Inclusion-Exclusion

容斥原理

Principle of Inclusion-Exclusion
容斥原理

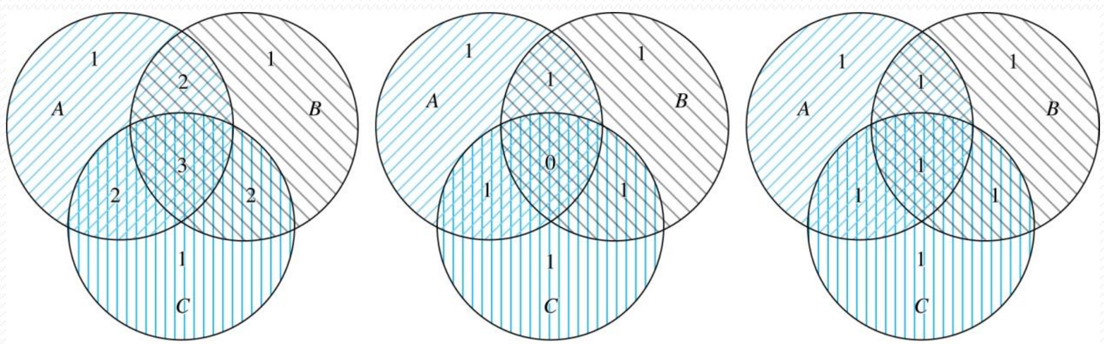
- In set theory, the basic principle of Inclusion-Exlusion is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Three Finite Sets

涉及到3个有限集合的容斥原理

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Three Finite Sets Continued

- Example: A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both
- French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Solution: Let S 表示选Spanish的集合，*F选*French的集合, and *R为选* Russian的， 那么有：

$|S| = 1232$, $|F| = 879$, $|R| = 114$, $|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$, and $|S \cup F \cup R| = 23$.

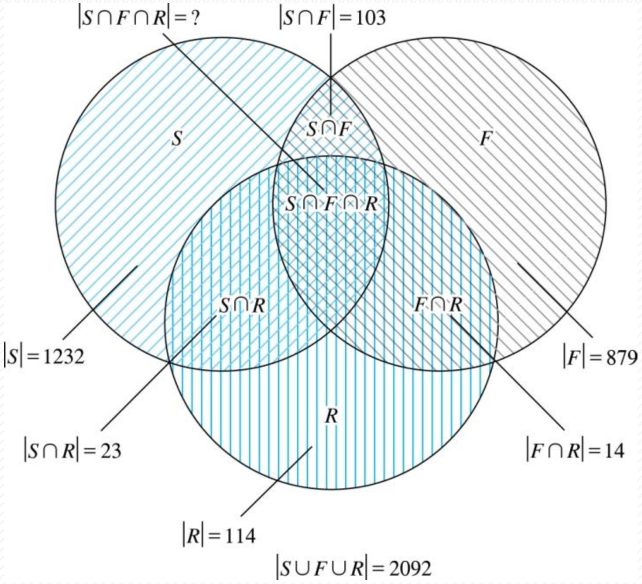
利用容斥原理有：

$$|S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| - |S \cap R| - |F \cap R| + |S \cap F \cap R|,$$

也即$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |S \cap F \cap R|$.

可以解出来 $|S \cap F \cap R| = 7$

Illustration of Three Finite Set Example



The Principle of Inclusion-Exclusion

一般化的容斥原理

Theorem 1. The Principle of Inclusion-Exclusion: Let $A_1, A_2, \ldots, A_n$ be finite sets. Then:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| +$$

$$\sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

建议同学们自己看看教材上的采用组合计数的证明方法，课堂内不讲证明过程。

容斥原理应用

The Number of Onto Functions 满射的数量

Example: How many onto functions are there from a set with 6 elements to a set with 3 elements?

Solution: Suppose that the co-domain is $\{b_1, b_2, b_3\}$. Let P_1, P_2, and P_3 be the properties that b_1, b_2, and b_3 are **not** in the range of the function(值域), respectively. The function is onto if none of the properties P_1, P_2, and P_3 hold.

Let $N(P_i)$ denotes the number of functions having property p_i,

$N(P'_1P'_2\,P'_3)$ denotes the number of onto functions $(\neg p_1 \wedge \neg p_2 \wedge \neg p_3)$

Assume A is the set with all functions which have property p_1,

B is the set with p_2, C is the set with p_3,

By the inclusion-exclusion principle:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

思考：上面这些集合的基数分别代表什么？

The Number of Onto Functions 满射的数量

$|A \cup B \cup C| + N(P'_1 P'_2\ P'_3)$ = the total number of functions N

So, the number of onto functions from a set with six elements to a set with three elements is (总数 减去 至少满足一个P_i的)

$N(P'_1 P'_2\ P'_3) = N - [N(P_1) + N(P_2) + N(P_3)] + [N(P_1P_2) + N(P_1P_3) + N(P_2P_3)] - N(P_1P_2P_3)$

Where N is the total number of functions from a set with six elements to $\{b_1, b_2, b_3\}$, and $N = 3^6$.

- The number of functions that do not have in the range is $N(P_1) = 2^6$. Similarly, $N(P_2) = N(3_1) = 2^6$.
- Note that $N(P_1P_2) = N(P_1P_3) = N(P_2P_3) = 1$ and $N(P_1P_2P_3) = 0$.

Hence, the number of onto functions from a set with six elements to a set with three elements is:

$$3^6 - 3 \cdot 2^6 + 3 = 729 - 192 + 3 = 540$$

The Number of Onto Functions (continued)

Theorem 1: Let m and n be positive integers with $m \geq n$. Then there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1) \cdot 1^m$$

onto functions from a set with *m* elements to a set with *n* elements.

Proof: follows from the principle of inclusion-exclusion (*see Exercise* 27).

An Alternative Form of Inclusion–Exclusion
容斥原理的另一种形式

- Let A_i be the subset containing the elements that have property P_i . The number of elements with all the properties $P_{i1}, P_{i2}, \ldots, P_{ik}$ will be denoted by $N(P_{i1}P_{i2} \ldots P_{ik})$.
- Writing these quantities in terms of sets, we have

$$|A_{i1} \cap A_{i2} \cap \cdots \cap A_{ik}| = N(P_{i1}P_{i2} \ldots P_{ik}).$$

- If the number of elements with none of the properties $P_1, P_2, \ldots, P_n$ is denoted by $N(P'_1P'_2 \ldots P'_n)$ and the number of elements in the set is denoted by N,

  it follows that $\boldsymbol{N(P'_1P'_2 \ldots P'_n) = N - |A_1 \cup A_2 \cup \cdots \cup A_n|}$.
- From the inclusion–exclusion principle, we see that

$$N(P'_1P'_2 \ldots P'_n) = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_iP_j) - \sum_{1 \le i < j < k \le n} N(P_iP_jP_k) + \cdots + (-1)^n N(P_1P_2 \ldots P_n).$$

Derangements错位排列

Definition: A *derangement(错位排列)* is a permutation of objects that leaves no object in the original position.

Example: The permutation of 21453 is a derangement of 12345 because no number is left in its original position. But 21543 is not a derangement of 12345, because 4 is in its original position.

Derangements错位排列(continued)

Theorem 2: The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right].$$

Proof (不做要求)： Let a permutation have property ***P_i* if it fixes element *i*.** The number of derangements is the number of permutations having none of the properties P_i for $i = 1, 2, \ldots, n$. This means that ***$Dn = N(P'_1 P'_2 \ldots P'_n)$***

Using the principle of inclusion–exclusion, it follows that:

$$\text{Dn} = N(P_1' P_2' \ldots P_n') = N - \sum_{1 \le i \le n} N(P_i) + \sum_{1 \le i < j \le n} N(P_i P_j) - \sum_{1 \le i < j < k \le n} N(P_i P_j P_k) + \cdots + (-1)^n N(P_1 P_2 \ldots P_n).$$

分别计算后面的每一个项，就可以完成证明。

Derangements 应用

The Hatcheck Problem帽子认领问题: A new employee checks the hats of n people at restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat.

Solution: The answer is the number of ways the hats can be arranged so that there is no hat in its original position divided by $n!$, the number of permutations of n hats.

Remark: It can be shown that the probability of a derangement approaches $1/e$ as n grows without bound.

$$\frac{D_n}{n!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right]$$

TABLE 1 The Probability of a Derangement.

n	2	3	4	5	6	7
$D_n/n!$	0.50000	0.33333	0.37500	0.36667	0.36806	0.36786

练习

- 第7版4.5节 T7

- English Edition 7th, 全版本P565 T16 (也即下面这道题）

题目： A group of *n* students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?