Paper Notes

2023 年 9 月 28 日

目录

1 Optimal error estimates of a linearized backward Euler FEM for the Landau-Lifshitz equation

考虑带交换场的 Landau-Lifshitz 方程如下所示

$$\begin{cases} \dfrac{\partial m}{\partial t} = \gamma m \times \Delta m - \lambda m \times (m \times \Delta m) & x \in \Omega,\ t \in (0, T] \\ m = m_0 & x \in \Omega,\ |m_0| = 1 \\ \dfrac{\partial m}{\partial \vec{n}} = 0 \end{cases}$$

同样, 我们可以考虑 Dirichlet 边界条件 $m = g$, 只需要要求边界条件的函数满足 $|g| = 1$. 为了后续讨论方便, 我们在此仅考虑齐次 Neumann 边界条件. 针对上面的方程, 我们可以发现其显然满足守恒律,给出如下的证明过程.

证明. 在方程两端同时点乘 m 得

$$m \cdot m_t = \gamma m \cdot (m \times \Delta m) - \lambda m \cdot (m \times (m \times \Delta m))$$

由混合积的性质得, $m \cdot (m \times \Delta m) = \Delta m \cdot (m \times m) = 0$, $m \cdot (m \times (m \times \Delta m)) = (m \times \Delta m) \cdot (m \times m)$

因此有 $\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} |m|^2 = m \cdot m_t = 0$, 也就是说 $|m|^2$ 是一个常数. □

为了后续讨论方便, 我们先对 LL 方程做一个变形, 变形过程主要依赖于守恒律以及 $A\times(B\times C)=B\,(A\cdot C)-C\,(A\cdot B)$. 我们分析 $m\times(m\times\Delta m)$ 这一项, 其变形过程如下所示,

$$m\times(m\times\Delta m)=m(m\cdot\Delta m)-\Delta m(m\cdot m)=m(\nabla(m\cdot\nabla m)-|\nabla m|^2)-\Delta m$$

而此时由于 $m\cdot\nabla m=\frac{1}{2}\nabla|m|^2=0$, 所以上式等价于 $m\times(m\times\Delta m)=-|\nabla m|^2 m-\Delta m$

那么 LL 方程就可以改写成如下形式

$$m_t-\gamma m\times\Delta m-\lambda\Delta m=\lambda|\nabla m|^2 m,\ |m|=1$$

本文得创新点在于提出了对于 $\gamma m\times\Delta m$ 的线性化处理技巧, 每一步计算刚度矩阵可以减少计算量

$$\gamma(m\times\Delta m,\phi)=-\gamma(\nabla m\times\nabla m,\phi)-\gamma(m\times\nabla m,\nabla\phi)\approx-\gamma(m_h^j\times\nabla m_h^{j+1},\nabla\phi)$$

现在给出一些记号, 记 $\{t_j\}_{j=0}^J$ 为时间划分, $t_j=j\tau$, $\tau=T/J$ 且 $m^j=m(\cdot,t_j)$, $D_\tau f^{j+1}=\frac{f^{j+1}-f^j}{\tau}$, $j=0,\ldots,J-1$. 同时我们给出一个向量叉乘的计算公式, 若 $f=(f_1,f_2,f_3)$, $g=(g_1,g_2,g_3)$, 则

$$\nabla f\times\nabla g=\begin{pmatrix}\nabla f_2\cdot\nabla g_3-\nabla f_3\cdot\nabla g_2\\ \nabla f_3\cdot\nabla g_1-\nabla f_1\cdot\nabla g_3\\ \nabla f_1\cdot\nabla g_2-\nabla f_2\cdot\nabla g_1\end{pmatrix}$$

在忽略数值离散条件下,我们得到有限元格式为

$$(m_t,\phi)+\lambda(\nabla m,\nabla\phi)+\gamma(m\times\nabla m,\nabla\phi)=\lambda(|\nabla m|^2 m,\phi),\ \forall\phi\in H^1(\Omega).$$

在上式引入线性化向后 Euler 格式就可以得到

$$(D_\tau m_h^{j+1},\phi)+\lambda(\nabla m_h^{j+1},\nabla\phi)+\gamma(m_h^j\times\nabla m_h^{j+1},\nabla\phi)=\lambda(|\nabla m|^2 m,\phi),\ m_h^0=\Pi_h m^0$$

上式则是相当于下面方程的有限元逼近,

$$D_\tau m_h^{j+1}-\lambda\Delta m^{j+1}-\gamma m^j\times\Delta m^{j+1}-\gamma\nabla m^j\times\nabla m^{j+1}=\lambda\left|\nabla m^j\right|^2 m^j$$

为了引出后续的误差分析, 我们先给出相应的正则性条件,

$\alpha>0, 2D\ case$

$\|m\|_{L^\infty(0,T;W^{2,2+\alpha})}+\|m_t\|_{L^2(0,T;H^2)}+\|m_t\|_{L^\infty(0,T;H^1)}+\|m_{tt}\|_{L^2(0,T;L^2)}\le K,$

$3D\ case$

$\|m\|_{L^\infty(0,T;W^{2,4})}+\|m_t\|_{L^2(0,T;H^2)}+\|m_t\|_{L^\infty(0,T;H^1)}+\|m_{tt}\|_{L^2(0,T;L^2)}\le K,$

为了后续讨论简单, 我们仅考虑计算区域是三维的情况.下面我们给出收敛性定理,

Thm 1.1. *Let $T > 0$ be a given constant and suppose that the LL equation has a unique solution $m : (0, T) \times \Omega \to \mathbb{R}^3$ satisfying the regularity conditions. Then the finite element system admits a unique solution m_h^{j+1}. If a quasi-uniform partition with mesh size h and a uniform time step τ are used, then there exist two positive constants τ_0 and h_0 such that when $\tau \le \tau_0$ and $h \le h_0$,*

$$\max_{0 \le j \le J} \left\| m_h^j - m^j \right\|_{L^2} \le C_0(\tau + h^2)$$

and

$$\max_{0 \le j \le J} \left\| m_h^j - m^j \right\|_{H^1} \le C_0(\tau + h)$$

where C_0 is a positive constant which only depends on physical parameters T, Ω, m_0, γ and λ.

有限元解不一定满足守恒律, 但是我们可以给出其与守恒之间的误差关系.

Cor 1.1. *Under the conditon of the Theorem, the finite element solution $\{m_h^j\}_{j=0}^J$ satisfies*

$$\max_{0 \le j \le J} \left\| 1 - \left| m_h^j \right|^2 \right\|_{L^2} \le \hat{C}_0(\tau + h^2)$$

where C_0 is a positive constant which only depends on physical parameters.

证明. 根据前面的分析我们知道真解是满足守恒律的, 也就是说 $\left| m^j \right|^2 = 1$ 成立, 则

$$\begin{aligned}
\left\| 1 - \left| m_h^j \right|^2 \right\| &= \left\| \left| m^j \right|^2 - \left| m_h^j \right|^2 \right\| \\
&= \left\| (m^j + m_h^j) \cdot (m^j - m_h^j) \right\| \\
&\le \left\| m^j + m_h^j \right\|_{L^\infty} \left\| m^j - m_h^j \right\|_{L^2} \\
&\le C \left\| m^j - m_h^j \right\|_{L^2} \le C(\tau + h^2)
\end{aligned}$$

□

本文后续将会采用时空分裂的技巧证明. 因此我们需要考虑时间半离散格式, 假设 M^{j+1} 是时间半离散格式的解, 那时间半离散格式可以写成

$$D_\tau M^{j+1} - \lambda \Delta M^{j+1} - \gamma M^j \times \Delta M^{j+1} - \gamma \nabla M^j \times \nabla M^{j+1} = \lambda \left| \nabla M^j \right|^2 M^j$$

$$(D_\tau M^{j+1}, \phi) + \lambda(\nabla M^{j+1}, \nabla \phi) + \gamma(M^j \times \nabla M^{j+1}, \nabla \phi) = \lambda(\left| \nabla M^j \right|^2 M^j, \phi)$$

误差就可以分裂成

$$\|m_h^j - m^j\| \le \|e^j\| + \|\theta_h^j\| + \|e_h^j\|$$

其中

$$e^j = M^j - m^j(\text{时间}),\ \theta_h^j = R_h^j M^j - M^j(\text{投影}),\ e_h^j = m_h^j - R_h^j M^j(\text{空间})$$

在后续证明之前, 我们先给出两个重要的引理

Lemma 1.1 (Gagliardo-Nirenberg inequality)**.** *Let u be a function defined on Ω and $\partial^s u$ be any partial derivative of u of order s, then*

$$\|\partial^j u\|_{L^p} \le C\|\partial^m u\|_{L^r}^a \|u\|_{L^q}^{1-a} + C\|u\|_{L^q}$$

for $0 \le j < m$ and $\frac{j}{m} \le a \le 1$ with

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$$

except $1 < r < \infty$ and $m - j - \frac{d}{r}$ is a nonnegative integer, in which case the above estimate holds only for $\frac{j}{m} \le a < 1$

Lemma 1.2 (discrete Gronwall's inequality)**.** *Let τ, B and a_k, b_k, c_k, γ_k for integer $k \ge 0$, be nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \tau\sum_{k=0}^{n}\gamma_k a_k + \tau\sum_{k=0}^{n} c_k + B,\ n \ge 0$$

Suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then

$$a_n + \tau\sum_{k=0}^{n} b_k \le \exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right)\left(\tau\sum_{k=0}^{n} c_k + B\right),\ n \ge 0$$

1.1 Temporal error estimates

Thm 1.2. *Let $T > 0$ be a given constant and suppose that the LL equation has a unique solution $m : (0,T) \times \Omega \to \mathbb{R}^3$ satisfying the regularity conditions. Then the temporal semi-discrete elliptic system with homogeneous Neumann boundary condition admits a unique solution M^{j+1} such that when $\tau \le \tau_1$ for some $\tau_1 > 0$,*

$$\max_{0\le j\le J}\|M^j\|_{W^{2,4}} + \max_{0\le j\le J}\|D_\tau M^j\|_{H^1} + \tau\sum_{j=1}^{J}\|D_\tau M^j\|_{H^2}^2 \le C$$

and

$$\max_{0\le j\le J}\left(\|e^j\|_{H^1}^2 + \tau\sum_{n=0}^{j}\|e^n\|_{H^2}^2\right) \le \frac{C_0^2}{16}\tau^2$$

证明. 我们可以将时间半离散方程写成如下形式

$$M^{j+1}-\tau\lambda\Delta M^{j+1}-\tau\gamma M^{j}\times\Delta M^{j+1}-\tau\gamma\nabla M^{j}\times\nabla M^{j+1}=M^{j}+\tau\lambda\left|\nabla M^{j}\right|^{2}M^{j}$$

为了说明该方程有解, 由于其关于 M^{j+1} 是线性的, 我们可以对其左边与 M^{j+1} 做内积得到

$$\begin{aligned}&(M^{j+1}-\tau\lambda\Delta M^{j+1}-\tau\gamma M^{j}\times\Delta M^{j+1}-\tau\gamma\nabla M^{j}\times\nabla M^{j+1},M^{j+1})\\&=(M^{j+1},M^{j+1})+\tau\lambda(\nabla M^{j+1},\nabla M^{j+1})\\&\geq\min(1,\tau\lambda)\left\|M^{j+1}\right\|_{H^{1}}\end{aligned}$$

其中用到了

$$\begin{aligned}&(M^{j}\times\Delta M^{j+1},M^{j+1})+(\nabla M^{j}\times\nabla M^{j+1},M^{j+1})\\&=(\nabla(M^{j}\times\nabla M^{j+1}),M^{j+1})\\&=-(M^{j}\times\nabla M^{j+1},\nabla M^{j+1})=0\end{aligned}$$

因此根据 Lax-Milgram 定理得知其存在唯一解.

下面我们利用数学归纳法来给出相应的误差证明, 由于 $e^{0}=0$, 因此误差估计在 $j=0$ 时显然成立, 因此我们假设误差估计在 $0\leq j\leq k-1$ 时成立, 下面我们只需要证明其在 $0\leq j\leq k$ 时成立.

为此我们利用 LL 方程与时间半离散格式作差即可得到误差方程,其需要涉及的变形过程如下,

$$\begin{aligned}&M^{j}\times\Delta M^{j+1}+\nabla M^{j}\times\nabla M^{j+1}-m^{j+1}\times\Delta m^{j+1}-\nabla m^{j+1}\times\nabla m^{j+1}\\&=M^{j}\times\Delta M^{j+1}+\nabla M^{j}\times\nabla M^{j+1}-m^{j}\times\Delta m^{j+1}-\nabla m^{j}\times\nabla m^{j+1}\\&-(m^{j+1}-m^{j})\times\Delta m^{j+1}-\nabla(m^{j+1}-m^{j})\times\nabla m^{j+1}\\&=\nabla(M^{j}\times\nabla M^{j+1})-\nabla(m^{j}\times\nabla m^{j+1})=\nabla(M^{j}\times\nabla M^{j+1}-m^{j}\times\nabla m^{j+1})\\&=\nabla\left(M^{j}\times\nabla M^{j+1}-M^{j}\times\nabla m^{j+1}+M^{j}\times\nabla m^{j+1}-m^{j}\times\nabla m^{j+1}\right)\\&=\nabla M^{j}\times\nabla e^{j+1}+M^{j}\times\Delta e^{j+1}+\nabla e^{j}\times\nabla m^{j+1}+e^{j}\times\Delta m^{j+1}\end{aligned}$$

上式过程, 我们其实只推了其中几项的化简, 未讨论的部分我们将会将其归入截断误差, 因此无需考虑. 同样, 我们可以通过增减项来得到

$$\left|\nabla M^{j}\right|^{2}M^{j}-\left|\nabla m^{j+1}\right|^{2}m^{j+1}=\left|\nabla M^{j}\right|^{2}M^{j}-\left|\nabla m^{j}\right|^{2}m^{j}+\left|\nabla m^{j}\right|^{2}m^{j}-\left|\nabla m^{j+1}\right|^{2}m^{j+1}$$

误差方程具体形式如下

$$\begin{aligned}D_{\tau}e^{j+1}-\lambda\Delta e^{j+1}=&\gamma M^{j}\times\Delta e^{j+1}+\gamma e^{j}\times\Delta m^{j+1}+\gamma\nabla M^{j}\times\nabla e^{j+1}\\&+\gamma\nabla e^{j}\times\nabla m^{j+1}-\lambda(\left|\nabla m^{j}\right|^{2}m^{j}-\left|\nabla M^{j}\right|^{2}M^{j})-R_{tr}^{j+1}\end{aligned}$$

其中截断误差为

$$
\begin{aligned}
R_{tr}^{j+1} = & D_\tau m^{j+1} - \frac{\partial m(\cdot, t_{j+1})}{\partial t} + \gamma(m^{j+1} - m^j) \times \Delta m^{j+1} \\
& + \gamma \nabla (m^{j+1} - m^j) \times \nabla m^{j+1} + \lambda(\left|\nabla m^{j+1}\right|^2 m^{j+1} - \left|\nabla m^j\right|^2 m^j)
\end{aligned}
$$

利用前文我们给出的正则性条件, 我们可以得到如下结果

$$
\tau \sum_{j=0}^{J-1} \left\|R_{tr}^{j+1}\right\|_{L^2}^2 \leq C\tau^2
$$

由于

$$
(D_\tau e^{j+1}, e^{j+1}) = \frac{1}{\tau}\left[\left\|e^{j+1}\right\|^2 - (e^j, e^{j+1})\right] \geq \frac{1}{2} D_\tau \left\|e^{j+1}\right\|^2
$$

和

$$
\gamma(M^j \times \Delta e^{j+1}, e^{j+1}) + \gamma(\nabla M^j \times \nabla e^{j+1}, e^{j+1}) = \gamma(\nabla(M^j \times \nabla e^{j+1}), e^{j+1}) = 0
$$

, 因此我们对方程两边分别与 e^{j+1}, $-\Delta e^{j+1}$ 做内积得到,

$$
\begin{aligned}
& \frac{1}{2} D_\tau \left\|e^{j+1}\right\|^2 + \lambda \left\|\nabla e^{j+1}\right\|^2 \\
& \leq \gamma(e^j \times \Delta m^j, e^{j+1}) + \gamma(\nabla e^j \times \nabla m^{j+1}, e^{j+1}) \\
& + \lambda(\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j, e^{j+1}) - (R_{tr}^{j+1}, e^{j+1}) \\
& := \sum_{n=1}^{4} I^n(e^{j+1})
\end{aligned}
$$

同理, 我们可以得到

$$
\begin{aligned}
& \frac{1}{2} D_\tau \left(\left\|\nabla e^{j+1}\right\|^2\right) + \lambda \left\|\Delta e^{j+1}\right\|^2 \\
& \leq -\gamma(e^j \times \Delta m^{j+1}, \Delta e^{j+1}) - \gamma(\nabla M^j \times \nabla e^{j+1}, \Delta e^{j+1}) \\
& - \gamma(\nabla e^j \times \nabla m^{j+1}, \Delta e^{j+1}) + (R_{tr}^{j+1}, \Delta e^{j+1}) \\
& - \lambda(\left|\nabla m^j\right|^j m^j - \left|\nabla M^j\right|^2 M^j, \Delta e^{j+1}) := \sum_{n=5}^{9} I^n(e^{j+1})
\end{aligned}
$$

接下来我们要对 $I^n(e^{j+1})$, $n = 1, \ldots, 9$ 一项项进行分析. 我们先罗列一下一些较为显然的估计如下,

$$
\begin{aligned}
\left|I^1\right| & \leq C \left\|e^{j+1} \times \Delta m^{j+1}\right\|_{L^2} \cdot \left\|e^j\right\|_{L^2} \\
& \leq C \left\|e^{j+1}\right\|_{L^6} \left\|\Delta m^{j+1}\right\|_{L^3} \left\|e^j\right\|_{L^2} \\
& \leq \epsilon \left\|e^{j+1}\right\|_{H^1}^2 + \epsilon^{-1} C \left\|e^j\right\|_{L^2}^2 \\
\left|I^2\right| & \leq \epsilon \left\|e^j\right\|_{H^1}^2 + \epsilon C \left\|e^{j+1}\right\|_{L^2}^2 \\
\left|I^4\right| & \leq C \left\|e^{j+1}\right\|_{L^2}^2 + C \left\|R_{tr}^{j+1}\right\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
\left|I^5\right| &\le \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|e^j\right\|_{H^1}^2 \\
\left|I^7\right| &\le \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2 \\
\left|I^9\right| &\le \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|R_{tr}^{j+1}\right\|_{L^2}^2
\end{aligned}
$$

分析 $|I^3|$ 的估计,

$$
\begin{aligned}
\left|I^3\right| &= \left|\left(\lambda(\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j), e^{j+1}\right)\right| \\
&\le C \left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \left\|e^{j+1}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \\
&= \left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla m^j\right|^2 M^j + \left|\nabla m^j\right|^2 M^j - \left|\nabla M^j\right|^2 M^j\right\| \\
&= \left\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j + \nabla M^j)\cdot \nabla e^j M^j\right\| \\
&= \left\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) M^j\right\| \\
&= \Big\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) M^j - (\nabla M^j \cdot \nabla e^j) m^j + (\nabla M^j \cdot \nabla e^j) m^j \\
&- (\nabla m^j \cdot \nabla e^j) M^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla m^j \cdot \nabla e^j) M^j - (\nabla m^j \cdot \nabla e^j) M^j\Big\| \\
&= \left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) m^j - (\nabla m^j \cdot \nabla e^j) m^j\right\| \\
&= \left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) e^j + (\nabla M^j \cdot \nabla e^j) m^j + (\nabla m^j \cdot \nabla e^j) m^j\right\| \\
&= \left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) e^j + \left|\nabla e^j\right|^2 m^j + 2(\nabla m^j \cdot \nabla e^j) m^j\right\|
\end{aligned}
$$

$$
\begin{aligned}
|I^3| &\le C \left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \left\|e^{j+1}\right\| \\
&\le \epsilon^{-1} C \Bigg(\left\|\left|\nabla m^j\right|^2 e^j\right\|^2 + \left\|\left|\nabla e^j\right|^2 e^j\right\|^2 + \left\|2(\nabla m^j \cdot \nabla e^j) e^j\right\| + \left\|\left|\nabla e^j\right|^2 m^j\right\|^2 \\
&+ \left\|2(\nabla m^j \cdot \nabla e^j) m^j\right\|^2\Bigg) + \epsilon \left\|e^{j+1}\right\|^2 \\
&\le \epsilon^{-1} C \Bigg(\left\|e^j\right\|^2 + \left\|e^j\right\|_{L^\infty}^2 \left\|\nabla e^j\right\|_{L^6}^2 \left\|\nabla e^j\right\|_{L^3}^2 + 2\left\|\nabla e^j\right\|_{L^6}^2 \left\|e^j\right\|_{L^6}^2 + \left\|\nabla e^j\right\|_{L^6}^2 \left\|\nabla e^j\right\|_{L^6}^2 \\
&+ \left\|e^j\right\|_{H^1}^2\Bigg) + \epsilon \left\|e^{j+1}\right\|^2
\end{aligned}
$$

我们给出一些不等式对上式进行化简,

$$
\begin{aligned}
&\left\|\nabla e^j\right\|_{L^6} \le \left\|\nabla e^j\right\|_{H^1} \le \left\|e^j\right\|_{H^2} \\
&\left\|e^j\right\|_{L^3} \le C\left(\left\|e^j\right\|_{L^2} \left\|e^j\right\|_{L^6}\right)^{\frac{1}{2}} \le C\left(\left\|e^j\right\|_{L^2} \left\|e^j\right\|_{H^1}\right)^{\frac{1}{2}} \le C\left(\left\|e^j\right\|_{H^2} \left\|e^j\right\|_{H^1}\right)^{\frac{1}{2}} \\
&\left\|\nabla e^j\right\|_{L^3} \le C\left(\left\|\nabla e^j\right\|_{L^2} \left\|\nabla e^j\right\|_{L^6}\right)^{\frac{1}{2}} \le C\left(\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2}\right)^{\frac{1}{2}}
\end{aligned}
$$

因此上式可以改为

$$\begin{aligned}\left|I^3\right| &\le \epsilon\left\|e^{j+1}\right\|^2+\epsilon^{-1}C\left(\left\|e^j\right\|^2+\left\|e^j\right\|_{H^1}^2+\frac{C_0^2}{16}\tau^{\frac{3}{2}}\left\|e^j\right\|_{H^2}^2\right)\\ &\le \epsilon\left\|e^{j+1}\right\|^2+\epsilon^{-1}C\left\|e^j\right\|_{H^1}^2+\epsilon\left\|e^j\right\|_{H^2}^2\end{aligned}$$

上面引入 $\frac{C_0}{16}\tau^{\frac{3}{2}}$ 的原因,

$$\left\|\nabla e^j\right\|_{L^6}^2\left\|\nabla e^j\right\|_{L^3}^2 \le C\left(\left\|e^j\right\|_{H^1}\left\|e^j\right\|_{H^2}\right)\left\|e^j\right\|_{H^2}^2$$

而此处的 j 满足 $j\in[0,k-1]$, 因此有下式成立,

$$\begin{aligned}\left\|e^j\right\|_{H^1}\left\|e^j\right\|_{H^2}\tau^{\frac{1}{2}} &\le \frac{1}{2}(\left\|e^j\right\|_{H^1}^2+\tau\left\|e^j\right\|_{H^2}^2)\\ &\le \frac{1}{2}\max\left(\left\|e^j\right\|_{H^1}^2+\tau\sum_{n=0}^{j}\left\|e^n\right\|_{H^2}^2\right)\\ &\le \frac{C_0^2}{32}\tau^2\end{aligned}$$

故

$$\left\|e^j\right\|_{H^1}\left\|e^j\right\|_{H^2}\tau^{\frac{1}{2}} \le \frac{C_0^2}{16}\tau^{\frac{3}{2}}C$$

所以对 $|I^3|$ 的估计, 其实暗含了对 ϵ 的限制,

$$\epsilon^{-1}C\frac{C_0^2}{16}\tau^{\frac{3}{2}} \le \epsilon \Rightarrow \epsilon \ge \frac{\sqrt{C}C_0}{4}\tau^{\frac{3}{4}}$$

按照同样的证明过程,我们可以得到,

$$\left|I^8\right| \le 2\epsilon\left\|e^j\right\|_{H^2}^2+\epsilon^{-1}C\left\|e^j\right\|_{H^1}^2$$

下面给出 $|I^6|$ 的估计,

$$\begin{aligned}\left|I^6\right| &= \left|C\left(\nabla M^j\times\nabla e^{j+1},\Delta e^{j+1}\right)\right|\\ &= \left|C\left(\nabla e^j\times\nabla e^{j+1},\Delta e^{j+1}\right)+C\left(\nabla m^j\times\nabla e^{j+1},\Delta e^{j+1}\right)\right|\\ &\le C\left|\left(\nabla e^j\times\nabla e^{j+1},\Delta e^{j+1}\right)\right|+C\left|\left(\nabla m^j\times\nabla e^{j+1},\Delta e^{j+1}\right)\right|\\ &\le C\left\|\nabla e^j\right\|_{L^3}\left\|\nabla e^{j+1}\right\|_{L^6}\left\|\Delta e^{j+1}\right\|_{L^2}+C\left\|m^j\right\|_{W^{1,\infty}}\left\|\nabla e^{j+1}\right\|_{L^2}\left\|\Delta e^{j+1}\right\|_{L^2}\\ &\le C\left(\left\|e^j\right\|_{H^1}\left\|e^j\right\|_{H^2}\right)^{\frac{1}{2}}\left\|\Delta e^{j+1}\right\|_{L^2}^2+\epsilon\left\|e^{j+1}\right\|_{H^2}^2+\epsilon^{-1}C\left\|e^{j+1}\right\|_{H^1}^2\\ &\le \frac{CC_0}{4}\tau^{\frac{3}{4}}\left\|\Delta e^{j+1}\right\|_{L^2}^2+\epsilon\left\|e^{j+1}\right\|_{H^2}^2+\epsilon^{-1}C\left\|e^{j+1}\right\|_{H^1}^2\\ &\le 2\epsilon\left\|e^{j+1}\right\|_{H^2}^2+\epsilon^{-1}C\left\|e^{j+1}\right\|_{H^1}^2\end{aligned}$$

同样我们要求 $\epsilon \ge \frac{CC_0}{4}\tau^{\frac{3}{4}}$. 依托上面的逐项分析, 我们将不等式求和可以得到如下形式

$$\begin{aligned}D_\tau\left(\left\|e^{j+1}\right\|_{H^1}^2\right)+\lambda\left\|e^{j+1}\right\|_{H^2}^2 &\le \epsilon\left\|e^j\right\|_{H^2}^2+\epsilon\left\|e^{j+1}\right\|_{H^2}^2+\varepsilon^{-1}C\left\|e^j\right\|_{H^1}^2\\ &\quad+\varepsilon^{-1}C\left\|e^{j+1}\right\|_{H^1}^2+\epsilon^{-1}C\left\|R_{tr}\right\|_{L^2}^2\end{aligned}$$

依据 Grownwall 不等式可以得到

$$\left\|e^{j+1}\right\|_{H^1}^2+\tau\sum_{n=0}^{j+1}\left\|e^n\right\|_{H^2}^2\leq\exp(2TC)\tau^2$$

从误差不等式中我们可以得到

$$\max_{0\leq j\leq J}$$

$\square$