# Paths and Connectivity
# 路与连通性

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# Connectivity (连通性）

无向图中的连通性：

DEF: Let *G* be a pseudograph(undirected 无向). Let *u* and *v* be vertices.

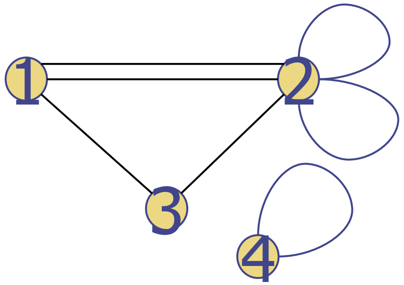
*u* and *v* are ***connected (连接的）*** *to each other* if there is a path in *G* which starts at *u* and ends at *v*.

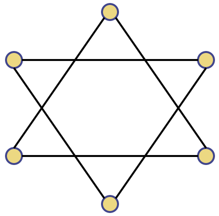
*G* is said to be ***connected*** (连通的) if all vertices are connected to each other *(pair-wise connected)*.

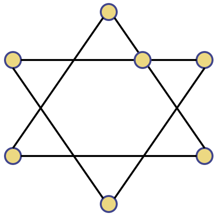
1. Note: Any vertex is automatically connected to itself via the empty path.
2. Note: 后面会有相应的有向图的连通性定义

# Connectivity 连通性

Q: Which of the following graphs are connected?

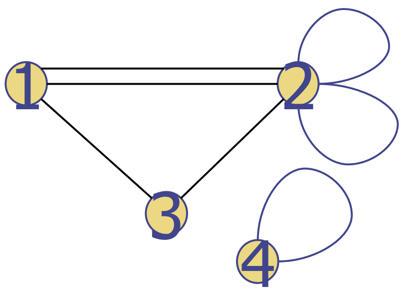


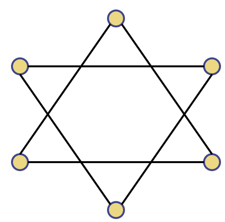



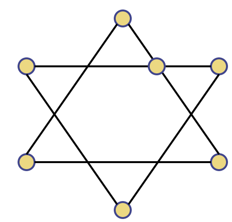


# Connectivity

A: First and second are disconnected. Last is connected.





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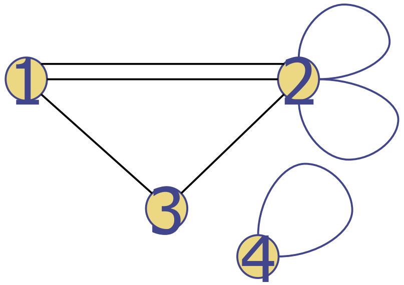


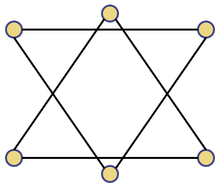

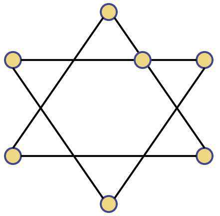

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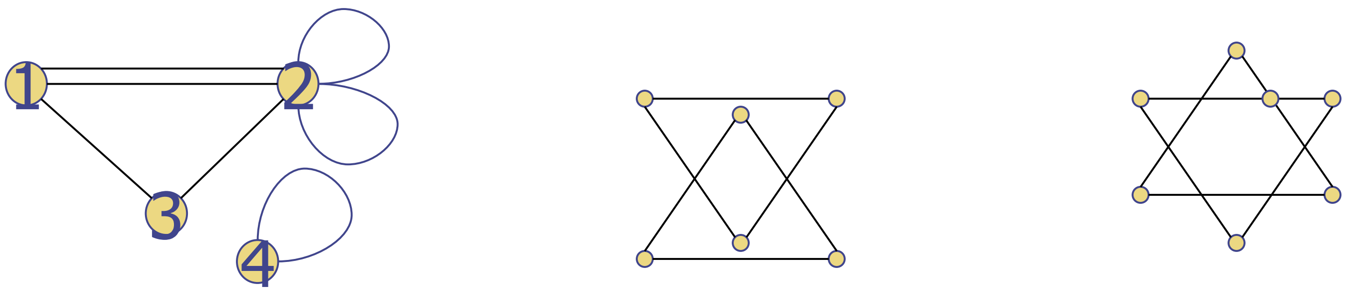

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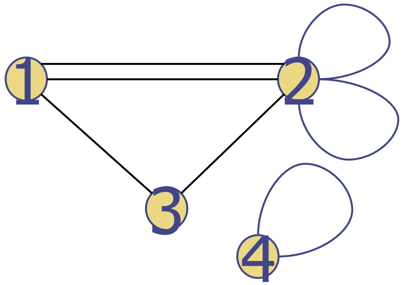


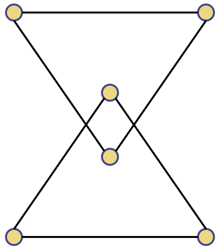

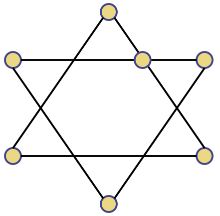

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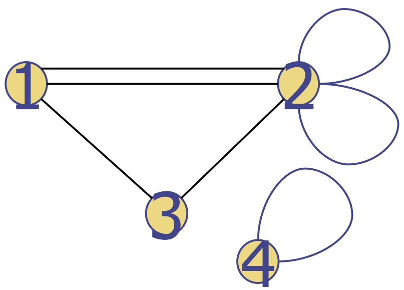


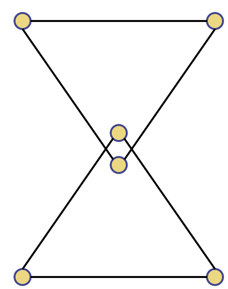

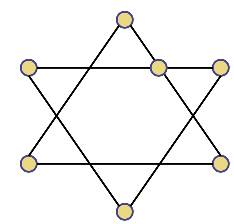

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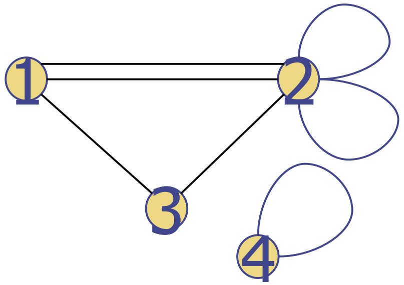
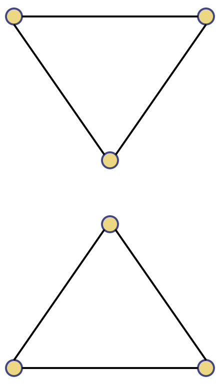
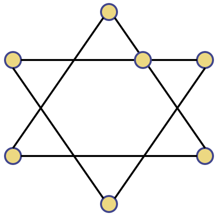




# Questions about connectivity

- Question: in a computer network with n terminals, can the computers send message each other?

- Question: in a transportation network, can a person get another place from any one of the places?

# Connectivity连通性定理

- Theorem: There is a simple path between every pair of distinct vertices of a connected undirected graph.
- 定理：无向连通图中的任何两个不同的结点之间都一定有一条简单路。
- Why?

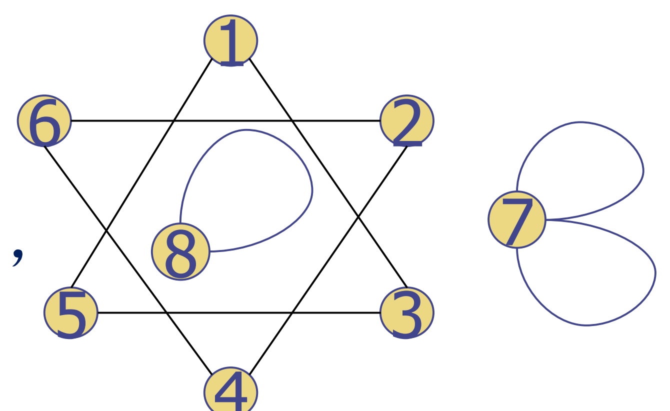
# Connected Components
# 连通分支 or 分图

DEF: 一个连通分支 in a graph *G* is a subgraph of G such that all its vertices in this subgraph are connected to each other and every possible connected vertex is included in this subgraph.

- Or: **the maximally connected subgraphs of G 最大连通子图**

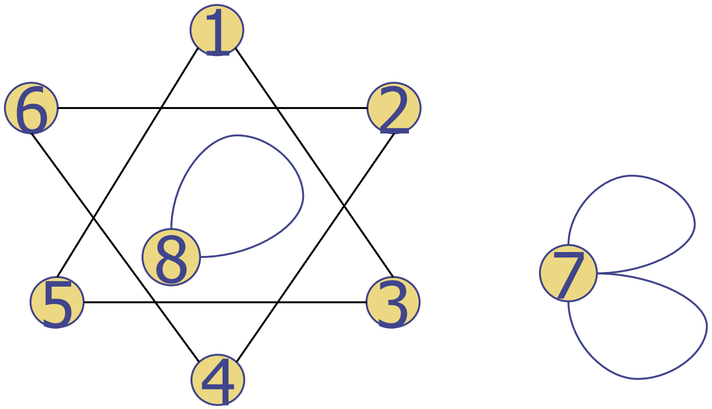
Q: What are the connected components of the following graph?

注：图的每一个结点都必然会在其中的一个分支中。与这个点连接的所有结点以及这些点关联的所有边形成一个子图，该子图就是所在的连通分支。



# Connected Components

A: The components are {1,3,5},{2,4,6},{7} and {8} as one can see visually by pulling components apart:



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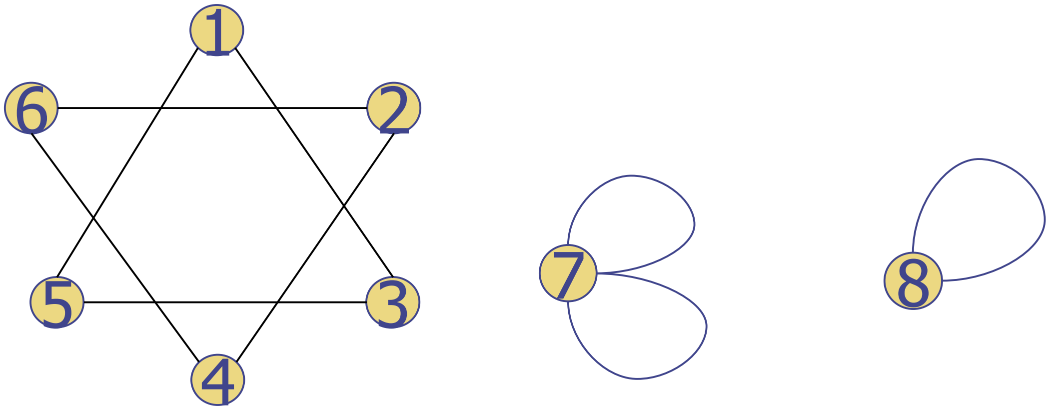

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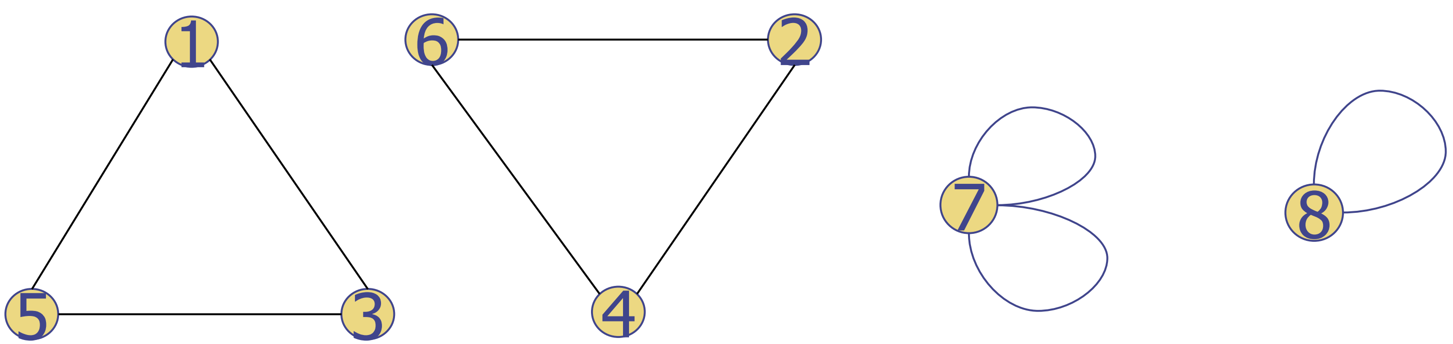


# Question about Connected Components

- Is there any path between two different connected components? Why?

- How many components are there in a connected graph?

# *cut vertex and cut edge* 割边(弦,桥)和割点

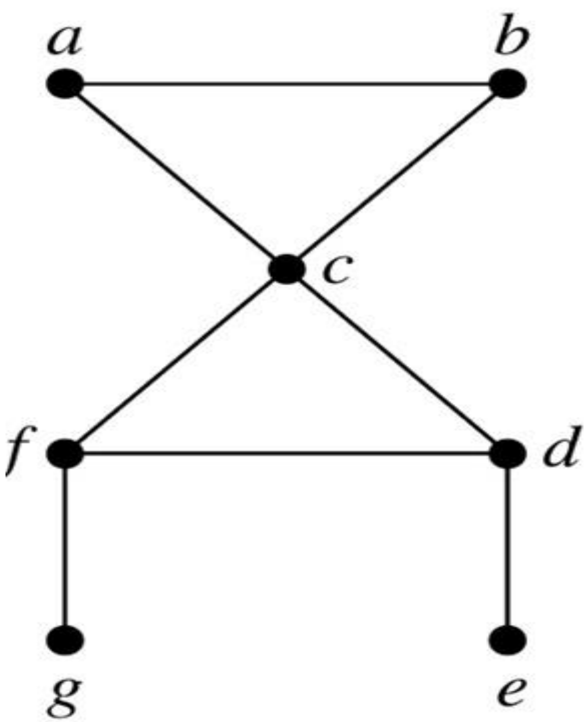
- Def: 去掉该点就会导致图的分支数增加，那么这样的结点称为割点

- Def: 类似，去掉该边能导致分图数增加，这样的边称为割边 (**弦,桥**）

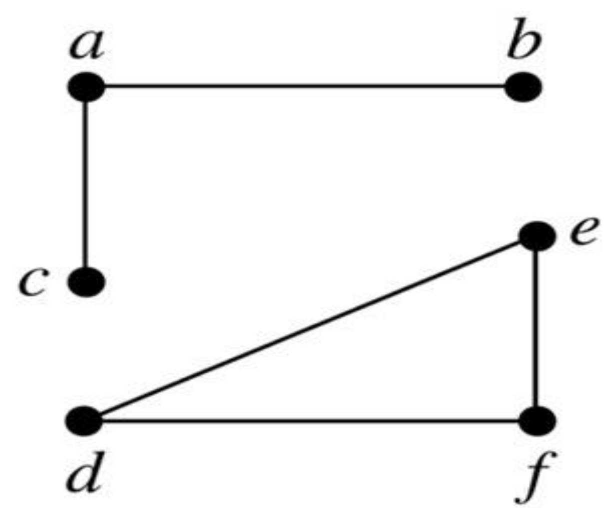
# Example

Please find out the cut vertices and cut edges from the following graphs:






$G_1$

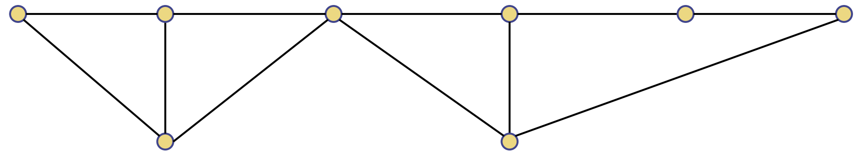



$G_2$

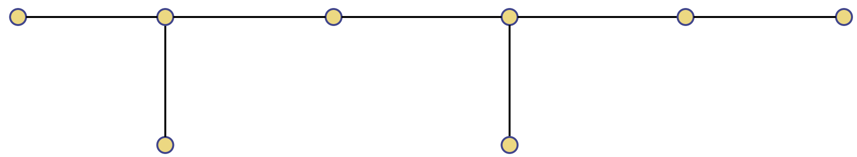
# *N*-Connectivity N-连通

Q: Rate following graphs in terms of their design value for computer networks:

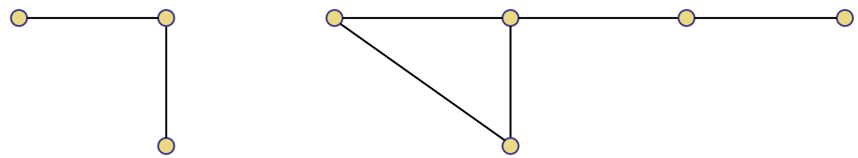
1)



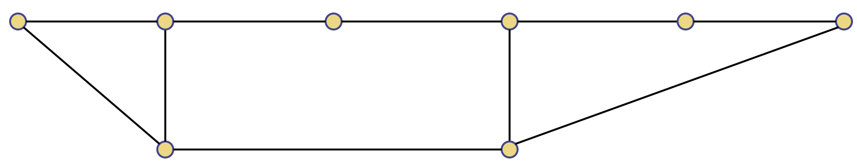
2)



3)



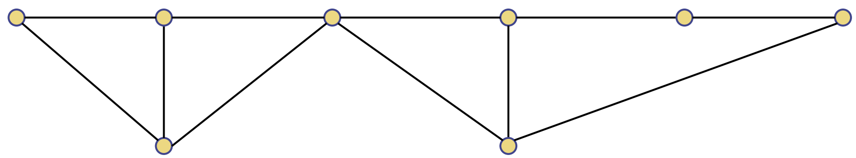
4)



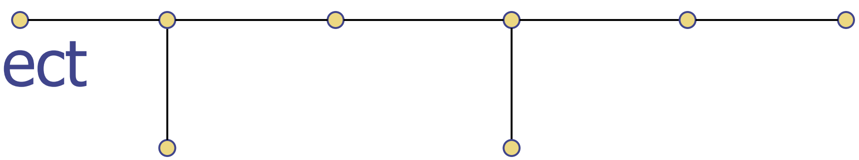
# *N*-Connectivity

A: Want all computers to be connected, even if 1 computer goes down:

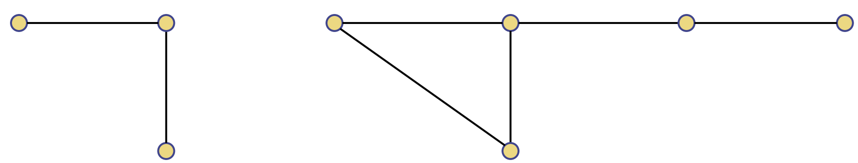
1) 2nd best. However, there's a weak link— "cut vertex"



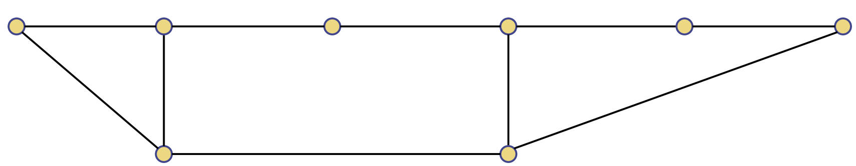
2) 3rd best. Connected but any computer could disconnect



3) Worst! Already disconnected

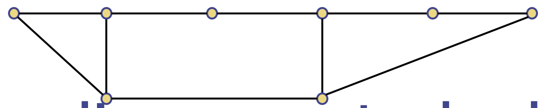


4) Best! Network dies only with 2 bad computers



- Think about the importance to build the redundant (冗余的) network

# *N*-Connectivity N-连通

The network  is best because it can only become disconnected when 2 vertices are removed. In other words, it is 2- connected.

Formally:

DEF: 一个至少3个点的简单连通图中，如果去掉任何一个点以及与之关联的边，仍然还是连通的，但如果去掉两个点就可能不连通了，就称为

2***-connected*。 *或者说至少要去掉两个点才能不连通。***

# *N*-Connectivity 连通度

There is also a notion of *N*-Connectivity. Think about the definition of N-connectivity.

*N*-Connectivity N-连通: a connected graph where we require at least *N* vertices to be removed to disconnect the graph. 这里的N称为图的点连通度

**边连通度**：将一个连通图变成不连通图需要删除的最少的边数

# Connectivity in Directed Graphs
# 有向图的连通性

In directed graphs may be able to find a path from *a* to *b* but not from *b* to *a*. However, connectivity was a symmetric concept for undirected graphs. So how to define directed Connectivity is non-obvious:

1) Should we ignore directions?
2) Should we insist that can get from *a* to *b* in actual digraph?
3) Should we insist that can get from *a* to *b* and that can get from *b* to *a as well*?

# Connectivity in Directed Graphs

分情况定义：

1) ***Weakly connected*** *弱连接*: can get from *a* to *b* in underlying undirected graph

2) ***Strongly connected*** *强连接*: can get from *a* to *b* AND from *b* to *a* in the digraph

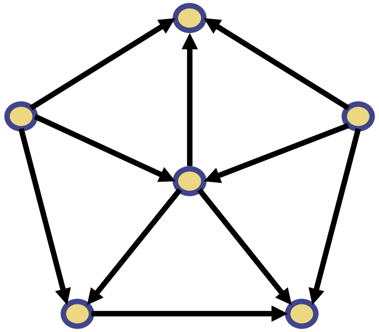
DEF: A graph is ***strongly*** connected if every pair of vertices is strongly connected.

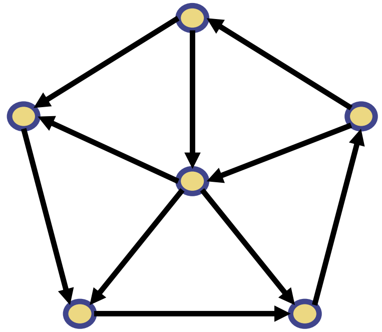
A graph is ***Weakly*** connected if every pair of vertices is weekly connected.

想想交通网络中有些是单行道的情况

# 有向图连通性

Q: Classify the connectivity of each graph.

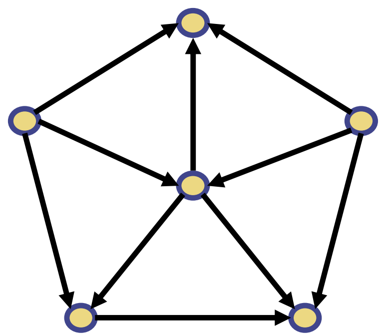




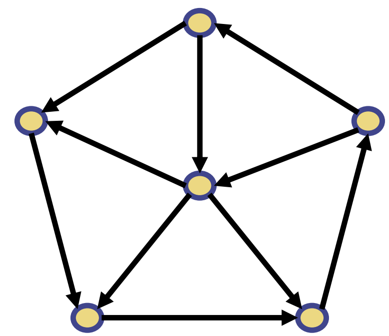
# 有向图连通性

A:

*weak*



*strong*



# Counting paths between vertices
# 结点间的路的数目

对图中任意给定的两个结点，思考如下问题：

Q1: are they connected? (or exists path between them)

Q2: how many paths between them?

Q3: which path is the shortest in a regular/weighed graph? (later on)

# 回忆结论：结点间的路的数目

- **Theorem:** 如果M 是图G的邻接矩阵. 那么$M^r$ 的(i, j) 项就是 从结点i到结点j的长度为r的路的数目. *(Could be proved using math induction)*

- *Note: This is the standard power of M, not the boolean product.*

# 回忆以下问题的解决

- Question: if vertex u and v are connected, is there shortest past between u and v?

- How to get the distance between u and v?

# Is a graph connected?
# 如何判断图的连通性

- 思考: how to determine whether a graph is connected (or not) based on the adjacency matrix?
- 怎样根据邻接矩阵来判断一个无向图是否是连通的？

# 图的连通性判断方法

- 假设M是(n,m)图G的邻接矩阵，分别计算$M^1$,$M^2$, $M^{n-1}$. 然后考察路的情况。

- 进一步的问题：想想，能否想出办法用邻接矩阵判断出一个简单无向图G是否是偶图？

- 介绍连接矩阵的定义...

- 偶图判断定理：一个没有单边环的图为偶图的充要条件是任意的回路都是偶数长
- 充分性证明：

# 课外练习

- 6.4节
- T11, T19(a)  T21(选做）T23(2)