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Stochastic Growth Models and Their Econometric Implications

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This article considers the consequences of explicitly allowing for stochastic technological progress and stochastic labor input in the discrete-time Solow-Swan and *AK* growth models. It shows that the capital-output ratio, but not output per capita, is ergodic irrespective of whether there is a unit root in technology, and thus is the more appropriate measure to use in the cross-sectional analysis of the growth process. Furthermore, the article derives the cross-sectional and time-series implications of the stochastic Solow-Swan model and contrasts these to those of its deterministic counterpart. Among these implications are that the mean of the capital-output ratio depends in a precise way not only on the saving rate and the growth rate of labor input, but also on the variance and higher-order cumulants of the capital-output ratio. Using the Summers-Heston data for seventy-two countries from 1960 to 1992, strong support is found for the predictions of the stochastic Solow-Swan model as compared to those of its deterministic counterpart (as well as those of the *AK* model), including a significant negative cross-sectional relationship between the mean and the variance of the capital-output ratio.

Keywords: stochastic growth models, ergodicity, econometric implications, cross-country growth regressions

JEL classification: C10, E20, O40

1. Introduction

The past decade has seen a resurgence of interest in the analysis of economic growth, from theoretical as well as empirical perspectives. The main focus of the empirical work has been on studying cross-country differences in steady-state levels of output per capita, convergence of output growth rates to their steady-state values, and the differentials in the rates of return across countries. Almost all of this empirical work has been based on *deterministic* growth models, with a random component added to the deterministic solution for econometric analysis. In this article, we consider a Solow-Swan growth model where technology and labor input are modeled as *stochastic* processes. We argue that working with such a stochastic model overcomes a variety of deficiencies of the deterministic approach and can help to gain a better understanding of the growth process across countries.[1]

In the case where technology is generated by a unit root process, output per capita is nonergodic and cross-sectional analysis based on output per capita can be misleading.

This, of course, cannot be seen from a deterministic specification, rendering the latter a poor guide as to the correct measure to be used in the cross-sectional analysis of the growth process. We show that irrespective of whether there is a unit root in technology, the capital-output ratio is ergodic and is the appropriate variable to use for cross-country analysis of the growth process.[2] We also derive two important implications of the stochastic Solow-Swan model for the analysis of economic growth in cross-sections: (1) We establish that the mean of the capital-output ratio depends in a precise way not only on the saving rate and the growth rate of labor input (the standard variables in the existing empirical growth literature), but also on the variance and all higher-order cumulants of the capital-output ratio, which in turn depend on the parameters of the technology and labor input processes. (2) We show that the mean of the steady-state distribution of the capital-output ratio under the stochastic model exceeds the value of the steady-state capital-output ratio under the corresponding deterministic model.

Using the Summers-Heston data for seventy-two countries from 1960 to 1992, we find strong support for both of these predictions of the stochastic Solow-Swan model in full sample and a subsample of fifty-eight "intermediate" countries, including a significant negative cross-sectional relationship between the mean and the variance of the capital-output ratio.[3] For a subsample comprising twenty OECD countries, the evidence for the first of these two predictions is inconclusive. This could be due to insufficient cross-country sampling variations in the innovations of the technology and labor input processes across OECD countries, partly reflecting the sample-selection bias inherent in the way OECD membership is determined.

Finally, we point to the reasons why a stochastic specification is also important for the time-series (and the panel data) analysis of the growth process. The stochastic Solow-Swan model implies a nonlinear vector autoregression in the logarithms of output per capita, the capital-output ratio, and labor input. This is in contrast to the deterministic Solow-Swan model that predicts a linear autoregressive equation in the logarithm of output per capita.

In addition to the stochastic Solow-Swan growth model, we also consider a stochastic variant of the *AK* endogenous growth model.[4] As for the stochastic Solow-Swan model, we consider the conditions under which the capital-output ratio in the stochastic *AK* model is ergodic and derive the model's cross-sectional implications. Unlike the Solow-Swan model, the *AK* model does not imply that the variance and higher-order cumulants of the capital-output ratio should help in understanding cross-sectional differences in the mean of the capital-output ratio. Again using the Summers-Heston data set, we find that the cross-sectional implications of the stochastic *AK* model are clearly rejected.[5]

While almost none of the recent empirical growth literature has dealt with uncertainty concerning technology or labor input, there is a distinguished theoretical literature that has analyzed asymptotic properties of stochastic variants of the Solow-Swan growth model or the Cass-Koopmans optimal growth model. Contributions analyzing stochastic variants of the continuous-time or discrete-time Cass-Koopmans growth model include Mirrlees (1965), Brock and Mirman (1972), Bourguignon (1975), Merton (1975), Donaldson and Mehra (1983), Majumdar and Radner (1983), Marimon (1989), Hopenhayn and Prescott (1992), and Duffie and Singleton (1993). Stochastic variants of the discrete-time Solow-Swan model as we consider in this article have been analyzed in particular by Mirman (1972, 1973), who studies a Solow-Swan model with deterministic labor input growth but allows technology to be generated as an identically and independently distributed random variable with compact support on the positive real line. Mirman proves convergence of the capital-labor ratio in his model to a globally attracting steady-state probability distribution function. For the case where technology follows a first-order Markov process with a globally attracting steady-state probability distribution function and with compact support on the positive real line, Mirman (1972) proves the existence of a steady-state probability distribution function for the capital-labor ratio. We go beyond the existing theoretical literature in that we allow for unit roots in the stochastic processes generating technology and labor input and establish the conditions under which the capital-output ratio in the stochastic Solow-Swan and *AK* models is ergodic in moments.[6] By ergodicity in moments, we mean that a variable converges to a globally attracting steady-state probability distribution function and that the moments of this probability distribution function exist and can be consistently estimated using long time averages. We also provide a tight link between ergodicity in moments and the empirical analysis of economic growth in cross sections.

The remainder of this article is organized as follows. Section 2 sets out the stochastic versions of the Solow-Swan and *AK* growth models. The conditions under which the capital-output ratio and its logarithm in these models are ergodic in moments are established in Section 3. Section 4 derives the cross-sectional implications of the stochastic Solow-Swan and *AK* models for the steady-state distribution of the (logarithm of the) capital-output ratio both when the stochastic process generating technology has a unit root and when technology follows a stationary process. A brief analysis of the time-series implications of the stochastic Solow-Swan model is presented in Section 5. In particular, a nonlinear vector autoregression (NVAR) in output per capita, labor input, and the capital-output ratio is derived. In Section 6, the NVAR specification is used to estimate some of the model parameters, including those of the technology and the labor input processes. These estimates are then employed to test the empirical validity of the cross-sectional implications derived in Section 4. Section 7 summarizes and concludes with some suggestions for future research.

2. Stochastic Solow-Swan and *AK* Growth Models

Consider a set of economies, each economy is indexed by i, in which aggregate production possibilities are described by a constant returns to scale production function of the form

$$Y_{it} = \mathcal{F}(A_{it}, K_{it}, H_{it}, L_{it}), \tag{1}$$

where, in standard notation, Y_{it} represents output, K_{it} the physical capital stock at the beginning of period t, H_{it} the human capital stock also at the beginning of period t, L_{it} is labor input, and A_{it} the state of technology. We assume that $\mathcal{F}(\cdot)$ is twice continuously differentiable in all its arguments. We further assume that technology broadly defined is exogenously given by

$$\log(A_{it}) = a_{i0} + g_i t + u_{iat}, \tag{2}$$

where a_{i0} is the economy-specific initial endowment of technology, and[7]

$$\Delta u_{iat} = -(1-\rho_{ia})u_{ia,t-1} + \varepsilon_{iat}, \quad |\rho_{ia}| \leq 1, \varepsilon_{iat} \sim iid(0, \sigma_{ia}^2). \tag{3}$$

The evolution of labor input is assumed to be exogenous as well—namely,

$$\log(L_{it}) = l_{i0} + n_i t + u_{ilt}, \tag{4}$$

where l_{i0} is an economy-specific initial endowment of labor input, and

$$\Delta u_{ilt} = -(1-\rho_{il})u_{il,t-1} + \varepsilon_{ilt}, \quad |\rho_{il}| \leq 1, \varepsilon_{ilt} \sim iid(0, \sigma_{il}^2). \tag{5}$$

We assume that ε_{iat} and ε_{ilt} are distributed independently across i but allow for a nonzero contemporaneous correlation between ε_{iat} and ε_{ilt} within a given economy. We further discuss the distributions of ε_{iat} and ε_{ilt}, particularly regarding the nature of their support, in Section 3 below. The physical capital stock depreciates in each period at a constant rate δ_{iK} and obeys the linear law of motion

$$K_{i,t+1} = (1-\delta_{iK})K_{it} + I_{iKt}, \quad \delta_{iK} \in (0,1). \tag{6}$$

Similarly, the human capital stock depreciates in each period at a constant rate δ_{iH} and obeys the linear law of motion

$$H_{i,t+1} = (1-\delta_{iH})H_{it} + I_{iHt}, \quad \delta_{iH} \in (0,1). \tag{7}$$

The economy is closed by assuming that households' aggregate saving is given by

$$S_{it} = s(A_{it}, K_{it}, H_{it}, L_{it})Y_{it}, \tag{8}$$

where the only restriction on the saving function $s(A_{it}, K_{it}, H_{it}, L_{it})$ at this point is that $S_{it}/Y_{it} \in (0,1)$, and in equilibrium we have

$$S_{it} = I_{iKt} + I_{iHt}. \tag{9}$$

Under the assumption of identical depreciation rates for physical and human capital, the growth model given by (1) to (9) may be simplified and written in terms of physical capital only. To see this, suppose that for given saving rates s_{it}, $t = 1, 2, \ldots$, and initial capital stocks K_{i0} and H_{i0}, it is desired to

$$\max_{\{K_{it}, H_{it}\}_{t=1}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta_i^t \mathcal{F}(A_{it}, K_{it}, H_{it}, L_{it}) \mid \Omega_{i0}\right], \tag{10}$$

subject to

$$s_{it}Y_{it} = K_{i,t+1} + H_{i,t+1} - (1-\delta_{iK})K_{it} - (1-\delta_{iH})H_{it}, \quad t = 0, 1, \ldots, \tag{11}$$

where Ω_{i0} denotes the information set at time 0. Using the stochastic maximum principle and denoting the Lagrangian multiplier by λ_{it}, the first-order conditions of (10) and (11) with respect to K_{it} and H_{it} are given by

$$E\left[\beta_i \frac{\partial \mathcal{F}_{it}}{\partial K_{it}}(1+\lambda_{it}s_{it}) + \beta_i\lambda_{it}(1-\delta_{iK}) - \lambda_{i,t-1} \mid \Omega_{i0}\right] = 0, \tag{12}$$

and

$$E\left[\beta_i \frac{\partial \mathcal{F}_{it}}{\partial H_{it}}(1+\lambda_{it}s_{it}) + \beta_i\lambda_{it}(1-\delta_{iH}) - \lambda_{i,t-1} \mid \Omega_{i0}\right] = 0, \tag{13}$$

$t = 1, 2, \ldots$. Thus, if there is a common rate of depreciation of physical and human capital, $\delta_{iH} = \delta_{iK} = \delta_i$, then at the optimum it clearly must be the case that $\partial\mathcal{F}_{it}/\partial K_{it} = \partial\mathcal{F}_{it}/\partial H_{it}$, for all i, t.[8]

Our interest in this article is on constant returns to scale production functions of one of the following two forms, both special cases of (1). First, for the Solow-Swan growth model we consider

$$Y_{it} = A_{it}L_{it}\mathcal{G}(\kappa_{it}, \eta_{it}), \tag{14}$$

where κ_{it} denotes the physical capital-labor ratio in terms of effective units of labor input—that is,

$$\kappa_{it} = \frac{K_{it}}{A_{it}L_{it}},$$

and η_{it} denotes the human capital-labor ratio in terms of effective units of labor input—that is

$$\eta_{it} = \frac{H_{it}}{A_{it}L_{it}}.$$

A specific example of (14) widely used in the recent empirical growth literature, which we also pay particular attention to, is the Cobb-Douglas specification, for which (14) is replaced by

$$Y_{it} = A_{it}^{1-\gamma_i-\phi_i}K_{it}^{\gamma_i}H_{it}^{\phi_i}L_{it}^{1-\gamma_i-\phi_i}, \quad \gamma_i \in (0,1), \phi_i \in (0,1), \gamma_i+\phi_i \in (0,1). \tag{15}$$

The second form we consider is for the *AK* growth model:

$$Y_{it} = A_{it}K_{it}^{\gamma_i}H_{it}^{1-\gamma_i}, \quad \gamma_i \in (0,1). \tag{16}$$

Imposing the optimality condition that $\partial\mathcal{F}_{it}/\partial K_{it} = \partial\mathcal{F}_{it}/\partial H_{it}$ in the case of the Cobb-Douglas production function (15) yields

$$H_{it} = \frac{\phi_i}{\gamma_i}K_{it}, \tag{17}$$

and (15) may therefore be rewritten as

$$Y_{it} = A_{it}^{1-\alpha_i}K_{it}^{\alpha_i}L_{it}^{1-\alpha_i}, \tag{18}$$

where $\alpha_i = \gamma_i + \phi_i$, and where, for simplicity of notation, we have redefined the initial endowment of technology, a_{i0}.[9] Note that α_i represents the combined share of physical and human capital in production. For the more general case of (14), note that on imposing the optimality condition $\partial\mathcal{F}_{it}/\partial K_{it} = \partial\mathcal{F}_{it}/\partial H_{it}$, one can substitute η_{it} in terms of κ_{it} as long as $\partial\mathcal{G}_{it}/\partial\kappa_{it}$ can be uniquely solved for κ_{it} as a function of η_{it}.

A similar argument can, of course, be made for the *AK* model. Imposing the optimality condition that $\partial\mathcal{F}_{it}/\partial K_{it} = \partial\mathcal{F}_{it}/\partial H_{it}$ in the case of the *AK* production function (16) yields[10]

$$H_{it} = \frac{1-\gamma_i}{\gamma_i} K_{it}, \tag{19}$$

and (16) may therefore be rewritten as

$$Y_{it} = A_{it} K_{it}, \tag{20}$$

where, for simplicity of notation, we have again redefined the initial endowment of technology, a_{i0}.[11]

Our analysis from now on will assume that the depreciation rates on physical and human capital are the same and be based on the production functions

$$Y_{it} = A_{it} L_{it} f(\kappa_{it}), \tag{21}$$

for the Solow-Swan growth model, which in the special case of the Cobb-Douglas specification reduces to (18), and (20) for the *AK* growth model. Also, to keep the notation simple, let the saving function $s(\kappa_{it})$ from now on be defined such that $s(\kappa_{it})Y_{it}$ denotes the portion of households' aggregate saving in period t that is invested in physical capital. Note from (6), (8), and (9) that under $\delta_{iK} = \delta_{iH} = \delta_i$ we then have

$$K_{i,t+1} = (1-\delta_i)K_{it} + s(\kappa_{it})Y_{it}. \tag{22}$$

We refer to K_{it} simply as the capital stock in what follows.

In addition to being twice continuously differentiable, we assume that f is strictly increasing and strictly concave, and satisfies the condition $f(0) = 0$, as well as the Inada conditions $\lim_{\kappa\to 0} f'(\kappa) = +\infty$, and $\lim_{\kappa\to\infty} f'(\kappa) = 0$, for any given value of $\kappa_{it} = \kappa$.[12] The Inada conditions in turn imply that[13]

$$\lim_{\kappa\to\infty} f(\kappa)/\kappa = 0, \tag{23}$$

and that

$$\lim_{\kappa\to 0} f(\kappa)/\kappa = +\infty. \tag{24}$$

It will be useful for what follows to introduce some additional notation. The output-labor ratio will be denoted by y_{it}, the capital-labor ratio by k_{it}, and the capital-output ratio by v_{it}:

$$y_{it} = \frac{Y_{it}}{L_{it}}, \quad k_{it} = \frac{K_{it}}{L_{it}}, \quad \text{and } v_{it} = \frac{K_{it}}{Y_{it}}.$$

Using these definitions, note that we can rewrite the Cobb-Douglas production function (18) as

$$y_{it} = A_{it}\kappa_{it}^{\alpha_i} = A_{it}v_{it}^{\frac{\alpha_i}{1-\alpha_i}}. \tag{25}$$

The main difference between the Solow-Swan and *AK* models can then be seen in their implications for the capital-output ratio, v_{it}. Under the Solow-Swan model with Cobb-Douglas production technology, it is given by $v_{it} = \kappa_{it}^{1-\alpha_i}$, while under *AK* technology it is given by $v_{it} = A_{it}^{-1}$. Note therefore that the Solow-Swan and *AK* models are not nested, and it is not possible to derive the implications of the *AK* model from those of the Solow-Swan theory by letting α_i, the exponent of the capital stock in (18), approach unity.

The stochastic difference equation describing the law of motion for the effective capital-labor ratio $\kappa_{i,t+1}$ is now easily obtained for both growth models. For the Solow-Swan growth model described by (21), (22), and (2) to (5), we have that [14]

$$\kappa_{i,t+1} = \exp(-\tau_i - \Delta u_{i,t+1})[s(\kappa_{it})f(\kappa_{it}) + (1-\delta_i)\kappa_{it}], \tag{26}$$

and for the AK growth model described by (20), (22), and (2) to (5), we have that

$$\kappa_{i,t+1} = \exp(-\tau_i - \Delta u_{i,t+1})[s(\kappa_{it})A_{it}\kappa_{it} + (1-\delta_i)\kappa_{it}], \tag{27}$$

where $\tau_i = n_i + g_i$, and

$$\Delta u_{i,t+1} = -(1-\rho_{ia})u_{iat} - (1-\rho_{il})u_{ilt} + \varepsilon_{i,t+1}, \quad \varepsilon_{it} \sim iid(0, \sigma_i^2), \tag{28}$$

with $u_{it} = u_{iat} + u_{ilt}$, $\varepsilon_{it} = \varepsilon_{iat} + \varepsilon_{ilt}$, $\sigma_i^2 = \sigma_{ia}^2 + \sigma_{il}^2 + 2\sigma_{ial}$, and $\sigma_{ial} = \text{cov}(\varepsilon_{iat}, \varepsilon_{ilt})$. To keep the notation simple, we suppress the country subscript i whenever our concern is with the time-series dimension only.[15]

3. Steady-State Distributions and Their Properties

In this section, we examine the asymptotic behavior of the stochastic difference equations (26) and (27) and derive the conditions under which the effective capital-labor ratio process $\{\kappa_t\}$ and the capital-output ratio process $\{v_t\}$ are ergodic in moments—namely, whether they converge to globally attracting steady-state probability distribution functions and whether the moments of these probability distribution functions exist and can be consistently estimated using long time averages. As is shown in detail below, not all the variables in the

Solow-Swan and AK growth models are ergodic, and the analysis of their cross-sectional implications has to be carried out with care. It is, therefore, important to identify the variables or their transformations that remain ergodic under fairly general assumptions concerning the technology and labor input processes.

It is useful to distinguish between the cases where the technology and labor input processes are stationary and where they contain unit roots. Accordingly, we distinguish between four cases:

- The Solow-Swan growth model with $\rho_a = 1$ and $\rho_l = 1$ (the unit root case),
- The Solow-Swan growth model with $|\rho_a| < 1$ and $|\rho_l| < 1$ (the stationary case),
- The AK growth model with $\rho_a = 1$ and $\rho_l = 1$, and
- The AK growth model with $|\rho_a| < 1$ as well as $|\rho_l| < 1$.

The following notation is used in this section: We denote the random variable underlying the steady-state probability distribution of $\{\kappa_t\}$ (or $\{\upsilon_t\}$) by κ_∞ (or υ_∞), if the distribution exists. A realization of κ_∞ (or υ_∞) is denoted by $\widetilde{\kappa}_\infty$ (or $\widetilde{\upsilon}_\infty$).

3.1. Case 1: Solow-Swan Growth Model with Unit Roots in Technology and Labor Input ($\rho_a = \rho_l = 1$)

In this case the stochastic difference equation describing the law of motion for the effective capital-labor ratio (26) is given by

$$\kappa_{t+1} = \exp(-\tau - \varepsilon_{t+1})[s(\kappa_t) f(\kappa_t) + (1-\delta)\kappa_t]. \tag{29}$$

We now establish the conditions under which $\{\kappa_t\}$ is ergodic in the first r moments. Our analysis involves three separate steps. In the first step, we show that with probability one $\{\kappa_t\}$ does not get absorbed by the bounds of zero or positive infinity for any strictly positive and finite initial condition κ_0. The second step under some further conditions establishes that with probability one the dependence of $\{\log \kappa_t\}$ on the initial condition $\log \kappa_0$ vanishes asymptotically and that $\{\log \kappa_t\}$ can in the limit be "enveloped" by a covariance-stationary first-order linear autoregressive process with innovations $\{\varepsilon_t\}$. Thus $\log \kappa_t$ as generated by (29) has a globally attracting, time-invariant limiting probability distribution function with moments up to order r so long as the shocks ε_t have moments up to that order. Finally, in step three, we show that if $\log \kappa_\infty$ has finite moments up to order r, then so does κ_∞, and thus $\{\kappa_t\}$ is ergodic in its first r moments.

In what follows, the results are set out in propositions, and all the proofs are given in a mathematical appendix. In the appendix, we also refer to some of the mathematics literature on establishing ergodicity for nonlinear discrete-time Markov processes.

Proposition 1 (Ergodicity in rth Moment; Unit Root Forcing Variables): *Suppose that*

(i) $\dfrac{s(\kappa)f(\kappa)}{\kappa}$ *is monotonically decreasing in* κ, (30)

(ii) $\dfrac{s(\kappa)f(\kappa)}{\kappa} \to 0$ *as* $\kappa \to \infty$, (31)

$$\frac{s(\kappa)f(\kappa)}{\kappa} \to +\infty \text{ as } \kappa \to 0, \tag{32}$$

(iii) $F_\varepsilon[\log(1-\delta) - n - g] = 0$, (33)

where F_ε *denotes the cumulative probability distribution function of* ε_t,

(iv) $(1-\delta) + s(\kappa)f'(\kappa) + s'(\kappa)f(\kappa) > 0$, (34)

and

$$s(\kappa)f(\kappa)/\kappa > s'(\kappa)f(\kappa) + s(\kappa)f'(\kappa), \tag{35}$$

for all finite, strictly positive values of κ,

(v) The absolute shocks, $|\varepsilon_t|$, $t = 1, 2, \ldots,$ *are identically and independently distributed,*

and

(vi) The moment generating function of $|\varepsilon_t|$, *denoted by* $M_\varepsilon(\theta)$, *exists for all* $|\theta| \leq r$.

Then for the discrete-time stochastic Solow-Swan growth model given by (21), (22), and (2) to (5) with $\rho_a = \rho_l = 1$ *the capital-labor ratio in terms of effective units of labor input* $\{\kappa_t\}$ *is ergodic in its* r*th moment. Furthermore, under the Cobb-Douglas specification (18) the capital-output ratio* $\{\upsilon_t\}$ *is also ergodic in its* r*th moment.*

Remark 3.1. From (21), it is readily seen that output per capita y_t is not ergodic if $\rho_a = 1$.

Condition (30) is a basic possibility requirement and ensures that for any admissible realization of ε, one can uniquely solve for κ in

$$\kappa = \exp(-\tau - \varepsilon)[s(\kappa)f(\kappa) + (1-\delta)\kappa].$$

Conditions (31) to (33) are required so that $\{\kappa_t\}$ is bounded away from zero and positive infinity with probability one.[16] Condition (33) is satisfied if the distribution of ε_t is truncated at the lower tail, so that large negative shocks are ruled out. However, note that (33) does not impose any restrictions on large positive shocks.[17] It is worth pointing out that condition (31) can be somewhat relaxed to allow for the possibility of endogenous, steady-state growth, but only at the expense of a tightening of the truncation condition (33). For example, if we allow $s(\kappa)f(\kappa)/\kappa$ to tend to a non-zero positive constant c as $\kappa \to \infty$, then we must have $F_\varepsilon[\log(c+1-\delta) - n - g] = 0$. An example where $s(\kappa)f(\kappa)/\kappa$ tends to a nonzero positive constant as $\kappa \to \infty$ is given by the constant-elasticity-of-substitution (CES) production function with the elasticity of substitution being larger than unity (see, for example, Barro and Sala-i-Martin, 1995, p. 44).

To show that, asymptotically, $\{\log \kappa_t\}$ with probability one does not depend on the initial condition κ_0 and is "enveloped" by a covariance-stationary first-order linear autoregressive process with innovations $\{\varepsilon_t\}$, we assume that (34) and (35) hold. If saving is a constant fraction $s \in (0, 1)$ of output, and $f(\kappa) = \kappa^\alpha$, $\alpha \in (0, 1)$, the Cobb-Douglas case, both conditions are satisfied for all t, as may be trivially verified. In the Cobb-Douglas case where the saving rate is not necessarily constant, the conditions (34) and (35) become

$$(1-\delta) + \alpha s(\kappa)\kappa^{\alpha-1} + s'(\kappa)\kappa^\alpha > 0, \tag{36}$$

and

$$\frac{s'(\kappa)\kappa}{s(\kappa)} < (1-\alpha), \tag{37}$$

for all finite, strictly positive values of κ. The first condition is satisfied if the saving rate is a nondecreasing function of the capital stock per unit of effective labor input. The second condition imposes a further restriction on the rate at which the saving function may vary positively with κ.

To establish that not only $\log \kappa_t$ has finite moments but that κ_t itself also does (and hence, in the Cobb-Douglas case, the capital-output ratio, $v_t = \kappa_t^{1-\alpha}$), we assume that conditions v and vi hold. As will be clear from the proof given in the appendix, the condition that $|\varepsilon_t|$, $t = 1, 2, \ldots$, have an identical distribution can be relaxed without much further complication.

3.2. *Case 2: Solow-Swan Growth Model with Stationary Technology and Labor Input ($|\rho_a| < 1$, $|\rho_l| < 1$)*

The method of analysis in this case is complicated by the fact that in general $\{\kappa_t\}$ now depends on the initial value of the shock, u_0. We first establish that for any pair of finite initial conditions κ_0 ($\kappa_0 > 0$) and u_0, $\{\kappa_t\}$ with probability one does not get absorbed by the bounds of zero or positive infinity if the conditions (30) to (32) are satisfied, and if the truncation condition

$$F_\zeta[\log(1-\delta) - n - g] = 0 \tag{38}$$

is satisfied, where $F_\zeta(\cdot)$ is the limiting cumulative distribution function of the composite shocks

$$\zeta_{t+1}(\rho_a, \rho_l) = -(1-\rho_a)u_{at} - (1-\rho_l)u_{lt} + \varepsilon_{t+1}. \tag{39}$$

Then, in a second step, under the further conditions (34) and (35) we show that asymptotically $\{\log \kappa_t\}$ with probability one does not depend on the initial condition κ_0 and can be "enveloped" by a covariance-stationary first-order linear autoregressive process with innovations $\{\varepsilon_t\}$. Thus $\{\log \kappa_t\}$ converges to a globally attracting steady-state probability distribution function possessing moments up to order r so long as the shocks ε_t have moments up to the same order. Finally, in the third step, we show that the rth order moment of the limiting distribution κ_∞ (and hence of v_∞ in the Cobb-Douglas case) also exists if $\log \kappa_\infty$ has finite moments up to order r—namely, $\{\kappa_t\}$ is ergodic in its rth moment.

Proposition 3.2 (Ergodicity in rth Moment; Stationary Forcing Variables): *If the conditions (30) to (32), (38) as well as (34) and (35) are satisfied, ρ_a, ρ_l, and γ_t (defined by (113)) are distinct for all t, the absolute shocks $|\varepsilon_t|$, $t = 1, 2, \ldots$, are identically and independently distributed, and $M_\varepsilon(\theta)$ exists for all $|\theta| \leq r$, then for the discrete-time stochastic Solow-Swan growth model given by (21), (22), and (2) to (5) with $|\rho_a| < 1$ and $|\rho_l| < 1$ the capital-labor ratio in terms of effective units of labor input, $\{\kappa_t\}$, is ergodic in its rth moment. Furthermore, under the Cobb-Douglas specification (18) the capital-output ratio $\{v_t\}$ is also ergodic in its rth moment.*

3.3. *Case 3: AK Growth Model with Unit Roots in Technology and Labor Input ($\rho_a = \rho_l = 1$)*

In this case neither $\{\log \kappa_t\}$ nor $\{\log v_t\}$ are ergodic. The proof for $\{\log \kappa_t\}$ is provided in the appendix. For $\{\log v_t\}$, from (20) and (2) it is immediate that

$$\log v_t = -a_0 - gt - u_{at}, \tag{40}$$

and thus, regardless of whether g is zero or not, $\{\log v_t\}$ is not ergodic, since $\{u_{at}\}$ contains a unit root. It is also interesting to note that the same conclusion holds for $\{\Delta \log y_t\}$. From (22) we have that

$$\Delta \log K_{t+1} = \log[s(\kappa_t)\exp(a_0 + gt + u_{at}) + (1 - \delta)], \tag{41}$$

which, when substituted into (20), yields

$$\begin{aligned} \Delta \log y_{t+1} &= \Delta \log A_{t+1} - \Delta \log L_{t+1} + \log[s(\kappa_t)\exp(a_0 + gt + u_{at}) + (1 - \delta)] \\ &= g - n + \log[s(\kappa_t)\exp(a_0 + gt + u_{at}) + (1 - \delta)] \\ &\quad + \Delta u_{a,t+1} - \Delta u_{l,t+1}. \end{aligned} \tag{42}$$

As $\{u_{at}\}$ contains a unit root, this implies that $\{\Delta \log y_t\}$ is not ergodic, regardless of the functional form of $s(\kappa_t)$, and whether or not g is zero.

3.4. *Case 4: AK Growth Model with Stationary Technology and Labor Input ($|\rho_a| < 1, |\rho_l| < 1$)*

In this last case we show that $\{\kappa_t\}$ may or may not be ergodic in moments, whereas $\{v_t\}$ is ergodic in moments, provided $g = 0$.

We again analyze $\{\kappa_t\}$ first. It is shown in the appendix that to apply the same kind of "envelope argument" as for the Solow-Swan growth model in Cases 1 and 2, the following conditions (in addition to $g = 0$) need to hold:

$$(1-\delta) + \left[s(\kappa) + \frac{1}{2}s'(\kappa)\kappa\right]\exp(a_0 + u_a) > 0, \tag{43}$$

and

$$s'(\kappa) < 0, \tag{44}$$

for all finite, strictly positive values of κ and all finite values of u_a. If these conditions do not hold, $\{\log \kappa_t\}$ is not stationary and does not converge to a globally attracting steady-state probability distribution function.

As far as $\{\log v_t\}$ is concerned, again using (40),

$$\log v_t = -a_0 - gt - u_{at},$$

it is clear that $\{\log v_t\}$ will be ergodic in moments if $g = 0$ and the shocks ε_{at} have finite moments. If $g \neq 0$, $\{\log v_t\}$ would have a trend component, which seems implausible.

As in Case 3, it is also interesting to note that the same conclusion holds for $\Delta \log y_t$. As before (see (42)), we have

$$\Delta \log y_{t+1} = g - n + \log[s(\kappa_t)\exp(a_0 + gt + u_{at}) + (1-\delta)] + \Delta u_{a,t+1} - \Delta u_{l,t+1}.$$

Thus, $\{\Delta \log y_t\}$ is ergodic in moments if $g = 0$ and $s(\kappa_t) = s$ for all t, or if $g = 0$ and the conditions (43) and (44) hold, so long as the shocks ε_{at} and ε_{lt} have finite moments.[18]

4. Stochastic Steady State and Cross-Sectional Implications

We now use the results obtained in the previous section on the ergodic properties of the (logarithm of the) capital-output ratio v_{it} to derive a number of cross-sectional implications of the stochastic Solow-Swan and AK growth models. We assume that ε_{it} is distributed independently both across time and across economies.

4.1. Stochastic Solow-Swan Growth Model

We begin by considering the cross-sectional implications of the stochastic Solow-Swan growth model. All derivations in this section are, for compatibility with the recent empirical growth literature (for example, Mankiw, Romer, and Weil, 1992), based on a saving rate s_i that is fixed over time and the Cobb-Douglas production function (18).

4.1.1. Cross-Sectional Implication 1 To derive our first cross-sectional implication, we consider the case of unit roots in technology and labor input ($\rho_{ia} = \rho_{il} = 1$). We assume that the combined technology and labor input innovation, $\varepsilon_{it} = \varepsilon_{iat} + \varepsilon_{ilt}$, is distributed as one-sided truncated normal with mean zero, variance σ_i^2, and a truncation point at

$b_i = \log(1 - \delta_i) - \tau_i$:

$$\varepsilon_{it} \sim \textit{iid Truncated } N(0, \sigma_i^2), \quad F_{\varepsilon_i}(b_i) = 0. \tag{45}$$

We shall denote the mean and the variance of the underlying nontruncated normal distribution by μ_i and ω_i^2. For each economy i, taking conditional expectations of both sides of (29), we have

$$E(\kappa_{i,t+1} \mid \kappa_{it}) = \psi(\kappa_{it}), \tag{46}$$

where

$$\psi(\kappa_{it}) = \varphi_i \exp(-\tau_i)[s_i \kappa_{it}^{\alpha_i} + (1 - \delta_i)\kappa_{it}], \tag{47}$$

and

$$\varphi_i = E[\exp(-\varepsilon_{i,t+1})]. \tag{48}$$

Using (45), it can be shown that[19]

$$\varphi_i = \exp(-\mu_i + \tfrac{1}{2}\omega_i^2) \left[\frac{1 - \operatorname{erf}\left(\frac{b_i + \omega_i^2 - \mu_i}{\sqrt{2}\omega_i} \right)}{1 - \operatorname{erf}\left(\frac{b_i - \mu_i}{\sqrt{2}\omega_i} \right)} \right], \tag{49}$$

where $\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x \exp(-t^2) dt$. Taking unconditional expectations of both sides of (46) we obtain

$$E(\kappa_{i,t+1}) = \varphi_i \exp(-\tau_i)[s_i E(\kappa_{it}^{\alpha}) + (1 - \delta_i) E(\kappa_{it})]. \tag{50}$$

As noted in Section 3, in the present special case where the saving rate s_i is fixed over time and the production function has the Cobb-Douglas form, the conditions (30) to (32) as well as (34) and (35) are clearly satisfied. Noting that under (45) condition (33) will also be satisfied and all moments of ε_{it} exist, then all moments of $\{\kappa_{it}\}$ also exist.

As before denoting the random variable underlying the steady-state distribution of κ_{it} by $\kappa_{i\infty}$, we therefore have $\lim_{t\to\infty} E(\kappa_{it}) = E(\kappa_{i\infty})$, and $\lim_{t\to\infty} E(\kappa_{it}^{\alpha}) = E(\kappa_{i\infty}^{\alpha})$. Using these results in (50) now yields:[20]

$$E(\kappa_{i\infty}) = \varphi_i \exp(-\tau_i)[s_i E(\kappa_{i\infty}^{\alpha_i}) + (1 - \delta_i) E(\kappa_{i\infty})], \tag{51}$$

or

$$w_i E[\exp(\log \kappa_{i\infty})] = s_i E[\exp(\alpha_i \log \kappa_{i\infty})], \tag{52}$$

where

$$w_i = \frac{\exp(\tau_i)}{\varphi_i} - (1 - \delta_i). \tag{53}$$

Denoting the moment generating function of $\log \kappa_{i\infty}$ by $M_{\log \kappa_{i\infty}}(\theta) = E[\exp(\theta \log \kappa_{i\infty})]$, we also have

$$\log \left[M_{\log \kappa_{i\infty}}(\theta) \right] = \log \left[\sum_{r=0}^{\infty} \frac{\theta^r}{r!} \mu_r'(\log \kappa_{i\infty}) \right] = \sum_{r=1}^{\infty} \frac{\theta^r}{r!} c_r(\log \kappa_{i\infty}), \tag{54}$$

where $\mu_r'(\log \kappa_{i\infty})$ is the rth moment of $\log \kappa_{i\infty}$, and $c_r(\log \kappa_{i\infty})$ is the rth cumulant of $\log \kappa_{i\infty}$. Therefore, the moment condition (52) can be written as

$$w_i M_{\log \kappa_{i\infty}}(1) = s_i M_{\log \kappa_{i\infty}}(\alpha_i). \tag{55}$$

Taking logarithms of both sides of this relation we obtain

$$\log[M_{\log \kappa_{i\infty}}(1)] - \log[M_{\log \kappa_{i\infty}}(\alpha_i)] = \log s_i - \log w_i. \tag{56}$$

Hence, in terms of the cumulants of $\log \kappa_{i\infty}$ we have (using (54))[21]

$$\sum_{r=1}^{\infty} \frac{1}{r!}(1 - \alpha_i^r) c_r(\log \kappa_{i\infty}) = \log s_i - \log w_i. \tag{57}$$

This cross-sectional moment condition differs from the corresponding relationship implied by the deterministic Solow-Swan model that has been used extensively in the empirical growth literature in two important aspects. First, since $c_1(\log \kappa_{i\infty}) = E(\log \kappa_{i\infty})$ and $c_2(\log \kappa_{i\infty}) = Var(\log \kappa_{i\infty})$, the mean of the steady-state distribution of $\log \kappa_{it}$ depends on the variance and higher-order cumulants of the distribution. Second, under the deterministic version of the model the definition of w_i is given by $w_i = \exp(\tau_i) - (1 - \delta_i)$, and not by (53).

Since κ_{it} is not directly observable, for empirical analysis one may want to exploit the result $v_{it} = \kappa_{it}^{1-\alpha_i}$ and write (57) in terms of the random variable $v_{i\infty}$ representing the steady-state distribution of the capital-output ratio:

Cross-Sectional Implication 1: *For the stochastic Solow-Swan growth model given by (21), (22), (2) to (5), and (45) with a constant saving rate s_i, and $\rho_a = \rho_l = 1$, the mean of the logarithm of the capital-output ratio is associated with the saving rate, the growth rate of labor input, the rate of depreciation, and the variance and all higher-order cumulants of the logarithm of the capital-output ratio through*

$$\begin{aligned} E(\log v_{i\infty}) \; = \; & \log s_i - \log w_i - \frac{1}{2}\left(\frac{1+\alpha_i}{1-\alpha_i}\right) Var(\log v_{i\infty}) \\ & - \sum_{r=3}^{\infty} \frac{1}{r!}\left(\frac{1-\alpha_i^r}{(1-\alpha_i)^r}\right) c_r(\log v_{i\infty}). \end{aligned} \tag{58}$$

Note that the variance and all higher-order cumulants of the logarithm of the capital-output ratio depend on the parameters of the technology and labor input processes. Cross-Sectional Implication 1 is, of course, not a surprising result. It holds generally true for stochastic nonlinear difference equations ergodic in moments that the mean of the steady-state probability distribution function is related to the variance and higher-order cumulants of the distribution. The contribution here is to derive the exact form of the cross-sectional relationship for the stochastic Solow-Swan model, which to our knowledge is novel.

4.1.2. Cross-Sectional Implication 2 The second cross-sectional implication we derive holds irrespective of whether the technology and labor input processes contain unit roots.[22] From (18) and the definition of y_{it} and κ_{it} we have

$$\Delta \log y_{i,t+1} = g_i + \alpha_i \Delta \log \kappa_{i,t+1} - (1 - \rho_{ia})u_{iat} + \varepsilon_{ia,t+1}. \tag{59}$$

Substituting (26) into (59), one obtains

$$\begin{aligned}\Delta \log y_{i,t+1} &= (g_i - \alpha_i \tau_i) + \alpha_i \log\left(\frac{s_i}{v_{it}} + 1 - \delta_i\right) - (1 - \alpha_i)(1 - \rho_{ia})u_{iat} \\ &\quad + \alpha_i(1 - \rho_{il})u_{ilt} + (1 - \alpha_i)\varepsilon_{ia,t+1} - \alpha_i \varepsilon_{il,t+1}.\end{aligned} \tag{60}$$

Taking unconditional expectations of (60) and noting from (59) that under $\{\log \kappa_{it}\}$ being ergodic $E(\Delta \log y_{i\infty}) = g_i$, as $t \to \infty$ we have

$$E\left\{\log\left[\frac{s_i}{v_{i\infty}} + (1 - \delta_i)\right]\right\} = \tau_i. \tag{61}$$

This result can also be written as

$$\log\left[\frac{s_i}{v_{i\infty}} + (1 - \delta_i)\right] = \tau_i + \zeta_i, \tag{62}$$

where ζ_i is a stochastic error term satisfying $E(\zeta_i) = 0$. Hence

$$v_{i\infty} = \frac{s_i}{\exp(\tau_i + \zeta_i) - (1 - \delta_i)}, \tag{63}$$

and since $s_i/[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]$ and $\log\{s_i/[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]\}$ are strictly convex functions of ζ_i for all realizations of ζ_i (and all strictly positive realizations of $v_{i\infty}$),[23] by Jensen's inequality we have the following relations between the mean of (the logarithm of) the steady-state distribution of the capital-output ratio under the stochastic model and (the logarithm of) the steady-state capital-output ratio under the deterministic model.

Cross-Sectional Implication 2:[24] *For the stochastic Solow-Swan growth model given by (21), (22), and (2) to (5) with a constant saving rate s_i, the truncation condition (38) satisfied, and $|\rho_{ia}| \le 1$, $|\rho_{il}| \le 1$, we have*

$$E(v_{i\infty}) > \frac{s_i}{\exp(\tau_i) - (1 - \delta_i)} = v_i^*, \quad \textit{and} \quad E(\log v_{i\infty}) > \log(v_i^*). \tag{64}$$

When the saving rate varies with κ_{it}, we can write

$$E\left\{\log\left[\frac{s_{i\infty}}{v_{i\infty}} + (1 - \delta_i)\right]\right\} = \tau_i, \tag{65}$$

where $s_{i\infty} = s(\kappa_{i\infty})$, and it follows, for all strictly positive realizations of $v_{i\infty}$, that

$$E\left(\frac{v_{i\infty}}{s_{i\infty}}\right) > \frac{v_i^*}{s_i^*}, \tag{66}$$

where s_i^* stands for the deterministic steady-state value of the saving rate.

4.2. Stochastic AK Growth Model

In this case, recall from (40) that

$$\log v_{it} = -a_{i0} - g_i t - u_{iat}.$$

When $g_i = 0$, $|\rho_{ia}| < 1$, and the shocks ε_{iat} have finite moments, $\{\log v_{it}\}$ is covariance stationary and ergodic in moments, and in a regression of $E(\log v_{i\infty})$ on s_i, n_i, and/or ω_i^2 (the variance of the nontruncated shocks ε_{it}), the latter variables should all be insignificant, unless it is assumed that a_{i0} varies with s_i, n_i, and/or ω_i^2. The case where $g_i \neq 0$ results in a trending v_{it}, which does not seem plausible.

Cross-Sectional Implication 3: *For the stochastic AK growth model given by (20), (22), and (2) to (5), with $g_i = 0$, $|\rho_{ia}| < 1$, and the shocks ε_{iat} having finite moments, in a regression of $E(\log v_{i\infty})$ on s_i, n_i, and/or ω_i^2, the latter variables should all be insignificant, unless it is assumed that a_{i0} varies with s_i, n_i, and/or ω_i^2.*

If there is a unit root in u_{iat}, for all i one may work with the first difference of $\log v_{it}$, $\triangle \log v_{it}$, as the latter is ergodic in moments. Again, s_i, n_i, and/or ω_i^2 should all be insignificant in a regression with $E(\triangle \log v_{i\infty})$ as the dependent variable. In this case, even if a_{i0} varied with s_i, n_i, and/or ω_i^2, we would not expect any of s_i, n_i, or ω_i^2 to be significant.

Cross-Sectional Implication 4: *For the stochastic AK growth model given by (20), (22), and (2) to (5), in a regression of $E(\triangle \log v_{i\infty})$ on s_i, n_i, and/or ω_i^2, the latter variables should all be insignificant.*

It is worth noting that the different cross-sectional implications obtained for the stochastic Solow-Swan and AK growth models are not due to the differences in our specification of technological progress under these two models. While technological progress is specified as labor augmenting in the Cobb-Douglas case, (18), and as factor-neutral in the AK technology case, (20), specifying technological progress as factor neutral in (18) would not affect Cross-Sectional Implication 1. As any production function that is multiplicative and strictly concave in all factors of production can be rewritten in a form such that technological progress is labor augmenting, all that is required for our analysis to apply to a Cobb-Douglas specification with factor-neutral technological progress is a redefinition of k_{it}.

5. Time-Series Implications

In this section, we derive a nonlinear vector autoregression (NVAR) in output per capita, labor input, and the capital-output ratio implied by the stochastic Solow-Swan growth model. This NVAR can, in principle, be used to study the interaction between stochastic technology and/or labor input and economic growth in time series on a single country or in data panels on several countries. While such an empirical analysis is beyond the scope of the current article, we use this NVAR in Section 6 below to obtain estimates of the parameters of the truncated normal distribution (45), for the innovations to technological progress ε_{at}, and to labor input ε_{lt}.[25] We assume in this section that the production function is Cobb-Douglas, (18), and that $\{u_{at}\}$ and $\{u_{lt}\}$ are generated by second-order autoregressive processes, which we write as

$$\Delta u_{at} = -(1-\rho_a)u_{a,t-1} - \rho_{a2}\Delta u_{a,t-1} + \varepsilon_{at}, \tag{67}$$

and

$$\Delta u_{lt} = -(1-\rho_l)u_{l,t-1} - \rho_{l2}\Delta u_{l,t-1} + \varepsilon_{lt}. \tag{68}$$

We allow the roots of $1-\rho_{a1}\zeta - \rho_{a2}\zeta^2 = 0$ and of $1-\rho_{a1}\zeta - \rho_{a2}\zeta^2 = 0$ to be on or outside the unit circle, where $\rho_{a1} = \rho_a - \rho_{a2}$ and $\rho_{l1} = \rho_l - \rho_{l2}$.

To derive the dynamic equation for the logarithm of output per capita, first note from (2) and (25) that

$$\triangle \log y_{t+1} = g + \alpha \triangle \log k_{t+1} - (1-\rho_a)u_{at} - \rho_{a2}\Delta u_{at} + \varepsilon_{a,t+1}. \tag{69}$$

Using the Cobb-Douglas specification in (26), we have

$$\triangle \log k_{t+1} = \log\left(\frac{s_t}{v_t} + 1 - \delta\right) - n - g - \Delta u_{t+1}. \tag{70}$$

Substituting this result in (69) and making use of equations (2), (4), and (25) to write the lagged disturbances in terms of parameters and observables, we obtain[26]

$$\begin{aligned}
\log y_{t+1} \;=\;& g(1-\alpha)(1+\rho_{a2}) - n\alpha(1+\rho_{l2}) \\
&+ a_0(1-\alpha)(1-\rho_a) - l_0\alpha(1-\rho_l) \\
&+ [\rho_{a1} + \alpha(1-\rho_{a1})]\log y_t + (1-\alpha)\rho_{a2}\log y_{t-1} \\
&+ \alpha(1-\rho_{l1})\log L_t - \alpha\rho_{l2}\log L_{t-1} \\
&+ \alpha(1-\rho_{a1})\log v_t - \alpha\rho_{a2}\log v_{t-1} \\
&+ \alpha\log\left[\frac{s_t}{v_t} + (1-\delta)\right] \\
&+ [g(1-\alpha)(1-\rho_a) - n\alpha(1-\rho_l)]t \\
&+ (1-\alpha)\varepsilon_{a,t+1} - \alpha\varepsilon_{l,t+1}.
\end{aligned} \tag{71}$$

Using (4) and (68), one obtains the equation for the logarithm of labor input

$$\begin{aligned}\log L_{t+1} &= n(1+\rho_{l2}) + l_0(1-\rho_l)\\ &\quad + \rho_{l1}\log L_t + \rho_{l2}\log L_{t-1}\\ &\quad + n(1-\rho_l)t + \varepsilon_{l,t+1}. \end{aligned} \tag{72}$$

Finally, recalling that $\log v_t = (1-\alpha)\log\kappa_t$, and using (26), we obtain the following equation for the logarithm of the capital-output ratio:

$$\begin{aligned}\log v_{t+1} &= -g(1-\alpha)(1+\rho_{a2}) - n(1-\alpha)(1+\rho_{l2})\\ &\quad - a_0(1-\alpha)(1-\rho_a) - l_0(1-\alpha)(1-\rho_l)\\ &\quad + (1-\alpha)(1-\rho_{a1})\log y_t - (1-\alpha)\rho_{a2}\log y_{t-1}\\ &\quad + (1-\alpha)(1-\rho_{l1})\log L_t - (1-\alpha)\rho_{l2}\log L_{t-1}\\ &\quad + [1-\alpha(1-\rho_{a1})]\log v_t + \alpha\rho_{a2}\log v_{t-1}\\ &\quad + (1-\alpha)\log\left[\frac{s_t}{v_t} + (1-\delta)\right]\\ &\quad - [g(1-\alpha)(1-\rho_a) - n(1-\alpha)(1-\rho_l)]t\\ &\quad - (1-\alpha)(\varepsilon_{a,t+1} + \varepsilon_{l,t+1}). \end{aligned} \tag{73}$$

Under the assumption of an exogenously given saving rate, equations (71) to (73) constitute a nonlinear VAR model in the logarithm of output per capita, the logarithm of labor input, and the logarithm of the capital-output ratio. The only nonlinear term in the model is given by $\log[s_t/v_t + (1-\delta)]$. Note that for the deterministic Solow-Swan model with constant saving rate s, the steady-state capital output ratio is constant, given by $s/[\exp(\tau) - (1-\delta)]$. Thus, while the stochastic Solow-Swan model implies a nonlinear vector autoregression in the logarithms of output per capita, the capital-output ratio, and labor input, its deterministic counterpart predicts a linear autoregressive equation in the logarithm of output per capita. The above NVAR model is also subject to important testable parametric restrictions. However, in the remainder of this article we focus our attention on the cross-sectional implications, and estimate an unrestricted version of the NVAR model only for the purpose of obtaining estimates of the variances and the covariances of ε_{at} and ε_{lt} (namely, σ_a^2, σ_l^2, and σ_{al}) that are needed in such a cross-sectional analysis. We do not expect that the cross-sectional results will be much affected by using estimates of σ_a^2, σ_l^2, and σ_{al} that allow for the parametric restrictions implied by the NVAR model.

6. Cross-Sectional Evidence

Our empirical evaluation of the cross-sectional implications derived in Section 4 is based on a sample of seventy-two countries from the Summers and Heston (1991) Penn World Tables, Version 5.6. The data are annual and cover the period from 1960 to 1992. The seventy-two countries (which we refer to in what follows as the "full sample") are those among the ninety-eight countries in the sample of Mankiw, Romer, and Weil (1992) for which complete data for the above period are available.[27] Analogously to Mankiw, Romer, and Weil (1992), we do not only present results for these seventy-two countries but also present results for two subsets of this sample. The first subset comprises twenty OECD countries, and the second subset contains the fifty-eight "intermediate" countries in the Mankiw, Romer, and Weil (1992) study that are also contained in our full sample.[28]

In line with the recent empirical growth literature, our analysis in this section is based on a saving rate s_i that is fixed over time and, for the stochastic Solow-Swan growth model, the Cobb-Douglas production function (18). We also assume that the combined technology and labor input innovation ε_{it} follows the truncated normal distribution (45) and is distributed independently across countries.

We construct time series on capital stock $\{K_{it}\}$ from 1961 to 1992 using the familiar perpetual inventory formula

$$K_{it} = (1-\delta_i)^t K_{i0} + \sum_{j=0}^{t-1} (1-\delta_i)^j I_{i,t-j-1}, \quad t = 1, 2, \ldots, \tag{74}$$

where I_{it} is gross investment in country i at time t. For the initial values, K_{i0} (the capital stock in 1960), we use the simple backcasting "steady-state" values of the averages of gross investment over the first five years in the sample, divided by the rates of depreciation.

We begin our empirical analysis with the Cross-Sectional Implication 1, derived in Section 4. Truncating (58) at $r = 2$, we have[29]

$$E(\log v_{i\infty}) \approx \log s_i - \log w_i - \frac{1}{2}\left(\frac{1+\alpha_i}{1-\alpha_i}\right) Var(\log v_{i\infty}). \tag{75}$$

Given the ergodicity results of Section 3, the population moments $E(\log v_{i\infty})$ and $Var(\log v_{i\infty})$ can be consistently estimated by long time averages. Such estimates of $E(\log v_{i\infty})$ and $Var(\log v_{i\infty})$ are given by

$$\hat{E}(\log v_{i\infty}) = \frac{1}{19} \sum_{t=1974}^{1992} \log v_{it}, \tag{76}$$

and

$$\widehat{Var}(\log v_{i\infty}) = \frac{1}{18} \sum_{t=1974}^{1992} \left[\log v_{it} - \hat{E}(\log v_{i\infty})\right]^2, \tag{77}$$

respectively. To reduce the impact of incorrect estimates of K_{i0} (and thus v_{i0}) on our results we drop all observations before 1974 when constructing cross-sectional estimates involving capital-output ratios.[30] To investigate the empirical validity of (75) (as well as Cross-Sectional Implications 2 to 4) we also need estimates of the saving rate s_i, the growth rate of labor input n_i, the deterministic rate of technological progress g_i, the rate of depreciation δ_i, the combined share of physical and human capital α_i, and the mean μ_i and variance ω_i^2 of the nontruncated normal distribution underlying the truncated normal distribution for the combined technology and labor input innovation ε_{it}. (Recall that n_i, g_i, δ_i, α_i, μ_i, and ω_i^2 all enter w_i, defined by (53).)

The estimates of s_i and n_i are obtained for each country as long-run averages of the share of investment in gross domestic product and the growth rate of the working-age population, respectively:

$$\hat{s}_i = \frac{1}{33}\sum_{t=1960}^{1992} s_{it}, \quad \text{and} \quad \hat{n}_i = \frac{1}{32}\sum_{t=1961}^{1992} n_{it}. \tag{78}$$

The cross-country means of the estimates $\hat{s}_i$ and $\hat{n}_i$ across the three country groups and a number of other summary statistics regarding these estimates are reported in Table 1.

The estimates of μ_i and ω_i^2, $\hat{\mu}_i$ and $\hat{\omega}_i^2$ (which enter w_i through the parameter φ_i defined by (48)) are obtained using the following two-step procedure. In the first step, the NVAR model given by (71) to (73) is estimated, without imposing the parametric restrictions implied by the stochastic Solow-Swan model either on the regression coefficients or on the variance-covariance matrix. Given our assumption that ε_{it} has a truncated normal distribution, the residuals of this VAR are realizations of a truncated normal distribution with mean zero and variance σ_{iy}^2, σ_{il}^2, and σ_{iv}^2, respectively. We now show how to compute an estimate of σ_i^2 from the estimates of σ_{iy}^2, σ_{il}^2, and $\sigma_{iyl} = Cov(\varepsilon_{iyt}, \varepsilon_{ilt})$. Note that the error in the first equation of the NVAR, (71), is given by

$$\varepsilon_{iyt} = (1-\alpha_i)\varepsilon_{iat} - \alpha_i\varepsilon_{ilt}, \tag{79}$$

with variance

$$\sigma_{iy}^2 = \alpha_i^2\sigma_{il}^2 + (1-\alpha_i)^2\sigma_{ia}^2 - 2\alpha_i(1-\alpha_i)\sigma_{ial}, \tag{80}$$

where $\sigma_{ial} = Cov(\varepsilon_{iat}, \varepsilon_{ilt})$. The second equation is the NVAR, (72) yields $\{\varepsilon_{ilt}\}$.[31] Using $\{\varepsilon_{iyt}\}$ and $\{\varepsilon_{ilt}\}$, one may obtain σ_{ial} from

$$\sigma_{iyl} = -\alpha_i\sigma_{il}^2 + (1-\alpha_i)\sigma_{ial}, \tag{81}$$

where $\sigma_{iyl} = Cov(\varepsilon_{iyt}, \varepsilon_{ilt})$. Also, from (80), we have

$$\sigma_{ia}^2 = \frac{\sigma_{iy}^2 - \alpha_i^2\sigma_{il}^2 + 2\alpha_i(1-\alpha_i)\sigma_{ial}}{(1-\alpha_i)^2}. \tag{82}$$

Using the definition

$$\sigma_i^2 = \sigma_{ia}^2 + \sigma_{il}^2 + 2\sigma_{ial},$$

σ_i^2 is now readily computed.

In the second step, we compute the mean μ_i and variance ω_i^2 of the underlying non-truncated normal distribution of ε_{it}, given that ε_{it} itself has mean zero and variance σ_i^2.

Table 1. Summary statistics for different country groupings.

	OECD Sample ($N = 20$)	Intermediate Sample ($N = 58$)	Full Sample ($N = 72$)
$\hat{s}_i$			
Mean	.2603	.1897	.1675
Median	.2543	.1809	.1633
Maximum	.3443	.3443	.3443
Minimum	.1805	.0140	.0140
Standard deviation	.0408	.0737	.0828
Interquartile range	.0407	.1051	.1345
$\hat{n}_i$			
Mean	.0081	.0196	.0206
Median	.0071	.0235	.0238
Maximum	.0239	.0388	.0388
Minimum	.0030	.0030	.0030
Standard Deviation	.0052	.0101	.0095
Interquartile range	.0055	.0177	.0141
$\hat{\mu}_i(\hat{\alpha}_{ML})$			
Mean	−.0006	−.0205	−.0336
Median	−.0003	−.0044	−.0119
Maximum	−.0001	−.0001	−.0001
Minimum	−.0036	−.1555	−.2907
Standard deviation	.0009	.0339	.0524
Interquartile range	.0003	.0252	.0432
$\hat{\omega}_i(\hat{\alpha}_{ML})$			
Mean	.0324	.0583	.0675
Median	.0326	.0497	.0619
Maximum	.0479	.1406	.1815
Minimum	.0208	.0208	.0208
Standard deviation	.0066	.0294	.0361
Interquartile range	.0070	.0429	.0545
$\hat{g}_i$			
Mean	.0282	.0227	.0193
Median	.0264	.0213	.0196
Maximum	.0563	.0654	.0654
Minimum	.0107	−.0200	−.0200
Standard deviation	.0102	.0149	.0163
Interquartile range	.0101	.0155	.0189

Note: $\hat{s}_i$: estimate of saving rate of country i, measured as the sample mean of the share of investment in gross domestic product; $\hat{n}_i$: average growth rate of labor input in country i, measured as the sample mean of the growth rate of the working-age population; $\hat{g}_i$: estimate of deterministic rate of technological progress in country i; $\hat{\mu}_i(\hat{\alpha}_{ML})$ and $\hat{\omega}_i(\hat{\alpha}_{ML})$ denote the estimates of the mean and standard deviation, respectively, of the nontruncated normal distribution underlying the combined technology and labor input innovations, computed at the ML estimate of α, $\hat{\alpha}_{ML}$. See the text for details of the computations.

This may be accomplished by solving the following equations for μ_i and ω_i^2 in terms of $b_i = \log(1-\delta_i) - n_i - g_i$ and σ_i^2:[32]

$$0 = \frac{\frac{1}{2}\mu_i\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega_i}\right)\right] + \frac{\omega_i}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\mu_i-b_i}{\omega_i}\right)^2\right]}{\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega_i}\right)\right]}, \tag{83}$$

and

$$\begin{aligned}\sigma_i^2 \; = \; & \frac{\frac{1}{2}(\mu_i^2+\omega_i^2)\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega_i}\right)\right] + \frac{\omega_i(\mu_i+b_i)}{\sqrt{2\pi}}\exp\left[\text{-}\frac{1}{2}\left(\frac{\mu_i-b_i}{\omega_i}\right)^2\right]}{\frac{1}{2}\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega}\right)\right]} \\ & - \frac{\left\{\frac{1}{2}\mu_i\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega_i}\right)\right] + \frac{\omega_i}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{\mu_i-b_i}{\omega_i}\right)^2\right]\right\}^2}{\frac{1}{4}\left[1+\mathrm{erf}\left(\frac{\mu_i-b_i}{\sqrt{2}\omega_i}\right)\right]^2}.\end{aligned} \tag{84}$$

To implement this two-step procedure, we also need estimates of g_i, δ_i, and α_i. Initially, we follow the recent empirical growth literature and assume that $g_i = g$, $\delta_i = \delta$, and $\alpha_i = \alpha$ for all countries in the sample. This is in line with the convergence hypothesis in deterministic growth models where the same technology is assumed to be available to all countries in the long run. In line with the empirical growth literature we also set $g = .02$ and $\delta = .05$. For the combined share of physical and human capital α, as there is little evidence on its value available in the literature, we estimate it by the maximum likelihood (ML) method using a grid-search procedure. The maximized log-likelihood value, concentrated in terms of α, is given by

$$\ell(\alpha) = -\frac{N}{2}\left[1+\log(2\pi)+\log\left(\frac{\mathbf{e}(\alpha)'\mathbf{e}(\alpha)}{N}\right)\right], \tag{85}$$

where N is the number of countries and $\mathbf{e}(\alpha)$ is a residual vector, with its ith element $e_i(\alpha)$ defined by[33]

$$e_i(\alpha) = \hat{E}(\log v_{i\infty}) - \log\hat{s}_i + \log\hat{w}_i(\alpha) + \frac{1}{2}\left(\frac{1+\alpha}{1-\alpha}\right)\widehat{Var}(\log v_{i\infty}), \tag{86}$$

where $\hat{E}(\log v_{i\infty})$, $\widehat{Var}(\log v_{i\infty})$, and $\hat{s}_i$ are given in (76) to (78), and $\hat{w}_i(\alpha)$ is given by

$$\hat{w}_i(\alpha) = [\exp(\hat{n}_i + g)/\hat{\varphi}_i(\alpha)] - (1-\delta), \tag{87}$$

with $\hat{n}_i$ given in (78), and

$$\hat{\varphi}_i(\alpha) = \exp[-\hat{\mu}_i(\alpha) + \tfrac{1}{2}\hat{\omega}_i^2(\alpha)]\left[\frac{1-\mathrm{erf}\left(\frac{\hat{b}_i+\hat{\omega}_i^2(\alpha)-\hat{\mu}_i(\alpha)}{\sqrt{2}\hat{\omega}_i}\right)}{1-\mathrm{erf}\left(\frac{\hat{b}_i-\hat{\mu}_i(\alpha)}{\sqrt{2}\hat{\omega}_i(\alpha)}\right)}\right]. \tag{88}$$

For the full sample of $N = 72$ countries the ML estimate of α, which we denote by $\hat{\alpha}_{ML}$, is approximately equal to .49, which seems quite reasonable, although it is not very precisely estimated.[34] This is best seen in Figure 1, where the values of the log-likelihood ratio statistic

$$\chi^2(\alpha) = 2[\ell(\hat{\alpha}_{ML}) - \ell(\alpha)] \tag{89}$$

are shown for various values of α in the range (0,1). Although the values of α greater or equal than .67 are clearly rejected at the 95 percent significance level ($\chi^2(\alpha) > 3.84$, for $\alpha \geq .67$), the same cannot be said for values of α less than .49. In fact, it is not possible to reject the hypothesis that $\alpha = .2$, or even less. There seems to be an important degree of asymmetry in the confidence interval around the ML estimate, $\hat{\alpha}_{ML}$. Considering that values of $\alpha < .2$ are implausible on a priori grounds, we base our empirical evaluation of Cross-Sectional Implication 1 on values of α in the range [.20, .66]. To assess whether the empirical evidence is consistent with Cross-Sectional Implication 1, we estimate the following unrestricted version of (86):

$$\begin{aligned}\hat{E}(\log v_{i\infty}) \;=\; & \gamma_0 + \gamma_1 \log \hat{s}_i + \gamma_2 \log \hat{w}_i(\alpha) \\ & + \gamma_3 \left[\frac{1}{2} \left(\frac{1+\alpha}{1-\alpha} \right) \widehat{Var}(\log v_{i\infty}) \right] + \xi_{i1},\end{aligned} \tag{90}$$

where ξ_{i1} is an error term. Cross-Sectional Implication 1 predicts that $\gamma_1 = 1$, $\gamma_2 = -1$, and $\gamma_3 = -1$. For the full sample of seventy-two countries we estimate (90) for different values of α in the range [.20, 66]. The intercept term γ_0 turns out to be statistically insignificant for all values of α in this range. The estimates of γ_1 and γ_2 are also extremely robust to the choice of $\alpha \in [.20, .66]$. The variations in $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are confined to the ranges (.80, .83), and $(-.93, -.91)$, respectively. The ranges of their associated t-ratios are (29.3, 29.8) and $(-44.1, -43.2)$. While the estimates of γ_3 are sensitive to variations in α, ranging from -4.6 to -1.5 as α varies from .20 to .66, the standard errors of these estimates vary almost proportionately with the coefficient estimates, implying very little variation in the t-ratios on $\hat{\gamma}_3$: the latter all fall in the range $(-3.5, -3.2)$. Thus the qualitative conclusions regarding the empirical validity of Cross-Sectional Implication 1 are not sensitive to the choice of α over the range [.20, .66]. In what follows we therefore present results based only on the ML estimate, $\hat{\alpha}_{ML} = .49$.

The cross-sectional estimates of (90) for the three country groupings under $\hat{\alpha}_{ML} = .49$ are summarized in Table 2a. χ^2_H is a diagnostic statistic for detecting heteroskedasticity and is distributed as a chi-square variate with one degree of freedom. OLS standard errors are reported in round brackets, and p-values are reported in square brackets.

The estimates of the coefficients on $\log \hat{s}_i$ and $\log \hat{w}_i(\alpha)$ (namely, γ_1 and γ_2) are statistically highly significant and have the correct signs for all three country groupings.[35] However, the theory's predictions that $\gamma_1 = 1$ and $\gamma_2 = -1$ are rejected both individually and jointly at the 95 percent significance level, using the full sample. While not all of these

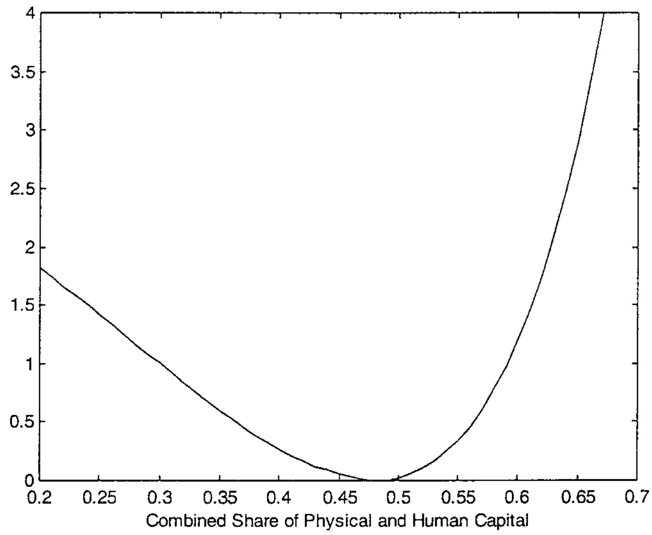

Figure 1. Log-likelihood ratio profile for the cross-section regression (eq. (86)).

restrictions are rejected in the intermediate and OECD samples, it has to be noted that in the intermediate sample γ_2 and in the OECD sample both γ_1 and γ_2 are estimated much less precisely than in the full sample and the associated tests are likely to be less powerful.

The variance of the logarithm of the capital-output ratio enters significantly for the full and intermediate samples and seems to be empirically an important factor for the dispersion of the logarithm of the mean of the capital-output ratio across economies. The variance term is not, however, significant in the case of the OECD countries. This could be due to lack of adequate variations across OECD economies, itself a result of the way OECD membership is determined. This can be seen clearly from the summary statistics presented in Table 1 for the three country groups. As to be expected, the dispersion of all the key statistics reported in Table 1 are much smaller for the OECD countries as compared to the other two country groups. In particular, there seems to be too little cross-sectional variation in the estimates of $\omega_i(\alpha)$ in the OECD sample for the variance of the capital-output ratio to contribute significantly to the cross-country variation of the mean of the capital-output ratio.[36]

In distinguishing the stochastic Solow-Swan growth model from its deterministic counterpart, it is also important to bear in mind that the definition of $\hat{w}_i(\alpha)$ differs between the stochastic and the deterministic specifications. Defining $\hat{w}_i^d(\alpha) = [\exp(\hat{n}_i + g)/\hat{\varphi}_i^d(\alpha)] - (1 - \delta)$, then under the stochastic specification $d = 1$, while under the

Table 2a. Cross-Sectional Implication 1 under homogeneous per capita growth rate (Solow-Swan growth model).

Dependent Variable $\hat{E}(\log v_{i\infty})$	OECD Sample ($N = 20$)	Intermediate Sample ($N = 58$)	Full Sample ($N = 72$)
$\hat{\gamma}_0$	3.7808	3.5448	—
	(1.2532)	(.6651)	
$\hat{\gamma}_1$	.8155	.8217	.8153
	(.1367)	(.0399)	(.0274)
$\hat{\gamma}_2$	$-.9093$	-1.0158	$-.9185$
	(3.433)	(.2124)	(.0209)
$\hat{\gamma}_3$	1.3293	-1.8526	-2.4604
	(9.3570)	(.7880)	(.7149)
S.E. of regression	.0903	.1341	.1477
$\overline{R}^2$	.7108	.9370	.9513
χ^2_H	1.0652	.6169	1.1426
	[.302]	[.432]	[.285]
F_2	.9913	11.8921	22.0612
	[.393]	[.000]	[.000]
F_3	.8023	8.2424	15.3791
	[.511]	[.000]	[.000]
W_{sd}	$-.9439$	1.0237	.8818
	[.345]	[.306]	[.378]
W_{ds}	.9378	-2.9373	-3.9844
	[.348]	[.003]	[.000]

Note: The underlying regression equation is given by (90),

$$\begin{aligned}\hat{E}(\log v_{i\infty}) &= \gamma_0 + \gamma_1 \log \hat{s}_i + \gamma_2 \log \hat{\omega}_i(\hat{\alpha}_{ML}) \\ &\quad + \gamma_3 \left[\frac{1}{2} \left(\frac{1 + \hat{\alpha}_{ML}}{1 - \hat{\alpha}_{ML}} \right) \widehat{Var}(\log v_{i\infty}) \right] + \zeta_{1i},\end{aligned}$$

where ζ_{1i} is an error term. The estimates in this table assume the same per capita growth rates across countries. The dependent variable is country-specific time averages of $\log(v_{it})$ over the period 1974 to 1992. $\hat{s}_i$ and $\hat{n}_i$ (that enters $\hat{\omega}_i(\hat{\alpha}_{ML})$) are based on country-specific time averages of s_{it} and n_{it} computed over the entire available sample, 1960 to 1992. Other parameters entering the construction of $\hat{\omega}_i(\hat{\alpha}_{ML})$ are fixed at $\hat{\alpha}_{ML} = .49$, $g = .02$, and $\delta = .05$; $\hat{\mu}_i(\hat{\alpha}_{ML})$ and $\hat{\omega}_i(\hat{\alpha}_{ML})$ are estimated along the lines set out in Section 6. χ^2_H is a diagnostic statistic for detecting heteroskedasticity and is distributed as a chi-square variate with one degree of freedom. F_2 and F_3 are the F-statistics for testing the theory restrictions $(\gamma_1, \gamma_2) = (1, -1)$ and $(\gamma_1, \gamma_2, \gamma_3) = (1, -1, -1)$, respectively. W_{sd} (W_{ds}) is the Godfrey-Pesaran nonnested W statistic for testing the stochastic (deterministic) versus the deterministic (stochastic) specifications, given by (91) and (92). OLS standard errors are reported in round brackets, and p-values are reported in square brackets.

deterministic specification $d = 0$. Also, it is worth noting that under the deterministic specification $\hat{w}_i^d(\alpha)$ does not depend on α and is given by the familiar expression $\hat{w}_i^0 = \exp(\hat{n}_i + g) - (1 - \delta) \approx \hat{n}_i + g + \delta$. Denoting the stochastic and the deterministic specifications by M_s and M_d, we have

$$M_s\colon\ \hat{E}(\log v_{i\infty}) \;=\; \gamma_0 + \gamma_1 \log \hat{s}_i + \gamma_2 \log \hat{w}_i^1(\hat{\alpha}_{ML}) + \gamma_3 \left[\frac{1}{2}\left(\frac{1+\hat{\alpha}_{ML}}{1-\hat{\alpha}_{ML}}\right)\widehat{Var}(\log v_{i\infty})\right] + u_{is}, \tag{91}$$

and

$$M_d\colon\ \hat{E}(\log v_{i\infty}) = \gamma_0 + \gamma_1 \log \hat{s}_i + \gamma_2 \log \hat{w}_i^0 + u_{id}, \tag{92}$$

where u_{is} and u_{id} are error terms. It is now easily seen that these specifications are nonnested and cannot be tested against one another using standard classical tests. We therefore use the Cox-type nonnested W-test developed in Godfrey and Pesaran (1983), which is known to have good small sample properties. We computed W statistics for testing M_s against M_d and *vice versa* for all the three country groupings. These statistics are denoted by W_{sd} and W_{ds}, respectively, and are given in the last two rows of Table 2a.[37] In the case of the full and intermediate samples, the deterministic specification is clearly rejected against the stochastic specification, but not the reverse. In the OECD sample, however, neither specification can be rejected against one another.[38]

While the results in Table 2a provide some support for the predictions of the stochastic Solow-Swan model in the full and intermediate samples, the fact that the theory's predictions regarding the values of γ_1, γ_2, and γ_3 are rejected is disconcerting and requires further investigations. Since there is a fair amount of evidence in the literature that the deterministic rates of technological progress, g_i, differ across countries,[39] our assumption so far that $g = .02$ across all countries would seem one potential source for these rejections. To investigate whether this is in fact the case, we compute country-specific estimates of g_i as $\hat{\psi}_{i0}/(1-\hat{\psi}_{i1})$, where $\hat{\psi}_{i0}$ and $\hat{\psi}_{i1}$ are the OLS estimates of ψ_{i0} and ψ_{i1} in[40]

$$\triangle \log y_{it} = \psi_{i0} + \psi_{i1} \triangle \log y_{i,t-1} + \xi_{i2}, \tag{93}$$

where ξ_{i2} is an error term. The last row of Table 1 reports the cross-country means and measures of dispersion of the resulting estimates $\hat{g}_i$ across the three country groups, suggesting significant degrees of cross-country heterogeneity in the deterministic rates of technological progress. The cross-sectional estimates of (90) allowing for country-specific estimates of g_i are summarized in Table 2b.[41] Comparing the results in Table 2b to those in Table 2a, the improvement in the fit (as measured by the standard error of the regression) of (90) across all three country groups is striking. Furthermore, in the full sample the restrictions that $\gamma_1 = 1$ and $\gamma_2 = -1$ cannot be rejected either individually or jointly at the conventional significance levels. Even the joint restriction that $\gamma_1 = 1$, $\gamma_2 = -1$, and $\gamma_3 = -1$ is not rejected at the 95 percent significance level. Allowing for cross-country heterogeneity in the deterministic rates of technological progress therefore further strengthens the empirical support for the stochastic Solow-Swan model. This holds true for the intermediate sample also.[42]

The above results also provide further support for the cross-sectional evidence obtained in the literature that relates the logarithm of output per capita to the saving rate and the growth rate of labor input but have the added advantage of being robust to the possibility

Table 2b. Cross-Sectional Implication 1 under heterogeneous per capita growth rate (Solow-Swan growth model).

Dependent Variable $\hat{E}(\log v_{i\infty})$	OECD Sample ($N = 20$)	Intermediate Sample ($N = 58$)	Full Sample ($N = 72$)
$\hat{\gamma}_0$	5.7301	5.0850	.2876
	(.3818)	(.1511)	(.1518)
$\hat{\gamma}_1$	1.0955	1.0380	1.0182
	(.0688)	(.0208)	(.0207)
$\hat{\gamma}_2$	−.8299	−.9503	−.9472
	(.0881)	(.0624)	(.0693)
$\hat{\gamma}_3$	1.7519	−.8731	−1.5608
	(4.9426)	(.4749)	(.4978)
S.E. of regression	.0423	.0695	.0881
$\overline{R}^2$	.9365	.9831	.9827
χ^2_H	2.5227	2.0867	2.9550
	[.112]	[.149]	[.086]
F_2	4.4684	3.1937	1.5875
	[.029]	[.049]	[.212]
F_3	2.9938	2.2238	2.4110
	[.062]	[.096]	[.074]
W_{sd}	−.1949	−1.4051	−1.0176
	[.846]	[.160]	[.309]
W_{ds}	.6077	−2.0764	−2.6278
	[.543]	[.038]	[.009]

Note: The regression equation is given by (90),

$$\hat{E}(\log v_{i\infty}) = \gamma_0 + \gamma_1 \log \hat{s}_i + \gamma_2 \log \hat{w}_i(\hat{a}_{ML}) + \gamma_3 \left[\frac{1}{2} \left(\frac{1+\hat{a}_{ML}}{1-\hat{\alpha}_{ML}} \right) \widehat{Var}(\log v_{i\infty}) \right] + \zeta_{i2}.$$

The estimates in this table allow for differences in per capita growth rates g_i across countries. The country-specific growth rates are estimated using the regressions (93). The ML estimate of α in this case is determined as $\hat{\alpha}_{ML} = .44$. The estimates of $\hat{\mu}_i(\hat{\alpha}_{ML})$ and $\hat{\omega}_i(\hat{\alpha}_{ML})$ are also computed taking into account the country-specific estimates of g_i. See also the notes to Table 2a.

of a unit root in the logarithm of output per capita. But it should be stressed that we do not view the underlying cross-section regressions as identifying causal mechanisms for differing long-run economic performances across countries. Rather, all our cross-section regressions should be interpreted as shedding light on the quantitative strength of the association of a country's long-run economic performance with variables such as the saving rate, the rate of labor input growth, and the volatility of shocks to technology and labor input.

We now turn our attention to the second important cross-sectional implication of the stochastic specification of the Solow-Swan growth model—namely, that for each country the mean of the capital-output ratio exceeds its deterministic counterpart v_i^* given by $s_i/[\exp(n_i+g_i)-(1-\delta_i)]$ (see 64)). The empirical validity of this implication is considered

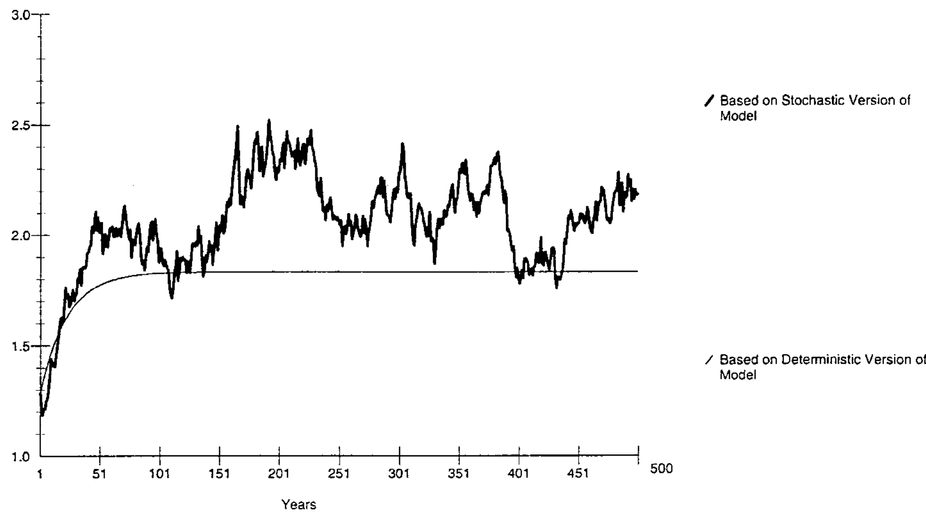

Figure 2. Capital-output ratio based on the stochastic and deterministic versions of the Solow-Swan growth model: a simulation exercise.

in Figures 2 and 3. Figure 2 shows the convergence of paths of the capital-output ratio under the deterministic and stochastic versions of the Solow-Swan growth model. These paths are generated using the parameter values/estimates $g = .02$, $\delta = .05$, $\hat{\alpha} = .49$, $\hat{s} = .1675$, $\hat{n} = .0206$, and $\rho_a = \rho_l = 1$, with ε_t drawn from the truncated normal distribution (45) with a truncation point at $\hat{b} = \log(1-\delta) - \hat{n} - g = -.0919$. The mean and standard deviation of the underlying nontruncated normal distribution are set to $\hat{\mu} = -.0336$ and $\hat{\omega} = .0675$, respectively. Both trajectories are simulated from an initial capital-output ratio equal to one half of the estimate of the deterministic steady-state capital-output ratio, given by

$$\hat{v}^* = \frac{\hat{s}}{\exp(\hat{n} + g) - (1 - \delta)} = 1.8319.$$

Figure 2 clearly shows that the stochastically simulated values of the capital-output ratio exhibit considerable persistence even after the corresponding deterministic values have reached their estimated steady-state value of 1.8319. This also suggests that the standard procedure where the capital-output ratio of the stochastic nonlinear model is approximated by adding a serially uncorrelated disturbance term to the deterministic steady state capital-output ratio can be very misleading, even if the underlying stochastic model is globally ergodic.[43] The second important difference between the stochastic and deterministic series lies in the fact that the capital-output ratios simulated from the stochastic model generally tend to exceed those obtained from the deterministic model, as predicted by Cross-Sectional Implication 2. Therefore, the standard approximation procedure in the analysis of nonlinear

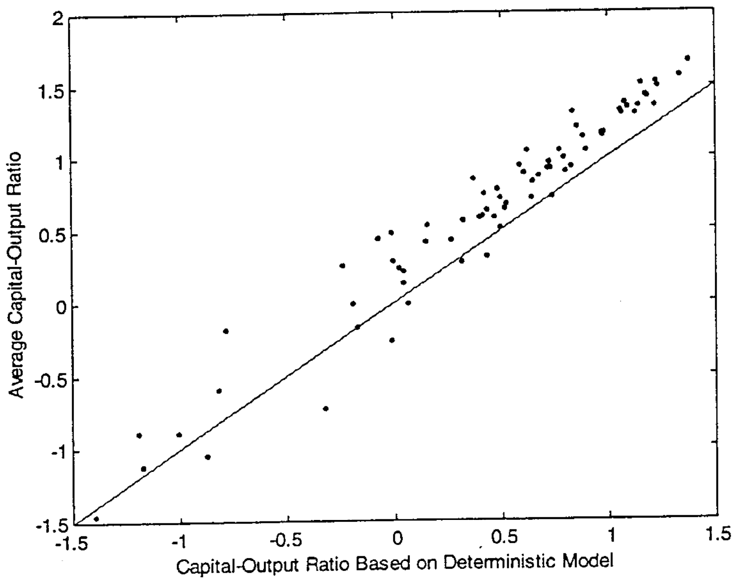

Figure 3. Scatter plot of the sample averages of the capital-output ratios and their counterparts based on the deterministic version of the Solow-Swan growth model for 72 countries using a logarithmic scale.

stochastic models—namely, to work with a Taylor-series expansion around the deterministic steady state—can also be misleading insofar as it carries out the approximation around an inappropriate steady-state value.

The Cross-Sectional Implication 2 is further explored in Figure 3. This figure compares, on a logarithmic scale, the sample averages of the capital-output ratios computed over the period 1960 to 1992 to their counterparts based on the deterministic version of the Solow-Swan growth model, computed as $\hat{v}_i^* = \hat{s}_i/[\exp(\hat{n}_i + \hat{g}_i) - (1 - \delta)]$, for all the seventy-two countries in our full sample; $\hat{v}_i^*$ is computed using the baseline value $\delta = .05$, the estimates $\hat{s}_i$ and $\hat{n}_i$, as well as country-specific estimates of g_i obtained using (93). The prediction of the stochastic Solow-Swan growth model that the points in the scatter plot of the mean capital-output ratio of the stochastic model against the deterministic capital-output ratio should lie above the 45 degree line is satisfied for sixty-three out of seventy-two countries. The nine countries for which (64) is violated are Burkina Faso, Burundi, Cameroon, Madagascar, Rwanda, Paraguay, Uruguay, Indonesia, and Sri Lanka.[44] Under the deterministic version of the Solow-Swan model such a systematic pattern for the scatter points would not be expected.

Cross-Sectional Implications 3 and 4 (the implications for the AK growth model) are evaluated in Tables 3 and 4.[45] Table 3 shows that the saving rate and the growth rate of labor input tend to significantly enter a cross-sectional regression with the logarithm of the capital-output ratio as the dependent variable, in contrast to Cross-Sectional Implication 3. Thus, this implication of the stochastic AK growth model is rejected rather decisively. While

Table 3. Cross-Sectional Implication 3 (AK endogenous growth model).

Dependent Variable $\hat{E}(\log v_{i\infty})$	OECD Sample $(N = 20)$	Intermediate Sample $(N = 58)$	Full Sample $(N = 72)$
$\hat{\phi}_0$	5.7355 (.8662)	5.5326 (.3536)	1.6270 (.1882)
$\hat{\phi}_1$	.8468 (.1355)	.8480 (.0406)	.8437 (.0330)
$\hat{\phi}_2$	−.0880 (.0379)	−.1437 (.0341)	−.1518 (.0362)
$\hat{\phi}_3$	−.0167 (.0546)	−.0035 (.0256)	.0029 (.0252)
S.E. of regression	.0920	.1393	.1592
$\overline{R}^2$	.7001	.9320	.9434
χ^2_H	.4704 [.493]	.8775 [.349]	2.2074 [.137]

Note: The regression equation is given by

$$\hat{E}(\log v_{i\infty}) = \phi_0 + \phi_1 \log \hat{s}_i + \phi_2 \log \hat{n}_i + \phi_3 \log \hat{\omega}_i^2(\hat{\alpha}_{ML}) + \zeta_{i3}.$$

See also the notes to Table 2a.

these results are not holding up for the corresponding regression with the first difference of the logarithm of the capital-output ratio as the dependent variable (Table 4, Cross-Sectional Implication 4), the results from the latter regression should be viewed with caution as they have very little explanatory power.

7. Conclusion

In this article we have derived and tested the consequences of allowing for uncertainty in the form of stochastic technology and labor input in the Solow-Swan and AK growth models. We have derived the conditions under which the capital-output ratio in these models is ergodic in moments—namely, converges to globally attracting steady-state probability distribution functions—and the moments of these probability distribution functions exist and can be consistently estimated using long time averages. Using these results, we have examined the cross-sectional implications of the stochastic Solow-Swan and AK growth models. In particular, we have derived the precise way in which the mean of the steady-state distribution of the capital-output ratio depends not only on the saving rate and the growth rate of labor input (the standard variables in the literature), but also on the variance and higher-

Table 4. Cross-Sectional Implication 4 (*AK* endogenous growth model).

Dependent Variable $\hat{E}(\Delta \log v_{i\infty})$	OECD Sample $(N = 20)$	Intermediate Sample $(N = 58)$	Full Sample $(N = 72)$
$\hat{\lambda}_0$	.0427	.0409	.0096
	(.0525)	(.0395)	(.0202)
$\hat{\lambda}_1$	.0035	.0037	$-.0004$
	(.0082)	(.0045)	(.0035)
$\hat{\lambda}_2$	.0032	.0025	.0028
	(.0023)	(.0038)	(.0039)
$\hat{\lambda}_3$	.0009	.0009	$-.0006$
	(.0033)	(.0029)	(.0027)
S.E. of regression	.0056	.0156	.0171
$\overline{R}^2$	$-.0248$	$-.0328$	$-.0326$
χ^2_H	2.2892	.1746	.4077
	[.130]	[.676]	[.523]

Note: The regression equation is given by

$$\begin{aligned}\hat{E}(\Delta \log v_{i\infty}) &= \lambda_0 + \lambda_1 \log \hat{s}_i + \lambda_2 \log \hat{n}_i \\ &\quad + \lambda_3 \log \hat{\omega}_i^2(\hat{\alpha}_{ML}) + \zeta_{i4}.\end{aligned}$$

See also the notes to Table 2a.

order cumulants of the steady-state distribution of the capital-output ratio, which in turn depends on the parameters of the technology and labor input processes. For our full sample of seventy-two countries and a subsample of fifty-eight intermediate countries from the Summers-Heston (1991) data set we found that the empirical evidence is largely consistent with the stochastic Solow-Swan growth model's ergodic properties, and that stochastic technological progress and stochastic labor input contribute significantly to differences in the logarithms of steady-state capital-output ratios. In particular, countries with a higher volatility of the capital-output ratio *ceteris paribus* have a lower mean capital-output ratio, as predicted by the stochastic Solow-Swan growth model. The deterministic Solow-Swan growth model and the (deterministic and stochastic) AK growth model, on the other hand, are rejected by our cross-sectional tests in the full and intermediate samples. For the subsample comprising twenty OECD countries the regression results are inconclusive, which could reflect inadequate cross-country sampling variations in the innovations of the technology and labor input processes across the OECD countries. In fact, it could be argued that the OECD country group is subject to important sample-selection bias due to the criteria by which group membership is determined. Our results provide further support for the cross-sectional evidence obtained in the literature that relates the logarithm of per capita output to the growth rate of labor input and the saving rate, but establish the mean-variance relationship as an important stylized fact for future work in this area to ponder, and have the added advantage of being robust to the possibility of a unit root in the logarithm of output per capita.

The results, however, should be viewed as a first step toward a more comprehensive empirical analysis. Further work using the NVAR time-series specification derived in Section 5 and pooling of the time-series observations across countries would be a desirable next step to ascertain the statistical significance of the effects of volatility on output per capita.[46] To embark on a dynamic panel analysis using the NVAR it has to be noted, though, that while the recent econometrics literature has developed a number of important insights into the study of dynamic linear panel data models under heterogeneous slope, the nonlinearity of the VAR framework of Section 5 necessitates further econometric work before such an applied analysis is feasible.

Certain forms of nonlinearities in the cross-country growth process have been considered in the work of Durlauf and Johnson (1995) and Quah (1997), for example, who argue that the emergence of steady-state "convergence clubs" is an important aspect of the cross-country growth process. It would be interesting to learn whether their findings would be affected by allowing the cross-sectional distribution of the capital-output ratio to depend on the variance (and higher-order cumulants) of the capital-output ratio as in the analysis of this article.

Finally, it is clearly also of interest to investigate the sources of the cross-country variations of the volatilities of the "technology" and "labor input" shocks. Such analysis could provide further insights into how a country's long-run economic performance depends on different sources of uncertainty, including those that are at least partly under government control.

Mathematical Appendix

Introduction

There are a number of approaches available in the mathematics literature to establish ergodicity for nonlinear discrete-time Markov processes of the form

$$x_{t+1} = m(x_t) + \varepsilon_{t+1}, \qquad \varepsilon_{t+1} \sim iid(0, \sigma_\varepsilon^2), \tag{94}$$

or

$$x_{t+1} = h(x_t, \varepsilon_{t+1}), \qquad \varepsilon_{t+1} \sim iid(0, \sigma_\varepsilon^2). \tag{95}$$

A variety of these methods that have found use in nonlinear time-series econometrics (for example, the methods of Doukhan and Ghindés, 1980, and of Lasota and Mackey, 1989, also reviewed in Granger and Teräsvirta, 1993, and Doukhan, 1994) are not applicable to the stochastic Solow-Swan growth model.[47] While our set of sufficient conditions may conceivably be modified to apply some of the ergodicity theorems provided in Meyn and Tweedie (1993), in this article we provide a direct, intuitive, and, to our knowledge, novel proof of ergodicity using the ideas outlined in Section 3.

Proof of Proposition 3.1

We begin by showing that $\{\kappa_t\}$ with probability one does not get absorbed by the bounds of zero or positive infinity for any strictly positive and finite initial condition at time zero κ_0. Using (29), it is readily seen that the probability that κ_{t+1} falls into the open interval (κ_L, κ_U) for a given value of κ is given by

$$\begin{aligned} &\Pr[\kappa_L < \exp(-\tau - \varepsilon_{t+1})\Psi(\kappa) < \kappa_U] \\ &\quad = \Pr\left[\tau + \log\left(\frac{\kappa_L}{\Psi(\kappa)}\right) < -\varepsilon_{t+1} < \tau + \log\left(\frac{\kappa_U}{\Psi(\kappa)}\right)\right] \\ &\quad = F_\varepsilon(B_L) - F_\varepsilon(B_U), \end{aligned} \tag{96}$$

where

$$B_L = -\tau - \log\left(\frac{\kappa_L}{\kappa}\right) + \log\left(\frac{\Psi(\kappa)}{\kappa}\right), \tag{97}$$

$$B_U = -\tau + \log\left(\frac{\kappa}{\kappa_U}\right) + \log\left(\frac{\Psi(\kappa)}{\kappa}\right), \tag{98}$$

and

$$\Psi(\kappa) = s(\kappa) f(\kappa) + (1 - \delta)\kappa. \tag{99}$$

Therefore, $\{\kappa_t\}$ will not be absorbed by the bounds of zero and positive infinity if

$$F_\varepsilon(B_L) - F_\varepsilon(B_U) = 1, \tag{100}$$

for all values of κ contained in (κ_L, κ_U). We proceed to show that (100) is indeed satisfied for all possible values of κ whenever conditions (31) to (33) hold. But first note that

$$\frac{\Psi(\kappa)}{\kappa} = \frac{s(\kappa) f(\kappa)}{\kappa} + (1 - \delta), \tag{101}$$

and therefore (31) and (32) can be restated in terms of $\Psi(\kappa)/\kappa$ as

$$\frac{\Psi(\kappa)}{\kappa} \to 1 - \delta \qquad \text{as} \quad \kappa \to \infty, \tag{102}$$

and

$$\frac{\Psi(\kappa)}{\kappa} \to +\infty \qquad \text{as} \quad \kappa \to 0. \tag{103}$$

We distinguish three scenarios, and in every scenario we shall show that the appropriately defined limits of $F_\varepsilon(B_L)$ and $F_\varepsilon(B_U)$ are equal to one and zero, respectively, if the conditions (31) to (33) are satisfied. In Scenario 1, we take κ as fixed, but let $\kappa_L \to 0$ and $\kappa_U \to \infty$. Scenario 2 analyzes the situation that $\kappa \to 0$ and $\kappa_L \to 0$, with $\kappa_L/\kappa \to \omega \in (0, 1)$, and

$\kappa_U \to \infty$. In Scenario 3, we examine the situation where $\kappa \to \infty$ and $\kappa_U \to \infty$, with $\kappa/\kappa_U \to \lambda \in (0, 1)$, and $\kappa_L \to 0$. Turning to Scenario 1, we have

$$F_\varepsilon(B_L) \to 1 \quad \text{as } \kappa_L \to 0,\ \kappa \text{ fixed}, \tag{104}$$

and

$$F_\varepsilon(B_U) \to 0 \quad \text{as } \kappa_U \to \infty,\ \kappa \text{ fixed}. \tag{105}$$

Under Scenario 2, using (103) we have

$$F_\varepsilon(B_L) \to 1 \quad \text{as } \kappa, \kappa_L \to 0, \frac{\kappa_L}{\kappa} \to \omega. \tag{106}$$

Also note that B_U can be written as $B_U = -\tau - \log(\kappa_U) + \log[\Psi(\kappa)]$, and hence it is readily seen that

$$F_\varepsilon(B_U) \to 0 \quad \text{as } \kappa \to 0,\ \kappa_U \to \infty. \tag{107}$$

Under Scenario 3, rewriting B_L as $B_L = -\tau - \log(\kappa_L) + \log[\Psi(\kappa)]$, we have

$$F_\varepsilon(B_L) \to 1 \quad \text{as } \kappa \to \infty,\ \kappa_L \to 0, \tag{108}$$

and using (102) and the truncation condition (33) it follows that

$$F_\varepsilon(B_U) \to 0 \quad \text{as } \kappa,\ \kappa_U \to \infty,\ \frac{\kappa}{\kappa_U} \to \lambda. \tag{109}$$

Therefore, with probability one, the bounds of zero and positive infinity are not absorbing limits of $\{\kappa_t\}$.

We next show that since $\{\log \kappa_t\}$ is finite with probability one under conditions (34) and (35) the dependence of $\{\log \kappa_t\}$ on the initial condition $\log \kappa_0$ vanishes asymptotically, and $\{\log \kappa_t\}$ can be "enveloped" by a covariance-stationary first-order linear autoregressive process with innovations $\{\varepsilon_t\}$. Defining $z_t = \log \kappa_t$, (29) can be written in the following additive form in z_t and ε_t:

$$z_{t+1} = z_t + \chi(z_t) - \tau - \varepsilon_{t+1}, \tag{110}$$

where

$$\chi(z_t) = \log\{s[\exp(z_t)] f[\exp(z_t)] \exp(-z_t) + (1-\delta)\}. \tag{111}$$

By the mean value theorem, for all possible finite realizations of z_∞, $\tilde{z}_\infty$, we have that

$$\chi(z_t) = \chi(\tilde{z}_\infty) - (1-\gamma_t)(z_t - \tilde{z}_\infty), \tag{112}$$

where $\bar{z}_t \in (z_t, \tilde{z}_\infty)$, and

$$\gamma_t = \frac{1 - \delta + s'(\bar{\kappa}_t) f(\bar{\kappa}_t) + s(\bar{\kappa}_t) f'(\bar{\kappa}_t)}{\frac{s(\bar{\kappa}_t) f(\bar{\kappa}_t)}{\bar{\kappa}_t} + (1-\delta)}, \tag{113}$$

with $\bar{\kappa}_t = \exp(\bar{z}_t)$. Using (112) we can rewrite (110) as

$$z_{t+1} = z_t - (1-\gamma_t)(z_t - \tilde{z}_\infty) + \chi(\tilde{z}_\infty) - \tau - \varepsilon_{t+1}, \tag{114}$$

where $\chi(\tilde{z}_\infty) = \log\{s[\exp(\tilde{z}_\infty)] f[\exp(\tilde{z}_\infty)]\exp(-\tilde{z}_\infty) + (1-\delta)\}$. Defining $x_t = z_t - \tilde{z}_\infty$, and $c = \chi(\tilde{z}_\infty) - \tau$, (114) becomes

$$x_{t+1} = \gamma_t x_t + c - \varepsilon_{t+1}. \tag{115}$$

Solving (115) recursively forward from the initial condition x_0 we obtain

$$x_t = \left(\prod_{h=0}^{t-1}\gamma_h\right)x_0 + \sum_{j=1}^{t-1}\left(\prod_{h=1}^{j}\gamma_{t-h}\right)(c - \varepsilon_{t-j}) + c - \varepsilon_t. \tag{116}$$

It is now easily seen that under conditions (34) and (35) with probability one there exists an upper bound $\bar{\gamma} \in (0,1)$ such that $0 < \gamma_t < \bar{\gamma}$ for all realizations of $\bar{z}_t$. (Note that we have already established that $\bar{z}_t$ is finite with probability one.) Thus $|\prod_{h=0}^{t-1}\gamma_h| < \bar{\gamma}^t$ with probability one, and the limiting probability distribution function of x_t with probability one will be independent of the initial condition, x_0, as $t \to \infty$. Also, with probability one we have

$$|x_t| < \bar{\gamma}^t|x_0| + \left(\sum_{j=0}^{t-1}\bar{\gamma}^j\right)|c| + \sum_{j=0}^{t-1}\bar{\gamma}^j|\varepsilon_{t-j}|. \tag{117}$$

Consider now the positive stable first-order autoregressive process $|q_t|$ defined by

$$|q_{t+1}| = \bar{\gamma}|q_t| + |c| + |\varepsilon_{t+1}|, \tag{118}$$

with initial condition $|q_0| = 0$. Solving this equation recursively forward yields

$$|q_t| = \left(\sum_{j=0}^{t-1}\bar{\gamma}^j\right)|c| + \sum_{j=0}^{t-1}\bar{\gamma}^j|\varepsilon_{t-j}|, \tag{119}$$

which if substituted in (117) gives

$$|x_t| < \bar{\gamma}^t|x_0| + |q_t|. \tag{120}$$

Therefore, for all finite initial values x_0, the limit of $\{x_t\}$ (as $t \to \infty$) with probability one will be "enveloped" by the limit of $\{|q_t|\}$, in the sense that

$$\lim_{t\to\infty}|x_t| \le \lim_{t\to\infty}|q_t|.$$

But it is easily established that the limiting process of $|q_t|$ is ergodic in its rth moment, $r = 1, 2, \ldots,$ so long as $|\varepsilon_t|$ has finite moments of the same order. Under (29), it therefore follows that $\{|x_t|\}$, and hence $\{|z_t|\}$, will also be ergodic in their rth moment when the rth moment of $|\varepsilon_t|$ exists.[48]

Consider now the conditions under which $\kappa_t = \exp(z_t)$ has finite moments. Under assumptions v and vi, and using (119) and (120) we have

$$\lim_{t\to\infty} M_{x_t}(\theta) \leq \lim_{t\to\infty} M_{q_t}(\theta) = \exp\left(\frac{\theta|c|}{1-\bar{\gamma}}\right)\prod_{j=0}^{\infty} M_{\varepsilon}(\bar{\gamma}^{j}\theta), \tag{121}$$

where $M_{x_t}(\theta)$, $M_{q_t}(\theta)$, and $M_{\varepsilon}(\theta)$ are, respectively, the moment generating functions of $|x_t|$, $|q_t|$, and $|\varepsilon_t|$. However, under assumption vi

$$\log M_{\varepsilon}(\theta) = \sum_{m=1}^{\infty} \frac{\theta^m}{m!} c_m(\varepsilon), \qquad \text{for } |\theta| \leq r, \tag{122}$$

where $c_m(\varepsilon)$ is the mth cumulant of $|\varepsilon_t|$. Therefore,

$$\begin{aligned}
\log\left[\prod_{j=0}^{\infty} M_{\varepsilon}(\bar{\gamma}^{j}\theta)\right] &= \sum_{j=0}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\bar{\gamma}^{mj}\theta^m}{m!}\right) c_m(\varepsilon),\\
&= \sum_{m=1}^{\infty}\left(\frac{\theta^m}{m!(1-\bar{\gamma}^m)}\right) c_m(\varepsilon),\\
&= \log M_{\varepsilon}(\theta) + \sum_{m=1}^{\infty}\left(\frac{\bar{\gamma}^m\theta^m}{m!(1-\bar{\gamma}^m)}\right) c_m(\varepsilon).
\end{aligned} \tag{123}$$

Since $\bar{\gamma} \in (0, 1)$ with probability one, then with probability one there always exists a finite positive integer ρ_0 such that $\bar{\gamma}^p/(1-\bar{\gamma}^p) < 1$ for $p > \rho_0$. Hence

$$\begin{aligned}
\left|\sum_{m=1}^{\infty}\left(\frac{1}{m!}\right)\left(\frac{\bar{\gamma}^m\theta^m}{(1-\bar{\gamma}^m)}\right) c_m(\varepsilon)\right| &\leq \sum_{i=1}^{\rho_0}\frac{1}{m!}\left(\frac{\bar{\gamma}^m|\theta|^m}{1-\bar{\gamma}^m}\right)|c_m(\varepsilon)|\\
&\quad + \sum_{m=\rho_0+1}^{\infty}\frac{1}{m!}|\theta|^m|c_m(\varepsilon)|.
\end{aligned} \tag{124}$$

But by assumption the second term on the right-hand side of the above inequality exists for $|\theta| \leq r$, and it must therefore be the case that $\lim_{t\to\infty} M_{x_t}(\theta)$ also exists for $|\theta| \leq r$. Similarly, since $|z_t| < |x_t| + |\tilde{z}_\infty|$, and $\tilde{z}_\infty$ is finite, it follows that $\lim_{t\to\infty} M_{z_t}(\theta)$ exists for $|\theta| \leq r$, where $M_z(\theta)$ is the moment generating function of $|z_t| = |\log \kappa_t|$. Finally,

$$E\left(|\kappa_t|^{\lambda}\right) = E\left(e^{\lambda|z_t|}\right) = M_{z_t}(\lambda), \tag{125}$$

and hence as $t \to \infty$, $E\left(|\kappa_\infty|^{\lambda}\right)$ exists for all $|\lambda| \leq r$. Therefore, all moments of κ_∞ (and hence, in the Cobb-Douglas case, of υ_∞) up to order r exist so long as the moment generating function of $|\varepsilon_t|$, $M_{\varepsilon}(\theta)$, exists for $|\theta| \leq r$. Clearly, r could be a fraction. ■

Proof of Proposition 3.2

The proof again comprises the three steps involved in the proof of Proposition 3.1. The third step uses the same argument as its analog in the proof of Proposition 3.1 and is therefore not repeated.

To prove that $\{\kappa_t\}$ with probability one does not get absorbed by the bounds of zero or positive infinity, note from (26) that

$$\kappa_{t+1}(\rho_a, \rho_l) = \exp(-\tau - \Delta u_{t+1})\Psi[\kappa_t(\rho_a, \rho_l)], \tag{126}$$

where as before $\Psi[\kappa_t(\rho_a, \rho_l)] = s[\kappa_t(\rho_a, \rho_l)]f[\kappa_t(\rho_a, \rho_l)] + (1-\delta)\kappa_t(\rho_a, \rho_l)$, and $\Delta u_{t+1} = -(1-\rho_a)u_{at} - (1-\rho_l)u_{lt} + \varepsilon_{t+1}$. We therefore have that

$$\frac{\kappa_{t+1}(\rho_a, \rho_l)}{\Psi[\kappa_t(\rho_a, \rho_l)]} = \exp[-\tau + (1-\rho_a)u_{at} + (1-\rho_l)u_{lt} - \varepsilon_{t+1}]. \tag{127}$$

In the case where $|\rho_a| < 1$ and $|\rho_l| < 1$,

$$u_t = \rho_a^t u_{a0} + \rho_l^t u_{l0} + \sum_{j=0}^{t-1} \rho_a^j \varepsilon_{a,t-j} + \sum_{j=0}^{t-1} \rho_l^j \varepsilon_{l,t-j}, \tag{128}$$

and as $t \to \infty$, $\zeta_{t+1}(\rho_a, \rho_l) = -(1-\rho_a)u_{at} - (1-\rho_l)u_{lt} + \varepsilon_{t+1}$, converges to a time-invariant distribution, with a zero mean and variance

$$2\left(\frac{\sigma_a^2}{1+\rho_a} + \frac{\sigma_l^2}{1+\rho_l} + \frac{(2-\rho_a-\rho_l)\sigma_{al}}{1-\rho_a\rho_l}\right),$$

and is bounded away from negative and positive infinity, with probability one. Hence, $\kappa_{t+1}(\rho_a, \rho_l)/\Psi[\kappa_t(\rho_a, \rho_l)]$ must also converge to a time-invariant distribution bounded away from zero and positive infinity, with probability one. More specifically, we have

$$\lim_{t\to\infty} \frac{\kappa_{t+1}(\rho_a, \rho_l)}{\Psi[\kappa_t(\rho_a, \rho_l)]} = \exp(-\tau - \zeta_\infty), \tag{129}$$

where ζ_∞ is the random variable that underlies the steady-state distribution of $\zeta_t(\rho_a, \rho_l)$. Therefore, for large t,

$$\kappa_{t+1}(\rho_a, \rho_l) \approx \Psi[\kappa_t(\rho_a, \rho_l)]\exp(-\tau - \zeta_\infty), \qquad -\infty < \zeta_\infty < +\infty. \tag{130}$$

Applying the same methodology as in the proof of Proposition 3.1 to (130), it is now easily seen that as $t \to \infty$, $\kappa_t(\rho_a, \rho_l)$ with probability one does not get absorbed by the bounds of zero or positive infinity, if conditions (31), (32), and (38) are satisfied.

We next show that under conditions (30) to (32), (34), (35), and (38), $\{\log \kappa_t\}$ with probability one asymptotically does not depend on the initial condition κ_0 and can in the limit be "enveloped" by a linear covariance-stationary first-order autoregressive process with innovations $\{\varepsilon_t\}$. Defining $z_t = \log \kappa_t$, and $X_t = (z_t, u_{at}, u_{lt})^T$,[49] we can rewrite the stochastic difference equation (26) describing the law of motion for the effective capital labor ratio, κ_{t+1}, as

$$X_{t+1} = \Phi X_t + \Theta(X_t) + D\xi_{t+1}, \tag{131}$$

where

$$\Phi = \begin{pmatrix} 1 & 1-\rho_a & 1-\rho_l \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_l \end{pmatrix}, \Theta(X_t) = \begin{pmatrix} \chi(z_t) - \tau \\ 0 \\ 0 \end{pmatrix}, D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\xi_t = (\varepsilon_{at}, \varepsilon_{lt})^T$, and $\chi(z_t) = \log\{s[\exp(z_t)] f[\exp(z_t)] \exp(-z_t) + (1-\delta)\}$. By the mean value theorem, for all possible finite realizations of X_∞, $\tilde{X}_\infty$, we have that

$$\Theta(X_t) = \Theta(\tilde{X}_\infty) + \left(\frac{\partial\Theta}{\partial X}\right)^T \Big|_{X=\bar{X}_t} (X_t - \tilde{X}_\infty), \tag{132}$$

where $\bar{X}_t \in (X_t, \tilde{X}_\infty)$, element by element. It should be noted that

$$\left(\frac{\partial\Theta}{\partial X}\right)^T \Big|_{X=\bar{X}_t} = \begin{pmatrix} -(1-\gamma_t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{133}$$

with γ_t defined by (113). Under conditions (34) and (35), $\gamma_t \in (0, 1)$, and with probability one there exists an upper bound $\bar{\gamma} \in (0, 1)$ such that $\gamma_t < \bar{\gamma}$ for all t. Using (132) and (133), we can rewrite (131) as

$$X_{t+1} = \Phi_t^* X_t + \Theta_t^*(\tilde{X}_\infty) + \xi_{t+1}, \tag{134}$$

where

$$\Phi_t^* = \begin{pmatrix} \gamma_t & 1-\rho_a & 1-\rho_l \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_l \end{pmatrix}, \Theta_t^*(\tilde{X}_\infty) = \begin{pmatrix} (1-\gamma_t)\tilde{z}_\infty + \chi(\tilde{z}_\infty) - \tau \\ 0 \\ 0 \end{pmatrix},$$

and $\chi(\tilde{z}_\infty = \log\{s[\exp(\tilde{z}_\infty)] f[\exp(\tilde{z}_\infty)] \exp(-\tilde{z}_\infty) + (1-\delta)\}$. The eigenvalues of Φ_t^* are given by γ_t, ρ_a, and ρ_l, and in the present case with probability one all fall inside the unit circle for all possible finite realizations of X_∞ if (34) and (35) hold. Defining

$$C = \begin{pmatrix} \chi(\tilde{z}_\infty) - \tau \\ 0 \\ 0 \end{pmatrix}, \text{ and } W_t = (z_t - z_\infty, u_{at}, u_{lt})^T,$$

(134) becomes

$$W_{t+1} = \Phi_t^* W_t + C + \xi_{t+1}, \tag{135}$$

Assuming that for all t the roots of Φ_t^*, γ_t, ρ_a, and ρ_l, are distinct, we can diagonalize (135), and analyze the first equation in the resulting system of uncoupled equations in the same way as (115) in the proof of Proposition 3.1, noting that if (34) and (35) hold, with probability one $\gamma_t < \bar{\gamma}$ with $\bar{\gamma}$ as defined above, and replacing u_t by

$$u_t = \rho_a^t u_{a0} + \rho_l^t u_{l0} + \sum_{j=0}^{t-1} \rho_a^j \varepsilon_{a,t-j} + \sum_{j=0}^{t-1} \rho_l^j \varepsilon_{l,t-j}. \qquad \blacksquare$$

***AK* Growth Model with Unit Roots in Technology and Labor Input ($\rho_a = \rho_l = 1$)**

In this case we show that $\{\log \kappa_t\}$ cannot be covariance stationary. The stochastic difference equation describing the law of motion for the effective capital-labor ratio, (27), in this case becomes

$$\kappa_{t+1} = \exp(-\tau - \varepsilon_{t+1})[s(\kappa_t)A_t\kappa_t + (1-\delta)\kappa_t]. \tag{136}$$

Defining again $z_t = \log \kappa_t$, we can rewrite (136) as

$$z_{t+1} = z_t + \log\{s[\exp(z_t)]\exp(a_0 + gt + u_{at}) + (1-\delta)\} - \tau - \varepsilon_{t+1}. \tag{137}$$

Clearly, $\{z_t\}$ is trended if $g \neq 0$.

We show next that even if the condition $g = 0$ is satisfied, $\{\log \kappa_t\}$ will not be covariance stationary. Again defining $X_t = (z_t, u_{at}, u_{lt})^T$, we can rewrite the stochastic difference equation describing the law of motion for the logarithm of the effective capital-labor ratio, z_{t+1}, as

$$X_{t+1} = \Phi X_t + \Theta(X_t) + D\xi_{t+1}, \tag{138}$$

where

$$\Phi = I_3,\ \Theta(X_t) = \begin{pmatrix} \chi(z_t, u_{at}) - \tau \\ 0 \\ 0 \end{pmatrix},\ D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\xi_t = (\varepsilon_{at}, \varepsilon_{lt})^T$, $\chi(z_t, u_{at}) = \log\{s[\exp(z_t)]\exp(a_0 + u_{at}) + (1-\delta)\}$, and I_3 denotes the identity matrix of order 3. By the mean value theorem, for all possible finite realizations of X_∞, $\tilde{X}_\infty$ we have that

$$\Theta(X_t) = \Theta(\tilde{X}_\infty) + \left(\frac{\partial\Theta}{\partial X}\right)^T\Big|_{X=\bar{X}_t}(X_t - \tilde{X}_\infty), \tag{139}$$

where $\bar{X}_t \in (X_t, \tilde{X}_\infty)$, element by element. It should be noted that

$$\left(\frac{\partial\Theta}{\partial X}\right)^T\Big|_{X=\bar{X}_t} = \begin{pmatrix} \vartheta_{zt} & \vartheta_{u_at} & \vartheta_{u_lt} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{140}$$

where

$$\vartheta_{zt} = \frac{s'(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at})\bar{\kappa}_t}{s(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at}) + (1-\delta)}, \tag{141}$$

$$\vartheta_{u_at} = \frac{s(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at}) - s'(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at})\bar{\kappa}_t}{s(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at}) + (1-\delta)}, \tag{142}$$

and

$$\vartheta_{u_l t} = \frac{-s'(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at})\bar{\kappa}_t}{s(\bar{\kappa}_t)\exp(a_0 + \bar{u}_{at}) + (1-\delta)}. \tag{143}$$

Using (139) and (140), we can rewrite (138) as

$$X_{t+1} = \Phi_t^* X_t + \Theta_t^*(\tilde{X}_\infty) + \xi_{t+1}, \tag{144}$$

where

$$\Phi_t^* = \begin{pmatrix} 1 + \vartheta_{zt} & \vartheta_{u_a t} & \vartheta_{u_l t} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Theta_t^*(\tilde{X}_\infty) = \begin{pmatrix} -\vartheta_{zt}\tilde{z}_\infty - \vartheta_{u_a t}\tilde{u}_{a\infty} - \vartheta_{u_l t}\tilde{u}_{l\infty} + \chi(\tilde{z}_\infty, \tilde{u}_{a\infty}) - \tau \\ 0 \\ 0 \end{pmatrix},$$

and $\chi(\tilde{z}_\infty, \tilde{u}_{a\infty}) = \log\{s[\exp(\tilde{z}_\infty)]\exp(a_0 + \tilde{u}_{a\infty}) + (1-\delta)\}$. From (144) it is clear that at least two eigenvalues of Φ_t^* are on the unit circle for all possible finite realizations of X_∞.

AK Growth Model with Stationary Technology and Labor Input ($|\rho_a| < 1$, $|\rho_l| < 1$)

We derive the conditions under which $\{\log \kappa_t\}$ can be "enveloped" by a covariance-stationary linear first-order autoregressive process. The stochastic difference equation describing the law of motion for the effective capital-labor ratio, (27), now becomes

$$\kappa_{t+1} = \exp[-\tau + (1-\rho_a)u_{at} + (1-\rho_l)u_{lt} - \varepsilon_{t+1})][s(\kappa_t)A_t\kappa_t + (1-\delta)\kappa_t]. \tag{145}$$

Defining as before $z_t = \log \kappa_t$, we can rewrite (145) as

$$\begin{aligned} z_{t+1} &= z_t + \log[s(\kappa_t)\exp(a_0 + u_{at}) + (1-\delta)] - \tau + (1-\rho_a)u_{at} \\ &\quad + (1-\rho_l)u_{lt} - \varepsilon_{t+1}, \end{aligned} \tag{146}$$

imposing as in Case 3 the condition $g = 0$. Defining again $X_t = (z_t, u_{at}, u_{lt})^T$, the stochastic difference equation describing the law of motion for the logarithm of the effective capital labor ratio, z_{t+1}, may be rewritten as

$$X_{t+1} = \Phi X_t + \Theta(X_t) + \xi_{t+1}, \tag{147}$$

where

$$\Phi = \begin{pmatrix} 1 & 1-\rho_a & 1-\rho_l \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_l \end{pmatrix}, \Theta(X_t) = \begin{pmatrix} \chi(z_t, u_{at}) - \tau \\ 0 \\ 0 \end{pmatrix}, D = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\xi_t = (\varepsilon_{at}, \varepsilon_{lt})^T$, and $\chi(z_t, u_{at}) = \log[s(\exp(z_t))\exp(a_0+u_{at})+(1-\delta)]$. By the mean value theorem, for all possible finite realizations of X_∞, $\tilde{X}_\infty$ (assuming an invariant steady-state probability distribution existed), we have that

$$\Theta(X_t) = \Theta(\tilde{X}_\infty) + \left(\frac{\partial \Theta}{\partial X}\right)^T \Big|_{X=\bar{X}_t} (X_t - \tilde{X}_\infty), \tag{148}$$

where $\bar{X}_t \in (X_t, \tilde{X}_\infty)$, element by element. It should be noted that

$$\left(\frac{\partial \Theta}{\partial X}\right)^T \Big|_{X=\bar{X}_t} = \begin{pmatrix} \vartheta_{zt} & \vartheta_{u_a t} & \vartheta_{u_l t} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{149}$$

where ϑ_{zt}, $\vartheta_{u_a t}$, and $\vartheta_{u_l t}$ are defined by (141) to (143). Using (148) and (149), we can rewrite (147) as

$$X_{t+1} = \Phi_t^* X_t + \Theta_t^*(\tilde{X}_\infty) + \xi_{t+1}, \tag{150}$$

where

$$\Phi_t^* = \begin{pmatrix} 1+\vartheta_{zt} & 1-\rho_a+\vartheta_{u_a t} & 1-\rho_l+\vartheta_{lt} \\ 0 & \rho_a & 0 \\ 0 & 0 & \rho_l \end{pmatrix},$$

$$\Theta_t^*(\tilde{X}_\infty) = \begin{pmatrix} -\vartheta_{zt}\tilde{z}_\infty - \vartheta_{U_a t}\tilde{u}_{a\infty} - \vartheta_{u_l t}\tilde{u}_{l\infty} + \chi(\tilde{z}_\infty, \tilde{u}_{a\infty}) - \tau \\ 0 \\ 0 \end{pmatrix},$$

and $\chi(\tilde{z}_\infty, \tilde{u}_{a\infty}) = \log\{s[\exp(\tilde{z}_\infty)]\exp(a_0 + \tilde{u}_{a\infty}) + (1-\delta)\}$. From (150) it is clear that the eigenvalues of Φ_t^* are given by $1+\vartheta_{zt}$, ρ_a, and ρ_l. Therefore, if conditions (43) and (44) do not hold, $\{X_t\}$ is not covariance stationary.

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Notes

1. We follow the recent empirical growth literature in considering the Solow-Swan model (Solow, 1956; Swan, 1956) as the vehicle for our analysis. See, for example, Mankiw (1995) for further discussion and defense of the assumptions to model technology and labor input as exogenous processes. For the theoretical derivations of this article, we will go beyond the recent empirical growth literature and allow the saving function to be a general function of the capital-labor ratio.

2. Note that if a variable is ergodic in moments, then its moments can be consistently estimated using long time averages.
3. Empirical evidence on the link between output *growth* and output volatility has been found by Kormendi and Meguire (1985) and Ramey and Ramey (1995). Their analyses are not based on structural models of economic growth.
4. For a review of deterministic versions of the *AK* model see, for example, Barro and Sala-i-Martin (1995, pp. 14–56).
5. Our empirical findings favoring the stochastic Solow-Swan model relative to the stochastic *AK* model are consistent with earlier work based on deterministic versions of those models (for example, by Jones, 1995). For an alternative, entirely empirical, approach to evaluating endogenous growth models using cross-country variances of output per capita, see Evans (1996).
6. In doing so, we relax the standard assumptions regarding the supports of the shocks to technology and labor input, allowing these to be semi-infinite rather than restricting them to be compact on the positive real line.
7. We restrict ourselves here to first-order autoregressive processes as characterizing the stochastic nature of technology. This restriction (as well as the corresponding one for labor input introduced below) can be easily relaxed, however, as will become clear below.
8. Under further restrictions, this argument could be extended to a general saving function of the form of $s(A_{it}, K_{it}, H_{it}, L_{it})$ in (8). For a defense of the assumption of a common rate of depreciation of physical and human capital, see, for example, Mankiw, Romer, and Weil (1992).
9. The new initial endowment is equal to the sum of the initial endowment specified in (2) plus $\phi_i/(1-\alpha_i)\log(\phi_i/\gamma_i)$.
10. A similar argument has been made for a deterministic optimal growth model by Jones (1995).
11. The new initial endowment is equal to the sum of the initial endowment specified in (2) plus $(1-\gamma_i)\log[(1-\gamma_i)/\gamma_i]$.
12. As we indicate in Section 3, our ergodicity in moments results also extend to certain production technologies that do not satisfy the Inada conditions, a prominent example being the constant-elasticity-of substitution production function.
13. See, for example, Inada (1963), and Barro and Sala-i-Martin (1995, p. 52).
14. Note that $(A_{it}L_{it})/(A_{i,t-1}L_{i,t-1}) = \exp(n_i + g_i + \Delta u_{it})$.
15. This is the case in Sections 3 and 5.
16. For a saving function bounded between zero and unity, conditions (31) and (32) are satisfied for all "well-behaved" production functions that satisfy the Inada conditions. See (23) and (24). The monotonicity condition (30) is satisfied, for example, in the case of a Cobb-Douglas production function with a constant saving rate.
17. This distinction between the effects of large positive shocks and large negative shocks cannot be identified in a local analysis of small fluctuations around steady state. The local analysis, for example, commonly employed in the equilibrium business-cycle literature (see, for example, Binder and Pesaran, 1995) is valid only subject to a global analysis of the type carried out here.
18. Using the same type of argument as in Cases 1 and 2, it is then also possible to show that υ_t is ergodic in moments.
19. A proof is available from the authors on request.
20. Note that when there are no unit roots in technology and labor input ($|\rho_{ia}| < 1$, $|\rho_{il}| < 1$), the function $\Psi(\cdot)$ obtained on taking conditional expectations of (26) depends both on κ_{it} and u_{it}. A steady-state relationship as simple as (51) then does not hold any longer, since κ_{it} itself depends on u_{it}. Cross-Sectional Implication 1 below therefore holds only when there are unit roots in technology and labor input.
21. It should be noted that equations (53) and (57) impose a necessary condition for the mean of the steady-state distribution of $\log\kappa_{it}$ to be well defined—namely,

$$\log(\varphi_i) < \tau_i - \log(1-\delta_i).$$

In the Summers-Heston data set analyzed in Section 6, we found this condition to be satisfied under (45).
22. We assume that the truncation condition (38) is satisfied.

23. Let $h(\zeta_i) = s_i/[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]$. It is then easily seen that

$$h''(\zeta_i) = v_{i\infty} \frac{\exp(\tau_i + \zeta_i)[\exp(\tau_i + \zeta_i) + 1 - \delta_i]}{[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]^2},$$

which, for $0 < \delta_i < 1$, is strictly positive for $v_{i\infty}$ strictly positive. Furthermore, letting $\mathcal{H}(\zeta_i) = \log\{s_i/[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]\}$, we have

$$\mathcal{H}''(\zeta_i) = \frac{\exp(\tau_i + \zeta_i)(1 - \delta_i)}{[\exp(\tau_i + \zeta_i) - (1 - \delta_i)]^2},$$

which, for $0 < \delta_i < 1$, is strictly positive.

24. A similar result has also been obtained by Merton (1975) for the unit root case with normally distributed shocks in a continuous-time setting.
25. While the recent econometrics literature has developed a number of important insights into the study of dynamic linear panel data models under heterogeneous slope, nonlinear model features have not been addressed. For an empirical analysis of some of the time-series implications of our stochastic Solow-Swan growth model, particularly for the recent convergence debate, see Lee, Pesaran, and Smith (1997). For related work in the time-series dimension, see, for example, Bernard and Durlauf (1995, 1996), and Evans and Karras (1996).
26. Note that $u_{at} = \log y_t - [\alpha/(1-\alpha)] \log v_t - a_0 - g_t$, and $u_{lt} = \log L_t - l_0 - nt$.
27. We have excluded twenty-six countries from the full sample of Mankiw, Romer, and Weil (1992) as we want to make use of as long a capital-output ratio series as possible to estimate the moments of the steady-state distribution of the capital-output ratio.
28. Intermediate countries in the terminology of Mankiw, Romer, and Weil (1992) are those countries in the sample of Summers and Heston (1991) whose data receive a grade higher than D from Summers and Heston and whose population in 1960 was at least 1 million people. A listing of the countries included in the three samples used in this article is available from the authors on request.
29. Because of the data limitations, we consider only the effect of the variance term in our empirical analysis.
30. We have also used other procedures to compute initial values for the capital stock series. In particular, we considered imposing values of the capital-output ratio consistent with historical estimates of the latter in the literature. However, the empirical results reported below appear to be quite robust to the choice of the truncation horizon under the simple backcasting procedure.
31. It is clear that one could also use the error term of the third equation in the NVAR, (73), to estimate σ^2_{ia} and σ^2_{il}. Since we have not imposed any restrictions in the estimation of the NVAR, we choose not to do so here.
32. A note deriving (83) and (84) is available from the authors on request (see also Johnson, Kotz, and Balakrishnan, 1994).
33. Note that w_i depends on α through μ_i and ω_i.
34. The computation of the ML estimate of α, and the estimates shown in Figure 1 and in Tables 2 to 4 below are carried out using the estimates $\hat{\mu}_i(\alpha)$ and $\hat{\omega}_i(\alpha)$ obtained over the entire available sample period 1960 to 1992.
35. The results for the intermediate and OECD samples are based on regressions with an intercept term, as the intercept is clearly significant for these samples.
36. We have also estimated $\mu_i(\alpha)$ and $\omega_i(\alpha)$ using output per worker for y_{it} and workers for L_{it}, rather than output per capita for y_{it} and working-age population for L_{it}. The results differ only slightly, and in particular do not affect any of our qualitative conclusions. We obtain $\bar{\hat{\mu}}(\hat{\alpha}_{ML}) = -.0006(.0010)$ and $\bar{\hat{\omega}}(\hat{\alpha}_{ML}) = .0312(.0078)$ for the OECD sample, $\bar{\hat{\mu}}(\hat{\alpha}_{ML}) = -.0215(.0349)$ and $\bar{\hat{\omega}}(\hat{\alpha}_{ML}) = .0591(.0309)$ for the intermediate sample, and $\bar{\hat{\mu}}(\hat{\alpha}_{ML}) = -.0350(.0541)$ and $\bar{\hat{\omega}}(\hat{\alpha}_{ML}) = .0685(.0375)$ for the full sample. All results are therefore based on the estimates reported in Table 1.
37. The nonnested W statistics are computed using *Microfit 4.0* (see Pesaran and Pesaran, 1997).
38. We also computed the Akaike information criterion and the Schwarz Bayesian criterion for the choice between the stochastic and the deterministic specifications. Both criteria support the stochastic specification for the full and intermediate samples, and both criteria favor the deterministic specification for the OECD sample.
39. See, for example, Lee, Pesaran, and Smith (1997).
40. The lagged term $\triangle \log y_{i,t-1}$ is included only if it proves to be statistically significant.

41. Using the country-specific estimates of g_i, we find that $\ell(\alpha)$ as given by (85) is maximized at $\hat{\alpha}_{ML} = .44$. The log-likelihood ratio profile is very similar to the one depicted in Figure 1, except that the new profile is centered slightly to the left of that in Figure 1. Furthermore, all qualitative conclusions regarding the empirical validity of Cross-Sectional Implication 1 are again not sensitive to the choice of α in the range [.20, .66]. In what follows we therefore present results for $\hat{\alpha}_{ML} = .44$.
42. These conclusions are robust to allowing for heterogeneity of $\hat{\delta}$ across countries in the range [.4, .6].
43. The extent to which the capital-output ratios under the stochastic version of the Solow-Swan growth model deviate from their steady-state mean value is dominated by the persistence of the exogenous processes u_t and reflects the weak propagation mechanisms inherent in the neoclassical growth model on its own. The u_t-process need not exhibit an exact unit root, however, for there to be strong persistence in the deviations form the mean.
44. This result is quite robust to increasing δ and is further strengthened if δ is reduced below the baseline value of .05. For example, increasing δ to .06 or .07, there is only one additional country for which (64) is violated—namely, the Dominican Republic. Setting δ to .04 reduces the number of countries falling below the 45 degree line from nine to six.
45. The estimates of $\omega_i^2(\alpha)$ used in Tables 3 and 4 are again computed over the period 1960 to 1992 using the ML estimate $\hat{\alpha}_{ML} = .44$.
46. Note also that this NVAR does not impose saving to be (strictly) exogenous.
47. A note containing a more detailed discussion of this literature is available from the authors on request.
48. The existence of the rth moment of z_t immediately follows from the fact that $z_t = x_t + \tilde{z}_\infty$, and $\tilde{z}_\infty$ is a finite realization from the steady-state distribution of z_t—namely, z_∞. Note that $|z_t| < |x_t| + |\tilde{z}_\infty|$, and hence $E|z_t|^r < \infty$, so long as $E|x_t|^r < \infty$, for $r = 1, 2, \ldots$.
49. To keep the notation simple, we will not explicitly indicate in this appendix the dependence of κ_t on ρ_a and ρ_l. Also, $(\cdot)^T$ denotes the transpose of $(\cdot)$.

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