**ISSN** 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.9(2010) No.1,pp.104-111

# An Economic Growth Model with Renewable Resources and Stochastic Technology Change

Honglin Yang *, Weixing Tian, Lixin Tian, Zhanwen Ding
Faculty of Science, Jiangsu University, Zhenjiang 212013, P. R. China.
*(Received 7 September 2009, accepted 15 December 2009)*

**Abstract:**This paper suggests an economic growth model with stochastic technology change in which renewable natural resource is introduced. The optimal sustainable growth path of economy is studied and explicit solutions to all variables of the model are obtained. The paper also analyzes how economy system is affected by stochastic technology change and renewable rate of resources.

**Keywords:**renewable resources, optimal growth, stochastic technology change

**JEL classification:** O31 O41

## 1 Introduction

As we know, the production in any economy can not be independent of natural resources (renewable or non-renewable). However, according to most economic growth theories, economic growth is regarded as the function of capital, technology, employment, bank rate and other relevant rules and all kinds of resources could be mutually substituted or replaced by other production elements[1]. In other words, resource is one of the affecting factors, but not a determinative one. However, resources play an important role in our economy, and there would be no growth without the resource input. Then the question is that how the natural resources affect the modern economic growth. Many studies have been made about this problem. The results of these studies show that economy would keep positive growth rate even the stock of resource (non-renewable) is finite[2-5]. Furthermore, technological progress is necessary. Concerning the description of technological progress, there are currently two main modes. One is called "level innovation", featuring the enlargement of categories and the other is called "vertical innovation", featuring the improvement of quality and productivity. No matter what kind of technological innovation, technological progress is usually assumed to follow Poisson process. Andre Grimaud and Luc Rogue [6] studied the relationship among technology advancement ("vertical innovation" mode), non-renewable resource and economic growth. Gilles Lafforgue [7] studied the relationship among stochastic technical change, non-renewable resource and optimal sustainable growth. As far as China's present status in terms of resources and economic development is concerned, the substitution elasticity between capital and energy has great uncertainty, for which China's economic growth needs huge capital investment to improve energy efficiency and exploit new energy[8]. National economy's "bottlenecks" of resource shortage has not been fundamentally alleviated[9], and the global large-scale use of conventional fossil fuels has resulted in increasingly serious resource shortages and environmental problems. The most urgent task facing energy scientists worldwide is to promote the use of clean fossil energy and renewable energy, especially in the field of renewable energy development and industrialization. Therefore, with the introduction of renewable resource constraints in modern economic growth model, to discuss how to achieve sustainable economic development on the premise of sustainable use of resources has become one of the most popular issues in modern economics[11-14].

* **Corresponding author**. *E-mail address*: yhl812@ujs.edu.cn


IJNS.2010.02.15/323

In this paper, we focus on the relationship among stochastic technical change, renewable resource and optimal sustainable growth. Furthermore, we discuss the effects of the randomness of technological progress and the resource renewable rate on the economic growth.

This paper has the following structure. The model is described in Section 2, and the optimal growth path is suggested in Section 3. We also investigate the conditions under which the economy on average attains a sustainable growth path. Section 4 analyzes the effect of uncertainty on economy. Section 5 analyzes the effect of resource renewable rate on economy. Finally, we draw a conclusion of the study in Section 6.

## 2 Model

Suppose there are two sectors in the economy: the final production sector and the R&D sector. At each time t, the final output is produced by the production sector according to

$$Y_t = A_t l_t^{\alpha} R_t^{1-\alpha} \tag{1}$$

where $Y_t$ is the output at time $t$ and is assumed to be totally consumed. $A_t$ is the level of technology at time $t$, $l_t$ is the amount of labor, and $R_t$ is the amount of resource invested into production by the final production sector. $\alpha$ and$1-\alpha$ stand for the output elasticities of labor and resources respectively, while $0 < \alpha < 1$.

Suppose the stock of resource $S_t$ is at time $t$, and the renewable rate of resource is $\sigma$, then the dynamic equation of the stock of resource at time $t$ is

$$\dot{S}_t = \sigma S_t - R_t \tag{2}$$

Assuming that the labor$(L)$ supply is fixed, we standardize the total flow of labor to one $(L = 1)$. The labor possesses two competing functions: produce final production or do research. Suppose the labor used for research is $n_t$ at time $t$, then we have $l_t + n_t = 1$ at each time $t$.

At each time $t$, the stock of knowledge evolves randomly as follows

$$dA_t = a\varepsilon A_t dq_t \tag{3}$$

where $a > 0$ is the deterministic instantaneous growth rate of the total factor productivity (TFP), $\varepsilon > 0$ is a constant parameter which is to be described further in Section 4, and $q_t$ represents the uncertainly accumulated number of innovations that have occurred since the start until time $t$. For the sake of simplicity, we assume that $q_0 = 0$ and the increments $dq_t$ follow a time-dependent Poisson process[15] with an endogenous arrival rate $\lambda(n_t)$. This means that the variations of $q_t$ are randomly distributed according to the following probability function

$$P(q_{t+\Delta t} - q_t = k) = \frac{(\int_t^{t+\Delta t} \lambda(n_s)ds)^k}{k!} \exp(-\int_t^{t+\Delta t} \lambda(n_s)ds) \tag{4}$$

where $0 \leq \lambda(n_t) \leq 1$ , $\lambda^{'}(n_t) > 0, n(t) \in [0, 1]$ and $\lambda(0) = 0$. In other words, the probability of innovation success increases with the increase of R&D effort, and this effort is necessary for the innovation process. We define the function $\lambda(n_t)$ as follows

$$\lambda(n_t) = \frac{\lambda}{\varepsilon} n_t$$

where $\lambda > 0$ and $\frac{\lambda}{\varepsilon} \leq 1$ . Let $b$ denote the ratio $\frac{\lambda}{\varepsilon}$ , then the constant $b$ is the marginal probability of success of technological innovation. Then, we obtain $EdA_t = a\lambda n_t A_t dt$ and $E\frac{dA_t}{dt} = a\lambda n_t A_t$, where $E$ denotes the expectations of technological progress at time $t$.

The stochastic technological process above can be described as follows. During the interval of time $dt$, research can either be successful, so that $dq_t = 1$ with probability $\lambda(n_t)dt$, or unsuccessful, so that $dq_t = 0$ with probability $1 - \lambda(n_t)dt$. In the latter case, i.e. as long as there is no innovation or during any phase of research between two subsequent innovations, the stock of knowledge remains constant. When a technological improvement is achieved, at a random point in time, this stock jumps by $a\varepsilon$ percent. For example, if a new innovation occurs at time $\tau_i$, the size of the discrete upward jump in the trajectory of $A_t$ is measured by

$$\Delta A_{\tau_i} = A(\tau_i, q_{\tau_i} + 1) - A(\tau_i, q_{\tau_i}) = a\varepsilon A(\tau_i, q_{\tau_i}) \tag{5}$$

where $A(\tau_i, q_{\tau_i} + 1)$ is the TFP level at time $\tau_i$ decided by the newly accumulated innovations. Along an infinite continuous time horizon, repeating this random experience makes the trajectory of to $A_t$ piecewise continuous. From (3), the size of the jump increases with $A_t$. But the time interval between two subsequent innovations is not certain. In principle such intervals depend on $n_t$.

We assume that the utility function of consumer is $U(C_t) = \frac{C_t^{1-\theta}-1}{1-\theta}$, where $\theta > 0$ expresses the elasticity of marginal utility and $\frac{1}{\theta}$ is the intertemporal elasticity of substitution. And we assume the utility discount rate at different time is a constant and denote it by $\rho(\rho > 0)$. Given the initial knowledge $A_0$ and the initial resource stock $S_0$ , under the infinite time horizon, the social planner is to choose $n_t$ and $R_t$ to solve the maximization problem:

$$H(A_0, S_0) = \max E_0 \int_0^{+\infty} U(C_t)e^{-\rho t}dt \tag{6}$$

The problem (6) from any time $t$ can be written as

$$H(A_t, S_t; t) = \max E_t \int_t^{+\infty} U(C_s)e^{-\rho(s-t)}ds \tag{7}$$

# 3 Optimal sustainable growth

## 3.1 Optimal conditions

In the optimization problem above, the control variables are $n_t$ and $R_t$, and the state variables are $A_t$ and $S_t$. Let $H(A_t, S_t; t) = e^{-\rho t}H(A_t, S_t)$, we get the Hamilton-Bellman-Jacobi (HBJ) equation as follows:

$$\rho H(A_t, S_t) = \max\{U(C_t) + \frac{1}{dt}E_t dH(A_t, S_t)\} \tag{8}$$

According to (8), and the Ito's formula for Poisson processes[16], we can re-write the HBJ equation as follows:

$$\rho H(A_t, S_t) = \max\{\frac{(A_t l_t^{\alpha} R_t^{1-\alpha})^{1-\theta} - 1}{1-\theta} + (\sigma S_t - R_t)H_S(A_t, S_t) + b(1-l_t)\Delta H(A_t, S_t)\} \tag{9}$$

where $H_S(A_t, S_t) = \frac{\partial H(A_t,S_t)}{\partial S}$ is the shadow price of the resource and $\Delta H(A_t, S_t)$ is the increment in the social welfare caused by a success of R&D. The latter is defined as follows:

$$\Delta H(A_t, S_t) = H(A(t, q_t + 1), S_t) - H(A(t, q_t), S_t) = H((1 + a\varepsilon)A_t, S_t) - H(A_t, S_t) \tag{10}$$

The first-order optimality conditions of (9) are

$$\begin{aligned} H_R &= (1-\alpha)A_t^{1-\theta} l_t^{\alpha(1-\theta)} R_t^{(1-\alpha)(1-\theta)-1} - H_S(A_t, S_t) = 0 & (11) \\ H_l &= \alpha A_t^{1-\theta} l_t^{\alpha(1-\theta)-1} R_t^{(1-\alpha)(1-\theta)} - b\Delta H(A_t, S_t) = 0 & (12) \end{aligned}$$

Concluding from (11) and (12), we may have

$$\begin{aligned} R(A_t, S_t) &= A_t^{(1-\theta)/\theta}(\frac{\alpha}{b\Delta H(A_t, S_t)})^{\alpha(1-\theta)/\theta}(\frac{1-\alpha}{H_S(A_t, S_t)})^{1-\alpha(1-\theta)/\theta} & (13) \\ l(A_t, S_t) &= A_t^{(1-\theta)/\theta}(\frac{\alpha}{b\Delta H(A_t, S_t)})^{(\alpha(1-\theta)+\theta)/\theta}(\frac{1-\alpha}{H_S(A_t, S_t)})^{(1-\alpha)(1-\theta)/\theta} & (14) \end{aligned}$$

Substituting (13) and (14) into (9), then we have

$$\begin{aligned} \rho H(A_t, S_t) = & \frac{\theta A_t^{(1-\theta)/\theta}}{1-\theta}(\frac{\alpha}{b\Delta H(A_t, S_t)})^{\alpha(1-\theta)/\theta}(\frac{1-\alpha}{H_S(A_t, S_t)})^{(1-\alpha)(1-\theta)/\theta} + b\Delta H(A_t, S_t) \\ & +\sigma S_t H_S(A_t, S_t) \end{aligned} \tag{15}$$

(15) is a partial differential equation in $H(A_t, S_t)$, with a generic solution having the following form

$$\hat{H}(A_t, S_t) = BA_t^{\beta} S_t^{\gamma} \tag{16}$$

where$B$, $\beta$ and $\gamma$ are parameters to be determined. Then

$$\begin{aligned} \hat{H}(A_t, S_t) &= \gamma BA_t^{\beta} S_t^{\gamma-1} = \gamma\hat{H}/S_t & (17) \\ \Delta\hat{H}(A_t, S_t) &= ((1+a\varepsilon)^{1-\theta} - 1)\hat{H} & (18) \end{aligned}$$

Substituting (17) and (18) into (15), we have

$$\begin{aligned} \beta &= 1-\theta & (19) \\ \gamma &= (1-\alpha)(1-\theta) & (20) \\ B &= (1-\theta)^{\alpha(1-\theta)-1}(\frac{\theta}{\rho - b\xi - \sigma})^{\theta}(\frac{\alpha}{b\xi})^{\alpha(1-\theta)} & (21) \end{aligned}$$

where $\xi = (1+a\varepsilon)^{1-\theta} - 1$. Clearly $\theta < 1 \Rightarrow \xi > 0$ and $\theta > 1 \Rightarrow \xi < 0$ for all $a > 0$ and $\varepsilon > 0$. Now we obtain the closed-form solutions of the program (9).

$$\begin{aligned} \hat{l}(A_t, S_t) &= \frac{\alpha(1-\theta)(\rho - b\xi - (1-\alpha)(1-\theta)\sigma)}{\theta b\xi} & (22) \\ \hat{R}(A_t, S_t) &= (\frac{\rho - b\xi - (1-\alpha)(1-\theta)\sigma}{\theta})S_t & (23) \end{aligned}$$

Finally, the optimal consumption path derived from (1) is given by

$$\hat{c}(A_t, S_t) = (\frac{\rho - b\xi - (1-\alpha)(1-\theta)\sigma}{\theta})(\frac{\alpha(1-\theta)}{b\xi})^{\alpha} A_t S_t^{1-\alpha} \tag{24}$$

## 3.2 Optimal growth

Owing to the analytical framework that we have adopted in this paper, the optimal allocation of labor between the production sector and the R&D sector is uniquely determined. Namely $l^* = \hat{l}(A_t, S_t)$, $n^* = 1 - l^*$, where $l^*$ and $n^*$ are constants along the optimal growth path over time.

In order to obtain the explicit solution to various economic variables, let us identify the necessary conditions of existence of this solution. Firstly,$l^*$, $n^*$ and $\hat{R}$ must satisfy the non-negative constraints: $l^* > 0, n^* \geq 0, \hat{R}(A_t, S_t) > 0$. At the optimum, the R&D effort is constant over time. If $n^* = 0$, then $A_t = A_0$ and there is no technological progress in the economic system, this is clearly inconsistent with reality. Therefore, we are more concerned about the case $n^* > 0$. In this case, we find that an interior solution of (9) exists if and only if the utility discount rate satisfies the following condition

$$b\xi + (1-\alpha)(1-\theta)\sigma < \rho < (1 + \frac{\theta}{\alpha(1-\theta)})b\xi + (1-\alpha)(1-\theta)\sigma \tag{25}$$

and along the optimal growth path, we have

$$R_t^* = (\frac{\rho - b\xi - (1-\alpha)(1-\theta)\sigma}{\theta})S_0 e^{g_S t} \tag{26}$$

where $g_S = \sigma - \frac{\rho - b\xi - (1-\alpha)(1-\theta)\sigma}{\theta} = \frac{\theta + (1-\alpha)(1-\theta)}{\theta}\sigma - \frac{\rho - b\xi}{\theta}$. Then $g_S > 0$ if $\sigma > \frac{\rho - b\xi}{1-\alpha(1-\theta)}$, and $g_S < 0$ if $\sigma < \frac{\rho - b\xi}{1-\alpha(1-\theta)}$. If resource is non-renewable($\sigma = 0$), then $g_S < 0$. We can see that it is precisely because of the impact of renewable resources that the growth rate of resource will appear positive when the resource renewable rate reaches a certain threshold.

Given the stochastic characteristics of the innovation process, the dynamics of $A_t, t = 0, 1, 2\cdots$ is built up as a geometric series

$$A^*(t, q_t) = A_0(1+a\varepsilon)^{q_t} \tag{27}$$

Then, the random variables $q_t$ and $A^*$ approach to infinite in probability as $t$ increases infinitely.

Thus, the equation of the optimal consumption trajectory can be written as

$$c^*(t, q_t) = \hat{c}(A_0, S_0)(1 + a\varepsilon)^{q_t} \exp((1 - \alpha)g_S t) \tag{28}$$

where $\hat{c}(A_0, S_0)$ is defined by (24). Equation (28) determines the optimal consumption level as a function of time $t$ and the innovation number $q_t$. If $\sigma > \frac{\rho - b\xi}{1-\alpha(1-\theta)}$, then $(1 + a\varepsilon)^{q_t}$ and $e^{(1-\alpha)g_S t}$ are increasing with time $t$, and we obtain a consumption path which is monotonically increasing. In this case, the resource renewable rate is supposed to be large enough, which is unrealistic (e.g., in the current resource situation of China). If $\sigma < \frac{\rho - b\xi}{1-\alpha(1-\theta)}$, then $(1 + a\varepsilon)^{q_t}$ is positively increasing and the other term $e^{(1-\alpha)g_S t}$ increase negatively. The respective intensities of these two opposite driving forces will determine the feasibility of an optimal sustainable growth path.

Because of the randomness of technological progress, the optimal paths (27) and (28) are piecewise continuous. In order to smooth the optimal path, we compute the expectation values of TFP and consumption in the interval $[0, t]$, which is used to replace the values of TFP and consumption at time $t$. Assuming the initial condition: $A^*(0, q_0) = A_0$, $c^*(0, q_0) = \widehat{c}(A_0, S_0)$, we can obtain the values of TFP and consumption in time as follows:

$$\bar{A}_t{}^* = E_0 A^*(t, q_t) \tag{29}$$

$$\bar{c}_t{}^* = E_0 c^*(t, q_t) \tag{30}$$

Since $n^*$ is constant over time and $q_0 = 0$, (4) can be rewritten as

$$P_0(q_t = k) = \frac{(\lambda(n^*)t)^k}{k!} \exp(-\lambda(n^*)t) \tag{31}$$

This implies

$$E_0(1 + a\epsilon)^{q_t} = \sum_{k=0}^{\infty} (1 + a\varepsilon)^k \frac{(\lambda n^* t/\varepsilon)^k}{k!} \exp(\frac{-\lambda n^* t}{\varepsilon}) = \exp(a\lambda n^* t) \tag{32}$$

and then

$$\bar{A}_t^* = A_0 \exp(g_{\bar{A}} t), \text{ with } g_{\bar{A}} = a\lambda n^* \tag{33}$$

$$\bar{c}_t^* = \hat{c}(A_0, S_0) \exp(g_{\bar{c}} t), \text{ with } g_{\bar{c}} = g_{\bar{A}} + (1 - \alpha)g_S \tag{34}$$

### 3.3 Sustainable growth

According to the standard definition in Brundtland's Report, sustainable development must "meet the needs of the present generation without compromising the ability of future generations to meet their own needs." In this statement, we can draw the concepts of intergenerational equity, intragenerational equity and efficiency. In our model, we focus only on the intergenerational equity constraint. Asheim et al. [17] show that a sufficient condition for sustainability of utility paths, which capture the notion of intergenerational equity, is that such paths must be non-diminishing over time. Hence a sustainable growth path can be defined as any optimal consumption path which is non-decreasing over time, since the utility is an increasing and monotonous function of consumption. In order to maintain sustainable economic growth, we require that the growth rate of expected consumption $\bar{c}_t^*$ must be nonnegative. From (34), $\bar{c}_t^*$ is nonnegative if and only if

$$\rho \leq \frac{\lambda a[\theta b\xi + \alpha(1 - \alpha)(1 - \theta)^2\sigma]}{\lambda a\alpha(1 - \theta) + b\xi(1 - \alpha)} + b\xi \tag{35}$$

## 4 Effects of the randomness of technological progress on economy

We can increase the randomness of any random through maintaining its expectation value and increasing its variance. In this way, we can analyze the effect of the randomness on the optimal paths. Below we consider the parameter $\varepsilon$ as introduced in Section 2. From the equations in Section 2, we have

$$E_t(dq_t) = \frac{\lambda n_t}{\varepsilon} dt, \; D_t(dq_t) = E_t(dq_t)(1 - E_t(dq_t)) \tag{36}$$

From (36) and (3), it follows $EdA_t = a\lambda n_t A_t dt$, which is independent of $\varepsilon$, and $D_t(dA_t) = (aA_t)^2\lambda n_t dt(\varepsilon - \lambda n_t dt)$ is clearly increasing in $\varepsilon$. Thus, we can change the randomness of technological progress by changing $\varepsilon$.

We compute the partial derivative with respect to $\varepsilon$ of the various expressions characterizing the optimal solution of the model, and after simplification the values of these derivatives are as follows:

$$\frac{\partial l^*}{\partial \varepsilon} = \frac{\alpha(1-\theta)}{\lambda\theta}(\frac{\varepsilon}{\xi})^2((1-\alpha)(1-\theta)\sigma - \rho)\frac{\partial(\frac{\varepsilon}{\xi})}{\partial \varepsilon} \tag{37}$$

$$\frac{\partial g_{\bar{A}}}{\partial \varepsilon} = -a\lambda\frac{\partial l^*}{\partial \varepsilon} \tag{38}$$

$$\frac{\partial \hat{R}_0}{\partial \varepsilon} = -\frac{\lambda S_0}{\theta}\frac{\partial(\frac{\varepsilon}{\xi})}{\partial \varepsilon} \tag{39}$$

$$\frac{\partial g_S}{\partial \varepsilon} = \frac{\lambda}{\theta}\frac{\partial(\frac{\varepsilon}{\xi})}{\partial \varepsilon} \tag{40}$$

$$\frac{\partial \hat{c}_0}{\partial \varepsilon} = \frac{A_0 S_0^{1-\alpha}}{\theta}(\frac{\varepsilon\alpha(1-\theta)}{\lambda\xi})^\alpha(\lambda(1-\alpha) + \frac{\varepsilon\alpha(\rho-(1-\alpha)(1-\theta)\sigma)}{\xi})\frac{\partial(\frac{\varepsilon}{\xi})}{\partial \varepsilon} \tag{41}$$

$$\frac{\partial g_{\bar{c}}}{\partial \varepsilon} = \frac{1}{\theta}(\lambda(1-\alpha) + a\alpha\rho(1-\theta)(\frac{\varepsilon}{\xi})^2)\frac{\partial(\frac{\varepsilon}{\xi})}{\partial \varepsilon} \tag{42}$$

where$\frac{\partial(\frac{\varepsilon}{\xi})}{\partial\varepsilon} = -\frac{(1+a\varepsilon\theta)(1+a\varepsilon)^{-\theta}-1}{\varepsilon^2}$, and $\frac{\partial(\frac{\varepsilon}{\xi})}{\partial\varepsilon} < 0$ if $\theta < 1$, $\frac{\partial(\frac{\varepsilon}{\xi})}{\partial\varepsilon} > 0$, if $\theta > 1$, so we can determine the signs of the expressions (37)-(42) (see Table 1).

**Table 1:** Effects of the randomness on economy

| | | $\frac{\partial l^*}{\partial\varepsilon}$ | $\frac{\partial n^*}{\partial\varepsilon}$ | $\frac{\partial g_{\bar{A}}}{\partial\varepsilon}$ | $\frac{\partial \hat{R}_0}{\partial\varepsilon}$ | $\frac{\partial g_S}{\partial\varepsilon}$ | $\frac{\partial \hat{c}_0}{\partial\varepsilon}$ | $\frac{\partial g_{\bar{c}}}{\partial\varepsilon}$ |
|---|---|---|---|---|---|---|---|---|
| $\theta < 1$ | $\rho < x_1$ | $< 0$ | $> 0$ | $> 0$ | $> 0$ | $< 0$ | $< 0$ | $< 0$ |
| | $x_1 < \rho < x_2$ | $< 0$ | $> 0$ | $> 0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ |
| | $\rho > x_2$ | $> 0$ | $< 0$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ |
| $\theta > 1$ | $\rho < x_3$ | $> 0$ | $< 0$ | $< 0$ | $< 0$ | $> 0$ | $< 0$ | $< 0$ |
| | $\rho > x_3$ | $> 0$ | $< 0$ | $< 0$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ |

where$x_1 = (1-\alpha)(1-\theta)\sigma - (1-\alpha)\frac{b\xi}{\alpha}$, $x_2 = (1-\alpha)(1-\theta)\sigma$, $x_3 = -\frac{b(1-\alpha)\xi^2}{a\alpha\varepsilon(1-\theta)}$.

## 5 Effects of the resource renewable rate on economy

In this section, we discuss the effects of the resource renewable rate on the economy. We compute the partial derivative with respect to $\sigma$ of the various expressions characterizing the optimal solution of the model and after simplification the values of these derivatives are as follows:

$$\frac{\partial l^*}{\partial\sigma} = -\frac{\alpha(1-\theta)^2(1-\alpha)}{\theta b\xi} \tag{43}$$

$$\frac{\partial n^*}{\partial\sigma} = -\frac{\partial l^*}{\partial\sigma} \tag{44}$$

$$\frac{\partial g_{\bar{A}}}{\partial\sigma} = -a\lambda\frac{\partial l^*}{\partial\sigma} \tag{45}$$

$$\frac{\partial \hat{R}_0}{\partial\sigma} = -\frac{(1-\alpha)(1-\theta)}{\theta}S_0 \tag{46}$$

$$\frac{\partial \hat{c}_0}{\partial\sigma} = -A_0 S_0^{1-\alpha}(\frac{\alpha(1-\theta)}{b\xi})^\alpha(\frac{(1-\alpha)(1-\theta)}{\theta}) \tag{47}$$

$$\frac{\partial g_S}{\partial\sigma} = \frac{1-\alpha}{\theta} + \alpha \tag{48}$$

$$\frac{\partial g_{\bar{c}}}{\partial\sigma} = \frac{\partial g_{\bar{A}}}{\partial\sigma} + (1-\alpha)\frac{\partial g_S}{\partial\sigma} = \frac{\alpha(1-\theta)^2(1-\alpha)}{\theta b\xi} + (1-\alpha)(\frac{1-\alpha}{\theta} + \alpha) \tag{49}$$

According to the above related assumption, we can determine the signs of the expressions (43)-(49) (see Table 2).

**Table 2:** Effects of the resource renewable rate on economy

| | $\frac{\partial l^*}{\partial \sigma}$ | $\frac{\partial n^*}{\partial \sigma}$ | $\frac{\partial g_{\bar{A}}}{\partial \sigma}$ | $\frac{\partial \hat{R}_0}{\partial \sigma}$ | $\frac{\partial g_S}{\partial \sigma}$ | $\frac{\partial \hat{c}_0}{\partial \sigma}$ | $\frac{\partial g_{\bar{c}}}{\partial \sigma}$ |
|---|---|---|---|---|---|---|---|
| $\theta < 1$ | $< 0$ | $> 0$ | $> 0$ | $< 0$ | $> 0$ | $< 0$ | $> 0$ |
| $\theta > 1$ | $> 0$ | $< 0$ | $< 0$ | $> 0$ | $> 0$ | $> 0$ | uncertain |

Note that the impact of $\sigma$ on $g_{\bar{c}}$ is uncertain if $\theta > 1$. In fact, from (49), if $\theta < 1$, both $\frac{\partial g_{\bar{A}}}{\partial \sigma}$ and $\frac{\partial g_S}{\partial \sigma}$ are positive, and so $\frac{\partial g_{\bar{c}}}{\partial \sigma} > 0$ . If $\theta > 1$, $\frac{\partial g_{\bar{A}}}{\partial \sigma} < 0$ and $\frac{\partial g_S}{\partial \sigma} > 0$, which show that the sign of $\frac{\partial g_{\bar{c}}}{\partial \sigma}$ is uncertain. However, for this case we can make further discussion. Form the fact that $\xi = (1 + a\varepsilon)^{1-\theta} - 1 < 0$ if $\theta > 1$ and the fact that under the condition other parameters of the economic system remain unchanged and according to the trends of the first term of (49) right hand, we can ensure that there must exist a unique $\theta^*$, $\theta^* > 1$, and if $\theta < \theta^*$, then $\frac{\partial g_{\bar{c}}}{\partial \sigma} > 0$; if $\theta > \theta^*$, then $\frac{\partial g_{\bar{c}}}{\partial \sigma} < 0$.

# 6 Conclusions

In this paper, we introduce the renewable resource to the economic growth model with stochastic technology change, and focus on the optimal sustainable growth path. With precise mathematical deduction, we obtain the explicit solutions to all the variables of the economic system. Furthermore, we discuss the effects of technological progress randomness and the resource renewable rate on the economy. According to the finding of this paper, in order to maintain the economic growth at a high speed, Government can take appropriate measures to indirectly regulate the parameters in the model. Such as through appropriate fiscal, monetary policy; by using economic levers to regulate consumer utility discount rate; by increasing recycling and re-utilization to increase the renewable rate of resource ; through the R&D investment and human capital development to improve the level of production technology to ensure the long-term stable growth of the economy.

# Acknowledgements

Research is supported by the National Nature Science Foundation of China (No. 90610031), Jiangsu Planned Projects for Postdoctoral Research Funds (No. 0901135C) and the High Talent Foundation of Jiangsu University (NO. 08JDG053, 07JDG023).

# References

[1] Zhou Hai-lin. Economic growth theory and the sustainable use of natural resources, Economic Review. 2001, 22(2) 35-38

[2] P.S. Dasgupta, G.M. Heal, The optimal depletion of exhaustible resources, Rev. Econom. Study. 1974, 41 3-28

[3] P.C. Garg, J.L. Sweeney, Optimal growth with deletable resources , Res. Energy. 1(1978) 43-56

[4] R.M. Solow, The economics of resources or the resources of economic, Amer. Econom. Stud. 41(1974) 29-45

[5] J. Stiglitz, Growth with exhaustible natural resources: () efficient and optimal growth, () the competitive economy, Rev. Econom. Stud. Symp. 41(1974) 123-152

[6] Andre Grimaud, Luc Rouge, Non-renewable resources and growth with vertical innovations: optimal, equilibrium and economic policies, Journal of Environmental Economics and Management.45(2003) 433-453

[7] Gilles Lafforgue. Stochastic technical change, non-renewable resource and optimal sustainable growth, Resource and Energy Economics , 2008, 30 540-554

[8] Zhaoning Zheng, Deshun Liu. Uncertainty of Capital-energy Substitution in ChinaOperations Research and Management Science, 2004, 13(2) 74-78

[9] Zhaoning Zheng, Deshun Liu. China's Trans-log Production Function Using Capital, Energy and Labor as Input. Systems Engineering - Theory & Practice, 2004, 24(5) 51-54

[10] Zhao Lixia, Wei Weixian. Research of Energy and Economic Growth model, Forecasting, 1998, 17 (6) 32-34

[11] Yang Honglin, Tian Lixin, Ding Zhanwen. Sustainable Economic Growth under Restraint of Energy, Systems Engineering, 2004, 23(4) 40-43

[12] Yang Honglin, Tian Lixin, Ding Zhanwen. Renewable Resources, Technological Progress and Economic Growth, International Journal of Nonlinear Science, 2006, 1(3) 149-154

[13] Yang Honglin, Ding Zhanwen, Tian Lixin. Equilibrium Growth of Economy with Renewable Resources, Systems Engineering-Theory& Practice, 2008, 28(11)9-15

[14] The Consumption Path in the Economic Growth Model with Renewable Energy, Systems Engineering-Theory &Practice, 2006, 26(6)13-17

[15] Y.H. Farzin, K.J.M. Huisman, P.M. Kort, Optimal timing of technology adoption, Journal of Economic Dynamics and Control. 1998, 22 779-799

[16] Sennewald, K. Wälde, K. "Ito's Lemma" and the Bellman equation for Poisson processes: an applied view, Journal of Economics, 2006, 89(1) 1-36

[17] Asheim, G.B., Buchholz, W., Tungodden, B. Justifying sustainability. Journal of Environmental Economics and Management, 2001, 41 (3) 252-268