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FORECAST THE USA STOCK INDICES WITH GARCH-TYPE MODELS

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Many thanks to my supervisor Johan Lyhagen.

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Abstract

GARCH-type models have been highly developed since Engle [1982] presented ARCH process 30 years ago. Different kinds of GARCH-type models are applicable to different kinds of research purposes. As documented by many literatures that short-memory processes with level shifts will exhibit properties that make standard tools conclude long-memory is present. Therefore, in this paper, we want to forecast with GARCH-type models and consider structural breaks and the long-memory characteristic.

We analyze structural breaks and use the FIGARCH [Baillie et al., 1996] model comparing with GARCH [Bollerslev, 1986] model and EGARCH [Nelson, 1991] model to forecast the conditional variance process of three USA stock indices: Dow Jones Industrials Average (DJIA) index, Standard & Poor 500 (S&P 500) index and NASDAQ Composite (NASDAQ) index by using different in-sample size, different error distributions and forecasting different steps. We find the FIGARCH model is sensitive to the changes of conditions, and forecast better than the other two GARCH-type models.

Keywords: Structural breaks; Long-memory; Stock Indices; Forecast; FIGARCH.

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Contents

1 Introduction

Economic time series exhibit unique characteristics and they are non-normal with excess kurtosis or fat tails. Sometimes they also exhibit skewness, what's more, they are volatile over time and their variances are not constant. Namely, economic variables are non-stationary. The traditional models are not suitable to analyze economic variables.

To solve these problems, a widely used class of stochastic process named Autoregressive Conditional Heteroskedasticity (ARCH) processes were introduced by Engle [1982] 30 years ago. These are white noise processes (mean zero, finite variance and serially uncorrelated) with non-constant variances conditional in the past. The conditional variances of economic time series are important to price derivatives, calculate measures of risk, and hedge. Bollerslev [1986] extended the ARCH model to the Generalized ARCH (GARCH) model by adding the past conditional variances items, therefore the conditional variances are also affected by their own past values. Exponential GARCH (EGARCH) model introduced by Nelson [1991] changes the conditional variances to the logarithm form and adds an item to analyze the data's different reaction to positive impact and negative impact.

It is widely documented that most of the daily and high frequency financial time series exhibit quite persistent autocorrelation in their squared returns, conditional variances, power transformations of absolute returns and other measures of volatility. Engle and Bollerslev [1986] introduced the Integrated GARCH (IGARCH) class of models to capture this effect, which provides a natural analog to the difference between stationary and a process that contains an autoregressive unit root, I (1) type processes for the conditional mean. However, IGARCH model can adequately capture the short-run volatility clustering and it is not good at the long-term situation. Therefore, Baillie et al. [1996] introduced the Fractionally Integrated GARCH (FIGARCH) model to improve this. The FIGARCH models are strictly stationary and ergodic for $0 \leq d \leq 1$. Bollerslev and Mikkelsen [1996] extended the FIGARCH model to the FIEGARCH model, to allow long-memory and leverage effect. Recently, Adaptive FIGARCH (A-FIGARCH) Baillie and Morana [2009] and FIEGARCH-in-mean (FIEGARCH-M) Christensen et al. [2010] model further developed the ARCH model.

These ARCH-type models are suitable for the unique variances, however, under the assumption of these models, they can only analyze stationary processes (white noise process is a class of weakly stationary processes). As stationarity has become a precondition of these most commonly used methods when analyzing economic data, Augmented Dickey-Fuller (ADF) test is widely used to test the unit root. However, Perron [1989, 1990] showed that when the data are stationary fluctuations around a trend function which contains a one-time break, ADF test is biased towards non-rejection null hypothesis, namely get a result that the data are non-stationary.

In this paper, we want to use GARCH-type models with different kinds of distributions to estimate stock indices and those with different in-sample size to do 1-step ahead and 5-step ahead forecast of the conditional variances. As it mentioned by Perron and Qu [2010], stock market volatility may be better characterized by a short-memory process affected by occasional level shifts. However, short-memory processes with level shifts will exhibit properties that make standard tools conclude long-memory is present [e.g. Granger and Ding, 1996]. Therefore, before forecast, we analyze whether there are structural breaks in our data first. And in order to analyze the long-memory characteristics we choose to use the AR(1)-FIGARCH(1,d,1) model. At the same time, we use AR(1)-GARCH(1,1) and AR(1)-EGARCH(1,1) to compare the analyzing results.

By comparing the forecasted conditional variances, we get the following conclusions: different in-sample sizes, error distributions and forecast horizons do not impact the forecast results of GARCH(1,1) and EGARCH(1,1) models much; the forecast results of GARCH(1,1) and EGARCH(1,1) models are similar when in-sample sizes are not larger than 1008, when the in-sample size is 1638, with error distribution of student t distribution or skewed-student distribution the forecast results of them maybe different; the FIGARCH(1,d,1) model is sensitive to the changes of all four kinds of conditions we used here; the forecast results of the FIGARCH(1,d,1) model are different with the forecast results of the other two models and they are most similar with the square of returns among the three GARCH-type models.

The structure of the paper is described below: Section 2 presents the data we used and their summary statistics. Section 3 introduces the test methods, the estimate models and forecast models, while estimation procedures are present in Section 4. The results and forecasting comparisons between our models are reported in Section 5. Section 6 offers brief conclusions.

2 Data

2.1 Data introduction

This paper uses the daily closing prices of Dow Jone's Industrials Average (DJIA), Standard & Poor 500 (S & P 500), and NASDAQ Composite (NASDAQ) indices. The data coverage is from 2005/01/03 to 2012/03/30, 1825 observations for each stock indices. The original data are shown in Figure 1.

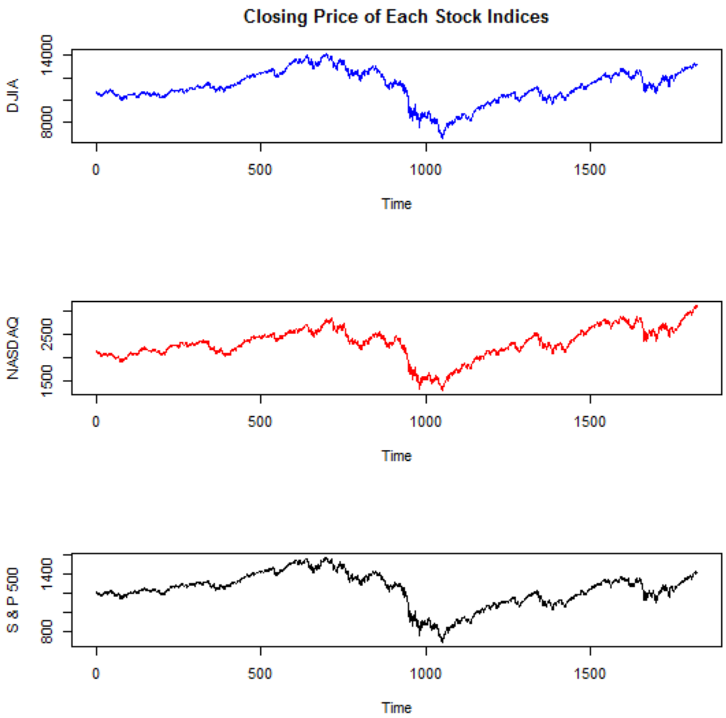

Figure 1: Closing Prices of Each Stock Indices.

As usual, we use the logarithm difference data of each closing price to analyze the index returns, namely $R_t = 100 \times (\ln(P_t) - \ln(P_{t-1}))$, where P_t means closing price at the period of time t, so that R_t is the percent return for the daily closing price from period $t-1$ to period t.

Figure 2 presents the percent return R_t, which shows there is little serial correlation in the returns. As discussed in Ding et al. [1993], although the returns themselves contain little serial correlation, the absolute value of returns has significantly positive serial correlation. Granger and Ding [1996] illustrate this, too. Therefore, here we also plot the absolute value of returns $|R_t|$ in Figure 3. It seems compare with small absolute returns, large absolute returns are more likely to be followed by a large absolute return. Figure 4 presents R_t^2, which contains the similar characteristic.

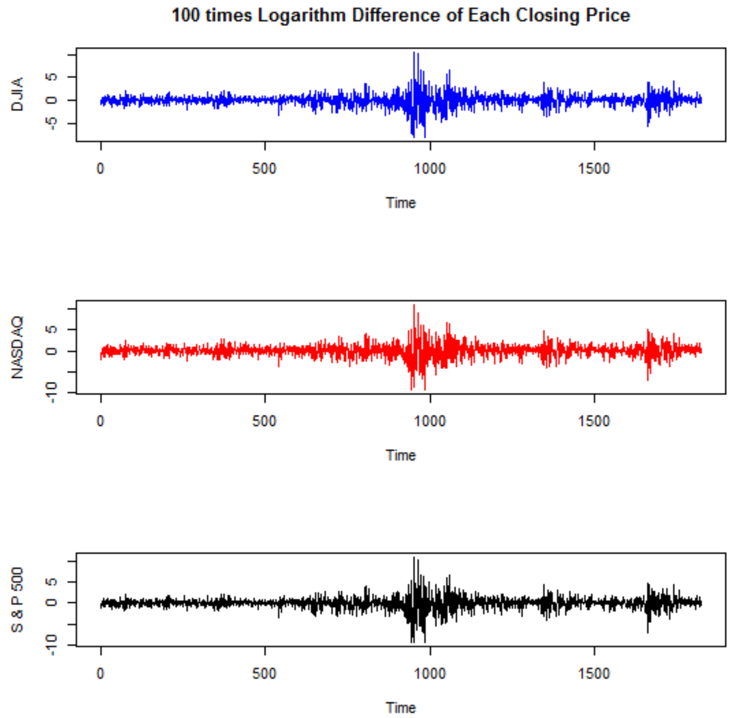

Figure 2: Returns of Each Stock Indices.

All p-values of ADF test are less than 0.01, which means all the index returns, the absolute index returns and square index returns are stationary. The results of autocorrelation function (ACF) and partial autocorrelation function (PACF) are shown in Figure 5, Figure 6 and Figure 7, lags equal to 2. It is obvious that the PACF of absolute daily index returns and square daily index returns approach to 0 very slowly, and this is a characteristic of long-memory.

2.2 Summary Statistics

We analyze some basic characteristics of daily index returns which are shown in Table 1. All three means are very close to zero and standard deviations are very small, which means there is no constant and all the data are around the mean. Absolute values of Skewness excess 0.5 indicate significant level of skewness. Thus, all the skewness values are negative and not excess. Excess kurtosis[2] values that exceed 1.0 in absolute value are

[2] Excess kurtosis = Sample kurtosis - 3.

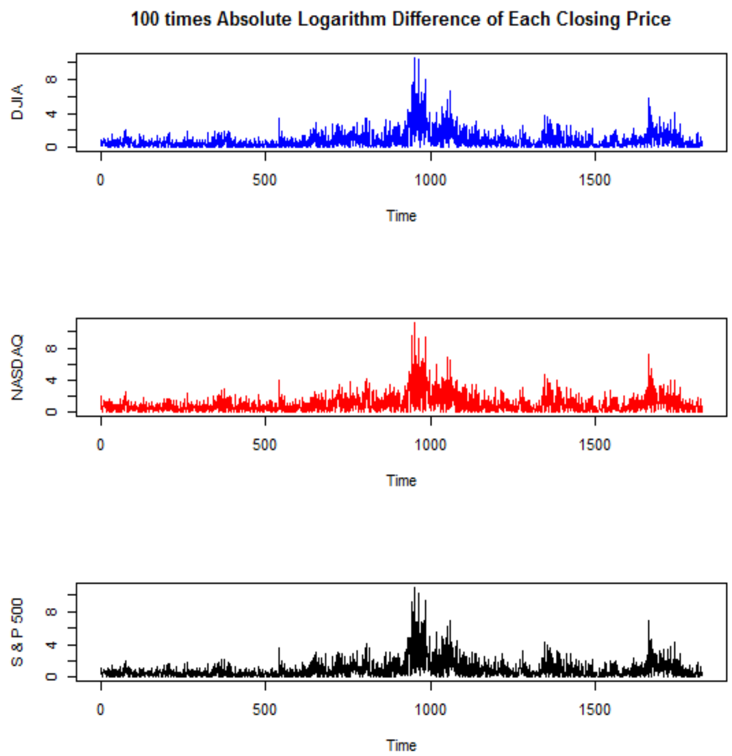

Figure 3: Absolute Returns of Each Stock Indices.

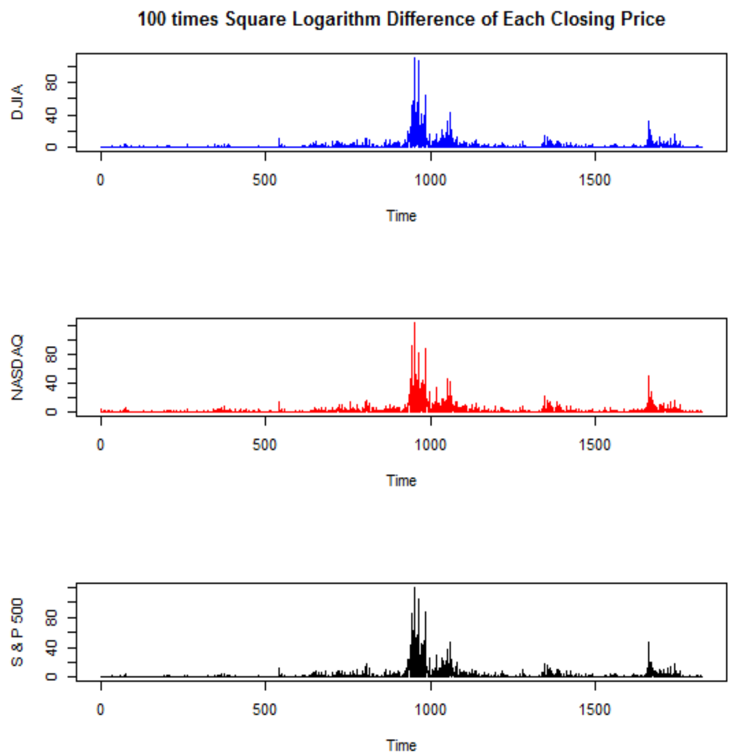

Figure 4: Square Returns of Each Stock Indices.

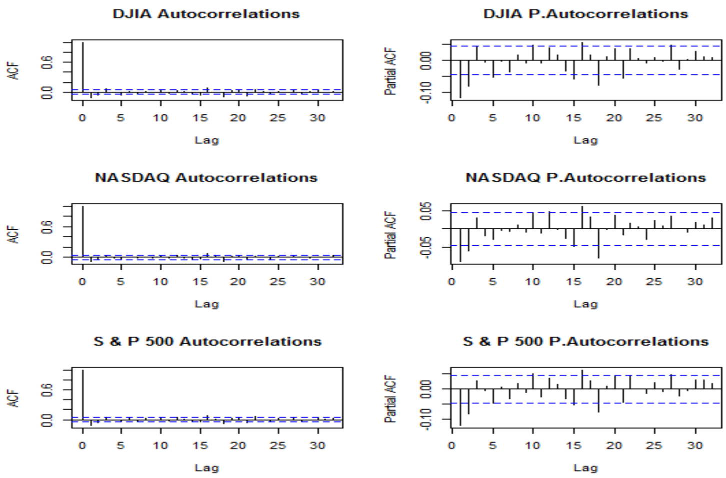

Figure 5: ACF and PACF of Daily Index Returns .

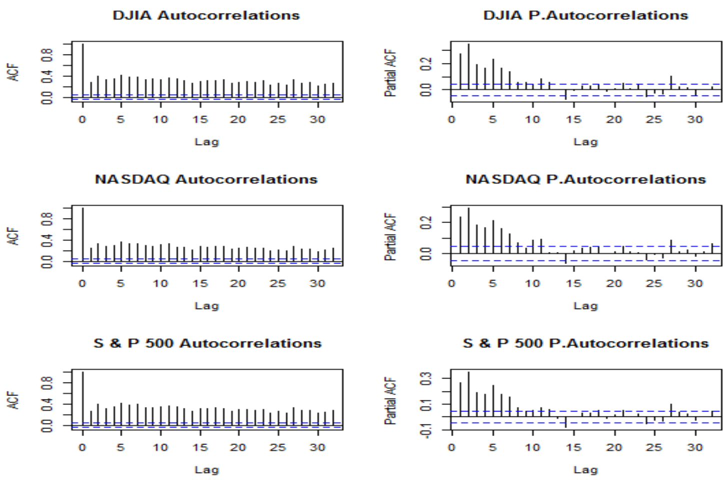

Figure 6: ACF and PACF of Absolute Daily Index Returns .

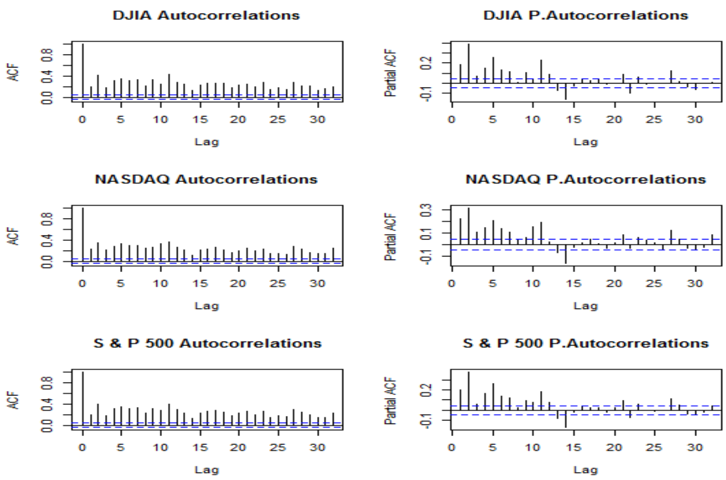

Figure 7: ACF and PACF of Square Daily Index Returns.

considered large. Therefore, all three curves have obviously more kurtosis than the normal distribution. In general, greater positive kurtosis and more negative skew in returns distributions indicates increased risk. Null hypotheses of Jarque-Bera Test, ARCH LM Test and Ljung-Box Test have been rejected under 5% significent level, therefore, Jarque-Bera test shows series are not unconditional normality distributed, ARCH-LM test shows there are ARCH effects, Ljung-Box test shows the data are not independently distributed. In order to check whether there is long-memory characteristic, we also use the modified rescaled range (R/S) test for square return data of DJIA, NASDAQ, and S&P 500, then get the results 2.8555**, 2.9076** and 2.9155**, which means null hypothesis of R/S test has been rejected under 1% significant level, thus there is long-term dependence for all three indices.

Table 1: **Summary Statistics of Daily Index Returns.**

	DJIA	S & P 500	NASDAQ
Mean	0.011	0.009	0.020
Maximum	10.508	10.957	11.159
Minimum	-8.201	-9.470	-9.588
Std. Dev.	1.316	1.441	1.521
Skewness	-0.061	-0.297	-0.210
Kurtosis	9.382	9.226	6.504
Jarque-Bera Test	6710.418*	6514.233*	3238.703*
ARCH LM Test	333.589*	321.198*	262.686*
Ljung-Box Test	33.847*	35.320*	19.501*
Number of obs.	1824	1824	1824

Note: This table shows some summary statistics of the square 100 times log-differences of Dow Jone's Industrials Average (DJIA) daily closing price, Standard & Poor 500 (S&P 500) daily closing price and NASDAQ daily closing price. The ARCH-LM test of Engle [1982] and Ljung-Box test are shown the χ^2 value, and are conducted using 2 lags. Asterisks (*) indicates a rejection of the null hypothesis at the 0.05 level.

3 Methodology

As we mentioned above, with structural breaks, short memory processes may have the long-memory characteristic. And time series with structural breaks cannot be forecasted well without considering these breaks. Therefore, in this paper, before using GARCH-type models to forecast, we first analyze whether there are structural breaks to check the number of the breaks and the break dates.

3.1 Estimate structural changes

We first briefly introduce the method we use to analyze structural breaks, details can be seen in Qu and Perron [2007]. We assume the total number of structural changes in the system is m, the break dates are denoted as $T = (T_1, \dots, T_m)$, with $T_0 = 1$ and $T_{m+1} = T$. A subscript j indexes a regime ($j = 1, \dots, m+1$), a subscript t indexes a temporal observation ($t = 1, \dots, T$). The model considered is

$$y_t = x_t'\beta_j + u_t \tag{1}$$

for $T_{j-1} \leq t \leq T_j$ ($j = 1, \dots, m+1$). In the matrix form is $Y = \bar{X}\beta + U$. The true values of the parameters are denoted with a 0 superscript, thus the data generating process is assumed to be $Y = \bar{X}^0\beta^0 + U$, where $\bar{X}^0$ is the diagonal partition of X using the partition $(T_1^0, \dots, T_m^0)$.

The method of estimation considered is restricted quasi-maximum likelihood (RQML) that assumes serially uncorrelated Gaussian errors. Conditional on a given partition of the sample $T = (T_1, \dots, T_m)$.

The basic idea to construct the quasi-maximum-likelihood estimate (QMLE) based on Normal serially uncorrelated errors is as follows, for any possible number of breaks, the overall value of the log-likelihood function is the sum of the values associated with a particular combination of $m+1$ segments. This is achieved by using a dynamic programming algorithm.

3.1.1 Log-likelihood ratio test

We using a likelihood ratio test for the null hypothesis of no change in any of the coefficients versus an alternative hypothesis with a prespecified number of changes, say m. With two assumptions, (1) we avoid the case where the marginal distribution of the regressors may change while the coefficients do not; (2) there is no serial correlation in the errors u_t.

Before testing whether there are structural changes across regimes, we first construct an AR(1) model, the coefficients of every regressor in all equations are allowed to change. We don't allow breaks in the covariance matrix of the errors. Under a given m partitions $T = (T_1, \ldots, T_m)$, we have

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t \quad \text{for } T_{j-1} + 1 \leq t < T_j \quad (j = 1, \ldots, m+1) \tag{2}$$

The log-likelihood function under the alternative hypothesis is

$$\log \hat{L}_T(T_1, \ldots, T_m) = -\frac{Tn}{2}(\log 2\pi + 1) - \frac{T}{2}\log|\hat{\Sigma}| \tag{3}$$

and the QMLE jointly solve the equations

$$\hat{\Sigma} = \frac{1}{T}\sum_{j=1}^{m+1}\sum_{t=T_{j-1}+1}^{T_j}(y_t - x_t'\hat{\beta}_j)(y_t - x_t'\hat{\beta}_j)', \tag{4}$$

$$\hat{\beta}_j = \left(\sum_{t=T_{j-1}+1}^{T_j} x_t\hat{\Sigma}^{-1}x_t'\right)^{-1}\sum_{t=T_{j-1}+1}^{T_j} x_t\hat{\Sigma}^{-1}y_t \tag{5}$$

3.1.2 Sequential Test

After estimating the break dates with the global maximization of the likelihood function, we can use a sequential test, [Bai and Perron, 1998]. The null hypothesis is the hypothesis that there are l breaks during the period, versus the alternative hypothesis that there are $l+1$ breaks.

The test is defined as

$$SEQ_T(l+1|l) = \max_{1\leq j\leq l+1} sup_{t\in\Lambda_{j,\varepsilon}} lr_T(\hat{T}_1, \ldots, \hat{T}_{j-1}, \tau, \hat{T}_j, \ldots, \hat{T}_l) - lr_T(\hat{T}_1, \ldots, \hat{T}_l),$$

where

$$\Lambda_{j,\varepsilon} = \{\tau; \hat{T}_{j-1} + (\hat{T}_j - \hat{T}_{j-1})\varepsilon \leq \tau \leq \hat{T}_j - (\hat{T}_j - \hat{T}_{j-1})\varepsilon\}$$

3.2 GARCH-type models

There are many different kinds of GARCH-type models applicable to different kinds of research purposes. In order to analyzing long-memory characteristic, here we use the

FIGARCH model, and use two simple GARCH-type models, GARCH and EGARCH to compare with each other. The forecast method is rolling window regression. In this section, we introduce the three GARCH-type models, and the forecast methods we use.

3.2.1 GARCH

All three GARCH-type processes we used in this paper are based on Generalized Autoregressive Conditional Heteroskedasticity (GARCH) process introduced in Bollerslev [1986], which is a extension of ARCH process [Engle, 1982]. GARCH process defines that the conditional variance is not only impacted by the past sample variances, but also by the lagged conditional variances. In empirical, the most common used process is GARCH(1,1), which is expressed as

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \tag{6}$$

where $\varepsilon_t | \psi_{t-1} \sim D(0, \sigma_t^2)$, are from a AR(1) model in our case. $\{\varepsilon_t\}$ is serially uncorrelated, the conditional variance σ_t^2 is changing over time. ψ_{t-1} is the information set of all information through time $t-1$.

Here we constrain all the roots of $(1 - \alpha_1 - \beta_1)$ and $(1 - \beta_1)$ lie outside the unit circle to keep the stability and covariance stationery of the $\{\varepsilon_t\}$ process.

3.2.2 EGARCH

Since the future stock returns volatility has a characteristic that they asymmetric respond to negative and positive return innovations, here we choose to use Exponential GARCH (EGARCH) model originally introduced by Nelson [1991] to analyze the short-run serial dependence in these volatility processes. EGARCH process was re-expressed in Bollerslev and Mikkelsen [1996] as follows

$$\log(\sigma_t^2) = \omega + [1 - \beta(L)]^{-1}[1 + \alpha(L)]g(z_{t-1}) \tag{7}$$

where

$$g(z_t) \equiv \theta z_t + \gamma[|z_t| - E|z_t|] \tag{8}$$

by construction, $\{g(z_t)\}_{t=-\infty,\infty}$ is a zero-mean, i.i.d, random sequence. $E(\theta z_t) = E(\gamma[|z_t| - E|z_t|]) = 0$. z_t is a i.i.d with mean zero, variance one, and tail thickness parameter. For $0 < z_t < \infty$, $g(z_t)$ is linear in z_t with slope $\theta + \gamma$, and for $-\infty < z_t \leq 0$, $g(z_t)$ is linear with slope $\theta - \gamma$. Thus, compare with GARCH model, with the item $g(z_t)$, EGARCH model allows the conditional variance process σ_t^2 to respond asymmetrically to positive or negative impact in stock indices. $E|z_t|$ depends on the assumption made on the unconditional density of z_t.

3.2.3 FIGARCH

As the inverse of the largest autoregressive root for $\ln(\sigma_t^2)$ is always very close to unity, it is highly suggestive of a unit root in the conditional variance equation. Therefore, Engle and Bollerslev [1986] proposed the Integrated GARCH (IGARCH) class of models, which assume $d = 1$. However, although IGARCH model can adequately capture the short-run volatility clustering, it is not good at the long-term situation. And as shown in Figure 6 and Figure 7, PACF parts slowly approach to zero, which is the long-memory characteristic. Baillie et al. [1996] proposed the Fractionally Integrated GARCH (FIGARCH) model to analyze the long-memory possibility. The conditional variance of the FIGARCH(p,d,q) model is defined by

$$\begin{aligned}\sigma_t^2 &= \omega + \beta_1 \sigma_{t-1}^2 + (1 - \beta_1 L - (1 - \phi_1 L)(1 - L)^d)\varepsilon_t^2 \\ &= \omega(1 - \beta_1 L)^{-1} + \lambda(L)\varepsilon_t^2\end{aligned} \tag{9}$$

where the roots of $\phi_1 L$ and $(1 - \beta_1 L)$ lie outside of the unit circle, $\lambda(L) = (1 - (1 - \phi_1 L)(1 - L)^d)[1 - \beta_1(L)]^{-1}$. d is the fractional differencing parameter, and $0 < d < 1$. For $d = 0$, Equation (9) is a GARCH model, for $d = 1$, Equation (9) is a IGARCH model, and for $0 < d < 1$, the effect of a impact to the forecast of σ_{t+T}^2 dissipates at a slow hyperbolic rate of decay.

3.3 Forecasting method of GARCH-type processes

In economics, volatility is used slightly more formally to describe the variability of the random component of a time series, which is often defined as the standard deviation (or σ) of the random Wiener-driven component in a continuous-time diffusion model.

In this paper, we use the forecast method introduced in Chapter 15 of Elliott et al. [2006], and make a rolling window regression forecast.

3.3.1 GARCH forecast model

To express GARCH(1,1) model as

$$\sigma^2_{t|t-1} = \omega + \alpha\varepsilon^2_{t-1} + \beta\sigma^2_{t-1|t-2}, \tag{10}$$

By recursive substitution, the GARCH(1,1) model may alternatively be expressed as an ARCH(∞) model,

$$\sigma^2_{t|t-1} = \omega(1-\beta)^{-1} + \alpha\sum_{i-1}^{\infty}\beta^{i-1}\varepsilon^2_{t-i}. \tag{11}$$

The one-step ahead variance forecasts equals $\sigma^2_{t+1|t}$. In order to do the longer run forecast, $\sigma^2_{t+h|t}$ for $h > 1$, we first set the conditional mean is constant and equal to zero, $\mu_{t|t-1} = 0$ and $\alpha + \beta < 1$, therefore the unconditional variance of the process exists

$$\sigma^2 = \omega(1-\alpha-\beta)^{-1}, \tag{12}$$

The h-step ahead forecast is then expressed as

$$\sigma^2_{t+h|t} = \sigma^2 + (\alpha+\beta)^{h-1}(\sigma^2_{t+1|t} - \sigma^2). \tag{13}$$

and the forecasts revert to the long-run unconditional variance at an exponential rate dictated by the value of $\alpha + \beta$.

3.3.2 EARCH forecast model

The EGARCH(1,1) model is

$$\log(\sigma^2_{t|t-1}) = \omega + \beta\log(\sigma^2_{t-1|t-2}) + \alpha(|z_{t-1}| - E(|z_{t-1}|)) + \gamma z_{t-1} \tag{14}$$

where $z_t = \varepsilon_t/\sigma_t$.

As mentioned in Ederington and Guan [2005], $E_t[ln(\sigma^2_{t+2})] = \omega + \beta E_t[\ln(\sigma^2_{t+1})]$ since the step-head values of both of the last two terms in Equation (14) are zero. So for $h > 1$,

$$E_t[\ln(\sigma^2_{t+h})] = \omega \sum_{j=0}^{h-2} \beta^j + \beta^{h-1} E_t[\ln(\sigma^2_{t+1})] \tag{15}$$

3.3.3 FIGARCH forecast model

The actual forecasts of the FIGARCH(1,d,1) model are most easily constructed by recursive substitution in

$$\sigma^2_{t+h|t+h-1} = \omega(1-\beta)^{-1} + \lambda(L)\sigma^2_{t+h-1|t+h-2} \tag{16}$$

with $\sigma^2_{t+h|t+h-1} \equiv \varepsilon^2_t$ for $h < 0$, and the coefficients in $\lambda(L) \equiv 1 - (1-\beta L)^{-1}(1 - \alpha L - \beta L)(1-L)^d$ calculated from the recursions,

$$\begin{aligned}
\lambda_j &= \beta\lambda_{j-1} + \delta_j - \phi\delta_{j-1}, \quad j = 2, 3, \ldots \\
\delta_j &= \frac{j-1-d}{j}\delta_{j-1}, \quad j = 2, 3, \ldots \\
\lambda_1 &= \phi - \beta + d \\
\delta_1 &= d
\end{aligned}$$

After getting the series of the forecast variances, we use Mincer−Zarnowitz volatility regression [Mincer and Zarnowitz, 1969] to check the forecast quality. That is, the squared observation of the returns has the property of being (conditionally) unbiasedness, or $E_t[y^2_{t+1}] = \sigma^2_{t:t+1}$. The regression is:

$$R^2_{t+1} = a + (b+1)\hat{\sigma}^2_{t:t+1|t} + \varepsilon_{t+1} \tag{17}$$

where we expect $a = b = 0$.

3.3.4 The forecast accuracy Test

As introduced in Mariano [2007], usually, there are three significance tests of forecast accuracy, Morgan-Granger-Newbold (MGN) Test, Meese-Rogoff (MR) Test [Meese and Rogoff, 1988] and Diebold-Mariano (DM) Test [Diebold and Mariano, 1995] with the same null hypothesis which is equivalent to equality of the two forecast error variances.

There are some limitations of the first two tests, like MGN test considers the following assumptions:

A1. Loss is quadratic.

A2. The forecast errors are

a. zero mean,

b. Gaussian,

c. serially uncorrelated,

d. contemporaneously uncorrelated.

MR test considers A1, A2.a and A2.b. However, DM test are applicable to non-quadratic loss functions, multi-period forecasts and forecast errors that are non-Gaussian, non-zero-mean, serially correlated and contemporaneously correlated. Therefore, comparing with the other tests, DM test is better for our data.

Assume the actual values are R_t^2 (square of return) : $t = 1, 2, 3, \ldots T$, and the two forecasts are $\hat{\sigma}_{it}^2 : t = 1, 2, 3, \ldots T$ and $\hat{\sigma}_{jt}^2 : t = 1, 2, 3, \ldots T$. Forecast errors are $e_{it} = \sigma_{it}^2 - R_t^2$ for $i = 1, 2$. The loss associated with forecast i depends on forecast and actual values only through the forecast error:

$$g(R_t^2, \hat{\sigma}_{it}^2) = g(\hat{\sigma}_{it}^2 - R_t^2) = g(e_{it})$$

The loss differentials between the two forecasts are

$$d_t = g(e_{1t}) - g(e_{2t})$$

The DM test is based on the sample mean of $d_t : t = 1, 2, \ldots, T$, meanwhile, all the assumptions A1 to A2.d need not to hold, but assuming covariance stationarity and short-term memory on the process $\{d_t\}$, then

$$\sqrt{T}(\bar{d} - \mu) \xrightarrow{d} N(0, 2\pi f_d(0)) \tag{18}$$

where $f_d(\cdot)$ is the spectral density of $\{d_t\}$, $\bar{d}$ is the sample mean differential, and

$$f_d(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_d(k) \exp(ik\lambda) \quad \text{for } \lambda \in [-\pi, \pi], \tag{19}$$

$$\begin{aligned} \gamma_d(k) &= \text{autocovariance of } d_t \text{ at displacement } h \qquad (20) \\ &= E(d_t - \mu)(d_{t-k} - \mu) \end{aligned}$$

The Diebold-Mariano Test statistic is

$$DM = \frac{\bar{d}}{\sqrt{\frac{2\pi \hat{f}_d(0)}{T}}} \xrightarrow{d} N(0,1), \text{ under } H_0 \tag{21}$$

where $\hat{f}_d(0)$ is a consistent estimate of $f_d(0)$. One consistent estimate is

$$\begin{aligned} 2\pi \hat{f}_d(0) &= \sum_{\tau=-(T-1)}^{T-1} l(\tau/S(T))\hat{\gamma}_d(\tau) \\ l(\omega) &= \begin{cases} 1 & \text{for } |\omega| \leq 1 \\ 0 & \text{otherwise} \end{cases} \\ S(T) &= \text{truncation lag} \\ \hat{\gamma}_d(\tau) &= \frac{1}{T} \sum_{t=|\tau|+1}^{T} (d_t - \bar{d})(d_{t-|\tau|} - \bar{d}) \end{aligned}$$

Consistent estimators of $f_d(0)$ can be of the form

$$\hat{f}_d(0) = \frac{1}{2\pi} \sum_{h=-S(T)}^{S(T)} \kappa \left(\frac{h}{S(T)} \right) \hat{\gamma}_d(h)$$

where

$$\hat{\gamma}_d(h) = \frac{1}{p} \sum_{t=|h|+1}^{T} (d_t - \bar{d})(d_{t-|h|} - \bar{d})$$

$$S(T) = \text{bandwidth or lag truncation}$$

$$\kappa(\cdot) = \text{weighting scheme or kernel}$$

$$\begin{aligned} \kappa(\omega) &= I(\omega) \\ &= \begin{cases} 1 & \text{if } |\omega| < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

4 Analyzing procedure

The previous methodology section states the method we use to analyze structural breaks, and the different method to model the data and to forecast. In this section, we introduce the details of the procedures.

4.1 Analyze structural changes procedure

In the structural break part, we use the method mentioned in Qu and Perron [2007] and the GAUSS code wrote by them, which can be download from Pierre Perron's Homepage. The procedures of the structural breaks analyst are as following,

- ⋆ First, use a dynamic programming algorithm to estimate an unrestricted AR(1) model. "Unrestricted" means that we allow both of the two coefficients to change, but we restrict the variance of the error term to be constant. Then we obtain the structural break dates.
- ⋆ Second, based on the break dates obtained in the first step, estimate the coefficients.
- ⋆ Third, use a dynamic programming algorithm to find the combination of segments which maximizes the global likelihood function.
- ⋆ Forth, repeat the step 2 to 3, until convergence.

4.2 Estimate and Forecast procedure

In order to estimate and forecast our data with GARCH-type models, we use "fGarch" package for GARCH model, "rugarch" package [Ghalanos, 2012] for EGARCH model, and MFE Toolbox [Weron et al., 2007] for the FIGARCH model. The steps of analyzing with GARCH-type models and forecasting are as follows, we also give an example of GARCH(1,1) model with $\varepsilon_t|\psi_{t-1} \sim N(0, \sigma_t^2)$ to explain it.

- ◇ First choose a period T as the known sample (in-sample) set. Here, we choose to compare $T = 504$ (2 years), $T = 1008$ (4 years) and $T = 1638$ (6 years and a half).
- ◇ Second, use this sample set to do the regression by a GARCH-type model with a certain distribution to get the coefficients. For GARCH(1,1), use Equation (10). After that, we get a set of coefficients.
- ◇ Third, use the Tth value of the sample to calculate the $T+1th$ one, like substitution ε_{t-1}^2 and $\sigma_{t|t-1}^2$ in Equation (10), to get $\sigma_{t+1|t}^2$.
- ◇ Forth, repeat step 2 and 3 by moving the T sample one period and one period forward to get a series of the one-step ahead value. In our case, the total observation number of returns is 1824, therefore, we need to do $1824 - 504 = 1320$ or $1824 - 1008 = 816$ or $1824 - 1638 = 186$ times one-step ahead forecast.
- ◇ Fifth, calculate the h-step ahead value. We choose $h = 5$, and use Equation (12) and Equation (13) to get a series of $\sigma_{t+h|t}^2$.
- ◇ Sixth, use Equation (17) to check the forecast quality.

For EGARCH model, we use the similar steps with GARCH, but a little different in step 5. We use Equation (15) to forecast the variance every day from day $t + 2$ through day $t + 5$, then average all 5 days forecast variance to get the σ_{t+5}^2.

5 Results

We use a part of our paper to introduce all the method we used and the procedure we did. Form this section, we will state analyzing results we get and the comparison between different methods and distributions.

5.1 Structural breaks

Before we forecast with the GARCH-type models, we first check whether there are structural breaks during the period of our data. Because as we mentioned in the introduction part, structural breaks are important impact for the results, and the figure of the returns changes much in the middle of the period we choose.

We set the maximum number of breaks (m) equals to 4, and did the likelihood ratio test first. As shown in Table 2, for $m = 1$, we fail to reject the null hypothesis $m = 0$ for each three indices, which means, there is no structural break in the regression coefficients.

Although the Sequential Test reject null hypothesis for H_0: $m = 1$ versus H_1: $m = 2$, it is on the base of there is one structural break, which has been rejected by the log-likelihood ratio test. Therefore, there is no structural break in our data.

Table 2: **The Results for Structural Changes in the Regression Coefficients.**

		Test target	Test value	Critical Values			
				10%	5%	2.5%	1%
DJIA	SupLR Test	$m = 1$	1.878	9.536	11.174	12.695	14.805
		$m = 2$	11.413	15.284	17.456	19.484	22.202
		$m = 3$	11.696	19.924	22.606	25.100	28.425
		$m = 4$	8.975	23.370	26.479	29.344	33.179
	Seq $(l+1 \mid l)$ Test	seq $(2 \mid 1)$	6.599	11.030	12.826	14.491	16.626
		seq $(3 \mid 2)$	0.908	11.939	13.713	15.346	17.412
		seq $(4 \mid 3)$	0.000	12.546	14.309	15.913	17.936
S&P 500	SupLR Test	$m = 1$	0.672	9.536	11.174	12.695	14.805
		$m = 2$	24.875	15.284*	17.456*	19.484*	22.202*
		$m = 3$	25.331	19.924*	22.606*	25.100*	28.425
		$m = 4$	24.469	23.370	26.479	29.344	33.179
	Seq $(l+1 \mid l)$ Test	seq $(2 \mid 1)$	16.978	11.030*	12.826*	14.491*	16.626*
		seq $(3 \mid 2)$	2.856	11.939	13.713	15.346	17.412
		seq $(4 \mid 3)$	0.000	12.546	14.309	15.913	17.936
NASDAQ	SupLR Test	$m = 1$	2.082	9.536	11.174	12.695	14.805
		$m = 2$	20.656	15.284*	17.456*	19.484*	22.202
		$m = 3$	21.817	19.924*	22.606	25.100	28.425
		$m = 4$	22.220	23.370	26.479	29.344	33.179
	Seq $(l+1 \mid l)$ Test	seq $(2 \mid 1)$	13.942	11.030*	12.826*	14.491	16.626
		seq $(3 \mid 2)$	3.751	11.939	13.713	15.346	17.412
		seq $(4 \mid 3)$	0.000	12.546	14.309	15.913	17.936

5.2 Forecasting and Comparison

After checking the structural breaks, we use AR(1)-GARCH(1,1), AR(1)-EGARCH(1,1) and AR(1)-FIGARCH(1,d,1) model with different kinds of distributions of the residuals, normal distribution, Student t distribution and Skewed-Student distribution, different in-sample sizes $T = 504$, $T = 1008$ and $T = 1638$ for each indices to do 1-step ahead forecast and 5-step ahead forecast. Therefore, for each index we get 27 1-step ahead forecast series and 27 5-step ahead forecast series.

Figures 8 - 10 show 1, 5, 10 and 20-step ahead forecast value of DJIA conditional variances using the AR(1)-GARCH(1,1) model, the AR(1)-EGARCH(1,1) model and the AR(1)-FIGARCH(1,d,1) with student t distribution and in-sample size equals to 1638.

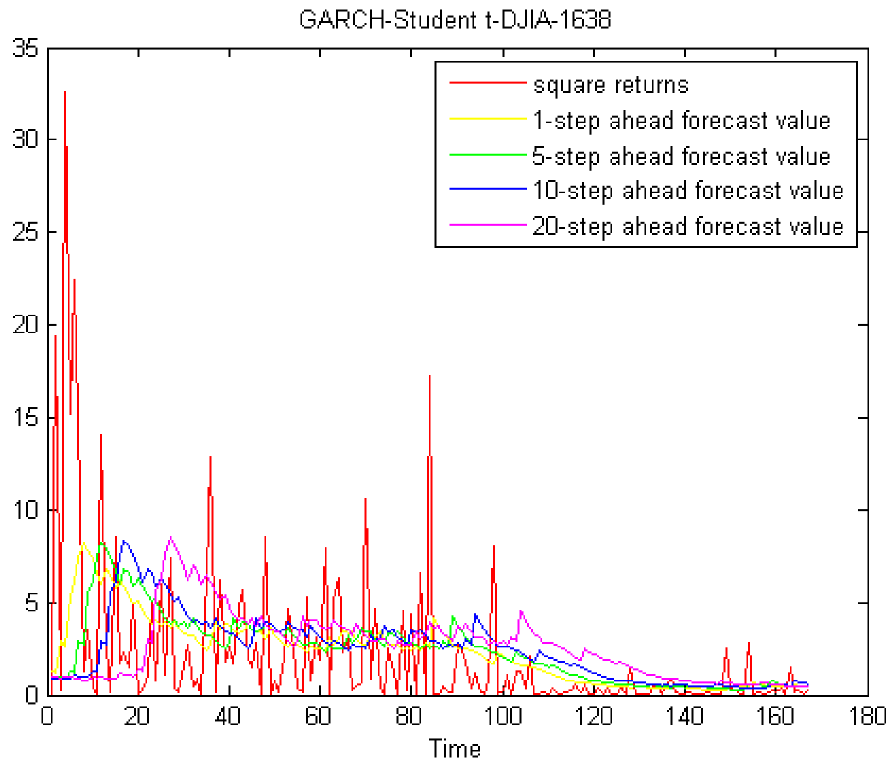

Figure 8: GARCH 1, 5, 10, 20-step forecast (DJIA, T=1638, Student t distribution).

For the GARCH(1,1) model, Figure 8 shows there is no obvious difference between shorter forecast horizon and longer forecast horizon, which can be explained by the GARCH forecast Equation (13), to forecast with different forecast horizons, we only change the power of $(\alpha + \beta)$. What's more, the shapes between square of returns and the forecasted conditional variances are very different, which means, GARCH(1,1) model is not suitable for long forecast horizon. Although they are curves for different forecast

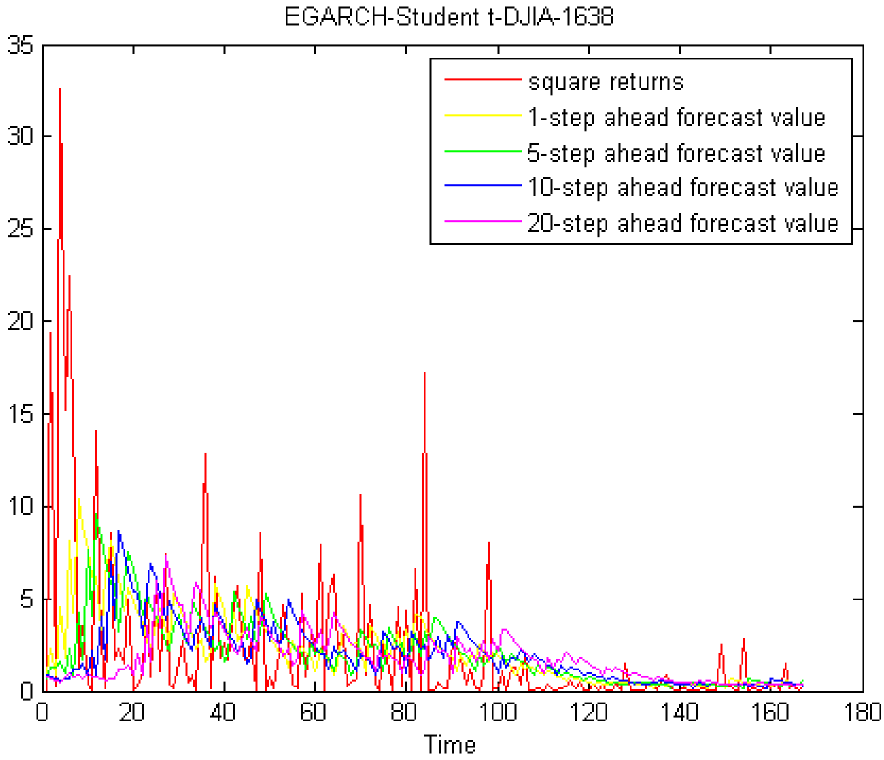

Figure 9: EGARCH 1, 5, 10, 20-step forecast (DJIA, T=1638, Student t distribution).

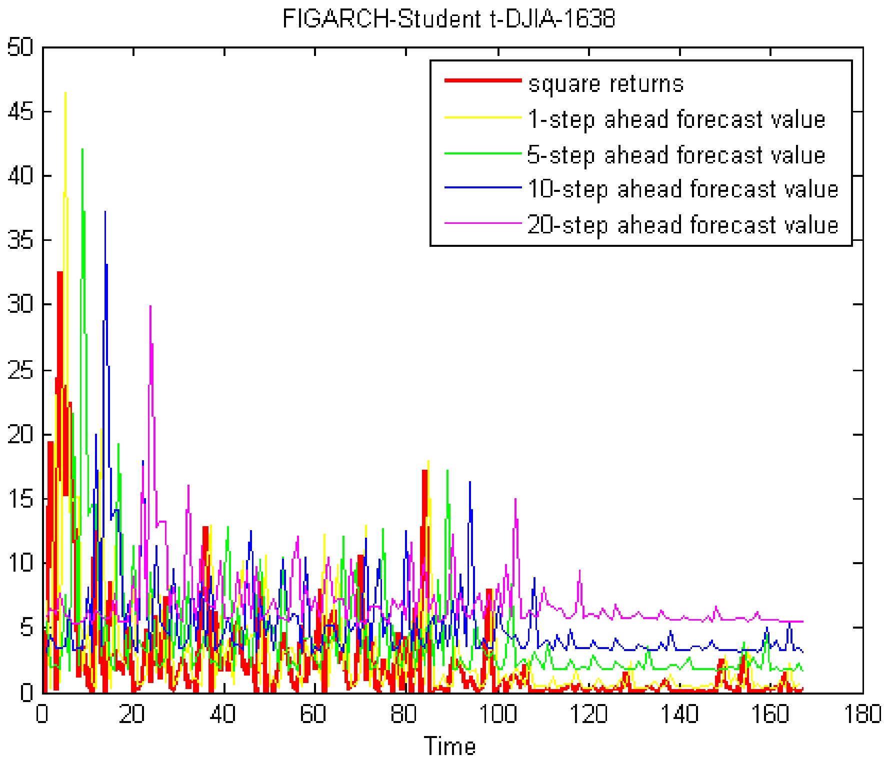

Figure 10: FIGARCH 1, 5, 10, 20-step forecast (DJIA, T=1638, Student t distribution).

horizons, they own the similar shape, therefore, the longer forecast horizon results are not satisfactory.

For the EGARCH(1,1) model, when the forecast horizons are longer, the forecast curves are more smooth in the high volatility period, and the forecast values are not obvious different between shorter and longer forecast horizons when the curves are flat. Althought the shape between square of returns and the forecst conditional variances are similar than GARCH(1,1) model, they are still very different. Therefore, as the GARCH(1,1) model, the EGARCH(1,1) model is not suitable for long forecast horizon either.

For the FIGARCH(1,d,1) model, compare with the other two models, the forecast h_t curves of the FIGARCH(1,d,1) model own the similar shape to the square returns. When the forecast horizon is larger, during the high volatility period, the forecast values are similar to the square returns, however, the means become larger and larger, which can be explained by the FIGARCH forecast Equation (16) that in the recursive substitution calculation way, the constant item has been added several times, and makes the means larger. And there is the same problem as the GARCH(1,1) and the EGARCH(1,1) model that although the shape is similar, they are not at the right time.

The FIGARCH(1,d,1) model is the main model we discussed in this paper. Therefore, we also make a comparison table of the estimate coefficients. Table 3 shows the statistic summaries of these coefficients we got during the rolling window forecast procedures. In Equation (9), $0 < d < 1$, ϕ and β are positive, ϕ values are very small.

Figures 8 - 10 give us a visual impression of the forecast values, but not accurate. Therefore, we use Mincer-Zarnowitz volatility regression to test the forecast qualities. The results of the regression test for DJIA, NASDAQ and S&P 500 are shown in Table 4, 5 and 6.

As mentioned in the methodology part, we expect in Equation (17), $a = b = 0$. Thus for each cases we make t test with null hypothesis $a = 0$ and $b + 1 = 0$, F test with null hypothesis $a = b + 1 = 0$, and show the AIC. Usually, the coefficients are $A = a$ and $B = b + 1$, when $A = B = 0$ the regression is failed, and we expect to reject the null hypothesis. In our case, we hope $b = 0$, but not $B = b + 1 = 0$, thus for F test we expect to reject null hypothesis, and for t test, we expect to reject $b + 1 = 0$ and fail to reject $a = 0$. And we highlight these forecast results which are satisfied our expect with

Table 3: The FIGARCH Coefficients.

		norm[1]			std			sstd		
		504	1008	1638	504	1008	1638	504	1008	1638
ω	min	0.0000	0.0328	0.0954	0.0000	0.0306	0.0788	0.0000	0.0329	0.0827
	max	0.2995	0.1859	0.1138	0.2940	0.1266	0.0933	0.2796	0.1303	0.0973
	mean	0.1195	0.1144	0.1084	0.1031	0.0880	0.0878	0.0999	0.0931	0.0916
	median	0.1096	0.1112	0.1091	0.0848	0.0833	0.0886	0.0866	0.0895	0.0927
ϕ	min	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	max	0.3837	0.1112	0.0179	0.2863	0.0906	0.0180	0.2806	0.0800	0.0124
	mean	0.0865	0.0260	0.0013	0.0538	0.0234	0.0013	0.0507	0.0203	0.0010
	median	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
d	min	0.1160	0.6483	0.7501	0.3717	0.6806	0.7543	0.3857	0.6969	0.7550
	max	0.8288	0.8299	0.7988	0.7957	0.8054	0.7845	0.8023	0.7984	0.7815
	mean	0.6060	0.7376	0.7617	0.6803	0.7392	0.7641	0.6843	0.7452	0.7639
	median	0.6535	0.7498	0.7597	0.7067	0.7436	0.7638	0.7081	0.7489	0.7638
β	min	0.3685	0.6837	0.7501	0.4921	0.7009	0.7543	0.4842	0.7002	0.7550
	max	0.8029	0.8300	0.7990	0.8181	0.8093	0.7845	0.8194	0.8131	0.7822
	mean	0.6914	0.7635	0.7630	0.7332	0.7626	0.7653	0.7342	0.7656	0.7649
	median	0.7465	0.7767	0.7597	0.7527	0.7712	0.7646	0.7606	0.7755	0.7638

[1] "norm" means normal distribution, "std" means Student t distribution, "sstd" means Skewed student distribution.

lightcyan colour. Therefore, according to the regression test, most of the GARCH(1,1) and EGARCH(1,1) models are satisfied, forecasts with larger in-sample size and shorter forecast horizon are more easy to pass the test. On the other hand, only a few of the FIGARCH(1,d,1) models are what we expect.

After using the regression test for each forecast series, we use Diebold-Mariano Test to analyze whether the forecast accuracy of two forecast series are equal. In our case, the forecast series have different (1) in-sample sizes, (2) error distributions, (3) models, and (4) forecast horizons. From Figure 8-10, we have already discussed the differences between different forecast horizons for each GARCH-type model, the differences are obviously shown on these figures. Therefore, here we don't compare with them any more, but to choose the 5-step ahead forecast conditional variances, as we want to forecast 1 week ahead, and to compare the other three conditions. The results are shown in Table 7, 8 and 9.

Table 7 shows for a certain GARCH-type model, whether forecast results are equal when use different error distributions. Under 5% significant level, the FIGARCH(1,d,1) models forecast results almost reject all the null hypotheses of $T = 504$ and $T = 1008$,

Table 4: Forecast regression test for DJIA.

T	h [1]	a	t	p value	b	t	p value	F	p value	AIC
GARCH-Normal distribution										
504	1	0.2824	1.4570	0.1510	-0.1082	19.2870	0.0000 ***[2]	371.9700	0.0000 ***	8479.9000
	5	0.3031	1.5440	0.1230	-0.1226	18.7100	0.0000 ***	350.0500	0.0000 ***	8471.9000
1008	1	0.1618	0.9290	0.3530	-0.1515	11.0710	0.0000 ***	122.5600	0.0000 ***	4286.1000
	5	0.2324	1.2820	0.2000	-0.2110	9.8790	0.0000 ***	97.6030	0.0000 ***	4288.3000
1638	1	0.3523	0.7520	0.4530	-0.1433	4.7900	0.0000 ***	22.9430	0.0000 ***	1048.5000
	5	1.2520	2.4300	0.0161	-0.5719	2.1590	0.0322	4.6608	0.0322 *	1046.2000
GARCH-Student t distribution										
504	1	0.4800	2.5510	0.0108 *	-0.3286	19.4020	0.0000 ***	376.4400	0.0000 ***	8476.4000
	5	0.6192	3.2920	0.0010 **	-0.4340	18.6590	0.0000 ***	348.1600	0.0000 ***	8472.3000
1008	1	0.2074	1.2140	0.2250	-0.2378	11.1120	0.0000 ***	123.4800	0.0000 ***	4286.6000
	5	0.2939	1.6690	0.0955 .	-0.3192	9.9340	0.0000 ***	98.6930	0.0000 ***	4287.8000
1638	1	0.4078	0.8670	0.3870	-0.2318	4.6240	0.0000 ***	21.3780	0.0000 ***	1049.9000
	5	1.2629	0.5114	2.4700	-0.6196	0.1763	2.1570	4.6543	0.0323 *	1046.2000
GARCH-Skewed student distribution										
504	1	0.4340	2.2870	0.0224 *	-0.2712	19.2800	0.0000 ***	371.7100	0.0000 ***	8480.1000
	5	0.5418	2.8490	0.0045 **	-0.3543	18.6400	0.0000 ***	347.4300	0.0000 ***	8476.2000
1008	1	0.2049	1.1950	0.2330	-0.2122	11.0430	0.0000 ***	121.9600	0.0000 ***	4288.0000
	5	0.2806	1.5860	0.1130	-0.2820	9.9440	0.0000 ***	98.8880	0.0000 ***	4287.6000
1638	1	0.3950	0.8410	0.4010	-0.2109	4.6690	0.0000 ***	21.8000	0.0000 ***	1049.5000
	5	1.2731	2.4960	0.0135 *	-0.6138	2.1410	0.0336 *	4.5839	0.0336 *	1046.3000
EGARCH-Normal distribution										
504	1	-0.0891	-0.4550	0.6490	0.2613	21.1790	0.0000 ***	448.5600	0.0000 ***	8409.9000
	5	0.1874	0.8840	0.3770	0.1809	16.3900	0.0000 ***	268.6400	0.0000 ***	8540.1000
1008	1	0.1069	0.6350	0.5260	-0.0569	12.0920	0.0000 ***	146.2100	0.0000 ***	4267.1000
	5	0.2766	1.5470	0.1220	-0.1506	9.8020	0.0000 ***	96.0880	0.0000 ***	4290.1000
1638	1	0.3438	0.7890	0.4310	-0.0672	5.3950	0.0000 ***	29.1050	0.0000 ***	1043.1000
	5	1.1063	2.2810	0.0237 *	-0.4431	2.7650	0.0063 **	7.6442	0.0063 **	1043.3000
EGARCH-Student *t* distribution										
504	1	0.3229	1.7360	0.0828 .	-0.1340	21.0090	0.0000 ***	441.4000	0.0000 ***	8415.3000
	5	0.5957	3.0300	0.0025 **	-0.2148	16.7600	0.0000 ***	280.8700	0.0000 ***	8530.0000
1008	1	0.1804	1.1100	0.2670	-0.1548	12.3000	0.0000 ***	151.3800	0.0000 ***	4262.7000
	5	0.3574	2.0940	0.0366 *	-0.2428	10.0070	0.0000 ***	100.1500	0.0000 ***	4286.5000
1638	1	0.4418	1.0340	0.3020	-0.1987	5.2960	0.0000 ***	28.0440	0.0000 ***	1044.0000
	5	1.1706	2.4930	0.0136 *	-0.5236	2.7420	0.0067 **	7.5184	0.0067 **	1043.4000
EGARCH-Skewed student distribution										
504	1	0.1086	0.5710	0.5680	0.0412	21.2820	0.0000 ***	452.9400	0.0000 ***	8406.6000
	5	0.3839	1.8870	0.0593 .	-0.0375	16.7200	0.0000 ***	279.5500	0.0000 ***	8531.1000
1008	1	0.1538	0.9370	0.3490	-0.1105	12.2960	0.0000 ***	151.1800	0.0000 ***	4262.9000
	5	0.3344	1.9270	0.0543 .	-0.2055	9.8910	0.0000 ***	97.8350	0.0000 ***	4288.5000
1638	1	0.4007	0.9400	0.3490	-0.1548	5.4320	0.0000 ***	29.5110	0.0000 ***	1042.7000
	5	1.1507	2.4390	0.0157 *	-0.4999	2.7740	0.0061 **	7.6952	0.0061 **	1043.3000
FIGARCH-Normal distribution										
504	1	0.5240	1.7110	0.0874 .	2.9780	6.9900	0.0000 ***	48.8620	0.0000 ***	8760.0000
	5	0.6873	2.2710	0.0233 *	2.7102	6.4910	0.0000 ***	42.1310	0.0000 ***	8743.4000
1008	1	3.6820	7.6240	0.0000 ***	-5.5300	-4.4790	0.0000 ***	20.0620	0.0000 ***	4382.0000
	5	3.8742	8.0020	0.0000 ***	-5.9991	-4.8910	0.0000 ***	23.9180	0.0000 ***	4357.5000
1638	1	-6.5210	-2.1530	0.0326 *	20.1570	2.8610	0.0047 **	8.1861	0.0047 **	1062.3000
	5	-11.1520	-3.0670	0.0025 **	32.8400	3.6670	0.0003 ***	13.4480	0.0003 ***	1037.8000
FIGARCH-Student *t* distribution										
504	1	0.9573	3.0690	0.0022 **	2.1709	5.1030	0.0000 ***	26.0390	0.0000 ***	8782.2000
	5	1.0887	3.4920	0.0005 ***	1.9459	4.6210	0.0000 ***	21.3500	0.0000 ***	8763.7000
1008	1	3.0236	5.2440	0.0000 ***	-5.0167	-2.5450	0.0111 *	6.4776	0.0111 *	4395.4000
	5	3.8722	6.2750	0.0000 ***	-7.4705	-3.7840	0.0002 ***	14.3160	0.0002 ***	4366.9000
1638	1	-9.0190	-2.8340	0.0051 **	29.7760	3.5100	0.0006 ***	12.3230	0.0006 ***	1058.3000
	5	-13.1930	-2.8950	0.0043 **	41.8270	3.3710	0.0009 ***	11.3600	0.0009 ***	1039.8000
FIGARCH-Skewed student distribution										
504	1	0.5506	1.8180	0.0693 .	3.2286	7.0050	0.0000 ***	49.0740	0.0000 ***	8759.7000
	5	0.7213	2.3920	0.0169 *	2.9079	6.3880	0.0000 ***	40.8060	0.0000 ***	8744.7000
1008	1	1.3904	9.6920	0.0000 ***	-0.8971	2.8330	0.0047 **	8.0272	0.0047 **	4393.9000
	5	0.9732	5.5140	0.0000 ***	-0.8223	4.8730	0.0000 ***	23.7430	0.0000 ***	4357.7000
1638	1	-12.6160	-3.1650	0.0018 **	38.3840	3.7020	0.0003 ***	13.7060	0.0003 ***	1057.0000
	5	-10.3870	-1.4940	0.1370	32.8360	1.8020	0.0732 .	3.2477	0.0732 .	1047.6000

[1] "h" means forecast horizon. 1 means this row is the regression test results of the one-step ahead variances series. Similar, 5 means five-step ahead. The following rows are the same, first row is the results of one-step ahead, and the second row is the results of five-step ahead.

[2] Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 " " 1

Table 5: Forecast regression test for NASDAQ.

T	h [1]	a	t	p value	b	t	p value	F	p value	AIC
GARCH-Normal distribution										
504	1	0.3710	1.5640	0.1180	-0.1084	18.3150	0.0000 ***[2]	335.5000	0.0000 ***	8878.3000
	5	0.4314	1.7830	0.0748 .	-0.1328	17.3770	0.0000 ***	302.0000	0.0000 ***	8881.1000
1008	1	0.1670	0.7010	0.4830	-0.1218	11.6470	0.0000 ***	135.6600	0.0000 ***	4753.3000
	5	0.3122	1.2550	0.2100	-0.1997	10.1160	0.0000 ***	102.3400	0.0000 ***	4757.9000
1638	1	0.4649	0.6830	0.4950	-0.1301	4.6800	0.0000 ***	21.9000	0.0000 ***	1180.4000
	5	1.5257	2.0500	0.0419 *	-0.4903	2.4760	0.0142 *	6.1311	0.0142 *	1173.1000
GARCH-Student t distribution										
504	1	0.4699	2.0120	0.0444 *	-0.2041	18.4270	0.0000 ***	339.5000	0.0000 ***	8875.0000
	5	0.5881	2.4930	0.0128 *	-0.2645	17.5010	0.0000 ***	306.3000	0.0000 ***	8877.6000
1008	1	0.2133	0.9060	0.3650	-0.1615	11.6110	0.0000 ***	134.8000	0.0000 ***	4754.0000
	5	0.3572	1.4610	0.1440	-0.2425	10.1690	0.0000 ***	103.4100	0.0000 ***	4757.0000
1638	1	0.5151	0.7610	0.4480	-0.1791	4.6280	0.0000 ***	21.4210	0.0000 ***	1180.9000
	5	1.5488	2.1040	0.0368 *	-0.5229	2.4810	0.0140 *	6.1535	0.0140 *	1173.0000
GARCH-Skewed student distribution										
504	1	0.4619	1.9720	0.0489 *	-0.1851	18.3360	0.0000 ***	336.2100	0.0000 ***	8877.7000
	5	0.5787	2.4430	0.0147 *	-0.2406	17.4010	0.0000 ***	302.7800	0.0000 ***	8880.4000
1008	1	0.2198	0.9330	0.3510	-0.1565	11.5760	0.0000 ***	134.0000	0.0000 ***	4754.8000
	5	0.3596	1.4730	0.1410	-0.2339	10.1860	0.0000 ***	103.7500	0.0000 ***	4756.7000
1638	1	0.5047	0.7400	0.4600	-0.1632	4.5980	0.0000 ***	21.1400	0.0000 ***	1181.1000
	5	1.5485	2.0920	0.0379 *	-0.5135	2.4590	0.0149 *	6.0475	0.0149 *	1173.1000
EGARCH-Normal distribution										
504	1	0.0593	0.2420	0.8090	0.1868	18.6690	0.0000 ***	348.5000	0.0000 ***	8861.8000
	5	0.2757	1.0580	0.2900	0.1563	15.6830	0.0000 ***	245.9500	0.0000 ***	8927.5000
1008	1	0.2082	0.8690	0.3850	-0.0750	11.3480	0.0000 ***	128.7800	0.0000 ***	4754.1000
	5	0.3641	1.4460	0.1490	-0.1354	9.6740	0.0000 ***	93.5920	0.0000 ***	4765.7000
1638	1	0.6314	0.9550	0.3410	-0.1267	4.5910	0.0000 ***	21.0700	0.0000 ***	1181.2000
	5	1.4286	1.9780	0.0495 *	-0.4035	2.7610	0.00636 **	7.6230	0.0064	1171.6000
EGARCH-Student t distribution										
504	1	0.2834	1.1930	0.2330	0.0212	18.7610	0.0000 ***	351.9800	0.0000 ***	8859.1000
	5	0.4960	1.9880	0.0470 *	-0.0046	15.9740	0.0000 ***	255.1600	0.0000 ***	8919.8000
1008	1	0.2887	1.2360	0.2170	-0.1293	11.3990	0.0000 ***	129.9400	0.0000 ***	4753.1000
	5	0.4544	1.8630	0.0628 .	-0.1892	9.7220	0.0000 ***	94.5250	0.0000 ***	4764.9000
1638	1	0.6894	1.0610	0.2900	-0.1883	4.6100	0.0000 ***	21.2540	0.0000 ***	1181.0000
	5	1.4901	2.1130	0.0360 *	-0.4527	2.7580	0.0064 **	7.6075	0.0064 **	1171.6000
EGARCH-Skewed student distribution										
504	1	0.1602	0.6710	0.5020	0.0881	19.1760	0.0000 ***	367.7000	0.0000 ***	8846.7000
	5	0.3683	1.4570	0.1450	0.0655	16.1720	0.0000 ***	261.5000	0.0000 ***	8914.5000
1008	1	0.2706	1.1470	0.2520	-0.1082	11.3080	0.0000 ***	127.9000	0.0000 ***	4754.9000
	5	0.4316	1.7480	0.0809 .	-0.1694	9.6430	0.0000 ***	92.9800	0.0000 ***	4766.3000
1638	1	0.6549	1.0060	0.3160	-0.1721	4.6610	0.0000 ***	21.7280	0.0000 ***	1180.6000
	5	1.4802	2.0890	0.0381 *	0.5518	2.7550	0.0065 **	7.5887	0.0065 **	1171.6000
FIGARCH-Normal distribution										
504	1	0.8979	2.3240	0.0203 *	2.6128	6.2740	0.0000 ***	39.3590	0.0000 ***	9138.7000
	5	1.0666	2.7710	0.0057 **	2.3555	5.8200	0.0000 ***	33.8770	0.0000 ***	9119.8000
1008	1	4.1266	6.3160	0.0000 ***	-3.8068	-2.9450	0.0033 **	8.6734	0.0033 **	4870.5000
	5	4.8394	7.3120	0.0000 ***	-4.9431	-4.0430	0.0000 ***	16.3430	0.0000 ***	4838.3000
1638	1	2.8631	0.5130	0.6080	-0.8335	0.0200	0.9840	0.0004	0.9841	1201.3000
	5	30.5610	4.9940	0.0000 ***	-42.9490	-4.5140	0.0000 ***	20.3770	0.0000 ***	1159.6000
FIGARCH-Student t distribution										
504	1	1.2863	3.3790	0.0007 ***	2.0674	5.1920	0.0000 ***	26.9580	0.0000 ***	9150.9000
	5	1.4613	3.8560	0.0001 ***	1.7896	4.7040	0.0000 ***	22.1240	0.0000 ***	9131.3000
1008	1	3.4859	4.1720	0.0000 ***	-3.1159	-1.4900	0.1370	2.2198	0.1366	4876.9000
	5	5.3774	5.9700	0.0000 ***	-6.4930	-3.5330	0.0004 ***	12.4810	0.0004 ***	4842.1000
1638	1	3.0402	0.6230	0.5340	-1.1108	-0.0140	0.9890	0.0002	0.9892	1201.3000
	5	22.0550	3.9550	0.0001 ***	-33.5090	-3.4270	0.0008 ***	11.7440	0.0008 ***	1167.6000
FIGARCH-Skewed student distribution										
504	1	0.9742	2.5970	0.0095 **	2.5989	6.3020	0.0000 ***	39.7160	0.0000 ***	9138.4000
	5	1.1547	3.0810	0.0021 **	2.3202	5.7660	0.0000 ***	33.2430	0.0000 ***	9120.4000
1008	1	1.8355	9.4270	0.0000 ***	-0.8497	4.2460	0.0000 ***	18.0290	0.0000 ***	4861.2000
	5	1.2832	5.2320	0.0000 ***	0.1770	5.3400	0.0000 ***	28.5120	0.0000 ***	4826.4000
1638	1	1.1070	0.2180	0.8280	2.0600	0.3690	0.7130	0.1358	0.7129	1201.2000
	5	22.9080	3.8210	0.0002 ***	-34.1300	-3.3290	0.0012 **	11.0850	0.0011 **	1168.3000

[1] "h" means forecast horizon. 1 means this row is the regression test results of the one-step ahead variances series. Similar, 5 means five-step ahead. The following rows are the same, first row is the results of one-step ahead, and the second row is the results of five-step ahead.

[2] Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 " " 1

Table 6: Forecast regression test for S&P 500.

T	h [1]	a	t	p value	b	t	p value	F	p value	AIC
GARCH-Normal distribution										
504	1	0.3507	1.5150	0.1300	-0.1038	19.3050	0.0000 ***[2]	372.6700	0.0000 ***	8938.7000
	5	0.3600	1.5340	0.1250	-0.1087	18.7820	0.0000 ***	352.7700	0.0000 ***	8930.0000
1008	1	0.1793	0.8360	0.4040	-0.1340	11.4560	0.0000 ***	131.2500	0.0000 ***	4638.1000
	5	0.2398	1.0760	0.2820	-0.1797	10.4150	0.0000 ***	108.4700	0.0000 ***	4635.4000
1638	1	0.4293	0.7210	0.4720	-0.1422	4.7170	0.0000 ***	22.2520	0.0000 ***	1141.5000
	5	1.5102	2.3220	0.0214 *	-0.5624	2.1900	0.0298 *	4.7976	0.0298 *	1136.6000
GARCH-Student t distribution										
504	1	0.6195	2.7670	0.0057 **	-0.3421	19.4090	0.0000 ***	376.7200	0.0000 ***	8935.5000
	5	0.8039	3.6020	0.0003 ***	-0.4528	18.7160	0.0000 ***	350.2900	0.0000 ***	8931.9000
1008	1	0.2272	1.0730	0.2840	-0.2226	11.4250	0.0000 ***	130.5400	0.0000 ***	4638.7000
	5	0.2963	1.3560	0.1760	-0.2899	10.4330	0.0000 ***	108.8400	0.0000 ***	4635.1000
1638	1	0.4876	0.8150	0.4160	-0.2262	4.5650	0.0000 ***	20.8370	0.0000 ***	1142.8000
	5	1.5380	2.3720	0.0187 *	-0.6152	2.1480	0.0330 *	4.6141	0.0331 *	1136.8000
GARCH-Skewed student distribution										
504	1	0.5454	2.4110	0.0160 *	-0.2731	19.3270	0.0000 ***	373.5300	0.0000 ***	8938.0000
	5	0.6754	2.9890	0.0029 **	-0.3546	18.8070	0.0000 ***	353.7100	0.0000 ***	8929.2000
1008	1	0.2314	1.0900	0.2760	-0.1940	11.3600	0.0000 ***	129.0800	0.0000 ***	4639.9000
	5	0.2887	1.3210	0.1870	-0.2476	10.4680	0.0000 ***	109.5900	0.0000 ***	4634.4000
1638	1	0.4815	0.8060	0.4220	-0.2138	4.5820	0.0000 ***	20.9920	0.0000 ***	1142.6000
	5	1.5272	2.3590	0.0194 *	-0.6045	2.1740	0.0310 *	4.7251	0.0310 *	1136.6000
EGARCH-Normal distribution										
504	1	0.2718	1.1160	0.2650	0.1336	17.6240	0.0000 ***	310.6000	0.0000 ***	8982.0000
	5	0.0786	0.3120	0.7550	0.3080	17.3850	0.0000 ***	302.2300	0.0000 ***	8970.5000
1008	1	0.2149	1.0270	0.3050	-0.0766	11.7340	0.0000 ***	137.6800	0.0000 ***	4632.5000
	5	0.4146	1.8710	0.0616 .	-0.1648	9.5450	0.0000 ***	91.0990	0.0000 ***	4650.9000
1638	1	0.5611	0.9900	0.3240	-0.1393	4.8150	0.0000 ***	23.1820	0.0000 ***	1140.6000
	5	1.4253	2.2930	0.0230 *	-0.4825	2.5340	0.0121 *	6.4237	0.0121 *	1135.0000
EGARCH-Student t distribution										
504	1	1.9838	8.8040	0.0000 ***	-0.7593	10.4000	0.0000 ***	108.1700	0.0000 ***	9157.2000
	5	1.9808	8.7350	0.0000 ***	-0.7394	10.1920	0.0000 ***	103.8800	0.0000 ***	9142.8000
1008	1	0.2937	1.4550	0.1460	-0.1689	11.9960	0.0000 ***	143.9200	0.0000 ***	4627.2000
	5	0.5103	2.4090	0.0162 *	-0.2560	9.7360	0.0000 ***	94.7890	0.0000 ***	4647.6000
1638	1	0.6168	1.1070	0.2700	-0.2386	4.8270	0.0000 ***	23.2960	0.0000 ***	1140.5000
	5	1.4820	2.4450	0.0155 *	-0.5513	2.5230	0.0125 *	6.3659	0.0125 *	1135.0000
EGARCH-Skewed student distribution										
504	1	2.1036	9.1360	0.0000 ***	-0.7759	8.0600	0.0000 ***	64.9560	0.0000 ***	9197.8000
	5	1.9487	8.4920	0.0000 ***	-0.6942	9.7780	0.0000 ***	95.6060	0.0000 ***	9150.5000
1008	1	0.2937	1.4550	0.1460	-0.1689	11.9960	0.0000 ***	143.9200	0.0000 ***	4627.2000
	5	0.5103	2.4090	0.0162 *	-0.2560	9.7360	0.0000 ***	94.7890	0.0000 ***	4647.6000
1638	1	0.5781	1.0400	0.3000	-0.2042	4.9330	0.0000 ***	24.3360	0.0000 ***	1139.6000
	5	1.4623	2.4050	0.0172 *	-0.5323	2.5530	0.0115 *	6.5165	0.0115 *	1134.9000
FIGARCH-Normal distribution										
504	1	0.4875	1.3070	0.1910	4.0425	7.3550	0.0000 ***	54.0990	0.0000 ***	9214.3000
	5	0.5844	1.5660	0.1180	3.8654	7.0700	0.0000 ***	49.9810	0.0000 ***	9193.8000
1008	1	3.7718	7.6580	0.0000 ***	-4.4958	-3.8070	0.0002 ***	14.4930	0.0002 ***	4745.7000
	5	4.1955	8.3890	0.0000 ***	-5.3786	-4.6510	0.0000 ***	21.6300	0.0000 ***	4716.0000
1638	1	-2.0760	-0.4420	0.6590	9.0280	0.9980	0.3190	0.9967	0.3194	1161.7000
	5	6.6210	0.7150	0.4760	-9.6710	-0.4310	0.6670	0.1860	0.6667	1141.2000
FIGARCH-Student t distribution										
504	1	1.0596	2.7400	0.0062 **	3.0027	5.2040	0.0000 ***	27.0860	0.0000 ***	9240.5000
	5	1.1977	3.0970	0.0020 **	2.7087	4.8020	0.0000 ***	23.0590	0.0000 ***	9220.0000
1008	1	3.2548	4.8040	0.0000 ***	-4.3235	-1.9150	0.0558 .	3.6690	0.0558 .	4756.4000
	5	5.0897	6.6010	0.0000 ***	-9.2710	-4.1120	0.0000 ***	16.9110	0.0000 ***	4720.7000
1638	1	-3.3690	-0.8260	0.4100	14.5910	1.4710	0.1430	2.1638	0.1430	1160.5000
	5	21.8380	2.3220	0.0213 *	-51.9600	-2.0450	0.0424 *	4.1804	0.0424 *	1137.2000
FIGARCH-Skewed student distribution										
504	1	0.4582	1.2070	0.2280	4.4979	7.2620	0.0000 ***	52.7330	0.0000 ***	9215.6000
	5	0.6511	1.7120	0.0872 .	4.0718	6.6650	0.0000 ***	44.4230	0.0000 ***	9199.2000
1008	1	1.6889	9.5320	0.0000 ***	-0.8720	3.5260	0.0004 ***	12.4330	0.0004 ***	4747.7000
	5	1.1991	5.6760	0.0000 ***	-0.8003	5.4170	0.0000 ***	29.3440	0.0000 ***	4708.5000
1638	1	-3.6180	-0.8440	0.4000	14.6150	1.4570	0.1470	2.1235	0.1468	1160.6000
	5	25.4000	2.3930	0.0177 *	-59.0600	-2.1470	0.0331 *	4.6099	0.0331 *	1136.8000

[1] "h" means forecast horizon. 1 means this row is the regression test results of the one-step ahead variances series. Similar, 5 means five-step ahead. The following rows are the same, first row is the results of one-step ahead, and the second row is the results of five-step ahead.

[2] Signif. codes: 0 "***" 0.001 "**" 0.01 "*" 0.05 "." 0.1 " " 1

Table 7: DM Test Results[1] (Compare different distributions).

In-Sample Size	Distribution[2]	Stock Index	GARCH[3]	EGARCH	FIGARCH
504	norm-std	DJIA	-1.5926	0.0854	-2.5932*[4]
		NASDAQ	-0.8396	0.8523	-2.7724*
		S&P 500	-1.5172	-1.3553	-2.8063*
	norm-sstd	DJIA	-1.4015	1.0690	-2.5719*
		NASDAQ	-0.8755	1.1856	-2.7494*
		S&P 500	-1.2160	-0.9414	-2.7891*
	std-sstd	DJIA	1.7908	0.6728	2.8293*
		NASDAQ	0.6874	0.5294	-1.3721
		S&P 500	1.8088	0.3382	2.8868*
1008	norm-std	DJIA	-1.9492	-0.5239	7.5644*
		NASDAQ	-1.2617	-0.6440	8.2835*
		S&P 500	-2.1113*	-0.6384	8.3702*
	norm-sstd	DJIA	-1.6540	-0.5239	6.6896*
		NASDAQ	-0.8644	-0.9759	7.5253*
		S&P 500	-1.6086	-1.4098	8.2862*
	std-sstd	DJIA	2.1364*	NaN	2.4513*
		NASDAQ	1.5993	-0.1322	-5.8546*
		S&P 500	2.3102*	0.0082	-7.7591*
1638	norm-std	DJIA	-1.5214	-1.3320	-1.5503
		NASDAQ	-1.3311	-1.1986	4.4354*
		S&P 500	-1.7361	-1.4475	-1.5814
	norm-sstd	DJIA	-1.5214	-1.1341	1.2627
		NASDAQ	-1.4110	-1.1053	4.6346*
		S&P 500	-1.6786	-1.2313	-1.7099
	std-sstd	DJIA	NaN	1.7093	2.0731*
		NASDAQ	1.0626	0.9874	-2.8349*
		S&P 500	1.8190	1.8623	-2.3194*

[1] All data in this table are the results of Diebold-Mariano Test statistic.
[2] "std" means Student t distribution, "sstd" means Skewed student distribution.
[3] Here we compare the AR(1)-GARCH(1,1) model, the AR(1)-EGARCH(1,1) model and the AR(1)-FIGARCH(1,d,1) model.
[4] "*" means the DM test result reject null hypothesis under 5% significant level, DM test results are outside interval [-1.96, 1.96].

Table 8: DM Test Results[1] (Compare Different In-Sample Sizes).

Distribution[2]	Stock Index	In-Sample Size	GARCH[3]	EGARCH	FIGARCH
Norm	DJIA	504-1008	0.1442	-0.5035	-2.5514*[4]
		504-1638	1.1156	-0.4976	1.5223
		1008-1638	1.2595	-0.2620	2.1621*
	NASDAQ	504-1008	-0.5909	0.3193	-2.1074*
		504-1638	0.2633	0.0842	-2.7975*
		1008-1638	1.1525	-0.4892	1.3710
	S&P 500	504-1008	-0.0893	-0.2271	-2.1691*
		504-1638	0.6697	-0.252	1.6439
		1008-1638	0.8657	-0.1549	1.8713
std	DJIA	504-1008	1.2513	-0.9205	2.1653*
		504-1638	1.4089	-0.8317	2.1464*
		1008-1638	1.4184	0.0567	1.9814*
	NASDAQ	504-1008	0.0736	-0.1617	-2.1124*
		504-1638	0.8769	-0.2038	-3.1839*
		1008-1638	1.1852	-0.6168	1.6713
	S&P 500	504-1008	0.7132	-0.7253	1.9716*
		504-1638	1.0216	-0.7521	1.9601*
		1008-1638	0.9574	-0.3552	1.8340
sstd	DJIA	504-1008	0.8033	-0.9719	2.1143*
		504-1638	0.9741	-0.8023	2.0804*
		1008-1638	1.1246	0.6080	1.8510
	NASDAQ	504-1008	-0.6922	-0.1154	-2.0766*
		504-1638	-0.2767	-0.2487	-3.0401*
		1008-1638	1.0118	-0.7455	1.5554
	S&P 500	504-1008	-0.0931	-0.5861	1.9482
		504-1638	-0.1177	-0.6820	1.9318
		1008-1638	-0.0709	-0.9414	1.7659

[1] All data in this table are the results of Diebold-Mariano Test statistic.

[2] "std" means Student t distribution, "sstd" means Skewed student distribution.

[3] Here we compare AR(1)-GARCH(1,1), AR(1)-EGARCH(1,1) and AR(1)-FIGARCH(1,d,1) models.

[4] "*" means the DM test result reject null hypothesis under 5% significant level, DM test results are outside interval [-1.96, 1.96].

Table 9: DM Test Results[1] (Compare Different Models).

Distribution[2]	Stock index	GARCH-EGARCH[3]	GARCH-FIGARCH	EGARCH-FIGARCH
T=504				
normal	DJIA	-1.1719	-2.7539*[4]	-2.4893*
	NASDAQ	-1.2090	-2.8950*	-2.3395*
	S&P 500	-1.2944	-2.9423*	-2.6926*
std	DJIA	1.1505	-2.7547*	-2.6524*
	NASDAQ	-0.1355	-2.9577*	-2.6267*
	S&P 500	-1.2250	-2.9246*	-2.4616*
sstd	DJIA	0.5140	-2.7104*	-2.5980*
	NASDAQ	-0.2198	-2.9154*	-2.6009*
	S&P 500	-0.9503	-2.8839*	-1.9045
T=1008				
normal	DJIA	0.2418	-5.7613*	-5.8643*
	NASDAQ	-0.1710	-6.3925*	-6.3319*
	S&P 500	-0.9907	-5.8248*	-5.8726*
std	DJIA	1.0737	-4.9754*	-5.1355*
	NASDAQ	-0.0079	-5.6812*	-5.6416*
	S&P 500	0.0805	-4.8756*	-4.9796*
sstd	DJIA	0.5895	-5.1986*	-5.3713*
	NASDAQ	-0.1180	-5.7952*	-5.7533*
	S&P 500	-0.6105	-5.0399*	-5.1175*
T=1638				
normal	DJIA	4.5597	-2.4910*	-2.7101*
	NASDAQ	1.4446	-2.9135*	-3.0324*
	S&P 500	2.1722	-2.7167*	-3.0199*
std	DJIA	0 - 8.3850i*	-2.4324*	-2.6519*
	NASDAQ	1.4925	-2.7824*	-2.9096*
	S&P 500	2.7945*	-2.5016*	-2.7611*
sstd	DJIA	4.4008*	-2.4916*	-2.7845*
	NASDAQ	1.4419	-2.7960*	-2.9295*
	S&P 500	2.6688 *	-2.5067*	-2.7874*

[1] All data in this table are the results of Diebold-Mariano Test statistic.
[2] "std" means Student t distribution, "sstd" means Skewed student distribution.
[3] Here we compare AR(1)-GARCH(1,1), AR(1)-EGARCH(1,1) and AR(1)-FIGARCH(1,d,1) models.
[4] "*" means the DM test result reject null hypothesis under 5% significant level, DM test results are outside interval [-1.96, 1.96].

and for $T = 1638$ errors with student t distribution and Skewed student distribution are different. However, according to this test, different error distributions have no obvious difference for the EGARCH(1,1) models, for the GARCH(1,1) model three cases reject null hypotheses.

Table 8 shows for a certain GARCH-type model, whether forecast results are equal when use different in-sample sizes. It seems, we fail to reject there is no difference for the GARCH(1,1) and EGARCH(1,1) models, only the FIGARCH(1,d,1) models are impacted by different in-sample sizes, and when the error distribution is student t distribution it may be easier to be impacted. Use $T = 504$ as an in-sample size of stock indices maybe too small, for the null hypotheses of most of the comparisons between $T = 504$ and other in-sample sizes have been rejected.

Table 9 shows under the same in-sample sizes, error distributions and forecst horizons, whether different GARCH-type models have the equal forecast accuracy. The results show that for most of the cases we cannot reject the null hypotheses of the forecast accuracy of GARCH(1,1) and EGARCH(1,1) models are equal, except when in-sample size is $T = 1638$. Therefore, when in-sample size is small, e.g. $T \leq 1008$, there is no obvious difference between the conditional variances forecasted by the GARCH(1,1) or the EGARCH(1,1) models. All the null hypotheses about the FIGARCH(1,d,1) models have equal forecast accuracy with the other two models have been rejected. The forecast accuracy of the FIGARCH(1,1) model thus is different from GARCH(1,1) and EGARCH(1,d,1) model.

According to the above results, we can easily get the conclusion that FIGARCH(1,d,1) models are more sensitive to the changes of conditions, which makes it more accuracy to estimate.

6 Conclusion

In this paper, we use three GARCH-type models: AR(1)-GARCH(1,1), AR(1)-EGARCH(1,1) and AR(1)-FIGARCH(1,d,1), with three different distributions for the errors: Normal distribution, Student t distribution and Skewed student distribution, three in-sample sizes $T = 504$ (two years), $T = 1008$ (four years) and $T = 1638$ (six years and a half), and two forecast horizons, 1-step ahead and 5-step ahead, to forecast three USA stock indices: DJIA, NASDAQ and S&P 500. Before we forecast, we did a structural break test by using a dynamic programming algorithm, the likelihood ratio test and the sequential test to analyze whether there are structural changes in the three index returns during the interval we choose. If there are some, test the number of breaks and the occurred dates. We get the results that there is no structural break.

In the forecast part, we first use rolling window method with in-sample size ($T = 504$, $T = 1008$ or $T = 1638$) to do one-step ahead forecast. With these results, we make the 5-step ahead forecast. After that, we use Mincer-Zarnowitz volatility regression test to analyze the forecast quality of each forecast series. Finally, we use DM test to analyze under different conditions whether two forecast series have the same forecast accuracy.

The regression test results show that compare with the AR(1)-FIGARCH(1,1) models forecast series, more AR(1)-GARCH(1,1) models and AR(1)-EGARCH(1,1) models forecast series can satisfy our expect; Forecast series with shorter forecast horizons and longer in-sample sizes perform better than the opposite ones; Errors with different distributions don't impact the forecast quality of AR(1)-GARCH(1,1) and AR(1)-EGARCH(1,1) models; For AR(1)-GARCH(1,1) model and AR(1)-EGARCH(1,1) models, there is no obvious difference between forecast with different in-sample sizes. The last two conclusion are the same as we get from Diebold-Mariano Test.

In order to compare the forecast accuracy of different GARCH-type models, we did a Diebold-Mariano Test. Null hypothesis of the test is the forecast accuracy of the compared two series are equal, and the DM test statistic should be compared with a critical value of standard normal distribution under some significant levels. We choose the significant level equals to 5%. Accoding to our results, we get the conclusions: (a) For AR(1)-GARCH(1,1) and AR(1)-EGARCH(1,1) models themselves, when changing the conditions of in-sample sizes and errors distributions, we cannot reject the null hypothesis that the forecast accuracy are the same. Consider with the results we get from Figures 8-10, we further conclude that when forecast with GARCH(1,1) or EGARCH(1,1) models, there is no obvious difference when use different in-sample sizes, different error distributions or different forecast horizons. (b) Under the same conditions (the same in-sample size, error distributions and forecast horizons), we cannot reject the null hypothesis that the forecast accuracy of AR(1)-GARCH(1,1) model and AR(1)-EGARCH(1,1) model is equal.

We reject the forecast accuracy of the AR(1)-FIGARCH(1,d,1) is equal to both AR(1)-GARCH(1,1) and AR(1)-EGARCH(1,1) models. AR(1)-FIGARCH(1,d,1) models are more sensitive to the changes of conditions. At last, we have the inference that AR(1)-FIGARCH(1,d,1) model forecasts better than AR(1)-GARCH(1,1) model and AR(1)-EGARCH(1,1) model when analyze stock index returns.

We use the rolling window forecast method to forecast the index returns, the whole sample period we choose is seven and one fourth years, and the forecast results of $T = 1638$ are the best, using small in-sample size to forecast is not a good choice. Therefore, perhaps we can also try the other forecast method that in-sample sizes are with the same start time and add a certain period in every regression. The summary statistics show the skewness of the data is not excess, however, the DM test shows the forecast accuracy of FIGARCH(1,d,1) model with student t distribution and Skewed student distribution is different, when the in-sample size is larger, the result is more obvious. Therefore, we also analyze the skewness with longer period, which shows excess skewness characteristics, therefore, with a not long years interval and not excess skewness summary statistics results, to choose GARCH-type models with errors in Student t distribution is suitable and convenience, which is also correspond with the results of structural break test that the index returns are stable. We also notice that when in-sample size is large, for DJIA and S&P 500, we can not reject the forecast accuracy is equal between normal distribution and either the other two distributions.

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