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Stochastic migration theory and migratory phase transitions

Wolfgang WEIDLICH
Institut für Theoretische Physik, University of Stuttgart, Pfaffenwaldring 57/III, D-7000 Stuttgart, Fed. Rep. Germany

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1. Introduction

Migration processes are an example of socio-economic dynamics of particular interest for quantitative research: The underlying motivations for the migration of individuals are relatively well defined and they must always result in a clear individual decision to maintain or to change the location in a given interval of time. On the other hand, the population numbers and migratory fluxes can be 'measured' and compared with theory. It is then the objective of a quantitative migration theory to understand the migratory dynamics by connecting quantitatively the micro-level of motivations and decisions with the macro-level of the global migration process.

In this talk I give a short account only of the general scope of the theory, which describes the migratory dynamics on the stochastic level (master equation) and the quasi-deterministic level (meanvalue equations) as well. Then I shall focus in presenting the outlines of a dynamic model for a dramatic migratory process beginning in the last century: The sudden formation of huge metropolitan areas all over the world, in particular in developing countries. The model explains this process by a 'migratory phase transition'. By a phase transition one understands—as in physics—the global change of the macroscopic properties of a system, if some control parameters of that system cross certain 'critical' values. In particular, originally stable states of the system may become unstable and transform into new dynamic modes.

2. The migratory decision process and the equations of motion

The migratory system considered here consists of a homogeneous population of N members migrating between L regions. The 'socio configuration' $\boldsymbol{n} = \{n_1, n_2, \ldots, n_L\}$, with n_i = number of people in region i, where

$$\sum_{i=1}^{L} n_i = N \tag{1}$$

describes the 'state' of the system. We have to understand the dynamics of $\boldsymbol{n}(t)$! For this aim we start from the behaviour of individuals, whose decisions to migrate are governed by comparative considerations of the 'utility' of the origin and destination area of living.

Therefore let us introduce a utility function $u_i(n_i)$ for each region i as a measure for the attractivity of that region. This utility can and will in general depend on the number n_i of people living there. We assume

$$u_i(n_i) = \delta_i + \kappa_i n_i + \rho_i n_i^2 \tag{2}$$

where, according to the form (2), the trend parameters δ_i, κ_i, ρ_i have the following meaning:

$$\delta_i = \text{preference parameter}, \qquad \kappa_i = \text{agglomeration parameter}, \qquad \rho_i = \text{saturation parameter}, \tag{3}$$

Our further procedure deviates from that of many conventional approaches in social science: We do not 'maximize utilities' in order to determine the 'optimal state' of the system, which is then considered as the given (equilibrium) state. Instead we build the utility function into a dynamic framework yielding equations of motion.

The first step is the introduction of 'individual transition probabilities' $p_{ji}(u_j, u_i)$ to migrate from region i to region j. This transition probability will depend on the utilities u_i and u_j of the origin- and destination-area. Since p_{ij} must be a positive definite quantity by definition, and since one has to expect $p_{ji} > p_{ij}$ for $u_j > u_i$, the simplest and most plausible ansatz for p_{ji} is

$$p_{ji}(u_j, u_i) = \omega \cdot \exp\left[u_j(n_j+1) - u_i(n_i)\right] \tag{4}$$

where ω is a frequency factor scaling the time regime in which the process will take place.

It is then easy to construct 'global transition probabilities' $w_{ji}(n_j, n_i)$ for a transition from a socioconfiguration $\{n_1, \dots, n_j, \dots, n_i, \dots, n_L\}$ to a neighboring socioconfiguration $\{n_1, \dots, (n_j + 1), \dots, (n_i - 1), \dots, n_L\}$. Since the n_i individuals do migrate independently, we obtain

$$w_{ji}(n_j, n_i) = n_i \cdot p_{ji}\left(u_j(n_{j+1});\ u_i(n_i)\right). \tag{5}$$

As a next step we introduce the probability distribution:

$$P(n_1, n_2, \dots, n_L;\ t) \equiv P(\boldsymbol{n};\ t) \geqslant 0. \tag{6}$$

By definition this is the probability to find the socioconfiguration $\{n_1, n_2, \dots, n_L\}$ at time t.

The distribution $P(\boldsymbol{n};\ t)$ obeys an equation of motion, the master equation:

$$\left.\begin{aligned}
&\frac{\mathrm{d}P(\boldsymbol{n};\ t)}{\mathrm{d}t} = \sum_{i,\,j=1}^{L} \left\{ w_{ij}(\boldsymbol{n}^{(ji)}) P(\boldsymbol{n}^{(ji)};\ t) - w_{ji}(\boldsymbol{n}) P(\boldsymbol{n};\ t) \right\} \\
&\text{with} \qquad \boldsymbol{n} = \{n_1, \dots, n_i, \dots, n_j, \dots, n_L\}, \\
&\qquad\qquad \boldsymbol{n}^{(ji)} = \{n_1, \dots, n_{i-1}, \dots, n_{j+1}, \dots, n_L\}, \\
&\text{and} \qquad w_{ji}(\boldsymbol{n}) = \omega n_i \exp\left[u_j(n_j+1) - u_i(n_i)\right].
\end{aligned}\right\} \tag{7}$$

The master equation (7) has a suggestive interpretation: The change with time of $P(\boldsymbol{n};\, t)$ is caused by

(1) the probability flux from $\boldsymbol{n}$ to all $\boldsymbol{n}^{(ji)}$ (last term on RHS),

(2) the probability flux from all $\boldsymbol{n}^{(ji)}$ to $\boldsymbol{n}$ (first term on RHS).

The solutions of the master equation (7) are discussed in [1] and [2]. Here we use it to derive equations of motion for meanvalues.

The meanvalue of a function of $\boldsymbol{n}$ is defined by

$$\overline{f(\boldsymbol{n})} = \sum_{\boldsymbol{n}} f(\boldsymbol{n}) P(\boldsymbol{n};\, t) \tag{8}$$

in particular

$$\bar{n}_i(t) = \sum_{\boldsymbol{n}} n_i P(\boldsymbol{n};\, t), \tag{9}$$

where the sum extends over all socio configurations. Making use of (7) one obtains the equation of motion

$$\frac{\mathrm{d}\bar{n}_k}{\mathrm{d}t} = \sum_{\boldsymbol{n}} n_k \frac{\mathrm{d}P(\boldsymbol{n};\, t)}{\mathrm{d}t} = \sum_{i=1}^{L} \overline{w_{ki}(\boldsymbol{n})} - \sum_{j=1}^{L} \overline{w_{jk}(\boldsymbol{n})}. \tag{10}$$

If the probability distribution $P(\boldsymbol{n};\, t)$ is essentially unimodal, the approximation

$$\overline{w_{ki}(\boldsymbol{n})} \approx w_{ki}(\bar{\boldsymbol{n}}) \tag{11}$$

holds, which may be inserted into (10) to yield

$$\begin{aligned} \frac{\mathrm{d}\bar{n}_k}{\mathrm{d}t} &= \sum_{i=1}^{L} w_{ki}(\bar{\boldsymbol{n}}) - \sum_{j=1}^{L} w_{jk}(\bar{\boldsymbol{n}}) \\ &= \sum_{i=1}^{L} \omega \bar{n}_i\, \mathrm{e}^{u_k(\bar{n}_k) - u_i(\bar{n}_i)} - \sum_{j=1}^{L} \omega \bar{n}_k\, \mathrm{e}^{u_j(\bar{n}_j) - u_k(\bar{n}_k)}. \end{aligned} \tag{12}$$

3. The dynamic model for sudden urban growth [1]

The preceding frame of equations, in particular (12), is the starting point for a dynamic model for a migratory phase transition describing instabilities of the kind of sudden urban growth. The model assumptions are as simple as possible: The L regions are assumed to be primordially equivalent, which means that no natural preference of one region against the other exists, and that all regional agglomeration parameters κ_i coincide. This means, the following trend parameters and utility functions are assumed:

$$\delta_i = 0, \qquad \kappa_i = \kappa, \qquad u_i(n_i) = \kappa n_i. \tag{13}$$

Introducing the scaled variables

$$\tau = \omega t, \qquad \nu_l = \kappa n_l, \tag{14}$$

[1] For details of the derivations see [3].

the meanvalue equations (12) then assume the form

$$\mathrm{d}\nu_l/\mathrm{d}\tau = F_l(\nu_1, \ldots, \nu_L) \tag{15}$$

with

$$F_l(\nu) = \mathrm{e}^{\nu_l} \sum_{j=1}^{L} \nu_j \, \mathrm{e}^{-\nu_j} - \nu_l \, \mathrm{e}^{-\nu_l} \sum_{j=1}^{L} \mathrm{e}^{\nu_j} \tag{16}$$

while the constraint (1), which is compatible with eq. (15), now has the form

$$\sum_{l=1}^{L} \nu_l = \kappa N. \tag{17}$$

3.1. Stationary states of the model

The stationary states of (15), $\hat{\nu} = \{\hat{\nu}_1, \hat{\nu}_2, \ldots, \hat{\nu}_L\}$ are seen to obey the equations:

$$S\hat{\nu}_l = \mathrm{e}^{2\hat{\nu}_l}, \qquad l = 1, 2, \ldots, L, \quad S = \text{const.} \tag{18}$$

This equation has two solutions for $S > 2e$:

$$\hat{\nu} = \hat{\nu}_+(S), \qquad \hat{\nu} = \hat{\nu}_-(S). \tag{19}$$

Let us assume, that the p first $\hat{\nu}_l$ take the value $\hat{\nu}_+(S)$ and the $q = (L - p)$ last $\hat{\nu}_l$ take the values $\hat{\nu}_-(S)$. This corresponds to a stationary state of the L region system with p densely and q thinly populated regions, whether or not it is stable. The agglomeration parameter κ pertaining to that situation follows from (17):

$$\kappa = \kappa_p(S) = \frac{1}{N}(p\hat{\nu}_+(S) + q\hat{\nu}_-(S)). \tag{20}$$

3.2. Computer solutions of the model

Due to its simplicity, computer solutions as well as special analytical solutions of the model can be found. In this section we discuss computer solutions. For illustrative purposes we restrict ourselves to the case $L = 3$ in the first example. Because of the constraint (17) the space of variables is two-dimensional only in this case and can be represented as the interior of a triangle.

In Figs. 1, 2, 3 we show the fluxlines of the three-region system for a small value $\hat{\kappa}_\alpha$, an intermediate value $\hat{\kappa}_\beta$ and a large value $\hat{\kappa}_\gamma$ of the scaled agglomeration parameter $\hat{\kappa} = \kappa/\kappa_c$, with $\kappa_c = L/2N$. Even for this simple system it turns out, that the three representative values $\hat{\kappa}_\alpha < \hat{\kappa}_\beta < \hat{\kappa}_\gamma$ of the 'control parameter' $\hat{\kappa}$ lead to three globally different cases of dynamic behaviour. If $\hat{\kappa}$ crosses certain critical values, a phase transition from one to the other macro-dynamical phase takes place.

For small $\hat{\kappa} = \hat{\kappa}_\alpha$ (Fig. 1) the system has one stable stationary state, which corresponds to a homogeneous population of the regions. All nonequilibrium states evolve into this equilibrium state.

For intermediate $\hat{\kappa} = \hat{\kappa}_\beta$ (Fig. 2) has 4 stable stationary states, the homogeneous state and three other stable states, each with one densely and two thinly populated regions. It depends on initial conditions now, into which of the 4 equilibria the migratory system will evolve.

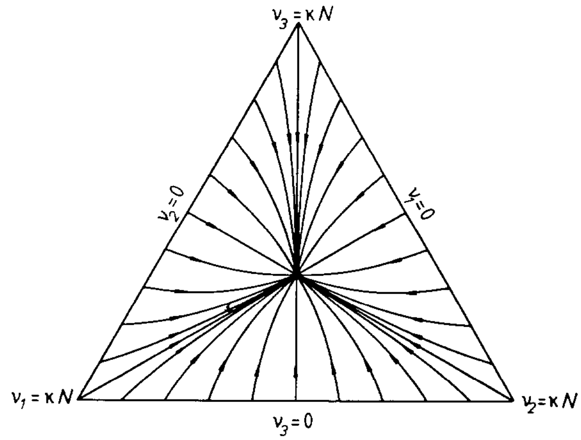

Fig. 1. Fluxlines for $L = 3$ and a small agglomeration parameter $\hat{\kappa}_\alpha < \hat{\kappa}_c$. There exists 1 stable stationary state with 1 basin of attraction.

For large $\hat{\kappa} = \hat{\kappa}_\gamma$ (Fig. 3) still another global situation emerges: The homogeneous state has now become unstable and only three stable equilibrium states persist, those with one dense and two thin regions. Again it depends on initial conditions, into which of the 3 equilibria the system evolves.

The fluxlines of the first example do not show, how the system traverses them with time. Therefore we represent this evolution with time in a second example for $L = 16$ in Figs. 4(a) and (b).

The two systems in Figs. 4(a) and (b) start at $t = 0$ with the same initial distribution of the populations over the regions. However, the further evolution is completely different due to different agglomeration trends.

In Fig. 4(a) a small agglomeration parameter $\hat{\kappa} = 0.7$ has been assumed, which finally leads to the equipartition of the population over all regions.

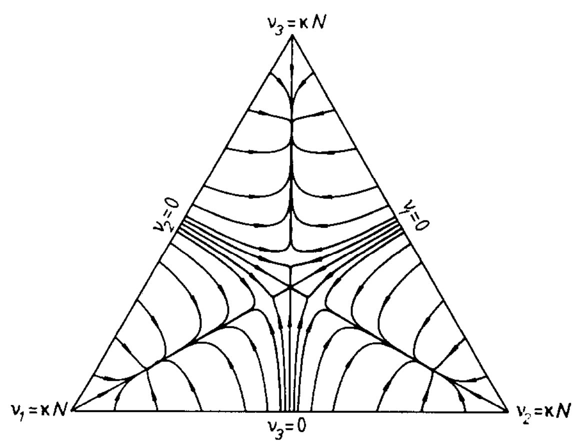

Fig. 2. Fluxlines for $L = 3$ and an intermediate agglomeration parameter $\hat{\kappa}_\beta$ with $\hat{\kappa}_c < \kappa_\beta < \hat{\kappa}_c$. There exist 3 unstable saddle points and 4 stable stationary states with 4 basins of attraction.

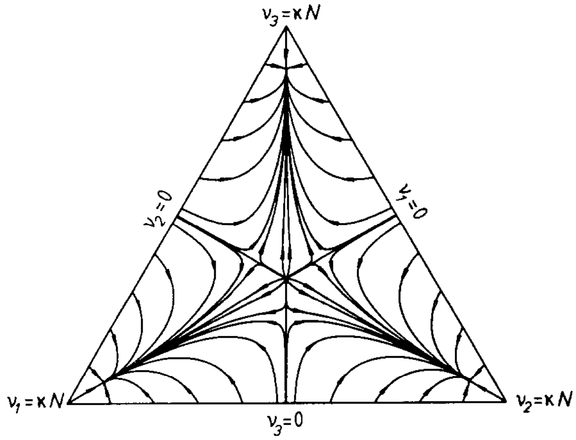

Fig. 3. Fluxlines for $L = 3$ and a large agglomeration parameter $\hat{\kappa} > \hat{\kappa}_c$. There exist 1 unstable state (node), 3 unstable saddle points and 3 stable stationary states with 3 basins of attraction.

In Fig. 4(b) on the other hand a large agglomeration parameter $\hat{\kappa} = 1.2$ is assumed, which now leads to the evolution of exactly one densely populated 'metropolitan area' while all other regions become depleted 'provincial areas' in the final equilibrium state.

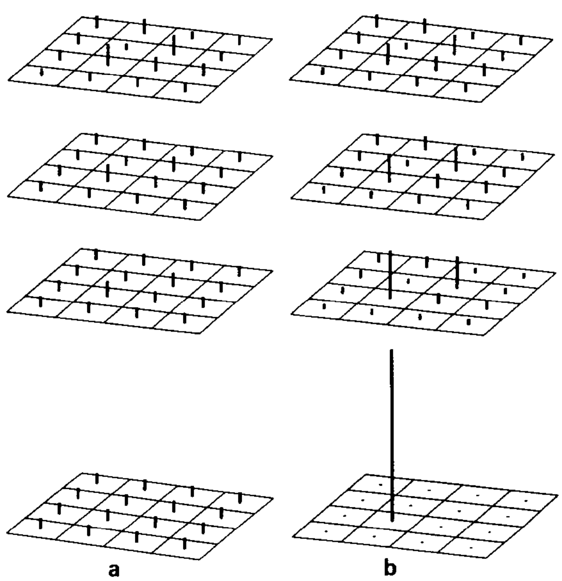

Fig. 4. (a) Relaxation of an initial population distribution over $L = 16$ regions into the homogeneous state for small agglomeration parameter $\hat{\kappa} = 0.7$ ($\tau = 0.0$; $\tau = 0.3$; $\tau = 0.4$; $\tau = 1.0$). (b) Evolution of an initial population distribution over $L = 16$ regions into one metropolitan area and 15 depleted regions for large agglomeration parameter $\hat{\kappa} = 1.2$ ($\tau = 0.00$; $\tau = 0.05$; $\tau = 0.10$; $\tau = 0.15$).

3.3. Analytic solutions of the model along symmetry paths

There exist special solutions of the equations of motion (15) along symmetry paths

$$\begin{aligned} &\nu_1(\tau) = \nu_2(\tau) = \cdots = \nu_p(\tau) = \nu_+(\tau), \\ &\nu_{p+1}(\tau) = \cdots = \cdots = \nu_L(\tau) = \nu_-(\tau), \end{aligned} \tag{21}$$

which can be treated analytically. An exact equation of motion can be derived from (15) for the difference variable

$$\nu(\tau) = \nu_+(\tau) - \nu_+(\tau). \tag{22}$$

It reads

$$\frac{\mathrm{d}\nu}{\mathrm{d}\tau} = Lf_p(\nu) = -L\frac{\partial V_p(\nu)}{\partial \nu} \tag{23}$$

with the 'evolution force'

$$f_p(\nu) = -\left(\tfrac{1}{2}\hat{\kappa} + \frac{q}{L}\nu\right)\mathrm{e}^{-\nu} + \left(\tfrac{1}{2}\hat{\kappa} - \frac{p}{L}\nu\right)\mathrm{e}^{\nu} \tag{24}$$

and 'evolution potential'

$$V_p(\nu) = (\hat{\kappa} + 1) - \left[\tfrac{1}{2}\hat{\kappa} + \frac{p}{L}(1-\nu)\right]\mathrm{e}^{\nu} - \left[\tfrac{1}{2}\hat{\kappa} + \frac{q}{L}(1+\nu)\right]\mathrm{e}^{-\nu}. \tag{25}$$

Remark: All stationary points determined in section 3.1 lie on symmetry paths.

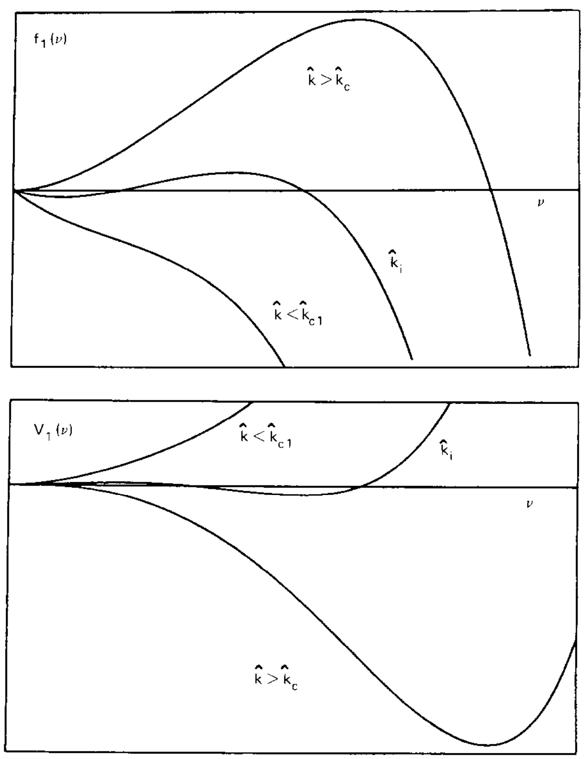

Fig. 5. The evolution force $f_1(\nu) = -\partial V_1(\nu)/\partial\nu$ and the evolution potential $V(\nu)$ for $L = 6$ and $p = 1$ an three values of $\hat{\kappa}$: $\hat{\kappa} < \kappa_c$; $\hat{\kappa}_c < \hat{\kappa}_i < \hat{\kappa}_c$; $\hat{\kappa} < \kappa_c$.

The evolution force and the evolution potential is depicted in Fig. 5 for $L = 6$ and $p = 1$ and different representative values of the agglomeration parameter $\hat{\kappa}$. For $\hat{\kappa} > \hat{\kappa}_c$ the force $f_p(\nu)$ drives the variable $\nu(\tau)$, according to (23), into the minimum of the potential. On the other hand for $\hat{\kappa} < \hat{\kappa}_c$ the variable $\nu(\tau)$ is driven back to the homogeneous state $\nu = 0$. For intermediate $\hat{\kappa}_i$ between κ_c and κ_c, however, the potential has a maximum at $\nu = \nu_{\mathrm{Max}}$ between $\nu = 0$ and its minimum value at $\nu = \hat{\nu}$. The force drives $\nu(\tau)$ back to $\nu = 0$ for $0 < \nu_0 < \nu_{\mathrm{Max}}$, and to $\nu = \hat{\nu}$ for $\nu < \nu_0 < \hat{\nu}$.

3.4. Analysis of small deviations along symmetry paths. (linear stability analysis)

Consider two neighboring solutions $\nu_l(\tau)$, $\tilde{\nu}_l(\tau) = \nu_l(\tau) + \epsilon_l(\tau)$, $l = 1, 2, \ldots, L$ of the equations of motion (15):

$$\dot{\nu}_l = F_l(\nu), \quad l = 1, 2, \ldots, L, \tag{26a}$$

$$\dot{\tilde{\nu}}_l = F_l(\tilde{\nu}), \quad l = 1, 2, \ldots, L. \tag{26b}$$

Linearizing the difference "(26b) minus (26a)" with respect to the small deviation $\epsilon(\tau)$, one obtains

$$\dot{\epsilon}_l = \sum_{k=1}^{L} F_{l|k}(\nu(\tau))\epsilon_k \tag{27}$$

or, in concise Dirac-notation

$$|\dot{\epsilon}\rangle = \Phi|\epsilon\rangle \tag{28}$$

The 'dynamic matrix' Φ has the form

$$\Phi = |\alpha_1\rangle\langle\beta_1| + |\alpha_2\rangle\langle\beta_2| + \Gamma \tag{29}$$

with

$$\left.\begin{aligned}
&|\alpha_1\rangle = \begin{pmatrix} -\nu_1\,\mathrm{e}^{-\nu_1} \\ \vdots \\ -\nu_L\,\mathrm{e}^{-\nu_L} \end{pmatrix}, \qquad |\alpha_2\rangle = \begin{pmatrix} \mathrm{e}^{\nu_1} \\ \vdots \\ \mathrm{e}^{\nu_L} \end{pmatrix}, \\
&\langle\beta_1| = \{\mathrm{e}^{\nu_1}, \ldots, \mathrm{e}^{\nu_L}\}, \qquad \langle\beta_2| = \{\mathrm{e}^{-\nu_1}(1-\nu_1), \ldots, \mathrm{e}^{-\nu_L}(1-\nu_L)\}, \\
&\Gamma = \begin{pmatrix} \gamma_1 & & & \\ & \gamma_2 & & 0 \\ & 0 & \ddots & \\ & & & \gamma_L \end{pmatrix}, \qquad \gamma_l = (\nu_l - 1)\,\mathrm{e}^{-\nu_l}\sum_{j=1}^{L}\mathrm{e}^{\nu_j} + \mathrm{e}^{\nu_l}\sum_{j=1}^{L}\nu_j\,\mathrm{e}^{-\nu_j}.
\end{aligned}\right\} \tag{30}$$

In the following we assume that $\nu(\tau)$ is a symmetry path solution (21).

The solution of (28) is facilitated, if the eigenvectors and eigenvalues of Φ are known. These can indeed be found. Furthermore it turns out, that only four different eigenvalues exist, namely γ_+, γ_-, λ and 0. The eigenvector equations read

$$\Phi|\eta_+^{(\sigma)}\rangle = \gamma_+|\eta_+^{(\sigma)}\rangle, \quad \sigma = 1, 2, \ldots, (p-1), \tag{31a}$$

$$\Phi|\eta_-^{(\sigma)}\rangle = \gamma_-|\eta_-^{(\sigma)}\rangle, \quad \sigma = 1, 2, \ldots, (q-1), \tag{31b}$$

$$\Phi|\eta^{(\lambda)}\rangle = \lambda|\eta^{(\lambda)}\rangle, \tag{31c}$$

$$\Phi|\eta^{(0)}\rangle = 0 \tag{31d}$$

with the explicit form of the eigenvalues, which are functions of the variables $\nu_+(\tau)$, $\nu_-(\tau)$ (where $\nu(\tau)=\nu_+(\tau)-\nu_-(\tau)$) along the symmetry path:

$$\gamma_+(\nu)=p(2\nu_+-1)+q\left[(\nu_+-1)\,\mathrm{e}^{-\nu}+\nu_-\,\mathrm{e}^{\nu}\right], \tag{32a}$$

$$\gamma_-(\nu)=q(2\nu_--1)+p\left[\nu_+\,\mathrm{e}^{-\nu}+(\nu_--1)\,\mathrm{e}^{\nu}\right], \tag{32b}$$

$$\lambda(\nu)=\left[L\nu_--p\right]\mathrm{e}^{\nu}+\left[L\nu_+-q\right]\mathrm{e}^{-\nu}=-L\frac{\partial^2 V_p(\nu)}{\partial\nu^2}. \tag{32c}$$

The explicit form of the eigenvectors is skipped here. We do however note the illustrative meaning of these deviation modes:

(a) The eigenvectors $|\eta_+^{(\sigma)}\rangle$ describe the deviation of two dense regions (in opposite directions) from the symmetry path. There exist $(p-1)$ linear independent deviation modes of this kind.

(b) The eigenvectors $|\eta_-^{(\sigma)}\rangle$ describe, correspondingly, deviations of two thin regions from the symmetry path. There exist $(q-1)=(L-p-1)$ linear independent modes of this kind.

(c) The eigenvector $|\eta^{(\lambda)}\rangle$ describes a collective deviation mode of all dense and thin regions. It means an acceleration/retardation on the symmetry path.

(d) $|\eta^{(0)}\rangle$ is a mode which cannot appear in the expansion (33) of $|\epsilon\rangle$, since this would lead to a forbidden deviation $|\epsilon\rangle$ with $\sum_{j=1}^{L}\epsilon_j\neq 0$.

Let us now expand the deviation vector $|\epsilon\rangle$ as a linear combination of the eigenvectors of the dynamic matrix Φ:

$$|\epsilon(\tau)\rangle=s_\lambda(\tau)\,|\eta^{(\lambda)}\rangle+\sum_{\sigma=1}^{p-1}s_\sigma^+(\tau)\,|\eta_+^{(\sigma)}\rangle+\sum_{\sigma=1}^{q-1}s_\sigma^-(\tau)\,|\eta_-^{(\sigma)}\rangle. \tag{33}$$

The omission of the mode $|\eta^{(0)}\rangle$ secures, that (33) satisfies the constraint

$$\sum_{j=1}^{L}\epsilon_j(\tau)=0 \tag{34}$$

which must hold, since $\nu(\tau)$ and $\tilde{\nu}(\tau)$ both fulfill (17).

The insertion of (33) into (29) now yields the decoupled equations of motion for the deviation amplitudes $s_\sigma^+(\tau)$, $s_\sigma^-(\tau)$ and $s_\lambda(\tau)$, which can easily be solved

$$\left.\begin{aligned}&\dot{s}_\sigma^\pm=\gamma_\pm(\tau)s_\sigma^\pm,\quad s_\sigma^\pm(\tau)=s_\sigma^\pm(0)\exp(\Gamma_\pm(\tau))\\&\text{with}\quad \Gamma_\pm(\tau)=\int_0^\tau\gamma_\pm(\tau')\,\mathrm{d}\tau',\end{aligned}\right\} \tag{35a, b}$$

$$\left.\begin{aligned}&\dot{s}_\lambda=\lambda(\tau)s_\lambda,\qquad s_\lambda(\tau)=s_\lambda(0)\exp(\Lambda(\tau))\\&\text{with}\quad \Lambda(\tau)=\int_0^\tau\lambda(\tau')\,\mathrm{d}\tau'.\end{aligned}\right\} \tag{35c}$$

The stability of a symmetry path solution now depends on the increase or decrease of the perturbation modes in $|\epsilon(\tau)\rangle$. This behaviour in turn depends on the sign of the eigenvalues $\gamma_\pm(\tau)$, $\lambda(\tau)$ belonging to $s_\pm(\tau)$, $s_\lambda(\tau)$, respectively.

In particular we are interested in the stability of the stationary points lying on the symmetry paths. Let us first consider the stationary point

$$\nu_+=\nu_-=\bar{\nu}=\frac{\kappa N}{L},\qquad \nu=\nu_+-\nu_-=0, \tag{36}$$

describing the homogeneous distribution of the population over the L regions. At this point the eigenvalues $\gamma_b + (\nu)$ and $\lambda(\nu)$ degenerate into

$$\left.\begin{array}{l} \lambda = \gamma_+ = \gamma_- = L(\hat{\kappa} - 1) \\ \text{with} \quad \hat{\kappa} = \kappa/\kappa_c \quad \text{where } \kappa_c = L/2N. \end{array}\right\} \tag{37}$$

Since the degenerate eigenvalue (37) changes sign if $\hat{\kappa}$ crosses the value 1, we conclude, that the homogeneous population of regions is stable for $\hat{\kappa} < 1$ (low agglomeration trend), but unstable for $\hat{\kappa} > 1$ (high agglomeration trend). In other words: a phase transition of the migratory system takes place, if $\hat{\kappa}$ crosses 1 from below, rendering unstable the originally stable homogeneous state.

In order to see, which of the other stationary states on symmetry paths are stable, we make use of the relation

$$e^{\hat{\nu}_+ - \hat{\nu}_-} = \sqrt{\hat{\nu}_+/\hat{\nu}_-} \tag{38}$$

valid at stationary states because of (18). The eigenvalues (32) then can be cast into the form

$$\gamma_+(\hat{\nu}) = (p\rho_+ + q\rho_-) \cdot (\rho_+ - 1/\rho_+) > 0, \tag{39a}$$

$$\gamma_-(\hat{\nu}) = (p\rho_+ + q\rho_-) \cdot (\rho_- - 1/\rho_-) < 0, \tag{39b}$$

$$\lambda(\hat{\nu}) = q\rho_-(\rho_+ - 1/\rho_+) + p\rho_+(\rho_- - 1/\rho_-) = -L\frac{\partial^2 V_p(\hat{\nu})}{\partial \hat{\nu}^2} \gtrless 0 \tag{39c}$$

with

$$\rho_+ = \sqrt{2\hat{\nu}_+} > 1, \qquad \rho_- = \sqrt{2\hat{\nu}_-} < 1. \tag{40}$$

Because of $\gamma_+(\hat{\nu}) > 0$ we conclude, that only stationary points with $p = 1$ (one dense region, $(L-1)$ thin regions) can be stable, since the corresponding expansion (33) does not contain the modes $s_\sigma^+(\tau) \,|\, \eta_+^\sigma \rangle$. Furthermore, only those stationary points on the symmetry path $p = 1$ are stable, which correspond to a minimum of the evolution potential, i.e. to $\partial^2 V_p(\nu)/\partial \hat{\nu}^2 > 0$, or to $\lambda < 0$ (see Fig. 5).

More generally we can conclude within the frame of this model, that
- the competition of cities (dense regions) leads to the survival of only one metropole, and
- the competition of provinces (thin regions) leads to their assimilation.

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