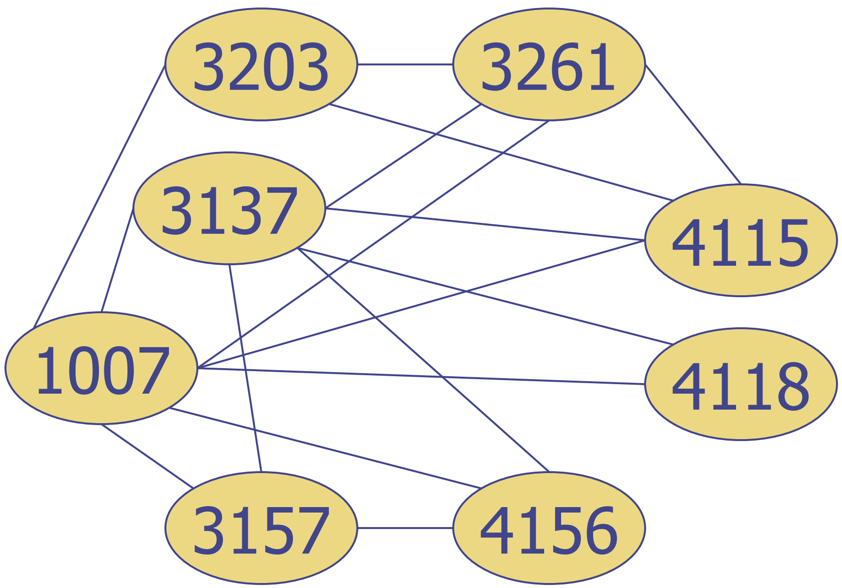
Graph Coloring and Schedules

Turn this into a graph coloring problem. Vertices are courses, and edges are courses which *cannot* be scheduled simultaneously because of possible students in common:



Graph Coloring and Schedules

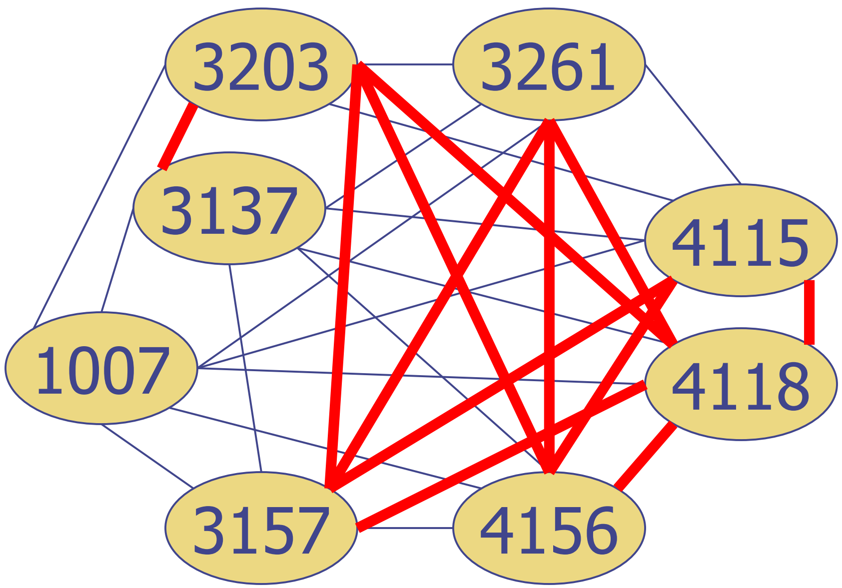
One way to do this is to put edges down where students mutually excluded (有冲突的）...



这个图中的边代表没有共同的学生，是可以安排在同一个时间段考试的

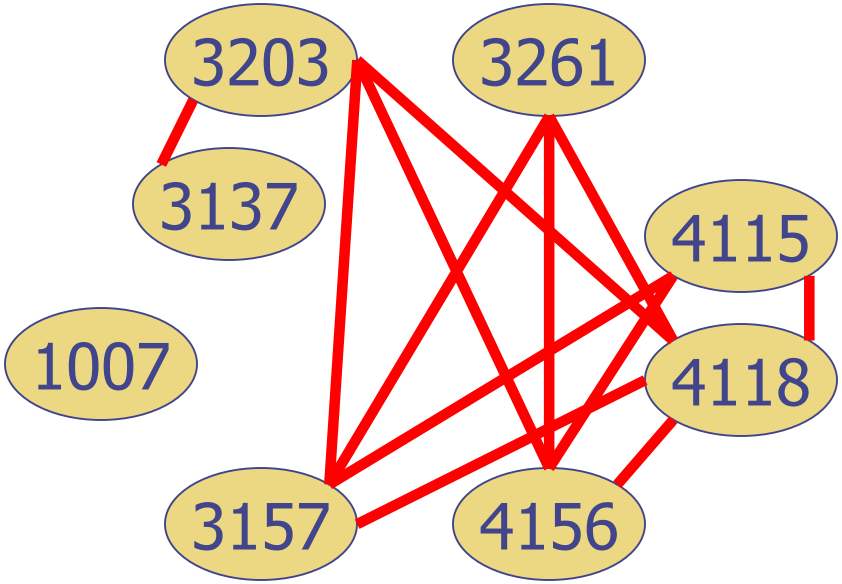
Graph Coloring and Schedules

…and then compute the complementary graph（补图）



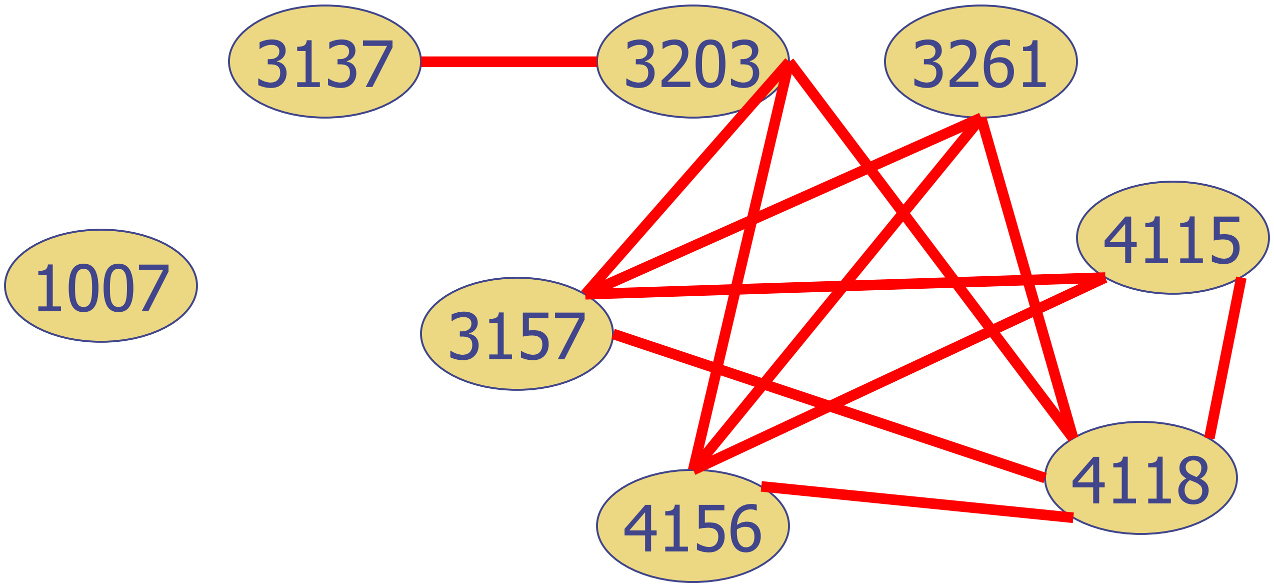
Graph Coloring and Schedules

…and then compute the complementary graph（补图）:



Graph Coloring and Schedules

Redraw:



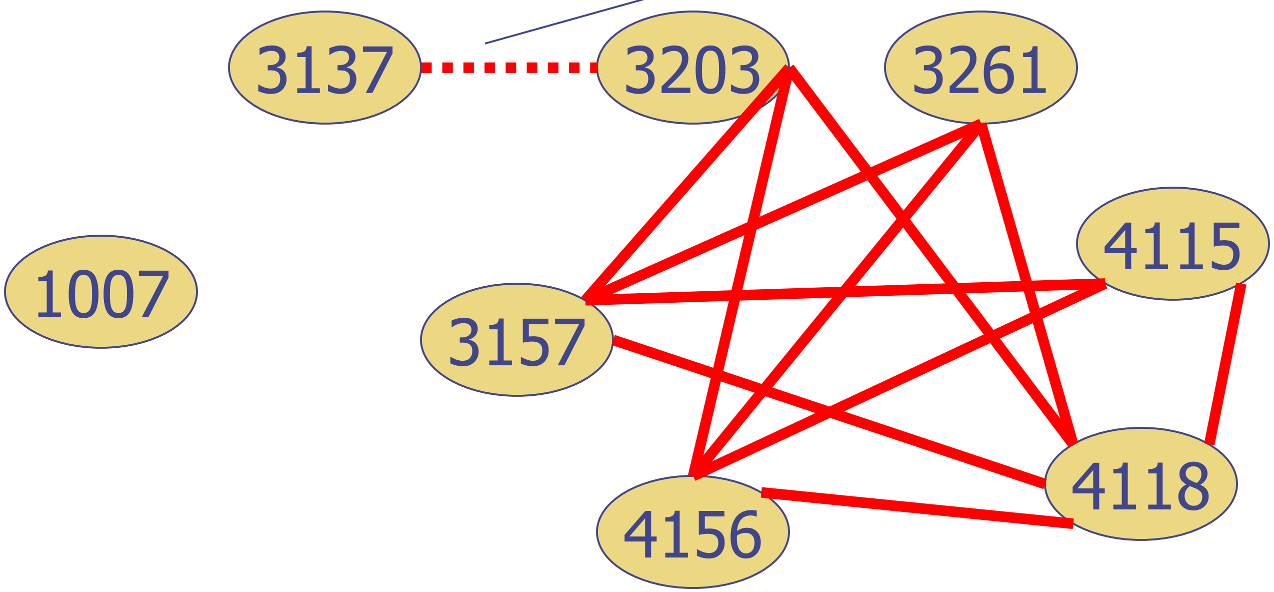
这个补图中的邻接的点对应的课程是不能同时间考试的，是有冲突的。

思考

- 能否用图着色的方法，寻找颜色数的方法来解决这个问题？

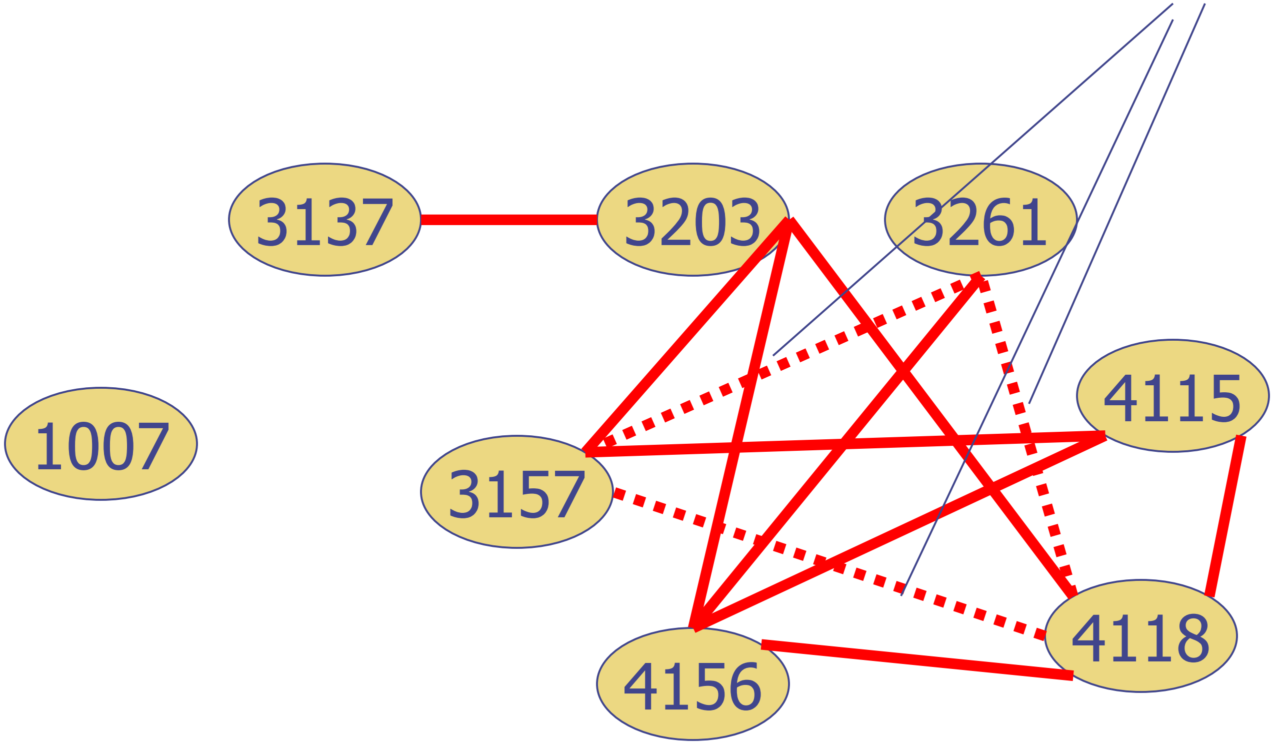
Graph Coloring and Schedules

Not 1-colorable because of edge



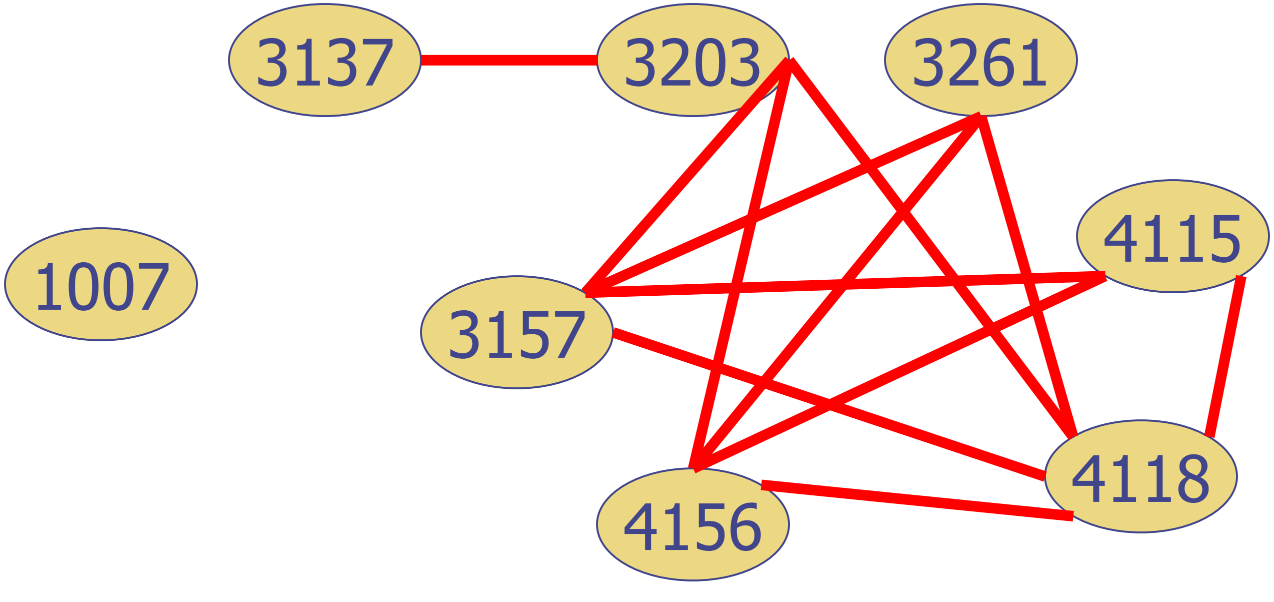
Graph Coloring and Schedules

Not 2-colorable because of <u>triangle</u>



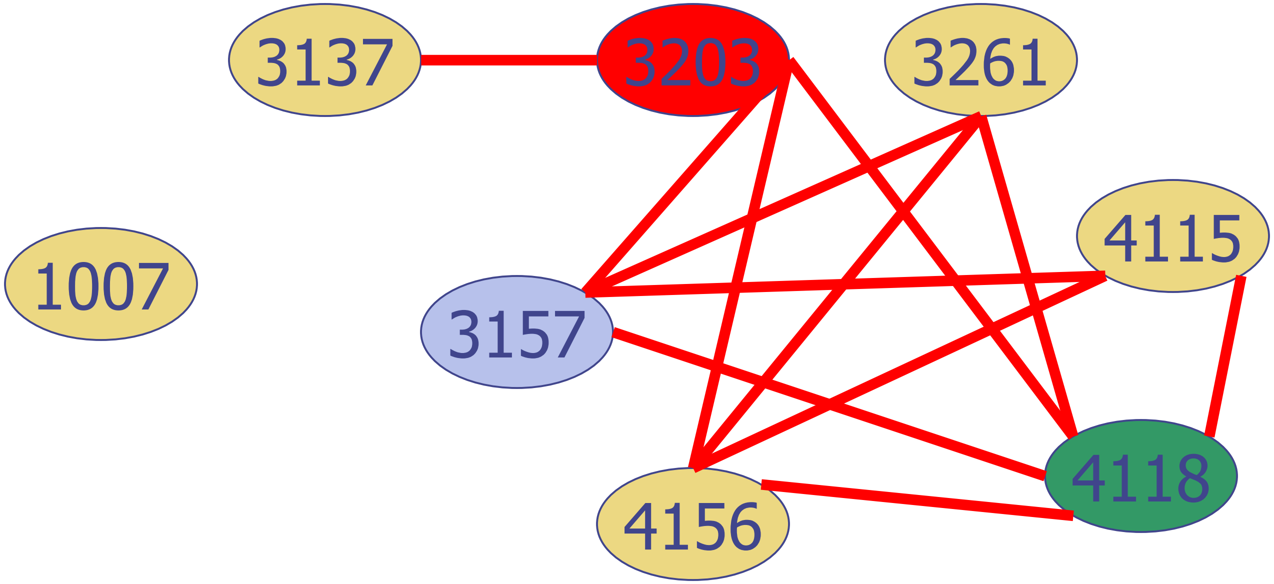
Graph Coloring and Schedules

Is 3-colorable. Try to color by Red, Green, Blue.



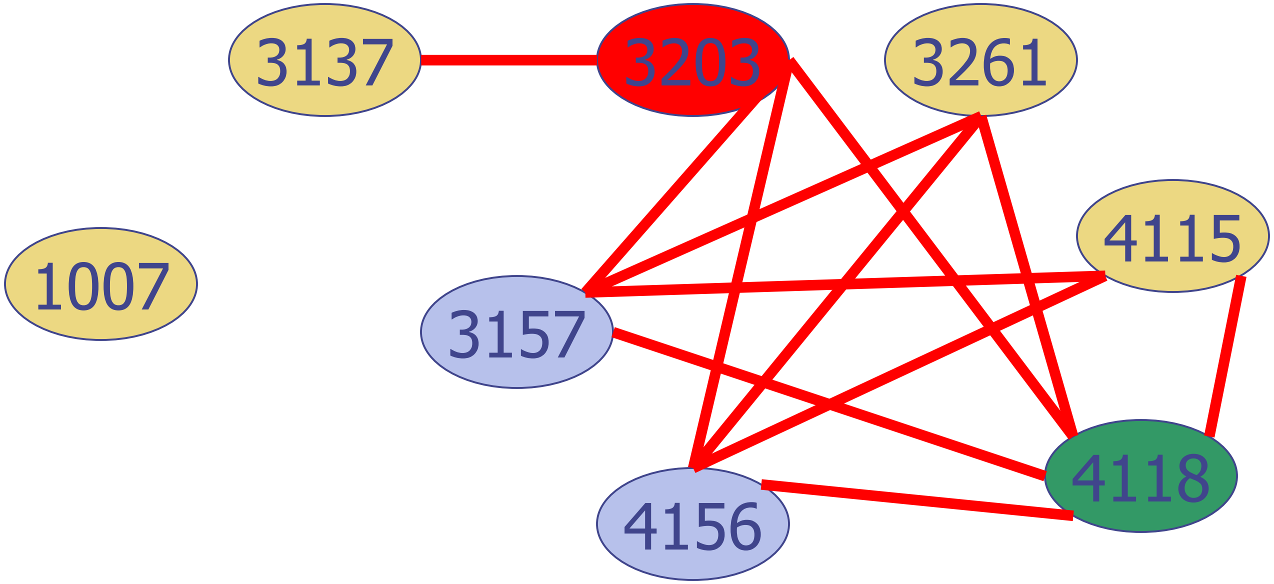
Graph Coloring and Schedules

假设：3203-Red, 3157-Blue, 4118-Green:



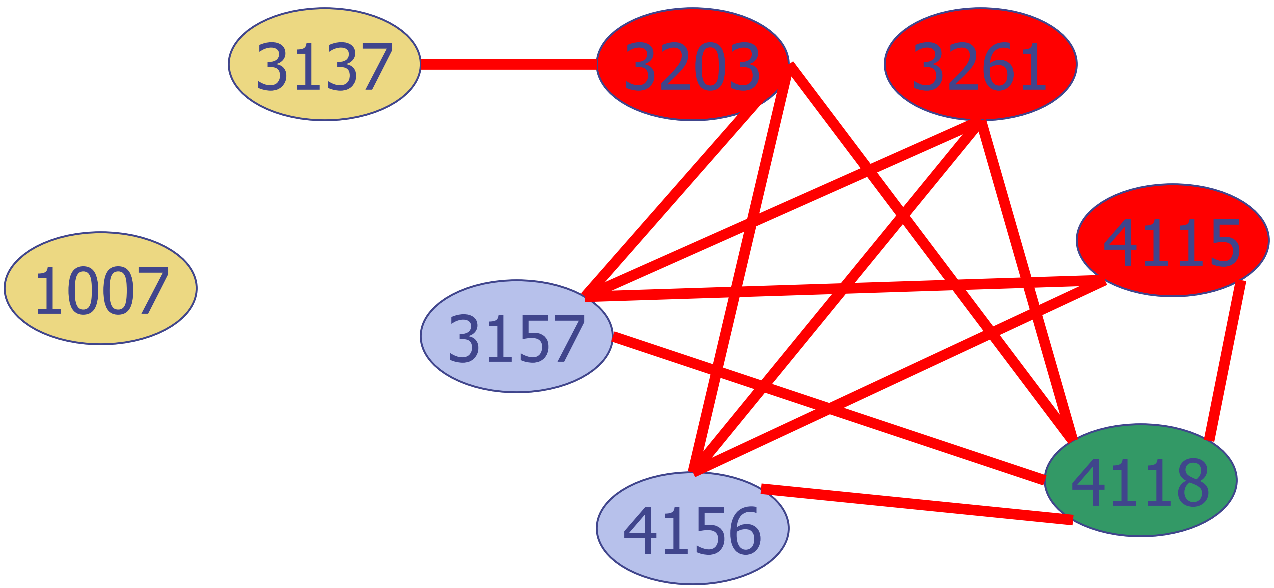
Graph Coloring and Schedules

So 4156 must be Blue:



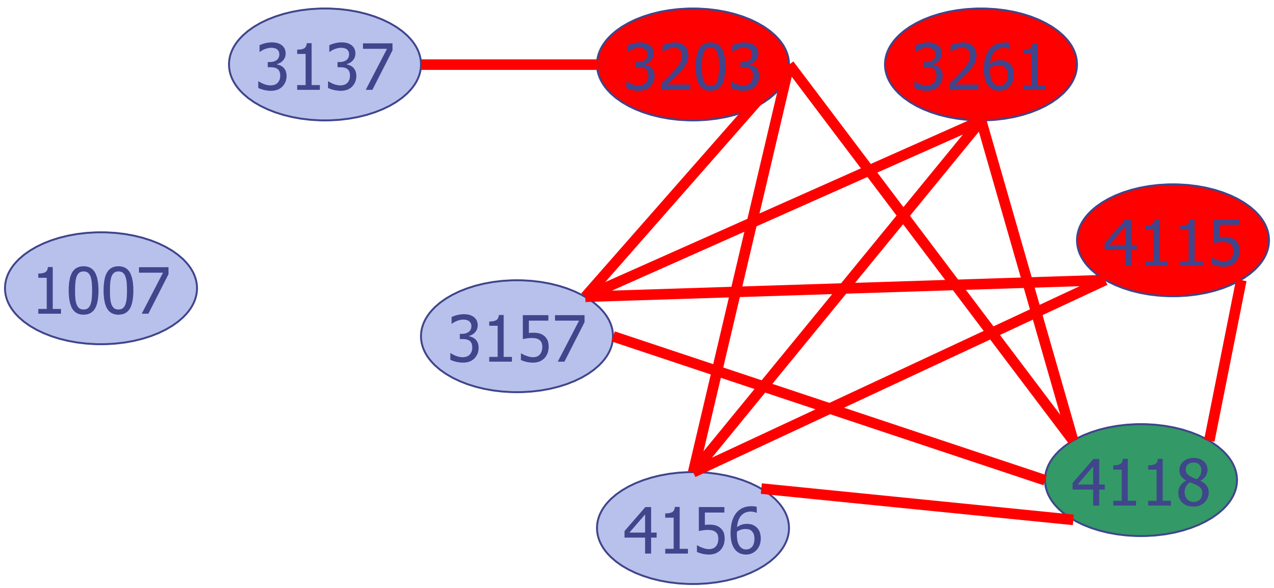
Graph Coloring and Schedules

So 3261 and 4115 must be Red.



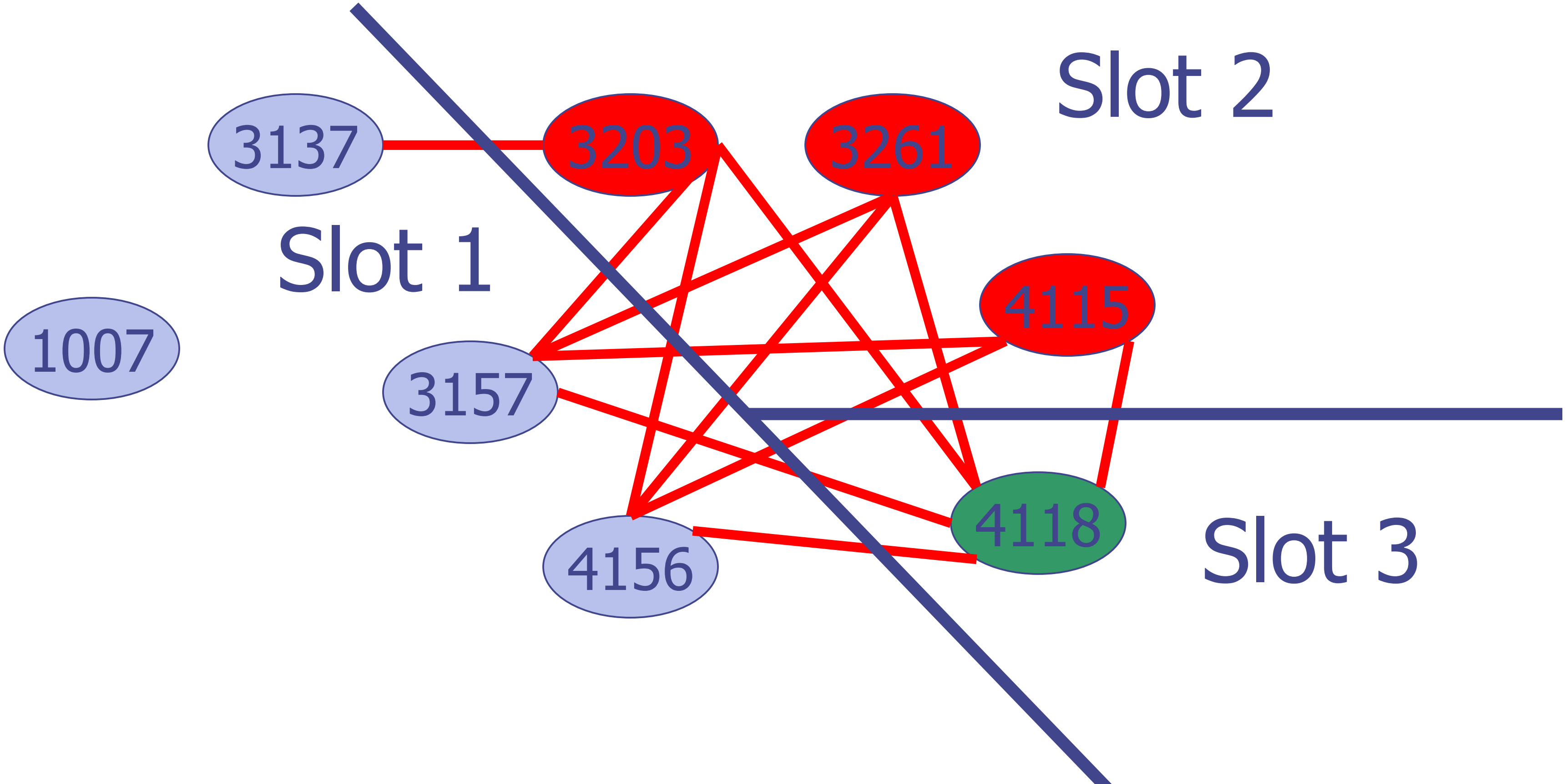
Graph Coloring and Schedules

3137 and 1007 easy to color.



Graph Coloring and Schedules

So need 3 exam slots:



图着色应用

- 例题2： 无线广播电台频率管制问题。

某些距离太近的点不能有相同的频率，要避免频率干扰，就需要合理规划频率。

Coloring Exercises

- 6.8节 T2， T10

习题选讲

- 习题选讲：
- 6.5节 T24
- 补充练习 T8 （思考简单图结论又如何？）