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Explanation of terms of grey forecasting models

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Abstract

Purpose – The purpose of this paper is to present the terms of grey forecasting models and techniques.
Design/methodology/approach – The definitions of basic terms about grey forecasting models and techniques are presented one by one.
Findings – The reader could know the basic explanation about the important terms about various grey forecasting models and techniques from this paper.
Practical implications – Many of the authors' colleagues thought that unified definitions of key terms would be beneficial for both the readers and the authors.
Originality/value – It is a fundamental work to standardise all the definitions of terms for a new discipline. It is also propitious to spread and universal of grey system theory.

Keywords Grey systems modelling and prediction, Grey forecasting, Original form, Even form, Even grey model (EGM), Origin, Shadow equation, Model GM(1, 1), Original difference (ODGM), Even difference (EDGM), Discrete grey model, Grey disaster forecasting

Paper type Conceptual paper

1. The original form of model GM(1, 1)

Let $X^{(0)}=(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$, $x^{(0)}(k)\geqslant 0$, $X^{(1)}$ is the 1-AGO sequence of $X^{(0)}$, that is:

$$X^{(1)} = \left(x^{(1)}(1), x^{(1)}(2), \ \ldots, \ x^{(1)}(n)\right),$$

where $x^{(1)}(k) = \sum_{i=1}^{k} x^{(0)}(i)$, $k=1, 2, \ldots n$. Then:

$$x^{(0)}(k)+ax^{(1)}(k) = b \tag{1}$$

is referred to as the original form of model GM(1, 1), and actually it is a difference equation (Deng, 1985; Liu *et al.*, 2016a).

The parameter vector $\hat{a} = [a, b]^T$ of formula (1) can be estimated using the least square method, which satisfies:

$$\hat{a} = \left(B^T B\right)^{-1} B^T Y \tag{2}$$

where:

$$B = \begin{bmatrix} -x^{(1)}(2) & 1 \\ -x^{(1)}(3) & 1 \\ \vdots & \vdots \\ -x^{(1)}(n) & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix} \tag{3}$$

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2. The even form of the model GM(1, 1) (Deng, 1985; Liu *et al.*, 2016a)

Let $X^{(0)}$, $X^{(1)}$ just as above, let:

$$Z^{(1)} = \left(z^{(1)}(2), z^{(1)}(3), \ldots, z^{(1)}(n)\right),$$

where $z^{(1)}(k) = 1/2(x^{(1)}(k)+x^{(1)}(k-1))$, then:

$$x^{(0)}(k) + az^{(1)}(k) = b \tag{4}$$

is referred to as the even form of the model GM(1, 1).

The even form of model GM(1, 1) is also essentially a difference equation. The parameter vector of formula (4) can also be estimated with formula (2), but it should be noted that the elements of matrix B are different from those in formula (3), which is:

$$B = \begin{bmatrix} -Z^{(1)}(2) & 1 \\ -Z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -Z^{(1)}(n) & 1 \end{bmatrix} \tag{5}$$

3. Shadow equation of GM(1, 1) (Deng, 1985; Liu *et al.*, 2016a)

The following differential equation:

$$\frac{dx^{(1)}}{dt} + ax^{(1)} = b \tag{6}$$

is called a shadow equation of the even form $x^{(0)}(k)+az^{(1)}(k) = b$ of the model GM(1, 1).

4. Even grey model (EGM)

Replace matrix B of formula (2) with (5), according to parameter vector $\hat{a} = [a, b]^T$ of the least squares estimator of (2) and the solution of shadow Equation (6), and modelling the difference, differential hybrid model of the time response formula of GM(1, 1). This is called the even hybrid form of model GM(1, 1), and is referred to as EGM for short (Deng, 1985, 1990; Liu *et al.*, 2015, 2016a).

The parameter $-a$ of Even GM(1, 1) is called development index and b is called grey actuating quantity. The development index reflects the trend of $\hat{x}^{(1)}$ and $\hat{x}^{(0)}$.

Even model GM(1, 1) is the grey prediction model proposed first by Professor Deng Julong, and is currently the most influential, widely used form. When researchers mention model GM(1, 1) they are often referring to EGM.

The time response sequence of the even model GM(1, 1) is as follow:

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)e^{-a(k-1)} + \frac{b}{a}, \quad k = 1, 2, \ldots, n \tag{7}$$

5. Original difference grey model (ODGM)

Based on the original form of model GM(1, 1) and the formula (2) to estimate the model's parameters, the model that takes the solution of the original difference Equation (1) as the time response formula is called the original difference form of model GM(1, 1), and it refers to ODGM for short (Liu *et al.*, 2015a, b, 2016a).

The time response formula of original difference model GM(1, 1) is:

$$\hat{x}^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1}{1+a}\right)^{k} + \frac{b}{a} \tag{8}$$

6. Even difference grey model (EDGM)

Based on the even form of model GM(1, 1) and estimated the model parameters, then the model that takes the solution of the even difference Equation (4) as the time response formula is called the even difference form of model GM(1, 1), and it refers to EDGM for short (Liu *et al.*, 2015a, b, 2016a).

The time response formula of even difference model GM(1, 1) is:

$$x^{(1)}(k) = \left(x^{(0)}(1) - \frac{b}{a}\right)\left(\frac{1-0.5a}{1+0.5a}\right)^{k} + \frac{b}{a} \tag{9}$$

7. Discrete grey model (DGM)

The difference equation as follows:

$$x^{(1)}(k+1) = \beta_1 x^{(1)}(k) + \beta_2 \tag{10}$$

is called a discrete form of model GM(1, 1), and it refers to DGM for short (Liu *et al.*, 2016a; Xie and Liu, 2009).

The time response formula of formula (7) of the discrete model GM(1, 1) is:

$$\hat{x}^{(1)}(k) = \left[x^{(0)}(1) - \frac{\beta_2}{1-\beta_1}\right]\beta_1^{k} + \frac{\beta_2}{1-\beta_1} \tag{11}$$

8. Metabolic GM(1, 1)

The GM(1, 1) model established on $X^{(0)} = (x^{(0)}(2), \ldots, x^{(0)}(n),\ x^{(0)}(n+1))$ with the new information piece $x^{(0)}(n+1)$ added and the oldest piece $x^{(0)}(1)$ of information removed is known as a metabolic GM(1, 1) (Deng, 1985; Liu *et al.*, 2016a).

9. The model of GM(0, *N*)

Assume that $X_1^{(0)} = (x_1^{(0)}(1),\ x_1^{(0)}(2),\ \ldots,\ x_1^{(0)}(n))$ is a data sequence of a system's characteristic variable:

$$X_2^{(0)} = \left(x_2^{(0)}(1),\ x_2^{(0)}(2),\ \ldots,\ x_2^{(0)}(n)\right)$$

$$X_3^{(0)} = \left(x_3^{(0)}(1),\ x_3^{(0)}(2),\ \ldots,\ x_3^{(0)}(n)\right)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$X_N^{(0)} = \left(x_N^{(0)}(1), x_N^{(0)}(2), \ldots, x_N^{(0)}(n)\right)$$

The data sequences of relevant factors, and $X_i^{(1)}$ the accumulation generated sequence of $X_i^{(0)}$, $i = 2, 3, \ldots, N$. Then:

$$x_1^{(1)}(k) = a + b_2 x_2^{(1)}(k) + b_3 x_3^{(1)}(k) + \cdots + b_N x_N^{(1)}(k) \tag{12}$$

is called the model of GM(0, N) (Deng, 1985, 1990; Liu *et al.*, 2016a). Because this model does not contain any derivative, it is a static model. Although its form looks like a multivariate linear regression model, it is essentially different from any of the statistical models. In particular, the general multivariate linear regression model is established on the basis of the original data sequences, while the model of GM(0, N) is constructed on the accumulation generation of the original data.

10. The model of GM(1, *N*)

Assume that $X_i^{(0)}$ and $X_i^{(1)}$ ($i = 1, 2, \ldots, N$) as given in Definition 7.6.1. Let $X_i^{(1)}$ be the accumulated sequences of $X_i^{(0)}$, $i = 1, 2, ..., N$, and $Z_1^{(1)}$ the adjacent neighbour average sequence of $X_1^{(1)}$. Then:

$$x_1^{(0)}(k) + az_1^{(1)}(k) = \sum_{i=2}^{N} b_i x_i^{(1)}(k) \tag{13}$$

is called the model of GM(1, N) (Deng, 1985; Liu *et al.*, 2016a; Liu and Guo, 1991).

The constant ($-a$) is known as the system's development coefficient, $b_i x_i^{(1)}(k)$ the driving term.

11. The power model of GM(1, 1)

The driving coefficient and $\hat{a} = [a, b_1, b_2, \ldots, b_N]^T$ the sequence of parameters.

Assume that $X^{(0)}$ is a sequence of raw data, $X^{(1)}$ the accumulation sequence of $X^{(0)}$, and $Z^{(1)}$ the adjacent neighbour average sequence of $X^{(1)}$. Then:

$$x^{(0)}(k) + az^{(1)}(k) = b\left[z^{(1)}(k)\right]^{\alpha} \tag{14}$$

is known as the power model of GM(1, 1) (Deng, 1985; Liu *et al.*, 2016a; Liu and Lin, 1998). Also:

$$dx^{(1)}/dt + ax^{(1)} = b\left(x^{(1)}\right)^{\alpha} \tag{15}$$

is known as the shadow equation of the power model of GM(1, 1).

12. Grey Verhulst model

When the power $\alpha = 2$ in the power model of GM(1, 1), the resultant model:

$$x^{(0)}(k) + az^{(1)}(k) = b\left(z^{(1)}(k)\right)^2 \tag{16}$$

is known as the grey Verhulst model (Deng, 1985; Liu *et al.*, 2016a); and:

$$dx^{(1)}/dt + ax^{(1)} = b\left(x^{(1)}\right)^2 \tag{17}$$

is known as the shadow equation of the grey Verhulst model.

13. The model of GM(*r*, *h*)

Assume that $X_i^{(0)} = (x_i^{(0)}(1), x_i^{(0)}(2), \ldots, x_i^{(0)}(n))$, $i = 1, 2, ..., h$, where $X_1^{(0)}$ stands for a data sequence of a system's characteristic, and $X_i^{(0)}$, $i = 2, 3, \ldots, h$ data sequences of relevant factors. Let:

$$\alpha^{(1)}\hat{x}_1^{(1)}(k) = \hat{x}_1^{(1)}(k) - \hat{x}_1^{(1)}(k-1) = \hat{x}_1^{(0)}(k)$$

$$\alpha^{(2)}\hat{x}_1^{(1)}(k) = \alpha^{(1)}\hat{x}_1^{(1)}(k) - \alpha^{(1)}\hat{x}_1^{(1)}(k-1) = \hat{x}_1^{(0)}(k) - \hat{x}_1^{(0)}(k-1)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\alpha^{(r)}\hat{x}_1^{(1)}(k) = \alpha^{(r-1)}\hat{x}_1^{(1)}(k) - \alpha^{(r-1)}\hat{x}_1^{(1)}(k-1) = \alpha^{(r-2)}\hat{x}_1^{(0)}(k) - \alpha^{(r-2)}\hat{x}_1^{(0)}(k-1)$$

and $z^{(1)}(k)=1/2(x^{(1)}(k)+x^{(1)}(k-1))$, then:

$$\alpha^{(r)}\hat{x}_1^{(1)}(k)+\sum_{i=1}^{r-1}a_i\alpha^{(r-i)}x_1^{(1)}(k)+a_r z_1^{(1)}(k)=\sum_{j=1}^{h-1}b_j x_{j+1}^{(1)}(k)+b_h \tag{18}$$

is referred to as the model of GM(r, h) (Deng, 1985; Liu *et al.*, 2014, 2016a; Liu, 2016). The GM (r, h) model is an rth order grey model in h variables.

14. Time-varying grey forecasting model GM(1, 1)

Assume $x^{(0)}=(x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$, then:

$$x^{(0)}(k)+az^{(1)}(k)=b(k-0.5)+c$$

is referred to as the time-varying grey forecasting GM(1, 1), $(dx^{(1)}/dt)+ax^{(1)}=bt+c$ is the shadow equation of the time-varying grey forecasting GM(1, 1) (Liu *et al.*, 2016a, b).

15. Grey forecasting

Grey forecasting (Deng, 1985; Liu *et al.*, 2016a) is about making quantitative forecasts about the future states of systems. GM(1, 1) is the most commonly used forecasting model. There are also other models including interval forecasts, disaster forecasts, seasonal disaster forecasts, wave forecasts and system forecasts, etc.

16. Value domain

Let $X(t)$ be a zigzagged line. If there are smooth and continuous curves $f_u(t)$ and $f_s(t)$, satisfying that for any t, $f_u(t)<X(t)<f_s(t)$, then $f_u(t)$ is known as the lower bound function of $X(t)$, $f_s(t)$ the upper bound function, and $S=\{(t, X(t))|X(t)\in[f_u(t), f_s(t)]\}$ the value domain of $X(t)$ (Deng, 1985; Liu *et al.*, 2016a, c).

17. Grey disaster forecasting

The grey disaster forecasting (Deng, 1985; Liu *et al.*, 2016a, c) is the prediction for the abnormal values (either too large or too small) in a particular sequence (number series). To be more specific, all abnormal values are selected from the original sequence according to a given value to form a subsequence, from which the time distribution sequence is derived. Then a GM (1, 1) model is set based on this time distribution sequence to foretell when abnormal numbers may appear in the future. The sequence that consists of values that are greater than the given value ξ is referred to as an upper disaster sequence. Otherwise, they are known as a lower disaster sequence.

The basic idea of grey disaster forecasting is essentially the predictions of abnormal values of sequences.

18. ξ-Contour point

Assume that X is a zigzagged line, let:

$$\sigma_{\max}=\max_{1\leqslant k\leqslant n}\{x(k)\} \text{ and } \sigma_{\min}=\min_{1\leqslant k\leqslant n}\{x(k)\}.$$

Then:

(1) for any $\forall\xi\in[\sigma_{\min}, \sigma_{\max}]$, $X=\xi$ is known as the ξ-contour (line); and

(2) the solutions $(t_i, x(t_i))(i=1, 2, \ldots)$ of system of equations:

$$\begin{cases} X=\{x(k)+(t-k)[x(k+1)-x(k)]|k=1,2,\ \ldots,\ n-1\} \\ X=\xi \end{cases}$$

is called the ξ-contour points (Deng, 1985; Liu *et al.*, 2016; Liu and Guo, 1991). The ξ-contour point is the intersection of the zigzagged line X and the ξ-contour line.

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