Stochastic Neoclassical Growth Model

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Contents

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1 Introduction

One of the messages from the Solow model and the Neoclassical Growth Model is that without growth in productivity, it is impossible to achieve sustained growth in standard of living. Both models converge to steady state, when productivity is constant. In these notes we examine the importance of productivity for business cycles. In particular, we ask how much of the observed business cycles can be accounted for by random shocks to productivity. We create a computation experiment in which a stochastic version on the NGM serves as laboratory for simulated business cycles. The model is almost identical to the deterministic NGM, but we introduce stochastic (random) productivity. We then can compare the observes business cycles facts in the data with the simulated business cycles in the model. The model presented in these notes is the main workhorse for the study of business cycles. Matlab codes for solving and simulating this model are available on the course web page.

2 Stochastic NGM

Just as with the deterministic NGM, we can prove that stochastic NGM is Pareto efficient. Therefore, the equilibrium can be found by solving the corresponding social planner's problem. We assume that the social planner maximizes the *expected utility* of the representative household, subject to the feasibility constraints. Thus, the social planner's problem is:

$$
\begin{aligned}
& & & \max_{\{c_t,h_t,k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-h_t\right) \qquad (1) \\
& & & s.t. \\
\text{[Feasibility]} & & : \quad & c_t + k_{t+1} = A_t k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t \\
\text{[Productivity]} & & : \quad & A_t = A_0 \left(1+\gamma_A\right)^t e^{z_t},\ z_t = \rho z_{t-1} + \varepsilon_t \text{ and } \varepsilon_t \sim i.i.d.\ N\left(0, \sigma_{\varepsilon}^2\right) \\
k_0 & & > \quad & 0 \text{ given}
\end{aligned}
$$

Thus, the productivity consists of a deterministic part $A_0\left(1+\gamma_A\right)^t$, representing the growth trend, and stochastic part, e^{z_t}, which causes fluctuations around the trend. The variable z_t is the random productivity parameter, and ε_t is called white noise process (also known as *innovation* process). The notation E_0 is conditional expectation, given the information available at time 0. In general, E_t is the conditional expectation given the information avaialble at time t.[1] The information available at time t is k_t and A_t, and the information available at time 0 is k_0 and A_0.

The Lagrange function is

$$
\mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-h_t\right) - \sum_{t=0}^{\infty} \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\theta} h_t^{1-\theta} - (1-\delta) k_t\right] \right\}
$$

[1] We could write the conditional expectation in the usual way, $E\left(X_t | \Omega_t\right)$, where Ω_t is the information available at time t.

The first order conditions for h_t, c_t and k_{t+1} are, $\forall t = 0, 1, ...$

$$\begin{aligned}
[c_t] &: \beta^t u_1 (c_t, 1 - h_t) = \lambda_t && (2)\\
[h_t] &: \beta^t u_2 (c_t, 1 - h_t) = \lambda_t (1 - \theta) A_t k_t^{\theta} h_t^{-\theta} && (3)\\
[k_{t+1}] &: \lambda_t = E_t \lambda_{t+1} \left[\theta A_{t+1} k_{t+1}^{\theta - 1} h_{t+1}^{1-\theta} + 1 - \delta \right] && (4)
\end{aligned}$$

Notice the expectation E_t in the Euler equation. Since the optimal path $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ has to solve (1) starting at any time t, the information on which we condition is that of time t (i.e. k_t and A_t).

Dividing (3) by (2) gives the usual condition for optimal allocation of time (labor supply):

$$\frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} = (1 - \theta) A_t k_t^{\theta} h_t^{-\theta}$$

The left hand side is the marginal rate of substitution between leisure and consumption, and the right hand side is the real wage. Notice that (2) for period $t+1$ is $\beta^{t+1} u_1 (c_{t+1}, 1 - h_{t+1}) = \lambda_{t+1}$. Using (2) and the same equation for period $t+1$ together with (4), gives

$$\begin{aligned}
\lambda_t &= E_t \lambda_{t+1} \left[\theta A_{t+1} k_{t+1}^{\theta - 1} h_{t+1}^{1-\theta} + 1 - \delta \right]\\
\beta^t u_1 (c_t, 1 - h_t) &= E_t \beta^{t+1} u_1 (c_{t+1}, 1 - h_{t+1}) \left[\theta A_{t+1} k_{t+1}^{\theta - 1} h_{t+1}^{1-\theta} + 1 - \delta \right]\\
u_1 (c_t, 1 - h_t) &= \beta E_t u_1 (c_{t+1}, 1 - h_{t+1}) \left[\theta A_{t+1} k_{t+1}^{\theta - 1} h_{t+1}^{1-\theta} + 1 - \delta \right]
\end{aligned}$$

The last condition is called the *stochastic Euler equation*. The left hand side shows the *pain* from giving up one unit of consumption and investing it, while the right hand side is the *expected gain* in the next period from extra unit of capital. The necessary conditions for optimal $\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}$ are:

$$\begin{aligned}
\frac{u_2 (c_t, 1 - h_t)}{u_1 (c_t, 1 - h_t)} &= (1 - \theta) A_t k_t^{\theta} h_t^{-\theta} && (5)\\
u_1 (c_t, 1 - h_t) &= \beta E_t u_1 (c_{t+1}, 1 - h_{t+1}) \left[\theta A_{t+1} k_{t+1}^{\theta - 1} h_{t+1}^{1-\theta} + 1 - \delta \right] && (6)\\
c_t + k_{t+1} &= A_t k_t^{\theta} h_t^{1-\theta} + (1 - \delta) k_t && (7)\\
\text{where } A_t &= A_0 (1 + \gamma_A)^t e^{z_t},\ z_t = \rho z_{t-1} + \varepsilon_t \text{ and } \varepsilon_t \sim i.i.d.\ N\left(0, \sigma_{\varepsilon}^2\right)
\end{aligned}$$

In these notes we will assume that $\gamma_A = 0$, since our goal is to study business cycles (i.e. deviations from trend), and we remove the trend from the data, as well as from the model.

At this point you may be wondering how we ended up with E_t when the original problem had E_0. A very useful excersie is to start by deriving the first order conditions for h_0, c_0 and k_1, given the information known at time $t = 0$. Using the above Lagrange, we get the necessary conditions $\{c_0, h_0, k_1\}$:

$$\begin{aligned}
[c_0] &: u_1 (c_0, 1 - h_0) = \lambda_0\\
[h_0] &: u_2 (c_0, 1 - h_0) = \lambda_0 (1 - \theta) A_0 k_0^{\theta} h_0^{-\theta}\\
[k_1] &: \lambda_0 = E_0 \lambda_1 \left[\theta A_1 k_1^{\theta - 1} h_1^{1-\theta} + 1 - \delta \right]
\end{aligned}$$

Now that we know k_1 and A_1 and suppse that we solve the problem in period $t = 1$. The lagrange is

$$\mathcal{L} = E_1 \left\{ \sum_{t=1}^{\infty} \beta^t u\left(c_t, 1 - h_t\right) - \sum_{t=1}^{\infty} \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\theta} h_t^{1-\theta} - \left(1 - \delta\right) k_t\right] \right\}$$

The necessary conditions for $\{c_1, h_1, k_2\}$ are:

$$\begin{aligned} [c_1] &: u_1\left(c_1, 1 - h_1\right) = \lambda_1 \\ [h_1] &: u_2\left(c_1, 1 - h_1\right) = \lambda_1 \left(1 - \theta\right) A_1 k_1^{\theta} h_1^{-\theta} \\ [k_2] &: \lambda_1 = E_1 \lambda_2 \left[\theta A_2 k_2^{\theta - 1} h_2^{1-\theta} + 1 - \delta\right] \end{aligned}$$

Similarly, when we solve for $\{c_2, h_2, k_3\}$, we condition on the information known at $t = 2$, i.e. we know k_2 and A_2. The lagrange at $t = 2$ is

$$\mathcal{L} = E_2 \left\{ \sum_{t=2}^{\infty} \beta^t u\left(c_t, 1 - h_t\right) - \sum_{t=2}^{\infty} \lambda_t \left[c_t + k_{t+1} - A_t k_t^{\theta} h_t^{1-\theta} - \left(1 - \delta\right) k_t\right] \right\}$$

The necessary conditions for $\{c_2, h_2, k_3\}$ are:

$$\begin{aligned} [c_2] &: u_1\left(c_2, 1 - h_2\right) = \lambda_2 \\ [h_2] &: u_2\left(c_2, 1 - h_2\right) = \lambda_2 \left(1 - \theta\right) A_2 k_2^{\theta} h_2^{-\theta} \\ [k_3] &: \lambda_2 = E_2 \lambda_3 \left[\theta A_3 k_3^{\theta - 1} h_3^{1-\theta} + 1 - \delta\right] \end{aligned}$$

The abobe steps illustrate that decisions made at time t utilize the information available at that time, and therefore the expectation is taken based on information available at time t, i.e. E_t.

3 Productivity Process

The stochastic productivity is the only source of uncertainty in this economy and is the engine of the Real Business Cycle doctrine (RBC). It is therefore important to discuss in detail the properties of the probabilistic model that we adopt for modelling the process of productivity - A_t. Suppose that we assume that

$$\begin{aligned} A_t &= A_0 \left(1 + \gamma_A\right)^t e^{z_t} \\ \text{where } z_t &= \rho z_{t-1} + \varepsilon_t \text{ and } \varepsilon_t \sim i.i.d.\ N\left(0, \sigma_\varepsilon^2\right) \end{aligned}$$

First, observe that

$$\log\left(A_t\right) = \log\left(A_0\right) + t \log\left(1 + \gamma_A\right) + z_t$$

Thus, we assume that A_t fluctuates around a constant growth trend with average growth rate of γ_A. Therefore, the $\log(A_t)$ has a linear growth component $\log\left(A_0\right) + t \log\left(1 + \gamma_A\right)$ and a cyclical component z_t. The model that we constructed however does not have the growth component, i.e. we set $\gamma_A = 0$, since our goal is to generate cyclical data (deviations from trend). If we assumed $\gamma_A > 0$, we would have to rewrite the necessary conditions in terms of detrended variables. Therefore, the productivity in our model is just $A_t = A_0 e^{z_t}$. The parameter A_0 affects the levels of simulated time series in the model, but not the deviations from the steady state. Therefore, we typically set $A_0 = 1$ for simplicity. We have already practiced how to decompose the log of a time series into a linear trend and the cyclical part, so it should be clear by now how one can obtain data on z_t.

Next, we want to discuss the special structure of $z_t = \rho z_{t-1} + \varepsilon_t$. This stochastic process is called AR(1), i.e. first-order autoregressive process. It is autoregressive because it looks like a regression of z_t on itself, with one lag. An AR(2) process would be $z_t = \rho_1 z_{t-1} + \rho_2 z_{t-2} + \varepsilon_t$, etc. For now, the normality of ε_t is not important. We will need some distributional assumptions later, when we simulate the process of z_t. What we focus on is the assumptions that ε_t is *i.i.d.* (identically and independently distributed), that $E(\varepsilon_t) = 0$ and that $Var(\varepsilon_t) = \sigma_\varepsilon^2$. The last assumption states that the variance of ε_t is constant, and does not change with time. From these assumptions, we can derive certain properties of z_t.

3.1 Mean

First, we would like to find the mean and variance of z_t. For this purpose, we recursively substitute the expression of z_{t-1} into z_t:

$$\begin{aligned} z_t &= \rho(\rho z_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \rho^2 z_{t-2} + \rho\varepsilon_{t-1} + \varepsilon_t \end{aligned}$$

Next, substitute z_{t-2}

$$\begin{aligned} z_t &= \rho^2(\rho z_{t-3} + \varepsilon_{t-2}) + \rho\varepsilon_{t-1} + \varepsilon_t \\ &= \rho^3 z_{t-3} + \rho^2\varepsilon_{t-2} + \rho\varepsilon_t + \varepsilon_t \end{aligned}$$

This recursive substitution becomes in the limit

$$z_t = \lim_{k\to\infty} \rho^k z_{t-k} + \sum_{k=0}^{\infty} \rho^k \varepsilon_{t-k}$$

We will always assume that $|\rho| < 1$, so the first term vanishes. This assumption is called *stationarity condition*. Thus,

$$z_t = \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2\varepsilon_{t-2} + \ldots$$

Observe that z_t is a weighted average of all the past shocks $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$ This is called the *moving average* representation of z_t. Since $E(\varepsilon_t) = 0\ \forall t$, we have

$$E(z_t) = E(\varepsilon_t) + \rho E(\varepsilon_{t-1}) + \rho^2 E(\varepsilon_{t-2}) + \ldots = 0$$

3.2 Variance

$$\begin{aligned} Var(z_t) &= Var(\varepsilon_t) + \rho^2 Var(\varepsilon_{t-1}) + \rho^4 Var(\varepsilon_{t-2}) + ... \\ &= \sigma_\varepsilon^2 + \rho^2\sigma_\varepsilon^2 + \rho^4\sigma_\varepsilon^2 + ... \\ &= \sigma_\varepsilon^2 \sum_{t=0}^{\infty} \left(\rho^2\right)^t = \frac{\sigma_\varepsilon^2}{1-\rho^2} \end{aligned}$$

Notice that we used the assumption that the $\varepsilon' s$ are independent, which implies that the variance of the sum is the sum of the variances.

3.3 Covariance and Correlation

$$\begin{aligned} Cov(z_t, z_{t-1}) &= Cov(\rho z_{t-1} + \varepsilon_t, z_{t-1}) \\ &= \rho Cov(z_{t-1}, z_{t-1}) + Cov(\varepsilon_t, z_{t-1}) \\ &= \rho Var(z_t) + 0 \end{aligned}$$

The last term is zero because z_{t-1} depends on all the previous shocks $\varepsilon_{t-1}, \varepsilon_{t-2}, ...$, which are not correlated with the current shock ε_t.

$$\begin{aligned} Cov(z_t, z_{t-2}) &= Cov(\rho z_{t-1} + \varepsilon_t, z_{t-2}) \\ &= \rho Cov(z_{t-1}, z_{t-2}) \\ &= \rho^2 Var(z_t) \end{aligned}$$

Similarly,

$$\begin{aligned} Cov(z_t, z_{t-k}) &= \rho^k Var(z_t) \\ Corr(z_t, z_{t-k}) &= \frac{Cov(z_t, z_{t-k})}{\sqrt{Var(z_t)}\sqrt{Var(z_{t-k})}} = \rho^k \end{aligned}$$

The covariance tells us that the process is persistent, i.e. adjacent shocks are correlated, but this correlation diminishes when the two shock are far apart. For example, if $\rho = 0.95$, then $Corr(z_t, z_{t-1}) = 0.95$ and $Corr(z_t, z_{t-25}) = 0.95^{25} = 0.28$.

3.4 Prediction

Suppose that we know the value of the shock at time t and we want to make a prediction about the next shock. Thus, we want to find the conditional expectation of z_{t+1} given z_t.

$$\begin{aligned} E(z_{t+1}|z_t) &= E(\rho z_t + \varepsilon_{t+1}|z_t) \\ &= E(\rho z_t|z_t) + E(\varepsilon_{t+1}|z_t) \\ &= \rho z_t + 0 \end{aligned}$$

The last term is zero because $\varepsilon' s$ are independent of each other, so ε_{t+1} is independent of all the past $\varepsilon' s$. Conditional expectation, is equal to unconditional expectation, when we condition on independent variable. Thus, $E(\varepsilon_{t+1}|z_t) = E(\varepsilon_{t+1}) = 0$. Suppose that we want to predict two periods ahead:

$$\begin{aligned}
E(z_{t+2}|z_t) &= E(\rho z_{t+1} + \varepsilon_{t+2}|z_t) \\
&= \rho E(z_{t+1}|z_t) \\
&= \rho^2 z_t
\end{aligned}$$

Similarly,

$$E(z_{t+k}|z_t) = \rho^k z_t$$

For example, suppose that $\rho = 0.9$ and the shock today is $z_t = 3$. What is your prediction of z_{t+1} and z_{t+10}? Answer:

$$\begin{aligned}
E(z_{t+1}|z_t = 3) &= 0.9 \cdot 3 = 2.7 \\
E(z_{t+10}|z_t = 3) &= 0.9^{10} \cdot 3 = 1.046
\end{aligned}$$

Thus, we predict that the next period shock will be close to today's, but after 10 periods there is a decay due to $0 < \rho < 1$.

3.5 Summary of AR(1) properties

Suppose that

$$\begin{aligned}
z_t &= \rho z_{t-1} + \varepsilon_t \\
\text{where } \varepsilon_t &\sim i.i.d. \text{ with } E(\varepsilon_t) = 0,\ Var(\varepsilon_t) = \sigma_\varepsilon^2
\end{aligned}$$

Then

$$\begin{aligned}
\text{[Mean]} &: E(z_t) = 0 \\
\text{[Variance]} &: Var(z_t) = \frac{\sigma_\varepsilon^2}{1-\rho^2} \\
\text{[Autocovariance]} &: Cov(z_t, z_{t-k}) = \rho^k \left(\frac{\sigma_\varepsilon^2}{1-\rho^2}\right) \\
\text{[Autocorrelation]} &: Corr(z_t, z_{t-k}) = \rho^k \\
\text{[Prediction]} &: E(z_{t+k}|z_t) = \rho^k z_t
\end{aligned}$$

3.6 Estimation

The parameters of this probabilistic model are ρ and σ_ε. The simplest way to estimate ρ is running the regression

$$\begin{aligned}
z_t &= \rho z_{t-1} + \varepsilon_t \\
&\Rightarrow \hat{\rho}
\end{aligned}$$

and $\hat{\sigma}_\varepsilon$ can be found from the residuals of the above regression, i.e.

$$\hat{\sigma}_\varepsilon = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(z_t - \hat{\rho} z_{t-1})^2}$$

Alternatively, we can estimate

$$\widehat{Var\left(z_t\right)} = \frac{1}{n-1}\sum_{t=1}^{n}\left(z_t - E\left(z_t\right)\right)^2 = \frac{1}{n-1}\sum_{t=1}^{n} z_t^2$$

And then, use $Var\left(z_t\right) = \frac{\sigma_\varepsilon^2}{1-\rho^2}$ to estimate σ_ε

$$\begin{aligned} \widehat{Var\left(z_t\right)} &= \frac{\sigma_\varepsilon^2}{1-\hat{\rho}^2} \\ &\Rightarrow \hat{\sigma}_\varepsilon^2 = \widehat{Var\left(z_t\right)}\left(1-\hat{\rho}^2\right) \end{aligned}$$

4 Solving the Model

The codes posted on the course web page use the method called "extension of deterministic path". This is easy to implement method, but not very efficient in terms of computational time. Suppose that at time $t=0$ we observe a value z_0, and suppose that there are no more shocks in the fututre: $\varepsilon_t = 0\ \forall t > 0$. In this case the future values of z_t are as follows: $z_1 = \rho z_0$, $z_2 = \rho^2 z_0$,...,$z_t = \rho^t z_0$,... Thus, $z_t \to 0$ as $t \to \infty$, and $e^{z_t} \to 1$ as $t \to \infty$, and eventually the model will converge to the steady state, as figure 1 shows.

The above graphs are called *impulse response functions*, since they show the response of endogenous variables to a one-time exogenous shock to technology. Solving the deterministic model with no more shocks after the very first period, will give us the values of (c_0, h_0, k_1) that are approximately correct. This is because the household chooses (c_0, h_0, k_1) at time 0, and the optimal choice of these variables does not depend on future shocks. Thus, we can solve the deterministic model given $\{z_t\}_{t=0}^{T} = \{\rho^t z_0\}_{t=0}^{T}$, and T is some large final period when the system is close to steady state. Notice that by solving the deterministc $\rho^t z_0$ use an approximation of the Euler equation $E_t\left(f\left(z_{t+1}\right)\right) \approx f\left(E_t\left(z_{t+1}\right)\right)$, for some non-linear function f. By Jensen's inequality, we know that the above is only equal for linear functions f, and that is the reaosn for the $\approx$ sign. Thus, solving for the deterministic path gives approximately correct value of initial capital for the next period, k_1. Now, suppose that in the next period the economy is hit by another shock, $\varepsilon_1 \neq 0$. Then, $z_1 = \rho z_0 + \varepsilon_1$. If there will not be any shocks after $t=1$, the model will converge to a steady state, and we can solve for the deterministic path that would give an approximately correct solution for k_2. By repeatedly solving for deterministic path from time t onward, we obtain a sequence of endogenous variables for the model.

5 Evaluating the Model's Performance

In order to evaluate the model, we need to start with specific question, for example "how much of the observed fluctuations in the U.S. economy can be accounted for by shocks to productivity?". The first step in answering that question is to document some business cycle facts from the data. After that, the model is calibrated, and simulated to generate artificial data. Finally, when we have the facts from the data and the simulated observations from

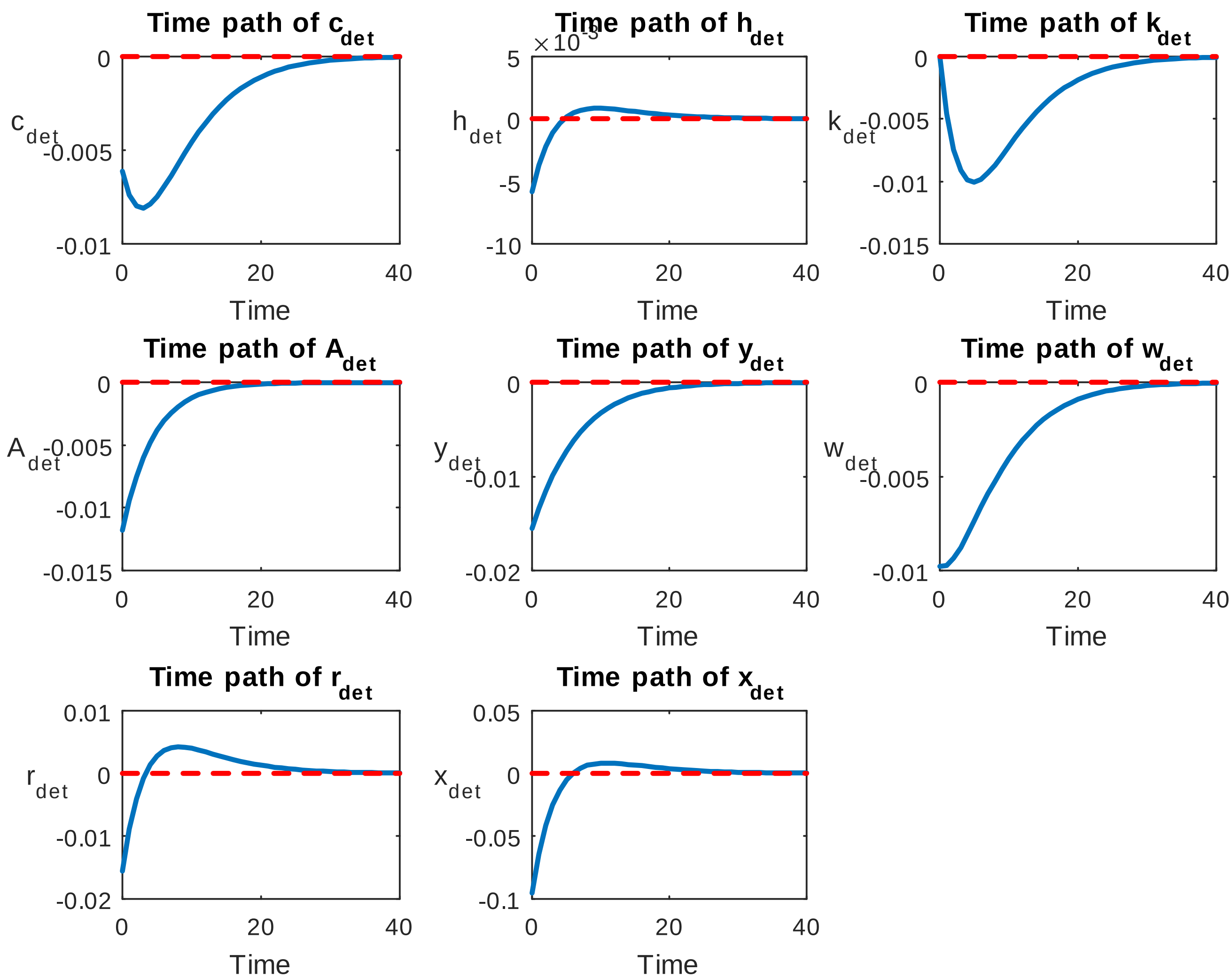


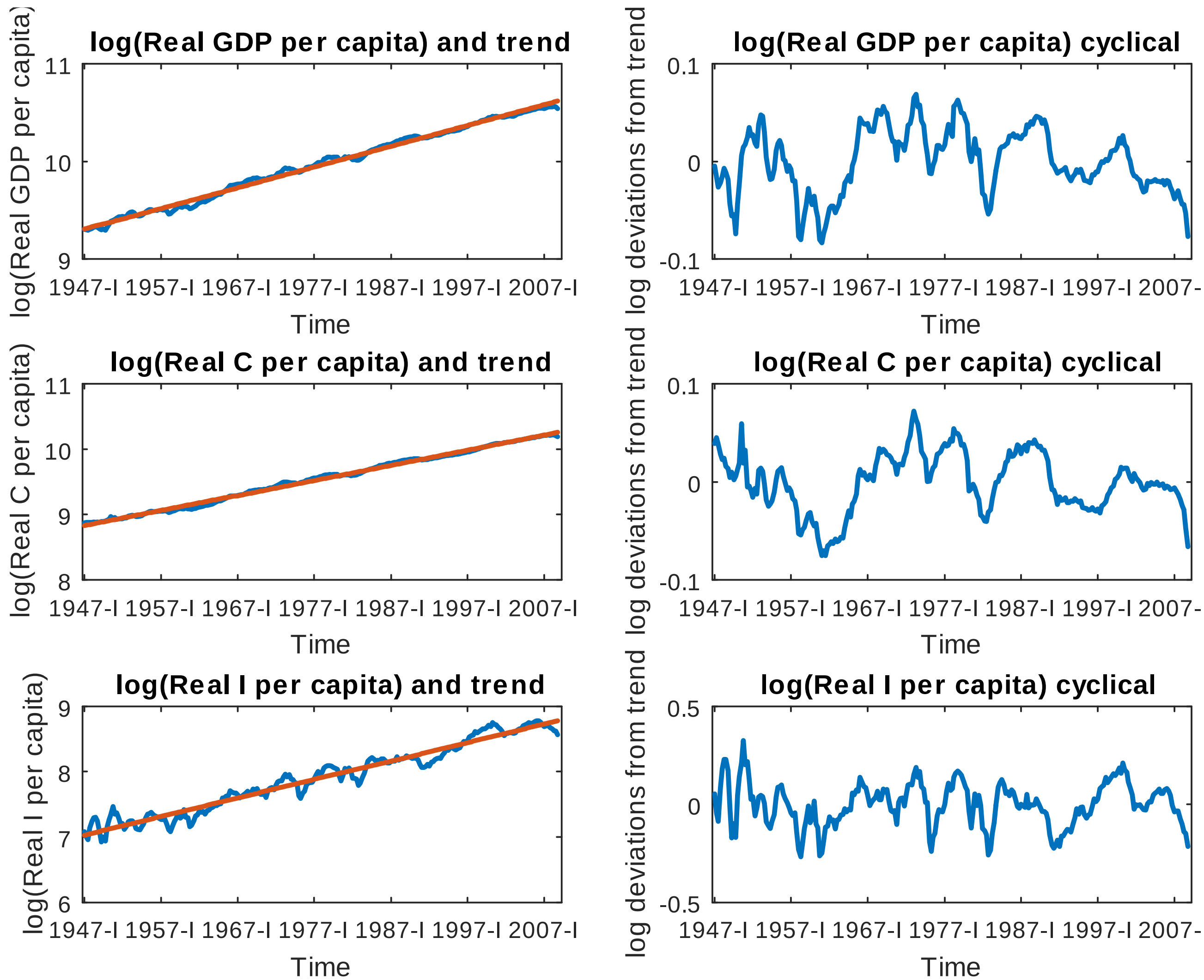
Figure 1: Impulse Response Functions

the model, we can evaluate the performance of the model. We can ask how well does the model replicate the facts from the data. The above procedure is called the computational experiment.

5.1 Facts from the data

There are many business cycles facts that one might want to document. At the most basic level, economists are interested in *volatility* and *comovement*. Volatility refers to the deviations of key economic variables from some trend. The next figure shows one way that economists use to decompose a time series into *trend* and *cyclical part*.

The top panel shows the natural log of real GDP, real personal consumption expenditures and real investment, together with a linear trend. The bottom graphs show the deviations of the log(data) from it's linear trend. The bottom graphs therefore depict the *detrended*



data or the *cyclical part* of the data. Economists are interested in measuring the volatility of the cyclical part, as measured by it's standard deviation. The numbers on the vertical axis give approximately the percentage deviation of the original variable from the trend. For example, notice that the deviations of investment from it's trend are much larger than the deviations of output from the trend. This is reflected in larger standard deviation of the detrended investment. We therefore say that investment is more volatile than output in the U.S. economy.

Comovement describes how the movement of key variables coincides with the movement of real GDP. If deviations from trend of some variable are in the same direction as those of real GDP, we say that the variable is *pro-cyclical*, and if the deviations are in the opposite direction of real GDP, the variable is called *counter-cyclical*. A convenient statistic for measuring comovement is the correlation coefficient.

5.2 Model economy

The main workhorse for the study of business cycles is the neoclassical growth model.

$$\begin{aligned}
& \max E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1-h_t\right) \\
& s.t. \\
\text{[Feasibility]} \quad : \quad & c_t + k_{t+1} = e^{z_t} k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t \\
\text{where } z_t \quad = \quad & \rho z_{t-1} + \varepsilon_t \text{ and } \varepsilon_t \sim i.i.d.\ N\left(0, \sigma_{\varepsilon}^2\right)
\end{aligned}$$

We calibrate the model economy to match some long-term moments in the data. For this purpose we need to assume that there are no technology shocks, so that the model will converge to a steady state. First we need to choose the functional form for the utility function:

$$u\left(c_t, 1-h_t\right) = \alpha \log c_t + (1-\alpha) \log\left(1-h_t\right)$$

The first order conditions become:

$$\begin{aligned}
\frac{1-\alpha}{\alpha} \frac{c_t}{1-h_t} &= (1-\theta) k_t^{\theta} h_t^{-\theta} \\
\frac{c_{t+1}}{c_t} &= \beta\left[\theta k_{t+1}^{\theta-1} h_{t+1}^{1-\theta} + 1 - \delta\right] \\
c_t + k_{t+1} &= k_t^{\theta} h_t^{1-\theta} + (1-\delta) k_t
\end{aligned}$$

At steady state:

$$\frac{1-\alpha}{\alpha} \frac{c}{1-h} = (1-\theta) \frac{y}{h} \tag{8}$$

$$1 = \beta\left[\theta \frac{y}{k} + 1 - \delta\right] \tag{9}$$

$$c + \delta k = y \tag{10}$$

From the NIPA, we know that $\theta = 0.35$. From equation (10) we have

$$\begin{aligned}
\frac{c}{k} + \delta &= \frac{y}{k} \\
\delta &= \frac{y}{k} - \frac{c}{k}
\end{aligned}$$

The value of δ depends on the frequency of the data (annual, quarterly). From equation (9) we find β.

$$\begin{aligned}
1 &= \beta\left[\theta \frac{y}{k} + 1 - \delta\right] \\
\beta &= \frac{1}{\theta \frac{y}{k} + 1 - \delta}
\end{aligned}$$

To calibrate α we need to use equation (8), and some evidence on the fraction of time endowment allocated to work. For example, if the discretionary time endowment is 100 hours per week and the average workweek is 40 hours, then we set $h = 0.4$ in equation (8) and solve for α.

$$
\begin{aligned}
\frac{1-\alpha}{\alpha}\frac{c}{1-h} &= (1-\theta)\,\frac{y}{h} \\
\frac{1-\alpha}{\alpha} &= (1-\theta)\,\frac{1-h}{h}\frac{y}{c} \equiv B \\
\alpha &= \frac{1}{1+B}
\end{aligned}
$$

In addition to the above parameters, we need to calibrate the parameters that govern the TFP, i.e. ρ and σ_ε. First, we need to obtain data on TFP, using

$$
\begin{aligned}
Y_t &= A_t K_t^{\theta} L_t^{1-\theta} \\
&\Rightarrow\ A_t = \frac{Y_t}{K_t^{\theta} L_t^{1-\theta}}
\end{aligned}
$$

This requires data on real output, real stock of physical capital and some measure of labor input - usually index of aggregate hours. Since we assume that $A_t = A_0\,(1+\gamma_A)^t\,e^{z_t}$, to obtain a series of $\{z_t\}$ one needs to take logs, and run the regression

$$
\begin{aligned}
\log(A_t) &= \log(A_0) + t\log(1+\gamma_A) + z_t \\
&\quad \text{or} \\
\log(A_t) &= \beta_0 + \beta_1 t + z_t
\end{aligned}
$$

The residuals of this regression give us an estimate of $\{z_t\}$. Finally, to estimate ρ we run the regression of z_t on its lag, i.e.

$$
z_t = \rho z_{t-1} + \varepsilon_t
$$

This gives the estimated value of ρ. The estimate of σ_ε is obtained simply as the standard deviation of the residuals from the last regression.

5.3 Running the experiments

The last stage in our computational experiment consists of using the calibrated model to generate artificial data. This can be done with codes like RBC.m that supplements this chapter. We compute the volatility and comovement statistics for the artificial data, and present it side by side with their data counterparts. The next table is an example of what the comparison of data with the model might look like.

Standard Deviation			
Variable	Data	Model	Model/Data
Y	0.0404	0.0304	0.7515
C	0.0346	0.0221	0.6392
I	0.0788	0.1144	1.4517
H	0.0290	0.0090	0.3119
w	0.0696	0.0247	0.3541

Standard Deviation relative to output		
Variable	Data	Model
Y	1.0000	1.0000
C	0.8548	0.7270
I	1.9499	3.7667
H	0.7170	0.2976
w	1.7224	0.8116

Cross correlation with output		
Variable	Data	Model
Y	1.0000	1.0000
C	0.8963	0.8836
I	0.5986	0.8383
H	0.6101	0.7242
w	0.9335	0.9678

In this example, the model generates about 75% of the observed volatility in output. Remember that the only source of fluctuations in the model is productivity shocks, while in the data fiscal and monetary policy shock might contribute to the observed fluctuations. The model also captures the consumption smoothing behavior observed in the data, in that consumption is less volatile than output and investment is more volatile. The model generates much greater volatility in investment than in the data. Moreover, the model generates much smaller volatility in the labor market variables than in the data.

6 Conclusions

We constructed a model economy to simulate business cycles and compare the simulated results to the data. The model captures some features from the data well and others not so well. The next step in business cycle research is to try and understand what features of the model are responsible for the failures and modify the model accordingly. For example, some economists introduce more elaborate structure into the labor market, in the form of sticky wages or search. One advantage of these extensions is that the model can generate patterns of unemployment, similar to the ones observed. These and other extensions are beyond the scope of this course.

7 Appendix A: Analytical Solution

A special case of the stochastic NGM has analytical solution. Consider a model with inelastic labor supply, log utility, $u(c_t) = \log(c_t)$, Cobb-Douglas production function, $y_t = A_t k_t^{\theta}$, and full depreciation, $\delta = 1$. We can show that the optimal policy for investment is $k_{t+1} = s y_t$

and for consumption $c_t = (1-s)\, y_t$. The Euler Equation becomes:

$$
\begin{aligned}
\frac{1}{c_t} &= \beta E_t \left[\frac{1}{c_{t+1}} \theta A_{t+1} k_{t+1}^{\theta-1} \right] \\
\text{or} \quad 1 &= \beta E_t \left[\frac{c_t}{c_{t+1}} \theta A_{t+1} k_{t+1}^{\theta-1} \right]
\end{aligned}
$$

Substitute the proposed policies into the Euler Equation:

$$
\begin{aligned}
1 &= \beta E_t \left[\frac{(1-s)\, A_t k_t^{\theta}}{(1-s)\, A_{t+1} k_{t+1}^{\theta}} \theta A_{t+1} k_{t+1}^{\theta-1} \right] \\
1 &= \beta E_t \left[\theta \frac{A_t k_t^{\theta}}{k_{t+1}} \right]
\end{aligned}
$$

Plug $k_{t+1} = s A_t k_t^{\theta}$, and simplify:

$$
\begin{aligned}
1 &= \beta E_t \left[\theta \frac{A_t k_t^{\theta}}{s A_t k_t^{\theta}} \right] \\
1 &= \beta E_t \left[\theta \frac{1}{s} \right] \\
&\Rightarrow \quad s = \beta\theta
\end{aligned}
$$

Thus, an optimal policy is $k_{t+1} = \beta\theta A_t k_t^{\theta}$, $c_t = (1-\beta\theta)\, A_t k_t^{\theta}$.

8 Appendix B: Two-Period model, State Space Approach

Consider a 2-period consumption and saving/investment problem under uncertainty. The current period state is known, but uncertainty about the second period is represented by different possible states of nature, indexed $s = 1, 2, ..., S$. Let π_s be the probability that state s occurs. The household receives wage w_t in the current period, and chooses consumption c_t, c_{t+1}^s at states $s = 1, 2, ..., S$, and assets k_{t+1}. The rate of return on assets in the next period r_{t+1} is unknown in period t, and depends in state of nature . Thus, we denote the rate of return in state s by r_{t+1}^s. We can assume that future wage is also uncertain, i.e. w_{t+1}^s is the next period wage in state s (although this generality does not affect our main point). The household's income in the second period, in state s, is $\left(1 + r_{t+1}^s\right) k_{t+1} + w_{t+1}^s$. The household's problem is:

$$
\begin{aligned}
& \max_{c_t, c_{t+1}^s, k_{t+1}} \sum_{s=1}^{S} \pi_s \left[u(c_t) + \beta u\left(c_{t+1}^s\right) \right] \\
& s.t. \\
[BC_t] \quad &: \quad c_t + k_{t+1} = w_t \\
[BC_{t+1}] \quad &: \quad c_{t+1}^s = \left(1 + r_{t+1}^s\right) k_{t+1} + w_{t+1}^s, \quad s = 1, 2, ..., S
\end{aligned}
$$

Notice that in the second period, there are S possible budget constraints, and only one of them will actually realize. But at time t, the household does not know which budget constraint will realize. The Lagrange function associated with the above problem is

$$\mathcal{L} = u(c_t) + \beta \sum_{s=1}^{S} \pi_s u\left(c_{t+1}^s\right) - \lambda_t \left[c_t + k_{t+1} - w_t\right] - \sum_{s=1}^{S} \lambda_{t+1}^s \left[c_{t+1}^s - \left(1 + r_{t+1}^s\right) k_{t+1} - w_{t+1}^s\right]$$

First order conditions:

$$\begin{aligned} [c_t] &: \; u'(c_t) - \lambda_t = 0 \\ \left[c_{t+1}^s\right] &: \; \beta \pi_s u'\left(c_{t+1}^s\right) - \lambda_{t+1}^s = 0, \quad s = 1, 2, ..., S \\ [k_{t+1}] &: \; -\lambda_t + \sum_{s=1}^{S} \lambda_{t+1}^s \left(1 + r_{t+1}^s\right) = 0 \end{aligned}$$

Notice that 1 unit of asset k pays return in all states of nature. Plugging λ_t and λ_{t+1}^s from the first two conditions into the Euler Equation, gives:

$$\begin{aligned} \lambda_t &= \sum_{s=1}^{S} \lambda_{t+1}^s \left(1 + r_{t+1}^s\right) \\ u'(c_t) &= \beta \sum_{s=1}^{S} \pi_s u'\left(c_{t+1}^s\right) \left(1 + r_{t+1}^s\right) \end{aligned} \tag{11}$$

Notice that the last equation can be written as

$$u'(c_t) = \beta E_t \left[u'\left(c_{t+1}^s\right)\left(1 + r_{t+1}^s\right)\right]$$

Compare this stochastic Euler Equation with the one in (6).

The above optimal investment condition can be derived from a slightly modified, equivalent Lagrange function:

$$\begin{aligned} \mathcal{L} &= u(c_t) + \beta \sum_{s=1}^{S} \pi_s u\left(c_{t+1}^s\right) - \lambda_t \left[c_t + k_{t+1} - w_t\right] - \sum_{s=1}^{S} \pi_s \frac{\lambda_{t+1}^s}{\pi_s} \left[c_{t+1}^s - \left(1 + r_{t+1}^s\right) k_{t+1} - w_{t+1}^s\right] \\ \mathcal{L} &= \sum_{s=1}^{S} \pi_s \left\{u(c_t) + \beta u\left(c_{t+1}^s\right) - \lambda_t \left[c_t + k_{t+1} - w_t\right] - \psi_{t+1}^s \left[c_{t+1}^s - \left(1 + r_{t+1}^s\right) k_{t+1} - w_{t+1}^s\right]\right\} \end{aligned}$$

where $\psi_{t+1}^s \equiv \lambda_{t+1}^s / \pi_s$ is scaled multiplier. Notice that we did not change the original Lagrange function, and that it can be written in expected value form:

$$\mathcal{L} = E_t \left\{u(c_t) + \beta u\left(c_{t+1}^s\right) - \lambda_t \left[c_t + k_{t+1} - w_t\right] - \psi_{t+1}^s \left[c_{t+1}^s - \left(1 + r_{t+1}^s\right) k_{t+1} - w_{t+1}^s\right]\right\}$$

The first order necessary conditions are

$$\begin{aligned} [c_t] &: \; u'(c_t) - \lambda_t = 0 \\ \left[c_{t+1}^s\right] &: \; \beta u'\left(c_{t+1}^s\right) - \psi_{t+1}^s = 0, \quad s = 1, 2, ..., S \\ [k_{t+1}] &: \; -\lambda_t + \sum_{s=1}^{S} \pi_s \psi_{t+1}^s \left(1 + r_{t+1}^s\right) = 0 \end{aligned}$$

Plugging ψ_t and ψ_{t+1}^s from the first two conditions into the Euler Equation, gives:

$$\begin{aligned}\lambda_t &= \sum_{s=1}^{S} \psi_{t+1}^s \left(1 + r_{t+1}^s\right) \\ u'(c_t) &= \beta \sum_{s=1}^{S} \pi_s u'\left(c_{t+1}^s\right)\left(1 + r_{t+1}^s\right)\end{aligned}$$

Notice that the last Euler Equation is identical to (11). Thus, we can derive the first order necessary conditions for a dynamic stochastic constrained optimization problem, from the Lagrange in expected value form, with all the constraints brought inside the expectation operator. The only difference is that the interpretation of λ_{t+1}^s is the same as $\pi_s \psi_{t+1}^s$, the effect of a 1 unit increase in income in state s on the maximized level of expected utility (objective function). The multiplier ψ_{t+1}^s on the other hand is the value of 1 unit increase in income in state s on utility in that state.

9 Appendix C: Equity Premium Puzzle

Based on the last section,

$$1 = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}(1 + r_{t+1})\right] = E_t\left[m_{t+1}(1 + r_{t+1})\right]$$

where $m_{t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ is stochastic discount factor. The right hand side can be written as

$$1 = cov_t\left[m_{t+1}, 1 + r_{t+1}\right] + E_t(m_{t+1}) E_t(1 + r_{t+1})$$

For risk-free asset, we have

$$1 = E_t\left[m_{t+1}\left(1 + r_{t+1}^f\right)\right] = E_t(m_{t+1})\left(1 + r_{t+1}^f\right)$$

Combining with r-sky asset

$$\begin{aligned}cov_t\left[m_{t+1}, 1 + r_{t+1}\right] + E_t(m_{t+1}) E(1 + r_{t+1}) &= E_t(m_{t+1})\left(1 + r_{t+1}^f\right) \\ E_t(1 + r_{t+1}) &= \left(1 + r_{t+1}^f\right) - \frac{cov_t\left[m_{t+1}, 1 + r_{t+1}\right]}{E_t(m_{t+1})} \\ E_t(r_{t+1}) &= r_{t+1}^f - \frac{cov_t\left[m_{t+1}, 1 + r_{t+1}\right]}{E_t(m_{t+1})} \\ E_t(r_{t+1}) &= r_{t+1}^f - \frac{cov_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}, (1 + r_{t+1})\right]}{E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)}\right]} \\ E_t(r_{t+1}) - r_{t+1}^f &= -\frac{cov_t\left[u'(c_{t+1}), (1 + r_{t+1})\right]}{E_t\left(u'(c_{t+1})\right)}\end{aligned}$$

The term on the right is *risk adjustment* or *risk premium* or *equity premium*. Thus, if an asset return is uncorrelated with consumption, its expected return is equal to the risk free return, and there is no premium. Since $u'(c)$ is diminishing, asset returns that are positively correlated with consumption are negatively correlated with marginal utility. Thus, assets with returns that are positively correlated with consumption, should have $cov_t(u'(c_{t+1})(1+r_{it+1})) < 0$, must promise higher expected return (because these assets make consumption more volatile, or increase risk). On the other hand, assets that are negatively correlated with consumption, and therefore $cov_t(u'(c_{t+1})(1+r_{t+1})) > 0$, can offer expected returns that are lower than the risk free return (because they help smooth consumption, or reduce risk - serve as insurance).

The *equity premium puzzle* refers to the phenomenon that observed returns on stocks over the past century are much higher than returns on government bonds. It is a term coined by Rajnish Mehra and Edward C. Prescott in 1985, and showed that either a large risk aversion coefficient or counterfactually large consumption variability was required to explain the means and variances of asset returns. Economists expect arbitrage opportunities would reduce the difference in returns on these two investment opportunities to reflect the risk premium investors demand to invest in relatively more risky stocks.