Chapter 15
Dynamic Optimization



Pierre de Fermat (1601?-1665) L. Euler Lev Pontryagin (1908-1988)

15.1 Dynamic Optimization

• In this chapter, we will have a *dynamic* system –i.e., evolving over time (either discrete or continuous time). Our goal: *optimize* the system.

• We will study an optimization problems with the following features:
1) Aggregation over time for the objective function.
2) Variables linked (constrained) across time.

• Analytical solutions are rare, usually numerical solutions are obtained.

• Usual problem: The cake eating problem
There is a cake whose size at time is W_t and a consumer wants to eat in T periods. Initial size of the cake is $W_0 = \varphi$ and $W_T = 0$. What is the optimal strategy, $\{W_t^*\}$?

15.1 Dynamic Optimization

• Typical econ example (Ramsey problem): A social planner's problem – Maximization of consumption over time:

$$Max_{C(t)} \int_0^T e^{-rt} C(t)dt \qquad s.t. \quad \dot{K} = \frac{dK}{dt} = F(K) - C - \delta K$$

• Notation:
K(*t*) = capital (only factor of production)
F(*K*) = well-behaved output function –i.e., $F_K>0$, $F_{KK}<0$, for $K>0$
C(*t*) = consumption
I(*t*) = investments = *F*(*K*) – *C(t)*
δ = constant rate of depreciation of capital.

• We are not looking for a single optimal value C* , but for values C*(t)* that produce an optimal value for the integral (or aggregate discounted consumption over time).

• We are looking for an optimal {C*(t)*}*.

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15.1 Dynamic Optimization

• We will use dynamic optimization methods in different environments:
- Continuous time and discrete time
- Finite horizons and infinite horizons
- Deterministic and stochastic

• Several ways to solve these problems:

1) Discrete time methods (Lagrange. Optimal control theory, Bellman equations, Numerical methods)

2) Continuous time methods (Calculus of variations, Optimal control theory, Bellman equations, Numerical methods).

Usual Applications: Asset-pricing, consumption, investments, I.O., etc.

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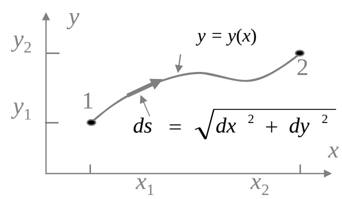
15.1 Calculus of variations – Classic Example

- The *calculus of variations* involves finding an extremum (maximum or minimum) of a quantity that is expressible as an integral.
- Question: What is the shortest path between two points in a plane? You know the answer -a straight line- but you probably have not seen a proof of this: the calculus of variations provides such a proof.
- Consider two points in the x-y plane, as shown in the figure.



- An arbitrary path joining the points follows the general curve $y = y(x)$, and an element of length along the path is

$$ds = \sqrt{dx^2 + dy^2}.$$

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15.1 Calculus of variations – Classic Example

- We can rewrite this as: $ds = \sqrt{1 + y'(x)^2}\,dx,$

  which is valid because $dy = \frac{dy}{dx}dx = y'(x)dx.$ Thus, the length is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,dx.$$

- Note that we have converted the problem from an integral along a path, to an integral over x:

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\,dx.$$

- We have thus succeeded in writing the problem down, but we need some additional mathematical machinery to find the path for which L is an extremum (a minimum in this case).

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15.1 Calculus of variations – Classic Example

- In our usual minimizing or maximizing of a function $f(x)$, we would take the derivative and find its zeroes These points of zero slope are *stationary points* –i.e., the function is *stationary* at those points, meaning for values of x near such a point, the value of the function does not change (due to the zero slope).
- Similarly, we want to be able to find solutions to these integrals that are stationary for infinitesimal variations in the path. This is called *calculus of variations*.
- The methods we will develop are called *variational methods*.
- These principles are common, and of great importance, in many areas of physics (such as quantum mechanics and general relativity) and economics.

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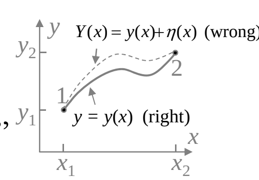
15.1 Euler-Lagrange Equations

- We will try to find an extremum (to be definite, a minimum) for an as yet unknown curve joining two points x_1 and x_2, satisfying the integral relation:

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]\,dx$$

- The function f is a function of three variables, but because the path of integration is $y = y(x)$, the integrand can be reduced to a function of just one variable, x.
- To start, let's consider two curves joining points 1 and 2, the "right" curve $y(x)$, and a "wrong" curve: $Y(x) = y(x) + \eta(x); \quad \eta(x_1) = \eta(x_2) = 0.$ $Y(x)$ that is a small displacement from the "right" curve, as shown in the figure.



- We call the difference between these curves as some function $h(x)$.

15.1 Euler-Lagrange Equations

- There are infinitely many functions *h*(*x*), that can be "wrong." We require that they each be longer than the "right" path. To quantify how close the "wrong" path can be to the "right" one, let's write $Y = y + \alpha h$, so that

$$S(\alpha) = \int_{x_1}^{x_2} f[Y, Y'(x), x]dx$$
$$= \int_{x_1}^{x_2} f[y + \alpha\eta, y' + \alpha\eta', x]dx.$$

- Now, we can characterize the shortest path as the one for which the derivative $dS/d\alpha = 0$ when $\alpha = 0$. To differentiate the above equation with respect to α, we need the partial derivative $\partial S/\partial\alpha$ via the chain rule

$$\frac{\partial f(y+\alpha\eta, y'+\alpha\eta', x)}{\partial\alpha} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'},$$

so $dS/d\alpha = 0$ gives

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2}\frac{\partial f}{\partial\alpha}dx = \int_{x_1}^{x_2}\left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right)dx = 0$$

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15.1 Euler-Lagrange Equations

- The second term in the equation can be integrated by parts:

$$\int_{x_1}^{x_2}\eta'\frac{\partial f}{\partial y'}dx = \left[\eta(x)\frac{\partial f}{\partial y'}\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx,$$

but the first term of this relation (the end-point term) is zero because *h*(*x*) is zero at the endpoints.

- Our modified equation is:

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2}\eta(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)dx = 0.$$

- This leads us to the *Euler-Lagrange* equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- Key: the modified equation has to be zero for any *h*(*x*).

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15.1 Euler-Lagrange Equations

- Let's go over what we have shown. We can find a minimum (more generally, a stationary point) for the path S if we can find a function for the path that satisfies:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

- The procedure for using this is to set up the problem so that the quantity whose stationary path you seek is expressed as

$$S = \int_{x_1}^{x_2} f[y(x), y'(x), x]dx,$$

where $f[y(x),y'(x),x]$ is the function appropriate to your problem.

- Then, write down the Euler-Lagrange equation, and solve for the function $y(x)$ that defines the required stationary path. 11

15.1 Euler-Lagrange Equations – Example I

- Find the shortest path between two points:

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + y'(x)^2}\, dx.$$

- The integrand contains our function $f(y, y', x) = \sqrt{1 + y'(x)^2}.$
- The two partial derivatives in the Euler-Lagrange equation are:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \quad => \quad \frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}.$$

- The Euler-Lagrange equation gives us: $\dfrac{d}{dx}\dfrac{\partial f}{\partial y'} = \dfrac{d}{dx}\dfrac{y'}{\sqrt{1 + y'^2}} = 0.$

- This says that $\dfrac{y'}{\sqrt{1 + y'^2}} = C, \quad \text{or} \quad y'^2 = C^2(1 + y'^2).$

=> y'^2 = constant (call it m^2), so $y(x) = mx + b$

=> A straight line is the shortest path.

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15.1 Euler-Lagrange Equations – Example II

- Intertemporal utility maximization problem:

$$\max \int_{x_1}^{x_2} [B - u(c(t))]dx \quad s.t. \quad c(t) = f(k(t)) - k'(t)$$

B: bliss level of utility
$c(t) = c(k(t), k'(t))$

- Use substitution method. Substitute constraint into integrand:
  $g(x(t),x'(t),t) = B - u(c(k(t),k'(t)) = V(k(t),k'(t))$

- Euler-Lagrange equation: $V_k = \frac{dV}{dk} = \frac{d}{dt}\frac{dV}{dk'} = \frac{d}{dt}V_{k'}$

$$V_k = -u'(c)f'(k); \quad V_{k'} = V_c \frac{dc}{dk'} = -u'(c)(-1) = u'(c)$$

$$V_k = \frac{d}{dt}V_{k'} \quad \Rightarrow -u'(c)f'(k) = \frac{d}{dt}u'(c)$$

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15.1 Euler-Lagrange Equations – Example II

- Repeating Euler-Lagrange equation:

$$V_k = \frac{d}{dt}V_{k'} \quad \Rightarrow -u'(c)f'(k) = \frac{d}{dt}u'(c)$$

- If we are given functional forms:
  $f(k(t)) = k(t)^{\alpha}$
  $u(c(t)) = \ln(c(t))$

Then,

$$-u'(c)f'(k) = \frac{d}{dt}u'(c)$$

$$-\frac{1}{c(t)}\alpha k(t)^{\alpha-1} = \frac{d}{dt}\left(\frac{1}{c(t)}\right) = -\frac{\dot{c}(t)}{c(t)^2}$$

$$\Rightarrow \alpha k(t)^{\alpha-1} = \frac{\dot{c}(t)}{c(t)}$$

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15.1 Dynamic Optimization – Calculus of Variations: Summary

• In the general economic framework, *F*(.) will be the objective function, the variable *x* will be time, and *y* will be the variable to choose over time to optimize *F*(.). Changing notation:

$$\max_x \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))dt \qquad \text{s.t. } x(t_0) = x_0, x(t_1) = x_1$$

• Necessary Conditions: Euler-LaGrange Equation

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0.$$

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15.1 Dynamic Optimization – Limitations

• Method gives extremals, but it does not tell maximum or minimum:
- Distinguishing mathematically between max/min is more difficult.
- Usually have to use geometry to setup the problem.

• Solution curve *x(t)* must have continuous second-order derivatives
- Requirement from integration by parts.

• We find stationary states, which vary only in space, not in time.
-Very few cases in which systems varying in time can be solved.
- Even problems involving time (e.g., brachistochrones) do not change in time.

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15.2 Optimal Control Theory

- *Optimal control*: Find a control law for a given system such that a certain optimality criterion is achieved.
- Typical example: Minimization of a cost functional that is a function of state and control variables.
- More general than Calculus of Variations.
- Handles Inequality restrictions on Instruments and State Variables
- Similar to Static Methods (Lagrange Method and KKT Theory)
- Long history in economics: Shell (1967), Arrow (1968) and Shell (1969).

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15.2 Optimal Control Theory

- The dynamic system is described by a *state equation* represented by:

$$\dot{x}(t) = g(t, x(t), u(t))$$

where $x(t)$ = *state variable*, $u(t)$ = *instrument* or *control variable*

- The control aim is to maximize the *objective functional:*

$$\int_{t_0}^{t_1} f(t, x(t), u(t))dt$$

- Usually the control variable $u(t)$ will be constrained as follows:

$$u(t) \in \Omega(t), \qquad t \in [t_0, t_1]$$

- Boundary Conditions: t_0, t_1, $x(t_0) = x_0$ fixed. Sometimes, $x(t_1) = x_T$.
- Sometimes, we will also have additional constraints. For example,

$h(x(t),u(t),t) \geq 0$

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15.2 Optimal Control Theory

• The most common dynamic optimization problems in economics and finance have the following common assumptions:

- *x(t),* the state variable, is usually a stock (if discrete, measured at the beginning of period *t*.)
- *u(t),* the control variable, is a flow (if discrete, measured at the end of period *t*)
- the *objective functional:* usually involves an intertemporal utility function is additively separable, stationary, and involves time-discounting.
- *f*(.), *g*(.) are well behaved (continuous and smooth) functions.
- there is a non-Ponzi scheme condition: $\lim_{t\to\infty} \varphi^t x_t \geq 0$ (φ: discount factor)
- the solution $u^*(t) = h(x(t))$ is called the *policy function.* It gives an optimal rule for changing the optimal control, given the state of the economy.

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15.2 Optimal Control Theory

• We can convert the calculus of variation problem into a control theory problem.

• Calculus of variation problem:

$$\max \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\,dt$$

Define: $x'(t)=u(t)$, $x(t_0)=x_0$, $x(t_1)=x_1$

Replace in calculus of variation problem:

$$\max \int_{t_0}^{t_1} F(t, x(t), u(t))\,dt$$

where *x(t)* is the state variable, and *u(t)* is the control variable.

• Notation: $x(t) = x_t = x^t$

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15.2 Optimal Control Theory

• Form a Lagrangian-type expression, the *Hamiltonian*

$$H(t,x(t),u(t),\lambda(t)) = f(t,x(t),u(t)) + \lambda\, g(t,x(t),u(t))$$

Necessary Conditions (f.o.c.), called the *Maximum Principle*):

$$\frac{dH}{du} = \frac{df}{du} - \lambda \frac{dg}{du} = 0$$

$$-\frac{dH}{dx} = \dot{\lambda} \qquad \text{(adjoint equation)}$$

$$\frac{dH}{d\lambda} = \dot{x}(t) = g(t, x(t), u(t))$$

Boundary (Transversality) Conditions: $x(t_0) = x_0$, $\lambda(t_1) = 0$

The Hamiltonian multiplier, $\lambda(t)$, is called the *co-state* or *adjoint* variable: It measures the imputed value of stock (state variable) accumulation.

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15.2 Optimal Control Theory – Pontryagin's Maximum principle

• Conditions for necessary conditions to work –i.e.,: u(t) and x(t) maximize:

$$\int_{t_0}^{t_1} f(t, x(t), u(t))dt \qquad \text{s.t. } \dot{x}(t) = g(t, x(t), u(t))$$

- Control variable must be piecewise continuous (some jumps, discountinuities OK).
- State variable must be continuous and piecewise differentiable.
- f(.) and g(.) first-order differentiable w.r.t. state variable and t, but not necessarily w.r.t. control variable.
- Initial condition finite for state variable.
- If no finite terminal value for state variable, then $\lambda(t_1) = 0$.

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15.2 Optimal Control Theory – Pontryagin's maximum principle

• *Sufficiency*: if *f*(.) and *g*(.) are strictly concave, then the necessary conditions are sufficient, meaning that any path satisfying these conditions does in fact solve the problem posed.

Lev Pontryagin (1908–1988, Russia/URSS)

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15.2 Optimal Control Theory – Example

• Let's go back to our first example:

$$\text{Max} \int_0^T e^{-rt} C_t\, dt \quad \text{s.t. } \dot{K} = Q - C - \delta K$$

$$\text{where } Q = Q(K) \text{ and } \frac{\partial Q}{\partial K} > 0 \text{ and } \frac{\partial^2 Q}{\partial K^2} < 0$$

• Form the Hamiltonian:

$$H = e^{-rt} C_t + \lambda\left[Q - C - \delta K\right] + \dot{\lambda} K \qquad (1) \text{ (boundaries: } K_0 \text{ and } K_T.)$$

F.o.c.:

$$\frac{\partial H}{\partial C} = e^{-rt} - \lambda = 0 \qquad => \; e^{-rt} = \lambda \qquad (2)$$

$$\frac{\partial H}{\partial K} = \lambda\left(\frac{\partial Q}{\partial K} - \delta\right) + \dot{\lambda} = 0 \qquad => \; \frac{\partial Q}{\partial K} = -\frac{\dot{\lambda}}{\lambda} + \delta \qquad (3)$$

$$\frac{\partial H}{\partial \lambda} = \dot{K} = Q - C - \delta K \qquad (4)$$

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15.2 Optimal Control Theory – Example

From (3) and (2) $\frac{\partial Q}{\partial K}(K^*) = -\frac{\dot{\lambda}}{\lambda} + \delta = r + \delta$ (5)

=> At the optimal (K*), the marginal productivity of capital should equal the total user cost of capital, which is the sum of the interest and the depreciation rates.

The capital stock does not change when it reaches K*.

We have the following dynamics:

C_t should be reduced if $K_t < K^*$ (dQ/dK too low)
C_t should be increased if $K_t > K^*$ (dQ/dK too high)

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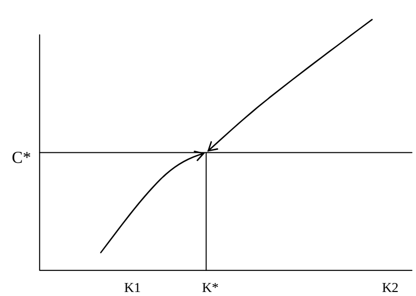
15.2 Optimal Control Theory – Example

- Dynamics of the system
  Following dynamics:
  C_t should be reduced if $K_t < K^*$ (dQ/dK too low)
- C_t should be increased if $K_t > K^*$ (dQ/dK too high)



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15.2 OCT – More State and Control Variables

• We can add to the problem more state and control variables. Analogy can be made to multivariate calculus.

• Objective Functional:

$$\int_{t_0}^{t_1} f(t, x_1(t), x_2(t), u_1(t), u_2(t))dt$$

Constraints:

$$\dot{x}_1(t) = g^1(t, x_1(t), x_2(t), u_1(t), u_2(t))$$
$$\dot{x}_2(t) = g^2(t, x_1(t), x_2(t), u_1(t), u_2(t))$$

Boundary Conditions: t_0, t_1, $x_1(t_0)=x_{10}$, $x_2(t_0) = x_{20}$; fixed
Free Endpoints for x_1 and x_2

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15.2 OCT – More State and Control Variables

• Form Hamiltonian:

$$H(t, x_1(t), x_2(t), u_1(t), u_2(t), \lambda_1(t), \lambda_2(t)) = f(t, x_1(t), x_2(t), u_1(t), u_2(t)) +$$
$$+ \lambda_1 g^1(t, x_1(t), x_2(t), u_1(t), u_2(t)) + \lambda_2 g^2(t, x_1(t), x_2(t), u_1(t), u_2(t))$$

F.o.c.:

$$u_i : \frac{\partial f}{\partial u_i} + \lambda_1 \frac{\partial g^1}{\partial u_i} + \lambda_2 \frac{\partial g^2}{\partial u_i} = 0 \qquad i = 1,2$$

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} = -\left\{\frac{\partial f}{\partial x_i} + \lambda_1 \frac{\partial g^1}{\partial x_i} + \lambda_2 \frac{\partial g^2}{\partial x_i}\right\}$$

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}$$

Boundary (Transversality) Conditions:

$$x_1(t_0)=x_{10},\ x_2(t_0)=x_{20},\ \lambda_1(t_1)=\lambda_2(t_1)= 0$$

Fixed Endpoint Problem: Add the boundary condition: $x(t_1) = x^*$

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15.2 Optimal Control Theory – General Problem

• More General Problem: Let's add a terminal-value function and inequality restrictions.

Objective Functional:

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt + \varphi(t, x(t_1)) \quad \text{s.t.}$$

$$\dot{x}_i(t) = g_i(t, x(t), u(t)), \quad x_i(t_0) = x_{i0}, \quad i = 1,..., n$$

$$x_i(t_1) = x_{i1}, \qquad i = 1,..., q$$

$$x_i(t_1) \text{ free}, \qquad i = q+1,..., r$$

$$x_i(t_1) \geq 0, \qquad i = r+1,..., s$$

$$K(x_1(t),..., x_n(t)) \geq 0, \quad \text{at } t_1$$

$$1 \leq q \leq r \leq s \leq n$$

$\boldsymbol{x}$ = n-dimensional vector, $\boldsymbol{u}$ = m-dimensional vector

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15.2 Optimal Control Theory – General Problem

F.o.c.:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} = g_i(t, x, u), i = 1, \ldots, n$$

$$\dot{\lambda}_i = -\left\{\frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \lambda_i \frac{\partial g_j}{\partial x_i}\right\}, j = 1, \ldots, n$$

$$\frac{\partial f}{\partial u_j} + \sum_{k=1}^{n} \lambda_k \frac{\partial g_k}{\partial u_j} = 0, j = 1, \ldots, m$$

Note: H(t,x*,u,λ) is maximized by u=u*

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15.2 Optimal Control Theory – General Problem

Transversality Conditions

(i) $x_i(t_1)$ free: $$\lambda_i(t_1) = \frac{\partial \varphi}{\partial x_i}$$

(ii) $x_i(t_1) \geq 0$: $$\lambda_i(t_1) \geq \frac{\partial \varphi}{\partial x_i}, \quad x_i(t_1)\left[\lambda_i(t_1) - \frac{\partial \varphi}{\partial x_i}\right] = 0$$

(iii) $K(x_q(t_1),\ldots,x_n(t_1)) \geq 0$: $$\lambda_i(t_1) = \frac{\partial \varphi}{\partial x_i} + p\frac{\partial K}{\partial x_i}, \quad i = q,\ldots,n$$
$$p \geq 0,\ pK = 0$$

(iv) $K(x_q(t_1),\ldots,x_n(t_1)) = 0$: $$\lambda_i(t_1) = \frac{\partial \varphi}{\partial x_i} + p\frac{\partial K}{\partial x_i}, \quad i = q,\ldots,n$$

(v) t_1 is free: at t: $$f + \sum_{i=1}^{n} \lambda_i g_i + \varphi_t = 0$$

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15.2 Optimal Control Theory – General Problem

(vi) $T \geq t_1$: $$f + \sum_{i=1}^{n} \lambda_i g_i + \varphi_i \geq 0$$

at t_1, with strict equality if $T > t_1$, if $T - t_1 > 0$ is required

(vii) $K(x_q(t_1),\ldots,x_n(t_1),t_1) \geq 0$: $$\lambda_i(t_1) = \frac{\partial \varphi}{\partial x_i} + p\frac{\partial K}{\partial x_i}, \quad i = q,\ldots,n$$
$$f + \sum_{i=1}^{n} \lambda_i g_i + \varphi_t + p\frac{\partial K}{\partial t_1} = 0$$
$$p \geq 0,\ K \geq 0,\ pK = 0,\ t = t_1$$

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15.2 Optimal Control Theory – General Problem

- State Variable Restrictions: $k(t,x) \geq 0$

$$\max \int_{t_0}^{t_1} f(t, x(t), u(t))dt + \varphi(x(t_1)) \qquad \text{s.t.}$$

$$\text{s.t.} \ \dot{x}(t) = g(t, x(t), u(t)), \quad x(t_0) = x_0, \quad k(t,x) \geq 0$$

- Form Hamiltonian:

$H = f(t,x,u) + \lambda g(x,u,t) + \eta k(t,x)$

Optimality Conditions:

$$u: H_u = f_u + \lambda g_u = 0$$
$$\dot{\lambda} = -H_x = -(f_x + \lambda g_x + \eta k_x)$$
$$\lambda(t_1) = \varphi_x(x(t_1)), \eta \geq 0, \eta k = 0$$

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15.2 OCT – Current Value Hamiltonian

- Often, in economics, we have to optimized over discounted sums, subject to our usual dynamic constraint:

$$\int_0^T e^{-rt} f(t,x,u)dt$$
$$\dot{x} = g(t,x,u), x(0) = x_0$$

Form Hamiltonian:

$H = e^{-rt} f(t,x,u) + \lambda\, g(t,x,u)$

Optimality Conditions:

$$u: H_u = e^{-rt} f_u + \lambda g_u$$
$$\dot{\lambda} = -H_x = -e^{-rt} f_x - \lambda g_x$$
$$\lambda(T) = 0$$

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15.2 OCT – Current Value Hamiltonian

• Sometimes, it is convenient to eliminate the discount factor. The resulting system involves current, rather than, discounted values of various magnitudes.

Hamiltonian: $H = e^{-rt} f(t,x,u) + \lambda(t)\, g(t,x,u)$

Define $m(t) = e^{rt}\lambda(t)$ => $\dot{m} = re^{rt}\lambda + e^{rt}\dot{\lambda} = rm + e^{rt}\dot{\lambda}$

Current Value Hamiltonian: $\tilde{H} = e^{rt}H = f(t,x,u) + mg(t,x,u)$

Optimality Conditions: $\frac{\partial \tilde{H}}{\partial u} = f_u + mg_u = 0, \; \dot{m} - rm = -f_x - mg_x$

Note: If $f(.)$ and $g(.)$ are autonomous, this substitution leads to autonomous transition equations describing the dynamics of the system.

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15.2 OCT – Current Value Hamiltonian - Example

• Using the usual notation, we want to maximize utility over time:

$$\max\; J = \int_{t_0}^{t_1} e^{-\rho t} U(C(t))dt \qquad \text{s.t.}$$

$$\dot{K}(t) = F(K) - C - \partial K, \quad K(0) = K_0, K(T) = K_T, \quad C(t,x) \geq 0$$

• Form Current Value Hamiltonian:

$$\tilde{H} = e^{\rho t} H = U(C(t)) + m(F(K) - C - \partial K)$$

• Optimality Conditions: $\frac{d\tilde{H}}{dC} = \frac{dH}{dC} = U'(C) - m = 0$

$$\dot{m} - \rho m = -U'(C) - m(F'(K) - \delta)$$

• Note: The current value Hamiltonian consists of two terms: 1) utility of current consumption, and 2) net investment evaluated by price m, which reflects the marginal utility of consumption.

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15.2 OCT – Current Value Hamiltonian - Example

• The static efficiency condition:

$$U'(C(t))=m(t),$$

maximizes the value of the Hamiltonian at each instant of time myopically, provided $m(t)$ is known.

• The dynamic efficiency condition:

$$\dot{m} - \rho m = -U'(C) - m(F'(K) - \delta)$$

forces the price m of capital to change over time in such a way that the capital stock always yields a net rate of return, which is equal to the social discount rate ρ. That is,

$$dm = [-U'(C) - m(F'(K) - \delta)]dt + \rho m dt$$

• There is a long-run foresight condition which establishes the terminal price $m(T)$ of capital in such a way that exactly the terminal capital stock $K(T)$ is obtained at T.

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15.2 OCT – Application

• *Equilibria in Infinite Horizon Problems*

$$\max \int_{t_0}^{\infty} e^{-rt} f(x(t), u(t))dt \text{ s.t. } \dot{x}(t) = g(x(t), u(t)), \quad x(t_0) = x_0$$

Hamiltonian: $\tilde{H}$ = f(x,u) + m g(x,u)

Optimality Conditions: $H_u = 0, \quad \dot{m} - rm = -H_x$

Transversality Conditions: $\lim_{t\to\infty} m(t)x(t) = 0$

$$\lim_{t\to\infty} e^{-rt} m(t)x(t) = 0$$

Problems of this sort, if they are of two dimensions, lead to phase plane analysis.

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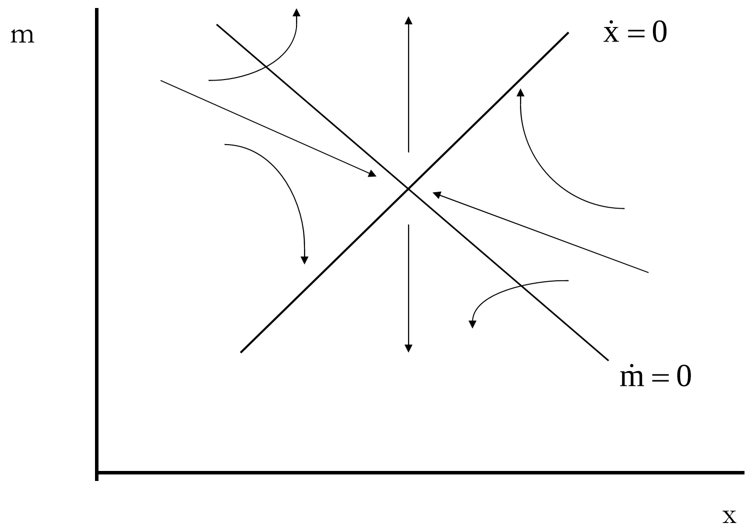
15.2 OCT – Application



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15.2 OCT – Sufficiency

- Sufficiency: Arrow and Kurz (1970)

If the maximized Hamiltonian is strictly concave in the state variables, any path satisfying the conditions above will be sufficient to solve the problem posed.

Hamiltonian: $\tilde{H} = f(x,u) + m\, g(x,u)$

Optimality Conditions: $H_u = 0, \quad \dot{m} - rm = -H_x$

$$\Rightarrow u = \hat{u}(m, x)$$

Maximized Hamiltonian:

$$H^* = f(x, \hat{u}(m, x)) + mg(x, \hat{u}(m, x))$$

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15.2 OCT – Applications

• Example I: Nerlove-Arrow Advertising Model

Let $G(t) \geq 0$ denote the stock of goodwill at time t .

$$\dot{G} = u - \partial G, \qquad G(0) = G_0$$

where $u = u(t) \geq 0$ is the advertising effort at time t measured in dollars per unit time. Sales S are given by

$$S = S(p,G,Z),$$

where p is the price level and Z other exogeneous variables.

Let $c(S)$ be the rate of total production costs, then, total revenue net of production costs is:

$$R(p,G,Z) = p\, S(p,G,Z) - c(S)$$

Revenue net of advertising expenditure is: $R(p,G,Z) - u$

Kenneth Joseph Arrow (1921, USA)

15.2 OCT – Applications

• The firm wants to maximize the present value of net revenue streams discounted at a fixed rate ρ:

$$\max_{u \geq 0, p \geq 0} \left\{ J = \int_0^{\infty} e^{-\rho t} [R(p,G,Z) - u]\, dt \right\}$$

subject to $\dot{G} = u - \partial G, \qquad G(0) = G_0$

• Note that the only place p occurs is in the integrand, which we can maximize by first maximizing R w.r.t p holding G fixed, and then maximized the result with respect to u. Thus,

$$\frac{\partial R(p,G,Z)}{\partial p} = S + p\frac{\partial S}{\partial p} - c_S \frac{\partial S}{\partial p} = 0, \qquad (7.5)$$

• Implicitly, we get $p^*(t) = p(G(t),Z(t))$

• Define $\pi(G,Z)=R(p^*,G,Z)$. Now, J is a function of G and Z only. For convenience, assume Z is fixed.

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15.2 OCT – Applications

• Solution by the Maximum Principle

$$H = \pi(G) - u + \lambda[u - \delta G]$$

$$\frac{dH}{du} = 0$$

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{\partial \pi}{\partial G}$$

$$\lim_{t \to +\infty} e^{-\rho t}\lambda(t) = 0.$$

The adjoint variable $\lambda(t)$ is the shadow price associated with the goodwill at time t . The Hamiltonian can be interpreted as the dynamic profit rate which consist of two terms:

(i) the current net profit rate $\pi(G) - u$.

(ii) the value $\lambda\dot{G} = \lambda[u - \delta G]$ of the new goodwill created by advertising at rate u.

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15.2 OCT – Applications

The second equation corresponds to the usual equilibrium relation for investment in capital goods:

$$\dot{\lambda} = \rho\lambda - \frac{\partial H}{\partial G} = (\rho + \delta)\lambda - \frac{\partial \pi}{\partial G}$$

It states that the marginal opportunity cost of investment in goodwill, $d\lambda := \dot{\lambda}dt$, should equal the sum of the marginal profit $(\partial\pi/\partial G)dt$ from increased goodwill and the capital gain, $\lambda(\rho + \delta)dt$.

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15.2 OCT – Applications

Define $\beta = (G/S)(\partial S/\partial G)$ as the elasticity of demand with respect to goodwill and after some algebra, we can derive:

$$G^* = \frac{\beta p S}{\eta[(\rho+\delta)\lambda - \dot{\lambda}]}.$$

We also can obtain the optimal long-run stationary equilibrium $\{\bar{G}, \bar{u}, \bar{\lambda}\}$ Again, after some simple algebra, we obtain: as

$$\bar{G} = G^s = \frac{\beta p S}{\eta(\rho+\delta)}$$

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15.2 OCT – Applications

The property of $\bar{G}$ is that the optimal policy is to go to as fast $\bar{G}$ possible.

If $G_0 < \bar{G}$, it is optimal to jump instantaneously to $\bar{G}$ by applying an appropriate impulse at t =0 and set $u^*(t) = \bar{u} = \delta\bar{G}$ for t >0.

If $G_0 > \bar{G}$ the optimal control $u^*(t)$=0 until the stock of goodwill depreciates to the level $\bar{G}$, at which time the control switches to $u^*(t) = \delta\bar{G}$ and stays at this level to maintain the level $\bar{G}$ of goodwill. See Figure.

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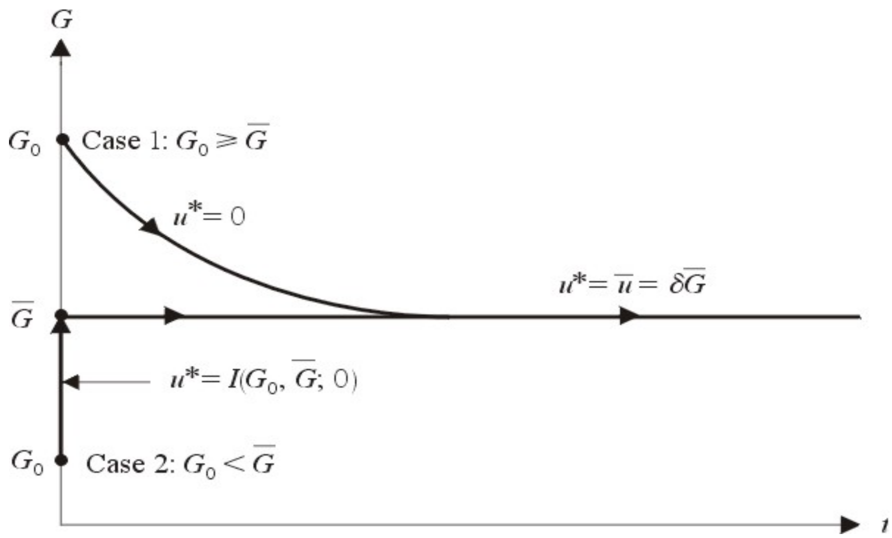
15.2 OCT – Applications



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15.2 OCT – Applications

- Example II: Neoclassical Growth Model (Robert Solow)

Preferences: $\int_0^\infty e^{-\rho t} \frac{C_t^{1-\sigma}}{1-\sigma} dt$

Capital accumulation: $\dot{K}_t = Y_t - N_t C_t - \delta K_t$

Technology: $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$ assume $A_t = 1 \quad N_t = 1$

Form current value Hamiltonian:

$$H(c, K, \theta) = \frac{C_t^{1-\sigma}}{1-\sigma} + \theta_t \left[K_t^\alpha - C_t - \delta K_{t-1}\right] \quad (1)$$

C is control, K is state variable, θ is adjoint variable.

First order conditions: $\frac{\partial H}{\partial C_t} = 0 \;=>\; C_t^{-\sigma} = \theta_t \quad (2)$

$$\dot{\theta}_t = \rho\theta_t - \frac{\partial H_t}{\partial K_t} \;=>\; \dot{\theta}_t = \rho\theta_t - \theta_t\left[\alpha K_t^{\alpha-1} - \delta\right] \quad (3)$$

$$\dot{K}_t = K_t^\alpha - C_t - \delta K_t \quad (4)$$

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15.2 OCT – Applications

Transversality condition $\lim_{t \to \infty} e^{-\rho t}\theta_t K_t = 0$ (5)

Characterization of the balanced growth path: Capital stock, consumption and the shadow price of capital remain constant in the balanced growth path $\frac{\dot{C}}{C} = g_c$; $\frac{\dot{K}}{K} = g_K$ and $\frac{\dot{\theta}_t}{\theta_t} = g_\theta$. From (3)

$\frac{\dot{\theta}_t}{\theta_t} = \rho - \left[\alpha K_t^{\alpha-1} - \delta\right]$ => $\alpha K_t^{\alpha-1} = \rho - \frac{\dot{\theta}_t}{\theta_t} + \delta$ (6)

Since the RHS is constant , therefore LHS also should be constant $\frac{\dot{K}}{K} = 0$. If capital stock is not growing output is not growing $\frac{\dot{Y}}{Y} = 0$ and consumption is not growing $\frac{\dot{C}}{C} = 0$.

From (2) $\frac{\dot{\theta}_t}{\theta_t} = -\sigma\frac{\dot{C}_t}{C_t}$ => $\frac{\dot{\theta}_t}{\theta_t} = 0$

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15.2 OCT – Applications

Recall that from (2): $C_t^{-\sigma} = \theta_t$ => $C_t = \theta_t^{\sigma}$

Then, we have a 2x2 non-linear system of differential equations:

$$\dot{K}_t = K_t^{\alpha} - \theta_t^{\sigma} - \delta K_t$$

$$\frac{\dot{\theta}_t}{\theta_t} = \rho - \left[\alpha K_t^{\alpha-1} - \delta\right] = (\rho + \delta) - \alpha K_t^{\alpha-1}$$

This system can be shown as a phase diagram in the (θ_t, K_t) space, to analyze the transition dynamics of the shadow price θ_t and capital.

From (6), we can calculate the steady state value for K_t:

$$K^* = [(\rho + \delta)/\alpha]^{\alpha-1} \qquad (\alpha<1)$$

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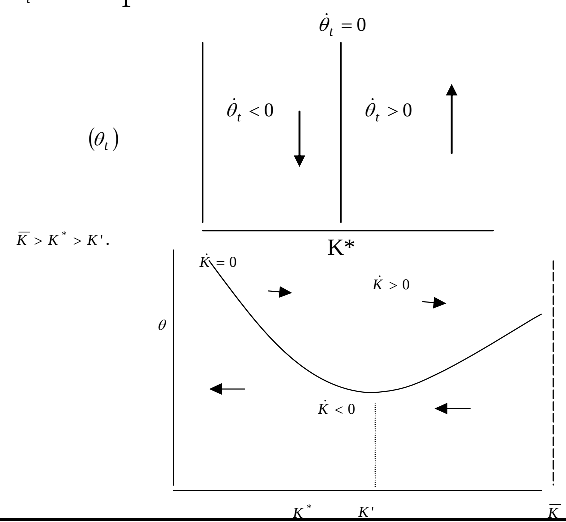
15.2 OCT – Applications

In (θ_t, K_t) space the transition dynamics of the shadow price θ_t and capital.



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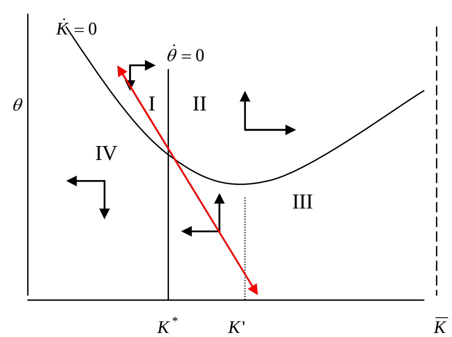
15.2 OCT – Applications

Putting all these things together the convergence to the steady state can be summarised in the following diagram.



Convergence to the steady state lies in region I and III as shown by the double arrow red line.

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15.3 Discrete Time Optimal Control

• We change from continuous measured variables to discrete measured variables.

• State Variable Dynamics: $\Delta x_t = x_{t+1} - x_t = f(x_t, u_t, t)$, x_0 given

• Objective Functional: $\max J = \sum_{t=0}^{t=T-1} u(x_t, t) + S(x_T)$

• Form a Lagrangean:

$$L = \sum_{t=0}^{T-1} u(x_t, t) + S(x_T) + \sum_{t=0}^{T-1} \lambda_t [f(x_t, u_t, t) - x_{t+1} + x_t]$$

• Define the Hamiltonian H^t:

$$H^t = H(x_t, u_t, \lambda_{t+1}, t) = u(x_t, u_t, t) + \lambda_{t+1} f(x_t, u_t, t)$$

• Now, re-write the Lagrangean:

$$L = S(x_T, T) + \sum_{t=0}^{T-1} \{H^t - \lambda_{t+1}(x_{t+1} - x_t)\}$$

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15.3 Discrete Time Optimal Control

• F.o.c.:

$$\frac{dL}{du_t} = \frac{dH^t}{du_t} = 0$$

$$\frac{dL}{d\lambda_{t+1}} = \frac{dH^t}{d\lambda_{t+1}} - (x_{t+1} - x_t) = 0$$

$$\frac{dL}{dx_t} = \frac{dH^t}{dx_t} + \lambda_{t+1} - \lambda_t = 0$$

$$\frac{dL}{dx_T} = \frac{dS}{dx_T} - \lambda_T = 0$$

• Optimality Conditions: $H_u = 0$

$$\Delta x_t = x_{t+1} - x_t = dH/d\lambda_{t+1} = f(x_t, u_t, t)$$

$$\Delta \lambda_t = -dH/dx_t = -du/dx_t - \lambda_{t+1}\, df/dx_t$$

Boundary Conditions: x_0 given and $\lambda_T = dS/dx_T$

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15.3 Discrete Time Optimal Control - Example

• Consider an production-inventory discrete problem.

Let I^k, P^k and S^k be the inventory, production, and demand at time k, respectively. Let I^0 be the initial inventory, let $\hat{I}$ and $\hat{P}$ be the goal levels of inventory and production, and let h and c be inventory and production cost coefficients.

The problem is:

$$\max_{P^k \geq 0} \left\{ J = \sum_{k=0}^{T-1} -\frac{1}{2}[h(I^k - \hat{\mathrm{I}})^2 + c(P^k - \hat{P})^2] \right\}$$

subject to

$$\triangle I^k = P^k - S^k, I^0 \text{ given.}$$

Form the Hamiltonian:

$$H^k = -\frac{1}{2}[h(I^k - \hat{\mathrm{I}})^2 + c(P^k - \hat{P})^2] + \lambda^{k+1}(P^k - S^k),$$

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15.3 Discrete Time Optimal Control - Example

The adjoint variable satisfies:

$$\triangle \lambda^k = -\frac{\partial H^k}{\partial I^k} = h(I^k - \hat{\mathrm{I}}),\ \lambda^T = 0.$$

To maximize the Hamiltonian, let us differentiate w.r.t. to production:

$$\frac{\partial H^k}{\partial P^k} = -c(P^k - \hat{P}) + \lambda^{k+1} = 0.$$

Since production must be nonnegative, we obtain the optimal production as

$$P^{k*} = \max\,[0, \hat{P} + \lambda^{k+1}/c].$$

These expressions determine a two-point boundary value problem. For a given set of data, it can be easily solved numerically. If the constraint $P^k \geq 0$ is dropped it can be solved analytically.

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15.3 Discrete Time Optimal Control

- Discrete Calculus of Variations
- We have an intertemporal problem: $\max \sum_{t=0}^{\infty} \beta^t g(x_{t+1}, x_t)$

Trick: Differentiate the objective function with respect to x_t at time t and t-1.

Objective Function at time t: $\beta^t g(x_{t+1},x_t)$

Optimality Condition at time t: $\beta^t g_x(x_{t+1},x_t)$

Optimality Condition at time t-1: $\beta^{t-1} g_x(x_t,x_{t-1})$

Complete Condition: $\beta^t g_{x_t}(x_{t+1}, x_t) + \beta^{t-1} g_{x_t}(x_t, x_{t-1}) = 0$

$$g_{x_t}(x_{t+1}, x_t) + \beta^{-1} g_{x_t}(x_t, x_{t-1}) = 0$$

Transversality Condition: $\lim_{T\to\infty} \beta^T \frac{\partial g}{\partial x_T} x_T = 0$

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15.3 Discrete Time Optimal Control

- Example I: Cash Flow Maximization by a Firm

$$\max \sum_{t=0}^{\infty} \beta^t \left[P[f(L_t) - C(L_{t+1} - L_t)] - WL_t \right], 0 < \beta < 1$$

P: Profit function, L_t: Labor at time t, W: wage rate.

Euler Equation (f.o.c.):

$$\beta^t \left[P[f'(L_t) + C'(L_{t+1} - L_t)] - W \right] - \beta^{t-1} PC'(L_t - L_{t-1}) = 0$$

$$P[f'(L_t) + C'(L_{t+1} - L_t)] - W - \beta^{-1} PC'(L_t - L_{t-1}) = 0$$

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15.3 Dynamic Programing (DP)

• "Programming" = planning (engineering context)

• Idea: A problem can be broken into recursive sub-problems, which we solve and then combine the solutions of the sub-problems.

Example: Shortest route Houston-Chicago.

You know that the shortest route passes includes the segment Houston-Dallas. Then, finding the shortest Houston-Chicago route is simplified to finding the shortest Dallas-Chicago route.

We can reduce calculation if a problem can be broken down into recursive parts.

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15.3 Dynamic Programing (DP)

• In general, it is quite efficient. Moreover, many problems are presented as DP problems, because DP is computationally efficient.

Example: Calculation of Fibonacci numbers.

We can calculate iteratively $f(n) = f(n\text{-}1) + f(n\text{-}2)$.

This function grows as n grows. The run time doubles as n grows and is order $O(2^n)$. Not efficient; there are a lot of repetitive calculations.

If we setup the problem as a recursive problem, time can be saved: Recursive problems have no memory!

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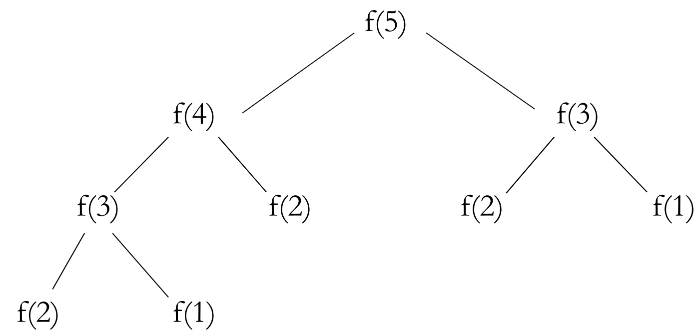
15.3 Dynamic Programing (DP)

• We can pose the calculations as a DP problem:



• Dynamic programming calculates from bottom to top.
• Values are stored for later use.
• This reduces repetitive calculation.

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15.3 DP – Typical Problem

- DP provides an alternative way to solve intertemporal problems
- Equivalent in many contexts to methods already seen
- We decompose the problem into sub-problems, which we solve and then combine the solutions of the sub-problems.
- Ramsey problem:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t. } \mathrm{c_t} + k_{t+1} - (1-\delta)k_t = f(k_t)$$

which can be rewritten as a recursive problem:

$$V(k_t) = \max_{c_t, k_{t+1}} \sum_{i=0}^{\infty} \beta^i u(k_{t+i}) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta \sum_{i=0}^{\infty} \beta^i u(c_{t+i+1})\}$$

$$= \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$$

$$\text{s.t. } k_{t+1} = f(k_t) - c_t - (1-\delta)k_t = g(k_t, c_t)$$

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15.3 DP – Bellman (HJB) Equation

$$V(k_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$$

• The previous equation is usually called *Bellman equation*, also called *Hamilton-Jacobi-Bellman* (*HJB*) equation.

• The Bellman Equation expresses the value function as a combination of a flow amount *u*(.) and a discounted continuation payoff.

• The current value function is $V(k_T)$. The continuation value function is $V(k_{T+1})$.

• The solution to the Bellman Equation is *V(.)*. In general, it is difficult to obtain (usual methods: guess a solution, value-function iterations).

• The key to get *V(.)* is a proper f.o.c.. Several ways to do it: Lagrangean, tracing the dynamics of $V(k_t)$, and using envelope theorem.

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15.3 DP – Extension: HJB under Uncertainty

• DP can easily be extended to maximization under uncertainty.

$$\max_{\{c_t, k_{t+1}\}} E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \qquad \text{s.t. } k_{t+1} = \varepsilon_t f(k_t) - c_t$$

where the production $f(k_t)$ is affected by a random shock, ε_t -an *i.i.d.* process-, realized at the beginning of each period *t,* so that the value of output is known when consumption takes place.

• The *HJB* equation becomes:

$$V(k_t.\varepsilon_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta E_t[V(k_{t+1}, \varepsilon_{t+1})]\}$$

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15.3 DP – Lagrangean and f.o.c.

• Using Lagrangean equation on the Ramsey problem, with finite *T*:

$$\sum_{t=0}^{T}\beta^t u(c_t)+\tilde{V}_0(k_{T+1})+\sum_{t=0}^{T}\tilde{\lambda}_t\left[f(k_t)+(1-\delta)k_t-c_t-k_{t+1}\right]$$

• Optimality Conditions (f.o.c.):

$$c_t:\beta^t u'(c_t)-\tilde{\lambda}_t=0$$
$$k_{t+1}:\tilde{\lambda}_{t+1}\left[f'(k_{t+1})+(1-\delta)\right]-\tilde{\lambda}_t=0, 0<t<T$$
$$k_{T+1}:\tilde{V}_0'(k_{T+1})-\tilde{\lambda}_t\le 0,\left[\tilde{V}_0'(k_{T+1})-\tilde{\lambda}_t\right]k_{T+1}=0$$
$$c_t+k_{t+1}-(1-\delta)k_t=f(k_t)$$

• Eliminate the multiplier to get

$$\beta u'(c_{t+1})\left[f'(k_{t+1})+(1-\delta)\right]=u'(c_t)$$
$$\tilde{V}_0'(k_{T+1})-\tilde{\lambda}_t\le 0,\left[\tilde{V}_0'(k_{T+1})-\tilde{\lambda}_t\right]k_{T+1}=0$$
$$c_t+k_{t+1}-(1-\delta)k_t=f(k_t)$$

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15.3 DP – Recursive Solution

Problem can be solved recursively, a sequence of two-period problems:

- Step (1): First solve the problem at t = T

Choose c_T and k_{T+1} to maximize: $\beta^T u(c_T)+\tilde{V}_0(k_{T+1})$

subject to $c_T+k_{T+1}-(1-\delta)k_T=f(k_T)$ k_T given

$$\Rightarrow c_T=c_T(k_T), k_{T+1}=k_{T+1}(k_T)$$

- Step (2): Now solve the period T-1 problem

Choose c_{T-1} and k_T to maximize

$$\beta^{T-1}u(c_{T-1})+\beta^T u(c_T(k_T))+\tilde{V}_0(k_{T+1}(k_T))$$

subject to $c_{T-1}+k_T-(1-\delta)k_{T-1}=f(k_{T-1})$ k_{T-1} given

- Step (3): Now solve the period T-2 problem

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15.3 DP – Recursive Solution

Continue solving backwards to time 0.

The same optimality conditions arise from the problem:

$$V_1(k_T) = \max_{c_T, k_{T+1}} u(c_T) + \beta V_0(k_{T+1})$$

subject to $c_T + k_{T+1} - (1-\delta)k_T = f(k_T)$

$$V_0(k_{T+1}) = \tilde{V}_0(k_{T+1}) / \beta^{T+1}$$

Optimality Conditions:

$$u'(c_T) = \lambda_T$$
$$\beta V_0'(k_{T+1}) - \lambda_T \le 0, \left[\beta V_0'(k_{T+1}) - \lambda_T\right] k_{T+1} = 0$$
$$c_T + k_{T+1} - (1-\delta)k_T = f(k_T)$$

These are the same conditions as before if we define $\lambda_T = \tilde{\lambda}_T / \beta^T$ 67

15.3 DP – Recursive Solution

Envelope Theorem implies $V_1'(k_T) = \lambda_T \left[f'(k_T) + (1-\delta)\right]$

Given the constraint and k_T, $V_1(k_T)$ is the maximized value of

$$u(c_T) + \beta V_0(k_{T+1})$$

Period T-1 problem is equivalent to maximizing

$$u(c_{T-1}) + \beta V_1(k_T)$$

with the same constraint at T-1 and k_{T-1} given:

$$V_2(k_{T-1}) = \max_{c_{T-1}, k_T} u(c_{T-1}) + \beta V_1(k_T)$$

subject to $c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$

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15.3 DP – Recursive Solution

Optimality Conditions:

$$u'(c_{T-1}) = \lambda_{T-1}$$
$$\beta V_1'(k_T) = \lambda_{T-1}$$
$$c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$$

The envelope theorem can be used to eliminate V_1'

$$u'(c_{T-1}) = \lambda_{T-1}$$
$$\beta\lambda_T\left[f'(k_T) + (1-\delta)\right] = \lambda_{T-1}$$
$$c_{T-1} + k_T - (1-\delta)k_{T-1} = f(k_{T-1})$$

The period T-1 envelope condition is

$$V_2'(k_{T-1}) = \lambda_{T-1}\left[f'(k_{T-1}) + (1-\delta)\right]$$

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15.3 DP – Bellman's Principle of Optimality

This process can be continued giving the following HJB Equation:

$$V_{J+1}(k_{T-j}) = \max_{c_t,k_t}\{u(c_{T-J}) + \beta V_0(k_{T-j+1})\}$$
$$\text{s.t.} \quad c_{T-J} + k_{T-J+1} - (1-\delta)k_{T-J} = f(k_{T-J}) \qquad k_{T\text{-}j} \text{ given}$$

• The general *HJB* equation, can be written as

$$V(k_T) = \max_{c_t,k_t}\{u(c_T) + \beta V(k_{T+1})\}$$

• Bellman's Principle of Optimality

The fact that the original problem can be written in this recursive way leads us to Bellman's Principle of Optimality:

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." 70

15.3 DP – HJB Equation - Summary

• The general *HJB* equation, can be written as

$$V(k_T) = \max_{c_t, k_t}\{u(c_T) + \beta V(k_{T+1})\}$$

• Using the Lagrangean approach we rewrite $V(k_T)$ as:

$$V(k_t) = \max_{c_t, k_t}\{u(c_t) + \beta V(k_{t+1}) + \lambda_t[g(k_t, c_t) - k_{t+1}]\}$$

with f.o.c.: $u'(c_t) + \lambda_t \dfrac{dg(k_t, c_t)}{dc_t} = 0$

$\beta V'(k_{t+1}) - \lambda_t = 0$

Envelope Theorem: $V'(k_t) = \lambda_t \dfrac{dg(k_t, c_t)}{dc_t}$

Euler equation: $u'(c_t) + \beta V'(k_{t+1}) \dfrac{dg(k_t, c_t)}{dc_t} = 0$

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15.3 DP – Solving HJB Equations

• Since t is arbitrarily taken, the Euler equation must also hold if we take one period backward:

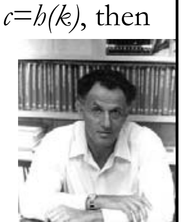
$$u'(c_{t-1}) + \beta V'(k_t) \frac{dg(k_{t-1}, c_{t-1})}{dc_{t-1}} = 0$$

Note: An Euler equation is a difference or differential equation that is an intertemporal f.o.c. for a dynamic choice problem. It describes the evolution of economic variables along an optimal path.

• If the Euler equation allows us to find a policy function $c=h(k)$, then the HJB equation becomes:

$$V(k_t) = \{u[h(k_t)] + \beta V[f(k_t) - h(k_t) + (1-\delta)k_t\}$$

• This equation has no explicit solution for generic utility and production functions.



Richard E. Bellman (1920 –1984, USA)

15.3 DP – Solving HJB Equations - Induction

• Let's put some structure. Suppose $u(c) = ln(c)$; $f(k) = k^\alpha$; & $\delta=1$.

Then, $$u'(c_t) + \beta V'(k_{t+1})\frac{dg(k_t, c_t)}{dc_t} = 0$$

$$\frac{1}{c_t} + \beta\frac{\alpha k_{t+1}^{\alpha-1}}{c_{t+1}}(-1) = 0 \qquad \Rightarrow \frac{1}{c_t} = \beta\frac{\alpha k_{t+1}^{\alpha-1}}{c_{t+1}}$$

$$\frac{k_{t+1}}{c_t} = \alpha\beta\frac{k_{t+1}^{\alpha}}{c_{t+1}} \qquad \Rightarrow \frac{k_t^\alpha - c_t}{c_t} = \alpha\beta\frac{k_{t+1}^{\alpha}}{c_{t+1}}$$

$$\frac{k_t^\alpha}{c_t} = \alpha\beta\frac{k_{t+1}^{\alpha}}{c_{t+1}} + 1$$

• This is a 1st-order difference equation. The forward solution, assuming $\alpha\beta$ small (as $T\to\infty$):

$$\frac{k_t^\alpha}{c_t} = \frac{1}{1-\alpha\beta} \qquad \Rightarrow c_t = (1-\alpha\beta)k_t^\alpha$$

$$\Rightarrow k_{t+1} = k_t^\alpha - c_t = \alpha\beta k_t^\alpha$$

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15.3 DP – Solving HJB Equation – Guess

• Guess a solution

Suppose $u(c) = ln(c)$; $f(k) = Ak^\alpha$; and $\delta=1$.

Guessed solution: $V(k) = B_0 + B_1\, ln(k)$.

Then: $V(k) = max\,\{ln(c_t) + \beta\,[B_0 + B_1\, ln(k_{t+1})]\}$

• After some work from f.o.c., we get:

$$k_{t+1} = \frac{\beta B_1}{1+\beta B_1}k_t^\alpha$$

$$c_t = h(k_t) = \frac{Ak_t^\alpha}{1+\beta B_1}$$

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15.3 DP – Solving HJB Equation – Guess

• By substitution in the HJB equation:

$$\left[B_0 + B_1 \ln(k_t)\right] = \ln\left(\frac{Ak_t^{\alpha} B_1}{1+\beta B_1}\right) + \beta\left[B_0 + B_1 \ln\left(\frac{Ak_t^{\alpha} B_1}{1+\beta B_1}\right)\right]$$

• After some algebra, we get $c_t^* = (1 - \alpha\beta)\ A\ k_t^{\alpha}$

$k^*_{t+1} = \alpha\beta\ A\ k_t^{\alpha}$

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Ken Arrow's office

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