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Stochastic technology shocks in an extended Uzawa–Lucas model: closed-form solution and long-run dynamics

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Abstract We add stochastic technological progress, modelled as a geometric Brownian motion with drift, to an augmented Uzawa–Lucas growth model. Under a particular combination of parameters we derive a closed form solution to the model and analytical expressions which show that uncertainty reduces the optimal levels of consumption and increases the proportion of human capital devoted to producing new human capital.

Keywords Economic growth · Physical capital accumulation · Human capital accumulation · Stochastic technological progress

JEL Classification O41 · O33 · C61 · C63 · J24

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1 Introduction

The Uzawa–Lucas model (Uzawa 1965, Lucas 1988) is one of the most celebrated endogenous growth models.[1] It is a two-sector growth model which, in its original formulation, was set in a purely deterministic framework and where the long-run engine of real per-capita income growth is human capital accumulation. The main objective of the present paper is to derive a closed-form solution to a stochastic version of the Uzawa–Lucas model in which we use a more general aggregate production function which postulates that the level of (labor-augmenting) technology follows a geometric Brownian motion with drift. Under a particular combination of parameters, our model subsumes the original deterministic Uzawa–Lucas model as a special case. The aggregate production function we employ resembles the one used by Mankiw et al. (1992, p. 416, Eq. (8)) in their path-breaking work, with two important differences: the first is that we assume, following Uzawa–Lucas, that the production of human capital is an economic activity being relatively intensive in human capital as an input; the second consists of including exogenous stochastic technological progress.

Closed-form solutions of continuous-time models with uncertainty pervade the economics and finance literature [Wälde (2011) provides an incomplete, but still very long, list of such models]. Well known classical examples of stochastic growth models with complete depreciation, logarithmic preferences and Cobb–Douglas technology include Mirman and Zilcha (1975), Danthine and Donaldson (1981), Long and Plosser (1983) and McCallum (1989).[2] More recently, Bethmann (2007) has extended a basic discrete time, stochastic, one-sector growth model with logarithmic preferences and full depreciation of physical capital [that is, the textbook McCallum (1989) real business cycle model] to the case with two capital goods (physical and human capital) and Smith (2007) has provided an analytical solution to the continuous time version of a stochastic growth model with isoelastic utility and Cobb–Douglas technology. With respect to Bethmann (2007), our model differs in that it is set in continuous time, the level of technology evolves according to a geometric Brownian motion (in Bethmann's model the logarithm of total factor productivity follows a first-order autoregressive process), and we do not assume full depreciation of human and physical capital. With respect to Smith (2007), the most significant difference concerns the fact that we add a separate human capital accumulation sector.

In another recent paper, Palokangas (2008) has studied the growth effects of competition within a multi-industry framework in which economic growth is generated by creative destruction and systematic investment risk cannot be eliminated by diversification. His analysis is characterized by the simultaneous presence of three ingredients: imitation, which leads to oligopolistic competition; innovation, which restores a monopoly in the market, and non-diversifiable risk which has two effects: it makes positive profits in the oligopoly-stages of the product-cycle necessary for technological change to occur, while leading to a positive correlation between the intensity of

[1] Boucekkine and Ruiz-Tamarit (2008) illustrate the most interesting properties of this model.

[2] Since the pioneering contribution of Brock and Mirman (1972), the stochastic growth model has gained a central role in economic growth theory. Olson and Roy (2006) provide an excellent recent survey on the topic.

competition and economic growth (provided that the degree of competition exceeds a critical level).[3] In Palokangas (2008), however, there is no human capital investment by agents; his model is more in line with the neo-Schumpeterian growth literature than with the alternative belief that skill-acquisition by forward-looking agents is at the heart of sustainable, long-run growth in real per-capita incomes.

This paper is structured as follows. Section 2 sets out and solves the model, developing the analysis through the Hamilton–Jacobi–Bellman equation. We compare the results of the stochastic model with its deterministic counterpart and illustrate, using numerical simulation, the effects of technology shocks on the optimal policy rules for consumption, the share of human capital employed in goods production and the long run dynamics of physical and human capital stocks. Section 3 discusses the results of the model and suggests ideas for future research. Additional technical material, together with the presentation of a method of solution for a stochastic version of the model in which the instantaneous utility function is logarithmic and the parameter value restrictions which permit a closed form solution are not imposed, is presented in an appendix.

2 The model

The economy is closed. There is a single good which is produced by combining physical capital $K(t)$, human capital $H(t)$ and labor in efficiency units $A(t)L(t)$, where $A(t)$ is the level of technology whose growth is subject to random shocks and $L(t)$ is raw labor. Output $Y(t)$ is allocated either to consumption or to (gross) physical capital investment. We follow Uzawa–Lucas in postulating that the total stock of human capital, $H(t)$, is allocated in proportion $u(t)$ to the production of goods and in proportion $1-u(t)$ to the acquisition of new human capital. The aggregate production function, which is similar to the one used by Mankiw et al. (1992, page 416, Eq. (8)), is therefore:

$$Y(t) = [A(t)L(t)]^{\gamma}[u(t)H(t)]^{\alpha}K(t)^{1-\alpha-\gamma}, \tag{1}$$

with $\gamma \in (0,1), \alpha \in (0,1)$ and $\alpha+\gamma \in (0,1)$. In order to focus on the impact of technology shocks on optimal consumption and the sectoral allocation of human capital, we treat L as being constant over time and equal to one. The production function displays constant returns to scale to the three factor-inputs, jointly considered, and diminishing returns to each of them. Since output is produced under perfectly competitive conditions, each input is remunerated according to its own marginal productivity: γ, α and $1-\alpha-\gamma$ are the shares of income accruing to $A(t)$, $u(t)H(t)$ and $K(t)$, respectively.

When $\gamma = 0$ our model gives rise to that of Uzawa–Lucas with no disembodied technological progress and a constant level of technology (normalized to one). When $\gamma = 1$, aggregate technology is $Y(t) = A(t)(\frac{u(t)H(t)}{K(t)})^{\alpha}$. Following Howitt (1999),

[3] We thank an anonymous referee for bringing to our attention the Palokangas' (2008) paper.

the term $(\frac{u(t)H(t)}{K(t)})^{\alpha}$ can be interpreted as capturing the fact that production tends to become more human capital intensive through time as physical capital accumulates.[4]

The laws of motion of $K(t)$ and $H(t)$, together with their initial conditions, are:

$$\dot{K}(t) = A(t)^{\gamma}[u(t)H(t)]^{\alpha}K(t)^{1-\alpha-\gamma} - \beta_K K(t) - C(t), \quad K(0) = K_0; \tag{2}$$
$$\dot{H}(t) = (\eta(1-u(t)) - \beta_H)H(t), \quad H(0) = H_0. \tag{3}$$

In (2), $C(t)$ is the flow of consumption of the single homogeneous good existing in the economy, and $\beta_K \in [0, 1]$ is the constant rate of depreciation of physical capital. In (3), $\eta \geq 0$ is the productivity of human capital in the production of new human capital and $\beta_H \in [0, 1]$ is the rate of depreciation of human capital. Equation (3) shows that the rate of change of human capital depends solely on the productivity and effort devoted to its accumulation, $\eta(1-u(t))$, and depreciation.

We assume that the level of technology evolves according to the following geometric Brownian motion:

$$dA(t) = \mu A(t)dt + \sigma A(t)dW(t), \quad A(0) = A_0, \tag{4}$$

where $\mu A(t)$ is the drift rate, $\sigma > 0$ is the variance parameter and $dW(t)$ is the increment of a Wiener process such that $E[dW(t)] = 0$ and $\mathrm{var}(dW(t)) = dt$. The deterministic version of the model sets σ to 0.

We assume that the representative individual gets utility from $C(t)$. The optimal inter-temporal decision problem can then be formulated as:

$$\max_{\{C(t),u(t)\}} E\left[\int_0^{+\infty} U(C(t))e^{-\rho t}\,dt\right], \tag{5}$$

where $\rho > 0$ is the rate of time preference, subject to (2), (3) and (4).

With a CIES utility function, we have $U(C(t)) \equiv \frac{C(t)^{1-\phi}-1}{1-\phi}$, where $\phi > 0$. Define $J(H, K, A)$ as the maximum expected value associated with the stochastic optimisation problem. The Hamilton–Jacobi–Bellman (HJB) equation is:

$$\rho J = \max_{C(t),u(t)} \left\{\frac{C(t)^{1-\phi}-1}{1-\phi} + J_K\dot{K}(t) + J_H\dot{H}(t) + J_A\mu A(t) + \frac{J_{AA}\sigma^2 A(t)^2}{2}\right\}, \tag{6}$$

where $\dot{K}(t)$ and $\dot{H}(t)$ are defined in (2) and (3) and subscripts denote partial derivatives of J with respect to the relevant variables of interest.

Dropping the ts for clarity, the optimal policy rules for the control variables are found by solving the derivatives of (6) with respect to C and u, postulating a suitable form for the value function and applying the transversality condition associated with

[4] In other words, for given A, the economy would need to accumulate human capital $u(t)H(t)$ when physical capital $K(t)$ increases in order for economic growth to be sustainable in the long run.

the verification theorem (Chang 2004). Differentiating (6) with respect to the control variables gives:

$$C = J_K^{-\frac{1}{\phi}}, \tag{7}$$

$$u = \frac{K^{\frac{1-\gamma-\alpha}{1-\alpha}}}{H}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}}. \tag{8}$$

Substituting these equations back into (6) gives:

$$\begin{aligned}
0 &= \frac{J_K^{-\frac{1-\phi}{\phi}}-1}{1-\phi} - \rho J + J_K\left(A^{\gamma} K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}-1} - \beta_K K - J_K^{-\frac{1}{\phi}}\right)\\
&\quad + J_H\left(\eta H - \eta K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}} - \beta_H H\right) + J_A \mu A + \frac{J_{AA}\sigma^2 A^2}{2}\\
&= \left(\frac{\phi}{1-\phi}\right) J_K^{1-\frac{1}{\phi}} - \frac{1}{1-\phi} - \rho J + J_K\left(K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}}\left[\frac{\eta J_H}{\alpha J_K}\right] - \beta_K K\right)\\
&\quad + J_H\left(\eta H - \eta K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}} - \beta_H H\right) + J_A \mu A + \frac{J_{AA}\sigma^2 A^2}{2}.
\end{aligned} \tag{9}$$

We postulate a value function separable in the state variables of the problem:

$$J(H, K, A) = T_H H^{\theta_1} + T_K K^{\theta_2} + T_A A^{\theta_3} + T_J, \tag{10}$$

where T_H, T_K, T_A and T_J are constant parameters. We have the following result which shows that a closed form solution to the problem exists under a particular combination of parameter values.

Theorem 1 *Assume that $\phi = 1 - \alpha - \gamma$ and $\eta = \rho + \beta_H$. Then the solution to (6) is given by*:

$$J(H, K, A) = T_H H + T_K K^{\alpha+\gamma} + T_A A^{\frac{\gamma}{1-\alpha}} + T_J, \tag{11}$$

where:

$$T_H = T_K\left[\frac{\left(-\rho + \frac{\mu\gamma}{1-\alpha} + \frac{\sigma^2}{2}\frac{\gamma}{1-\alpha}\left(\frac{\gamma}{1-\alpha}-1\right)\right) H_0}{-A_0^{\frac{\gamma}{1-\alpha}}\eta}\right]^{\alpha-1}\frac{\alpha(\alpha+\gamma)}{\eta} \tag{12}$$

$$T_K = \left[\frac{\left(\frac{1-\alpha-\gamma}{\alpha+\gamma}\right)\left[(\alpha+\gamma)\right]^{\frac{-\alpha-\gamma}{1-\alpha-\gamma}}}{\rho + (\gamma+\alpha)\beta_K}\right]^{1-\alpha-\gamma} \tag{13}$$

$$T_A = \frac{T_H\left(\eta - \frac{\eta}{\alpha}\right)\left[\frac{\alpha(\alpha+\gamma)T_K}{\eta T_H}\right]^{\frac{1}{1-\alpha}}}{-\rho + \frac{\gamma}{1-\alpha}\mu + \frac{\frac{\gamma}{1-\alpha}\left(\frac{\gamma}{1-\alpha}-1\right)\sigma^2}{2}} \quad \text{and} \tag{14}$$

$$T_J = -\frac{1}{\rho(1-\phi)}. \tag{15}$$

The optimal rules for the levels of the control variables are given by:

$$C = \frac{K}{\left[(\alpha+\gamma)T_K\right]^{\frac{1}{1-\alpha-\gamma}}}, \quad u = \frac{A^{\frac{\gamma}{1-\alpha}}}{H}\left[\frac{\alpha(\alpha+\gamma)T_K}{\eta T_H}\right]^{\frac{1}{1-\alpha}}. \tag{16}$$

The optimal levels of the state variables are:

$$K(t) = e^{-\Omega_2 t}\left[\Omega_1(\alpha+\gamma)\int_0^t e^{\Omega_2(\alpha+\gamma)s}A(s)^{\frac{\gamma}{1-\alpha}}\,ds + K_0^{\alpha+\gamma}\right]^{\frac{1}{\alpha+\gamma}} \tag{17}$$

where:

$$\Omega_1 = \left[\frac{\alpha(\alpha+\gamma)T_K}{\eta T_H}\right]^{\frac{\alpha}{1-\alpha}} \quad \textit{and} \quad \Omega_2 = \beta_K + \frac{1}{\left[(\alpha+\gamma)T_K\right]^{\frac{1}{1-\alpha-\gamma}}}$$

and:

$$H(t) = e^{(\eta-\beta_H)t}\left[H_0 - \eta\Gamma\int_0^t e^{-(\eta-\beta_H)s}A(s)^{\frac{\gamma}{1-\alpha}}\,ds\right], \tag{18}$$

where $\Gamma = \left[\alpha(\alpha+\gamma)T_K/\eta T_H\right]^{\frac{1}{1-\alpha}}$.

Proof See Appendix A.

Theorem 1 provides a result similar to that of the AK-model [see Barro and Sala-i-Martin, 2004, p. 208, Eq. (4.15)] in that, for all t, there exists a linear relationship between the optimal level of consumption and capital.

The theorem also shows that the optimal levels of the state variables K and H are functions of A, a random variable whose equation of motion is defined by (4). Setting $\sigma = 0$ in (4) [and hence in (17) and (18)] yields the levels of the state variables in the deterministic version of the model, from which the levels of the control variables in the deterministic model may be derived [via (7) and (8)]. By making reference to Jensen's inequality for a strictly concave function of $A(t)$, we are able to compare the expected levels of the state variables in the stochastic version of the model with those of the deterministic version.

Consider first the stochastic model. Finding the expressions for $E[K(t)]$ and $E[H(t)]$ using (17) and (18) involves taking the expectation of the integrand, which includes a strictly concave function of $A(t)$, since the restrictions on α and γ imply that $0 < \gamma/(1-\alpha) < 1$. Referring to (22), when $\sigma > 0$, $e^{\frac{\sigma^2 t}{2}(\xi^2-\xi)} < 1$, this result confirms Jensen's inequality for the random process $A(t)$ under a strictly concave transformation, that is, $E[A(t)^{\xi}] < (E[A(t)])^{\xi}$. Noting that, in the deterministic model, $E[A(t)] = A_D(t)$, where the subscript denotes the deterministic result, it follows that $E[A(t)^{\frac{\gamma}{1-\alpha}}] < (A_D(t))^{\frac{\gamma}{1-\alpha}}$. This allows us to state the results in Proposition 1, which contrast the optimal levels of state and control variables in the stochastic and deterministic versions of the model.

Proposition 1 *Under the assumptions of Theorem 1, we have, for all* $t = 0, \ldots, \infty$:

$$\begin{aligned} E[A(t)^{\frac{\gamma}{1-\alpha}}] &\leq (A_D(t))^{\frac{\gamma}{1-\alpha}}, \\ E[K^{\gamma+\alpha}(t)] &\leq (K_D(t))^{\gamma+\alpha} \\ E[H(t)] &\geq H_D(t), \\ E[C(t)^{\gamma+\alpha}] &\leq (C_D(t))^{\gamma+\alpha} \end{aligned} \tag{19}$$

Proof See Appendix B.

It is quite difficult to obtain a similar result comparing the deterministic path of u_D with that expected from the stochastic model due to the fact that the expression for u depends on the random variables A and H. However, as Fig. 1 shows, a simulation imposing the parameter value restrictions of Theorem 1 illustrates that the optimal expected value of u is lower in the stochastic model than it is in the deterministic model (the parameter values chosen are: $\alpha = 1/3, \gamma = 1/3, \rho = 4/100, \beta_H = 5/100, \beta_K = 5/100, \mu = 2/100, \eta = 9/100, \sigma = 0$ (deterministic model) and $\sigma = 0.148$ (stochastic model) and we compared the deterministic paths with the averages obtained over 1,000 iterations of discretised versions of the stochastic differential equations).[5] The simulation confirms the results of proposition 1, that is, that the stochastic technological progress, modelled as a geometric Brownian motion with drift, reduces the expected optimal levels of C and K and increases H. The simulation also shows that it reduces the optimal expected level of u.

Since all of the above results are derived under a particular restriction on the parameter values of the model, an interesting question is whether the results are generalisable to cases when these restrictions do not hold. Appendix C shows that, for the case of a logarithmic instantaneous utility function, although it is difficult to obtain a closed form solution, through the 'guess and verify' method it is possible to split the HJB partial differential equation into nonlinear separated differential equations which can be solved through numerical simulation.

[5] According to Mankiw et al. (1992, p.432), with an aggregate production function employing physical capital, human capital and labor in efficiency units as inputs, the shares of GDP going to physical and human capital both equal 1/3. In our model this would imply $\alpha = 1/3$ and $1 - \alpha - \gamma = 1/3$ leading, in turn, to $\gamma = 1/3$. Moreover, Mankiw et al. (1992, footnote 6, p. 414, and p. 430) also suggest a value of 2% for the exogenous growth rate of A in the absence of shocks (in our model the parameter μ). Finally, the values of β_H and β_K (0.05), ρ (0.04) are those of Mulligan and Sala-i-Martin (1993, p. 761). We set $\sigma = 0.148$ to show clearly the difference between the results of the stochastic model and those of the deterministic version.

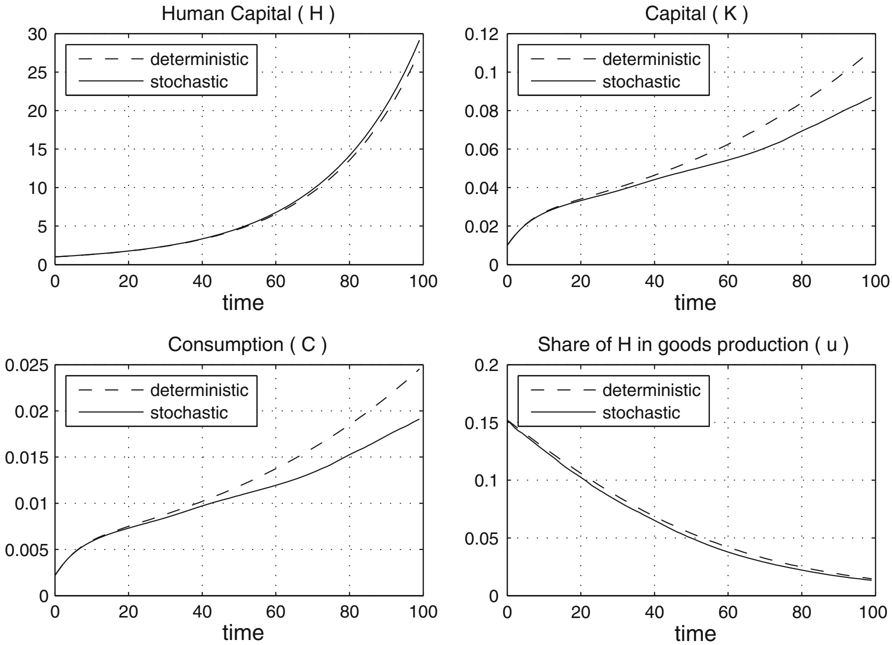

Fig. 1 Comparison of simulations for the deterministic and stochastic versions of the model with CIES utility

3 Discussion

Since uncertainty about the evolution of the level of technology is an important feature of real world decision-making, analyzing the main predictions of stochastic extensions to deterministic growth models can provide important insights into how uncertainty affects optimal policy rules. Our augmentation of the original Uzawa–Lucas model using a CIES utility function, including labor in efficiency-units and using a geometric Brownian motion for the evolution of technology, allows us to derive closed-form solutions for most of the policy variables under a particular combination of parameter values. These predict that uncertainty on the level of technology, on average, reduces the optimal levels of consumption, the share of human capital devoted to goods production (a numerical simulation result) and the stock of physical capital, while increasing the stock of human capital. These results can be explained as follows. More uncertainty suggests, on average, more savings for the future, which is compatible with a reduction in current consumption. As in the standard AK-model (where C and K are always linearly and positively related to each other), the decline in consumption reduces K as well. However, unlike the one-sector endogenous growth model (the AK-model), in order to prevent the adverse effects on total output Y of possible further negative technological shocks, agents decide to increase their own investment in human capital.

A natural question to ask is what happens when the above-mentioned combination of parameter values does not hold. In the case of a logarithmic instantaneous utility function, one can solve numerically the HJB partial differential equation and provide numerical approximations to the optimal paths. We present this analysis in Appendix C to this paper.

For future research, it would be interesting to analyze how the results of this paper would change in the presence of endogenous, rather than exogenous, stochastic technological progress driven by human capital [see, for instance, La Torre and Marsiglio (2010)]. In this case, skilled labor (human capital) would be employed not only to produce goods and to accumulate new human capital, but also to invent new ideas.

Acknowledgments Simulations are carried out using Matlab 7.9.0 and Maple 13. We thank three anonymous referees and Prof. K. Wälde for comments on a previous version of this paper. This work has been partially written during a research visit by DLT and CC to the Department of Economics and Related Studies of the University of York (UK). DLT and CC thank MF for this possibility.

Appendix A: Proof of Theorem 1

From (10) we have:

$$J_H = \theta_1 T_H H^{\theta_1 - 1}, \quad J_K = \theta_2 T_K K^{\theta_2 - 1},$$
$$J_A = \theta_3 T_A A^{\theta_3 - 1}, \quad J_{AA} = \theta_3 T_A (\theta_3 - 1) A^{\theta_3 - 2}$$

Substituting these into (6), noting (7) and (8), gives:

$$\begin{aligned}
0 = & \left(\frac{\phi}{1-\phi}\right)\left[\theta_2 T_K K^{\theta_2 - 1}\right]^{1-\frac{1}{\phi}} - \frac{1}{1-\phi} - \rho\left(T_H H^{\theta_1} + T_K K^{\theta_2} + T_A A^{\theta_3} + T_J\right) \\
& + \left(\frac{\eta}{\alpha} - \eta\right) K^{\frac{1-\gamma-\alpha}{1-\alpha}} \left[\frac{\alpha A^{\gamma} \theta_2 T_K K^{\theta_2 - 1}}{\theta_1 \eta T_H H^{\theta_1 - 1}}\right]^{\frac{1}{1-\alpha}} \theta_1 T_H H^{\theta_1 - 1} \\
& - \theta_2 T_K K^{\theta_2 - 1} \beta_K K + \theta_1 T_H H^{\theta_1 - 1} H\,(\eta - \beta_H) + \theta_3 T_A A^{\theta_3 - 1} \mu A \\
& + \frac{\theta_3(\theta_3 - 1) T_A A^{\theta_3 - 2} \sigma^2 A^2}{2},
\end{aligned}$$

that is:

$$\begin{aligned}
0 = & \left(\frac{\phi}{1-\phi}\right)\left[\theta_2 T_K\right]^{1-\frac{1}{\phi}} K^{\frac{(\theta_2 - 1)(\phi - 1)}{\phi}} - \frac{1}{1-\phi} - \rho\left(T_H H^{\theta_1} + T_K K^{\theta_2} + T_A A^{\theta_3} + T_J\right) \\
& + \left(\frac{\eta}{\alpha} - \eta\right) \theta_1 T_H \left[\frac{\alpha \theta_2 T_K}{\theta_1 \eta T_H}\right]^{\frac{1}{1-\alpha}} H^{\theta_1 - 1} K^{\frac{1-\gamma-\alpha}{1-\alpha}} A^{\frac{\gamma}{1-\alpha}} K^{\frac{\theta_2 - 1}{1-\alpha}} H^{\frac{1-\theta_1}{1-\alpha}} \\
& - \theta_2 T_K \beta_K K^{\theta_2} + \theta_1 T_H (\eta - \beta_H) H^{\theta_1} + \theta_3 T_A \mu A^{\theta_3} \\
& + \frac{\theta_3(\theta_3 - 1) T_A \sigma^2}{2} A^{\theta_3}.
\end{aligned}$$

Letting $\theta_1 = 1, \theta_2 = \alpha + \gamma, \theta_3 = \frac{\gamma}{1-\alpha}$ and $T_J = -\frac{1}{\rho(1-\phi)}$ ((15)) gives:

$$\begin{aligned}
0 = & \left(\frac{\phi}{1-\phi}\right) \left[(\alpha+\gamma)T_K\right]^{1-\frac{1}{\phi}} K^{\frac{(\alpha+\gamma-1)(\phi-1)}{\phi}} \\
& -\rho \left(T_H H + T_K K^{\alpha+\gamma} + T_A A^{\frac{\gamma}{1-\alpha}}\right) \\
& + \left(\frac{\eta}{\alpha} - \eta\right) T_H \left[\frac{\alpha(\alpha+\gamma)T_K}{\eta T_H}\right]^{\frac{1}{1-\alpha}} A^{\frac{\gamma}{1-\alpha}} \\
& -(\gamma+\alpha)T_K \beta_K K^{\gamma+\alpha} + T_H (\eta - \beta_H) H + \frac{\gamma}{1-\alpha} \mu T_A A^{\frac{\gamma}{1-\alpha}} \\
& + \frac{\frac{\gamma}{1-\alpha}\left(\frac{\gamma}{1-\alpha} - 1\right) T_A \sigma^2}{2} A^{\frac{\gamma}{1-\alpha}}.
\end{aligned}$$

If $\phi = 1 - \alpha - \gamma$ then:

$$\begin{aligned}
0 = & \left[\left(\frac{1-\alpha-\gamma}{\alpha+\gamma}\right) \left[(\alpha+\gamma)\right]^{\frac{-\alpha-\gamma}{1-\alpha-\gamma}} T_K^{-\frac{1}{1-\alpha-\gamma}} - \rho - (\gamma+\alpha)\beta_K\right] K^{\alpha+\gamma} T_K \\
& + (\eta - \rho - \beta_H) T_H H \\
& + \left[-\rho T_A + \left(\frac{\eta}{\alpha} - \eta\right) T_H \left[\frac{\alpha(\alpha+\gamma)T_K}{\eta T_H}\right]^{\frac{1}{1-\alpha}} + \frac{\gamma}{1-\alpha} T_A \mu \right. \\
& \left. + \frac{\frac{\gamma}{1-\alpha}\left(\frac{\gamma}{1-\alpha} - 1\right) T_A \sigma^2}{2}\right] A^{\frac{\gamma}{1-\alpha}}.
\end{aligned}$$

Since $\eta - \rho - \beta_H = 0$, the second term in this summation disappears. The equation then must be satisfied for all values of K and A, implying that the remaining two expressions in parentheses must equal zero. Equations (13) and (14) are obtained by setting the first and third terms in parentheses equal to zero.

We now obtain the first derivatives of (11) with respect to K and H and substitute them into (7) and (8), thereby deriving the optimal policy rules for C and u in (16).

Substituting the expressions for these policy rules into the constraints gives:

$$\dot{K}(t) = \Omega_1 A(t)^{\frac{\gamma}{1-\alpha}} K(t)^{1-\gamma-\alpha} - \Omega_2 K(t), \tag{20}$$

$$\dot{H}(t) = (\eta - \beta_H) H(t) - \eta \Gamma A(t)^{\frac{\gamma}{1-\alpha}}, \tag{21}$$

where Γ is defined in Theorem 1, which allows us to obtain the solutions for K and H in (17) and (18).

In order to obtain (12), thereby fully determining the system, we use the verification theorem (Chang 2004) which requires that $\lim_{t\to+\infty} e^{-\rho t} E[J(H(t), K(t), A(t))] = 0$. Replacing J in this expression with the right hand side of (11) and noting that:

$$E[A(t)^{\xi}] = (E[A(t)])^{\xi} e^{\frac{\sigma^2 t}{2}(\xi^2 - \xi)}, \tag{22}$$

where $\xi = \gamma/(1-\alpha)$, $0 < \xi < 1$, implies that $e^{-\rho t}E[K^{\alpha+\gamma}]$ and $e^{-\rho t}E[A^{\frac{\gamma}{1-\alpha}}]$ tend to zero as $t \to +\infty$ if the following condition holds:[6]

$$\rho > \frac{\gamma}{1-\alpha}\left(\mu + \frac{\sigma^2}{2}\left(\frac{\gamma}{1-\alpha} - 1\right)\right). \tag{24}$$

Recalling that $\rho = \eta - \beta_H$, the term $e^{-\rho t}E[H]$ tends to zero as $t \to +\infty$ when:

$$\lim_{t\to+\infty} H_0 - \eta\Gamma \int_0^t e^{-(\eta-\beta_H)s} E\left(A(s)^{\frac{\gamma}{1-\alpha}}\right) ds = 0. \tag{25}$$

Solving explicitly (25) with respect to the ratio $\frac{T_K}{T_H}$ one obtains:

$$\int_0^{+\infty} e^{-\rho s} E\left[A(s)^{\frac{\gamma}{1-\alpha}}\right] ds = \frac{H_0}{\eta\Gamma},$$

which may be solved to yield (12).

Appendix B: Proof of Proposition 1

Expressions for A, H and C are straightforward. For K, by Jensen's inequality we obtain:

$$\begin{aligned} E[K(t)^{\gamma+\alpha}] &= e^{-\Omega_2 t}\left[\Omega_1(\alpha+\gamma)\int_0^t e^{\Omega_2(\alpha+\gamma)s} E[A(s)^{\frac{\gamma}{1-\alpha}}]\, ds + K_0^{\alpha+\gamma}\right] \\ &\le e^{-\Omega_2 t}\left[\Omega_1(\alpha+\gamma)\int_0^t e^{\Omega_2(\alpha+\gamma)s} (E[A(s)])^{\frac{\gamma}{1-\alpha}}\, ds + K_0^{\alpha+\gamma}\right] = (K_D(t))^{\gamma+\alpha} \end{aligned}$$

[6] Given an arbitrary, deterministic, starting value $A(0)$, (4) has the analytical solution:

$$A(t) = A(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}. \tag{23}$$

Raising $A(t)$ to the power ξ and taking expectations gives:

$$E[A(t)^{\xi}] = E\left[A(0)^{\xi} e^{\left(\mu - \frac{\sigma^2}{2}\right)\xi t + \sigma\xi W_t}\right].$$

We can isolate $A(0)e^{\mu t} = A(t)$ on the right hand side of this expression and note that $E[e^{\sigma\xi W_t}] = e^{\frac{\sigma^2\xi^2}{2}t}$, which is (22).

Appendix C: A numerical method for the case $u(c) = \ln(c)$

In this section we focus on the case of a logarithmic instantaneous utility function to illustrate a numerical solution to the Bellman equation without the parameter value restrictions used to derive Theorem 1 (logarithmic utility being a particular case of the CIES utility function when ϕ tends to one). The first order conditions associated with this problem are (7) with $\phi = 1$ and (8). By substituting these into (6) we get:

$$\begin{aligned} 0 = -\ln(J_K) - \rho J + J_K\left(A^{\gamma} K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}-1} - \beta_K K\right) - 1 \\ + J_H\left(\eta H - \eta K^{\frac{1-\gamma-\alpha}{1-\alpha}}\left[\frac{\alpha A^{\gamma} J_K}{\eta J_H}\right]^{\frac{1}{1-\alpha}} - \beta_H H\right) + J_A \mu A + \frac{J_{AA}\sigma^2 A^2}{2} \end{aligned} \tag{26}$$

We now look for a solution of this form:

$$J(H(t), K(t), A(t)) = f\left(\frac{A(t)^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}(t)}{H(t)}\right) + \frac{\alpha}{\rho(\gamma+\alpha)} \ln(H(t)) + g(A(t)), \tag{27}$$

where f and g are two unknown functions to be determined. By computing:

$$\begin{aligned} J_H &= -f'\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H^2} + \frac{\alpha}{\rho(\gamma+\alpha)H}, \\ J_K &= \left(\frac{\gamma+\alpha}{\alpha}\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma}{\alpha}}}{H}, \\ J_A &= -\frac{\gamma}{\alpha} f'\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}-1} K^{\frac{\gamma+\alpha}{\alpha}}}{H} + g'(A), \\ J_{AA} &= \left(\frac{\gamma}{\alpha}\right)^2 f''\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\left(\frac{A^{-\frac{\gamma}{\alpha}-1} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)^2 \\ &\quad -\frac{\gamma}{\alpha}\left(-\frac{\gamma}{\alpha} - 1\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}-2} K^{\frac{\gamma+\alpha}{\alpha}}}{H} + g''(A) \end{aligned}$$

and substituting these into (26) we obtain:

$$\begin{aligned} 0 = &-\ln\left(\left(\frac{\gamma+\alpha}{\gamma+\alpha-\gamma}\right) f'\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma}{\alpha}}}{H}\right) \\ &-\rho\left(f\left(\frac{A^{-\frac{\gamma}{\alpha}} K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right) + \frac{\alpha}{\rho(\gamma+\alpha)} \ln(H) + g(A)\right) \end{aligned}$$

$$+J_K K\left(\left[\frac{\eta^{-1}(\gamma+\alpha)f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)}{-f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}+\frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1}-\beta_K\right)-1$$

$$+J_H H\left(\eta-\eta\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\left[\frac{\eta^{-1}(\gamma+\alpha)f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)}{-f'\left(\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}\right)\frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}+\frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}}-\beta_H\right)$$

$$+J_A\mu A+\frac{J_{AA}\sigma^2 A^2}{2}.$$

Now, let $x \equiv \frac{A^{-\frac{\gamma}{\alpha}}K^{\frac{\gamma+\alpha}{\alpha}}}{H}$ and $y \equiv A$. We get:

$$\begin{aligned}0 &= -\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)\right)-\rho f(x)-\frac{\gamma}{(\gamma+\alpha)}\ln(x)\\&\quad+\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)\,x\left(\left[\frac{\eta^{-1}(\gamma+\alpha)f'(x)}{-f'(x)\,x+\frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1}-\beta_K\right)-1\\&\quad+\left(-f'(x)\,x+\frac{\alpha}{\rho(\gamma+\alpha)}\right)\left(\eta-\eta x\left[\frac{\eta^{-1}(\gamma+\alpha)f'(x)}{-f'(x)\,x+\frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}}-\beta_H\right)\\&\quad-\frac{\gamma\mu}{\alpha}f'(x)\,x+\left(\frac{\sigma\gamma}{\sqrt{2}\alpha}\right)^2 f''(x)\,x^2-\frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right)f'(x)\,x\\&\quad+\left(\frac{\gamma}{\gamma+\alpha}\right)\ln(y)-\rho g(y)+g'(y)\mu y+\frac{g''(y)\sigma^2y^2}{2}.\end{aligned}\tag{28}$$

(26) can be split into the following ordinary differential equations:

$$\begin{aligned}&-\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)\right)-\rho f(x)-\frac{\gamma}{(\gamma+\alpha)}\ln(x)\\&+(\gamma+\alpha)\left(\frac{1}{\alpha}-1\right)f'(x)\,x\left(\left[\frac{\eta^{-1}(\gamma+\alpha)f'(x)}{-f'(x)\,x+\frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1}\right)\\&-\beta_K\left(\frac{\gamma+\alpha}{\alpha}\right)f'(x)\,x-1+\left(-f'(x)\,x+\frac{\alpha}{\rho(\gamma+\alpha)}\right)(\eta-\beta_H),\\&-\frac{\gamma\mu}{\alpha}f'(x)\,x+\left(\frac{\sigma\gamma}{\sqrt{2}\alpha}\right)^2 f''(x)\,x^2-\frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right)f'(x)\,x=0,\end{aligned}\tag{29}$$

and

$$\left(\frac{\gamma}{\gamma+\alpha}\right)\ln(y)-\rho g(y)+g'(y)\mu y+\frac{g''(y)\sigma^2y^2}{2}=0,\tag{30}$$

It follows that the values of C/K and u can be rewritten in terms of f and x as follows:

$$\frac{C}{K} = \frac{\alpha}{(\gamma+\alpha)\,x f'(x)} \quad \text{and} \quad u = \left[\frac{\eta^{-1}(\gamma+\alpha)x^{1-\alpha} f'(x)}{\left(\frac{\alpha}{\rho(\gamma+\alpha)} - x f'(x)\right)}\right]^{\frac{1}{1-\alpha}}.$$

Deriving explicit expressions for these control variables requires knowlege of f. Equation (29) can be written in normal form as:

$$f''(x) = \Gamma(x, f(x), f'(x)), \tag{31}$$

where:

$$\begin{aligned}\Gamma(x, f(x), f'(x)) := \frac{1}{-\left(\frac{\sigma\gamma}{\sqrt{2}\alpha}\right)^2 x^2}\Bigg[& -\ln\left(\left(\frac{\gamma+\alpha}{\alpha}\right) f'(x)\right) - \rho f(x) \\ & -\frac{\gamma}{(\gamma+\alpha)}\ln(x) + (\gamma+\alpha)\left(\frac{1}{\alpha}-1\right) f'(x)\, x \left(\left[\frac{\eta^{-1}(\gamma+\alpha) f'(x)}{-f'(x)\,x + \frac{\alpha}{\rho(\gamma+\alpha)}}\right]^{\frac{1}{1-\alpha}-1}\right) \\ & -\beta_K\left(\frac{\gamma+\alpha}{\alpha}\right) f'(x)\,x - 1 + \left(-f'(x)\,x + \frac{\alpha}{\rho(\gamma+\alpha)}\right)(\eta-\beta_H) - \frac{\gamma\mu}{\alpha} f'(x)\,x \\ & -\frac{\sigma^2\gamma}{2\alpha}\left(-\frac{\gamma}{\alpha}-1\right) f'(x)\,x\Bigg].\end{aligned}$$

Assume the initial condition $x_0 > 0$ such that $f_0' > 0$ and $x_0 f_0' < \frac{\alpha}{\rho(\gamma+\alpha)}$. The following result states a sufficient condition which guarantees the existence and uniqueness of a solution to (31). The proof easily follows by using standard differential equation arguments and by noting that Γ is a C^∞ function in a neighborhood of (x_0, f_0, f_0').

Proposition 2 *Given the above assumptions on x_0 and f_0', the Cauchy problem*:

$$\begin{cases} f''(x) = \Gamma(x, f(x), f'(x)), \quad f(x_0) = f_0, \\ f'(x_0) = f_0', \end{cases} \tag{32}$$

has a unique local solution.

Concerning (30), we have the following result:

Proposition 3 *The general solution to* (30) *is given by*:

$$g(y) = \left(\frac{\gamma}{\rho\gamma+\alpha}\right)\ln(y) + \left(\frac{\gamma}{\rho\gamma+\alpha}\right)\left[\mu - \frac{\sigma^2}{2}\right] + D_1 y^{\varrho_1} + D_2 y^{\varrho_2},$$

where D_1 are D_2 are arbitrary constants and $\varrho_1 > 0$ and $\varrho_2 < 0$ are the solutions to the following second order equation:

$$-\rho + \varrho\mu + \frac{\varrho(\varrho - 1)\sigma^2}{2} = 0.$$

Proof To find the complementary solutions, consider first the homogeneous equation

$$-\rho g(y) + g'(y)\mu y + \frac{g''(y)\sigma^2 y^2}{2} = 0 \tag{33}$$

Conjecture the solution $g(y) = y^{\varrho}$, where ϱ is a constant to be determined. Using this conjecture, Eq. (33) reduces to:

$$y^{\varrho}\left[-\rho + \varrho\mu + \frac{\varrho(\varrho - 1)\sigma^2}{2}\right] = 0 \tag{34}$$

The quadratic equation in brackets yields two roots ϱ_1, ϱ_2, one of which is positive and the other negative. To find the particular solution, conjecture:

$$g(y) = \lambda_0 \ln(y) + \lambda_1,$$

where λ_0 and λ_1 are constants to be determined. Using this conjecture, (33) becomes:

$$\left(\frac{\gamma}{\gamma + \alpha}\right)\ln(y) - \rho\left[\lambda_0 \ln(y) + \lambda_1\right] + \lambda_0\mu - \frac{\sigma^2}{2} = 0.$$

Collecting like terms reveals that:

$$\lambda_0 = \left(\frac{\gamma}{\rho\gamma + \alpha}\right)$$
$$\lambda_1 = \frac{\lambda_0}{\rho}[\mu - \frac{\sigma^2}{2}].$$

So the general solution to (33) should be:

$$g(y) = \left(\frac{\gamma}{\rho\gamma + \alpha}\right)\ln(y) + \left(\frac{\gamma}{\rho\gamma + \alpha}\right)\left[\mu - \frac{\sigma^2}{2}\right] + D_1 y^{\varrho_1} + D_2 y^{\varrho_2},$$

where D_1 are D_2 are arbitrary constants. □

The transversality condition requires that $\lim_{t\to+\infty} e^{-\rho t} E[J(H(t), K(t), A(t))] = 0$ which leads to the conclusion that $D_1 = D_2 = 0$. It is quite difficult to find a closed form solution to (32), so we proceed to provide a numerical solution, using the same parameter values as in Sect. 2. The behavior of $f(x)$, $xf'(x)$ and $u(x)$ against x are shown in Fig. 2.

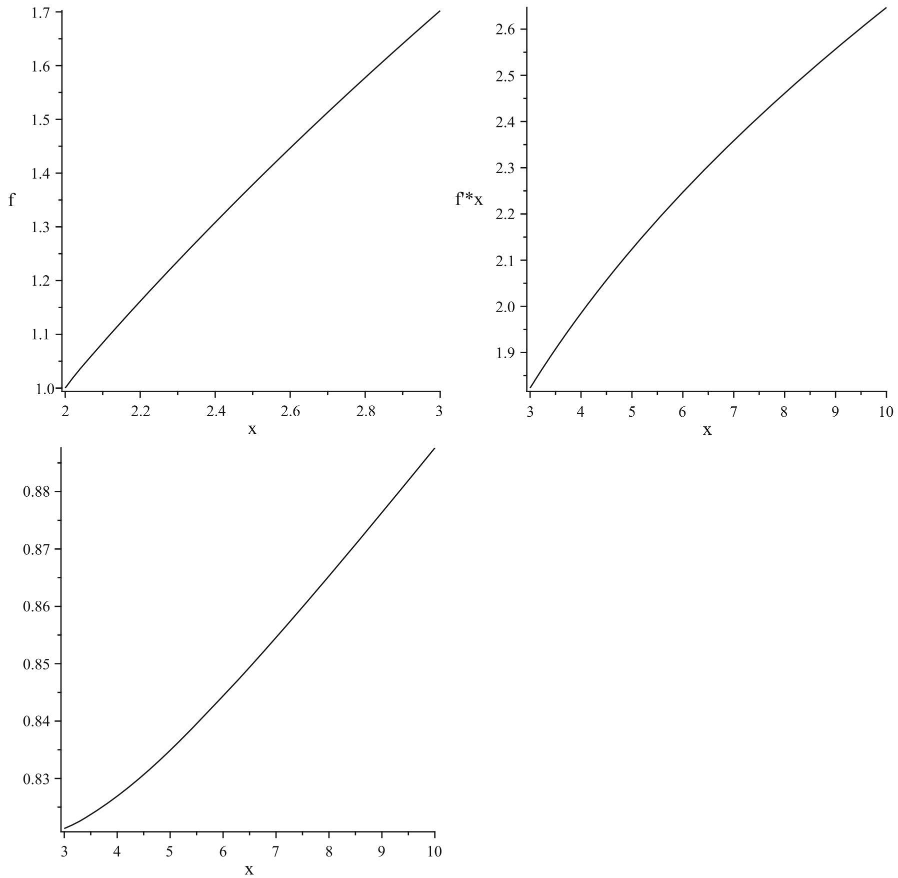

Fig. 2 Plots of f, $xf'(x)$ and u against x

References

Barro R, Sala-i-Martin X (2004) Economic growth, 2nd edn. MIT Press, Cambridge

Bethmann D (2007) A closed-form solution of the Uzawa–Lucas model of endogenous growth. J Econ 90(1):87–107

Boucekkine R, Ruiz-Tamarit JR (2008) Special functions for the study of economic dynamics: the case of the lucas-uzawa model. J Math Econ 44:33–54

Brock WA, Mirman LJ (1972) Optimal economic growth under uncertainty. J Econ Theory 4:497–513

Chang FR (2004) Stochastic optimization in continuous time. 1st edn. Cambridge University Press, Cambridge

Danthine J-P, Donaldson JB (1981) Stochastic properties of fast versus slow growing economies. Econometrica 49:1007–1033

Howitt P (1999) Steady endogenous growth with population and R&D input growing. J Political Econ 107(4):715–730

La Torre D, Marsiglio S (2010) Endogenous technological progress in a multi-sector growth model. Econ Model 27(5):1017–1028

Long JB, Plosser C (1983) Real business cycles. J Political Econ 91:36–69

Lucas RE (1988) On the mechanics of economic development. J Monet Econ 22(1):3–42

Mankiw N, Romer D, Weil DN (1992) A contribution to the empirics of economic growth. Q J Econ 107(2):407–437

McCallum BT (1989) Real business cycles. In: Robert B (ed) Modern business cycle theory. Harvard University Press, Cambridge, pp 16–50

Mirman LJ, Zilcha I (1975) On optimal economic growth under uncertainty. J Econ Theory 11:329–339

Mulligan CB, Sala-i-Martin X (1993) Transitional dynamics in two-sector models of endogenous growth. Q J Econ 108:739–773

Olson LJ, Roy S (2006) Theory of stochastic optimal economic growth. In: Le Van C, Dana R, Mitra T, Nishimura K (eds) Handbook of economic growth. The discrete time case, vol 1. Springer, Berlin

Palokangas T (2008) Competition and product cycles with non-diversifiable risk. J Econ 94(1):1–30

Smith WT (2007) Inspecting the mechanism exactly: a closed-form solution to a stochastic growth model. BE J Macroecon 7(1):Article 30

Uzawa H (1965) Optimum technical change in an aggregative model of economic growth. Int Econ Rev 6:18–31

Wälde K (2011) Production technologies in stochastic continuous time models. J Econ Dyn Control 35(4):616–622