ENGR30003: Numerical Programming for Engineers

Semester 2, 2019 / Assignment 2

Student name: Yuxuan Liang Student ID: 1010124

2 Rootfinding

2.1 Analytical solution for $\theta = 0°$

The equation relating θ, β and M is given by

$$\tan(\theta) = 2\cot(\beta)\frac{M^2\sin^2(\beta) - 1}{M^2(\gamma + \cos(2\beta)) + 2}$$

When $\theta = 0°$, we get

$$0 = 2\cot(\beta)\frac{M^2\sin^2(\beta) - 1}{M^2(\gamma + \cos(2\beta)) + 2}$$

Roots occur when

$$2\cot(\beta)\,(M^2\sin^2(\beta) - 1) = 0$$

One root at: $\cot(\beta) = 0$

$$\Longrightarrow \frac{\cos(\beta)}{\sin(\beta)} = 0$$

$$\Longrightarrow \cos(\beta) = 0$$

$$\Longrightarrow \beta_U = \frac{\pi}{2} = 90°$$

Another root at: $M^2\sin^2(\beta) - 1 = 0$

$$\Longrightarrow \sin^2(\beta) = \frac{1}{M^2}$$

$$\Longrightarrow \beta_L = \arcsin\left(\frac{1}{M}\right)$$

The two roots are:

$$\beta_L = \arcsin\left(\frac{1}{M}\right), \qquad \beta_U = 90°$$

2.2 Graphical solution

a. $M = 1.5$ and $\theta = 5°$, $10°$ and $15°$

When $M = 1.5$, two roots are observed in the diagram for $\theta = 5°$ and $10°$, but there is no intersection of $f(\beta)$ and $y = 0$ for $\theta = 15°$.

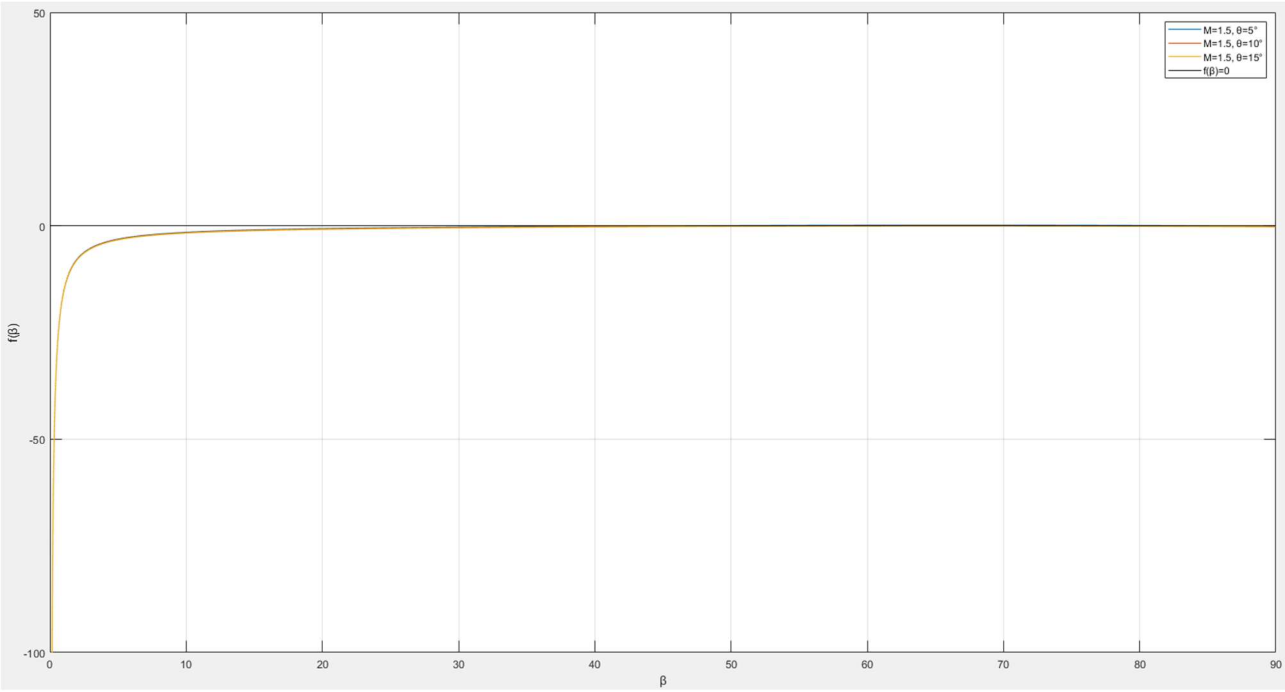


Figure 1: $\beta - f(\beta)$ diagram for $M = 1.5$ and $\theta = 5°$, $10°$ and $15°$

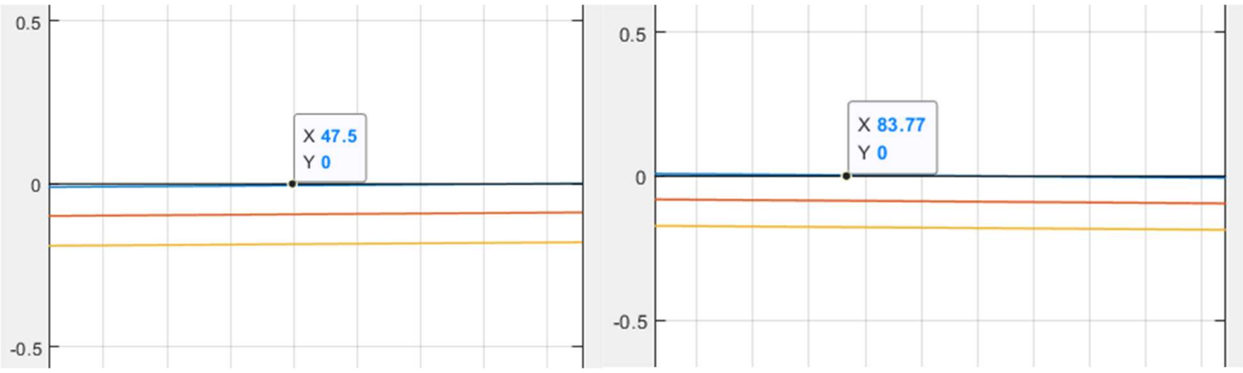


Figure 2: intersections of $f(\beta)$ and $y = 0$ for $\theta = 5°$

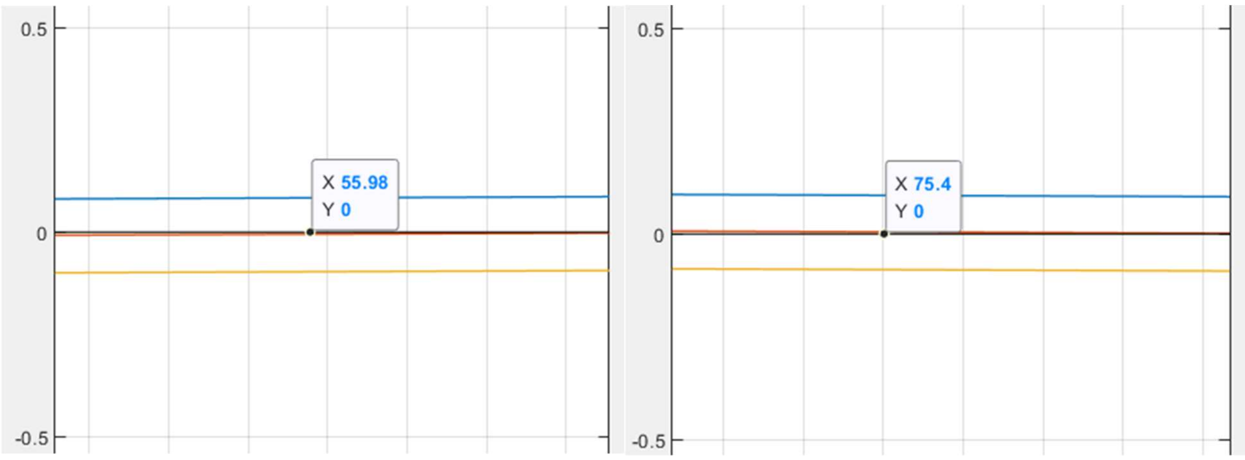


Figure 3: intersections of $f(\beta)$ and $y = 0$ for $\theta = 10°$

b. $M = 5.0$ and $\theta = 20°,\ 30°$ and $45°$

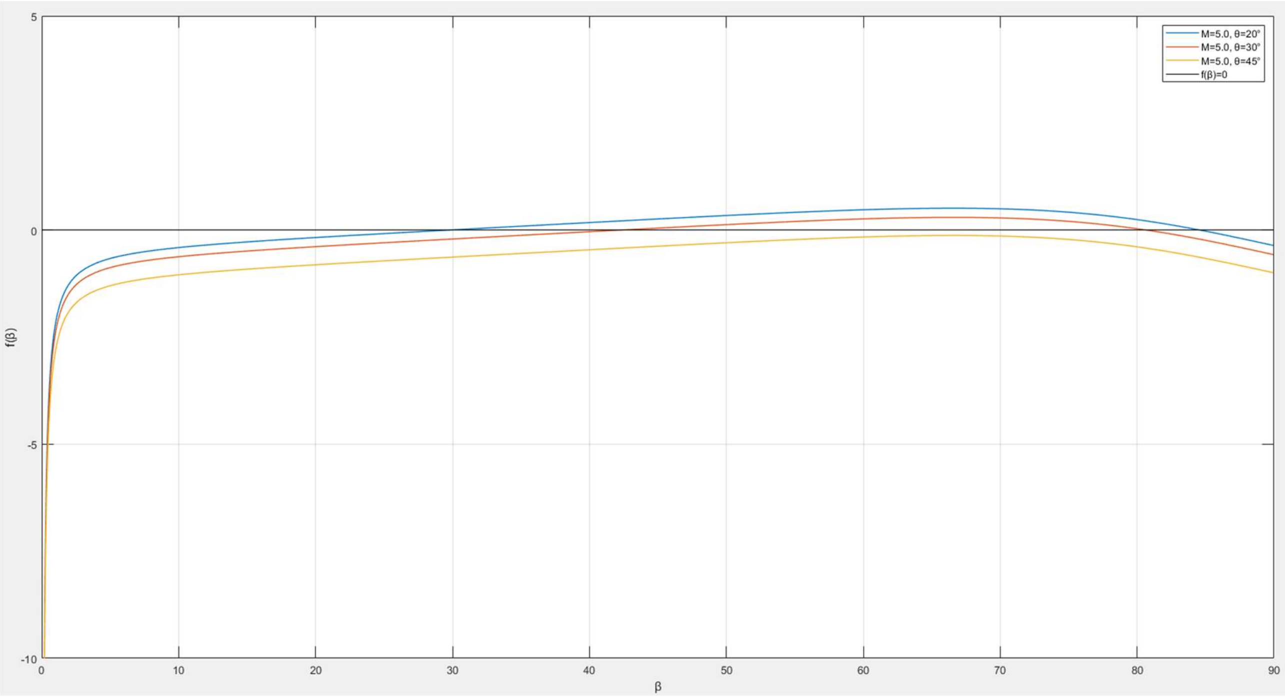


Figure 4: $\beta - f(\beta)$ diagram for $M = 5.0$ and $\theta = 20°,\ 30°$ and $45°$

Set a table to see how β_U and β_L change with θ and M

M	θ	β_L	β_U
1.5	5°	48°	90°
1.5	10°	56°	76°
1.5	15°	N/A	N/A
5.0	20°	30°	85°
5.0	30°	42°	81°
5.0	45°	N/A	N/A

The table demonstrates that for the same M, as θ increases, β_L increases while β_U decreases. But when θ increases to a certain extent, there is no β_L and β_U, since $f(\beta)$ never reaches 0. As for M, as it increases, θ_{max} increases. This means a larger maximum θ beyond which the solutions of β are not physically relevant.

For $M = 1.5$, the value of θ_{max} is between $10°$ and $15°$. The approximate value of θ_{max} would be $12°$.
For $M = 5.0$, the value of θ_{max} is between $30°$ and $45°$. The approximate value of θ_{max} would be $40°$.

2.3 C program to solve shock-wave equation

(a) Based on the graphical solution for $M = 5.0$ and $\theta = 20°$, an initial guess of $\beta_L = 30°$ and $\beta_U = 85°$ was chosen.

(b) Plot the results with Matlab.

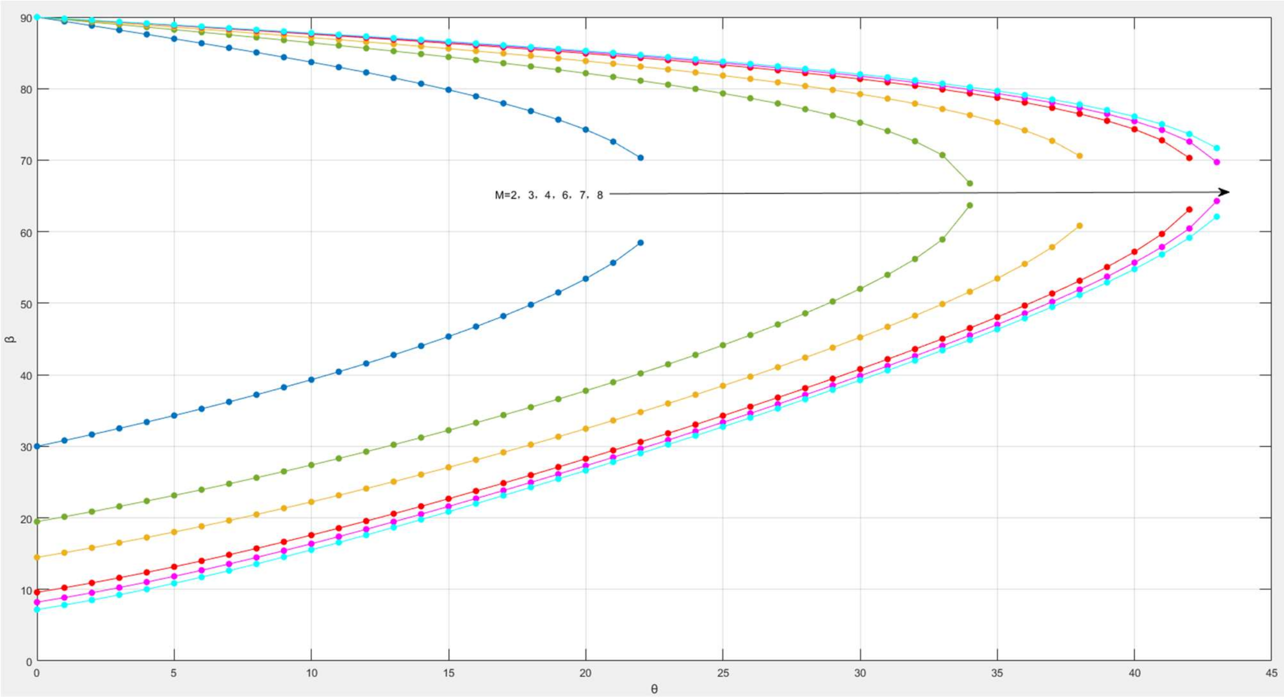


Figure 5: $\theta-\beta-M$ diagram for $M=2.0, 3.0, 4.0, 6.0, 7.0, 8.0$

We can see that as M increases, the curves of β_L and β_U extend to a larger θ. At the same θ, as M increases, β_L decreases and β_U increases.

3 Regression

The system of equations is in the form of $[A]\{X\}=\{C\}$, in which $\{X\}$ has to be solved to obtain *a* and *b*.

For a 2×2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So the inverse of [A] is:

$$[A]^{-1} = \frac{1}{N\cdot\sum_{i=1}^{N} x_i{}^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}\begin{bmatrix} N & -\sum\limits_{i=1}^{N} x_i \\ -\sum\limits_{i=1}^{N} x_i & \sum\limits_{i=1}^{N} x_i{}^2 \end{bmatrix}$$

Then *a* and *b* can be obtained by:

$$\begin{Bmatrix} a \\ b \end{Bmatrix} = [A]^{-1}\{C\} = \frac{1}{N\cdot\sum_{i=1}^{N} x_i{}^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}\begin{bmatrix} N & -\sum\limits_{i=1}^{N} x_i \\ -\sum\limits_{i=1}^{N} x_i & \sum\limits_{i=1}^{N} x_i{}^2 \end{bmatrix}\begin{Bmatrix} \sum\limits_{i=1}^{N} x_i y_i \\ \sum\limits_{i=1}^{N} y_i \end{Bmatrix}$$

$$= \begin{Bmatrix} \dfrac{N\cdot\sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N\cdot\sum_{i=1}^{N} x_i{}^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i} \\ \dfrac{-\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i + \sum_{i=1}^{N} x_i{}^2 \cdot \sum_{i=1}^{N} y_i}{N\cdot\sum_{i=1}^{N} x_i{}^2 - \sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i} \end{Bmatrix}$$

$$
= \left\{ \begin{array}{c} \dfrac{\sum_{i=1}^{N} x_i y_i - \dfrac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N}}{\sum_{i=1}^{N} {x_i}^2 - \dfrac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}} \\ \dfrac{\dfrac{-\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i}{N} + \dfrac{\sum_{i=1}^{N} {x_i}^2 \cdot \sum_{i=1}^{N} y_i}{N}}{\sum_{i=1}^{N} {x_i}^2 - \dfrac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}} \end{array} \right\}
$$

Start with *a*:

$$
a = \frac{\sum_{i=1}^{N} x_i y_i - \dfrac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N}}{\sum_{i=1}^{N} {x_i}^2 - \dfrac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}
$$

Simplify the numerator of *a*:

$$
\begin{aligned}
a_{numerator} &= \sum_{i=1}^{N} x_i y_i - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N} \\
&= \sum_{i=1}^{N} x_i y_i - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N} - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N} + \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N} \\
&= \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \cdot \frac{\sum_{i=1}^{N} y_i}{N} - \sum_{i=1}^{N} y_i \cdot \frac{\sum_{i=1}^{N} x_i}{N} + N \cdot \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N \times N} \\
&= \sum_{i=1}^{N} x_i y_i - \sum_{i=1}^{N} x_i \, \bar{y} - \sum_{i=1}^{N} y_i \, \bar{x} + N\bar{x}\bar{y} \\
&= \sum_{i=1}^{N} (x_i y_i - x_i \bar{y} - y_i \bar{x} + \bar{x}\bar{y}) \\
&= \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\end{aligned}
$$

Simplify the denominator of *a*:

$$
\begin{aligned}
a_{denominator} &= \sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N} \\
&= \sum_{i=1}^{N} {x_i}^2 - \sum_{i=1}^{N} x_i \cdot \frac{\sum_{i=1}^{N} x_i}{N} \\
&= \sum_{i=1}^{N} {x_i}^2 - \sum_{i=1}^{N} x_i \bar{x}
\end{aligned}
$$

$$= \sum_{i=1}^{N} {x_i}^2 - 2\sum_{i=1}^{N} x_i\bar{x} + \sum_{i=1}^{N} x_i\bar{x}$$

$$= \sum_{i=1}^{N} {x_i}^2 - 2\sum_{i=1}^{N} x_i\bar{x} + (N\bar{x})\bar{x}$$

$$= \sum_{i=1}^{N} ({x_i}^2 - 2x_i\bar{x} + \bar{x}^2)$$

$$= \sum_{i=1}^{N} (x_i - \bar{x})^2$$

So *a* is obtained:

$$a = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

Then simplify *b*:

$$b = \frac{\frac{-\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i y_i}{N} + \frac{\sum_{i=1}^{N} {x_i}^2 \cdot \sum_{i=1}^{N} y_i}{N}}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \frac{-\frac{\sum_{i=1}^{N} x_i}{N} \cdot \sum_{i=1}^{N} x_i y_i + \sum_{i=1}^{N} {x_i}^2 \cdot \frac{\sum_{i=1}^{N} y_i}{N}}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \frac{-\bar{x} \cdot \sum_{i=1}^{N} x_i y_i + \bar{y} \cdot \sum_{i=1}^{N} {x_i}^2}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \frac{\bar{y} \cdot \sum_{i=1}^{N} {x_i}^2 - \bar{x}\bar{y}\sum_{i=1}^{N} x_i - \bar{x} \cdot \sum_{i=1}^{N} x_i y_i + \bar{x}\bar{y}\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \frac{\left(\bar{y} \cdot \sum_{i=1}^{N} {x_i}^2 - \bar{y}\frac{\sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i}{N}\right) - \left(\bar{x} \cdot \sum_{i=1}^{N} x_i y_i - \bar{x}\frac{\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{N}\right)}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \frac{\bar{y}\left(\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i}{N}\right)}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}} - \frac{\bar{x}\left(\sum_{i=1}^{N} x_i y_i - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{N}\right)}{\sum_{i=1}^{N} {x_i}^2 - \frac{\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}{N}}$$

$$= \bar{y} - a\bar{x}$$

In Eq.5, the solution for *a* is:

$$a = \frac{\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N}(x_i - \bar{x})^2}$$

When

$$\sum_{i=1}^{N}(x_i - \bar{x})^2 = 0$$

the denominator equals to 0.

When $x_1 = x_2 = \cdots = x_N$, then

$$\Rightarrow x_i - \bar{x} = 0$$
$$\Rightarrow (x_i - \bar{x})^2 = 0$$
$$\Rightarrow \sum_{i=1}^{N}(x_i - \bar{x})^2 = 0$$

Therefore, when $x_1 = x_2 = \cdots = x_N$, linear regression will fail to find a solution.

4 Linear Algebraic Systems

The tri-diagonal system is given as:

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 \\ c_2 & a_2 & b_2 & 0 & \cdots & 0 \\ 0 & c_3 & a_3 & b_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & c_{N-1} & a_{N-1} & b_{N-1} \\ 0 & \cdots & 0 & 0 & c_N & a_N \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \\ \vdots \\ Q_N \end{Bmatrix}$$

which is in the form of $[A]\{X\} = \{C\}$.

First augment $[A]$ with $\{C\}$:

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & | & Q_1 \\ c_2 & a_2 & b_2 & 0 & \cdots & 0 & | & Q_2 \\ 0 & c_3 & a_3 & b_3 & \cdots & \vdots & | & Q_3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & | & \vdots \\ 0 & \cdots & 0 & c_{N-1} & a_{N-1} & b_{N-1} & | & \vdots \\ 0 & \cdots & 0 & 0 & c_N & a_N & | & Q_N \end{bmatrix}$$

For $k = 2, 3, \ldots, N+1$:

${a_{ij}}^{(k)} = {a_{ij}}^{(k-1)}$ for $i = k-1$ and $j = k-1\ to\ N+1$,

$$a_1^* = a_1^{(2)} = a_1^{(1)} = a_1$$

$$b_1^* = b_1^{(2)} = b_1^{(1)} = b_1$$

$$Q_1^* = Q_1^{(2)} = Q_1^{(1)} = Q_1$$

$$a_i^{(k)} = a_i^{(k-1)} \text{ for } i = 2, 3, \ldots, N$$

$$b_i^{(k)} = b_i^{(k-1)} \text{ for } i = 2, 3, \ldots, N-1$$

$$Q_i^{(k)} = Q_i^{(k-1)} \text{ for } i = 2, 3, \dots, N$$

For the zeros in the right upper triangle maintain as 0.

${a_{ij}}^{(k)} = 0$ for $i = k \; to \; N$ and $j = k - 1$, so

$$c_i^* = 0 \text{ for } i = 2, 3, \dots, N$$

$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \left(\frac{a_{i,k-1}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}}\right) a_{k-1,j}^{(k-1)}$ for $i = k \; to \; N$ and $j = k \; to \; N+1$, so

$$a_i^* = a_i - \left(\frac{c_i}{a_{i-1}^*}\right) b_{i-1} = a_i - c_i b_{i-1}/a_{i-1}^*$$

$$b_i^* = b_i - \left(\frac{c_i}{a_{i-1}}\right) \times 0 = b_i$$

$$Q_i^* = Q_i - \left(\frac{c_i}{a_{i-1}}\right) Q_{i-1}^* = Q_i - c_i Q_{i-1}^*/a_{i-1}^*$$

For the zeros, if $a_{ij}^{(k)} = 0$, it always has $a_{i,k-1}^{(k-1)} = 0$ or $a_{k-1,j}^{(k-1)} = 0$, so zeros maintain.

This change for each $a_i, b_i \; and \; Q_i$ will happen only once when it is the *i*^th^ elimination. And in the *i+1*^th^ elimination they will maintain because $a_i^{(k)} = a_i^{(k-1)}$ for $i = 2, 3, \dots, N$.

Now $a_i, b_i \; and \; Q_i$ are obtained:

$$a_i^* = \begin{cases} a_i, & i = 1 \\ a_i - c_i b_{i-1}/a_{i-1}^*, & i = 2, 3, \dots, N \end{cases}$$

$$b_i^* = b_i, \qquad i = 1, 2, 3, \dots, N$$

$$Q_i^* = \begin{cases} Q_i, & i = 1 \\ Q_i - c_i Q_{i-1}^*/a_{i-1}^*, & i = 2, 3, \dots, N \end{cases}$$

So the matrix can be rewritten as:

$$\begin{bmatrix} a_1^* & b_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2^* & b_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3^* & b_3 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & a_{N-1}^* & b_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & a_N^* \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_N \end{Bmatrix} = \begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \\ \vdots \\ \vdots \\ Q_N^* \end{Bmatrix}$$

Start with x_N:

$$x_N = \frac{a_{N,N+1}^{(N)}}{a_{N,N}^{(N)}} = \frac{Q_N^*}{a_N^*}$$

For $i = N-1, N-2, N-3, \dots, 2, 1$, the following equation can be derived:

$$a_i^* x_i + b_i x_{i+1} = Q_i^*$$

$$\Longrightarrow x_i = (Q_i^* - b_i x_{i+1})/a_i^*$$

So the solution to the original tri-diagonal matrix can be written as:

$$x_i = \begin{cases} Q_i^*/a_i^*, & i = N \\ (Q_i^* - b_i x_{i+1})/a_i^*, & i = N-1, N-2, \dots, 1 \end{cases}$$

5 Interpolation

Plot the data provided in the file in_interp.csv.

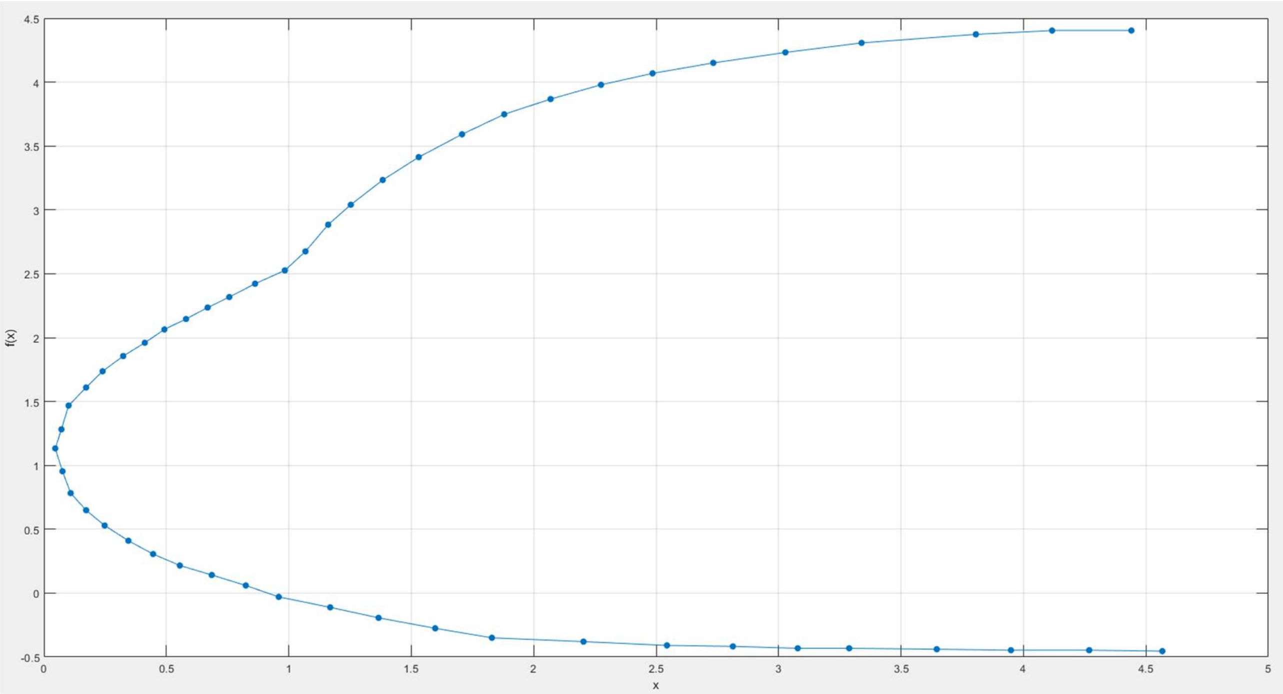


Figure 6: $x - f(x)$ diagram for provided data

Then plot the cubic spline interpolation function:

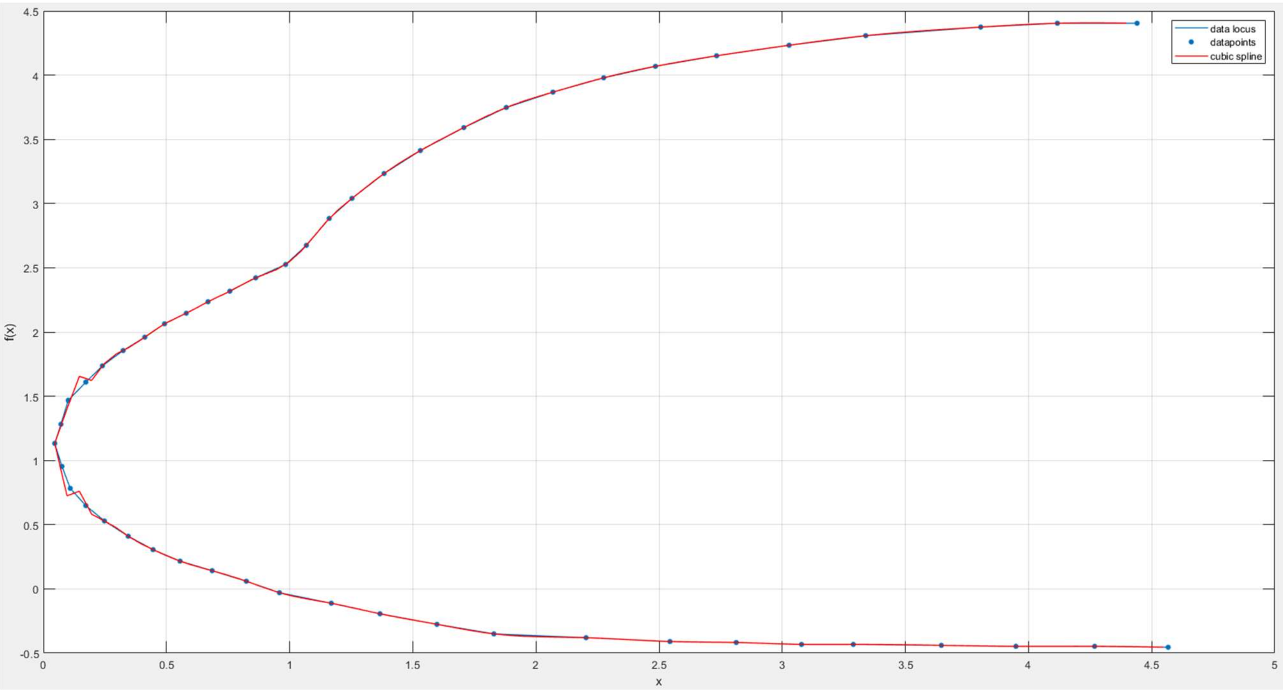


Figure 7: cubic spline interpolation comparing with the actual datapoints

Figure 7 shows that cubic spline interpolation tracks the actual datapoints well, except for the places where the absolute of the data locus' derivative is large (as shown in the left most of the diagram). The locus of the interpolation function is smoother, because the first and double derivative of the function is continuous the interior nodes.

6 Differentiation, differential equations

For 1st-order upwind finite-difference approximation, $N_x = 80 \Longrightarrow \Delta x = \frac{1-0}{N_x} = 0.0125$. The following three diagrams demonstrate the results with different CFL number (1, 0.75, 0.25) for the time-levels $t = 0.05, 0.1, 0.15, 0.2$ and the exact solution for comparison.

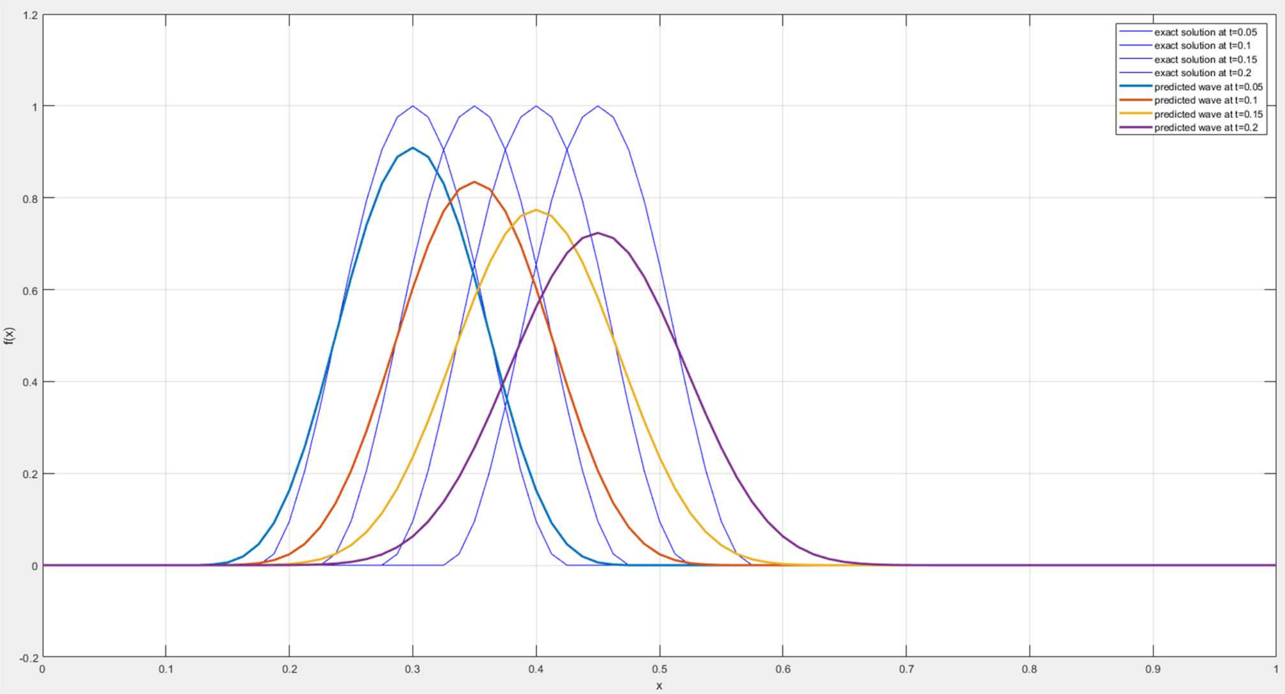


Figure 8: 1st-order upwind, $N_x = 80$, CFL = 1

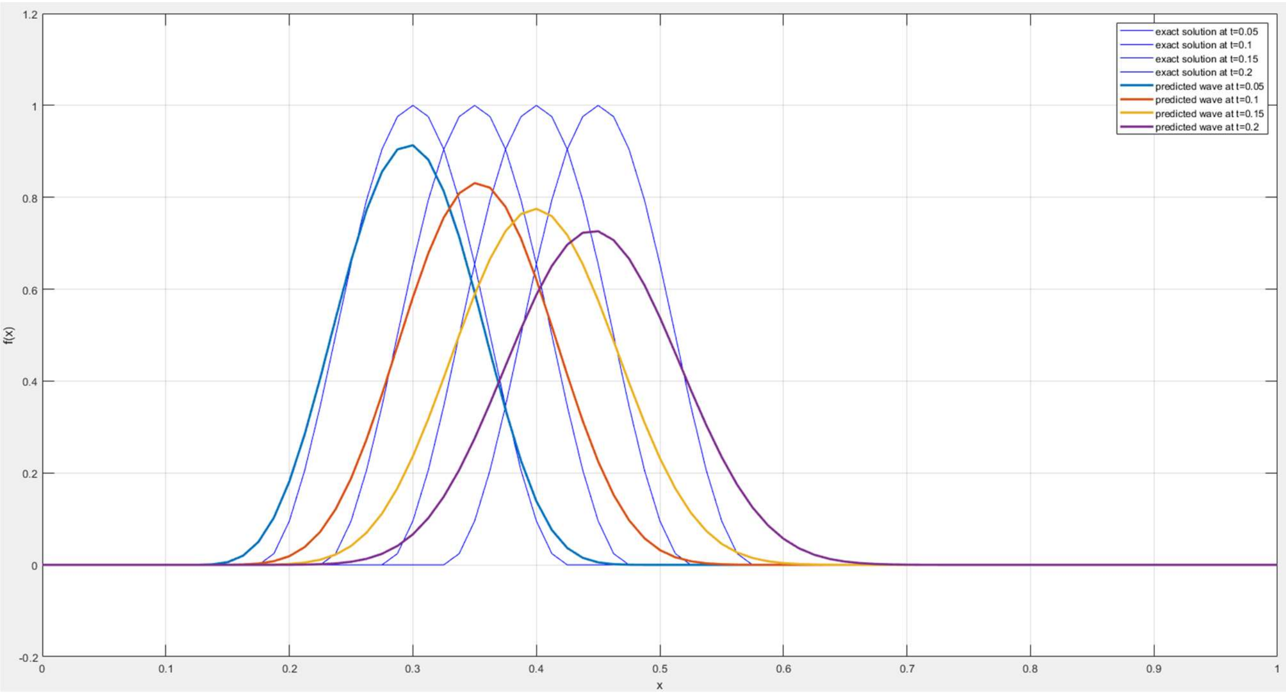


Figure 9: 1st-order upwind, $N_x = 80$, CFL = 0.75

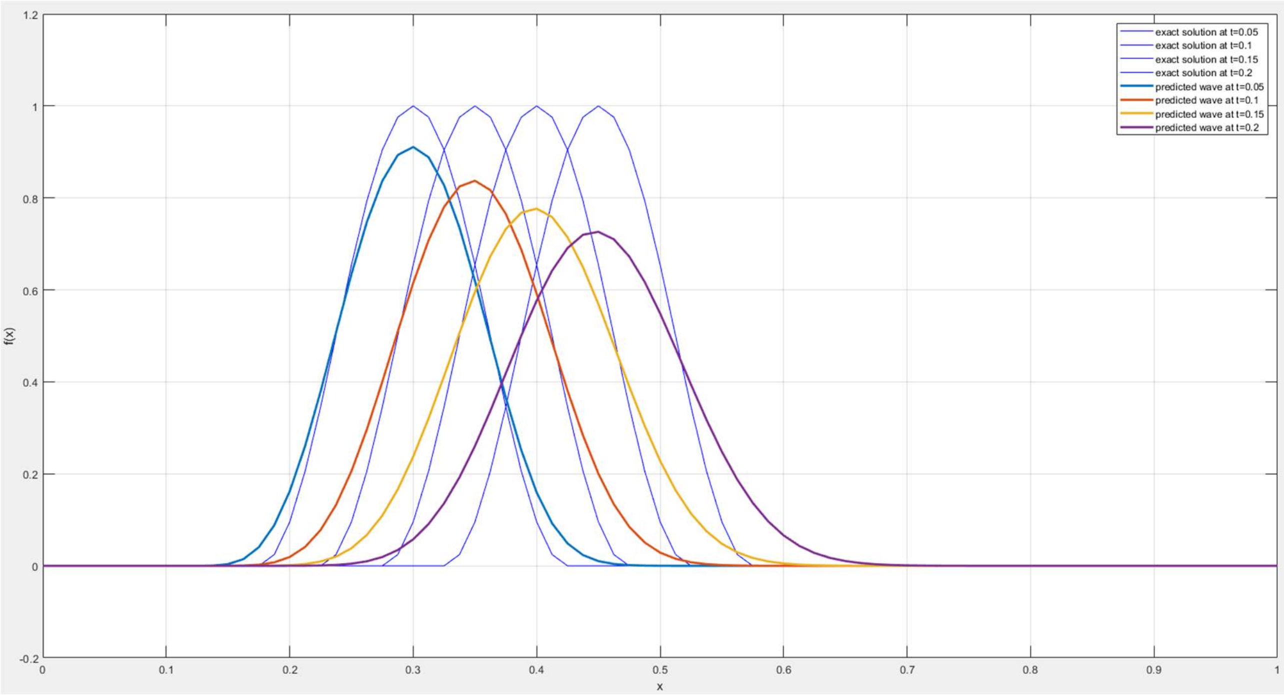


Figure 10: 1st-order upwind, $N_x = 80$, CFL = 0.25

From Figure 8, 9 and 10, we can see that the amplitude of the wave obtained by 1st -order upwind finite-difference approximation decreases as t increases. It lost about 0.28 of amplitude when $t = 0.2$ comparing with the initial condition whose amplitude is roughly 1.

In addition, the wave period increases as t increases. When $t = 0.2$, the wave period is 0.7-0.2=0.5 which is double as the initial condition (0.375-0.125=0.25). Increasing wave period means the frequency decreases, which means losing energy physically.

Plot the result of 1st-order upwind finite-difference approximation for $N_x = 200 \Longrightarrow \Delta x = 1/N_x = 0.005$:

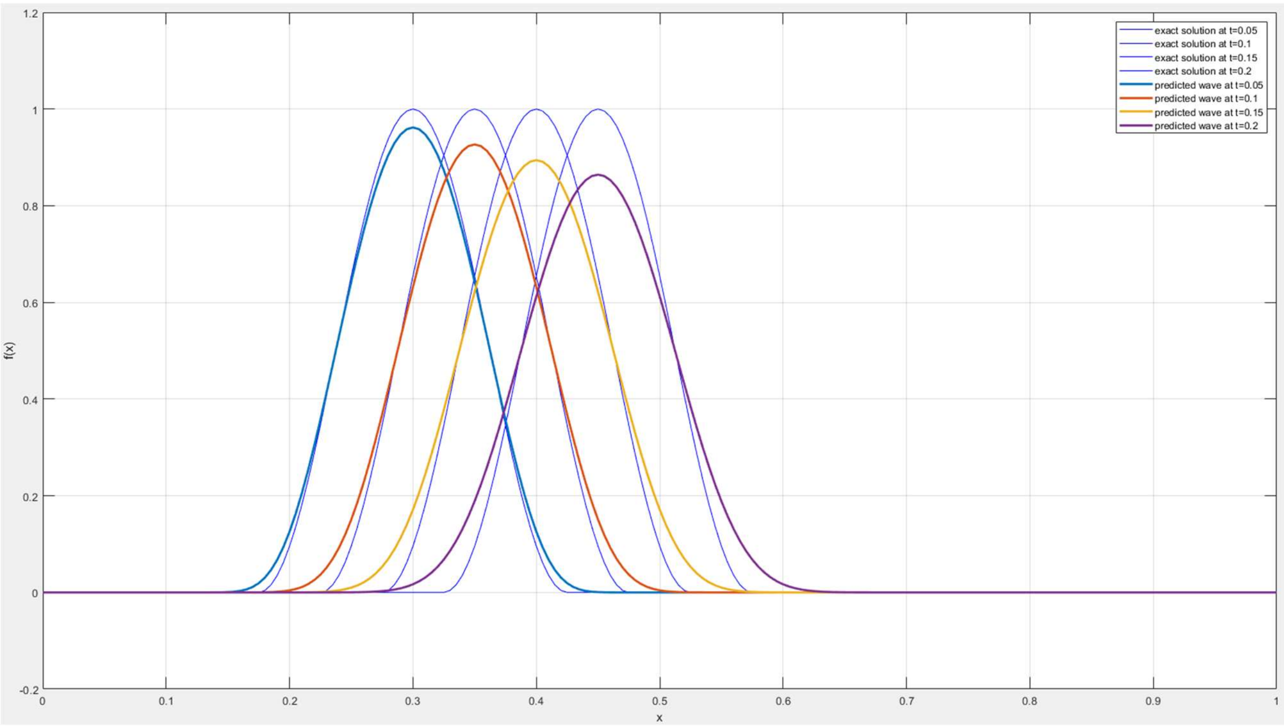


Figure 11: 1st-order upwind, $N_x = 200$, CFL = 1

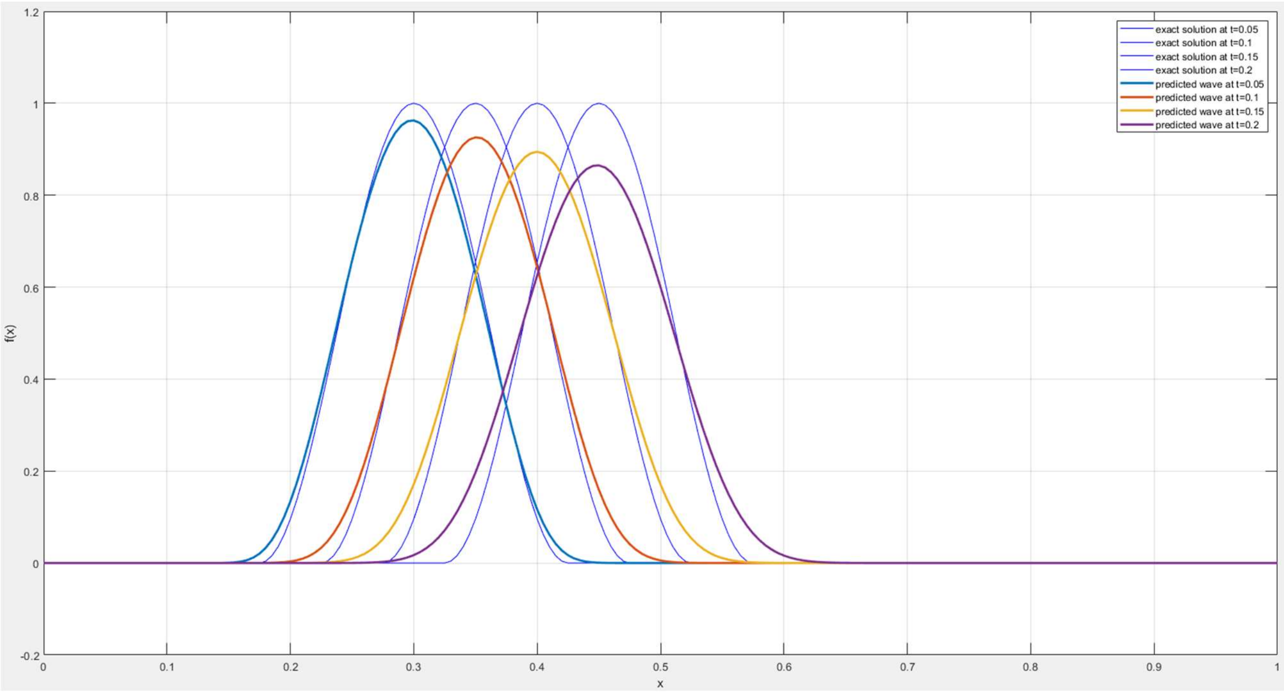


Figure 12: 1st-order upwind, $N_x = 200$, CFL = 0.75

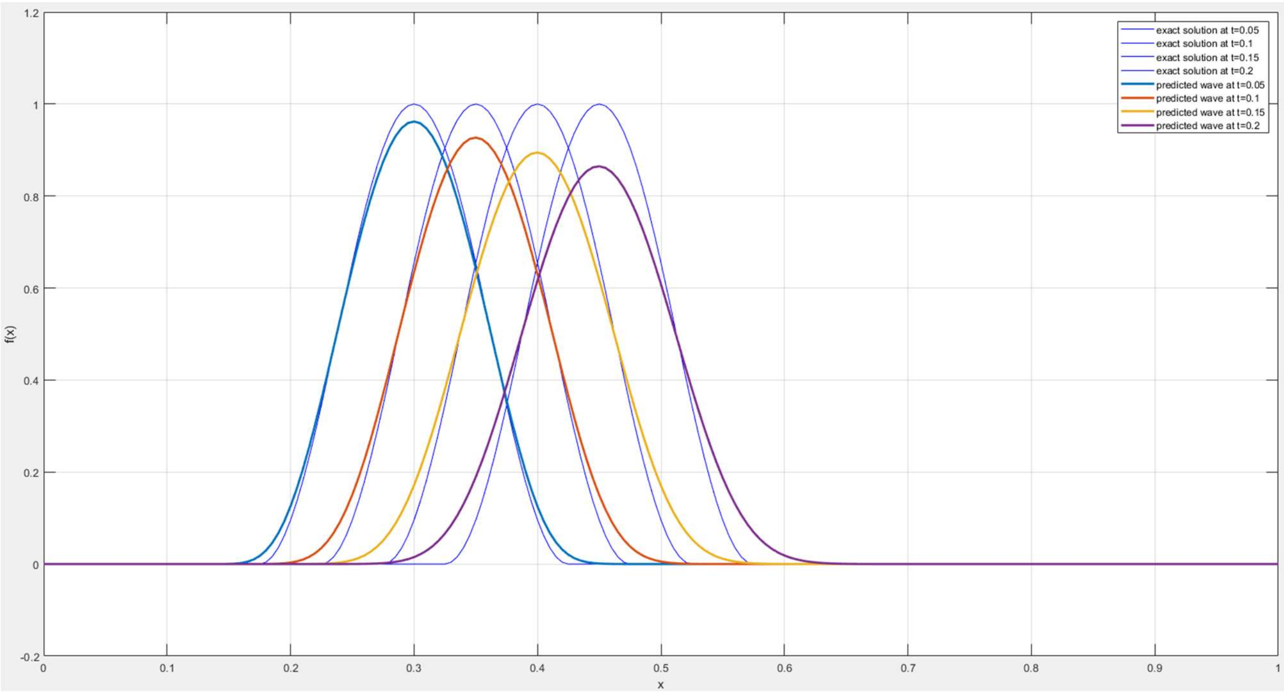


Figure 13: 1st-order upwind, $N_x = 200$, CFL = 0.25

***Δx* :** From Figure 11, 12, 13, we can see that the prediction for $N_x = 200$ and $t = 0.2$ has a smaller amplitude loss which is roughly 0.14, and a smaller wave period (0.63-0.26=0.37), comparing with the approximation using $N_x = 80$. And the overall locus is smoother because of the smaller Δx. Therefore, using a smaller Δx i.e. larger N_x, for approximation could get a better agreement with the exact solution.

CFL: In order to see how CFL affects the prediction wave, plot the solution at time-level $t = 0.15$ (because $t = 0.15$ can be always divided by Δt of CFL = 0.25, 0.75 and 1, so time-level will be accurate) for different CFL, for both $N_x = 80$ and 200. Magnify the circled area of Figure 14 to get Figure 15 and 16.

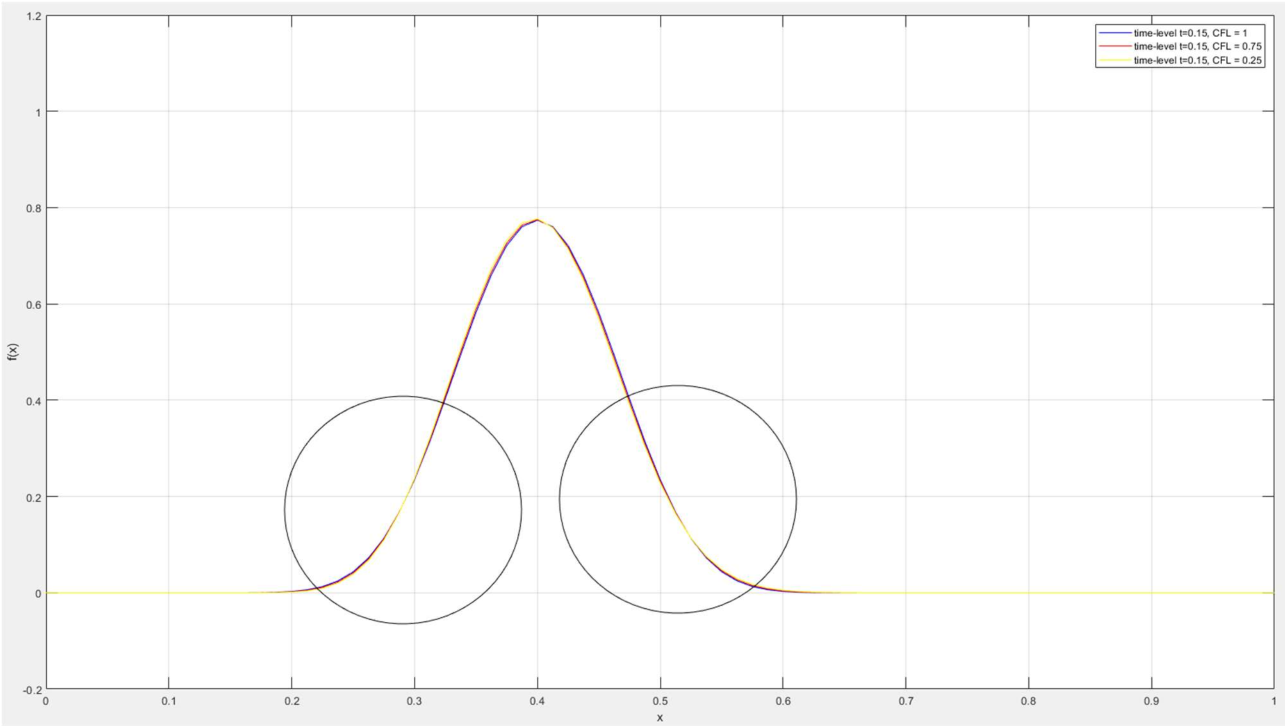


Figure 14: 1st-order upwind, $N_x = 80$, CFL = 1, 0.75, 0.25

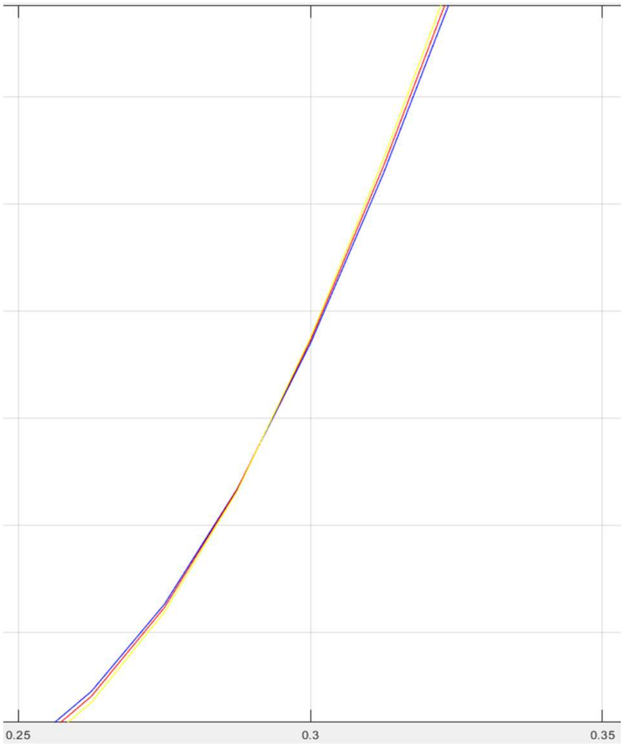


Figure 15: rising area (1st-order, $N_x = 80$)

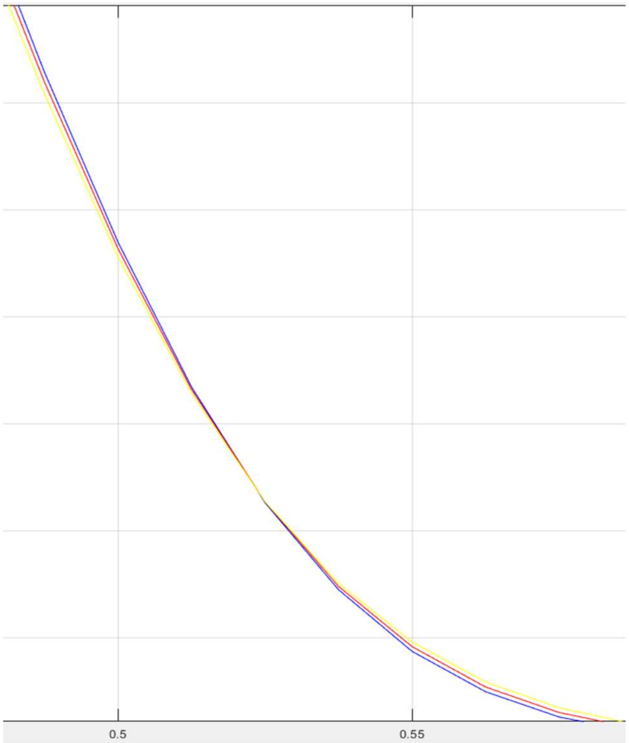


Figure 16: decreasing area (1st-order, $N_x = 80$)

When the wave is rising, in Figure 15, the yellow locus (CFL=0.25) is below the red (CFL=0.75) and the blue (CFL=1) locus. But after a certain x (roughly 0.29), the yellow locus becomes above the other two, which means the predicted wave with a small CFL climbs faster than the one with a large CFL. In Figure 16 the yellow locus is below the other two at first and then become above as x increases, which means the predicted wave with a small CFL decreases slower than the one with a large CFL.

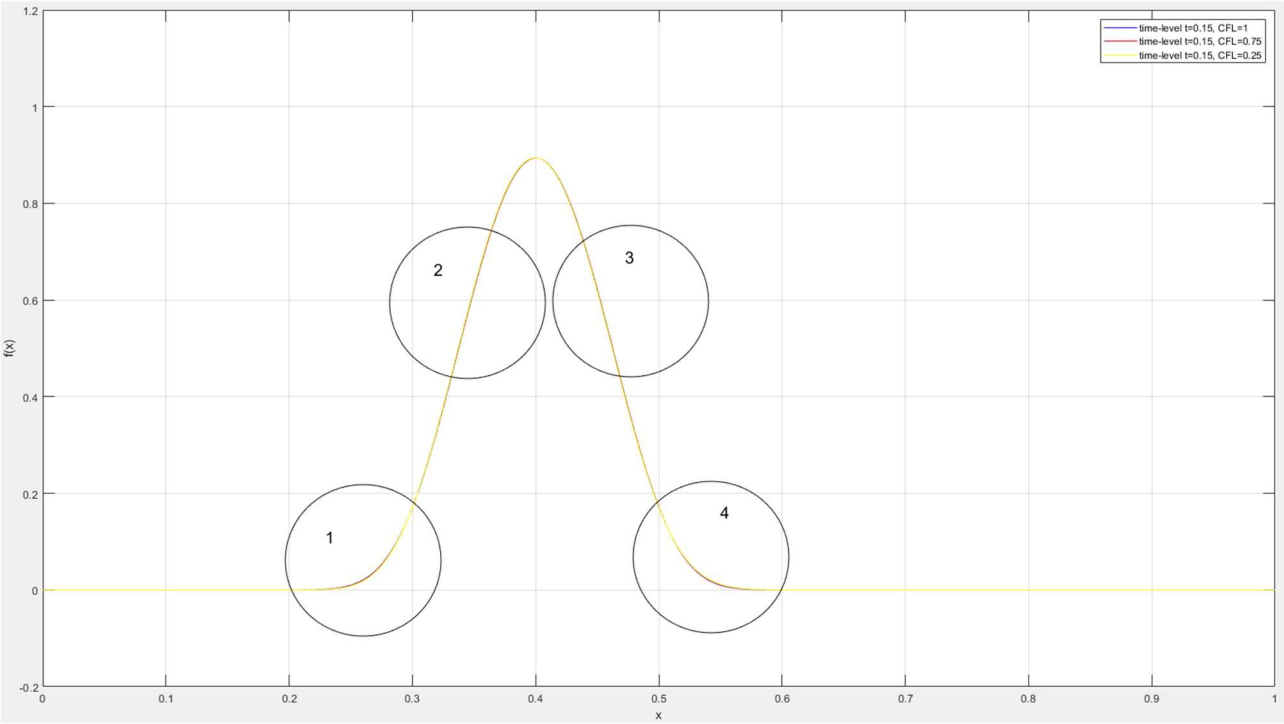


Figure 17: 1st-order upwind, $N_x = 200$, CFL = 1, 0.75, 0.25

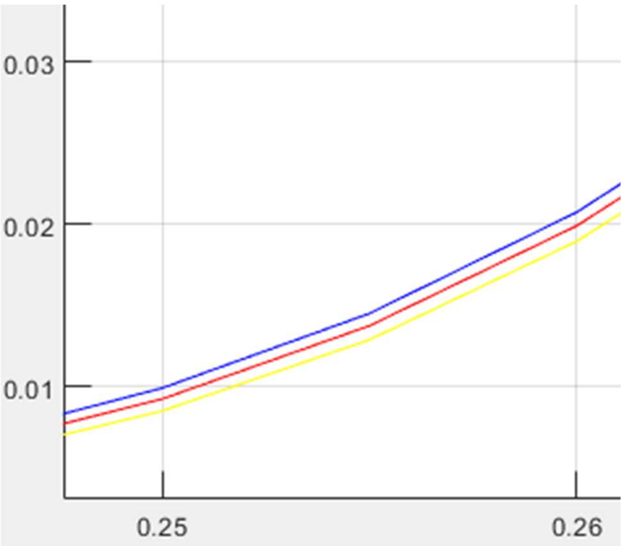


Figure 18: rising area 1(1st-order, $N_x = 200$)

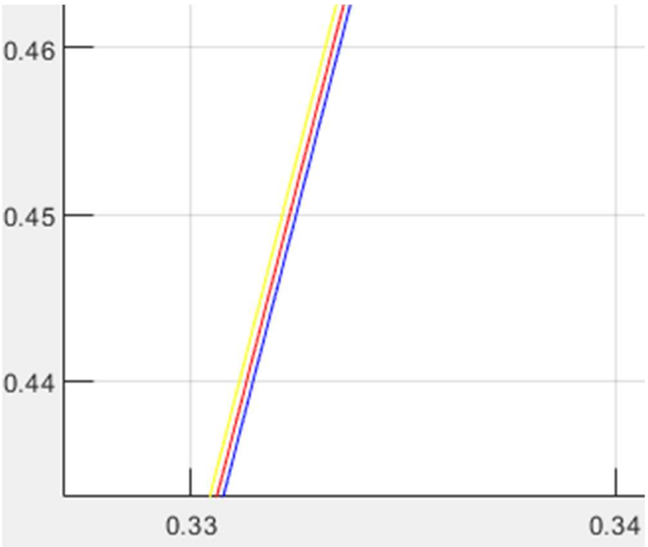
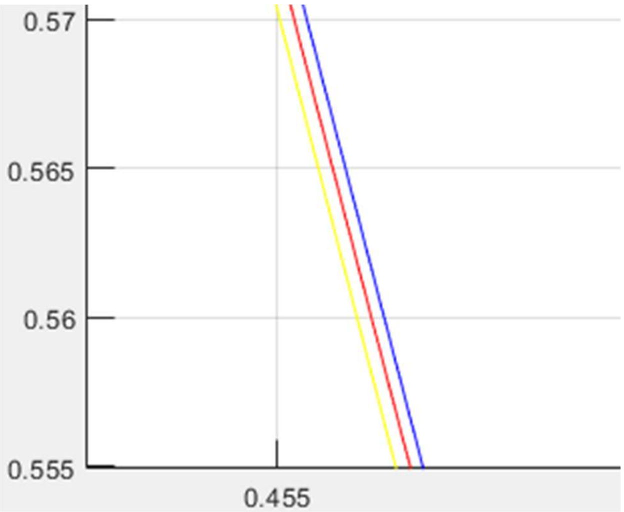


Figure 19: rising area 2(1st-order, $N_x = 200$)

The figures for $N_x = 200$ with different CFL also approve that small CFL results in faster rising and slower decreasing of the predicted wave. I also find that a smaller Δx makes the predicted waves of different CFL more fitting to each other, that's why I have to do higher-times magnification in order to see the slight difference in Figure 17.



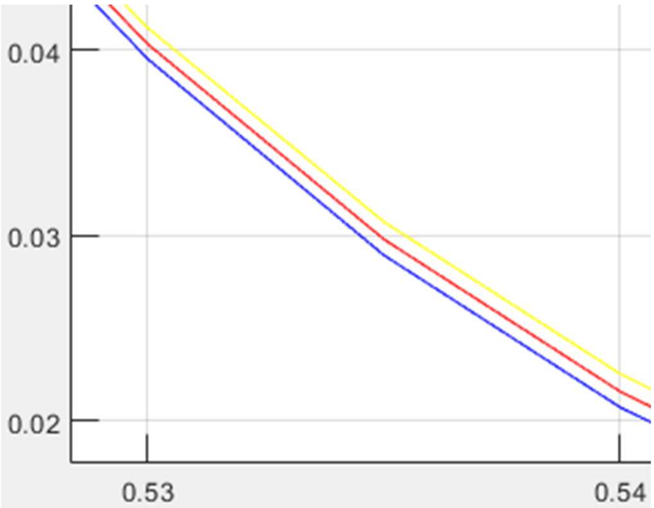


Figure 20: decreasing area 3(1st-order, $N_x = 200$) Figure 21: decreasing area 4(1st-order, $N_x = 200$)

Then plot the result of 2nd-order central finite-difference approximation:

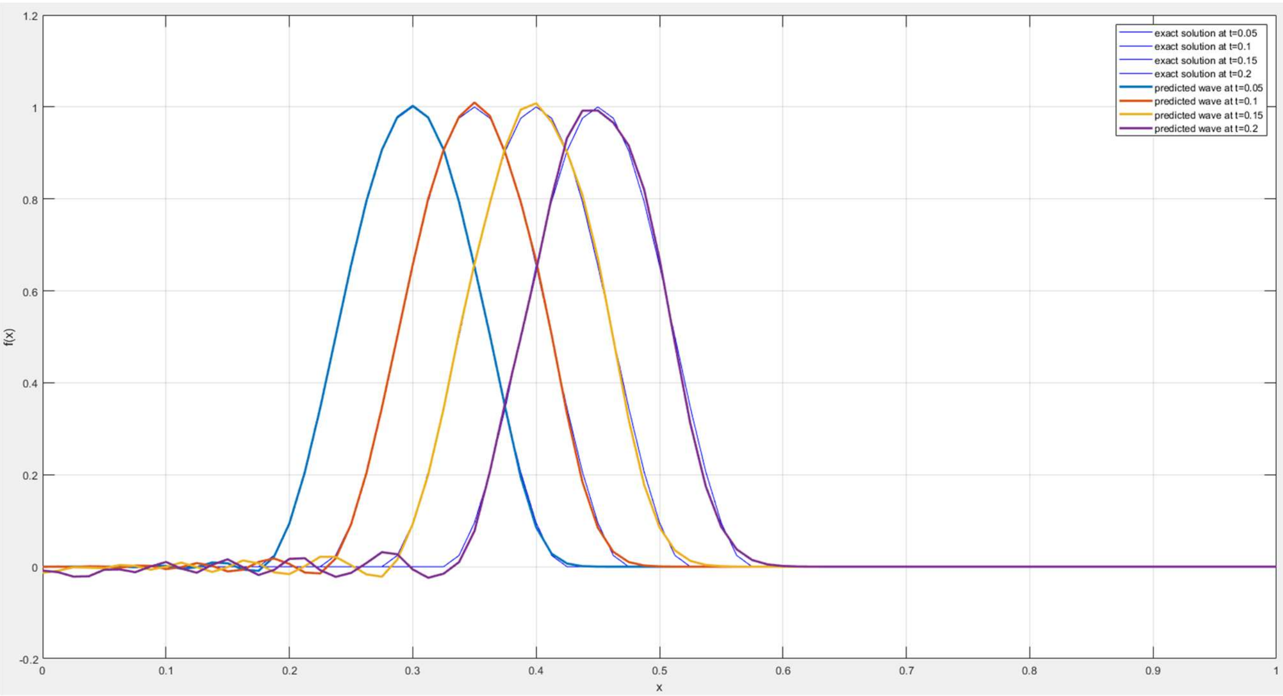


Figure 22: 2nd-order central, $N_x = 80$, CFL = 1

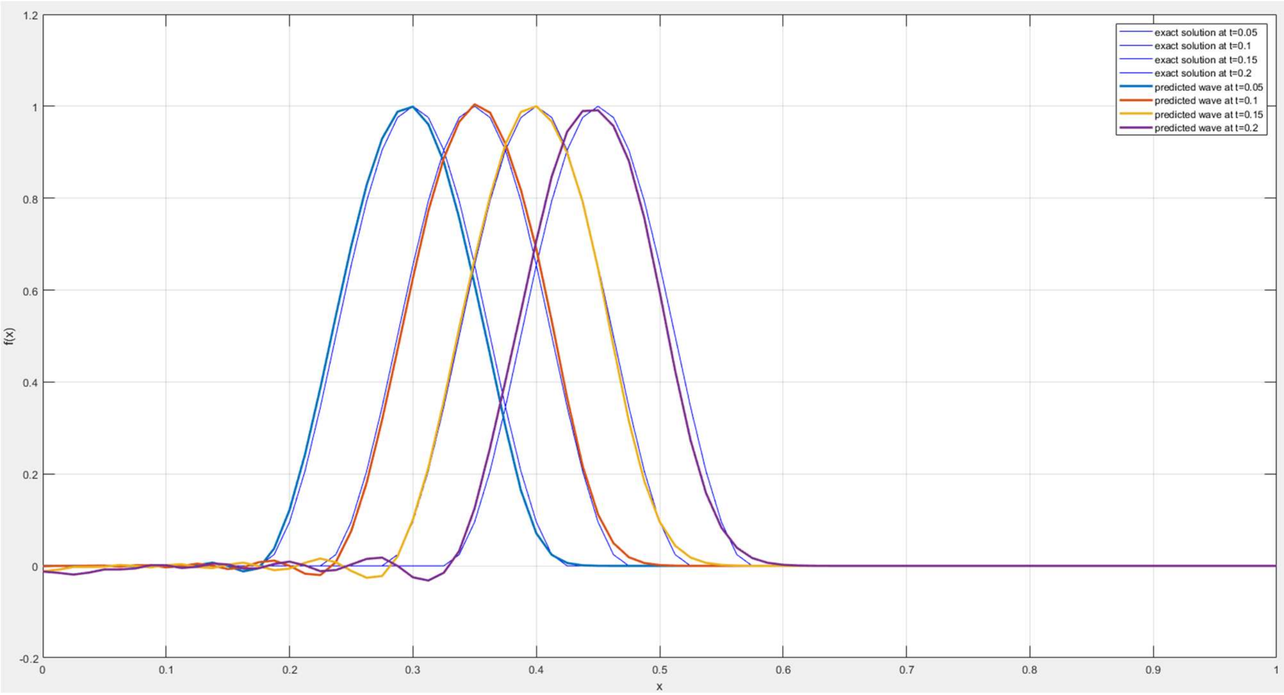


Figure 23: 2nd-order central, $N_x = 80$, CFL = 0.75

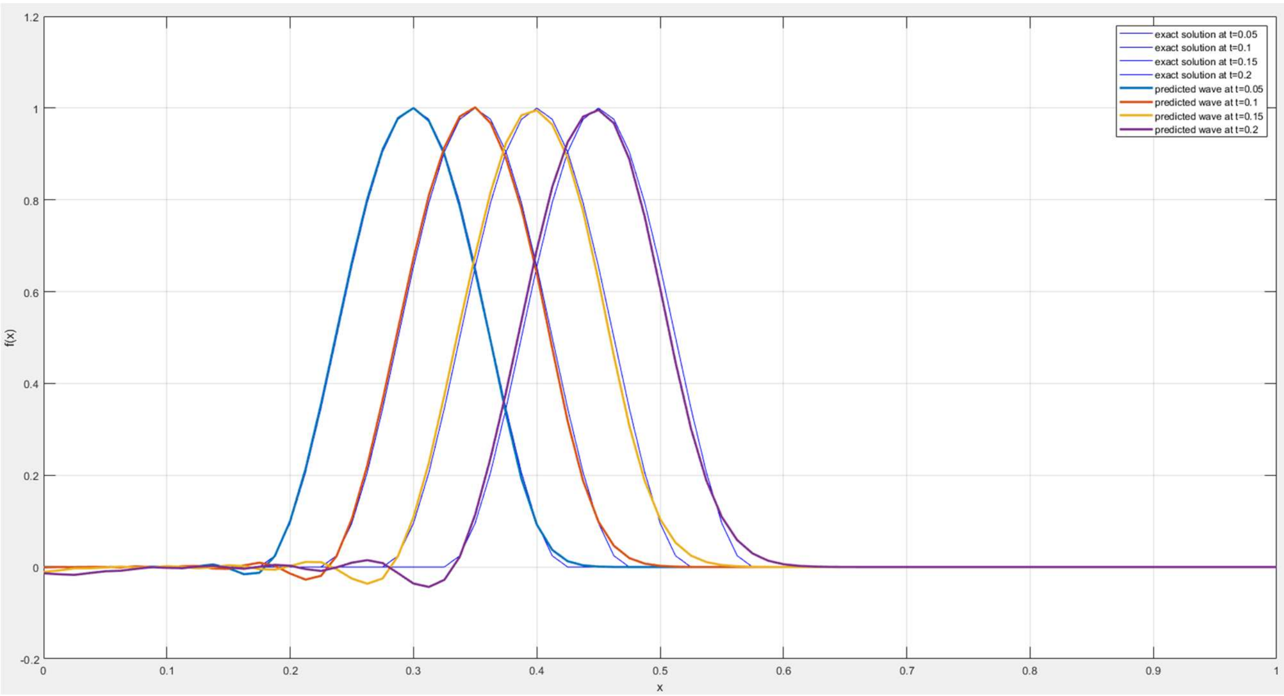


Figure 24: 2nd-order central, $N_x = 80$, CFL = 0.25

For $N_x = 80$, from Figure 22, 23 and 24, we can see that comparing with 1st -order upwind finite-difference approximation, there is no or just a very small amplitude decrease in 2nd-order central finite-difference approximation. And the wave period is almost the same as the initial condition, just a little bit larger than the exact solution. But unlike 1st -order upwind, there are some fluctuations before the predicted wave. The fluctuations become more drastic as t increases. As for CFL, as it decreases, which means a smaller Δt, the fluctuations have a lower frequency but more drastic, and the wave period becomes larger, as we can see that the predicted wave is not tracking the exact solution well from where f(x) is close to 0.

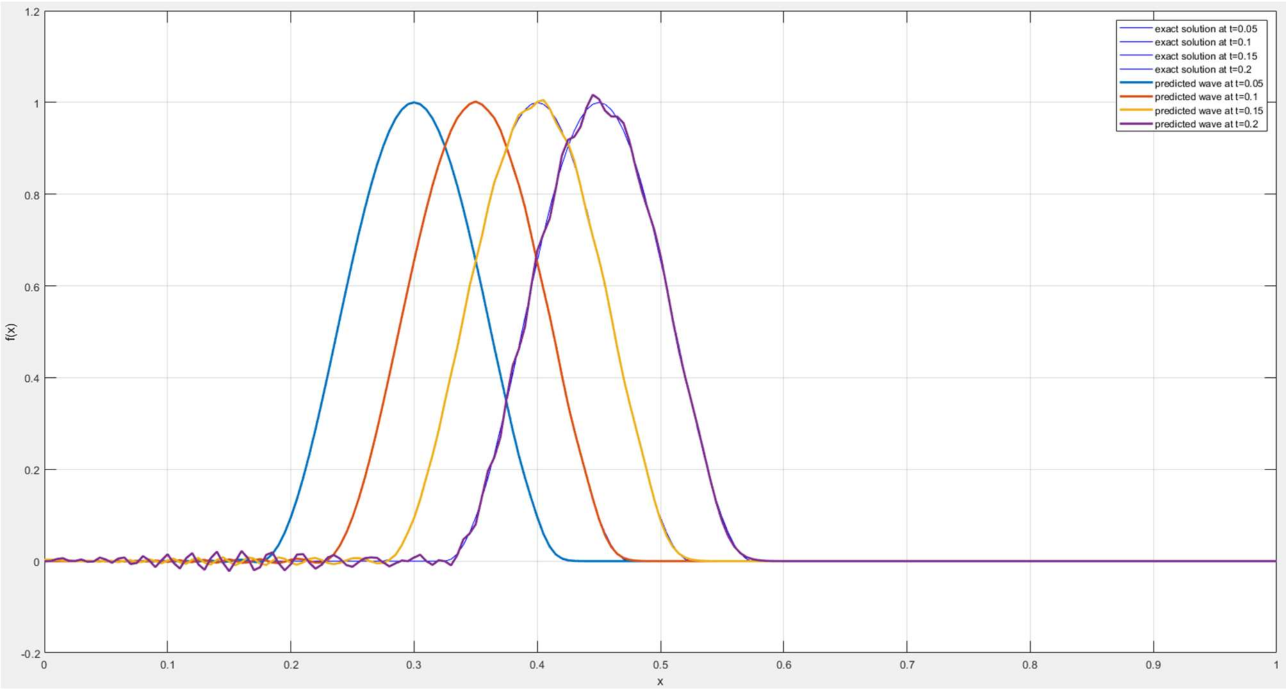


Figure 25: 2nd-order central, $N_x = 200$, CFL = 1

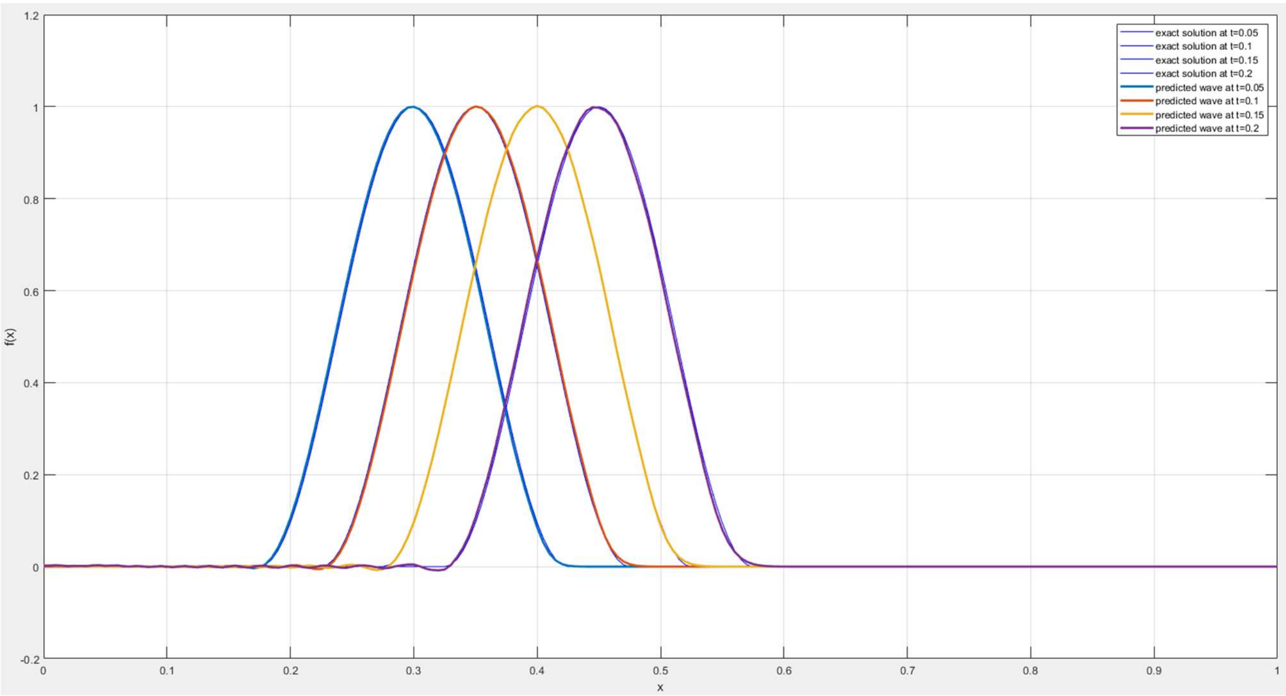


Figure 26: 2nd-order central, $N_x = 200$, CFL = 0.75

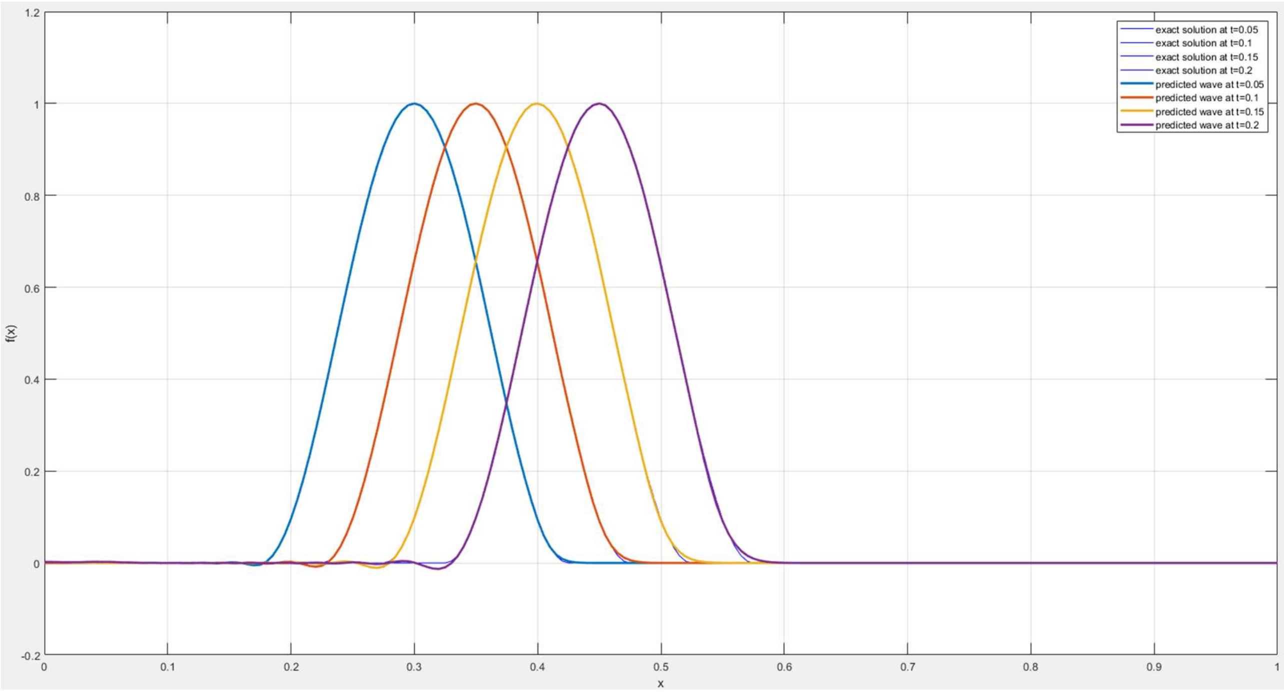


Figure 27: 2nd-order central, $N_x = 200$, CFL = 0.25

For $N_x = 200$, from Figure 26 and 27, the predicted locus is smoother and the fluctuations before the wave is slighter comparing with the one for $N_x = 80$. As for CFL, as it decreases, the frequency of the fluctuation becomes lower, the overall tracking of the exact solution is excellent but we can see some diversion when the wave starts to increase and when the wave decreases to 0.

However, from Figure 25 where CFL=1, the wave for $t = 0.15, 0.2$ is not stable. There are fluctuations with high frequency before the wave and during the wave. Therefore, I think small CFL results in lower fluctuation frequency, better tracking at most of time but larger error when the wave starts to rise from 0 and decreases to 0, while larger CFL results in small errors for all the time and less stability, but it tracks better when the wave starts to rise and decreases to 0 since the fluctuation frequency is high so it can adjust quickly.

Then plot the result for CFL>1. From Figure 29, we can see that the predicted wave becomes less stable with a CFL number larger than 1, though this is not obvious for the result of 1st-order upwind method in Figure 28.

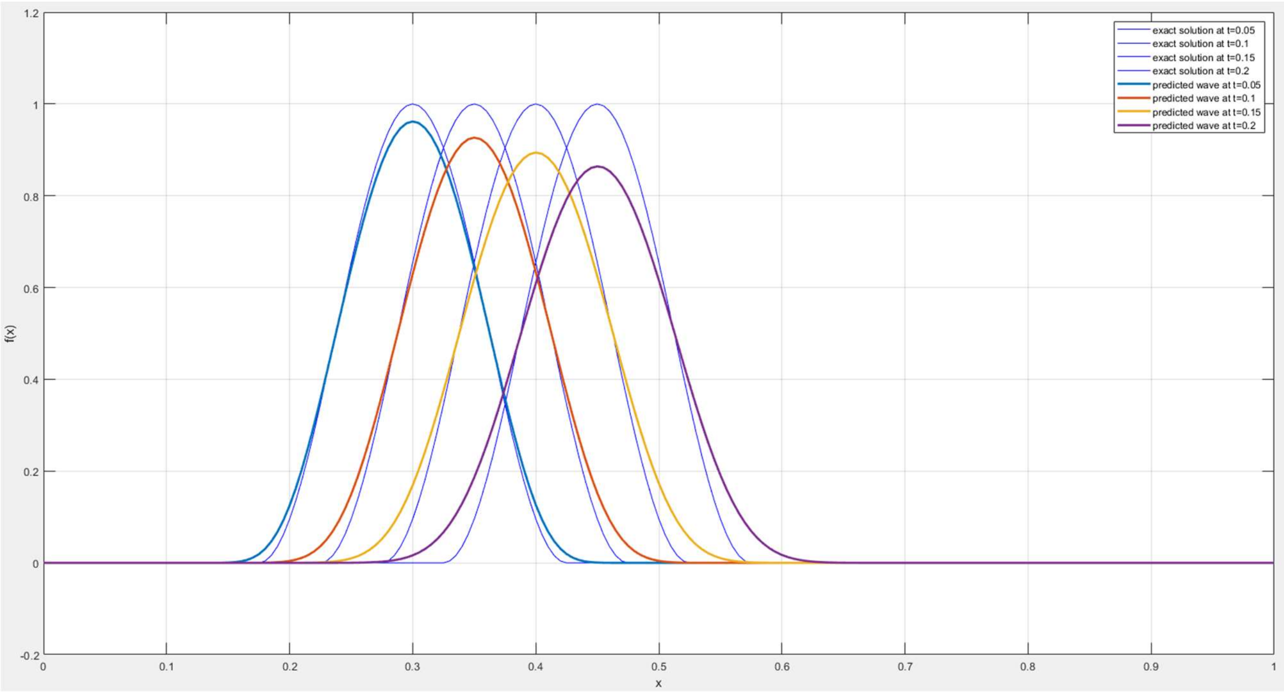


Figure 28: 1st-order upwind, $N_x = 200$, CFL = 1.002

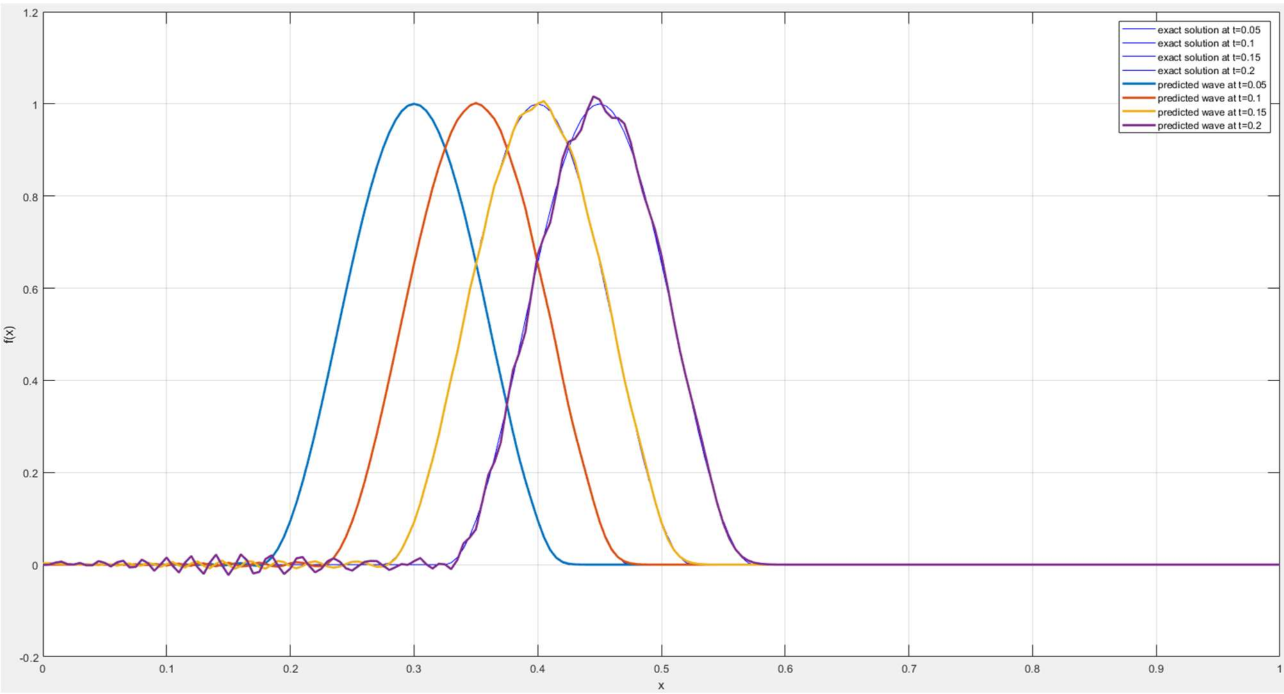


Figure 29: 2nd-order central, $N_x = 200$, CFL = 1.002

In order to see what happens when CFL>1, plot the result for CFL=1.1 (since the diagram for CFL=1.002 does not show much differences to the diagram for CFL=1). From Figure 30 and 31, the locus of the predicted wave for $t = 0.2$ obtained by either method is no longer smooth. There are apparent fluctuations in the locus of either method, while the fluctuations of 2nd-order central method is more drastic.

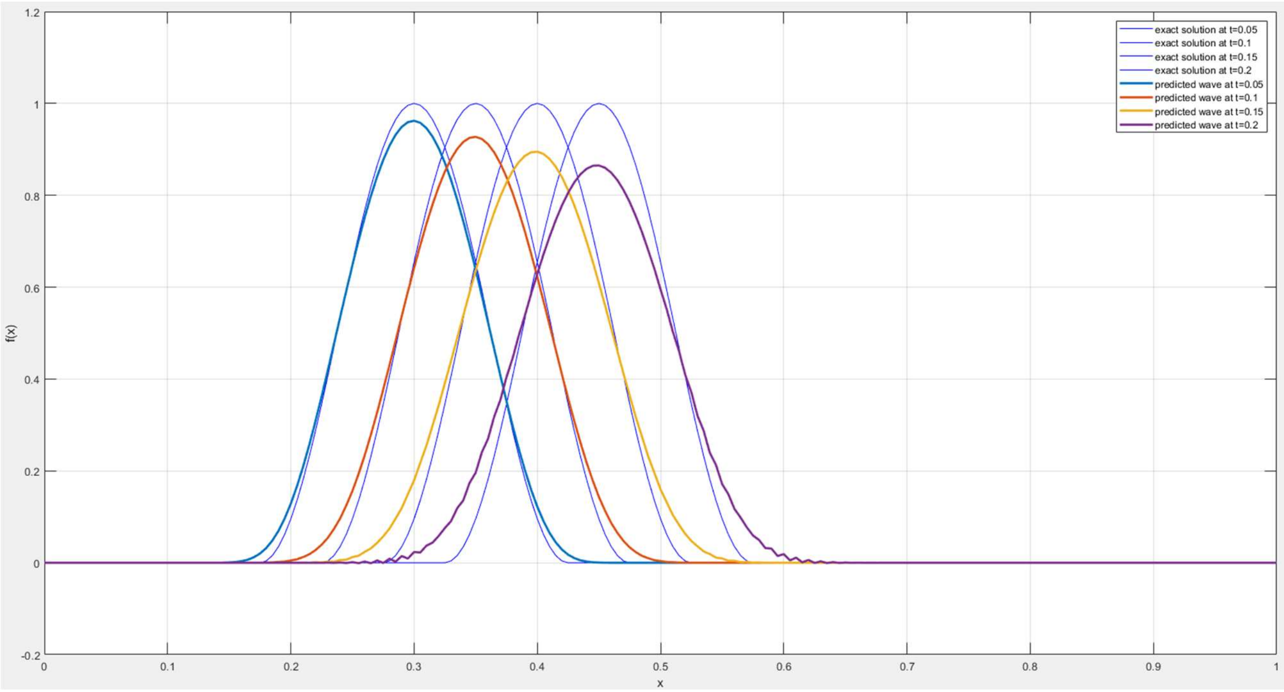


Figure 30: 1st-order upwind, $N_x = 200$, CFL = 1.1

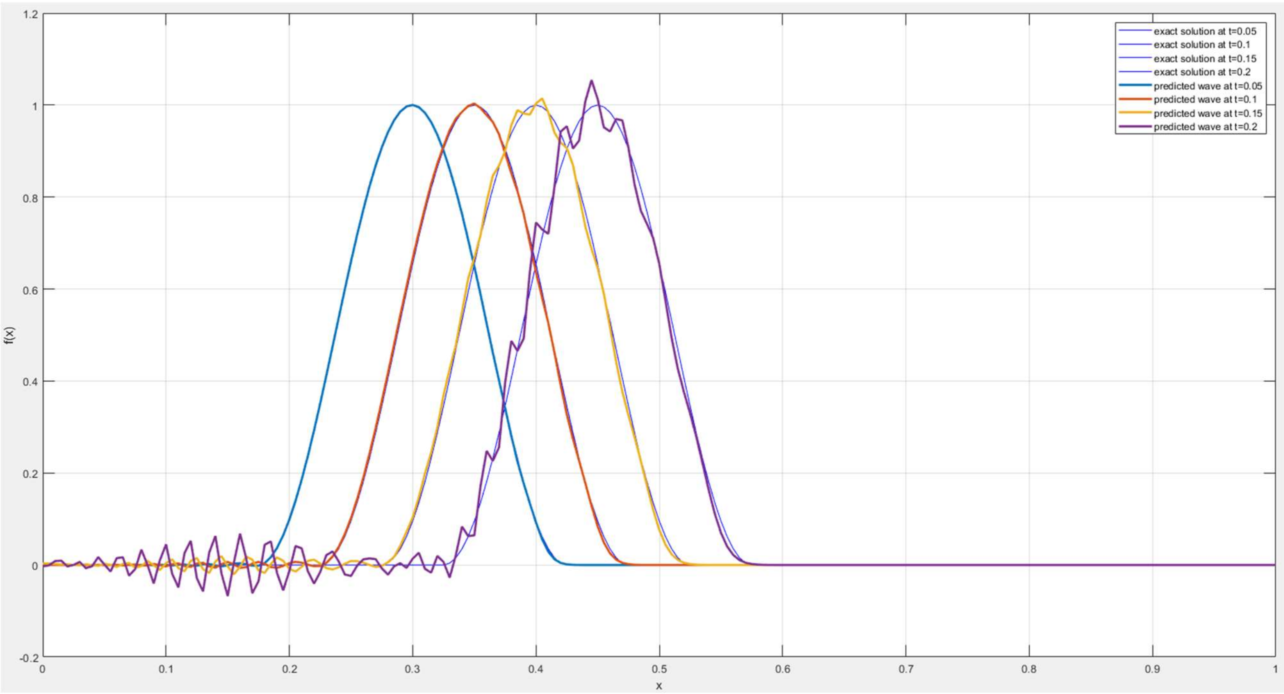


Figure 31: 2nd-order central, $N_x = 200$, CFL = 1.1

Then I plot the result for CFL = 1.25, which is further from 1. In Figure 32 and 33, the locus of the predicted wave no longer converges. The locus of prediction using 1st-order upwind method diverges to an amplitude of roughly 28. The locus of 2nd-order central method also diverges, but much more slightly than 1st-order upwind method, to an amplitude of roughly 1.3.

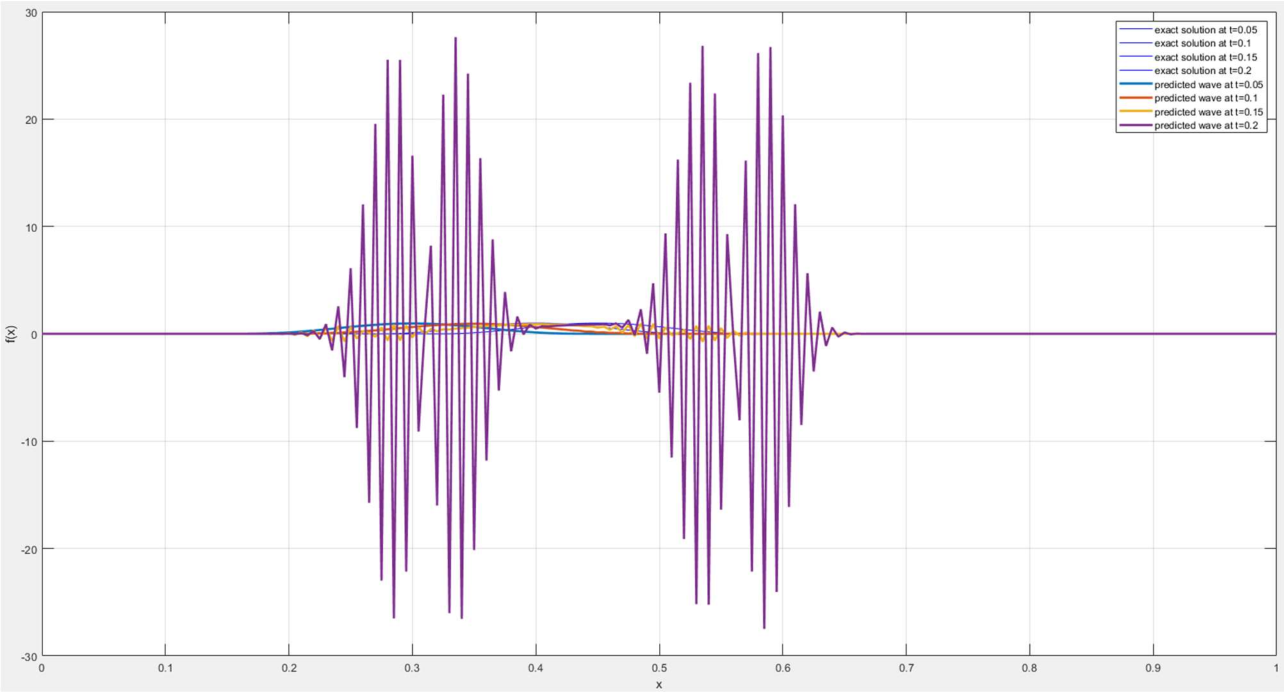


Figure 32: 1st-order upwind, $N_x = 200$, CFL = 1.25

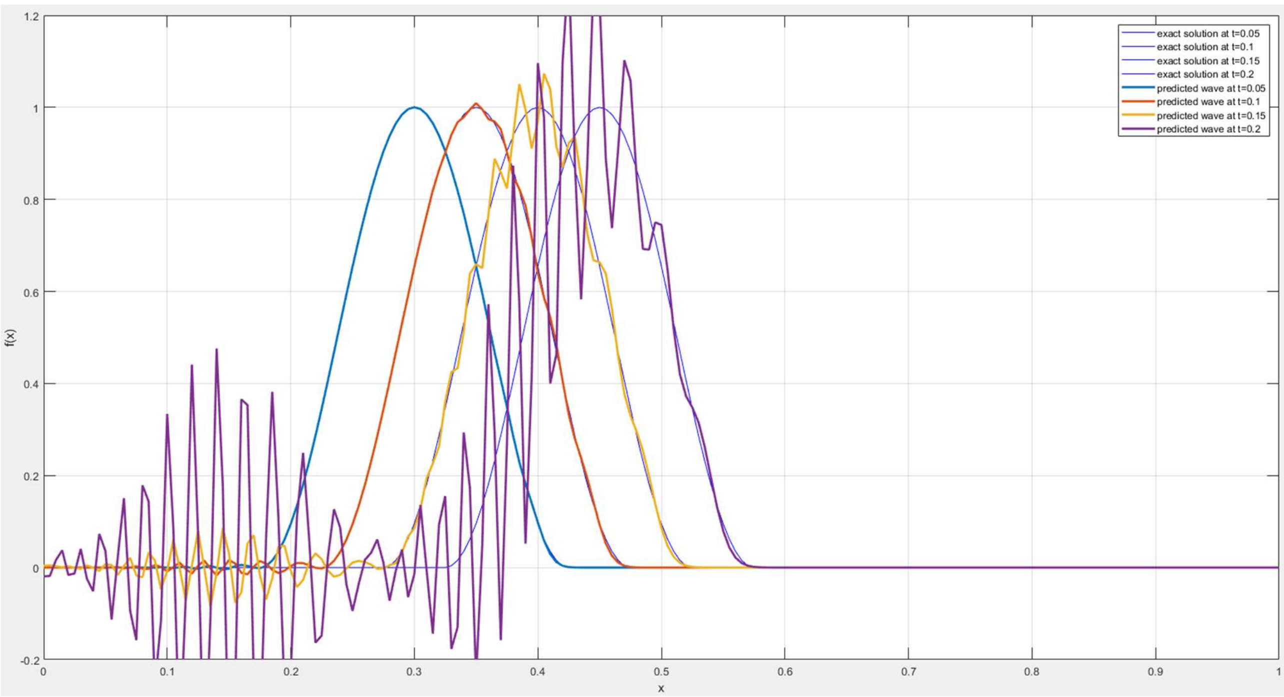


Figure 33: 2nd-order central, $N_x = 200$, CFL = 1.25

Conclusions

1. The agreement of the numerical prediction with the exact solution is good when using a small Δx. Smaller Δx gives better tracking of the exact solution and smoother locus.
2. The agreement of the numerical prediction with the exact solution is excellent with a small CFL which means a small Δt. Smaller CFL leads to slighter fluctuation. Larger CFL can track the exact solution well when the wave starts to rise and when it decreases to 0, but it leads to less stability.
3. The locus of the predicted wave no longer converges when CFL>1. It may diverge to a certain extent according to how far away it is from 1. The larger CFL leads to a quicker divergence.