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Accinelli Elvio
Universidad Autónoma de Sa…


Juan Gabriel Brida
Universidad de la República d…

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Population growth and the Solow-Swan model

Elvio Accinelli[1] **and Juan Gabriel Brida**[2]

[1]Facultad de Economía.
Universidad Autónoma de San Luis Potosí
San Luis Potosí- México
elvio@correo.xoc.uam.mx

[2]School of Economics and Management
Free University of Bolzano
Bolzano-Italy
JuanGabriel.Brida@unibz.it

ABSTRACT

One of the key elements in any standard economic growth theory is that population growth is exponential with a constant rate $n > 0$. This simple model can provide an adequate approximation to such growth *only* for the initial period because, growing exponentially, population approaches infinity when t goes to infinity, which is clearly unrealistic. The exponential model does not accommodate growth reductions due to competition for environmental resources such as food and habitat. In this paper we reformulate the neoclassical Solow model of economic growth by assuming that the law describing population growth verifies two stylized facts: 1) population is strictly increasing and bounded and 2) the rate of growth of population is strictly decreasing to zero. The main result of the paper is the proof of the convergence of capital per worker to a constant value independently of the initial condition. This constant value coincides with the steady state of the original Solow model with zero population growth rate.

Keywords: Solow-Swan model; population models.

JEL Classification: C62; O41.

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1 Introduction

In the neoclassical model of economic growth it is assumed that labour force L grows at a constant rate $n > 0$. In continuous time it is natural to define this growth rate as:

$$n = \frac{\dot{L}}{L} = \frac{\frac{\partial L}{\partial t}}{L} \tag{1.1}$$

which implies that the labour force growths exponentially and for any initial level L_0, at time t the level of the labour force is

$$L(t) = L_0 e^{nt} \tag{1.2}$$

The simple Malthusian model can provide an adequate approximation to such growth only for an initial period because, growing exponentially, labour force approaches infinity when t goes to infinity, which is clearly unrealistic. The exponential model does not accommodate growth reductions due to competition for environmental resources such as food and habitat. Verhulst (Verhulst, 1838) considered that a stable population would have a characteristic saturation level; this limit for the population size is usually called the *carrying capacity* of the environment[1]. To incorporate this numerical upper bound on the growth size he introduced the logistic equation as an extension of the exponential model[2].

It is a very well known stylized fact that since the 1950s, population growth rate is decreasing and it is projected to decrease to 0 during the next six decades. This decrease is particularly relevant in the group of developed countries but is also observable on a global scale. The decrease in the rate of growth is predominantly due to the aging of the population and, consequently, a dramatic increase in the number of deaths. From 2030 to 2050, the world population would grow more slowly than ever before in its history. (See (Day, 1996))

Then, as described in (Maynard Smith, 1998), a more realistic law of growth of the labour force $L(t)$ must verify the following properties:

1. when population is small enough in proportion to environmental carrying capacity L_∞, then L grows at a constant rate $n > 0$,
2. when population is large enough in proportion to environmental carrying capacity L_∞, the economic resources become more scarce and this affect negatively growth of the population,
3. population growth rate is decreasing to 0.

The logistic equation is one of the simplest realistic model of population dynamics verifying all these properties and any modern model of human population growth incorporates these stylized facts. In this paper we introduce a modification in the neoclassical economic growth model by assuming that population growth follows the properties defined above[3]. In section 2 we introduce the model and we obtain the dynamical equation describing how capital per worker varies over time. Section 3 describes the qualitative properties of the model. In particular, we show that economic growth accelerates when population growth rate decreases to zero and that per capita capital converges to a constant value (that is independent of the population growth law and of the initial per capita capital) as time tends to infinity. Conclusions and future developments are summarized in the last section.

[1]In (Arrow et al., 1995), (Cohen (1), 1995), (Cohen (2), 1995) and (Daily et al., 1992) the reader can find detailed information about the concept of carrying capacity of human population.

[2]In (Accinelli et al., 2005) and (Brida et al., 2006), (Ritelli et al., 2003) and (Ritschl, 1985) the reader can find different improved versions of the Solow model with population growth following logistic growth. These can be seen as preliminary exercises of the present paper.

[3](Donghan, 1998) is a previous paper in this direction that inspired the present work. We incorporate ideas from this paper for some of the proofs presented here.

2 The model

There are three key elements to the model:

- the production function, i.e. how the inputs of capital K and labour L are transformed into outputs $Y = F(K, L)$
- how the capital change over time
- how the labour force change over time

As usual, we shall assume that:

1. the production function $F(K, L)$ satisfy the following conditions:
   (a) $F(\lambda K, \lambda L) = \lambda F(K, L)$, $\forall \lambda, K, L \in R^+$ (constant return to scale)
   (b) $F(K, 0) = F(0, L) = 0$, $\forall K, L \in R^+$
   (c) $\frac{\partial F}{\partial K} > 0$, $\frac{\partial F}{\partial L} > 0$, $\frac{\partial^2 F}{\partial K^2} < 0$, $\frac{\partial^2 F}{\partial L^2} < 0$
   (d) $\lim\limits_{K \to 0} \frac{\partial F}{\partial K} = \lim\limits_{L \to 0} \frac{\partial F}{\partial L} = +\infty$; $\lim\limits_{K \to +\infty} \frac{\partial F}{\partial K} = \lim\limits_{L \to +\infty} \frac{\partial F}{\partial L} = 0$ (Inada conditions)
2. the capital stock changes equal the gross investment $I = sF(K, L)$ minus the capital depreciation δK :
$$\dot{K} = sF(K, L) - \delta K \tag{2.1}$$
3. the labour force $L(t)$ verify the following properties:
   (a) $L(0) = L_0 > 0, \dot{L}(t) > 0, \forall t \geq 0$ and $\lim\limits_{t \to +\infty} L(t) = L_\infty$ (population is strictly increasing and bounded)
   (b) If $n(t) = \frac{\dot{L}(t)}{L(t)}$ then $\dot{n}(t) < 0, \forall t \geq 0$ and $\lim\limits_{t \to +\infty} n(t) = 0$ (the rate of growth of population is strictly decreasing to zero)

The last assumption is the unique difference with the original Solow model as presented in (Solow, 1956).
If $k = \frac{K}{L}$ is the capital per worker and $f(k) = F\left(\frac{K}{L}, 1\right) = F(k, 1)$ is the production function in intensive form, we have that:

- $f(0) = 0;$
- $f'(k) > 0, \forall k \in R^+$
- $\lim\limits_{k \to +\infty} f'(k) = 0$
- $\lim\limits_{k \to 0^+} f'(k) = +\infty$
- $f''(k) < 0, \forall k \in R^+$

From $k = \frac{K}{L}$, we have that

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \tag{2.2}$$

and then

$$\frac{\dot{k}}{k} = \frac{sF\left(K,L\right) - \delta K}{K} - n(t) = s\frac{F\left(\frac{K}{L},1\right)}{\frac{K}{L}} - \delta - n\left(t\right) \tag{2.3}$$

From this, we obtain the equation of motion for the modified Solow model which describes how capital per worker varies over time:

$$\dot{k} = sf(k) - \left(\delta + n(t)\right)k \tag{2.4}$$

Note that in the original Solow model where labour force grows exponentially it is $n(t) = n$ (constant) and the equation of the motion is

$$\dot{k} = sf(k) - \left(\delta + n\right)k. \tag{2.5}$$

In this case there is a non trivial steady state $\hat{k}_n$: the unique positive solution of equation

$$sf(k) = \left(\delta + n\right)k. \tag{2.6}$$

This equilibrium point is globally asymptotically stable and the convergence of any solution of (2.5) to the value $\hat{k}_n$ is monotonic[4]. In particular, for $n = 0$ we denote by $\hat{k}_0$ be the unique positive solution of equation

$$sf(k) = \delta k. \tag{2.7}$$

Then we have that

$$\lim_{n \to 0^+} \hat{k}_n = \hat{k}_0 \tag{2.8}$$

and that $\hat{k}_n$ increases to $\hat{k}_0$ as n decreases to zero. Note that equation (2.4) is not autonomous and do not have a non zero steady state.

3 Qualitative analysis of the model

In this section we will prove that $\hat{k}_0$ is a global attractor of the equation of motion (2.4). In particular we will show that $\forall k_0 > 0$ there exists a unique solution $k(t)$ to the problem

$$\begin{cases} \dot{k} = sf(k) - \left(\delta + n(t)\right)k \\ k(0) = k_0 \end{cases} \tag{3.1}$$

and that this solution verifies

$$\lim_{t \to +\infty} k(t) = \hat{k}_0 \tag{3.2}$$

In addition, we will prove comparative results showing the variation of the solution of (3.1) when the initial condition k_0 (or the curve describing population growth) changes. Let start with the following theorem of existence and uniqueness of solutions.

[4]In (Simonovits, 2000) and (Solow, 1956) the reader can find a detailed description of the dynamical properties of the classical Solow model.

Theorem 3.1. *For any* $k_0 > 0,$ *there is a unique solution of the initial value problem*

$$\begin{cases} \dot{k} = sf(k) - (\delta + n(t))\, k \\ k(0) = k_0 \end{cases} \tag{3.3}$$

with domain $[0, +\infty)$.

Proof: Let $g\,(k,t) = sf(k) - (\delta + n(t))\, k$ and $A = \left[\frac{k_0}{2}, +\infty\right) \times [0, +\infty)$. Since

$$\left| \frac{\partial g\,(k,t)}{\partial k} \right| = \left| sf'(k) - \delta - n(t) \right| \leq sf'(\frac{k_0}{2}) + \delta + n(0), \forall\, (k,t) \in A, \tag{3.4}$$

g satisfies a Lipchitz condition with constant $L = sf'(\frac{k_0}{2}) + \delta + n(0)$. Then, being g continuous and uniformly Lipchitzian, there exists a unique solution of the initial value problem (S) with domain $[0, +\infty)$.

Theorem 3.2. *Let* $k_1(t)$ *and* $k_2(t)$ *be solutions of equation* $\dot{k} = sf(k) - (\delta + n(t))\, k$ *with initial conditions* $k_1(0)$ *and* $k_2(0)$ *respectively. If* $0 < k_1(0) < k_2(0)$*, then* $k_1(t) < k_2(t), \forall t \in [0, +\infty)$*.*

Proof: If there exist $\hat{t} \in [0, +\infty)$ such that $k_1(\hat{t}) \geq k_2(\hat{t})$, then by continuity $\exists \tilde{t} \in \left[0, \hat{t}\right]$ such that $k_1(\tilde{t}) = k_2(\tilde{t})$. Then we have two different solutions of $\dot{k} = sf(k) - (\delta + n(t))\, k$ with the same initial condition $k(\tilde{t}) = k_1(\tilde{t})$. This contradicts the uniqueness of the solutions and hence the result is proved.
Note that this result is also true for the classical model and implies that, if two economies have the same fundamentals, then the one with bigger initial capital per worker has bigger capital per worker forever.
Now we introduce the following comparison theorem that will be used later in this paper.

Theorem 3.3 (comparison)**.** *Let* $f(t)$ *and* $g\,(t)$ *be solutions of the differential equations* $\dot{x} = F(x,t)$ *and* $\dot{x} = G(x,t)$ *respectively on the strip* $a \leq t \leq b$*, with* $f(a) = g\,(a)$*. If* $F(x,t) \leq G(x,t)$ *and* F *(or* G*) is Lipchitzian on the strip* $a \leq t \leq b$*, then* $f(t) \leq g\,(t)\,, \forall t \in [a, b]$*.*

Proof: See (Waltman, 2004) page 213.

Theorem 3.4. *Let* $k_1(t)$ *and* $k_2(t)$ *be solutions of the differential equations*

$$\dot{k} = sf(k) - (\delta + n_1(t))\, k \tag{3.5}$$

and

$$\dot{k} = sf(k) - (\delta + n_2(t))\, k \tag{3.6}$$

with the same initial condition $k_1(0) = k_2(0) = k_0$*. If* $n_1(t) \leq n_2(t), \forall t \in [0, +\infty)$*, then* $k_1(t) \geq k_2(t), \forall t \in [0, +\infty)$*.*

Proof: This is a corollary of the previous comparison theorem.
This result has the following economic interpretation: if two economies have the same fundamentals and start with the same initial capital per capita, the one with population growth rate decreasing faster always has bigger capital per worker. But, as we will show later in this work, both economies converge to the same long run value of per capita capital.

Theorem 3.5. *Let $t_0 \geq 0$ be a fixed number and let $k_{1t_0}(t)$, $k_{2t_0}(t)$ and $k_{3t_0}(t)$ be the solutions of the differential equations*

$$(A): \ \dot{k} = sf(k) - (\delta + n(t_0))\, k, \tag{3.7}$$

$$(B): \ \dot{k} = sf(k) - (\delta + n(t))\, k \tag{3.8}$$

and

$$(C): \dot{k} = sf(k) - \delta k \tag{3.9}$$

respectively, and with the same initial condition

$$k_{1t_0}(t_0) = k_{2t_0}(t_0) = k_{3t_0}(t_0) = k_0. \tag{3.10}$$

If $\hat{k}_n$ and $\hat{k}_0$ are the equilibrium points of equations (A) and (C), respectively, then:

1. $k_{1t_0}(t) \leq k_{2t_0}(t) \leq k_{3t_0}(t), \forall t \in [t_0, +\infty)$
2. $\lim\limits_{t\to+\infty} k_{2t_0}(t) = \hat{k}_0, \forall t_0 \geq 0.$

Proof: 1. We have that: $\dot{n}(t) < 0, \forall t \geq 0$ and $\lim\limits_{t\to+\infty} n(t) = 0$. This implies that

$$sf(k) - (\delta + n(t_0))\, k \leq sf(k) - (\delta + n(t))\, k \leq sf(k) - \delta k, \forall t \in [t_0, +\infty)\,. \tag{3.11}$$

Then, the comparison Theorem implies that $k_{1t_0}(t) \leq k_{2t_0}(t) \leq k_{3t_0}(t)$, $\forall t \in [t_0, +\infty)\,.$
2. Let $\varepsilon > 0$; we want to prove that there exists $H > 0$ such that $\forall t \geq H, \left|k_{2t_0}(t) - \hat{k}_0\right| < \varepsilon.$. From $\lim\limits_{n\to 0^+} \hat{k}_n = \hat{k}_0$ we know that there exist $n_1 > 0$ such that

$$\forall n \geq n_1, \left|\hat{k}_n - \hat{k}_0\right| < \frac{\varepsilon}{3} \tag{3.12}$$

Let $\bar{n} \geq \max(n_1, n(t_0))$ and $t_1 \geq t_0$ such that $\bar{n} = n(t_1)$ and let $k_{1t_1}(t)$ and $k_{3t_1}(t)$ be the solutions of the differential equations (A) and (C), respectively with the initial condition

$$k_{1t_1}(t_1) = k_{3t_1}(t_1) = k_{2t_0}(t_1). \tag{3.13}$$

Then the previous item in this theorem implies that

$$k_{1t_1}(t) \leq k_{2t_0}(t) \leq k_{3t_1}(t), \forall t \in [t_1, +\infty)\,. \tag{3.14}$$

We know that $\lim\limits_{t\to+\infty} k_{3t_1}(t) = \hat{k}_0$ and then $\exists H_1 > 0$ such that

$$\forall t \geq H_1, \left|k_{3t_1}(t) - \hat{k}_0\right| < \frac{\varepsilon}{3} \tag{3.15}$$

We also have that $\lim\limits_{t\to+\infty} k_{1t_1}(t) = \hat{k}_{\bar{n}}$ and then $\exists H_2 > 0$ such that

$$\forall t \geq H_2, \left|k_{1t_1}(t) - \hat{k}_{\bar{n}}\right| < \frac{\varepsilon}{3} \tag{3.16}$$

Thus, $\forall t \geq H = \max(H_1, H_2)$ it is:

$$\hat{k}_0 - \frac{2\varepsilon}{3} < \hat{k}_{\bar{n}} - \frac{\varepsilon}{3} < k_{1t_1}(t) \leq k_{2t_0}(t) \leq k_{3t_1}(t) < \hat{k}_0 + \frac{\varepsilon}{3} \tag{3.17}$$

and this implies that

$$\left|k_{2t_0}(t) - \hat{k}_0\right| \leq \varepsilon, \forall t \geq H. \tag{3.18}$$

Remark 3.1. In the previous theorem, equations (A) and (C) represent the classical Solow model with constant rate of growth of population $n(t_0)$ and 0 respectively. In the first part we showed that an economy with rate of growth of population that decrease to zero improves the income of an economy with constant rate of growth of population. In the second, we showed that if an economy has growth rate of population strictly decreasing to zero then capital per worker converges to the positive steady state of the classical Solow model when the growth rate of population is zero.

Remark 3.2. It can be shown that the convergence of $k_2(t)$ to $\hat{k}_0$ is monotone. In particular we have that:

1. if $k_0 \leq \hat{k}_n$ then $k_2(t)$ is strictly increasing in $[0, +\infty)$
2. if $\hat{k}_n < k_0 \leq \hat{k}_0$ then $\exists \hat{t} \in [0, +\infty)$ such that $k_2(t)$ is strictly decreasing in $\left[0, \hat{t}\right]$ and is strictly increasing in $\left[\hat{t}, +\infty\right)$
3. if $\hat{k}_0 < k_0$ then $\exists \hat{t} \in [0, +\infty)$ such that $k_2(t)$ is strictly decreasing in $[0, +\infty)$ or $\exists \hat{t} \in [0, +\infty)$ such that $k_2(t)$ is strictly decreasing in $\left[0, \hat{t}\right]$ and is strictly increasing in $\left[\hat{t}, +\infty\right)$[5].

Theorem 3.6. *The solution of (3.1) is asymptotically stable.*

Proof: To prove the (Lyapunov) stability of the solution $k_0(t)$ of (S) we have to show that: $\forall \epsilon > 0, \exists \delta > 0$ such that for the solution $k_1(t)$ of (2.4) with initial condition $k_1 = k_1(0)$ verifying $|k_1 - k_0| < \delta$, we have that

$$|k_1(t) - k_0(t)| < \epsilon, \forall t \in [0, +\infty) . \tag{3.19}$$

Let $\epsilon > 0$, and $\varphi_1(t)$ and $\varphi_2(t)$ the solutions of (2.4) with initial conditions $\varphi_1(0) = \frac{3k_0}{2}$ and $\varphi_2(0) = \frac{k_0}{2}$ respectively. From the previous theorem we have that

$$\lim_{t \to +\infty} \varphi_1(t) = \lim_{t \to +\infty} \varphi_2(t) = \hat{k}_0 \tag{3.20}$$

and then $\exists t_0 > 0$ such that $|\varphi_1(t) - k_0(t)| < \epsilon$ and $|\varphi_2(t) - k_0(t)| < \epsilon, \forall t \in [t_0, +\infty)$. Then, from Theorem 2 we have that $\forall k_1 \in \left[\frac{k_0}{2}, \frac{3k_0}{2}\right]$, if $k_1(t)$ is the solution of (2.4) with initial condition k_1 we have that

$$\varphi_2(t) \leq k_1(t) \leq \varphi_1(t), \forall t \in [0, +\infty) . \tag{3.21}$$

Thus, $\forall k_1 \in \left[\frac{k_0}{2}, \frac{3k_0}{2}\right]$ the solution $k_1(t)$ verifies

$$|k_1(t) - k_0(t)| < \epsilon, \forall t \in [t_0, +\infty) . \tag{3.22}$$

Then by the Theorem of continuous dependence on initial conditions, $\forall \epsilon > 0, \exists \delta > 0$ (we can choose it such that $\delta < \frac{k_0}{2}$) such that if $k_1(t)$ is the solution of (2.4) with initial condition $k_1 = k_1(0)$ verifying $|k_1 - k_0| < \delta$, we have that

$$|k_1(t) - k_0(t)| < \epsilon, \forall t \in [0, t_0] . \tag{3.23}$$

[5] See (Donghan, 1998) for the proof.

Then δ verifies that if $k_1(t)$ is the solution of (2.4) with initial condition $k_1 = k_1(0)$ such that $|k_1 - k_0| < \delta$, then it is

$$|k_1(t) - k_0(t)| < \epsilon, \forall t \in [0, +\infty) . \tag{3.24}$$

This shows that the solution of (S) is (Lyapunov) stable. From the previous Theorem we have that for any solutions $k_1(t)$ and $k_2(t)$ of (2.4) it is

$$\lim_{t\to+\infty} k_1(t) = \lim_{t\to+\infty} k_2(t) = \hat{k}_0 \tag{3.25}$$

and then it is

$$\lim_{t\to+\infty} [k_1(t) - k_2(t)] = 0. \tag{3.26}$$

This show that the solution of (3.1) is asymptotically stable.

Remark 3.3. This means that solutions with close initial conditions remain close forever and also implies that $\hat{k}_0$ is a global attractor of equation (2.4)

Remark 3.4. $\hat{k}_0$ is the unique positive solution of equation

$$sf(k) = \delta k. \tag{3.27}$$

and then it depends only on s, f and δ and do not depend on n. Then, two economies with different population growth rates which are decreasing to zero but with the same technology f, fraction s of output that is saved and rate of capital depreciation δ will converge to the same long run value of capital per worker.

Remark 3.5. We can analyze the impact of technology (A) on growth through its impact on the environmental carrying capacity L_∞. We assume that technological development increases the carrying capacity of the environment, i.e. $L_\infty(A)$ is an increasing function of the variable A. Of course, if technology A just affects the environmental carrying capacity it does not impact the dynamics of the classical Solow model (with exponential population growth). However, it does impact the Solow model with population growth s.t. the rate of growth of population is strictly decreasing to zero. As technological development leads to greater population, and as the steady state equilibrium values of consumption per capita, capital per capita and output per capita remain constant when technology develops, it implies that technology increases the aggregate levels of consumption, capital and output.

4 Concluding remarks.

In growth theory it is usually assumed that population growth follows an exponential law. This is clearly unrealistic because, in particular, it implies that population goes to infinity when time goes to infinity. In this paper we suggest a more realistic approach by considering that population growth is strictly increasing and bounded and that its rate of growth is strictly decreasing to zero. The paper shows that there exists a constant (long run) value $\hat{k}_0$ that attracts any solution of the model as t tends to infinity. The unique positive solution $\hat{k}_0$ of $sf(k) = \delta k$ depends only on the technology f, the fraction s of output that is saved and the rate of capital depreciation δ and then the intrinsic rate of population growth $n(t)$ plays no role in determining the long run level of per capita output. Then two economies with different population growth rates which are decreasing to zero but with the same technology f, fraction s of output that is saved and rate of capital depreciation δ will converge to the same long run value $\hat{k}_0$. By the contrary, with exponential population growth an increase in the intrinsic rate of population growth leads to lower levels of long run output per capita. The paper also shows that long run values of per capita levels of consumption, capital and output are greater than those of the classical model. Thus, in the long run, economic growth is improved if labour force growth rate decreases. This is a motivation for policy makers to have an efficient population growth rate. Finally, being that population $L(t)$ is strictly increasing and bounded by L_∞, the long run level of aggregate capital is the finite value $L_\infty \, \hat{k}_0$. This makes another difference with the classical Solow model, where aggregate capital goes unrealistically to infinity as t tends to infinity.

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