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1 author:

Mizuki Tsuboi
University of Hyogo

Stochastic Accumulation of Human Capital and Welfare in the Uzawa-Lucas Model: An Analytical Characterization

Mizuki Tsuboi[a]

[a]*Graduate School of Economics, University of Hyogo, 8-2-1, Nishi-Ku, Kobe, 651-2197, Japan*

Abstract

Stochastic growth models are often solved numerically, because they are not tractable in general. However, recent several studies find the closed-form solution to the stochastic Uzawa-Lucas model in which technological progress or population dynamics follow a Brownian motion process with one or two parameter restriction(s). However, they assume that the return on the accumulation of human capital is deterministic, which is inconsistent with empirical evidence. Therefore, I develop the Uzawa-Lucas model in which the accumulation of human capital follows a mixture of a Brownian motion process and many Poisson jump processes, and obtain the closed-form solution. Moreover, I use it to examine the nexus between human capital uncertainty, technological progress, expected growth rate of human capital, and welfare.

Keywords: Human Capital, Welfare, Endogenous Growth, Uncertainty
JEL Classification: C61, J24, O33, O41

1. Introduction

Stochastic growth models, pioneered by Brock and Mirman (1972), have been the cornerstone in macroeconomics. They have made it possible to study the interaction between growth and uncertainty, and also led to the development of the real business cycle theory. Modern macroeconomic models are primarily based on these models, and they can be used to quantitatively examine how macroeconomic aggregates respond to exogenous shocks such as technology shocks.

One shortcoming of stochastic growth models, however, is their intractability. To a large extent, because of uncertainty, the model is analytically harder to solve than their deterministic counterparts, and they are often solved numerically around the steady state. Moreover, a case of closed-form solutions is by and large limited. For example, in order to obtain the closed-form solution, some authors use logarithmic utility function, instead of the more general constant relative risk aversion (CRRA) preferences for the household side.

Email address: `mizuki.zard.musiclover@gmail.com` (Mizuki Tsuboi)

Or, for the firm side, production technology has been limited to linear or the so-called AK type, instead of Cobb-Douglas technology[1].

This being said, the breakthrough of Bucci et al (2011) is substantial: they first find the closed-form solution to the continuous-time version of the two-sector optimal growth model of Uzawa (1965) and Lucas (1988) in which technological progress follows an exogenous Brownian motion process. Although Bucci et al (2011) assume the CRRA preference and generalized Cobb-Douglas production technology à la Mankiw et al (1992), without ignoring depreciation of physical and human capital, they successfully discover the explicit solution with *two* parameter restrictions, one of which has been standard since Xie (1991, 1994). Economically, they uncover that the larger technology shock reduces the optimal level of consumption and time devoted to the production of final goods. The former is obtained by virtue of Jensen's inequality, while the latter is confirmed by numerical simulation.

Subsequently, Hiraguchi (2013) revisits Bucci et al (2011) and finds that, the closed-form solution is, in fact, available with *one* parameter restriction of Xie (1991, 1994) only. Moreover, Hiraguchi (2013) proves that the value function of Bucci et al (2011) does not satisfy the optimality conditions, and unveils the new correct value function. It shows that technology shocks, de facto, have *nothing* to do with the optimal level of consumption and *increase* time devoted to the final goods production, both of which are in sharp contrast to Bucci et al (2011). Following the lead of them, Marsiglio and La Torre (2012a, 2012b) also get the closed-form solution to the Uzawa-Lucas model with stochastic population dynamics driven by, again, a Brownian motion process[2].

Four papers on the stochastic Uzawa-Lucas model above are innovative in that they *analytically*, not numerically, throw new light on the mechanism through which technology/demographic shocks affect the macroeconomy in a transparent way. At the same time, they leave some common possibilities to be studied. First, they only consider a Brownian motion process. While this process is widely used (Turnovsky, 1997, 2000), it is not the only stochastic process used in macroeconomics.

For instance, another stochastic process of the Poisson jump process is also used extensively by some authors such as Aghion and Howitt (1992), Steger (2005), Sennewald and Wälde (2006), Sennewald (2007), and more recently, Brunnermeier and Sannikov (2014). Moreover, Wälde (2011a) finds the closed-form solution to the stochastic AK model with a *combination* of a Brownian motion process and many Poisson jump processes, while Hiraguchi (2014) does so in the stochastic Ramsey model with leisure. To string studies cited above together, it is worth exploring whether we can find the closed-form solution to the stochastic Uzawa-Lucas model with a mixed Brownian and many Poisson jump processes. This is the first agenda of this paper.

[1]The other approach is to abstract from the depreciation of capital. See the introduction of Bucci et al (2011) and Wälde (2011a) for a list of studies that have tried getting the explicit solution to stochastic growth models.

[2]In virtue of a variety of mathematical techniques, Hiraguchi (2009), Naz et al (2016), and Chaudhry and Naz (2018) also find the closed-form solution to the Uzawa-Lucas model in continuous time, though in the deterministic setup.

Second, these four papers all assume that human capital, *the heart of the Uzawa-Lucas model*, accumulates in a *deterministic* way. However, it is hard to believe that the return on the accumulation of human capital is *not* uncertain. Since human capital uncertainty is known to have sizable impacts on, for example, estimates of rates of return (Levhari and Weiss, 1974), or on the decision of students on when to leave school and to enter the labor market (Bilkic et al, 2012), its stochastic nature has to be examined for the design of policy.

Notwithstanding its importance, the lack of human capital uncertainty has been frequently indicated, for instance, by Levhari and Weiss (1974) and Krebs (2003) in theory, and Hartog et al (2007) in practice. Specifically, Hartog et al (2007) construct a simulation model to replicate the situation in which agents *ex ante* face risks associated with education. They empirically demonstrate that investment in a college education is as risky as investment in the stock market with a portfolio of some 30 randomly chosen stocks, and hence the *stochastic* returns from human capital accumulation.

On the other hand, in order to understand the interactions between human capital uncertainty and major macroeconomic aggregates, Krebs (2003) develops a *discrete*-time, one-sector stochastic growth model with human capital in which heterogeneous households face undiversifiable idiosyncratic risks in an incomplete market. He shows that the larger human capital risk increases physical capital stock, while leads to human capital contraction, weakens the growth rate, and deteriorates welfare. These findings are robust even in the presence of risky physical capital. Since four papers on stochastic Uzawa-Lucas model above do not conduct an inspection of the relationship between welfare and (human capital) uncertainty, the finding of Krebs (2003) must be appreciated.

Following Krebs (2003), I not only find the closed-form solution, but also use that solution in studying nexus between uncertainty and welfare in the context of the stochastic Uzawa-Lucas model. Moreover, I analyze the effect of technological progress on welfare. Although Krebs (2003) abstracts from technological progress, some empirical studies such as Hojo (2003), and more recently, Madsen (2014) and Cinnirella and Streb (2017), point out that the true value of human capital can be well understood *only in conjunction with* technological progress. Furthermore, all my findings are completely characterized based on the closed-form solution obtained, and I will show that the larger human capital uncertainty has nothing to do with physical capital stock, and that it reduces expected growth rate of human capital. Krebs (2003) acknowledges that his results are primarily driven by market incompleteness, whereas here, I assume a complete market. In addition, I examine the welfare implications of the higher arrival rate and the larger jump size of many Poisson jump processes on expected growth rate of human capital and welfare. This is possible because, unlike Krebs (2003), I will deal with the Poisson jump process, as well as a Brownian motion process.

Summing up, the purpose of this paper is to find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of human capital follows a combination of a Brownian motion process and many Poisson jump processes, and in particular, to examine impacts of larger human capital uncertainty on expected growth rate of human capital and welfare.

This paper is organized as follows. Section 2 presents the baseline Uzawa-Lucas model with stochastic human capital accumulation driven by a Brownian motion process. Section

3 extends the baseline model by considering a mixture of a Brownian motion process and many Poisson jump processes. Section 4 provides the further extension which incorporates the risky physical capital (the reason for this extension will be clear in section 2). Concluding remarks appear in Section 5.

2. The Baseline Model

In this section, I construct a stochastic Uzawa-Lucas model in which the accumulation of human capital follows a Brownian motion process. I assume that the total number of workers L equals unity, so that per capita terms are equivalent to the aggregate terms. This assumption, also made in Bucci et al (2011) and Hiraguchi (2013), greatly simplifies the analysis below.

A representative household is endowed with one unit of time and uses all of that. It either works or learns. There is no other use of time. Let $u(t) \in (0, 1)$ denote the fraction of time spent working to produce final goods $Y(t)$. Correspondingly, $1 - u(t)$ represents the fraction of time spent learning to accumulate new human capital. The amount of leisure is fixed exogenously, so there is no choice about it[3].

2.1. Capital Accumulation and Household

The accumulation of human capital $H(t)$ is stochastically governed by the following rule

$$dH(t) = b(1 - u(t))H(t)dt - \delta_h H(t)dt + \sigma_h H(t)dz_h \tag{1}$$

where $b > 0$ is an exogenous parameter that indicates how efficient human capital accumulation is. $\delta_h \in (0, 1)$ is the depreciation rate of human capital. dz_h is the increment of a Brownian motion process such that the mean $\mathcal{E}(dz_h) = 0$ and variance $\mathcal{V}(dz_h) = dt$, and $\sigma_h > 0$ is the associated diffusion coefficient of human capital (if $\sigma_h = 0$, then we would recover the deterministic limit). The initial stock of human capital $H(0) = H_0 > 0$ is given, so that $H(t) > 0$ for all t with probability 1.

Note that the stochastic process (1) is the controlled diffusion process, that is, it contains one of key control variables in the Uzawa-Lucas model, $u(t)$. Bucci et al (2011) and Hiraguchi (2013) assume that technological progress is stochastic, while the process of human capital accumulation is deterministic ($\sigma_h = 0$), which is at odds with the empirical literature such as Hartog et al (2007), and is subject to the critique by Levhari and Weiss (1974) and Krebs (2003). Treating the control variable in the drift of a Brownian motion process makes it much harder to obtain the closed-form solution. However, as we will see, it enables us to examine the interplay among major macroeconomic variables, particularly the relationship between welfare and human capital uncertainty.

The economy-wide resource constraint is governed in the deterministic way

[3]An explicit incorporation of leisure makes it extremely difficult to obtain the analytical solution. Therefore, I abstract from it. See Ladrón-De-Guevara et al (1999) for the deterministic Uzawa-Lucas model with leisure. No study has found the closed-form solution to the stochastic Uzawa-Lucas model with leisure.

$$dK(t) = \underbrace{A(t)^{\gamma}(u(t)H(t))^{\alpha}K(t)^{1-\alpha-\gamma}}_{\equiv Y(t)}dt - \delta_k K(t)dt - C(t)dt \tag{2}$$

where $\gamma \in (0,1)$. $K(t)$ is physical capital, and δ_k is its depreciation rate. $\alpha \in (0,1)$ represents the human capital share of income in the Cobb-Douglas production function. This implies $\alpha + \gamma \in (0,1)$. $C(t)$ denotes consumption of the final good. The initial stock of physical capital $K(0) = K_0 > 0$ is also given. We will examine the stochastic version of (2) in Section 4.

$A(t)$ is technology, whose law of motion is simply

$$dA(t) = \mu A(t)dt \tag{3}$$

where $\mu > 0$. This is stochastically modelled in Bucci et al (2011) and Hiraguchi (2013). As the focus of this paper is on the stochastic accumulation of human capital, I keep (3) deterministic throughout. The initial stock of technology $A(0) = A_0 > 0$ is given as well. Finally, preferences of a representative household are given by the standard CRRA utility function:

$$E\int_0^{\infty} e^{-\rho t}\frac{C(t)^{1-\phi}-1}{1-\phi}dt \tag{4}$$

where E is the mathematical expectation operator with respect to the information set available to the representative household. $\rho > 0$ is the subjective discount rate, that is, the rate at which utility is discounted. $\phi > 0$ is the index of relative risk aversion (and $1/\phi$ is the elasticity of intertemporal substitution). When future consumption is uncertain, a larger ϕ makes future utility gain smaller, raising the value of additional future consumption. The representative household maximizes its expected utility (4) subject to the stochastic process (1) and laws of motion for physical capital (2) and for technology (3).

2.2. Optimization

In order to solve the above optimization problem, let $J(K, A, H)$ be the value function (or the indirect utility function). Then, the associated Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\rho J(K,A,H) = \max_{\{C_t,u_t\}}\left(\frac{C(t)^{1-\phi}-1}{1-\phi} + J_K\frac{dK}{dt} + J_A\frac{dA}{dt} + J_H\frac{dH}{dt} + \frac{J_{HH}}{2}\frac{(dH)^2}{dt}\right)$$

that is,

$$\begin{aligned}\rho J(K,A,H) = \max_{\{C,u\}}\Bigg(&\frac{C^{1-\phi}-1}{1-\phi} + J_K(uH)^{\alpha}K^{1-\alpha} - J_K\delta_k K - J_K C + J_A\mu A + J_H b(1-u)H \\ &- J_H\delta_h H + \frac{J_{HH}H^2\sigma_h^2}{2}\Bigg)\end{aligned} \tag{5}$$

where $J_K \equiv \partial J/\partial K$, $J_H \equiv \partial J/\partial H$, $J_A \equiv \partial J/\partial A$ and $J_{HH} \equiv \partial^2 J/\partial H^2$. Relevant first-order conditions with respect to C and u are respectively

$$C = J_K^{-\frac{1}{\phi}} \tag{6}$$

and

$$u = \left(\frac{\alpha J_K}{b J_H}\right)^{\frac{1}{1-\alpha}} \frac{A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}}}{H} \tag{7}$$

Substituting these first-order conditions (6) and (7) back to the HJB equation (5), and slightly rearranging, we get the maximized HJB equation of the form

$$\begin{aligned} \rho J(K,A,H) = \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} \\ - J_K \delta_k K + J_A \mu A + J_H b H - J_H \delta_h H + \frac{J_{HH} H^2 \sigma_h^2}{2} \end{aligned}$$

Note that this is the partial differential equation. In general, it is impossible to solve it analytically. As Wälde (2011b, p.277) nicely puts, "For a much larger class of models - which are then standard models - closed-form solutions cannot be found for general parameter sets. Economists then either go for numerical solutions...or they restrict the parameter set in a useful way. Useful means that with some parameter restriction, value functions can be found again and closed-form solutions are again possible." Several studies follow the latter approach (parameter restriction) and make a success of the closed-form solution with one or two restriction(s). Although the source of uncertainty is utterly different, it turns out that we can also find the closed-form solution with one parameter restriction. It can be summarized as follows:

Theorem 1. *If we impose the following parameter constraint originally used by Xie (1991),*

$$\phi = 1 - \alpha - \gamma \tag{8}$$

then there exists the closed-form representation of the value function (that satisfies the transversality condition, or TVC)

$$J(K,A,H) = \mathcal{Q}_X K^{\alpha+\gamma} + \mathcal{Q}_Y A^{\gamma} H^{\alpha} + \mathcal{Q}_Z \tag{9}$$

where

$$\mathcal{Q}_X \equiv \frac{1}{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{\rho+(\alpha+\gamma)\delta_k}\right)^{1-\alpha-\gamma}$$

$$\mathcal{Q}_Y \equiv \frac{1}{b^{\alpha}}\left(\frac{1-\alpha-\gamma}{\rho+(\alpha+\gamma)\delta_k}\right)^{1-\alpha-\gamma}\left(\frac{1-\alpha}{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_h^2}{2}\alpha(1-\alpha)}\right)^{1-\alpha} \tag{10}$$

$$\mathcal{Q}_Z \equiv -\frac{1}{\rho(\alpha+\gamma)}$$

Moreover, straightforward calculation shows that the control variables are expressed as

$$C = \frac{\rho + (\alpha+\gamma)\delta_k}{1-\alpha-\gamma} K \tag{11}$$

and

$$u = \frac{\rho - \mu\gamma - \alpha b + \alpha\delta_h + \frac{\sigma_h^2}{2}\alpha(1-\alpha)}{b(1-\alpha)} \tag{12}$$

Proof. See Appendix B[4].

2.3. Macroeconomic Implications

I in turn comment on main points in Theorem 1.

2.3.1. Parameter Restriction and Value Function

The parameter restriction (8) says that the risk aversion parameter equals the physical capital share of income. It allows us to find out the closed-form representation of the value function (9). Whether it holds true in practice is still open debate, because the estimate of ϕ is a task of great difficulty[5]. However, this restriction has been widely used by a number of authors in order to obtain the closed-form solution to their model. Xie (1991, 1994), Rebelo and Xie (1999), Smith (2007), Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b), Hiraguchi (2013), and Hiraguchi (2014), all use the restriction (8) to generate the insights that cannot be appreciated without the explicit solution. Following them, I also use (8).

One may wish to back down the parameter restriction (8) and hence the closed-form representation (9), and instead resort to, say, the value function iteration or (implicit) finite-difference method. Although that can be another approach, it too has some shortcomings. For instance, there is no guarantee that the value function would converge to the "true"

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[4]We can be sure that $u \in (0,1)$ as long as the inequality

$$\alpha b - \alpha\delta_h - \frac{\sigma_h^2}{2}\alpha(1-\alpha) + \mu\gamma < \rho < b - \alpha\delta_h - \frac{\sigma_h^2}{2}\alpha(1-\alpha) + \mu\gamma$$

holds. Moreover, one can establish that the TVC

$$\lim_{t\to\infty} E[e^{-\rho t}K^{\alpha+\gamma}] = \lim_{t\to\infty} E[e^{-\rho t}A^{\gamma}H^{\alpha}] = 0$$

is satisfied. The proof requires the verification theorem. See Chang (2004, Ch.4) for details of this theorem. Hiraguchi (2013, Appendix B) provides an excellent proof of the TVC for the stochastic Uzawa-Lucas model in which technology follows a geometric Brownian motion process. To save space, I will not mention the proof of TVC in what follows.

[5]For example, on the one hand, Lucas (2003) claims that ϕ ranges from 1 (log utility) to 4, but on the other, Smith (2007) says that ϕ should be smaller than 1.

one. Or, even if it does, without the analytical solution, it would be hard to see what is drawing one's findings. As we will see, the closed-form solution at our disposal is essential for perspicuously understanding the interaction among major macroeconomic variables. Therefore, the real problem is not the parameter restriction itself: what really matters is to evaluate which "cost" - the cost of imposing the parameter restriction and the cost of numerical approximation - is higher. Here, following many studies cited above, I proceed on grounds that the former is lower.

With one parameter restriction (8), the value function (9) can be obtained analytically. Here, we can see that physical capital K and the product of technology and human capital AH are separable. This is in sharp contrast to Bucci et al (2011), who find these three state variables are all separable. As noted in Hiraguchi (2013, p.137), the economic implication of nonseparability between A and H is that the long-run engine of stochastic endogenous growth models is a fusion of technology and human capital. This is consistent with recent empirical studies such as Madsen (2014) and Cinnirella and Streb (2017), which emphasize the importance of the *interaction* between technological progress and human capital accumulation. Moreover, in what follows, I use the value function (9) for the welfare analysis. As Turnovsky (1997, 2000) show, when the value function is expressed in an explicit form, we can use it to assess the effect of parameters or variables of interest on welfare, in particular the influence of uncertainty on welfare. It is only possible in stochastic growth models, because deterministic models have, by construction, nothing to say about the effect of uncertainty on welfare.

2.3.2. Control Variables and Expected Growth Rate

Equation (11) tells us that the consumption-capital ratio is constant. It is at odds that the optimal level of consumption C is neither dependent of human capital stock H nor of technology A. Nevertheless, on this point, Hiraguchi (2013, p.137) succinctly puts "We cannot not find the intuitive explanation why the current consumption level c is independent of the TFP level A and the human capital H. However, these values affect the physical capital accumulation and then they affect the *future* consumption. The independence result is consistent with Smith (2007) who obtains the closed-form solution to the one-sector neoclassical growth model." This property is also documented in Wälde (2011a)'s survey on one-sector stochastic growth models. Therefore, it is left for the future research.

But notice one more unpleasant property: *consumption-capital ratio is irrelevant to the shock term σ_h*. This is also seen in Hiraguchi (2013) - see his equation (30)[6] - and indicated by the horizontal dashed line in Figure 1. In what follows, I use the following "standard" parameter values: $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\rho = 0.05$, and $\delta_k = \delta_h = 0.03$[7]. This parameterization does not violate the inequality guaranteeing $u \in (0, 1)$, so it preserves

[6]See also Equation (16) in Bucci et al (2011), Equation (11) in Marsiglio and La Torre (2012a), and Equation (50) in Marsiglio and La Torre (2012b).

[7]Following the seminal paper of Mankiw et al (1992, p.432), I set the human capital share $\alpha = 1/3$. For physical capital share, it has been commonplace in macroeconomics to assume that it is also 1/3. However, as Karabarbounis and Neiman (2014) document, the labor share is declining (or, put differently, physical capital share is rising) globally. Therefore, I set $\gamma = 0.27$ so that the physical capital share roughly equals

some degree of reality. In Figure 1, we can see that consumption-capital ratio C/K is independent of the size of human capital shock σ_h. Do human capital, technology, and demographic shock have really nothing to do with the optimal level of consumption? This is the important point missed in Hiraguchi (2013) and others, and therefore we will explore this in Section 4.

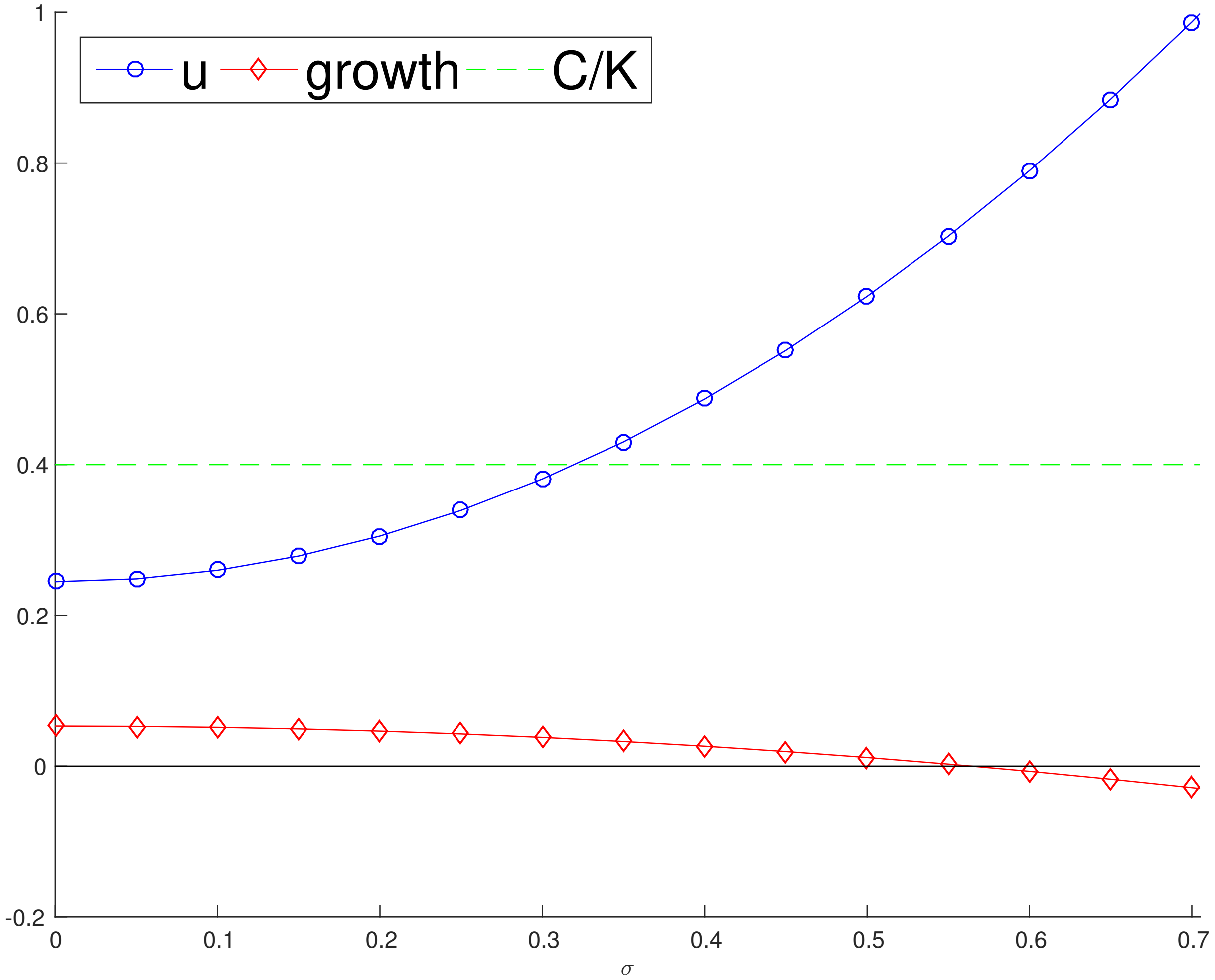

Figure 1: Human capital uncertainty, time devoted to working, expected growth rate of human capital, and consumption-capital ratio. As more time is devoted to working, the expected growth rate of human capital decreases, and eventually becomes negative. Consumption-capital ratio is irrelevant to the size of uncertainty. Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\rho = 0.05$, and $\delta_k = \delta_h = 0.03$. I use $K = 10$ for C/K.

Equation (12) says that the time spent in working is constant as well, again consistent

0.40, the value used by Ahn et al (2017). $b = 0.11$ is used when Barro and Sara-i-Martin (2004) simulate the Uzawa-Lucas model. I choose $\mu = 0.02$ and $\delta_k = \delta_h = 0.03$, again following Mankiw et al (1992). Finally, following Caballé and Santos (1993) and Moll (2014), I set $\rho = 0.05$. In Figure 1, I use $K = 10$ to make it transparent.

with Hiraguchi (2013). Note that it is increasing in σ_h. In other words, the larger σ_h causes people to spend their time in working more, and in parallel, less time in the accumulation of human capital. Consequently, more uncertainty about human capital leads to its contraction, consistent with Levhari and Weiss (1974) and Krebs (2003). This has an implication for the expected growth rate of human capital. Although stochasticity does not allow us to calculate the *actual* growth rate of human capital, we can nonetheless compute its *expected* growth rate, as u turns out to be constant[8]. One can show that the expected growth rate of human capital $\mathcal{G}_h$ is

$$\mathcal{G}_h \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right) = \frac{b - \rho - \delta_h - \frac{\sigma_h^2}{2}\alpha(1-\alpha) + \mu\gamma}{1-\alpha} \tag{13}$$

where $\dot{H} \equiv dH/dt$. We can see that it is decreasing in human capital uncertainty σ_h. This is because, as indicated above, the larger human capital shock discourages people to spend time in accumulating their human capital, and hence human capital contraction. It thus leads to lower expected growth rate of human capital. We can also see that, in the absence of technological progress ($\mu = \gamma = 0$), no depreciation ($\delta_h = 0$), and no uncertainty ($\sigma_h = 0$), the sign of growth rate is solely determined by the relative size of b and ρ. As discussed in Kuwahara (2017), it is the usual property of the deterministic Uzawa-Lucas model, and the model here retains that property, as it should.

It would be interesting to see whether $\mathcal{G}_h$ is positive or negative under reasonable parameterization. It is positive if the numerator of (13) is positive as well, i.e. as long as the inequality $\rho < b - \delta_h - \alpha(1-\alpha)\sigma_h^2/2 + \mu\gamma$ holds, and vice versa. It is illustrated by the line with diamonds in Figure 1 (the line with circles indicates the relationship between u and σ_h. It is also plotted, since more u is supposed to reduce $\mathcal{G}_h$. They should go in the opposite direction). We can see that, for the modest degree of uncertainty (roughly $\sigma_h < 0.55$), the expected growth rate of human capital is positive. However, it gets negative when uncertainty is larger than the threshold value. As such, when the human capital shock is sufficiently large, it is difficult to realize the positive human capital growth. It can be overcome though, for instance, by enhancing the technological grow rate μ via investment in R&D.

2.3.3. Welfare

Since we have the closed-form representation of the value function (9), the welfare analysis can be done by simply differentiating (9) with respect to the parameter of interest. This is the reason why the analytical solution is better than numerical solution; it allows us to reveal the welfare implications in a transparent way. Specifically, first, we have

$$\frac{\partial J}{\partial \mu} > 0 \tag{14}$$

[8] I thank one anonymous referee for pointing out this property.

in other words, technological progress is welfare-improving. This is an important point missed in Krebs (2003), and can be found by taking technological progress into account, at the cost of tractability. To grasp why technological progress improves welfare, note that the constant (10) is increasing in μ. Noting that A and H are multiplicative, technological progress strengthens the contribution of *both* A and H to the welfare J, as J is the positive function of state variables A and H.

Notice also that u is decreasing in μ, as seen in equation (12). This means that technological progress discourages people to spend time in working, or in parallel, encourages them to spend time in the accumulation of human capital. Since it increases the stock of human capital in the economy H, it leads to welfare enhancement. Through these two channels, welfare ameliorates because of technological progress[9].

Next, we can examine the nexus between human capital uncertainty and welfare. One can show that

$$\frac{\partial J}{\partial \sigma_h} < 0 \tag{15}$$

that is, the larger human capital shock reduces welfare, consistent with Krebs (2003). The reason for this can, again, be identified via the constant (10) and the control variable u. First, because the constant (10) is decreasing in σ_h, the larger shock reduces both the contribution of A and H to welfare J (due to nonseparativity). Second, since u is increasing in σ_h, the larger human capital shock encourages people to work more, or put differently, discourages them to spend time in the accumulation of human capital. This leads to human capital contraction, and thus welfare is deteriorated. Through these two channels, in contrast to technological progress, human capital uncertainty reduces welfare. This finding complements that of Krebs (2003).

2.3.4. Simple Simulation

This paper is concerned with the stochastic accumulation of human capital. Although studies cited above document the considerable amount of uncertainty associated with human capital (accumulation), it would be necessary to provide further empirical rationale for why I use the *stochastic*, not deterministic, differential equation (1)[10]. To motivate, see Figure 2, which displays measures of US human capital stock for people aged 15 to 64 in 1940s (in 5-year intervals), recently constructed by Lee and Lee (2016).

In most cases, human capital accumulates over time. However, Figure 2 shows that in 1940s, that is, before, during, and after World War II, there was human capital *contraction*. Although human capital had accumulated from 1940 to 1945, we can see its contraction from 1945 to 1950. Analyzing what caused human capital contraction would not be easy and it requires serious econometric test. We can at best gauge that it was caused by the large event such as World War II. In sum, over the 10 years in 1940s, the amount of US

[9] One more marginal channel through which μ increases A can be seen by solving the differential equation (3). However, this somewhat overlaps the first channel.

[10] I thank one anonymous referee for encouraging me to pursue this.

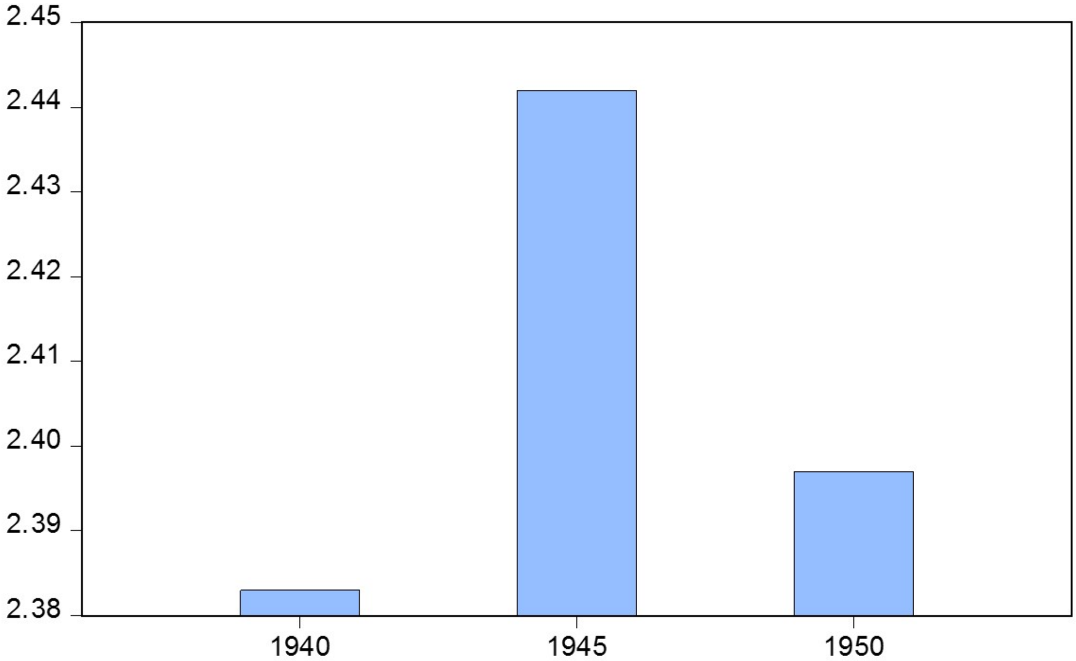

Figure 2: US human capital stock in 1940s. It increased between 1940 and 1945, while decreased between 1945 and 1950, thereby yielding the bell-shaped curve. *Source:* Data set accompanying Lee and Lee (2016).

human capital first went up, and then went down, that is, we can see the *bell-shaped* picture in Figure 2.

The purpose of the exercise here is to see whether the stochastic differential equation (1) can generate the bell-shaped picture in Figure 2. I admit that it is too simple to quantitatively "match the data" as in the quantitative macroeconomics literature. But the focus here is on the validity of adding *stochastic* elements. If stochastic differential equation can better mimic Figure 2 than its deterministic counterpart, it may justify why we need the stochastic elements in (1), in addition to the empirical studies such as Hartog et al (2007).

To this end, I present the discretized version of equation (1) in Figure 3[11]. It displays the simulated path for various degrees of human capital uncertainty σ_h, being 1%, 10%, 20%, and 30%. First, see the line with σ_h=1%, which virtually represents the deterministic path. Over the simulated interval, it is (ignoring tiny drops) always going up over time. For the reasonable parameter values, however, it neither replicates human capital contraction between 1945 and 1950 nor generates the bell-shaped curve in Figure 2. Thus, the

[11] Specifically, I use the solution to (1)

$$H(t) = H(0)e^{\left(b(1-u)-\delta_h-\frac{\sigma_h^2}{2}\right)t}e^{\sigma_h z_h}$$

for simulation. Higham (2001) provides a concise explanation of simulation technique for stochastic differential equaiton driven by a Brownian motion process.

deterministic differential equation fails to mimic Figure 2.

On the other hand, stochastic paths are successful in replicating the bell-shaped curves. For σ_h larger than 10%, we can first see the accumulation of human capital between $t = 0$ and $t = 5$ (albeit initial contraction at $t = 1$), and then the contraction between $t = 5$ and $t = 10$, hence generating the bell-shaped curve (especially for $\sigma_h = 20\%$ and $\sigma_h = 30\%$). It is true that simulated stochastic paths do not quantitatively replicate what we see in Figure 2. Nonetheless, it *does* generate the bell-shaped curve seen in Figure 2 *qualitatively*, while the deterministic counterpart fails. The bottom line of this exercise is that, it is appropriate to use the *stochastic* differential equation (1) rather than the deterministic differential equation to explain the accumulation process of human capital[12].

The findings in this section can be summarized as follows:

Proposition 1. *One parameter restriction commonly used in the literature makes it possible to find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of human capital follows a Brownian motion. The larger human capital shock does not affect consumption-capital ratio, increases time spent in working, reduces the expected growth rate of human capital, and deteriorates welfare.*

3. The Model with Jumps

In the previous section, we find the closed-form solution to the stochastic Uzawa-Lucas model with human capital accumulation following a Brownian motion only. Despite the above simulation exercise, it may be the case that the accumulation process of human capital may better be described by the *jump* process, rather than a Brownian motion process. Consequently, this section presents the extension of the previous section[13]. Specifically, as in Wälde (2011a) and Hiraguchi (2014), I consider the stochastic Uzawa-Lucas model with a *mixture* of a Brownian motion process and many Poisson jump processes.

In the context of endogenous growth models, Poisson jump processes are frequently used, for instance, in the creative destruction or Schumpeterian growth model of Aghion and Howitt (1992). They are also theoretically studied by Sennewald and Wälde (2006) and Sennewald (2007) in detail. However, the existing studies on the stochastic Uzawa-Lucas model, such as Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b) and Hiraguchi (2013), all analyze a Brownian motion case only. I provide the Poisson jump extension for this line of research. As such, in this section, I will examine whether we can still find the closed-form solution to the stochastic Uzawa-Lucas model with a combination of Brownian

[12] According to Lee and Lee (2016) data set on human capital stock, we can also see the "bell-shaped" pattern in US between 2000 and 2005. During this period, the number declined from 3.708 (in 2000) to 3.673 (in 2005). In fact, this phenomenon is not unique in US. For instance, in Switzerland between 1980 and 2000, the number consecutively declined from 3.103 (in 1980) to 2.758 (in 2000); in Spain between 1915 and 1920, the number declined from 1.403 (in 1915) to 1.389 (in 1920); in Portugal between 2000 and 2005, the number declined from 2.318 (in 2000) to 2.253 (in 2005). These evidence suggests that the above exercise can be applied not only to the period of unprecedentedly big events (such as World War II) or to the specific country, but also to other (possibly) disruptive events across time and space.

[13] I thank one anonymous referee for pointing out this possibility.

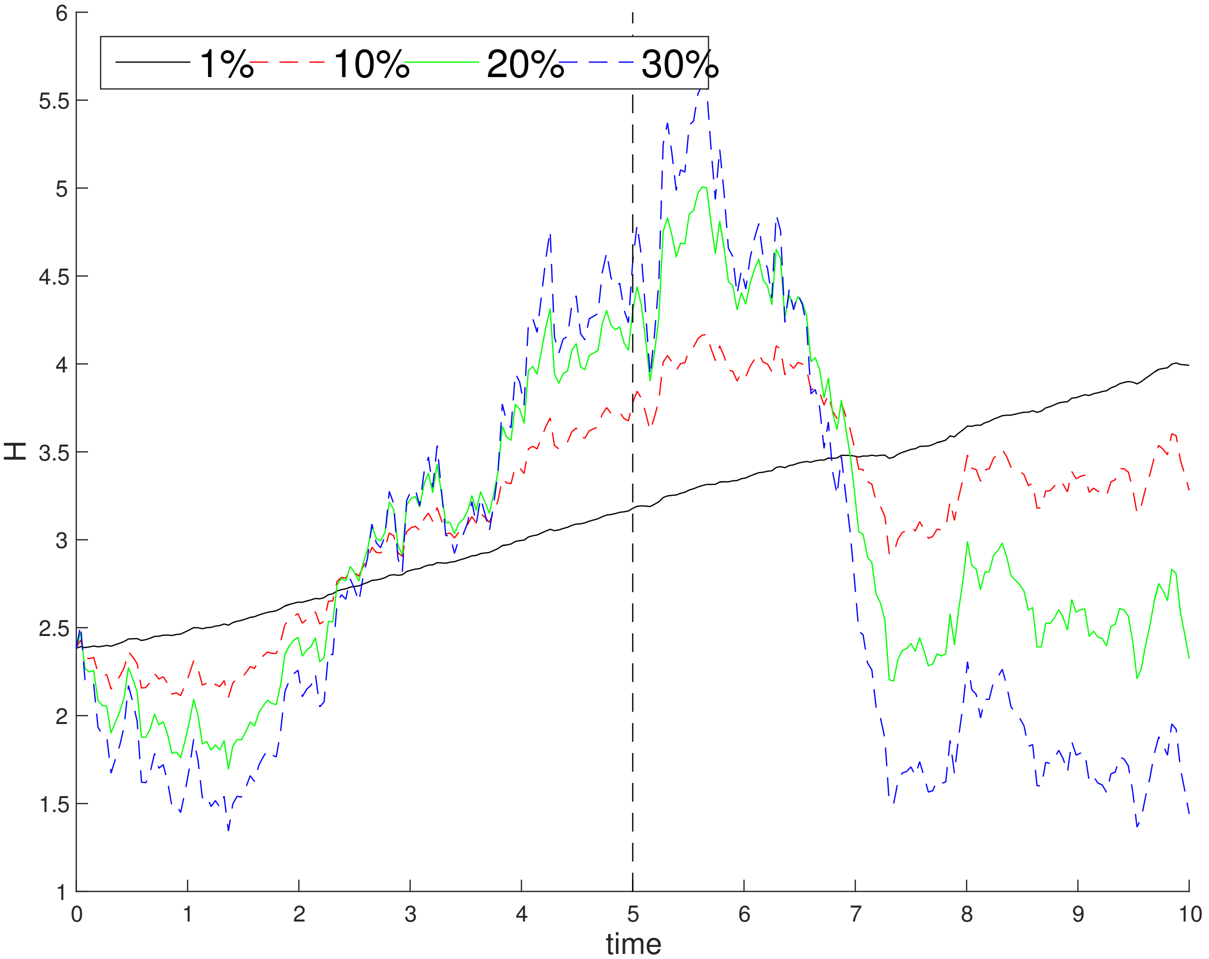

Figure 3: Simulation of stochastic differential equation (1). Parameters are $\alpha = 1/3$, $\gamma = 0.27$, $b = 0.11$, $\mu = 0.02$, $\rho = 0.05$, and $\delta_k = \delta_h = 0.03$. I use $\sigma_h = 0.01, 0.1, 0.2$, and 0.3. As σ_h gets larger, we can see the bell-shaped curves, which may look like Figure 2.

motion and many Poisson jump processes, and mainly discuss the welfare implications of Poisson jump processes[14].

3.1. Brownian Motion and Poisson Jump Process

Suppose that there are N independent Poisson jump processes q_i with the arrival rate λ_i that drive the accumulation of human capital, in addition to a Brownian motion process. Then, the stochastic differential equation (1) is modified as follows,

$$dH(t) = b(1 - u(t))H(t)dt - \delta_h H(t)dt + \sigma_h H(t)dz_h + \sum_{i=1}^{N} H(t)\beta_i dq_{it} \tag{16}$$

where $\beta_i > -1$ is the size of jumps. The rest of the model remains unchanged. A representative household maximizes its expected utility (4) subject to deterministic laws of motion for physical capital (2) and technology (3), and to the stochastic process (16). Since first-order conditions (6) and (7) are unchanged as well, the resulting maximized HJB equation is given by

$$\begin{aligned}\rho J(K, A, H) = {} & \frac{\phi}{1-\phi} J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} - J_K \delta_k K \\ & + J_A \mu A + J_H bH - J_H \delta_h H + \frac{J_{HH} H^2 \sigma_h^2}{2} + \sum_{i=1}^{N} \lambda_i \left(J(K, A, (1+\beta_i)H) - J(K, A, H)\right)\end{aligned}$$

where the last term stems from Poisson uncertainty. Because of a combination of Brownian motion and many Poisson jump processes, the analysis here becomes much harder than that of the previous section. Is it still possible to find the closed-form solution in this case? It can be summarized as follows:

Theorem 2. *If we impose the parameter constraint (8), then there exists the closed-form representation of the value function that satisfies the TVC of the form*

$$J(K, A, H) = \mathcal{O}_X K^{\alpha+\gamma} + \mathcal{O}_Y A^{\gamma} H^{\alpha} + \mathcal{O}_Z$$

where

$$\mathcal{O}_X (= \mathcal{Q}_X) \equiv \frac{1}{\alpha+\gamma} \left(\frac{1-\alpha-\gamma}{\rho + (\alpha+\gamma)\delta_k}\right)^{1-\alpha-\gamma}$$

[14] Steger (2005) compares a Brownian motion process with a Poisson jump process in the AK model. He shows that, to conduct a sensible comparison between these requires some unrealistic restrictions. Furthermore, even when they are imposed, he finds that insights from the comparison is quantitatively negligible. Following his findings, I will not do empirical simulation in what follows. In principle, with Poisson jump processes, we would see occasional jumps in Figure 3, in addition to random fluctuations driven by a Brownian motion process.

$$\mathcal{O}_Y \equiv \frac{1}{b^\alpha}\left(\frac{1-\alpha-\gamma}{\rho+(\alpha+\gamma)\delta_k}\right)^{1-\alpha-\gamma}\left(\frac{1-\alpha}{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_h^2}{2}\alpha(1-\alpha)-\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right)}\right)^{1-\alpha} \tag{17}$$

$$\mathcal{O}_Z (= \mathcal{Q}_Z) \equiv -\frac{1}{\rho(\alpha+\gamma)}$$

Moreover, straightforward calculation shows that the control variable C is still given by (11), while u is now expressed as

$$u = \frac{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_h^2}{2}\alpha(1-\alpha)-\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right)}{b(1-\alpha)} \tag{18}$$

Proof. See Appendix A[15].

3.2. Macroeconomic Implications

Theorem 2 shows that we can still obtain the closed-form solution to the stochastic Uzawa-Lucas model *even with a combination of a Brownian motion process and many Poisson jump processes.* This finding crucially differs from previous studies such as Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b) and Hiraguchi (2013), because they consider a Brownian motion process only. As in the previous section, I in turn comment on main points in Theorem 2. To save space, I will not repeat what we discussed there.

In (18), we can see that u is decreasing both in the arrival rate λ_i and jump size β_i. Therefore, increase in the arrival rate or jump size discourages people to spend time in working, and in parallel, encourages them to spend their time in human capital sector. This leads to the accumulation of human capital. Because the human capital stock in the economy increases, one can expect that the expected growth rate of human capital increases as well, and welfare is improved.

Formally, the expected growth rate of human capital with Poisson jump $\mathcal{G}_h^q$ is now given by

$$\mathcal{G}_h^q \equiv \mathcal{E}\left(\frac{\dot{H}}{H}\right) = \frac{b-\rho-\delta_h-\frac{\sigma_h^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i((1+\beta_i)^\alpha-1)+(1-\alpha)\sum_{i=1}^N \lambda_i\beta_i}{1-\alpha}$$

[15] We can be sure that $u \in (0,1)$ as long as the inequality

$$\alpha b-\alpha\delta_h-\frac{\sigma_h^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right) < \rho < b-\alpha\delta_h-\frac{\sigma_h^2}{2}\alpha(1-\alpha)+\mu\gamma+\sum_{i=1}^N \lambda_i\left((1+\beta_i)^\alpha-1\right) \tag{19}$$

holds. Moreover, one can establish that the appropriate TVC is satisfied. Sennewald (2007) provides the proof of the TVC for the Poisson jump case.

where I use the "fact" that $\mathcal{E}(dq_{it}) = \lambda_i dt$ (Sennewald and Wälde, 2006). We can immediately see that $\mathcal{G}_h^g$ gets higher as the arrival rate λ_i and jump size β_i increase, because these lead to human capital accumulation. In the same vein, it is straightforward to show that, for all i,

$$\frac{\partial J}{\partial \lambda_i} > 0$$

and

$$\frac{\partial J}{\partial \beta_i} > 0$$

in other words, the higher arrival rate and the larger jump improve welfare. As in the previous section, there are two underlying channels through which they improve welfare. The first is via the control variable u, which leads to human capital accumulation, and the second is via the constant (17), which increases the contribution of both technology A and human capital H to welfare J. It would be worth reiterating that these results are possible only by considering the Poisson jump process. The findings in this section can be summarized as follows.

Proposition 2. *One parameter restriction commonly used in the literature again makes it possible to find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of human capital follows a combination of a Brownian motion and many Poisson jump processes. The higher arrival rate and larger size of jump decrease time spent in working, strengthen the expected growth rate of human capital, and improve welfare.*

Although this section extends the previous section, for example, by revealing the relationship between the arrival rate and welfare, you may note that it nevertheless leaves one thing unresolved; as we saw in Figure 1, the consumption-capital ratio C/K is still independent of human capital uncertainty, the arrival rate, and the jump size. It remains a conundrum, and I will strive to solve it in the next section.

4. The Model with Risky Physical Capital

How can we solve the puzzle that consumption-capital ratio C/K is independent of major parameters of stochastic processes? The conjectural resolution would be obtained by realizing that, literally, consumption is the function of physical capital K. Therefore, if shocks had an impact on consumption, it would affect via K. For that reason, in this section, following Krebs (2003, Appendix 2), I assume that *both* human and physical capital accumulation follow the stochastic process. In Krebs (2003), although the purpose of this extension is to check the robustness of his findings, my purpose here is to solve a puzzle, and importantly, to find a closed-form solution to the stochastic Uzawa-Lucas model with risky physical and human capital accumulation[16].

4.1. Risky Capital and Closed-Form Solution

Suppose that there are n independent Poisson jump processes q_j^k with the arrival rate λ_j^k that drive the accumulation of *physical* capital, as well as that of human capital. Then, the deterministic differential equation (2) becomes the stochastic differential equation

$$dK(t) = \underbrace{A(t)^{\gamma}(u(t)H(t))^{\alpha}K(t)^{1-\alpha-\gamma}}_{\equiv Y(t)}dt - \delta_k K(t)dt - C(t)dt - \sigma_k K dz_k - \sum_{j=1}^{n} \beta_j^k K dq_{j\,t}^k \tag{20}$$

where, for simplicity, I assume that dz_h and dz_k are uncorrelated. σ_k is the diffusion coefficient of physical capital. $\beta_j^k < 1$ is the jump size of the Poisson process for physical capital. Equation (20) coincides with equation (12) in Wälde (2011a), if the production function $Y(t)$ is an AK type. As in Wälde (2011a), physical capital accumulation follows a combination of a Brownian motion process and many Poisson jump processes, which generalizes the model of Eaton (1981) and Rebelo and Xie (1999).

The rest of the model again remains unchanged. As such, a representative household maximizes its expected utility (4) subject to the deterministic law of motion for technology (3), and to two stochastic processes (16) and (20). As first-order conditions (6) and (7) are not changed, the maximized HJB equation reads

$$\begin{aligned}\rho J(K,A,H) = {} & \frac{\phi}{1-\phi}J_K^{\frac{\phi-1}{\phi}} - \frac{1}{1-\phi} + \alpha^{\frac{1}{1-\alpha}} b^{\frac{\alpha}{\alpha-1}} \left(\frac{1-\alpha}{\alpha}\right) A^{\frac{\gamma}{1-\alpha}} K^{\frac{1-\alpha-\gamma}{1-\alpha}} J_K^{\frac{1}{1-\alpha}} J_H^{\frac{\alpha}{\alpha-1}} - J_K\delta_k K \\ & + J_A \mu A + J_H bH - J_H \delta_h H + \frac{J_{HH}H^2\sigma_h^2}{2} + \sum_{i=1}^{N} \lambda_i \left(J(K,A,(1+\beta_i)H) - J(K,A,H)\right) \\ & + \frac{J_{KK}\sigma_k^2}{2} + \sum_{j=1}^{n} \lambda_j^k (J((1-\beta_j^k)K,A,H) - J(K,A,H))\end{aligned} \tag{21}$$

where $J_{KK} \equiv \partial J^2/\partial K^2$ and the last two terms emerge out of uncertainty about depreciation of physical capital. Before going on, the hard problems is that now the HJB equation (21) includes four terms relevant to uncertainty (that is, σ_h, σ_k, λ_i and λ_j^k). Previous studies of stochastic Uzawa-Lucas models, such as Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b), and Hiraguchi (2013), implicitly perceive that a closed-form solution is hard to get even just with one Brownian motion process. Here, with two kinds of Brownian motion and many Poisson jump processes, how come we can find the closed-form solution? In spite of

[16] The notion of stochastic physical capital accumulation is first proposed by Eaton (1981). He assumes that the depreciation rate of physical capital follows a Brownian motion process. Similarly, Rebelo and Xie (1999) assume that it follows both a Brownian motion process and one Poisson process.

the complexity of (21), nonetheless, it is yet available with one parameter restriction. It can be summarized as follows:

Theorem 3. *If we impose the parameter constraint (8), then there exists the closed-form representation of the value function that satisfies the TVC of the form*

$$J(K, A, H) = \mathcal{B}_X K^{\alpha+\gamma} + \mathcal{B}_Y A^{\gamma} H^{\alpha} + \mathcal{B}_Z$$

where

$$\mathcal{B}_X \equiv \frac{1}{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{\rho+(\alpha+\gamma)\delta_k+\frac{\sigma_k^2}{2}(\alpha+\gamma)(1-\alpha-\gamma)-\sum_{j=1}^{n}\lambda_j^k((1-\beta_j^k)^{\alpha+\gamma}-1)}\right)^{1-\alpha-\gamma} \tag{22}$$

$$\mathcal{B}_Y \equiv \frac{(\alpha+\gamma)\mathcal{B}_X}{b^{\alpha}}\left(\frac{1-\alpha}{\rho-\mu\gamma-\alpha b+\alpha\delta_h+\frac{\sigma_h^2}{2}\alpha(1-\alpha)-\sum_{i=1}^{N}\lambda_i((1+\beta_i)^{\alpha}-1)}\right)^{1-\alpha}$$

$$\mathcal{B}_Z(=\mathcal{Q}_Z) \equiv -\frac{1}{\rho(\alpha+\gamma)}$$

Besides, straightforward calculation shows that, while u is still given by (18), the control variable C is expressed as

$$C = \frac{\rho+(\alpha+\gamma)\delta_k+\frac{\sigma_k^2(\alpha+\gamma)(1-\alpha-\gamma)}{2}-\sum_{j=1}^{n}\lambda_j^k((1-\beta_j^k)^{\alpha+\gamma}-1)}{1-\alpha-\gamma}K \tag{23}$$

Proof. See Appendix A[17].

4.2. Macroeconomic Implications

Theorem 3 demonstrates that we can still find the closed-form solution to the stochastic Uzawa-Lucas model *even in the presence of two kinds of stochastic processes that follow a combination of a Brownian motion process and many Poisson jump processes.* As in the previous sections, I will not repeat what we discussed there, and focus on the implications for consumption and welfare.

First, equation (23) shows that consumption-capital ratio C/K is no longer independent of stochastic terms, as opposed to the horizontal dashed line in Figure 1. Here, it *does* depend positively on physical capital uncertainty σ_k, its arrival rate λ_j^k and the size of jump β_j^k[18]. Taken together with Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b) and

[17] We can be sure that $u \in (0, 1)$ as long as the inequality (19) holds. Moreover, one can establish that the appropriate TVC is satisfied. See Sennewald (2007) for the proof.

[18] See also Proposition 6 and 7 in Rebelo and Xie (1999) and equation (16) in Wälde (2011a).

Hiraguchi (2013), it seems that the only shock type that can *directly* affect consumption-capital ratio C/K in the stochastic Uzawa-Lucas model would be the one in stochastic process for physical capital. The reasonable guess then is that technology, demographic, and human capital shock have, in fact, nothing to do with the optimal ratio of consumption to physical capital.

Second, since the welfare is the function of the state variable K, which is now stochastic, we can investigate the relationship between welfare and uncertainty about physical capital accumulation. Unlike Krebs (2003), because households do not have choice about investment in physical capital, the channel through which shocks affect welfare can be identified via the constant (22)[19]. One can show that

$$\frac{\partial J}{\partial \sigma_k} < 0$$

and that, for all j,

$$\frac{\partial J}{\partial \lambda_j^k} < 0$$

and

$$\frac{\partial J}{\partial \beta_j^k} < 0$$

in other words, the larger physical capital shock, the higher arrival rate, and the larger size of jump all reduce welfare, because it leads to physical capital contraction. It is worth emphasizing that no studies have found the closed-form solution to the stochastic Uzawa-Lucas model with depreciation of physical capital following a Brownian motion (and many Poisson jump processes), and thus it is hoped that more studies on this direction will be accumulated. The findings in this section can be summarized as follows:

Proposition 3. *One parameter restriction commonly used in the literature again enables us to find the closed-form solution to the stochastic Uzawa-Lucas model in which the accumulation of both human and physical capital follows a combination of a Brownian motion and many Poisson jump processes. The larger physical capital shock, the higher arrival rate, and larger size of jump deteriorate welfare. More importantly, consumption-capital ratio depends on shock terms when the depreciation of physical capital is described by the stochastic process.*

5. Concluding Remarks

I analyze the Uzawa-Lucas model in which the accumulation of human capital is driven by a combination of one Brownian motion process and many Poisson jump processes. Imposing one parameter restriction originally proposed by Xie (1991), I obtain the closed-form

[19] To be precise, as $\mathcal{B}_X$ is contained in $\mathcal{B}_Y$, shock terms in (22) affect both A and H, and hence welfare J. However, this channel would be too obvious to explain in detail.

solution. I then analytically characterize the link between human capital uncertainty, technological progress, expected growth rate of human capital, and welfare, some of which are not discussed in Krebs (2003).

Specifically, for the case with a Brownian motion process only, which is akin to previous studies of Bucci et al (2011), Marsiglio and La Torre (2012a, 2012b), and Hiraguchi (2013), I show that larger human capital uncertainty increases time spent in working, reduces the expected growth rate of human capital, and reduces welfare. I then extend the model by adding many Poisson jump processes and present the first stochastic Uzawa-Lucas model with a mixture of one Brownian motion process and many Poisson jump processes. For this case, I demonstrate that the higher arrival rate and larger size of jumps decrease time spent in working, increase the expected growth rate of human capital, and improve welfare. These can be only confirmed by considering the Poisson jump process.

At that stage, it turned out that, together with the previous studies, human capital shocks, like technology and demographic shocks, have no direct impact on the optimal consumption-capital ratio. To solve this puzzle, I assume that physical capital accumulation is stochastic, as well as the accumulation of human capital. In this case, consumption-capital ratio is dependent on the physical capital shock. Moreover, I show that this shock, the higher arrival rate, and larger size of jump all deteriorate welfare. Besides, I establish that the closed-form solution to the Uzawa-Lucas model in which both the accumulation of human and physical capital follow an amalgamation of a Brownian motion process and many Poisson jump processes is available.

To summarize for the purpose of policy implications, given that the larger human capital uncertainty reduces its expected growth rate and deteriorates the welfare of households, policymakers should get rid of it by implementing the sound policy. The findings of this paper challenges the series of recent papers of Lester et al (2014), Cho et al (2015), and Xu (2017), who all argue that the larger uncertainty does not necessarily reduce welfare, and in some cases, rather improve welfare.

The obvious extension of this paper would be to develop the stochastic Uzawa-Lucas model in which technological progress or population dynamics is driven by a combination of Brownian and Poisson jump processes. Whether we can still find the closed-form solution is uncertain at this stage. Another possibility is to obtain the explicit solution without the parameter restriction. Although its use is common since Xie (1991, 1994), and it provides us with clear economic channel through which macroeconomic variables interact in a transparent way, it prevents us from seeing how the prediction of the model is (un)changed by varying the risk aversion parameter. This may be possible by assuming the more general utility function of Duffie and Epstein (1992) that disentangles the risk aversion parameter from the intertemporal elasticity of substitution. It is challenging, but the possibility is probably here to stay.

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Appendix A. Analytical Solution

This appendix briefly describes how to find the closed-form representation of the value function in Theorems 1, 2, and 3. For this purpose, postulate the tentative value function of the form

$$J(K, A, H) = xK^{\theta_1} + yA^{\theta_2}H^{\theta_3} + z$$

where x, y, z, θ_1, θ_2, and θ_3 are all unknown constants to be determined. The relevant partials are $J_K = x\theta_1 K^{\theta_1-1}$, $J_{KK} = x\theta_1(\theta_1-1)K^{\theta_1-2}$, $J_A = y\theta_2 A^{\theta_2-1}H^{\theta_3}$, $J_H = y\theta_3 A^{\theta_2}H^{\theta_3-1}$, and $J_{HH} = y\theta_3(\theta_3-1)A^{\theta_2}H^{\theta_3-2}$.

To obtain the explicit expression, substitute these partials into the maximized HJB equation (21). Then, set $\theta_1 = \alpha+\gamma$, $\theta_2 = \gamma$, and $\theta_3 = \alpha$. Finally, by imposing the parameter restriction (8), you can find the explicit expressions for x, y, and z, and consequently, those for control variables C and u and for the value function $J(K, A, H)$ in Theorem 3. Expressions in Theorem 2 are available by abstracting from the shock terms associated with the stochastic depreciation of physical capital, while those in Theorem 1 are obtained by abstracting from many Poisson jump processes in equation (16).

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