Paper Notes

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目录

1 Optimal error estimates of a linearized backward Euler FEM for the Landau-Lifshitz equation

考虑带交换场的 Landau-Lifshitz 方程如下所示

$$\begin{cases} \dfrac{\partial m}{\partial t} = \gamma m \times \Delta m - \lambda m \times (m \times \Delta m) & x \in \Omega,\ t \in (0, T] \\ \quad m = m_0 & x \in \Omega,\ |m_0| = 1 \\ \dfrac{\partial m}{\partial \vec{n}} = 0 \end{cases}$$

同样, 我们可以考虑 Dirichlet 边界条件 $m = g$, 只需要要求边界条件的函数满足 $|g| = 1$. 为了后续讨论方便, 我们在此仅考虑齐次 Neumann 边界条件. 针对上面的方程, 我们可以发现其显然满足守恒律,给出如下的证明过程.

证明. 在方程两端同时点乘 m 得

$$m \cdot m_t = \gamma m \cdot (m \times \Delta m) - \lambda m \cdot (m \times (m \times \Delta m))$$

由混合积的性质得, $m \cdot (m \times \Delta m) = \Delta m \cdot (m \times m) = 0$, $m \cdot (m \times (m \times \Delta m)) = (m \times \Delta m) \cdot (m \times m)$

因此有 $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|m|^2 = m \cdot m_t = 0$, 也就是说 $|m|^2$ 是一个常数. $\square$

为了后续讨论方便, 我们先对 LL 方程做一个变形, 变形过程主要依赖于守恒律以及 $A\times(B\times C) = B\,(A\cdot C) - C\,(A\cdot B)$. 我们分析 $m\times(m\times\Delta m)$ 这一项, 其变形过程如下所示,

$$m\times(m\times\Delta m) = m(m\cdot\Delta m) - \Delta m(m\cdot m) = m(\nabla(m\cdot\nabla m) - |\nabla m|^2) - \Delta m$$

而此时由于 $m\cdot\nabla m = \frac{1}{2}\nabla|m|^2 = 0$, 所以上式等价于 $m\times(m\times\Delta m) = -|\nabla m|^2 m - \Delta m$

那么 LL 方程就可以改写成如下形式

$$m_t - \gamma m\times\Delta m - \lambda\Delta m = \lambda|\nabla m|^2 m,\ |m| = 1$$

本文得创新点在于提出了对于 $\gamma m\times\Delta m$ 的线性化处理技巧, 每一步计算刚度矩阵可以减少计算量

$$\gamma(m\times\Delta m,\phi) = -\gamma(\nabla m\times\nabla m,\phi) - \gamma(m\times\nabla m,\nabla\phi) \approx -\gamma(m_h^j\times\nabla m_h^{j+1},\nabla\phi)$$

现在给出一些记号, 记 $\{t_j\}_{j=0}^J$ 为时间划分, $t_j = j\tau$, $\tau = T/J$ 且 $m^j = m(\cdot,t_j)$, $D_\tau f^{j+1} = \frac{f^{j+1}-f^j}{\tau}$, $j = 0,\ldots,J-1$. 同时我们给出一个向量叉乘的计算公式, 若 $f = (f_1,f_2,f_3)$, $g = (g_1,g_2,g_3)$, 则

$$\nabla f\times\nabla g = \begin{pmatrix} \nabla f_2\cdot\nabla g_3 - \nabla f_3\cdot\nabla g_2 \\ \nabla f_3\cdot\nabla g_1 - \nabla f_1\cdot\nabla g_3 \\ \nabla f_1\cdot\nabla g_2 - \nabla f_2\cdot\nabla g_1 \end{pmatrix}$$

在忽略数值离散条件下,我们得到有限元格式为

$$(m_t,\phi) + \lambda(\nabla m,\nabla\phi) + \gamma(m\times\nabla m,\nabla\phi) = \lambda(|\nabla m|^2 m,\phi),\ \forall\phi\in H^1(\Omega).$$

在上式引入线性化向后 Euler 格式就可以得到

$$(D_\tau m_h^{j+1},\phi) + \lambda(\nabla m_h^{j+1},\nabla\phi) + \gamma(m_h^j\times\nabla m_h^{j+1},\nabla\phi) = \lambda(|\nabla m|^2 m,\phi),\ m_h^0 = \Pi_h m^0$$

上式则是相当于下面方程的有限元逼近,

$$D_\tau m_h^{j+1} - \lambda\Delta m^{j+1} - \gamma m^j\times\Delta m^{j+1} - \gamma\nabla m^j\times\nabla m^{j+1} = \lambda\left|\nabla m^j\right|^2 m^j$$

为了引出后续的误差分析, 我们先给出相应的正则性条件,

$\alpha > 0, 2D\ case$

$\|m\|_{L^\infty(0,T;W^{2,2+\alpha})} + \|m_t\|_{L^2(0,T;H^2)} + \|m_t\|_{L^\infty(0,T;H^1)} + \|m_{tt}\|_{L^2(0,T;L^2)} \le K,$

$3D\ case$

$\|m\|_{L^\infty(0,T;W^{2,4})} + \|m_t\|_{L^2(0,T;H^2)} + \|m_t\|_{L^\infty(0,T;H^1)} + \|m_{tt}\|_{L^2(0,T;L^2)} \le K,$

为了后续讨论简单, 我们仅考虑计算区域是三维的情况.下面我们给出收敛性定理,

Thm 1.1. *Let $T > 0$ be a given constant and suppose that the LL equation has a unique solution $m : (0, T) \times \Omega \to \mathbb{R}^3$ satisfying the regularity conditions. Then the finite element system admits a unique solution m_h^{j+1}. If a quasi-uniform partition with mesh size h and a uniform time step τ are used, then there exist two positive constants τ_0 and h_0 such that when $\tau \le \tau_0$ and $h \le h_0$,*

$$\max_{0\le j\le J} \left\| m_h^j - m^j \right\|_{L^2} \le C_0(\tau + h^2)$$

and

$$\max_{0\le j\le J} \left\| m_h^j - m^j \right\|_{H^1} \le C_0(\tau + h)$$

where C_0 is a positive constant which only depends on physical parameters T, Ω, m_0, γ and λ.

有限元解不一定满足守恒律, 但是我们可以给出其与守恒之间的误差关系.

Cor 1.1. *Under the conditon of the Theorem, the finite element solution $\{m_h^j\}_{j=0}^J$ satisfies*

$$\max_{0\le j\le J} \left\| 1 - \left| m_h^j \right|^2 \right\|_{L^2} \le \hat{C}_0(\tau + h^2)$$

where C_0 is a positive constant which only depends on physical parameters.

证明. 根据前面的分析我们知道真解是满足守恒律的, 也就是说 $\left|m^j\right|^2 = 1$ 成立, 则

$$\begin{aligned}
\left\| 1 - \left| m_h^j \right|^2 \right\| &= \left\| \left| m^j \right|^2 - \left| m_h^j \right|^2 \right\| \\
&= \left\| (m^j + m_h^j) \cdot (m^j - m_h^j) \right\| \\
&\le \left\| m^j + m_h^j \right\|_{L^\infty} \left\| m^j - m_h^j \right\|_{L^2} \\
&\le C \left\| m^j - m_h^j \right\|_{L^2} \le C(\tau + h^2)
\end{aligned}$$

□

本文后续将会采用时空分裂的技巧证明. 因此我们需要考虑时间半离散格式, 假设 M^{j+1} 是时间半离散格式的解, 那时间半离散格式可以写成

$$D_\tau M^{j+1} - \lambda \Delta M^{j+1} - \gamma M^j \times \Delta M^{j+1} - \gamma \nabla M^j \times \nabla M^{j+1} = \lambda \left| \nabla M^j \right|^2 M^j$$

$$(D_\tau M^{j+1}, \phi) + \lambda(\nabla M^{j+1}, \nabla \phi) + \gamma(M^j \times \nabla M^{j+1}, \nabla \phi) = \lambda(\left| \nabla M^j \right|^2 M^j, \phi)$$

误差就可以分裂成

$$\left\|m_h^j - m^j\right\| \leq \left\|e^j\right\| + \left\|\theta_h^j\right\| + \left\|e_h^j\right\|$$

其中

$$e^j = M^j - m^j(\text{时间}),\ \theta_h^j = R_h^j M^j - M^j(\text{投影}),\ e_h^j = m_h^j - R_h^j M^j(\text{空间})$$

在后续证明之前, 我们先给出两个重要的引理

Lemma 1.1 (Gagliardo-Nirenberg inequality)**.** *Let u be a function defined on Ω and $\partial^s u$ be any partial derivative of u of order s, then*

$$\left\|\partial^j u\right\|_{L^p} \leq C \left\|\partial^m u\right\|_{L^r}^a \left\|u\right\|_{L^q}^{1-a} + C \left\|u\right\|_{L^q}$$

for $0 \leq j < m$ and $\frac{j}{m} \leq a \leq 1$ with

$$\frac{1}{p} = \frac{j}{d} + a\left(\frac{1}{r} - \frac{m}{d}\right) + (1-a)\frac{1}{q}$$

except $1 < r < \infty$ and $m - j - \frac{d}{r}$ is a nonnegative integer, in which case the above estimate holds only for $\frac{j}{m} \leq a < 1$

Lemma 1.2 (discrete Gronwall's inequality)**.** *Let $\tau,\ B$ and $a_k,\ b_k,\ c_k,\ \gamma_k$ for integer $k \geq 0$, be nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \leq \tau\sum_{k=0}^{n}\gamma_k a_k + \tau\sum_{k=0}^{n} c_k + B,\ n \geq 0$$

Suppose that $\tau\gamma_k < 1$, for all k, and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then

$$a_n + \tau\sum_{k=0}^{n} b_k \leq \exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right)\left(\tau\sum_{k=0}^{n} c_k + B\right),\ n \geq 0$$

1.1 Temporal error estimates

Thm 1.2. *Let $T > 0$ be a given constant and suppose that the LL equation has a unique solution $m : (0,T)\times\Omega \to \mathbb{R}^3$ satisfying the regularity conditions. Then the temporal semi-discrete elliptic system with homogeneous Neumann boundary condition admits a unique solution M^{j+1} such that when $\tau \leq \tau_1$ for some $\tau_1 > 0$,*

$$\max_{0\leq j\leq J}\left\|M^j\right\|_{W^{2,4}} + \max_{0\leq j\leq J}\left\|D_\tau M^j\right\|_{H^1} + \tau\sum_{j=1}^{J}\left\|D_\tau M^j\right\|_{H^2}^2 \leq C$$

and

$$\max_{0\leq j\leq J}\left(\left\|e^j\right\|_{H^1}^2 + \tau\sum_{n=0}^{j}\left\|e^n\right\|_{H^2}^2\right) \leq \frac{C_0^2}{16}\tau^2$$

证明. 我们可以将时间半离散方程写成如下形式

$$M^{j+1}-\tau\lambda\Delta M^{j+1}-\tau\gamma M^j\times\Delta M^{j+1}-\tau\gamma\nabla M^j\times\nabla M^{j+1}=M^j+\tau\lambda\left|\nabla M^j\right|^2M^j$$

为了说明该方程有解, 由于其关于 M^{j+1} 是线性的, 我们可以对其左边与 M^{j+1} 做内积得到

$$\begin{aligned}
&(M^{j+1}-\tau\lambda\Delta M^{j+1}-\tau\gamma M^j\times\Delta M^{j+1}-\tau\gamma\nabla M^j\times\nabla M^{j+1},M^{j+1})\\
&=(M^{j+1},M^{j+1})+\tau\lambda(\nabla M^{j+1},\nabla M^{j+1})\\
&\geq\min(1,\tau\lambda)\left\|M^{j+1}\right\|_{H^1}
\end{aligned}$$

其中用到了

$$\begin{aligned}
&(M^j\times\Delta M^{j+1},M^{j+1})+(\nabla M^j\times\nabla M^{j+1},M^{j+1})\\
&=(\nabla(M^j\times\nabla M^{j+1}),M^{j+1})\\
&=-(M^j\times\nabla M^{j+1},\nabla M^{j+1})=0
\end{aligned}$$

因此根据 Lax-Milgram 定理得知其存在唯一解.

下面我们利用数学归纳法来给出相应的误差证明, 由于 $e^0=0$, 因此误差估计在 $j=0$ 时显然成立, 因此我们假设误差估计在 $0\leq j\leq k-1$ 时成立, 下面我们只需要证明其在 $0\leq j\leq k$ 时成立.

为此我们利用 LL 方程与时间半离散格式作差即可得到误差方程,其需要涉及的变形过程如下,

$$\begin{aligned}
&M^j\times\Delta M^{j+1}+\nabla M^j\times\nabla M^{j+1}-m^{j+1}\times\Delta m^{j+1}-\nabla m^{j+1}\times\nabla m^{j+1}\\
&=M^j\times\Delta M^{j+1}+\nabla M^j\times\nabla M^{j+1}-m^j\times\Delta m^{j+1}-\nabla m^j\times\nabla m^{j+1}\\
&-(m^{j+1}-m^j)\times\Delta m^{j+1}-\nabla(m^{j+1}-m^j)\times\nabla m^{j+1}\\
&=\nabla(M^j\times\nabla M^{j+1})-\nabla(m^j\times\nabla m^{j+1})=\nabla(M^j\times\nabla M^{j+1}-m^j\times\nabla m^{j+1})\\
&=\nabla\left(M^j\times\nabla M^{j+1}-M^j\times\nabla m^{j+1}+M^j\times\nabla m^{j+1}-m^j\times\nabla m^{j+1}\right)\\
&=\nabla M^j\times\nabla e^{j+1}+M^j\times\Delta e^{j+1}+\nabla e^j\times\nabla m^{j+1}+e^j\times\Delta m^{j+1}
\end{aligned}$$

上式过程, 我们其实只推了其中几项的化简, 未讨论的部分我们将会将其归入截断误差, 因此无需考虑. 同样, 我们可以通过增减项来得到

$$\left|\nabla M^j\right|^2M^j-\left|\nabla m^{j+1}\right|^2m^{j+1}=\left|\nabla M^j\right|^2M^j-\left|\nabla m^j\right|^2m^j+\left|\nabla m^j\right|^2m^j-\left|\nabla m^{j+1}\right|^2m^{j+1}$$

误差方程具体形式如下

$$\begin{aligned}
D_\tau e^{j+1}-\lambda\Delta e^{j+1}&=\gamma M^j\times\Delta e^{j+1}+\gamma e^j\times\Delta m^{j+1}+\gamma\nabla M^j\times\nabla e^{j+1}\\
&+\gamma\nabla e^j\times\nabla m^{j+1}-\lambda(\left|\nabla m^j\right|^2m^j-\left|\nabla M^j\right|^2M^j)-R_{tr}^{j+1}
\end{aligned}$$

其中截断误差为

$$
\begin{aligned}
R_{tr}^{j+1} &= D_\tau m^{j+1} - \frac{\partial m(\cdot, t_{j+1})}{\partial t} + \gamma(m^{j+1} - m^j) \times \Delta m^{j+1} \\
&+ \gamma \nabla (m^{j+1} - m^j) \times \nabla m^{j+1} + \lambda(\left|\nabla m^{j+1}\right|^2 m^{j+1} - \left|\nabla m^j\right|^2 m^j)
\end{aligned}
$$

利用前文我们给出的正则性条件, 我们可以得到如下结果

$$
\tau \sum_{j=0}^{J-1} \left\|R_{tr}^{j+1}\right\|_{L^2}^2 \leq C\tau^2
$$

由于

$$
(D_\tau e^{j+1}, e^{j+1}) = \frac{1}{\tau}\left[\left\|e^{j+1}\right\|^2 - (e^j, e^{j+1})\right] \geq \frac{1}{2} D_\tau \left\|e^{j+1}\right\|^2
$$

和

$$
\gamma(M^j \times \Delta e^{j+1}, e^{j+1}) + \gamma(\nabla M^j \times \nabla e^{j+1}, e^{j+1}) = \gamma(\nabla(M^j \times \nabla e^{j+1}), e^{j+1}) = 0
$$

, 因此我们对方程两边分别与 e^{j+1}, $-\Delta e^{j+1}$ 做内积得到,

$$
\begin{aligned}
&\frac{1}{2} D_\tau \left\|e^{j+1}\right\|^2 + \lambda \left\|\nabla e^{j+1}\right\|^2 \\
&\leq \gamma(e^j \times \Delta m^j, e^{j+1}) + \gamma(\nabla e^j \times \nabla m^{j+1}, e^{j+1}) \\
&+ \lambda(\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j, e^{j+1}) - (R_{tr}^{j+1}, e^{j+1}) \\
&:= \sum_{n=1}^{4} I^n(e^{j+1})
\end{aligned}
$$

同理, 我们可以得到

$$
\begin{aligned}
&\frac{1}{2} D_\tau \left(\left\|\nabla e^{j+1}\right\|^2\right) + \lambda \left\|\Delta e^{j+1}\right\|^2 \\
&\leq -\gamma(e^j \times \Delta m^{j+1}, \Delta e^{j+1}) - \gamma(\nabla M^j \times \nabla e^{j+1}, \Delta e^{j+1}) \\
&- \gamma(\nabla e^j \times \nabla m^{j+1}, \Delta e^{j+1}) + (R_{tr}^{j+1}, \Delta e^{j+1}) \\
&- \lambda(\left|\nabla m^j\right|^j m^j - \left|\nabla M^j\right|^2 M^j, \Delta e^{j+1}) := \sum_{n=5}^{9} I^n(e^{j+1})
\end{aligned}
$$

接下来我们要对 $I^n(e^{j+1})$, $n = 1, \ldots, 9$ 一项项进行分析. 我们先罗列一下一些较为显然的估计如下,

$$
\begin{aligned}
\left|I^1\right| &\leq C \left\|e^{j+1} \times \Delta m^{j+1}\right\|_{L^2} \cdot \left\|e^j\right\|_{L^2} \\
&\leq C \left\|e^{j+1}\right\|_{L^6} \left\|\Delta m^{j+1}\right\|_{L^3} \left\|e^j\right\|_{L^2} \\
&\leq \epsilon \left\|e^{j+1}\right\|_{H^1}^2 + \epsilon^{-1} C \left\|e^j\right\|_{L^2}^2 \\
\left|I^2\right| &\leq \epsilon \left\|e^j\right\|_{H^1}^2 + \epsilon C \left\|e^{j+1}\right\|_{L^2}^2 \\
\left|I^4\right| &\leq C \left\|e^{j+1}\right\|_{L^2}^2 + C \left\|R_{tr}^{j+1}\right\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
\left|I^5\right| &\leq \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|e^j\right\|_{H^1}^2 \\
\left|I^7\right| &\leq \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2 \\
\left|I^9\right| &\leq \epsilon \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon^{-1} C \left\|R_{tr}^{j+1}\right\|_{L^2}^2
\end{aligned}
$$

分析 $|I^3|$ 的估计,

$$
\begin{aligned}
\left|I^3\right| &= \left|\left(\lambda(\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j), e^{j+1}\right)\right| \\
&\leq C \left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \left\|e^{j+1}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla m^j\right|^2 M^j + \left|\nabla m^j\right|^2 M^j - \left|\nabla M^j\right|^2 M^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j + \nabla M^j) \cdot \nabla e^j M^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) M^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) M^j - (\nabla M^j \cdot \nabla e^j) m^j + (\nabla M^j \cdot \nabla e^j) m^j\right. \\
&\left.-(\nabla m^j \cdot \nabla e^j) M^j + (\nabla m^j \cdot \nabla e^j) M^j + (\nabla m^j \cdot \nabla e^j) M^j - (\nabla m^j \cdot \nabla e^j) M^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) M^j + (\nabla M^j \cdot \nabla e^j) m^j - (\nabla m^j \cdot \nabla e^j) m^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) e^j + (\nabla M^j \cdot \nabla e^j) m^j + (\nabla m^j \cdot \nabla e^j) m^j\right\| \\
=&\left\|\left|\nabla m^j\right|^2 e^j + \left|\nabla e^j\right|^2 e^j + 2(\nabla m^j \cdot \nabla e^j) e^j + \left|\nabla e^j\right|^2 m^j + 2(\nabla m^j \cdot \nabla e^j) m^j\right\|
\end{aligned}
$$

$$
\begin{aligned}
|I^3| \leq& C \left\|\left|\nabla m^j\right|^2 m^j - \left|\nabla M^j\right|^2 M^j\right\| \left\|e^{j+1}\right\| \\
\leq& \epsilon^{-1} C \left(\left\|\left|\nabla m^j\right|^2 e^j\right\|^2 + \left\|\left|\nabla e^j\right|^2 e^j\right\|^2 + \left\|2(\nabla m^j \cdot \nabla e^j) e^j\right\| + \left\|\left|\nabla e^j\right|^2 m^j\right\|^2\right. \\
&\left.+ \left\|2(\nabla m^j \cdot \nabla e^j) m^j\right\|^2\right) + \epsilon \left\|e^{j+1}\right\|^2 \\
\leq& \epsilon^{-1} C \left(\left\|e^j\right\|^2 + \left\|e^j\right\|_{L^\infty}^2 \left\|\nabla e^j\right\|_{L^6}^2 \left\|\nabla e^j\right\|_{L^3}^2 + 2 \left\|\nabla e^j\right\|_{L^6}^2 \left\|e^j\right\|_{L^6}^2 + \left\|\nabla e^j\right\|_{L^6}^2 \left\|\nabla e^j\right\|_{L^6}^2\right. \\
&\left.+ \left\|e^j\right\|_{H^1}^2\right) + \epsilon \left\|e^{j+1}\right\|^2
\end{aligned}
$$

我们给出一些不等式对上式进行化简,

$$
\begin{aligned}
&\left\|\nabla e^j\right\|_{L^6} \leq \left\|\nabla e^j\right\|_{H^1} \leq \left\|e^j\right\|_{H^2} \\
&\left\|e^j\right\|_{L^3} \leq C \left(\left\|e^j\right\|_{L^2} \left\|e^j\right\|_{L^6}\right)^{\frac{1}{2}} \leq C \left(\left\|e^j\right\|_{L^2} \left\|e^j\right\|_{H^1}\right)^{\frac{1}{2}} \leq C \left(\left\|e^j\right\|_{H^2} \left\|e^j\right\|_{H^1}\right)^{\frac{1}{2}} \\
&\left\|\nabla e^j\right\|_{L^3} \leq C \left(\left\|\nabla e^j\right\|_{L^2} \left\|\nabla e^j\right\|_{L^6}\right)^{\frac{1}{2}} \leq C \left(\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2}\right)^{\frac{1}{2}}
\end{aligned}
$$

因此上式可以改为

$$
\begin{aligned}
\left|I^3\right| &\leq \epsilon \left\|e^{j+1}\right\|^2 + \epsilon^{-1} C \left(\left\|e^j\right\|^2 + \left\|e^j\right\|_{H^1}^2 + \frac{C_0^2}{16} \tau^{\frac{3}{2}} \left\|e^j\right\|_{H^2}^2 \right) \\
&\leq \epsilon \left\|e^{j+1}\right\|^2 + \epsilon^{-1} C \left\|e^j\right\|_{H^1}^2 + \epsilon \left\|e^j\right\|_{H^2}^2
\end{aligned}
$$

上面引入 $\frac{C_0}{16}\tau^{\frac{3}{2}}$ 的原因,

$$
\left\|\nabla e^j\right\|_{L^6}^2 \left\|\nabla e^j\right\|_{L^3}^2 \leq C \left(\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2} \right) \left\|e^j\right\|_{H^2}^2
$$

而此处的 j 满足 $j \in [0, k-1]$, 因此有下式成立,

$$
\begin{aligned}
\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2} \tau^{\frac{1}{2}} &\leq \frac{1}{2} (\left\|e^j\right\|_{H^1}^2 + \tau \left\|e^j\right\|_{H^2}^2) \\
&\leq \frac{1}{2} \max \left(\left\|e^j\right\|_{H^1}^2 + \tau \sum_{n=0}^{j} \left\|e^n\right\|_{H^2}^2 \right) \\
&\leq \frac{C_0^2}{32} \tau^2
\end{aligned}
$$

故

$$
\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2} \tau^{\frac{1}{2}} \leq \frac{C_0^2}{16} \tau^{\frac{3}{2}} C
$$

所以对 $|I^3|$ 的估计, 其实暗含了对 ϵ 的限制,

$$
\epsilon^{-1} C \frac{C_0^2}{16} \tau^{\frac{3}{2}} \leq \epsilon \Rightarrow \epsilon \geq \frac{\sqrt{C} C_0}{4} \tau^{\frac{3}{4}}
$$

按照同样的证明过程,我们可以得到,

$$
\left|I^8\right| \leq 2\epsilon \left\|e^j\right\|_{H^2}^2 + \epsilon^{-1} C \left\|e^j\right\|_{H^1}^2
$$

下面给出 $|I^6|$ 的估计,

$$
\begin{aligned}
\left|I^6\right| &= \left|C \left(\nabla M^j \times \nabla e^{j+1}, \Delta e^{j+1} \right)\right| \\
&= \left|C \left(\nabla e^j \times \nabla e^{j+1}, \Delta e^{j+1} \right) + C \left(\nabla m^j \times \nabla e^{j+1}, \Delta e^{j+1} \right)\right| \\
&\leq C \left|\left(\nabla e^j \times \nabla e^{j+1}, \Delta e^{j+1} \right)\right| + C \left|\left(\nabla m^j \times \nabla e^{j+1}, \Delta e^{j+1} \right)\right| \\
&\leq C \left\|\nabla e^j\right\|_{L^3} \left\|\nabla e^{j+1}\right\|_{L^6} \left\|\Delta e^{j+1}\right\|_{L^2} + C \left\|m^j\right\|_{W^{1,\infty}} \left\|\nabla e^{j+1}\right\|_{L^2} \left\|\Delta e^{j+1}\right\|_{L^2} \\
&\leq C \left(\left\|e^j\right\|_{H^1} \left\|e^j\right\|_{H^2} \right)^{\frac{1}{2}} \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon \left\|e^{j+1}\right\|_{H^2}^2 + \epsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2 \\
&\leq \frac{C C_0}{4} \tau^{\frac{3}{4}} \left\|\Delta e^{j+1}\right\|_{L^2}^2 + \epsilon \left\|e^{j+1}\right\|_{H^2}^2 + \epsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2 \\
&\leq 2\epsilon \left\|e^{j+1}\right\|_{H^2}^2 + \epsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2
\end{aligned}
$$

同样我们要求 $\epsilon \geq \frac{CC_0}{4}\tau^{\frac{3}{4}}$. 依托上面的逐项分析, 我们将不等式求和可以得到如下形式

$$
\begin{aligned}
D_\tau \left(\left\|e^{j+1}\right\|_{H^1}^2 \right) + \lambda \left\|e^{j+1}\right\|_{H^2}^2 \leq{}& \epsilon \left\|e^j\right\|_{H^2}^2 + \epsilon \left\|e^{j+1}\right\|_{H^2}^2 + \varepsilon^{-1} C \left\|e^j\right\|_{H^1}^2 \\
&+ \varepsilon^{-1} C \left\|e^{j+1}\right\|_{H^1}^2 + \epsilon^{-1} C \left\|R_{tr}\right\|_{L^2}^2
\end{aligned}
$$

依据 Grownwall 不等式可以得到

$$\left\|e^{j+1}\right\|_{H^1}^2+\tau\sum_{n=0}^{j+1}\|e^n\|_{H^2}^2\le\exp(2TC)\tau^2$$

从误差不等式中我们可以得到

$$\max_{0\le j\le J}\left\|M^j\right\|_{H^2}\le C,\max_{0\le j\le J}\left\|D_\tau M^j\right\|_{H^1}\le C,\tau\sum_{n=1}^{J}\|D_\tau M^n\|_{H^2}^2\le C$$

根据椭圆方程估计有

$$\begin{aligned}
\left\|M^j\right\|_{W^{2,4}} &\le C\left\|D_\tau M^j\right\|_{L^4}+C\left\|\nabla M^j\times\nabla M^{j+1}\right\|_{L^4}+C\left\|\left|\nabla M^j\right|^2M^j\right\|_{L^4}\\
&\le C\left\|\nabla M^j\times\nabla M^{j+1}\right\|_{L^4}+C\left\|\left|\nabla M^j\right|^2M^j\right\|_{L^4}+C\\
&\le C\left\|\nabla M^j\right\|_{L^6}\left\|\nabla M^{j+1}\right\|_{L^{12}}+C\left\|\nabla M^j\right\|_{L^6}\left\|\nabla M^{j+1}\right\|_{L^{12}}\left\|M^j\right\|_{L^\infty}+C\\
&\le C\left\|\nabla M^{j+1}\right\|_{L^{12}}+C\\
&\le C\left\|M^{j+1}\right\|_{W^{2,4}}^{5/7}\left\|M^{j+1}\right\|_{H^2}^{2/7}+C\\
&\le\frac{1}{2}\left\|M^j\right\|_{W^{2,4}}+C.
\end{aligned}$$

□

1.2 Spatial error estimates

定义 Ritz 投影 $R_h^j:H^1(\Omega)\to V_h$, 同时定义了如下的双线性函数形式

$$B^j(u,v)=\lambda(\nabla u,\nabla v)+\gamma(M^{j-1}\times\nabla u,\nabla v)+C_m(u,v)$$

其中 $C_m(u,v)$ 的作用是保证 B^j 的正定性, 从而保证 Lax-Milgram 定理成立.

我们定义 $R_h^0M^0\in V_h$ 使得 $B^1(M^0-R_h^0,\phi)=0,\forall\phi\in V_h$. 同时对于 $M^{j+1}\in H^1(\Omega)$, 我们可以定义 $R_h^{j+1}M^{j+1}\in V_h$ 使得

$$B^{j+1}(M^{j+1}-R_h^{j+1}M^{j+1},\phi)=0,\ \forall\phi\in V_h,\ j=0,\dots,J-1$$

我们在此罗列一些经典有限元的结果:

$$\begin{aligned}
&\left\|\theta_h^j\right\|_{L^4}+h\left\|\theta_h^j\right\|_{W^{1,4}}\le Ch^2\left\|M^j\right\|_{W^{2,4}}\\
&\left\|R_h^jM^j\right\|_{W^{1,p}}\le C\left\|M^j\right\|_{W^{1,p}}\le C,\ 2<p\le\infty\\
&\tau\sum_{n=1}^{J}\|D_\tau\theta_h^n\|_{L^2}^2\le Ch^4
\end{aligned}$$

Thm 1.3. 全离散有限元系统, 其初值为 $m_h^0=\Pi_h m_0$, 有唯一解. 且如果 $\exists \tau_2, h_0$, 使得 $\tau \leq \tau_2, h\leq h_0$, 则有

$$\max\left(\left\|e_h^j\right\|^2+\tau\sum_{n=1}^{j}\left\|e_h^n\right\|_{H^1}^2\right)\leq\frac{C_0^2}{16}h^4$$

$$\max\left\|e_h^j\right\|_{H^1}\leq C_0 h.$$

证明. 由于系数矩阵正定, 因此解存在且唯一. 下面我们同样采用数学归纳法证明. 由于 $M^0=m^0$, $m_h^0=\Pi_h m^0$, 故

$$\left\|e_h^0\right\|_{L^2}^2=\left\|\Pi_h m^0-R_h m^0\right\|^2\leq C\left\|\Pi_h m^0-m^0\right\|^2+C\left\|m^0-R_h^0 m^0\right\|^2\leq C_1 h^4$$

上述式子的放缩中其实涉及到了网格投影的不等式, 但由于网格投影无非是保证曲边三角形映射到标准区域的操作, 因此我们在此给出其相关的不等式, 并不对其进行详细描述,

$$\left\|v-\Pi_h v\right\|_{L^p}+h\left\|v-\Pi_h v\right\|_{W^{1,p}}\leq Ch^{r+1}\left\|v\right\|_{W^{1+r,p}}$$

其中要求 $r=0,1$ 和 $1\leq p\leq\infty$. 故根据上式, 我们可以得到 $C_0\geq 4\sqrt{C_1}$.

下面我们要求 $j\leq k-1$ 均成立, 证明当 $j=k$ 时成立. 在给出证明前, 我们先引入逆估计不等式,

$$\left\|u_h\right\|_{W^{k,p}}\leq Ch^{l-k+n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\|u_h\right\|_{W^{k,q}}$$

其中要求 $l\leq k$, $q\leq p$, n 为维数. 根据给出的逆估计不等式, 我们可以得到下面的两条式子,

$$\left\|u_h\right\|_{L^3}\leq Ch^{0-0+n\left(\frac{1}{3}-\frac{1}{2}\right)}\left\|u_h\right\|_{W^{0,2}}=Ch^{-\frac{n}{6}}\left\|u_h\right\|_{L^2}\leq\frac{CC_0}{4}h^{2-\frac{n}{6}}$$

$$\left\|u_h\right\|_{L^\infty}\leq Ch^{-\frac{n}{2}}\left\|u_h\right\|_{L^2}\leq\frac{CC_0}{4}h^{2-\frac{n}{2}}$$

我们考虑下面的全离散有限元系统和时间半离散系统的弱形式,

$$\left(D_\tau m_h^{j+1},\phi\right)+\lambda\left(\nabla m_h^{j+1},\nabla\phi\right)+\gamma\left(m_h^j\times\nabla m_h^{j+1},\nabla\phi\right)=\lambda\left(\left|\nabla m_h^j\right|^2 m_h^j,\phi\right)$$

$$\left(D_\tau M^{j+1},\phi\right)+\lambda\left(\nabla M^{j+1},\nabla\phi\right)+\gamma\left(M^j\times\nabla M^{j+1},\nabla\phi\right)=\lambda\left(\left|\nabla M^j\right|^2 M^j,\phi\right)$$

二者做差可得到

$$\begin{aligned}
&\left(D_\tau\left(m_h^{j+1}-M^{j+1}\right),\phi\right)=\left(D_\tau e_h^{j+1},\phi\right)+\left(D_\tau\theta_h,\phi\right)\\
&\lambda\left(\nabla\left(m_h^{j+1}-M^{j+1}\right),\nabla\phi\right)=\lambda\left(\nabla e_h^{j+1},\phi\right)+\lambda\left(\nabla\theta_h,\phi\right)\\
&\gamma\left(m_h^j\times\nabla m_h^{j+1},\nabla\phi\right)-\gamma\left(M^j\times\nabla M^{j+1},\nabla\phi\right)\\
&=\gamma\left(m_h^j\times\nabla m_h^{j+1}-m_h^j\times\nabla R_h^{j+1}M^{j+1}+m_h^j\times\nabla R_h^{j+1}M^{j+1}-M^j\times\nabla R_h^{j+1}M^{j+1}\right.\\
&\quad\left.+M^j\times\nabla R_h^{j+1}M^{j+1}-M^j\times\nabla M^{j+1},\nabla\phi\right)\\
&=\gamma\left(m_h^j\times\nabla e_h^{j+1},\nabla\phi\right)+\gamma\left(\left(e_h^{j+1}+\theta_h^{j+1}\right)\times\nabla R_h^{j+1}M^{j+1},\nabla\phi\right)+\gamma\left(M^j\times\nabla\theta_h^{j+1},\nabla\phi\right)
\end{aligned}$$

组合可得

$$
\begin{aligned}
&\left(D_\tau e_h^{j+1},\phi\right)+\left(D_\tau \theta_h^{j+1},\phi\right)+\lambda\left(\nabla e_h^{j+1},\nabla\phi\right)+\lambda\left(\nabla\theta_h^{j+1},\nabla\phi\right)\\
&+\gamma\left(m_h^j\times\nabla e_h^{j+1},\nabla\phi\right)+\gamma\left(\left(e_h^{j+1}+\theta_h^{j+1}\right)\times\nabla R_h^{j+1}M^{j+1},\nabla\phi\right)\\
&+\gamma\left(M^j\times\nabla\theta_h^{j+1},\nabla\phi\right)=\lambda\left(\left|\nabla m_h^j\right|^2 m_h^j-\left|\nabla M^j\right|^2 M^j,\phi\right)
\end{aligned}
$$

根据前文中提及的 $B^j(u,v)$ 的形式, 我们可以得到下面的式子成立,

$$
B^{j+1}(-\theta_h^{j+1},\phi)=-\lambda\left(\nabla\theta_h^{j+1},\nabla\phi\right)-\gamma\left(M^j\times\nabla\theta_h^j,\nabla\phi\right)-C_m\left(\theta_h^j,\phi\right)=0
$$

故而可得到

$$
\begin{aligned}
&\frac{1}{2}\left(D_\tau e_h^{j+1},\phi\right)+\lambda\left(\nabla e_h^{j+1},\nabla\phi\right)\\
&\leq-\left(D_\tau\theta_h^{j+1},\phi\right)-\gamma\left(m_h^j\times\nabla e_h^{j+1},\nabla\phi\right)+C_m\left(\theta_h^{j+1},\phi\right)\\
&-\gamma\left(\left(e_h^{j+1}+\theta_h^{j+1}\right)\times\nabla R_h^{j+1}M^{j+1},\nabla\phi\right)+\lambda\left(\left|\nabla m_h^j\right|^2 m_h^j-\left|\nabla M^j\right|^2 M^j,\phi\right)\\
&=:\sum_{n=1}^{5} I_h^n(\phi),\ \forall\phi\in V_h
\end{aligned}
$$

我们取 $\phi=e_h^{j+1}$ 并逐个估计 I_h^n,

$$
\begin{aligned}
\left|I_h^1\right|&=\left|\left(D_\tau\theta_h^{j+1},e_h^{j+1}\right)\right|\leq C\left\|e_h^{j+1}\right\|_{L^2}^2+\left\|D_\tau\theta_h^{j+1}\right\|_{L^2}^2\\
\left|I_h^2\right|&=0\\
\left|I_h^3\right|&\leq\epsilon\left\|\nabla e_h^{j+1}\right\|_{L^2}^2+\epsilon^{-1}C\left\|e_h^j+\theta_h^j\right\|_{L^2}^2\\
&\leq\epsilon\left\|\nabla e_h^{j+1}\right\|_{L^2}^2+\epsilon^{-1}C\left\|e_h^j\right\|^2+\epsilon^{-1}C\left\|\theta_h^j\right\|_{L^2}^2\\
&\leq\epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+\epsilon^{-1}C\left\|e_h^j\right\|^2+\epsilon^{-1}h^4\\
\left|I_h^4\right|&\leq C\left\|e_h^{j+1}\right\|_{L^2}^2+Ch^4.
\end{aligned}
$$

上述式子的推导过程中用到了

$$
\left\|R_h^{j+1}M^{j+1}\right\|_{W^{1,\infty}}\leq C\left\|M^{j+1}\right\|_{W^{1,\infty}}\leq C\left\|M^{j+1}\right\|_{W^{2,4}}\leq C
$$

而对于 I_h^5 的分析, 我们先对其改写,

$$
\begin{aligned}
I_h^5&=\lambda\left(\left|\nabla m_h^j\right|^2 m_h^j-\left|\nabla M^j\right|^2 m_h^j+\left|\nabla M^j\right|^2 m_h^j-\left|\nabla M^j\right|^2 M^j,e_h^{j+1}\right)\\
&=\lambda\left(\left|\nabla M^j\right|^2\left(e_h^j+\theta_h^j\right),e_h^{j+1},e_h^{j+1}\right)+\lambda\left(\left|\nabla m_h^j\right|^2 m_h^j-\left|\nabla M^j\right|^2 m_h^j,e_h^{j+1}\right)\\
&=\lambda\left(\left|\nabla M^j\right|^2\left(e_h^j+\theta_h^j\right),e_h^{j+1},e_h^{j+1}\right)+2\lambda\left(\left(\nabla M^j\cdot\nabla\left(e_h^j+\theta_h^j\right)\right)R_h^jM^j,e_h^{j+1}\right)\\
&+2\lambda\left(\left(\nabla M^j\cdot\nabla\left(e_h^j+\theta_h^j\right)\right)e_h^j,e_h^{j+1}\right)+\lambda\left(\left|\nabla\left(e_h^j+\theta_h^j\right)\right|^2 R_h^jM^j,e_h^{j+1}\right)\\
&\lambda\left(\left|\nabla\left(e_h^{j+1}+\theta_h^{j+1}\right)\right|^2 e_h^j,e_h^{j+1}\right):=\sum_{n=1}^{5}\Pi_h^n
\end{aligned}
$$

同样我们逐项分析,

$$
\begin{aligned}
\left|\Pi_h^1\right| &\leq C\left\|M^j\right\|_{W^{1,\infty}}^2\left\|e_h^j+\theta_h^j\right\|_{L^2}\left\|e_h^{j+1}\right\|_{L^2} \\
&\leq C\left\|e_h^{j+1}\right\|^2+C\left\|e_h^j\right\|^2+Ch^4 \\
\left|\Pi_h^2\right| &\leq C\left|\left(\nabla M^j\cdot\nabla\left(e_h^j+\theta_h^j\right)R_h^jM^j, e_h^{j+1}\right)\right| \\
&\leq C\left|\left(\left(\nabla M^j\cdot\nabla e_h^j\right)R_h^jM^j, e_h^{j+1}\right)\right|+C\left|\left(\left(\Delta M^j\cdot\theta_h^j\right)R_h^jM^j, e_h^{j+1}\right)\right| \\
&+C\left|\left(\left(\nabla M^j\cdot\nabla R_h^jM^j\right)\theta_h^j, e_h^{j+1}\right)\right|+C\left|\left(\left(\nabla M^j\cdot\nabla e_h^{j+1}\right)R_h^jM^j, \theta_h^j\right)\right| \\
&\leq C\left\|M^j\right\|_{W^{1,\infty}}^2\left\|\nabla e_h^j\right\|\left\|e_h^{j+1}\right\|+C\left\|M^j\right\|_{H^2}\left\|M^j\right\|_{L^\infty}\left\|\theta_h^j\right\|_{L^3}\left\|e_h^{j+1}\right\|_{L^6} \\
&+C\left\|M^j\right\|_{W^{1,\infty}}^2\left\|\theta_h^j\right\|\left\|e_h^{j+1}\right\|+C\left\|M^j\right\|_{W^{1,\infty}}^2\left\|\theta_h^j\right\|\left\|\nabla e_h^{j+1}\right\| \\
&\leq \epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+\epsilon\left\|e_h^j\right\|_{H^1}^2+\epsilon^{-1}C\left\|e_h^{j+1}\right\|^2+\epsilon^{-1}Ch^4. \\
\left|\Pi_h^3\right| &\leq C\left\|M^j\right\|_{W^{1,\infty}}\left\|\nabla\left(e_h^j+\theta_h^j\right)\right\|_{L^3}\left\|e_h^j+\theta_h^j\right\|\left\|e_h^{j+1}\right\|_{L^6} \\
&\leq C\left(\frac{C_0}{4}h^{1-\frac{d}{6}}+h\right)\left\|e_h^j+\theta_h^j\right\|\left\|e_h^{j+1}\right\|_{H^1} \\
&\leq \epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+\epsilon^{-1}C\left\|e_h^j\right\|_{H^1}^2+\epsilon^{-1}Ch^4 \\
\left|\Pi_h^4\right| &\leq C\left\|R_h^jM^j\right\|_{L^\infty}\left\|\nabla\left(e_h^j+\theta_h^j\right)\right\|\left\|\nabla\left(e_h^j+\theta_h^j\right)\right\|_{L^3}\left\|e_h^{j+1}\right\|_{L^6} \\
&\leq C\left(\left\|\nabla e_h^j\right\|+Ch\right)\left(\left\|\nabla e_h^j\right\|_{L^3}+Ch\right)\left\|e_h^{j+1}\right\|_{H^1} \\
&\leq C\left(\frac{C_0}{4}h^{1-\frac{d}{6}}+h\right)\left\|e_h^j\right\|_{H^1}\left\|e_h^{j+1}\right\|_{H^1}+Ch^2\left\|e_h^{j+1}\right\|_{H^1} \\
&\leq \epsilon\left\|e_h^j\right\|_{H^1}^2+\epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+\epsilon^{-1}Ch^4 \\
\left|\Pi_h^5\right| &\leq C\left\|e_h^j\right\|_{L^\infty}\left\|\nabla\left(e_h^j+\theta_h^j\right)\right\|\left\|\nabla\left(e_h^j+\theta_h^j\right)\right\|_{L^3}\left\|e_h^{j+1}\right\|_{L^6} \\
&\leq C\frac{C_0}{4}h^{2-\frac{d}{2}}\left(\frac{C_0}{4}h^{1-\frac{d}{6}}+h\right)\left(\left\|e_h^j\right\|_{H^1}+h\right)\left\|e_h^{j+1}\right\|_{H^1} \\
&= C\left(\frac{C_0}{16}h^{3-\frac{2}{3}d}+\frac{C_0}{4}h^{3-\frac{d}{2}}\right)\left(\left\|e_h^j\right\|_{H^1}+h\right)\left\|e_h^{j+1}\right\|_{H^1} \\
&\leq \epsilon\left\|e_h^j\right\|_{H^1}^2+\epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+Ch^4
\end{aligned}
$$

综上可得 $\left|I_h^5\right|$ 的估计,

$$
\left|I_h^5\right| \leq \epsilon\left\|e_h^{j+1}\right\|_{H^1}^2+\epsilon^{-1}C\left\|e_h^j\right\|_{L^2}^2+\epsilon^{-1}Ch^4.
$$

我们将上面的估计带回原式可得,

$$
\begin{aligned}
D_\tau\left\|e_h^{j+1}\right\|^2+\lambda\left\|e_h^{j+1}\right\|_{H^1}^2 \leq{}& \epsilon\left(\left\|e_h^j\right\|_{H^1}^2+\left\|e_h^{j+1}\right\|_{H^1}^2\right)+\epsilon^{-1}C\left\|e_h^{j+1}\right\|^2 \\
&+\epsilon^{-1}Ch^4+\left\|D_\tau\theta_h^{j+1}\right\|^2
\end{aligned}
$$

同样我们可以对上式进行求和得,

$$\begin{aligned}\left\|e_h^{j+1}\right\|^2+\frac{\lambda}{2}\tau\sum_{n=1}^{j+1}\left\|e_h^n\right\|_{H^1}^2 &\leq C\tau\sum_{n=0}^{j+1}\left\|e^n\right\|^2+C_1h^4+Ch^4+\tau\sum_{n=1}^{j+1}\left\|D_\tau\theta_h^{j+1}\right\|^2\\ &\leq C\sum_{n=0}^{j+1}\tau\left\|e_h^n\right\|^2+C_1h^4+Ch^4\end{aligned}$$

我们利用 Gronwall 不等式得到存在 $\tau_2>0$, 使得 $\tau\leq\tau_2$

$$\left\|e_h^{j+1}\right\|^2+\tau\sum_{n=1}^{j+1}\left\|e_h^n\right\|_{H^1}^2\leq C_1\exp\left(2TC\right)h^4$$

本定理的另一个结论用逆估计不等式 $\left\|e_h^j\right\|_{H^1}\leq Ch^{-1}\left\|e_h^j\right\|$ 给出. $\square$

2 ETD

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