Welfare Effects of Controlling Labor Supply: An Application of the Stochastic Ramsey Model

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Abstract

In this paper we extend Merton's (1975) classic stochastic version of the Ramsey model by allowing the goverment to control the expected growth rate of the labor supply. We characterize the solution to this control problem for general time-seperable preferences, and derive an analytical solution for the CRRA case. The results show to what extent the planner, or government, increases consumption and welfare by taking an active role in controlling the economy. We also explore the implications of goverment control of labor growth for the term structure of interest rates and the effects of taxes on capital.

JEL code: C68, D81, E21, E24, E43, E61.

Key words: Utility maximization, Ramsey model, stochastic control, interest rate dynamics, Malliavin calculus.

1 Introduction

In his seminal work, Ramsey (1928) analyzes the dynamics of consumption and capital formation. In the deterministic Ramsey economy, labor supply is proportional to the population and is assumed to grow exponentially. Bourguignon (1974), Merton (1975), and Malliaris and Brock (1982) introduce uncertainty by adding noise to the labor supply. Still relying on many assumptions and simplifications, this stochastic economy is more realistic, and in the above references many asymptotic properties of different economic variables are derived and compared to the steady-state certainty case. Recently, Jensen and Wang (1999) extend the above one-sector growth model to a two-sector trading model consisting of a capital good industry and a consumer good industry. They analyze the effects of different uncertainties on the capital to labor ratio, but do not solve any optimization problems regarding consumption or the saving rate.

In this paper we focus on the original one-sector economy, but go one step further and assume not only that the labor supply is augmented with exogenous shocks, but also that the growth rate of this stochastic process can be controlled. We show how the stochastic control problem can be solved by the use of Malliavin calculus, and derive the conditions on the growth rate that must be satisfied in order to maximize the expected utility. We characterize the solution for general utility functions and derive an analytical solution for the constant relative risk aversion (CRRA) case. Furthermore, we explore the similarity between the command optimum and the market solution, and study the effects of government taxation on capital.

Merton (1975) shows, by assuming a Cobb-Douglas production function and a constant saving rate, how a stochastic differential for the labor supply determines the stochastic process for the short term interest rate. The same idea is used here. Different utility functions implies different growth rate policies, and in combination with different production functions we derive a set of interest rate processes. Some nest earlier ad-hoc interest rate specifications, while others have functional forms that do not, and are to our knowledge unexplored in interest rate theory. In particular, we examine in detail how the increasing/decreasing return to scale properties affect the possibility for explosions of the interest rate processes.

Our setup is an extension of earlier results concerning the classic Ramsey problem. It has also many similarities with more standard stochastic control problems, such as the optimal portfolio choice in Merton (1990, chs. 4-6), and the flexible labor supply in Bodie et al. (1992), where the agents can choose their mix of risky and risk-free assets over time and decide their combinations of work and leisure, respectively. We emphasize that we study the effects of economic growth in an economy where labor supply is optimally controlled, but say nothing of how such policies should be implemented in reality. Nevertheless, one could say that we quantify the potential welfare gains of controlling the labor supply growth rate in this simplified economy optimally, an issue that might not be without practical relevance[1].

In section 2 we describe the stochastic Ramsey model. Section 3 presents the solution to the expected utility maximization problem and the optimal growth rate policies, and in section 4 we present some illustrative examples. In section 5 we analyze what interest rate models that are consistent with our model, and the impact of capital taxes. Section 6 concludes and summarizes the paper, and gives some suggestions for future extensions.

2 The stochastic Ramsey model

In order to model uncertainty in the Ramsey economy, we follow the concepts of Bourguignon (1974) and Merton (1975), and use Brownian motions. More precisely, consider a time interval $\mathcal{T} = [0, T]$ with $T > 0$, possibly infinite. On this time interval we let $\{W(t) : t \in \mathcal{T}\}$ denote a Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$. The filtration $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$, we take to be the natural filtration generated by the σ-fields $(W(s) : 0 \leq s \leq t)$ and completed by the P-null sets of $\mathcal{F}$.

Randomness is introduced via uncertainty in the labor supply, by defining the capital stock $K(t)$, and the labor force $L(t)$, to be the solutions to the following stochastic differential equations:

$$\begin{cases} dK(t) = (F(K(t), L(t)) - \lambda K(t) - C(t))\, dt, & K(0) = K_0 > 0 \\ dL(t) = n(t) L(t)\, dt + \sigma(t) L(t)\, dW(t), & L(0) = L_0 > 0, \end{cases} \tag{1}$$

where the production function F is homogeneous of degree one, λ is the nonnegative and constant depreciation rate, and $C(t)$ is the aggregated consumption. We assume that the drift term $n(t)$ is an $\mathbb{F}$-adapted positive processes, and that the diffusion term $\sigma(t)$ is a strictly positive deterministic function. We also assume that the drift and diffusion terms are sufficiently regular for strong solutions to exist.

In the references above, $n(t)$ and $\sigma(t)$ are assumed to be constants and the asymptotic properties of different economic quantities are compared to the certainty case, i.e. $n(t) \equiv n$, and $\sigma(t) \equiv 0$. Since the purpose of this paper is to investigate optimal growth rate policies, we do not impose such assumptions. Instead we assume

[1] The policy implications are not intended to resurrect the debate about goverment versus market intervention. Our results hinge upon the many assumptions underlying the Ramsey model, as well as the fact that the goverment is assumed to instantaneously and costlessly adjust the growth rate. In reality, this cannot be done without costs. Futhermore, it is assumed that *only* the goverment can intervene, and thus precludes any mechanism by which the private sector could do better.

that the volatility function $\sigma(t)$ is exogenously given (and in many of the examples it is assumed to be a constant, $\sigma > 0$), but that a central planner has the ability to choose $n(t)$ optimally. In this way, we try to mimic a government interference in the labor market, and to quantify the effects of its actions.

It is now straightforward to derive *the stochastic neoclassical differential equation for growth* by applying Itô's lemma to the capital per labor quantity, $k(t) \equiv K(t)/L(t)$:

$$dk(t) = \left(f(k(t)) - \left[n(t) - \sigma^2(t) + \lambda\right]k(t) - c(t)\right)dt - \sigma(t)\,k(t)\,dW(t), \qquad (2)$$

with $c(t) \equiv C(t)/L(t)$. Since $k(0) = k_0 = K_0/L_0 > 0$, we have a stochastic differential equation whose solution $k(t)$ is well defined given technical conditions on the positive production per labor function f and on the growth process $n(t)$.

The objective of the central planner is to determine the optimal consumption stream, and thereby also the optimal capital per labor quantity:

Problem 1 (The finite horizon stochastic Ramsey problem) *Find*

$$V(k_0) \equiv \sup_{c \in \mathcal{A}(k_0)} E\left[\int_0^T U_1(t, c(t))\,dt + U_2(k(T))\right] \qquad (3)$$

where the set $\mathcal{A}(k_0)$ is defined such that the dynamics of the capital per labor quantity $k(t)$ is given by (2), the initial value $k(0) = k_0 > 0$, and the solution $k(t) \geq 0$ a.s. for all $t \in \mathcal{T}$.

In order to get rid of trivial solutions, we assume from now on that $V(k_0)$ is finite for at least one value of k_0. In most applications the utility function U_1 is time separable and assumed to take the specific form $U_1(t,x) \equiv e^{-\rho t}U_1(x)$, where the discount factor $\rho \geq 0$. Both $U_1(x)$ and the bequest function, $U_2(x)$, are strictly increasing, strictly concave and continuously differentiable functions satisfying the Inada conditions, $U'_\cdot(\infty) \equiv \lim_{x\to\infty} U'_\cdot(x) = 0$ and $U'_\cdot(0+) \equiv \lim_{x\downarrow 0} U'_\cdot(x) = \infty$. These conditions preclude some preferences, such as the quadratic utility, and may be dropped later on.

The stochastic Ramsey problem is usually defined by setting $U_2(\cdot) = 0$ in (3), i.e. in the problem formulations in Bourguignon (1974), Merton (1975, 1990), and Malliaris and Brock (1982), the agents have no incentives to leave bequests. For simplicity, the above references analyze the problem in the limit of $T \to \infty$ only. Our definition of the problem therefore nests earlier work by also investigating a finite time horizon, while the analysis of the infinite horizon setting is done by letting $T \to \infty$, and adding the transversality condition $\lim_{T\to\infty} E\left[U_2(k(T))\right] = 0$ a.s.

3 The solution

In order to find a solution to the stochastic maximization problem stated in the previous section, basically two methods can be used. The standard procedure is to derive and solve the Hamilton-Jacobi-Bellman (HJB) partial differential equation, an approach used by Merton (1975, 1990), Bodie et al. (1992), Corsetti(1997), and Turnovsky (2000). The drawback with this technique is that the HJB-equation generally is nonlinear and extremely difficult to solve analytically.

The other method is often referred to as the duality approach and is based upon the Lagrange theory of optimization. This probabilistic approach has recently been used in many financial applications. Compared to the dynamic programming approach it is more general in the sense that we do not have to assume a specific underlying Markovian structure. It is also in many cases easier to use which motivates our choice of the method.

We start by introducing the local martingale $\{Z(t) : t \in \mathcal{T}\}$, defined by

$$Z(t) = \exp\left(-\frac{1}{2}\int_0^t \theta^2(s)\,ds + \int_0^t \theta(s)\,dW(s)\right) \tag{4}$$

where

$$\theta(t) \equiv \sigma^{-1}(t)\left(f(k(t))\,k^{-1}(t) - \left(n(t) - \sigma^2(t) + \lambda\right)\right), \tag{5}$$

and $\int_0^T \theta^2(s)\,ds < \infty$ a.s. by assumption. Since $Z(t)$ is positive it is according to Fatou's lemma also a supermartingale. The stochastic differential of $Z(\cdot)$ is given by $dZ(t) = \theta(t)\,Z(t)\,dW(t)$, hence by applying the Itô formula to the product $Z(t)\,k(t)$ and integrating it up, we obtain

$$Z(t)\,k(t) + \int_0^t Z(s)\,c(s)\,ds = k_0 + \int_0^t Z(s)\,k(s)\,(\theta(s) - \sigma(s))\,dW(s). \tag{6}$$

For an admissible consumption stream, $c \in \mathcal{A}(k_0)$, the left hand side is positive and consequently the Itô integral on the right hand side is not only a local martingale, but also a supermartingale. This implies the budget constraint

$$E\left[\int_0^T Z(s)\,c(s)\,ds + Z(T)\,k(T)\right] \le k_0, \tag{7}$$

which has the interpretation that the expected discounted future consumption plus the expected discounted terminal capital stock cannot exceed the initial capital endowment. Hence, the process $Z(t)$ will serve as a type of discounting factor, though a stochastic one.

Now, consider a utility function U satisfying the Inada conditions and let us denote by I the inverse of its derivative U'. Both these functions are continuous, strictly decreasing, and map $(0,\infty)$ onto itself with $I(0+) = U'(0+) = \infty$ and $I(\infty) = U'(\infty) = 0$. We introduce the convex dual $\tilde{U}$ of U by setting

$$\tilde{U}(y) \equiv \max_{0<x<\infty} [U(x) - xy] = U(I(y)) - yI(y), \tag{8}$$

and observe that $\tilde{U}$ is a convex decreasing function with $\tilde{U}(\infty) = U(0+)$ and $\tilde{U}(0+) = U(\infty)$. Furthermore, it is continuously differentiable on $(0,\infty)$ with $\tilde{U}'(y) = -I(y)$. In order to solve the stochastic Ramsey problem, we maximize (3) given the budget constraint (7). We see that

$$\begin{aligned} E&\left[\int_0^T U_1(t,c(t))\,dt + U_2(k(T))\right] + \gamma\left(k_0 - E\left[\int_0^T Z(s)\,c(s)\,ds + Z(T)\,k(T)\right]\right) \\ &= E\left[\int_0^T [U_1(t,c(t)) - \gamma Z(t)\,c(t)]\,dt\right] + E\left[U_2(k(T)) - \gamma Z(T)\,k(T)\right] + \gamma k_0 \\ &\leq E\left[\int_0^T \tilde{U}_1(t,\gamma Z(t))\,dt + \tilde{U}_2(\gamma Z(T))\right] + \gamma k_0. \end{aligned} \tag{9}$$

According to (8) we have equality if and only if

$$c(t) = I_1(t,\gamma Z(t)) \qquad \text{and} \qquad k(T) = I_2(\gamma Z(T)). \tag{10}$$

The Lagrange multiplier $\gamma > 0$ is obtained by substituting these quantities back into the budget constraint (7):

$$\chi_0(\gamma) \equiv E\left[\int_0^T Z(s)\,I_1(s,\gamma Z(s))\,ds + Z(T)\,I_2(\gamma Z(T))\right] = k_0. \tag{11}$$

If $\chi_0(\gamma) < \infty$ for all $\gamma > 0$, then χ_0 maps $(0,\infty)$ onto itself and is continuous, strictly decreasing with $\chi_0(0) = \infty$ and $\chi_0(\infty) = 0$. Hence, in this case γ is uniquely defined. Finally, in order to verify that the optimal consumption process in (10) belongs to $\mathcal{A}(k_0)$, we need to show that $k(t) \geq 0$ a.s. This follows if we can control the capital per labor process in such a way that (10) is satisfied. Then we have from (6) and (7) that the process $Z(t)\,k(t) + \int_0^t Z(s)\,c(s)\,ds$ is a martingale with a positive terminal value, and that the capital per labor process is formally given by[2]

$$k(t) = \frac{1}{Z(t)} E\left[Z(T)\,I_2(\gamma Z(T)) + \int_t^T Z(s)\,I_1(s,\gamma Z(s))\,ds \,|\mathcal{F}_t\right] \geq 0 \text{ a.s.} \tag{12}$$

[2]Note that the non-negativity of $k(t)$ depends on the fact that the inverse functions I_1 and I_2 take values on the positive real line. This somewhat motivates our strong definition of a utility function. In principle it would be possible to allow for more flexibility by also considering utility functions of the form $U(x) = 1 - \exp(-\delta x)$, with $\delta > 0$. See Korn (1997) for related results.

If the central planner can control the labor supply, then $k(t)$ can also be controlled, i.e. the terminal value can be attained. In order to see this let us consider the martingale

$$M(t) = E\left[Z(T)\, I_2(\gamma Z(T)) + \int_0^T Z(s)\, I_1(s, \gamma Z(s))\, ds \,|\mathcal{F}_t\right]. \tag{13}$$

According to the martingale representation theorem, there exists an $\mathbb{F}$-adapted process $\psi(t)$, with $\int_0^T \psi(t)^2 dt < +\infty$ a.s., such that $M(t) = M(0) + \int_0^t \psi(s)\, dW(s)$. It follows from (11) that $M(0) = k_0$, and from (10) and (12) that $M(t) = Z(t)\, k(t) + \int_0^t Z(s)\, c(s)\, ds$. Consequently we have according to (6) that

$$\psi(t) = Z(t)\, k(t)\, (\theta(t) - \sigma(t)). \tag{14}$$

We can also identify $\psi(t)$ using Malliavin calculus and the Clark-Ocone formula, see Nualart (1995) for details:

$$\psi(t) = E\left[D_t\left(Z(T)\, I_2(\gamma Z(T)) + \int_0^T Z(s)\, I_1(s, \gamma Z(s))\, ds\right) |\mathcal{F}_t\right], \tag{15}$$

where D_t denotes the Malliavin derivative. If we were to present a theorem striving for full generality, we would implicitly define $\theta(t)$ (and consequently $n(t)$) as the process that equales (14) and (15). However, for our purposes the following slightly less general result will be sufficient.

Proposition 2 *Let the process $Z(t)$ be given by (4), and assume that there exists a continuously differentiable deterministic function g such that*

$$g(t, Z(t)) = E\left[Z(T)\, I_2(\gamma Z(T)) + \int_t^T Z(s)\, I_1(s, \gamma Z(s))\, ds \,|\mathcal{F}_t\right]. \tag{16}$$

Then the optimal labor growth rate $n(t)$ is implicitly given by the equation

$$\theta(t) = \sigma(t) \frac{g(t, Z(t))}{g(t, Z(t)) - Z(t) g_z(t, Z(t))}, \tag{17}$$

where $\theta(t) = \sigma^{-1}(t) \left(f(k(t))\, k^{-1}(t) - \left(n(t) - \sigma^2(t) + \lambda\right)\right)$ from (5).

Proof. Under technical conditions, see Bermin (1999), Malliavin derivation and conditional expectation commute. Hence,

$$D_t g(t, Z(t)) = E\left[D_t\left(Z(T)\, I_2(\gamma Z(T)) + \int_t^T Z(s)\, I_1(s, \gamma Z(s))\, ds\right) |\mathcal{F}_t\right]. \tag{18}$$

Moreover, using standard Malliavin calculus it follows that $D_t \int_0^T \phi(s)\, ds = D_t \int_t^T \phi(s)\, ds$ for all sufficiently smooth processes $\phi(t)$, and that $D_t g(t, Z(t)) = g_z(t, Z(t))\, Z(t) \theta(t)$. This completes the proof. ∎

Corollary 3 *Suppose that the production function F is not homogenous of degree one. Then, proposition 2 still holds if $f(k(t))$ is replaced by $F(K(t), L(t))/L(t)$, which means that $F(K(t), L(t))/K(t)$ replaces $f(k(t))/k(t)$.*

Proof. The proof is straightforward and thus omitted. ■

The duality approach generates explicit solutions for the optimal consumption policy but it does not give any clues about the optimal portfolio weights (trading strategies), or in this case the optimal growth rate. To obtain these quantities, one has to go back to the HJB-equation, or, as we do here, use Malliavin calculus. The decision of which approach to take is often problem dependent, but the Malliavin approach has shown to be simpler in many important applications, see e.g. Karatzas and Ocone (1991).

4 Applications

Finding $\theta(t)$, and hence $n(t)$, from (17) seems to be straightforward, but the simplicity is illusive. $Z(t)$ is a process that depends on $\theta(t)$, and computing the conditional expectation of some transformation of $Z(t)$ is certainly not an easy task. Only in some greatly simplified cases can we calculate g and g'_z, and hence obtain analytical solutions. In this section we make such assumptions in order to investigate the effects of labor control on the capital formation in the Ramsey economy. Nevertheless, (17) can be used as a basis for more general numerical solutions.

The first example illustrates an important situation where the calculations are particularly easy to conduct.

Example 4 *Let us assume logarithmic utility, that is $U_1(t, x) = e^{-\rho t}\log x$ and $U_2(x) = e^{-\rho T}\log x$. Since $I_1(t, x) = e^{-\rho t}/x$ and $I_2(x) = e^{-\rho T}/x$, we have that*

$$g(t, Z(t)) = E\left[\frac{1}{\gamma}e^{-\rho T} + \int_t^T \frac{1}{\gamma}e^{-\rho s}ds\,|\mathcal{F}_t\right] = \frac{1}{\gamma}e^{-\rho T} + \frac{1}{\gamma\rho}\left(e^{-\rho t} - e^{-\rho T}\right). \quad (19)$$

Hence, $g_z(t, Z(t)) \equiv 0$ and proposition 2 implies that $\theta(t) = \sigma(t)$, or equivalently that

$$n(t) = \frac{f(k(t))}{k(t)} - \lambda. \quad (20)$$

In order to find the optimal Lagrange multiplier we note from (11) that $k_0 = g(0, Z(0))$, yielding $\gamma = \left(e^{\rho T} + \rho - 1\right)\frac{e^{-\rho T}}{k_0\rho}$. Moreover, from (10) we see that the optimal consumption and terminal capital are $c(t) = \frac{e^{-\rho t}}{Z(t)\gamma}$ and $k(T) = \frac{e^{-\rho T}}{Z(T)\gamma}$, where γ is as above and $Z(t)$ is defined by (4) with $\theta(t) = \sigma(t)$. From (12), we also *have by definition that* $g(t, Z(t)) = k(t) Z(t)$. *However, from (19),* g *is a purely deterministic process. Consequently, we can express* $Z(t)$ *in terms of* $k(t)$, *yielding*

$$c(t) = k(t) \frac{\rho}{1 - e^{-\rho(T-t)}(1-\rho)}. \tag{21}$$

The dynamics of the optimally controlled capital per labor will thus be

$$dk(t) = \left(\sigma^2(t) - \frac{\rho}{1 - e^{-\rho(T-t)}(1-\rho)}\right) k(t)\, dt - \sigma(t) k(t)\, dW(t), \tag{22}$$

which implies that $k(t)$ *is log-normally distributed.*

In the infinite horizon limit, $T \to \infty$, *we distinguish between two cases. If* $\rho > 0$, *we see that* $\gamma \to \frac{1}{k_0 \rho}$ *and* $c(t) \to \rho k(t)$. *If on the other hand* $\rho = 0$, $\gamma \to \infty$ *which implies that there exists no solution to the original problem.*

However, the case of $\rho = 0$ *works with a finite horizon, yielding* $\gamma = \frac{T+1}{k_0}$ *and* $c(t) = \frac{k(t)}{1+T-t}$.

We see that the optimal choice of $n(t)$ given by (20) has striking effects on economic growth. In the infinite horizon setting, the expected growth rate in (22) depends only on the government's idea of the agents' time preference and the variance of the labor supply growth process. Without the ability to control $n(t)$, the growth of $k(t)$ is given by (2), that is, the industrial properties of f, such as diminishing returns or the capital-labor intensity, influences the capital accumulation. This means that in the absence of continuing improvements in technology, per capita growth must eventually cease, which is reflected in the study of the steady-state distribution of $k(t)$ in Merton (1975, 1990). Unfortunately, this neoclassic prediction is not supported by the persistent long-term economic growth rate in many countries.

Under optimal control, the government completely offsets the impact of technology, and the capital stock approaches, with a constant σ, a geometric Brownian motion with drift term $\sigma^2 - \rho$, as $T \to \infty$. If $\sigma^2/2 > \rho$, the economy will follow a balanced growth path indefinitely like one with a stochastic AK technology, where the aggregate production function, but not the one of the representative firm, is linear in capital, as in Corsetti (1997) and Turnovsky (2000).

It also follows from (22) that the expected capital growth is an increasing function of $\sigma^2(t)$, contrary to the belief that greater uncertainty dampens economic growth. This is not only a feature of the controlled economy; the drift term in (2) also increases with uncertainty in the labor supply. For a review of the dispute of this relationship in the literature, see Turnovsky (2000).

In the case of logaritmic utility, explicit solutions of $n(t)$, $c(t)$ and $k(T)$ are easy to find, because $g'_z = 0$ in (17). Also, we do not have to assume $\sigma(t)$ to be a deterministic function. In fact, the volatility could be any $\mathbb{F}$-adapted process, as long as $\int_0^T \sigma^2(s)\, ds < \infty$ a.s.

For more general utility functions we have to add other restrictions. One convenient restriction is to assume a deterministic (constant) $\theta(t)$. If such a $\theta(t)$ solves (17), we also have an expression for $n(t)$, and this solution is then optimal in the class of deterministic (constant) $\theta(t)$. Note that with this assumption, we cannot guarantee that we have found the global optimum. However, the solution will generate a local optimum. Henceforth, we call such restricted strategies *simple rules* and below we exemplify its use in the power utility case.

Example 5 *Let us only consider simple rules, i.e. we assume that $\theta(t)$ is deterministic. Further assume for notational simplicity that the volatility is constant, i.e. $\sigma(t) = \sigma$. Let $U_1(t,x) = e^{-\rho t}x^{\alpha}/\alpha$ and $U_2(x) = e^{-\rho T}x^{\alpha}/\alpha$, where $\alpha < 1, \alpha \neq 0$. This implies that $I_1(t,x) = (e^{\rho t}x)^{\frac{1}{\alpha-1}}$ and $I_2(x) = \left(e^{\rho T}x\right)^{\frac{1}{\alpha-1}}$. Using (16), we have $g(t, Z(t)) = E\left[Z^{\frac{\alpha}{\alpha-1}}(T)\,|\mathcal{F}_t\right]\left(\gamma e^{\rho T}\right)^{\frac{1}{\alpha-1}} + \int_t^T E\left[Z^{\frac{\alpha}{\alpha-1}}(s)\,|\mathcal{F}_t\right](\gamma e^{\rho s})^{\frac{1}{\alpha-1}}\,ds$. Because of the Markov property of Z, we see that $g(t,Z(t)) \propto Z^{\frac{\alpha}{\alpha-1}}(t)$. This implies that, $g_z'(t, Z(t))\,Z(t) = \frac{\alpha}{\alpha-1}g(t,Z(t))$ and (17) reduces to $\theta(t) = (1-\alpha)\sigma \equiv \theta$. We get $g(t,Z(t)) = Z^{\frac{\alpha}{\alpha-1}}(t)\,\gamma^{\frac{1}{\alpha-1}}e^{\frac{\rho}{\alpha-1}t}\frac{1}{\beta}\left(e^{\beta(T-t)}(1+\beta)-1\right)$, where we have set $\beta = \frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha}$. From (5) the optimal growth rate is*

$$n(t) = \frac{f(k(t))}{k(t)} - \lambda + \alpha\sigma^2. \tag{23}$$

In order to compute $c(t)$ and $k(T)$ we need γ. From (11), $k_0 = g(0, Z(0))$ which implies that $\gamma = \left(\frac{\beta k_0}{e^{\beta T}(1+\beta)-1}\right)^{\alpha-1}$. The optimal consumption and terminal capital per labor are given by $c(t) = Z^{\frac{1}{\alpha-1}}(t)\,e^{\frac{\rho t}{\alpha-1}}\gamma^{\frac{1}{\alpha-1}}$ and $k(T) = Z^{\frac{1}{\alpha-1}}(T)\,e^{\frac{\rho T}{\alpha-1}}\gamma^{\frac{1}{\alpha-1}}$, respectively. By substituting $Z(t)$ for $k(t)$ as in the previous example, we obtain

$$c(t) = k(t)\,\frac{\beta}{e^{\beta(T-t)}(1+\beta)-1}. \tag{24}$$

The dynamics of the optimally controlled capital per labor will thus be

$$dk(t) = \left(\sigma^2(1-\alpha) - \frac{\beta}{e^{\beta(T-t)}(1+\beta)-1}\right)k(t)\,dt - \sigma k(t)\,dW(t)\,, \tag{25}$$

that is, $k(t)$ is again log-normally distributed.
In the limit $T \to \infty$, $\gamma^{\frac{1}{\alpha-1}} \to -k_0\beta > 0$ if $\beta < 0$, and $c(t) \to -\beta k(t)$. If, on the other hand, $\beta > 0$ then $\gamma^{\frac{1}{\alpha-1}} \to 0$, and there exists no solution to the original problem in this case. Similar to the previous example, the non-existence of a solution also occurs when $\beta \to 0$, since this implies that $\gamma \to \infty$.
However, with a finite horizon β can take any value. In particular, as $\beta \to 0$ we find that $\gamma^{\frac{1}{\alpha-1}} \to \frac{k_0}{T+1}$ and $c(t) \to \frac{k(t)}{1+T-t}$.

Remark 6 *Note that the solution is optimal in the class of all time-dependent* $\theta(t)$*, as long as* σ *is constant. If* σ *is a time-dependent function the expressions for* $n(t)$ *and* $c(t)$ *must be slightly modified. The general conclusions will remain, but notationally the expressions become more complicated.*

Remark 7 *If* U_1 *and* U_2 *have different risk coefficients, say* α_1 *and* α_2*, the solution can no longer be calculated as easily as in example 5. The reason is that then* $g'_z(t, Z(t))\, Z(t) \neq \frac{\alpha}{\alpha-1} g(t, Z(t))$*, and the optimal* $\theta(t)$ *is no longer constant.*

Apart from the fact that the risk preference of the agents now enter the solution, the conclusions drawn from examples 4 and 5 are very similar. When the volatility is a constant (or a time-dependent function) the logarithmic utility may be considered as a special case of the power utility with $\alpha = 0$. However, the case of logarithmic utility is interesting on its own since we can allow the volatility to be any $\mathbb{F}$-adapted process. We also see that the optimal consumption $c(t)$ and the optimal growth rate $n(t)$ are functionals of $k(t)$. In order to better understand the evolution of the capital per labor let us assume power utility, and a constant volatility. Solving the stochastic differential equation (25) yields

$$k(t) = k_0 \frac{e^{\beta T}(1+\beta) - e^{\beta t}}{e^{\beta T}(1+\beta) - 1} e^{\sigma^2(1-\alpha)t} \cdot \mathcal{E}(t, \sigma), \tag{26}$$

where $\mathcal{E}(t,\sigma) \equiv e^{-\frac{1}{2}\sigma^2 t - \sigma W(t)}$ is a standard stochastic exponential. Hence, $k(t)$ can be expressed as a time-dependent function strictly bounded away from zero from above, multiplied by a stochastic exponential. This shows that $k(t)$ can never hit the boundaries zero and infinity in the case of a finite time horizon. If we assume an infinite horizon, (25) reduces to a geometric Brownian motion with constant drift term. Following Karatzas and Shreve (1988, p. 349), we conclude that the boundaries are unattainable, i.e. $k(t) \in (0, \infty)$ a.s. for all $t \geq 0$.

Using (24) and (26), we have

$$c(t) = \frac{\left(\frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha}\right) k_0}{e^{\left(\frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha}\right)T}\left(1 + \frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha}\right) - 1} e^{\left(\frac{1}{2}\sigma^2(1-\alpha) - \frac{\rho}{1-\alpha}\right)t - \sigma W(t)}. \tag{27}$$

Hence, we can easily derive the flows of the optimal consumption (or the sensitivities) with respect to the parameters α, ρ, σ, and T. We use the notation $\frac{dc(t)}{d\alpha} = \phi_\alpha(t)\, c(t)$ and similarly for the other terms. Since $\phi_\alpha(t) = d\ln(c(t))/d\alpha$, (27) gives that the sensitivity functions are linear in t, with signs dependent on the parameter configuration.

In figure 1 we plot the deterministic functions ϕ_α and ϕ_ρ. Given our choice of parameters we see that the consumption is quite insensitive to changes in the relative risk aversion coefficient $(1-\alpha)$ for moderate values of α ($\alpha \lesssim 0.5$). At the beginning of the time period a positive change of α increases the consumption, with a larger increase for a larger α. As we approach the terminal date ϕ_α becomes negative. The explanation is that only a less risk averse agent would dare to increase his present consumption at the cost of lower future consumption. We also see that for high values of α, an additional increase will considerable decrease future consumption.

The sensitivity function $\phi_\rho(t)$ is also downward sloping but much more symmetric. Positive changes in the discount factor at the beginning of the time period will have a positive affect on present consumption, which is quite reasonable. The larger the discount factor the smaller the effect, since when ρ is large, we strongly favor immediate consumption to future consumption. An additional increase of ρ will then only have marginal effects on the present consumption, compared to if ρ is small. At the end of the period the effect is the opposite. A positive change in ρ reduces the importance of future consumption, hence a negative value of ϕ_ρ.

The calculation of ϕ_σ differs somewhat since it will depend on $W(t)$. Its expected value will, however, also be proportional, by a factor $\sigma(1-\alpha)$, to t. Hence, $E[\phi_\sigma(t)]$ is always an increasing function of t, with a steeper slope the more risk-averse agents. The reason that the consumption will increase (on average) with greater uncertainty follows from (25); the expected capital growth is an increasing function of σ^2, which allows for higher future consumption.

We also see directly from (27) that $\phi_T \gtreqless 0$ if and only if $1+\frac{1}{2}\alpha\sigma^2-\frac{\rho}{1-\alpha} \lesseqgtr 0$. Only for very high values of the risk parameter α ($\alpha \lesssim 1$) the optimal consumption increases as the time horizon increases. Agents with a low risk aversion are the only ones that are willing to increase $c(t)$ in this case. If $\frac{1}{2}\alpha\sigma^2-\frac{\rho}{1-\alpha}=-1$ (which in the case of logarithmic utility reduces to $\rho=1$), the optimal consumption is *neutral* in T. In this artificial case, we consume at any time all the existing per capita capital, i.e. $c(t)=k(t)$ according to (24). In general though, the parameters have an impact on the optimal consumption.

5 The market solution

The original Ramsey economy is entirely run by a central planner who maximizes the expected utility of the representative agent. There is, however, an equivalence to a decentralized economy with a labor market and a capital service market, and inhabited with identical agents. The optimization problem (3) could be viewed as a representative agent maximizing the consumption over his remaining expected life, saving a part if he should live longer than he expects, or if he wishes to leave bequests, but with an important addition. The agents can not influence the labor growth rate, $n(t)$, so when making their consumption choice, they have to assume that someone is adjusting $n(t)$ in an optimal manner. We therefore cannot assume a completely decentralized model. There must exist a central planner, or a government, who takes a dynamically active role and acts in the best interest of the agents, a fact the agents know and account for when deciding their consumption plan. If so, the optimal consumption of the agents will be identical to the expressions we found in the previous section. In figures 2 and 3 we show the impact of different strategies taken by the planner on the consumption of the individuals, given that they believe the planner will choose $n(t)$ optimally. We consider three cases, illustrated in figure 2a: in the first case the planner chooses an optimal labor supply growth rate, in the second case the planner is somewhat lazy and chooses $n(t)$ to be the expected value of the optimal one, while in the third case the planner just fixes a constant growth rate of the labor supply. In the simulations the time horizon is 20 years, and all the paths are simulated using an Euler approximation of (2) on a weekly basis. In figure 2b, we plot the implied realization of the consumption given that the individuals believe that the planner will act optimally, i.e. $c(t)$ is given by (24). We see that the consumption decreases drastically as the planner departs from the optimal rule; the consumption for the constant rule is for instance only about half the optimal consumption.

Since we have only plotted one realization, these results should be interpreted with care. In figure 3 we therefore plot the expected value of the cumulative utility

$$V(t) = E\left[\int_0^t e^{-\rho s} U_1(c(s))\, ds + e^{-\rho t} U_2(k(t))\right], \qquad t = 1, ..., T \tag{28}$$

based on 10000 realizations. At the terminal date, $t = T$, we get a numerical solution to the stochastic Ramsey problem. As expected, we see that the optimally controlled $n(t)$ outperforms the other two rules. In particular, it is clearly superior to the constant growth rate rule. The difference between the optimal and the expected rules is much smaller, but one should note that we still need the optimal rule in order to compute its expected value. The simulated terminal values are: $V(T) = 2939$ for the optimal rule, $V(T) = 2874$ using the expected value of the optimal rule, and $V(T) = 2398$ in the case of the constant rule. Although, it is hard to quantify the exact improvement in welfare, it follows, since all the agents are identical, that an optimal controlled growth rate is Pareto efficient.

From now on, we will therefore assume that the planner acts in the behalf of the agents. More specifically, we analyze how interest rates are determined, and the impact of government taxation. If we let $r(t)$ denote the real rate of interest, and $w(t)$ the wage per capita, equilibrium in a decentralized market implies that $w(t)L(t) + r(t)K(t) = \frac{dK(t)}{dt} + C(t) + \lambda K(t)$. Perfect competition and profit maximization among identical firms yield the familiar market clearing conditions

$$r(t) = f'(k(t)) \qquad \text{and} \qquad w(t) = f(k(t)) - k(t)\, f'(k(t)). \tag{29}$$

Since the market solution to the Ramsey problem is identical to the command optimum, these two equations give us the possibility to study how the interest rate and wage are influenced by controlling the growth rate of the labor supply.

5.1 Interest rate dynamics

When analyzing the evolution of the interest rate, $r(t) = f'(k(t))$, we see that a direct application of the Itô formula yields

$$\begin{aligned} dr(t) &= f''(k(t))\left(f(k(t)) - \left[n(t) - \sigma^2(t) + \lambda\right]k(t) - c(t)\right)dt \\ &\quad + \frac{1}{2}f'''(k(t))k^2(t)\,\sigma^2(t)\,dt - f''(k(t))k(t)\,\sigma(t)\,dW(t). \end{aligned} \tag{30}$$

Below we present some examples of different interest rate models that can be generated in the controlled Ramsey economy.

Example 8 *Suppose we have power utility and a constant volatility. Assume further a Cobb-Douglas production function, $f(x) = x^\delta/\delta$, where $0 < \delta < 1$. According to example 5, the optimal growth rate is $n(t) = \frac{1}{\delta}r(t) - \lambda + \alpha\sigma^2$, and the stochastic differential for $r(t)$ becomes*

$$\begin{aligned} dr(t) &= (1-\delta)\left(\left(\alpha - \frac{\delta}{2}\right)\sigma^2 + \frac{\beta}{e^{\beta(T-t)}(1+\beta) - 1}\right)r(t)\,dt \\ &\quad + (1-\delta)\,\sigma r(t)\,dW(t). \end{aligned} \tag{31}$$

with $\beta = \frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha}$ as before. This is the famous interest rate model of Dothan (1978). The non-exploding properties of $k(t)$ implies that the interest rate process $r(t) = 1/k^{1-\delta}(t)$ is also non-exploding. Hence, zero and infinity are unattainable boundaries in the finite horizon case.

Now, let us analyze the impact on interest rates when we no longer assume a constant return to scale. More closely, we investigate the possibilities of explosions in the case of an infinite horizon, see appendix A for details.

Example 9 *Let us assume a power utility, constant volatility, and an infinite time horizon. We use the same notation as in example 5, and recall that in this case we need $\beta = \frac{1}{2}\alpha\sigma^2 - \frac{\rho}{1-\alpha} < 0$. The production function is assumed to be of the form*

$$F(K(t), L(t)) = \frac{1}{\delta+\varepsilon}K^{\delta+\varepsilon}(t)\,L^{1-\delta}(t), \qquad 0 < \delta+\varepsilon < 1, 0 < \delta < 1, \tag{32}$$

such that there are increasing (decreasing) returns to scale if ε is strictly positive (negative). Note that $\varepsilon = 0$ reduces to the case of a constant return to scale, with $F(K(t), L(t))/L(t) = k^\delta(t)/\delta$. Profit maximization of the firms (with respect to *the capital stock) implies that* $r(t) = \frac{\partial F}{\partial K}(K(t), L(t)) = k^{\delta+\varepsilon-1}(t) L^{\varepsilon}(t)$. *Corollary 3, Itô's lemma with* $dk(t)$ *from (25),* $n(t)$ *from (23), and* $dL(t)$ *from (1) yield*

$$dr(t) = r(t)\left(\frac{1}{2}\sigma^2((1-\delta)(\alpha-\delta)+\varepsilon\alpha) + \frac{\rho}{1-\alpha}(1-\delta-\varepsilon) - \varepsilon\lambda\right)dt + \frac{\varepsilon}{\delta+\varepsilon}r^2(t)\,dt + (1-\delta)\sigma r(t)\,dW(t). \quad (33)$$

From the results presented in appendix A, it follows that zero is an unattainable boundary, and that infinity is an unattainable boundary if and only if $\varepsilon \le 0$. *If* $\varepsilon > 0$, *infinity is an absorbing boundary, which will be reached with probability one in finite time if and only if* $\frac{1}{2}\sigma^2(\varepsilon\alpha - (1-\delta)(1-\alpha)) + \frac{\rho}{1-\alpha}(1-\delta-\varepsilon) - \varepsilon\lambda \ge 0$. *Moreover, in the practically relevant case where the boundaries are unattainable, we have the mean-reversion property if* $\varepsilon < 0$ *(decreasing return to scale). If* $\Psi \equiv \frac{1}{2}\sigma^2((1-\delta)(\alpha-\delta)+\varepsilon\alpha) + \frac{\rho}{1-\alpha}(1-\delta-\varepsilon) - \varepsilon\lambda > 0$, *the interest rate will revert towards the mean level* $\Psi\frac{\delta+\varepsilon}{|\varepsilon|} > 0$.

See Merton (1975, 1990) for more information about the interest rate process in this example. Finally, let us show that it is rather straightforward to generate new interest rate processes. For simplicity, we assume that the utility function is logarithmic.

Example 10 *Let us assume a logarithmic utility, constant volatility, and an exponential per capita production function,* $f(x) = A\frac{1-e^{-\delta x}}{\delta}$, *where the constants* $\delta, A > 0$. *Since* $r(t) = Ae^{-\delta k(t)}$, *it follows that the interest rate is restricted so that* $0 < r(t) < A$ *a.s. for all* $t \ge 0$.
With a finite T, *the stochastic differential for* $r(t)$ *is*

$$dr(t) = r(t)\ln\frac{r(t)}{A}\left(\sigma^2 - \frac{\rho}{1-e^{-\rho(T-t)}(1-\rho)} + \frac{1}{2}\sigma^2\ln\frac{r(t)}{A}\right)dt - \sigma r(t)\ln\frac{r(t)}{A}dW(t), \quad (34)$$

while in the case of an infinite horizon, we have

$$dr(t) = r(t)\ln\frac{r(t)}{A}\left(\sigma^2 - \rho + \frac{1}{2}\sigma^2\ln\frac{r(t)}{A}\right)dt - \sigma r(t)\ln\frac{r(t)}{A}dW(t). \quad (35)$$

It is interesting to note that also this process can have a mean-reverting property. If $\rho < \sigma^2$, *the mean-reverting level is given by* $A\exp\left(2\left(\frac{\rho}{\sigma^2}-1\right)\right) < A$.

Again we stress that when working with logarithmic utility, we can in principle use any $\mathbb{F}$-adapted volatility process. This gives us the freedom to generate a large number of different interest rate processes.

5.2 The impact of government taxation

We end this section with a brief study of the impact of government taxation. Suppose the government is taxing the return of capital at a constant rate τ, and hence that the individuals are facing the aggregate budget equation $w(t)L(t)+(1-\tau)r(t)K(t) = \frac{dK(t)}{dt} + C(t) + \lambda K(t)$. With $r(t)$ and $w(t)$ determined from (29), Itô's lemma gives

$$\begin{aligned} dk(t) &= \left(f(k(t)) - \left[n(t) - \sigma^2(t) + \lambda + \tau f'(k(t))\right]k(t) - c(t)\right)dt \\ &\quad -\sigma(t)k(t)\,dW(t). \end{aligned} \tag{36}$$

We see immediately that the solution to the stochastic Ramsey problem in section 3 is unaffected if $\theta(t)$ in (5) is replaced by

$$\tilde{\theta}(t) \equiv \sigma^{-1}(t)\left(f(k(t))k^{-1}(t) - \left(n(t) - \sigma^2(t) + \lambda + \tau f'(k(t))\right)\right). \tag{37}$$

Hence, proposition 2 still holds once $\theta(t)$ is changed to $\tilde{\theta}(t)$. Consequently, the optimal policies in the examples in section 4 change, $n(t) \to n(t) - \tau f'(k(t))$, that is, the optimal growth rates of the labor supply decrease as the government introduces taxes. It trivially follows that $k(t)$ and $c(t)$ are unaffected, as long as the government takes the effect of taxes under consideration when regulating $n(t)$ optimally. For further results on the impact of government taxation in other models, see Turnovsky (1995).

6 Conclusions

In this paper we examine a stochastic Ramsey economy where the growth rate of the labor supply can be controlled. The purpose is to quantify the potential welfare gains of such regulations. We derive the optimal growth rate policies and investigate capital formation, consumption, and interest rate dynamics. The results we find are theoretical, in the sense that they show how labor supply should be regulated if optimal control were possible, but do not consider how to practically implement the policies.

Furthermore, we investigate which kind of interest rate models are consistent with our framework. Some interest rate processes are similar to earlier specifications, but we also show how new ones may be derived. In particular, we analyze how increasing/decreasing returns to scale affect the possibility of explosions.

We also study the impact of government taxation. We find that taxes on capital have no effect on capital accumulation and consumption, if the planner acts on the behalf of the individuals. If this is the case, the planner will automatically reduce the growth rate of the labor supply in such a way that the net effect is zero.

In earlier investigations of the stochastic Ramsey problem, the labor force is equal or proportional to the population. In our framework this might be a problem. The reason for this is that controlling the labor supply in our model implies that we also control the growth of the population. The latter is of course a difficult and undesirable task, although there is a parallel to birth control policies in fast growing population countries. However, we leave this problem as an open question for further research in this area.

A possible generalization of the model would be to augment the Ramsey economy with prices and money supply as suggested for example in Malliaris and Brock (1982, p. 206). Instead of controlling the growth of labor supply, policies for optimal money supply growth could be computed (or both). This is an ongoing research project.

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Appendix A

In this appendix, we analyze the behavior of the interest rate processes in section 5.1, based on the Feller test of explosions.

Define the interval $I \equiv (l, r)$, and let $X(t)$ be the solution to the diffusion

$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t), \qquad X(0) = x_0 \in I. \tag{38}$$

We assume that b and σ are functions from I to $\mathbb{R}$, such that $\sigma(x) > 0$ for all $x \in I$, and that these functions are locally integrable, see Karatzas and Shreve (1988, p. 343). We use the notations $p(x) \equiv \int_c^x \exp\left(-2\int_c^y \frac{b(z)}{\sigma^2(z)}dz\right)dy, c \in I$, for the scale function, and $m(dx) \equiv \frac{2dx}{p'(x)\sigma^2(x)}$, for the speed measure. We also set $v(x) = \int_c^x (p(x) - p(y))\,m(dy)$. The following results hold.

Proposition 11 *Let the process $X(t)$ be defined as the solution to*

$$dX(t) = \left(aX(t) + bX^2(t)\right)dt + \sigma X(t)\,dW(t), \qquad X(0) = x_0 \in (0, \infty). \tag{39}$$

Then, if $b \le 0$ the boundaries zero and infinity are unattainable, i.e.

$$P(\inf\{t \ge 0 : X(t) \neq (0, \infty)\} = \infty) = 1. \tag{40}$$

If $b > 0$, zero is an unattainable boundary, while infinity is an attainable boundary. In this case, infinity is an absorbing boundary which will be reached in finite time with positive probability. In particular,

$$P\left(\inf\left\{t \geq 0 : X\left(t\right) \neq \left(0, \infty\right)\right\} < \infty\right) = 1, \tag{41}$$

if and only if $a \geq \frac{1}{2}\sigma^2$.

Proof. Straightforward but tedious calculations show that $v\left(0\right) = \infty, v\left(\infty\right) = \infty, b \leq 0$ and $v\left(0\right) = \infty, v\left(\infty\right) < \infty, b > 0$. Using Feller's test of explosions, see Karatzas and Shreve (1988, p. 348), it follows that infinity is an attainable boundary for $b > 0$, while zero is always an unattainable boundary. In the case where $b > 0$, we calculate $\lim_{x\to\infty} \int_c^x m\left(dx\right) = \infty$ from which it follows that infinity is an absorbing boundary, see Karlin and Taylor (1981, p. 234). Finally, since $\lim_{x\to 0} p\left(x\right) = -\infty$ if and only if $a \geq \frac{1}{2}\sigma^2$, the proof concludes using proposition 5.32 in Karatzas and Shreve (1988, p. 350). ■

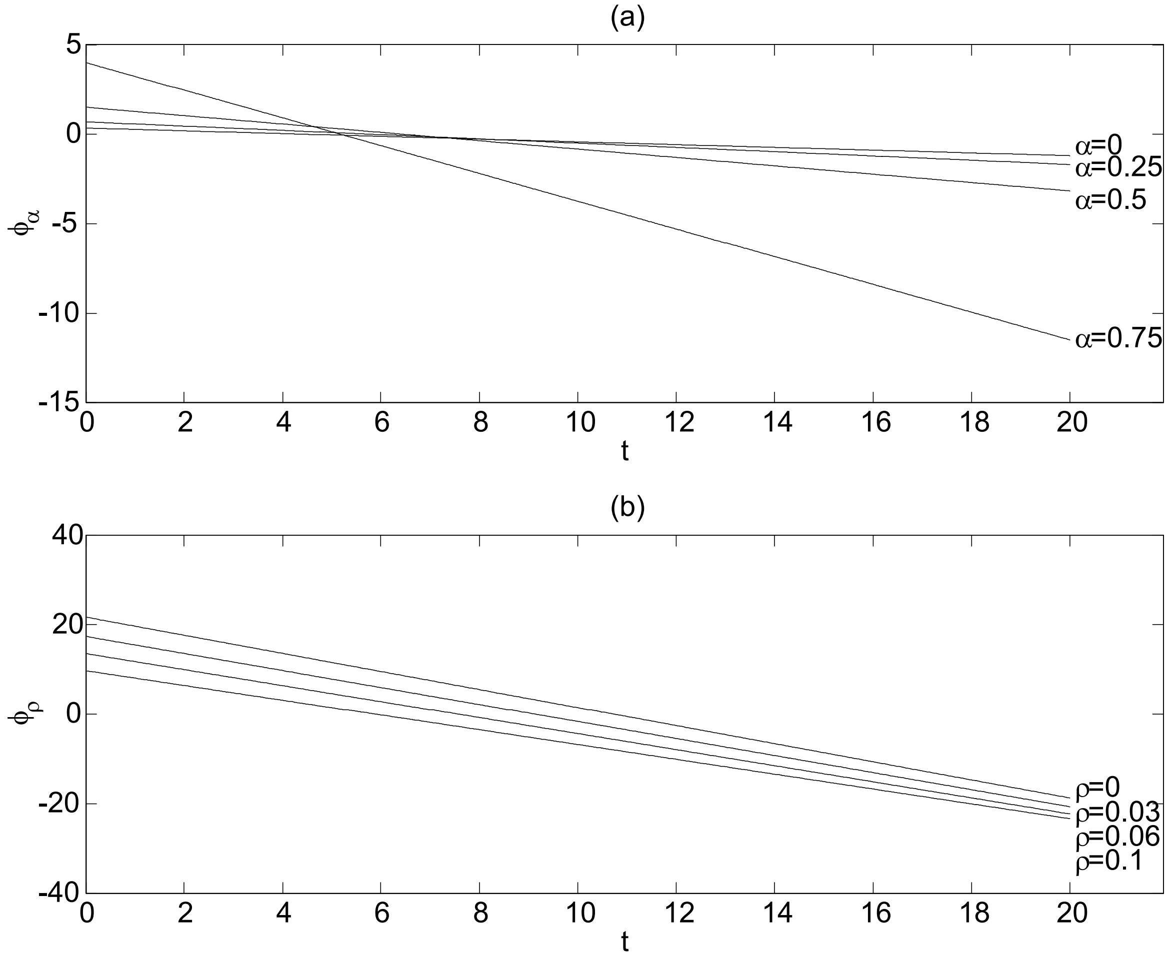

Figure 1: The deterministic functions ϕ_α and ϕ_ρ as defined by the relations $dc(t)/d\alpha = \phi_\alpha(t)\,c(t)$ and $dc(t)/d\rho = \phi_\rho(t)\,c(t)$ with $c(t)$ from (27). The common parameters used are: $k_0 = 100, \sigma = 0.2$, and $T = 20$ years. In (a) $\alpha = 0, 0.25, 0.5, 0.75$ and $\rho = 0.06$. In (b) $\alpha = 0.5$ and $\rho = 0, 0.03, 0.06$, and 0.1.

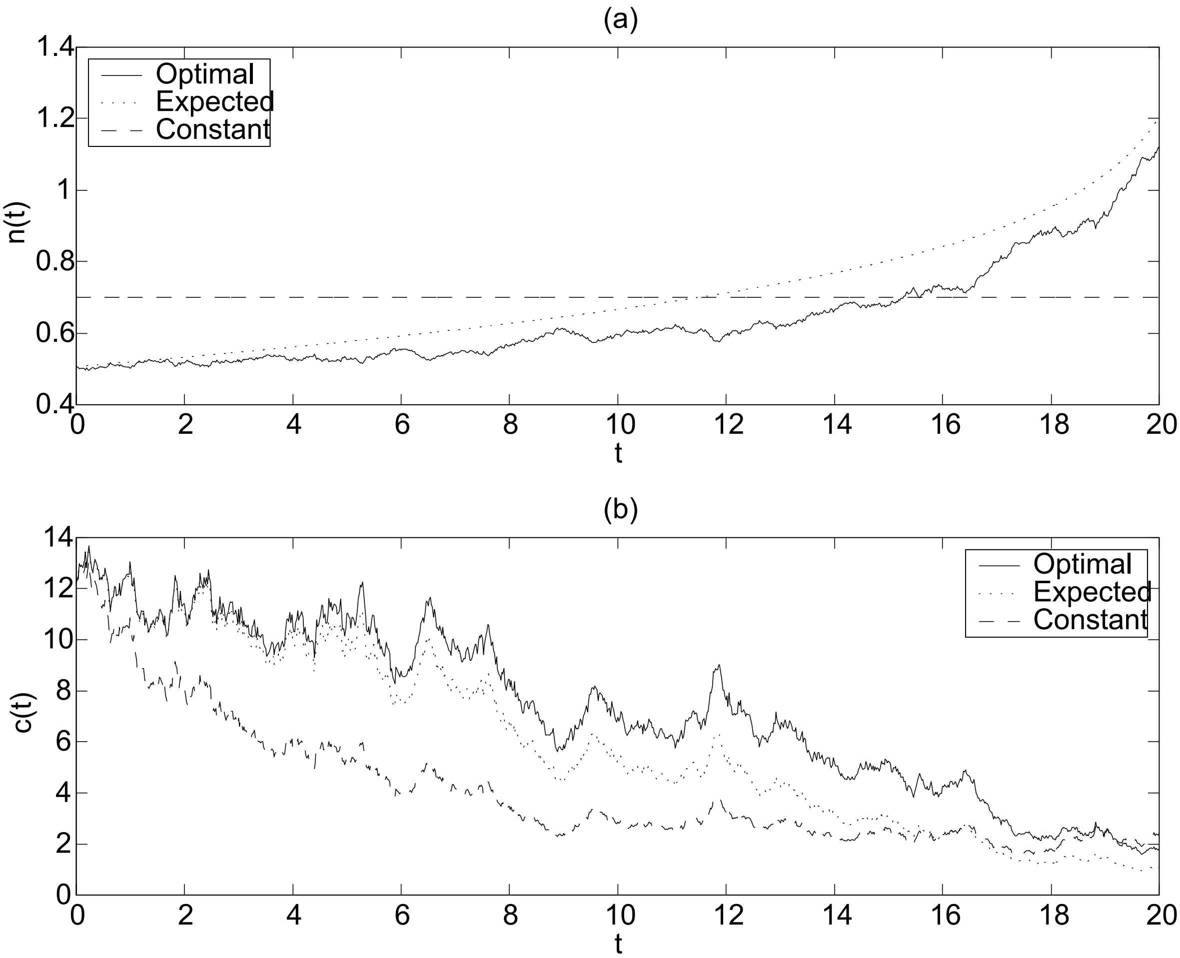

Figure 2: Typical paths of (a) $n(t)$ and (b) $c(t)$ in the case of a power utility, example 5, and a Cobb-Douglas production function $f(x) = x^{\delta}/\delta$. The optimal rule is given by (23), while for the other, $n(t)$ either equals the expected value of the optimal rule, $E[f(k(t))/k(t)] - \lambda + \alpha\sigma^2$, or the constant 0.7. The parameters used for the simulation are: $k_0 = 100, \alpha = 0.5, \delta = 0.8, \lambda = 0.01, \rho = 0.06, \sigma = 0.2$, and $T = 20$ years.

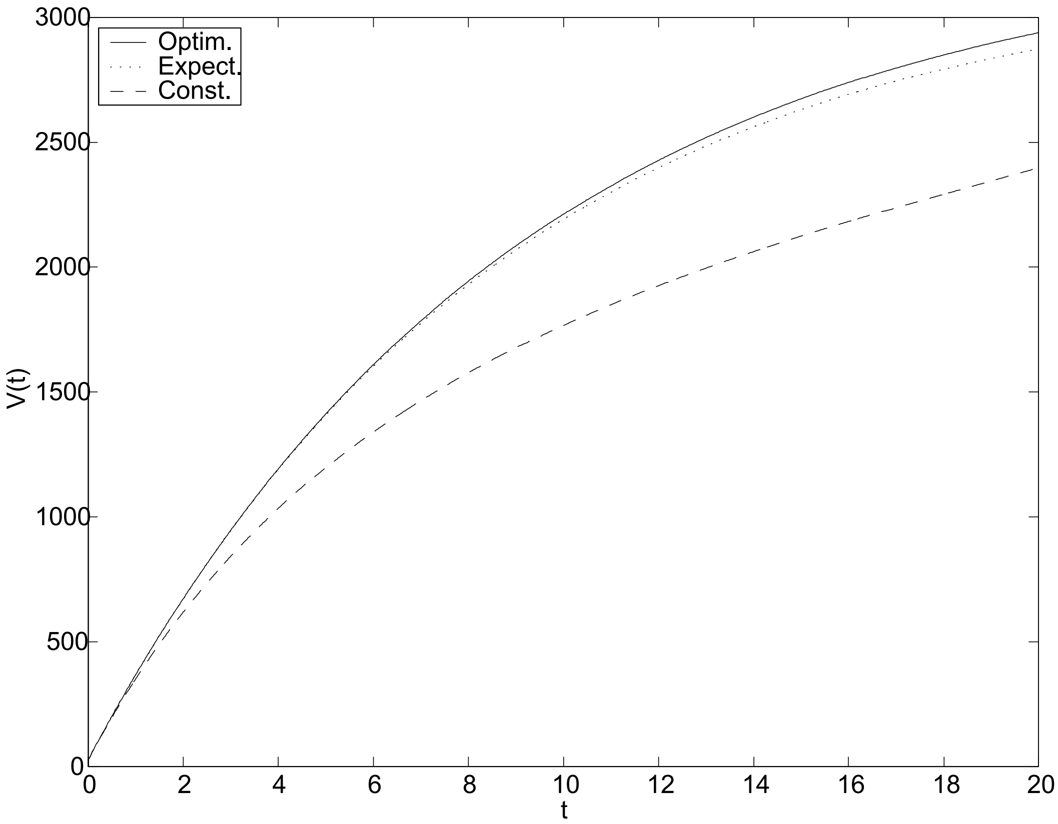

Figure 3: Numerical approximations of $V(t)$ in (28) based on 10000 realizations. The parameters are as in figure 2: $k_0 = 100, \alpha = 0.5, \delta = 0.8, \lambda = 0.01, \rho = 0.06, \sigma = 0.2$, and $T = 20$ years.