Chapter 2
The Grey Systems Theory Framework

Over the past 30 years, grey systems theory has developed as a scientific discipline with its own theoretical structure consisting of systems analysis, evaluation, modeling, prediction, decision-making, control, and techniques of optimization.

2.1 Grey Models and Framework

A grey number is a figure that represents a range of values rather than an exact value when the exact value for the said figure is not known. The range of a grey number can be an interval or a general number set. Grey numbers are usually expressed as the symbol "⊗", which is called grey. A grey number represents the degree of information uncertainty in a given system. As the basis of grey systems theory, research on grey numbers and grey measures has attracted increased attention over the past years.

Professor Julong Deng presented the basic thinking and models of grey system theory (GST) in his first paper, which was published in Systems and Control Letters (Deng 1982). In 1985, he presented the GST framework in his two books, which were published by National Defense Industry Press (Deng 1985a) and Huazhong University of Science and Technology Press (Deng 1985b), respectively. Such books addressed grey numbers and the operations of interval grey numbers, grey incidence analysis, grey generation, grey clustering, grey forecasting model, grey decision-making, and grey control. He confirmed the GST framework in his later publications, such as "A course on Grey System Theory" (Deng 1990) and "The Basis of Grey System Theory" (Deng 2002).

2.2   The Thinking, Models and Framework of Grey Systems Theory

2.2.1   Grey Numbers and Its Operations

In 2004, an axiomatic definition of the grey degree of a grey number was put forward by Sifeng Liu and Yi Lin (Liu and Lin 2004), and such a definition was based on a measure of grey number and its background or domain. This definition of the grey degree of a grey number satisfied the requirement of standardability, which provided the bedrock for us to cognize the uncertainty of grey information In 2010, the introduction of the unreduction axiom and a new definition of degree of greyness of grey numbers were put forward. This allowed the operations of grey numbers and grey algebraic system to be built based on the grey "kernel" and the degree of greyness of grey numbers (Liu et al. 2010). On these grounds, the operation of grey numbers has been transformed to the operation of real numbers. Thus, to a certain extent the problem of setting up the operation of grey numbers and the grey algebraic system has been solved.

In 2012, Sifeng Liu, Zhigeng Fang, Yingjie Yang and others developed the concept of general grey number as follows:

$$g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i, \bar{a}_i\right]$$

Among them, any interval grey numbers $\otimes_i \in [\underline{a}_i, \bar{a}_i] \subset \bigcup_{i=1}^{n} [\underline{a}_i, \bar{a}_i]$, which satisfy $\underline{a}_i, \bar{a}_i \in \Re$ and $\bar{a}_{i-1} \leq \underline{a}_i \leq \bar{a}_i \leq \underline{a}_{i+1}$, $g^- = \inf_{\underline{a}_i \in g^{\pm}} \underline{a}_i$, $g^+ = \sup_{\bar{a}_i \in g^{\pm}} \bar{a}_i$ are called the lower and upper limits of $g^{\pm}$.

Such researchers also found that summation and subtraction operations about the degree of greyness of grey numbers do not satisfy the introduction of the un-reduction axiom. Then they verified summation and subtraction operations about grey numbers and the grey synthesis axiom as specified below (Liu et al. 2012). The algebra system based on grey "kernel" and the degree of greyness of grey numbers is shown in Fig. 2.1.

In addition, Zhigeng Fang and colleagues put forward the concept of standard interval grey number and offered the algorithm of standard interval grey numbers (Fang and Liu 2005), while Qiaoxing Li and colleagues raised grey number rules based on numerical coverage (Li and Liu 2012). Such researchers carried out beneficial exploration around grey numbers and related operations.

In the grey algebra system based on the grey "kernel" and the degree of greyness of grey numbers, the principles of "take the bigger one" are still used to calculate the degree of greyness of the "multiplication" and "division" operation outcome, according to the introduction of the un-reduction axiom. To reveal the inherent law of synthesis of degree of greyness in the process of operations of
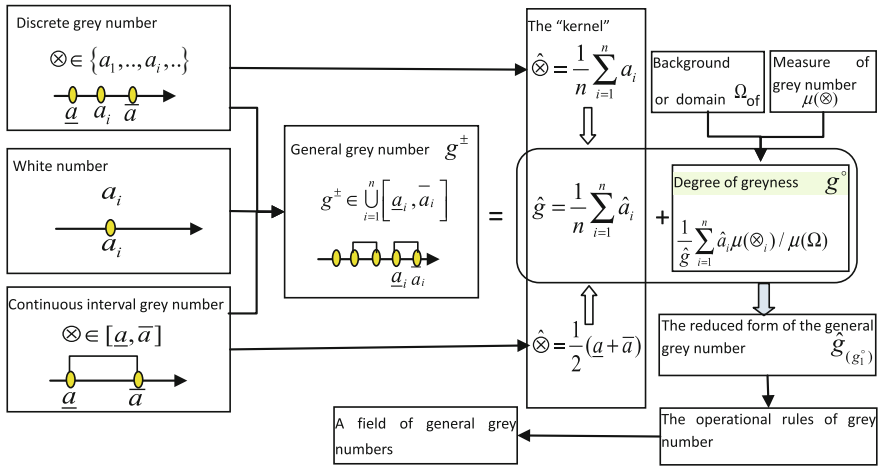
Fig. 2.1 Algebra system based on grey "kernel" and degree of greyness of grey numbers

"multiplication" and "division" and structure a more exquisite principle for operations of "multiplication" and "division" is a critical problem waiting to be solved.

2.2.2 The Grey Sequence Operator

In order to solve the prediction problem of shock-disturbed systems, Sifeng Liu put forward the concept of buffer operator developed the axioms for the buffer operator system, and constructed several practical buffer operators (Liu 1991). One of most used weakening buffer operators with acceptable properties is presented below:

$$x(k)d = \frac{1}{n-k+1}[x(k)+x(k+1)+\cdots+x(n)]; k=1,2,\cdots,n \qquad (2.1)$$

Research on buffer operators has become very active since 1991 and significant new studies have emerged. For example, Yaoguo Dang (Dang et al. 2004), Zhengpeng Wu (Wu et al. 2009), Jie Cui (Cui and Dang 2009), Lizhi Cui (Cui and Liu 2010), Yeqing Guan (Guan and Liu 2008), Xiao-Li Hu (Hu et al. 2013), Yan Gao (Gao et al. 2013), Zhengxin Wang (Wang et al. 2009), Xuemei Li (Li et al. 2012) and Wenqiang Dai (Dai and Su 2012), among others, have developed a variety of different weakening and strengthening buffer operators based on three buffer operator axioms. In 2011, Yong Wei brought forth the general formula of buffer operator:

$$x(k)d = x(k) \cdot [x(k)/\frac{1}{\sum_{i=k}^{n}\omega_i}\sum_{i=k}^{n}\omega_i x(i)]^{\alpha} \qquad (2.2)$$

Yong Wei proved that the buffer operator given in Eq. (2.1) can express weakening buffer operator, strengthening buffer operator and the identity operator, respectively, according to the different values of α (Wei 2011).

2.2.3 The Grey Prediction Models

The grey prediction model is a type of grey model that has been widely researched and used. In 2005, Naiming Xie built on Model GM (1, 1) by Professor Julong Deng (Deng 1982), and proposed the discrete grey model as well as its properties (Xie and Liu 2005). Later, Lifeng Wu developed the fractional accumulation discrete grey model and completed the perturbation problem of grey model (Wu et al. 2013). Xiangdong Chen and Jun Xia set up the DHGM (2, 2) coupled equations, combining grey differential equation and self-memory principle based on the power system of self-memory principle (Chen and Xia 2009). Xiaojun Guo proposed the interval grey number for the self-memory prediction model based on the degree of greyness of synthesis grey numbers; he then studied the self-memory prediction model from different perspectives (Guo et al. 2014).

Various developments and derived models have emerged in recent years. For example, Yaoguo Dang came up with the GM (1, 1) model based on x(n) as the initial condition (Dang and Liu 2004). Xican Li proposed the GM (1, 1, β) model, studied the content type and parameter set form of the model, analyzed several properties of the GM (1, 1, β) model, then developed its optimization algorithm (Li et al. 2014). Zhengxin Wang provided several kinds of forms of GM (1, 1) power model and studied the characteristics of their time response function (Wang 2008). Xinping Xiao studied generalized accumulation grey model and proposed a combined optimization method (Xiao 2000). Wuyong Qian developed the grey GM (1, 1, t$^\alpha$)model with the time power item and studied the process of modeling and parameter estimation (Qian and Dang 2000). Wanmei Tang proposed a new prediction model based on the grey supporting vector machine (Tang 2006). Ke Zhang put forward the multi variable discrete grey model based on driving control (Zhang 2014). Bo Zeng came up with the random oscillation sequence prediction model taking the smooth operator compress random oscillation amplitude (Zeng et al. 2001). Qishan Zhang used the particle swarm algorithm and provided a new method of increasing grey GM (1, 1) precision through the optimization of background value interpolation coefficient and boundary value (Zhang 2001). Tianxiang Yao studied the parameter characteristics of the new information discrete GM (1, 1) model and fit properties of the geometric sequence, then put forward a new information discrete GM (1, 1) model with sectional correction (Yao and Liu 2009). Liyun Wu and Zhengpeng Wu constructed the twice time varying parameter discrete grey model with features of the white index law coincidence, linear law coincidence, twice law coincidence and stretching transformation consistency (Wu et al. 2013). Carmona Benitez improved the GM model and forecasted long-term trends in the passenger flow of the American air transport industry using the

improved model, which led to satisfying results (Benitez et al. 2013). Mark Evans proposed a more general grey Verhulst model and forecasted changes in the strength of British steel (Evans 2014). Jie Yang and Wenguo Weng made further improvements to the unbiased grey model and forecasted the amount of gas supply in a few cities (Yang and Weng 2014). Additionally, Naiming Xie studied the prediction problems of grey number sequence (Xie and Liu 2004).

In 2015, Sifeng Liu and colleagues determined four kinds of GM (1, 1) basic models including the even GM (1, 1) model, discrete GM (1, 1) model, even difference GM (1, 1) model, and original difference GM (1, 1) model. Such models were determined through simulation experiments, which helped to determine suitable types of sequences for the different models (Liu 2015). The spectrum diagram of grey prediction models is as follows (Fig. 2.2).

In the field of big data analysis, grey system prediction based on small data mining is emerging as an effective tool to extract valuable information from masses of data.

2.2.4   Grey Incidence Analysis Models

Grey incidence analysis models are used to assess whether different data sequences are closely associated or not, according to the geometric shapes of their sequence curves. Early grey incidence analysis models measured similarity based on proximity. Examples include Deng's grey incidence model, which is based on the point incidence coefficients (Deng 1985), and Liu's grey incidence model based on the whole or global perspective (Liu and Guo 1991). Such models include absolute
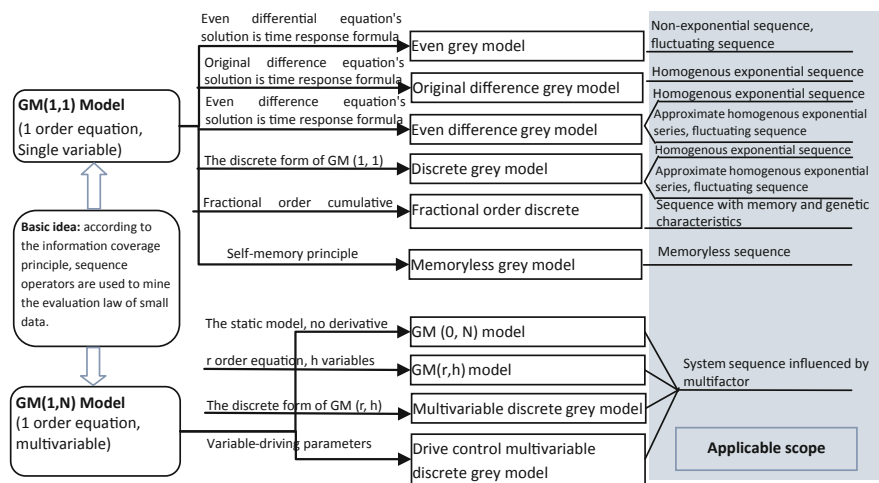

Fig. 2.2 The spectrum diagram of grey prediction models

degree of grey incidence, relative degree of grey incidence and synthetic degree of grey incidence, as follows:

$$\varepsilon_{ij} = \frac{1 + |s_i| + |s_j|}{1 + |s_i| + |s_j| + |s_i - s_j|} \tag{2.3}$$

$$r_{ij} = \frac{1 + |s_i'| + |s_j'|}{1 + |s_i'| + |s_j'| + |s_i' - s_j'|} \tag{2.4}$$

$$\rho_{ij} = \theta \varepsilon_{ij} + (1 - \theta) r_{ij}, \ \theta \in [0, 1] \tag{2.5}$$

In 2011, Sifeng Liu and Naiming Xie built a new grey incidence analysis model based on the perspective of similarity and proximity, respectively, as follows (Xie 2011):

$$\varepsilon_{ij} = \frac{1}{1 + |s_i - s_j|} \ , \ s_i - s_j = \int_1^n (X_i^0 - X_j^0) dt \tag{2.6}$$

$$\rho_{ij} = \frac{1}{1 + |S_i - S_j|} \ , \ S_i - S_j = \int_1^n (X_i - X_j) dt. \tag{2.7}$$

Ke Zhang and colleagues proposed a two-dimensional grey incidence degree model based on absolute incidence degree and double integral. With their new model, the research object became the relationship between surface analysis and curve analysis (Zhang and Liu 2010):

$$\varepsilon_{pq} = \frac{1 + |s_p| + |s_q|}{1 + |s_p| + |s_q| |s_p - s_q|}, \tag{2.8}$$

where $s_p = \iint X_p^0 dx dy$, $s_q = \iint X_q^0 dx dy$, $s_p - s_q = \iint (X_p^0 - X_q^0) dx dy$.

The spectrum diagram of grey incidence analysis models is as follows (Fig. 2.3).

2.2.5 Grey Clustering Evaluation Models

In 1993, the fixed weight clustering model (Liu 1993) and grey clustering evaluation model using end-point triangular possibility functions were proposed by professor Sifeng Liu (Liu and Zhu 1993).

In 2011, Sifeng Liu and Naiming Xie improved the triangular possibility function model proposed in 1993 and constructed the grey evaluation method based
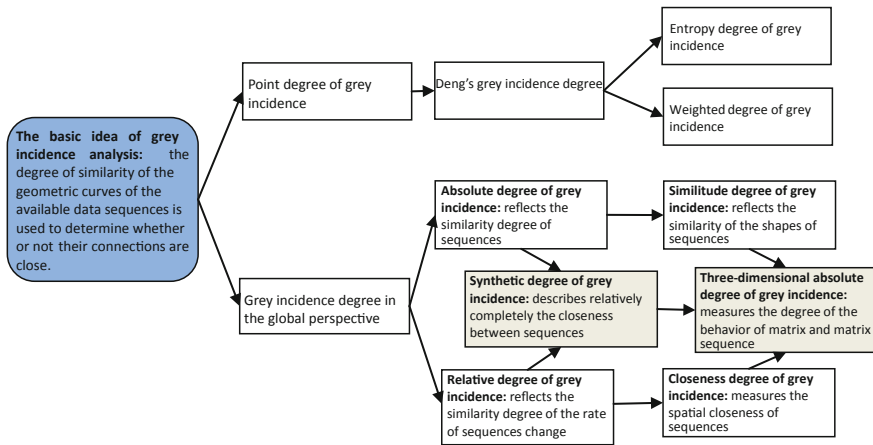
Entropy degree of grey incidence

Point degree of grey incidence

Deng's grey incidence degree

Weighted degree of grey incidence

The basic idea of grey incidence analysis: the degree of similarity of the geometric curves of the available data sequences is used to determine whether or not their connections are close.

Grey incidence degree in the global perspective

Absolute degree of grey incidence: reflects the similarity degree of sequences

Similitude degree of grey incidence: reflects the similarity of the shapes of sequences

Synthetic degree of grey incidence: describes relatively completely the closeness between sequences

Three-dimensional absolute degree of grey incidence: measures the degree of the behavior of matrix and matrix sequence

Relative degree of grey incidence: reflects the similarity degree of the rate of sequences change

Closeness degree of grey incidence: measures the spatial closeness of sequences

Fig. 2.3 The spectrum diagram of grey incidence analysis models

on center-point triangular possibility functions (Liu and Xie 2011). The new method reduced the base of the triangular possibility function of class k, which became the straight line joining the two center points of class k − 1 and class k + 1, and replaced the straight line joining the left end-point of class k − 1 and the right end-point of class k + 1. The multiple cross phenomenon existing in the original triangular possibility function clustering model is avoided effectively, and the clustering vector satisfied the requirement of normalization.

In 2014, Sifeng Liu and Zhigeng Fang set the weight function of grey class 1 to the possibility function of lower measure, set the weight function of grey class s to the possibility function of upper measure, and then put forward the grey clustering evaluation model based on center-point and end-point mixed triangular possibility functions (Liu and Fang 2014); see Figs. 2.4 and 2.5 for details. To extend the value bound of each clustering index, the grey clustering evaluation model based on mixed end-point triangular possibility function is suitable for situations in which all
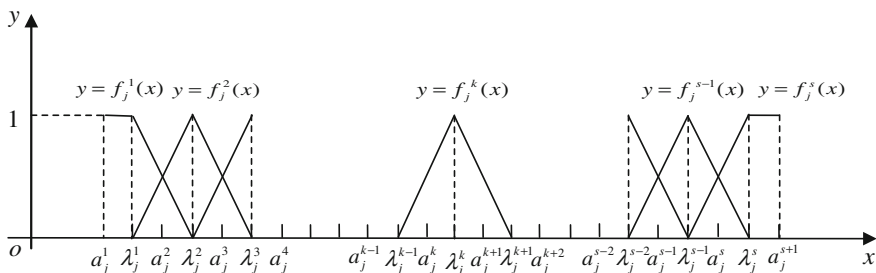
$y = f_j^1(x)$ $y = f_j^2(x)$ $y = f_j^k(x)$ $y = f_j^{s-1}(x)$ $y = f_j^s(x)$

a_j^1 λ_j^1 a_j^2 λ_j^2 a_j^3 λ_j^3 a_j^4 a_j^{k-1} λ_j^{k-1} a_j^k λ_j^k a_j^{k+1} λ_j^{k+1} a_j^{k+2} a_j^{s-2} λ_j^{s-2} a_j^{s-1} λ_j^{s-1} a_j^s λ_j^s a_j^{s+1}

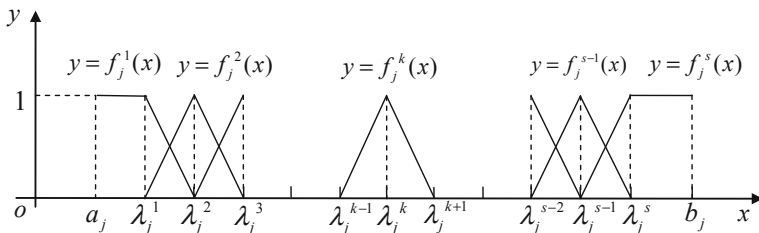**Fig. 2.4** End-point mixed possibility function

Fig. 2.5 Center-point mixed possibility function

grey boundaries are clear, but the points belonging to each grey class are most likely unknown. The grey clustering evaluation model based on mixed center-point triangular possibility function is suitable for situations where the points belonging to each grey class are most likely clear, but the grey boundaries are unknown. The mixed triangular possibility functions are more suitable for solving problems of clustering evaluation with poor information.

Grey clustering models based on mixed triangular possibility functions are applicable to evaluation and classification of poor information objects, which have broad application prospects.

2.2.6 Grey Decision-Making Models

In 2010, Sifeng Liu and colleagues put forward the multi-attribute intelligent grey target decision model (Liu et al. 2010), based on Julong Deng's (Deng 1986) grey target model ideas.

As the basis of the new model, a grey target is defined as a region a decision maker wants to reach, with an inside ideal point across multiple objectives. To facilitate the uniform distance measure of a decision strategy to the pre-defined grey target, four kinds of measure procedures are designed: the effect measures for benefit type objectives and cost type objectives; the lower effect measure for moderate type objectives; and the upper effect measure for moderate type objectives. Such procedures are designed according to three types of decision objective including benefit objective, cost objective, and non-monotonic objective with a most preferred middle value. Then, a matrix of synthetic effect measures can be easily obtained based on the uniform distance measure of a decision strategy to the grey target over different objectives. Based on the obtained matrix information, different decision strategies can be evaluated comprehensively The proposed method has a clear physical meaning as missing target, hitting target, as well as hitting performance.

Measurements with good properties satisfy the requirement of normalization and dimensionless, and the greater the effect, the greater the value measure. In 2009,

Sifeng Liu first reported the initial idea of intelligent grey target and gained validation from Professor Deng Julong and other experts.

In 2014, Sifeng Liu and colleagues constructed a new decision model entailing a two-stage grey comprehensive measure. The comparison between the maximum components δ_i^k and δ_j^k of decision coefficient vector δ_i and δ_j may conflict with the comparison between δ_i and δ_j, which is decision paradox that has been solved (Liu et al. 2014).

Indeed, in 2014, Sifeng Liu and colleagues defined a weight vector group of kernel clustering and **the weighted coefficient vector of kernel clustering for decision-making**. Then a novel two-stage decision model with weight vector group of kernel clustering and **the weighted coefficient vector of kernel clustering for decision-making** was put forward, and several functional **weight vector group of kernel clustering** were developed. This method can effectively solve the decision paradox and produce consistent results (Liu 2014).

Additionally, Dang Luo studied different types of grey decision models and obtained a series of significant results (Luo and Wang 2012; Luo and Wang 2012). Huan Guo and Xinping Xiao researched grey double layers and multi-objective linear programming and also solved problems (Guo and Xiao 2014). Finally, Jie Cui and colleagues developed the weighting formula evaluation value of each detection stage based on the new information priority principle, which provided a new process for solving grey decision problems consisting of multiple stages (Cui et al. 2012).

2.2.7 Combined Grey Models

Combined grey models include the Grey-Econometrics Combined Model (Liu and Zhu 1996), Grey Cobb-Douglas Model (Liu et al. 2004), Grey DEA model (Wang and Liu 2009), Grey Markov Model (Liu et al. 2014), Grey Rough Model (Jian et al. 2011), among others. Zhigeng Fang and colleagues opened a new direction for combined grey models based on grey game model. The researchers carried out effective research on economic decision applications using grey game model (Fang et al. 2006; Fang et al. 2010). In 2008, Qiao-Xing Li and Si-Feng Liu studied the grey matrix and grey input-output models (Li and Liu 2008). Based on such models, the authors put forward the Enterprise Grey Input-output analytical model in 2012 (Li et al. 2012).

2.2.8 Grey Control Models

In the 1980s, Professor Julong Deng and Chaoshun Zhou (Deng and Zhou 1986; Zhou and Deng 1986; Deng 1988) worked on grey forecasting control, and addressed the stability of grey linear systems and sufficient conditions for the stability of inter-connected dynamic systems. Furthermore, Chunhua Su and

colleagues studied the robust stability problem of grey stochastic time-delay systems, especially the distribution type, neutral type and neutral-distribution type exponential robust stability problem of grey stochastic time-delay systems. In order to do so, Chunhua Su and colleagues used several methods such as the Lyapuonv function, Lyapunov-Krasovskii function, model transformation, combined Itô formula, matrix inequality, Holder inequality, Schur complement, decomposition technique of continuous grey matrix cover and other mathematical tools. As a result, the researchers' work provided effective criteria for robust stability, and obtained several useful achievements (Su and Liu 2008, 2009).

It has been 31 years since the successful development of the first grey controller in 1985. During the 1980s, the conditions to establish a more effective control system seemed ripe, given the integration of a variety of grey control methods and models. However, the promotion and establishment of a new control method is not a short-term endeavor.

2.3 The New Framework and Main Components of Grey Systems Theory

Since 1982, grey systems theory has matured and seen the development of a new, generally accepted system structure. The new framework and main components of GST are shown in Table 2.1.

The grey systems theory course offered at Nanjing University of Aeronautics and Astronautics was selected as the national perfect curriculum and national excellent resource sharing, and the book Grey System Theory and Its Application, 4th and 6th edition, has been selected as "Eleventh Five Years" and "Twelfth Five Years" national programmes respectively.

A recent search through the China Knowledge Net (CNKI) using eight key terms such as grey system, grey theory, grey model, GM (1, 1), grey incidence analysis, grey clustering, grey prediction and grey decision making yielded 10,180 full text PhD theses, and 48,185 Masters' degree theses containing such terms. Also, a search of PhD and Masters' theses based on such search terms as key words resulted in 2,873 and 13,463 thesis, respectively. Full text journals containing such search terms amounted to 69,276 and journal papers based on those search terms as key words totalled 39,544 papers. International publishers Springer-Verlag and Taylor & Francis Group have launched a number of grey systems works in English, and Science Press has officially approved a book series on Grey Systems; the first of the 22 volumes is already on its way. Furthermore, the grey system modeling software version 8.0 written by Professor Bo Zeng and Professor Yang Shen contains applications of commonly used grey systems models. Interested readers can go to the website of the Institute for Grey System Studies at Nanjing University of Aeronautics and Astronautics (http://igss.nuaa.edu.cn/), to download the software free of charge.

Table 2.1 The new GST framework

	Main contents	Detailed thinking, methods, and models of GST
Basic thinking and models	Operations of grey number and grey algebra system	Grey number, the algorithms of interval grey number, the kernel of grey number concept and general grey number, the operation axiom of grey number, as well as the algebra system based on grey "kernel" and degree of greyness
	Sequence operator	Average generation operator, accumulating generation operator, inverse accumulating generation operator, shock-disturbed system, axiom system of buffer operator, series of accumulating and inverse accumulating generators
	Grey incidence model	Series grey incidence analysis model including Deng's grey incidence model, absolute degree of incidence, relative degree of incidence, synthetic degree of incidence, nearness degree of incidence, similitude degree of incidence, three-dimensional grey incidence degree, among others
	Grey clustering evaluation model	Variable weight grey clustering model, fixed weight grey clustering model, grey clustering evaluation model based on mixed end-point, and center-point triangular possibility function
	Family model of GM and grey system forecasting	Even GM (1, 1) model, even difference GM (1, 1) model, original difference GM (1, 1) model, discrete GM (1, 1) model, fractional order grey model, memoryless grey model, grey Verhulst model, multivariable discrete grey model, discrete grey model with approximate non-homogenous exponential law, sequence grey forecasting, interval forecasting, catastrophe forecasting, grey wave forecasting, and system forecasting
	Grey decision model	Grey target decision, four kinds of uniform effect measure function which are able to characterize factors for positive point and negative point, multi-attribute intelligent grey target decision model, two-stage grey synthetic measure decision model
Advanced thinking and models	Grey equation and grey matrix	Grey algebraic equation, grey differential equation, grey matrix, and matrix equation
	Combined grey model	Grey-Econometrics Combined Model, grey Cobb-Douglas Model, grey DEA model, grey Markov Model, and grey rough model
	Grey game model	Grey matrix game model based on pure strategy, grey matrix game model based on mixed strategy, duopoly strategy output-making model based on bounded knowledge and bounded rationality, the paradox of centipede game model (a new model of grey structured algorithm of forwards induction)

Table 2.1 (continued)

	Main contents	Detailed thinking, methods, and models of GST
	Grey input and output	P-F theorems of grey non-negative matrices, regional input-output model, enterprise grey input-output, grey input-output optimization model
	Grey programming	Linear programming models with grey parameters, grey linear programming of prediction type, drift grey linear programming, grey 0–1 programming, grey multiple objective programming, grey non-linear programming
	Grey control model	Grey control model, controllability and observability of grey systems, robust stability of grey systems, grey linear time-delay systems, grey stochastic linear time-delay systems

Although grey systems theory has been applied successfully to many contexts and in many different countries, it is still in its infancy and, thus, has much scope for further development and improvement. Colleagues who are interested in grey systems research must welcome constructive criticism, and continuously explore the potential to further advance grey systems theory.