Behrens, Axel

Working Paper

Optimal growth under stochastic resource supply

Kiel Working Paper, No. 438

Provided in Cooperation with:
Kiel Institute for the World Economy (IfW)

Kieler Arbeitspapiere
Kiel Working Papers

Kiel Working Paper No. 438

Optimal Growth under Stochastic Resource Supply

by

Axel Behrens

September 1990

Institut für Weltwirtschaft an der Universität Kiel

The Kiel Institute of World Economics

ISSN 0342 - 0787

Kiel Institute of World Economics
Düsternbrooker Weg 120
2300 Kiel
Federal Republic of Germany

Kiel Working Paper No. 438

Optimal Growth under Stochastic Resource Supply

by

Axel Behrens

September 1990

The author himself, not the Kiel Institute of World Economics, is solely responsible for the contents and distribution of each Kiel Working Paper.

Since the series involves manuscripts in a preliminary form, interested readers are requested to direct criticisms and suggestions directly to the author and to clear any quotations with him.

Optimal Growth under Stochastic Resource Supply

Axel BEHRENS
Kiel Institut of World Economics
Düsternbrooker Weg 120
2300 Kiel
FRG

Abstract

If an economy faces stochastic fluctuations of the resource availability, for example in the case of resource imports, it is optimal in the long run to increase the domestic savings rate in contrast to a deterministic world. Uncertainty is partly substituted by capital accumulation. But this is only true in the long run. The short run effects depend on the capital stock so far accumulated. It is shown that in some cases it may be optimal to lessen the domestic savings rate temporarily when uncertainty increases.

1. INTRODUCTION

The sharp and sustained increase in fluctuations of oil prices in the seventies and after the Iraq intervention emerge as a central issue influencing the growth prospects of resource-importing countries. Such countries have responded to the crises with several instruments. Stockpiling, diversifikation of imports, import tariffs, and many others.

This paper is a preliminary attempt to consider the influence of stochastic resource supply to the optimal investment decision in a growth context, i.e. how much more or less should a country save and invest if there is uncertainty with respect to oil or resource markets. The second question we will pose is: Can the short run effects differ from the long run steady-state effects?

The neoclassical theory of economic growth, under certainty for both given and optimal savings functions, has been extensively discussed in the last three decades. There have been extensions of the model developed by Solow [7], for example by Merton [4] and Chang/Malliaris [2] who introduced uncertainty in a continous time framework using Itô's lemma.

In this paper a one-sector neoclassical growth model is used with an exogenous resource supply. The dynamics can be described by a diffusion-type stochastic process. This can be interpreted as a problem of an economy that is vulnerable to supply disruptions, for example in the case of resource imports. We will concentrate on a stochastic "Ramsey-Problem" and develop the optimal savings policy for a country that might face supply disturbances.[1]

2. THE MODEL

The source of uncertainty in this model is the amount of resources $R(t)$ available at a certain point of time. A reasonable formulation of the stochastic process for $R(t)$ can be found in analogy to population processes.[2] Let h be the time between two deliveries of resources and $x_i(t+h)$ denote a random variable that represents the "net-followed-deliveries" of the i-th resource supplier, where N is the number of firms that are on the market. It is assumed that the expected number of the "net-followed-deliveries" per supplier grows with a constant rate m. It is further assumed that the (random) derivation from the expected value can be described as the sum of two independent components. Firstly, a systematic component $\sigma\eta(t,h)$, representing the random influences that affect all firms in the same way, for example speculative attacts on the resource market. It is assumed that these components are identically, independently distributed over time. Secondly, a nonsytematic component $\nu_i \; \varepsilon_i(t,h)$ that describes random effects to firm i. This yields:

$$x_i(t+h) = mh + \sigma\eta(t,h) + \nu_i\varepsilon_i(t,h) \qquad i = 1,2,\ldots,N \tag{1}$$

To get a stochastic differential equation, note that:

$$R(t+h) - R(t) = \sum_{i=1}^{N} x_i(t+h) \tag{2}$$

$$\Rightarrow \qquad R(t+h) - R(t) = mRh + \sigma R\eta(t,h) + \sum^{i} \nu_i\varepsilon_i(t,h) \; ,$$

[1] To deal with supply disturbances the economy may use "Buffer Stocks". This point has been extensively discussed in the literature, for example by Newbery/Stiglitz [6], Newbery [5] and Hallett [3], and will not be discussed here.

[2] Cf. Cox/Miller [2].

n, σ and ν_i are constant. For a sufficient large number of firms, we can neglect the non-systematic components. Taking the limit for $h \to 0$ this yields the stochastic differential equation for the resource dynamics:

$$d\,R(t) = m\,R(t)\,dt + \sigma\,R(t)\,dZ, \tag{3}$$

where Z stands for a "Wiener-Process" with the instantaneous mean and variance $m\,R$ and $\sigma^2 R^2$ per unit of time, respectively. For the production of the economy's single output Y there exists a constant returns to scale, strictly concave and linear homogenous production function F.

$$Y = F\,(\,K(t),\,L(t),\,R(t)\,), \tag{4}$$

where $K(t)$ denotes the capital stock, $L(t)$ the labor force, and $R(t)$ the amount of available resources. We consider a small economy and the resources that are available follow an exogenous given growth path that is subject to stochastic fluctuations, described by equation *(3)*.[3] Note that in this model no current uncertainty exists, but only future uncertainty. Hence, competitive factor shares are the same as in the certainty case and well defined. Flexible factor prices ensure market clearing at each point of time. The labor force L grows with an exogenously given growth rate n.

$$dL = n\,L\,dt. \tag{5}$$

Capital can be accumulated by private savings S, with the marginal savings rate s, where[4] $0 < s < 1$.

$$\dot{K} = S = s\;F(\,K\,,\,L\,,\,R\,). \tag{6}$$

What follows is the consideration of the per capita case, defining $y := Y/L$, $k := K/L$ and $r := R/L$, the usual procedure gives us the per capita production function f.

$$y = f\,(\,k,\,r\,),\quad with\quad f_k\,,\,f_r > 0\;;\;f_{kk}\,,\,f_{rr} < 0\;; \tag{7}$$

$$f_{kk}\,f_{rr} - (\,f_{kr}\,)^2 > 0$$

---

[3] This is also consistent with the rational expectations hypotheses.

[4] A dot above a variable denotes its partial derivative with respect to time. A subscript denotes a partial derivative. Time index is omitted when there is no ambiguity.

The per capita capital accumulation can be written as:

$$\dot{k} = s\, f(k, r) - n\, k. \tag{8}$$

Using Itô´s lemma the change of the per capita resource supply is given by (9), where $o(dt)$ is defined as: $\lim\ o(dt)/dt = 0$ as $dt \to 0$.

$$dr = (m - n)\, r\, dt + \sigma\, r\, dZ + o(dt) \tag{9}$$

Per capita consumption is obtained immediately from (6) as $c = (1 - s)\, f(k, r)$. The country is assumed to have a strictly concave "Neumann-Morgenstern-Type" utility function u, which depends only on consumption.

$$u = u\{c\} = u\{(1-s)\, f(k,r)\}. \tag{10}$$

3. THE OPTIMAL SAVINGS RATE UNDER UNCERTAINTY

We are now able to determine the optimal savings rate under uncertainty. In the stochastic approach the country maximizes its expected discounted utility over time $(0,T)$, with δ as the social discount rate.

$$\max_{s}\ E \int_0^T e^{-\delta t}\, u(s, k, r)\, dt \tag{11}$$

$$\begin{aligned} \text{s.t.}\quad & dk = [\, s\, f(k,r) - nk\,]\, dt \\ & dr = (m - n)\, r\, dt + \sigma\, r\, dZ + o(dt) \\ & k(0) = k_0 \\ & r(0) = r_0 \end{aligned}$$

To get the first order conditions, we define $J(k, r, t)$ as the optimal value function. Expanding (11) around (k, r, t), taking the expectation, dividing by dt and letting $dt \rightarrow 0$ gives us the desired "dynamic programming" equation:

$$0 = \max_s \left[e^{-\delta t} u(k, r, s) + \frac{1}{dt} E\, dJ \right] \tag{12}$$

$$\Rightarrow \quad 0 = J_t + \max_s \{ e^{-\delta t} u(k, r, s) + J_k [s f(k,r) - n k] + J_r (m - n) r + 0.5 J_{rr} \sigma^2 r^2 \}$$

Maximizing over s yields:

$$e^{-\delta t} u'(c) = J_k \tag{13}$$

(13) is the usual and well known result. The marginal costs of saving should equal the shadow price of the capital stock at each point of time. To develop the dynamics of the savings rate over time we need and expression for $1/dt\, E\, ds$. Differentiating (12) with respect to k gives:

$$e^{-\delta t} \frac{\partial u}{\partial k} + \frac{1}{dt} E\, dJ_k = 0 \tag{14}$$

Itô's "differential generator" is applied to both sides of (13).

$$\frac{1}{dt} E\, d(e^{-\delta t} u'(c)) = \frac{1}{dt} E\, dJ_k \tag{15}$$

With (14) and (15) this yields:

$$\frac{1}{dt} E\, du'(c) = u'(c) [\delta - (f_k - s f_k - s_k f)] \tag{16}$$

Now we need an expression for the left hand side of (16). Again making use of Itô 's lemma it follows:

$$
\begin{aligned}
\frac{1}{dt} E\, du'(c) = {} & u''\,(f_k - s_k f - s f_k)\;(s f(\,) - n k) \\
& + u''\,(f_r - s_r f - s f_r)\;(m - n)\, r \\
& - u'' f\;(1/dt)\, E\, ds \\
& + 0.5\;[\; u'''\,(f_r - s_r f - s f_r)^2 \\
& \qquad + u''\,(f_{rr} - s f_{rr} - 2 s_r f_r - s_{rr} f)\;]\, \sigma^2 r^2 \\
& + 0.5\; u'''\, f^2 s_r^2\, \sigma^2 r^2 \\
& - [\; u'''\,(f f_r - s_r f^2 - s f f_r) + u'' f_r\,]\, s_r\, \sigma^2 r^2
\end{aligned}
\tag{17}
$$

Substituting (17) into (16), dividing by $f u''$, rearranging yields:

$$
\begin{aligned}
\frac{1}{dt} E\, ds = {} & - \frac{u'}{u''}\,\frac{1}{f}\,(\delta - (f_k - s f_k - s_k f)) \\
& + \frac{1}{f}\,(f_k - s_k f - s f_k)\,(sf - nk) \\
& + \frac{1}{f}\,(f_r - s_r f - s f_r)\,(m - n)\, r \\
& + \frac{1}{2f} \left\{ \frac{u'''}{u''}\,(f_r - s_r f - s f_r)^2 \right. \\
& \qquad \left. + (f_{rr} - s f_{rr} - 2 s_r f_r - s_{rr} f) \right\} \sigma^2 r^2 \\
& + \frac{1}{2}\,\frac{u'''}{u''}\, f s_r^2\, \sigma^2 r^2 \\
& - \left\{ \frac{u'''}{u''}\,(f_r - s_r f - s f_r) + \frac{f_r}{f} \right\} s_r\, \sigma^2 r^2
\end{aligned}
\tag{18}
$$

(18) deviates from the certainty case $(\sigma = 0)$ by the last three factors. Putting together these terms we obtain an expression T:

$$T = \Bigg[\underset{\ominus}{\frac{u'''}{u''}} \underset{\oplus}{\Big\{ f_r^2 \left((s-1)^2 + 4 s_r^2 \right) \Big\}} + \{ \underset{\ominus}{f_{rr}} \underset{\oplus}{(1-s)} - \underset{\oplus}{s_{rr}} f \} + 4 \underset{\ominus}{s_r} f_r \Big\{ \frac{u'''}{u''} f (s-1) - 1) \Big\} \Bigg] \sigma^2 \tag{19}$$

T is unambiguously negative if the term in the last brackets of (19) is nonnegative, i.e if $e(u'', c) \leq -1$ with: $e(u'', c) = du'' c / dc\ u''$. If the elasticity of the variation of the marginal utility is sufficiently large, the introduction of uncertainty will lead to an optimal path of s that is less steep than in the certainty case. This, however, is only true when uncertainty is introduced.[5] To interpret that, it is meaningful to go back to the first order conditions. The marginal costs of savings $MC(s)$ should equal the marginal utility of savings $MU(s)$. In the dynamic sense it depends on how these values develop. Due to the properties of the utility function, it can be seen that $MC(s)$ and $MU(s)$ are negatively sloped and convex to the origin. Taking the expectation, we see that expected marginal costs will be higher in the future. This leads to an incentive to reduce the optimal savings rate in the future. Likewise, the expected marginal utility of savings are higher in the future. This, however, leads to a higher savings rate in the future. The net effect depends on the accumulated capital stock. With a low capital stock, the marginal utility that can be gained from building up the stock is very high. Therefore, at the beginning of the program it can be optimal to raise the savings rate. Over time capital is accumulated and the marginal returns are reduced. Finally, the economy reaches point A in Figure 1.

[5] Before the whole path of s under uncertainty can be determined, we have to check the properties of the optimal path under certainty. It is a standard result that the optimal savings rate under certainty will converge monotonically to its steady state level s^{**} from below or from above.

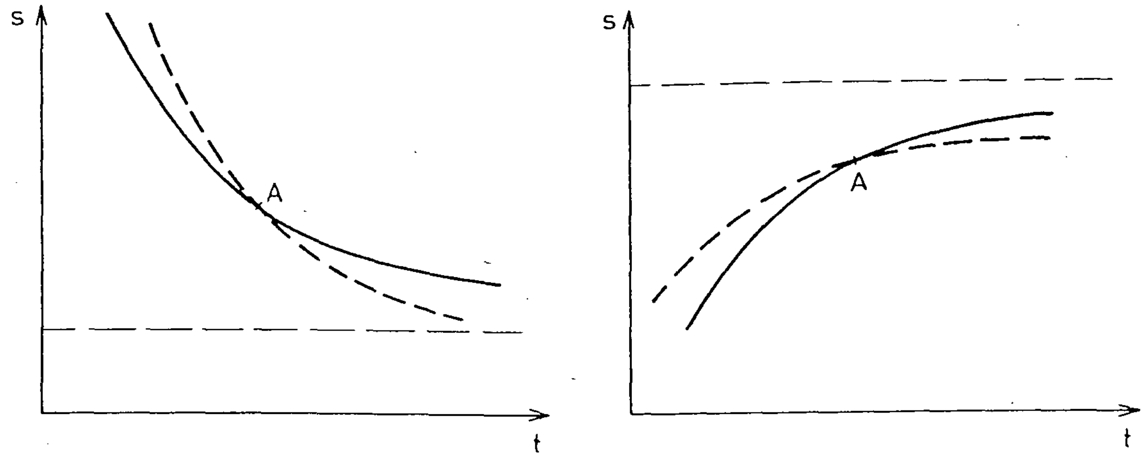
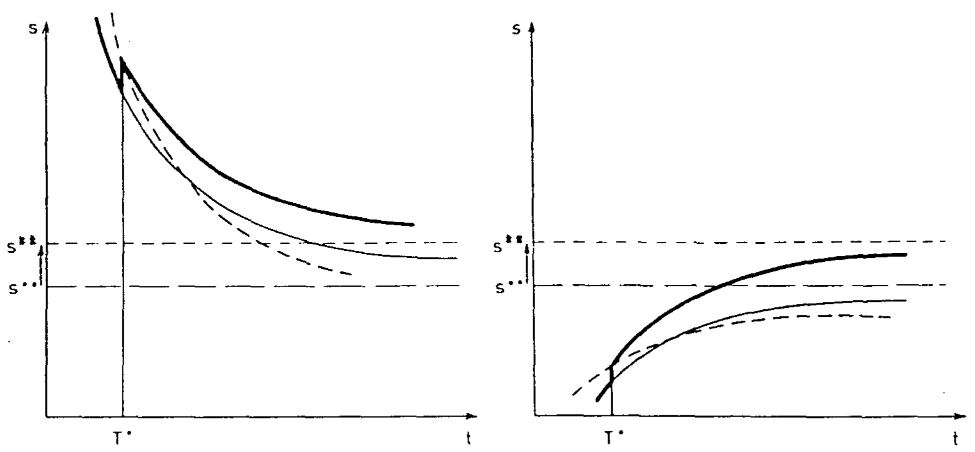
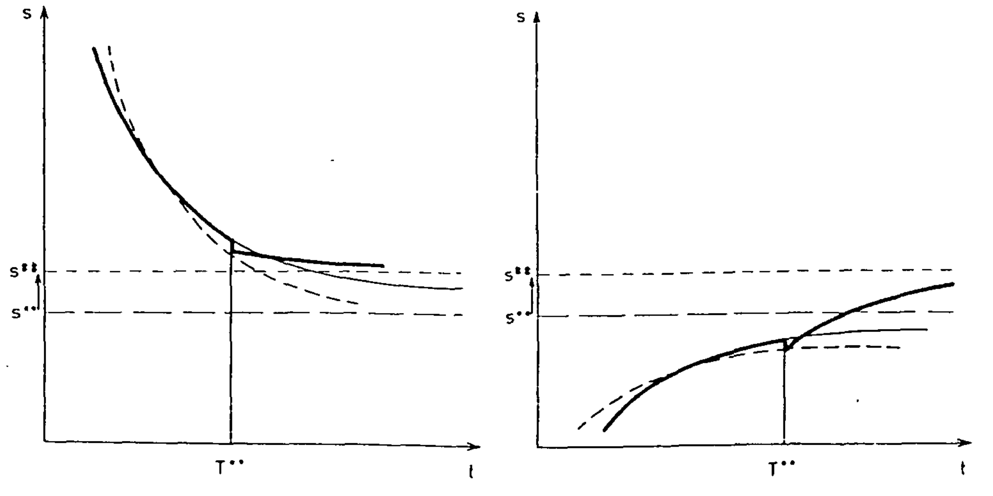


Figure 1:

At time t uncertainty is introduced. The further development of the optimal savings rate depends on the long run steady state distribution of k. Inspection of (18) for $(1/dt)\, E\, ds = 0$ shows that the negative influence of uncertainty in the long run can only be compensated by a higher savings rate. This implies that the steady state savings rate under uncertainty $s^{\#\#}$ is higher than the rate s^{**} under certainty. For the development of s, four cases have to be distinguished: a) Introduction of uncertainty takes place before (after) point A in Figure 1. b) The savings rate is monotonically decreasing or increasing:



Figures 2/3:



Figures 4/5:

Before uncertainty is introduced, the country is on the certainty path (thin line). Once uncertainty occurs (T^* or T^{**}) the economy jumps to the optimal uncertainty path. In situations where the capital stock is small (T^*), s jumps up and converges to the higher steady state level $s^{\#\#}$. In this case uncertainty leads to a higher savings rate. If the accumulated stock is large (T^{**}), it is optimal to reduce the savings rate for some time. In the long run s is again higher than in the certainty case.

4. CONCLUDING COMMENTS

If an economy faces stochastic fluctuations of the resource availability, it is optimal in the long run to increase the domestic savings rate. Uncertainty is partly "substituted" by higher domestic capital accumulation. The short run effects depend crucially on the capital stock so far accumulated. With a high capital stock it may be optimal to lessen the savings rate temporarily. In this model we abstract of the possibility of "Buffer Stocks". In reality the optimal policy will be a mixture of stockpiling and savings rate adjustment.

An application of this model might be found if the time of the introduction (T^* or T^{**}) of uncertainty is interpreted as the beginning of the first oil crisis when countries learned about their vulnerability. For economies that are highly industrialized, i.e. that are endowed with a high capital stock (T^{**}), it was optimal to lessen their savings rate temporarily, as they did. On the other hand, less developed countries (T^{*}) should have increased their savings rate, but failed to do so. Today it is seen that in such countries not enough capital was accumulated in the past, and thus the debt crisis becomes worse and worse. Maybe the reason for this is the *non*-optimal behaviour of the less developed countries after the first oil crisis.

REFERENCES

[1] Chang, F.-R. and A.G. Malliaris, 1987, Asymptotic Growth under Uncertainty: Existence and Uniqueness, Review of Economic Studies 54, 169-174.

[2] Cox, D.A. and H.D. Miller, 1968, The Theory of Stochastic Processes (John Wiley, New York).

[3] Hallett, A.J.H., 1984, Optimal Stockpiling in a high-risk Commodity Market, Journal of Economic Dynamics and Control 8, 211-238.

[4] Merton, R.C., 1975, An Asymptotic Theory of Growth under Uncertainty, Review of Economic Studies 42, 375-393.

[5] Newbery, D.M.G., 1984, Commodity Price Stabilization in Imperfect or Cartelized Markets, Econometrica 52, 563-578.

[6] Newbery, D.M.G. and J.E. Stiglitz, 1982, Optimal Commodity Stockpiling Rules, Oxford Economic Papers 34, 403-427.

[7] Solow, R., 1956, A Contribution to the Theory of Economic Growth, Quarterly Journal of Economics 70, 65-94.

Research leading to this paper was supported by the "Deutsche Forschungsgemeinschaft" (SFB 178) and that support is gratefully acknowledged. I also wish to thank Bernd Fitzenberger, Jesko Hentschel, Michael Koop, Ernst Mohr, Michael Rauscher, Horst Siebert and Timothy Worrall for helpful comments.