STOCHASTIC MODELS OF NATURAL GAS PRICES

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1.0 INTRODUCTION

Over the past two decades, natural gas plays a very essential role in the energy market due to relatively the cleanest burning fossil fuel as well as growing concern over air pollution control, which gives the market further growth potential. The gas market has experienced radical changes in North America and Europe over a deregulation period and elimination of stage monopolies, and the corresponding financial markets have also gradually furnished in the last decades. The futures market successfully adapted itself to this dynamic and highly competitive market resulted from counterparty performance risk, and as a solution for essential risk management necessity. Consequently, the successful futures market then becomes the foundation for many other forms of derivatives trade, such as options and swaps. Now, the energy market has become a fairly liberal market. Moreover, a number of fundamental price drivers, such as issues of extraction, storage, transportation, weather, policies, technological advances, and etc., cause extremely complex gas prices behaviour.The natural gas spot prices have several important properties.

First, the natural gas spot prices have been historically considered to be a mean-reverting process. This means that the prices move up and down frequently, but oscillate around an equilibrium level (long-run mean) from a long term viewpoint. This is just the effect of mean reversion, i.e. the prices mean revert to a long-term mean. The mean-reversion behaviour of natural gas prices is related to their reactions to events such as floods, summer heat waves, and other breaking news-making events, which can create new and unexpected supply-and-demand imbalances in the market. These events include such as the temperatures reverting to their average seasonal levels, which tend to cause the natural gas prices to come back to their typical levels.

Another very important characteristic of gas spot prices is strong cyclic in nature over a year due to seasonal variation in supply and demand. Seasonality results from mainly demand fluctuations. This is largely due to natural gas being a main source of heating homes and businesses. Heat usage increases in the winter and goes down in the summer, and so due to the market forces of supply and demand, the price of natural gas has a general upward price movement in the winter and downward movement in the summer. This seasonality is also seen in the price of natural gas forwards/futures. On the other hand, the difficulty of storage and the limitation of transmission capacity make the supply side becoming not elastic enough to match very quickly the suddenly increased demand side. Hence, seasonal fluctuation of gas prices is the inevitable result of the seasonal imbalances between demand

and supply. The seasonality effects can be seen not only through historical spot prices, but also through futures and forward prices.

In section 2, we shall deliberate on current outlook on stochastic models of natural gas spot prices. Ultimately, section 3 discusses some summary and conclusion.

2.0 Stochastic Models of Natural Gas Spot Prices

In this work, we shall present different spot price models used in natural gas spot pricing in two separate frameworks: mean-reversion models and regime-switching models, and then explain other new price-modelling techniques based on these two categories. The typical feature of many commodities such as natural gas is that of mean reversion and this is captured by an Ornstein-Uhlenbeck (OU) process. The commonly OU process is used in a single-factor model with a Wiener process as the risk term. In modelling the forward curves, Schwartz [1] showed that this is insufficient due to a cost of carry and its effects on the drift term. To overcome this drift adjustment, convenience yield was taken into the models. The interest rate also was considered a third stochastic factor by Schwartz [1], but this did not yield any qualitative advantage over the two factor model. Thus in commodity modelling literature, stochastic interest rates are rarely ever considered. In Pilipovic [2, 3] a long-run stochastic mean also is proposed in the second factor model. Now in the case of natural gas, there is seasonality being exhibited in the price dynamics.

Xu [4] modified Philipovic's model [2, 3] to include seasonality via a positioning term, which is a sum of two sinusoids with different periods, whose parameters are obtained from the forward curve. Chen and Forsyth [5] used a one-factor regime-switching model to simulate the natural gas price that supposedly imitates the two-factor convenience yield model from Gibson-Schwartz [6]. The partial differential equation (PDE) method of pricing by Chen and Forsyth [5] is much more efficient in computing the value of storage. Hikspoors and Jaimungal [7] proposed a two-factor model with stochastic long-run mean-reversion and a seasonal component g_t in spot price process.

To capture the appropriate mathematical models for gas prices, there is an important test that spot models must withstand, i.e., how well does the model fit the futures curve. Since the futures price is equal to the discounted expected value of the spot price at its expiry, the proposed spot model must be able to match the futures curve when taking its expectation. Thus in the literature, it’s important that the spot price process chosen has an explicit form for its expectation, so as to determine the futures price.

We shall introduce a series of stochastic processes (because of the random behaviour of gas prices) appropriate for natural gas spot prices based on available literature.

2.1 A One-Factor Model - Ornstein-Uhlenbeck (OU) process

The commonly used process to model natural gas behaviour is the mean-reverting Ornstein-Uhlenbeck (OU) process. This is the most popular one-factor model in natural gas spot simulation. The OU process, S_t (relating to the natural gas spot price) is defined by the stochastic differential equation (SDE)

$$dS_t = \theta(\mu - S_t)dt + \sigma dW_t \tag{1}$$

where θ is the speed of mean reversion, μ is the value that the spot price reverts to, σ is the diffusion term and W_t is a Wiener process or Brownian motion. The expectation, variance and covariance of S_t are given by

$$E(S_t) = S_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) \tag{2}$$

$$Var(S_t) = \frac{\sigma^2}{2\theta} \tag{3}$$

$$Cov(S_s, S_t) = E[(S_s - E[S_s])(S_t - E[S_t])] = \frac{\sigma^2}{2\theta e^{-\theta(s+t)}(e^{2\theta(s\wedge t)} - 1)} \tag{4}$$

This process is used as the standard spot price model for pricing the natural gas storage in Boogert and Jong [8], Chen and Forsyth [5], Thompson *et al*. [9] and Bringedal [10]. The disadvantage of this approach is that the spot price evolution cannot be accurately accounted for. But the advantages are the ease of calibration, and the simple form for the futures price, which follows from equations (2) and (3). The futures price is given by

$$F(t, T, S_t) = E[S_T | S_t] \tag{5}$$

$$F(t, T, S_t) = S_t e^{-\theta(T-t)} + \mu\left(1 - e^{-\theta(T-t)}\right) \tag{6}$$

where $F(t, T, S_t)$ is the futures price at time t, for a contract expiring at time T given that the spot price is S_t.

The appropriate model for industry practitioners, who have to take positions every day with respect to injection and withdrawal of gas from the storage, is that of Li [11] which is relatively simple but can directly simulate the expected spot price process with respect to the futures price. Li [11] takes the spot price to assume the following process,

$$S_T = \begin{cases} S_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\epsilon_T}, & for\ the\ valuation\ month \\ F_{0,i} e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\epsilon_T} x, & for\ the\ i^{th}\ month\ contract \end{cases} \tag{7}$$

where S_T is the spot price at time T in the future, S_o is the spot price on the valuation date (current) and σ the spot price volatility. $F_{0,i}$ is the price of the futures contract as of today based on an expiry date i. Since the futures price is the expected value of the spot price, then for every subsequent i^{th}month, to begin with the futures price expiring in i^{th}month, the spot price process is set for the month. This approach facilitates the expected spot dynamics and includes the forward curve with computationally less expensive.

2.2 Multi-factor models in commodity pricing

In this section, a variety of two-factor models with their third-factor extensions are described. The most popular two-factor model and the first in the class of convenience yield models is that of Gibson-Schwartz [6]. To represent the existing two driving noise terms in market movements, two-factor models can be used to indicate these two random sources of volatility. In Carmona and Ludkovski [12], a stochastic market price of risk term is introduced to fit the implied convenience yield for different maturities. We shall now discuss the necessity of convenience yield in energy market before discussing about the models based on convenience yield.

2.2.1 Why is convenience yield necessity in energy markets?

If supply constraints show shock, then demand exerts its own fundamental price drivers. In energies, demand drivers introduce the issues of convenience yield and seasonality that have no parallel in money markets**.** Sometimes, due to irregular market movements such as an inverted market, the holding of an underlying good or security may become more profitable than owning the contract or derivative instrument, due to its relative scarcity versus high demand. The amount of benefit or premium associated with holding an underlying product or physical good, rather than the contract or derivative product is called convenience yield. The convenience yield is derived by the difference between the first purchase price of the underlying asset and its price after the shock. In other words, when an asset is easy to come by, an investor does not have the need to own the actual asset at that time, and can buy or sell as he please. When there is a shortage of the particular asset, it is better to own the asset rather than to own its contract or have to purchase it during the shortage period because it is likely to be at a higher price due to the demand. In energy related assets, storage costs of energy along with the other industrial management costs influence the true value of the assets. Although the holder of the energy contract has the option of consumption flexibility and has no risk in the event of commodity shortage, cost flow implied by the storage expenses affects on the holder's decision to postpone consumption. This driven net flow is the convenience yield and is represented by δ where

$$\delta = \text{Convenience yield} = \text{Benefit of direct access - cost of carry.}$$

The standard pricing for forward contracts with maturity T in markets is that of discounted spot price, i.e.

$$F(t,T) = S_t E\left[e^{\int_t^T r_s ds}\right] \tag{8}$$

where T is the time of exercise, r is the riskless-interest rate, S_t is the spot price and $F(t, T)$ is the forward price at time t with exercise time T. However, the forward contract price that includes the convenience yield is obtained by

$$F(t,T) = S_t E_Q\left[e^{\int_t^T (r_s - \delta_s) ds}\right] \tag{9}$$

where δ is the convenience yield. Q is the risk-neutral measure, and so this implies that S_t can be inferred as a drift correction term in the spot price process.

2.2.2 Gibson-Schwartz model

The first spot convenience yield model was introduced by Gibson and Schwartz in 1990. The spot price has the convenience yield δ_t added to the drift and is assumed to be a mean-reverting process that drives the geometric Brownian motion commodity spot price S_t.

Let (Ω, F, P) be a probability space under a filtration $\{F_t\}_{t \geq 0}$. According to the Gibson- Schwartz model, under the risk-neutral measure Q,

$$The\ first\ factor: \quad dS_t = (r_t - \delta_t)S_t dt + \sigma S_t dW_t{}^1 \tag{10}$$

$$The\ second\ factor: \quad d\delta_t = \kappa(\theta - \delta_t)dt + \gamma dW_t{}^2 \tag{11}$$

where $W_t{}^1$ and $W_t{}^2$ are correlated Wiener processes with $dW_t{}^1 dW_t{}^2 = \rho dt.$

In many energy commodities, mean-reverting process is typically accounted for spot prices of energy assets, but in 1990 Gibson-Schwartz [6] argued that the convenience yield affects the spot price process and induces mean-reversion to it. Unlike interest rate models, it makes sense that convenience yields can take positive or negative values so such proposed model seems logical.

Schwartz [1] in 1997 compared the one, two and three-factor spot models in calibrating forward curves. The one-factor model just has the mean-reverting OU spot price process, the two-factor model is the above-mentioned model and the three-factor model has stochastic interest rates. It is shown that there is no qualitative improvement in assuming a stochastic interest rate, and so stochastic interest rates are not included in this article. In Schwartz [1], it is shown that the futures price for the above spot prices is

$$F(t, T, S_t) = S_t e^{\int_t^T r_s ds} e^{B(t,T)\delta_t + A(t,T)} \tag{12}$$

where

$$B(t, T) = \frac{e^{\kappa T} - 1}{\kappa} \tag{13}$$

$$A(t, T) = \frac{\kappa\theta + \rho\sigma\gamma}{\kappa^2}\left(1 - e^{-\kappa(T-t)} - \kappa(T-t)\right) + \frac{\gamma^2}{\kappa^3}\left(2\kappa(T-t) - 3 + 4e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}\right). \tag{14}$$

We can observe that the affine form in $B(t, T)\delta_t + A(t, T)$ is with respect to convenience yield δ_t in the futures price model in equation (12).

Runggaldier [13] in 2003 developed another two-factor model by introducing another OU process for the spot price S_t and exerting an affine structure with respect to market price of risk λ_t, the differential between the actual return that an asset pays against the risk-free rate, normalized by the asset's volatility. The market price of risk λ_t , which follows the following OU process (16) and can take positive or negative values, is the risk-neutral measure of the spot price process, i.e.

$$dW_t{}^1 = d\widetilde{W}_t{}^1 - \lambda_t dt \tag{15}$$

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda dW_t{}^3 \tag{16}$$

In 2004, Carmona and Ludkovski [14] used Runggaldier's idea to improve the Gibson-Schwartz two-factor model, and for better calibration of the futures curve. The Wiener process of the spot price process is substituted by equation (15), and the extra stochastic factor λ_t is added as a third factor to the model, as given by equations (10) and (11);

$$\textit{The first factor}: \qquad dS_t = (r_t - \delta_t - \sigma\lambda_t)S_t dt + \sigma S_t d\widetilde{W}_t^{\,1} \tag{17}$$

$$\textit{The second factor}: \qquad d\delta_t = \kappa(\theta - \delta_t)dt + \gamma dW_t^{\,2} \tag{18}$$

$$\textit{The third factor}: \qquad d\lambda_t = \kappa_\lambda\big(\bar{\lambda} - \lambda_t\big)dt + \sigma_\lambda dW_t^{\,3} \tag{19}$$

This three-factor model of the spot price process provides another approach to the model of forward curve, in a form of the stochastic differential equation (20), and gives rise to an explicit form for the futures price due to the affine nature of S_t as

$$dF(t,T_i) = \left(r_t + \sigma\lambda_t + \rho\gamma\frac{e^{-\kappa T_i} - 1}{\kappa}\lambda_t\right)F(t,T_i)dt \tag{20}$$

$$+\,\sigma F(t,T_i)\, d\widetilde{W_t}^{1} + \gamma F(t,T_i)\frac{e^{-\kappa T_i - 1}}{\kappa} dW_t^{\,2} + \alpha dW_t^{\,F^i}.$$

Carmona three-factor model, i.e. equations (17) through (19), is specific to natural gas or energy and can be used for general commodities.

For energy commodities, there are the strong seasonal forces of supply and demand not only in the spot price process but also in futures and forward curves which create complicated characteristics for them. We shall now discuss models where a seasonality term is introduced and characterized in the movement of natural gas. The estimation of parameters of the seasonality term is typically done through the forward curves. Note that the British thermal unit (Btu) is a traditional unit of energy equal to about 1055 joules. It is the amount of energy needed to heat one pound of water by one degree Fahrenheit. In natural gas, by convention 1 MMBtu (1 million Btu)=1.054615 GJ. So, 1 MMBtu = 28.263682 m^3 of natural gas at defined temperature and pressure.

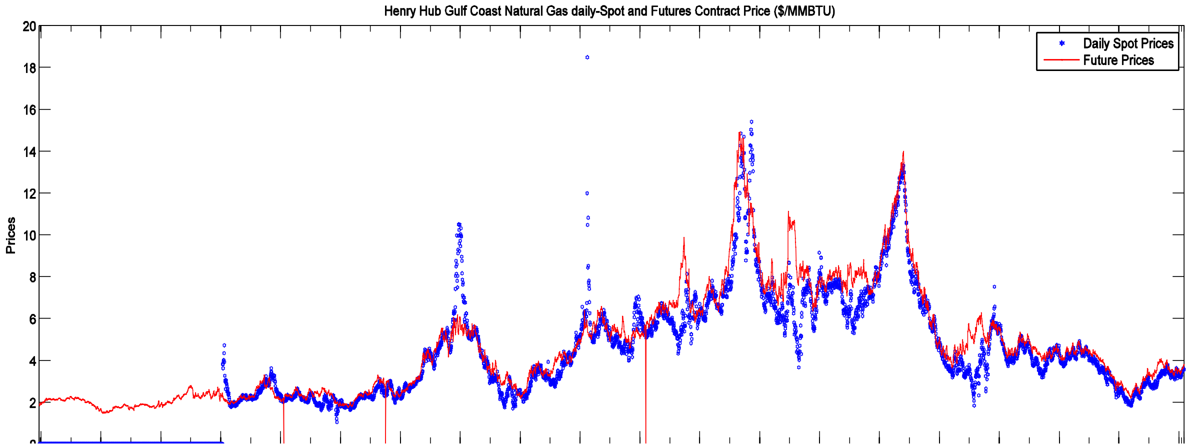

Figure 1. Henry Hub Gulf Coast Natural Gas Daily-Spot and Futures Contract Prices Dollars per Million British Thermal Unit (mmBTu).

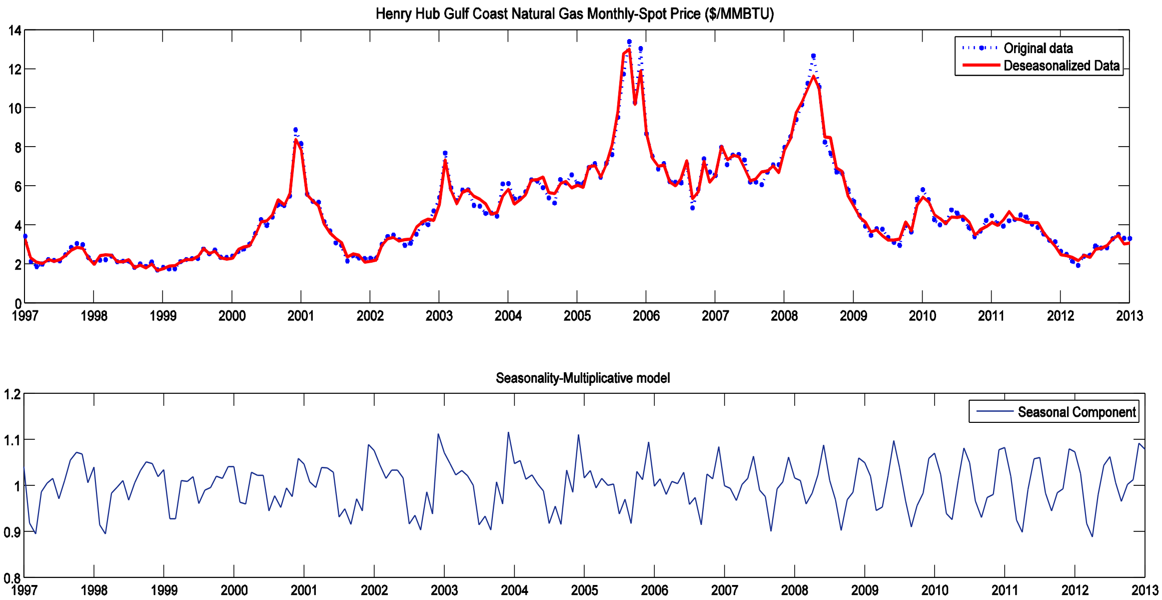

Figure 2. Henry Hub Gulf Coast Natural Gas Monthly-Spot Prices with Deseasonalized price process (above) and its multiplicative seasonality component (below).

Figures 1 and 2 are examples of the behaviour of natural gas spot and futures prices, and show a general trend of high gas prices in the winter months and lower gas prices in the summer months on both spot and futures prices.

Remark 2.1 The data source is provided from Energy Information Administration (eia), web page http://www.eia.gov/dnav/ng. Also, the software Matlab R2012a is used to plot the data and estimate the multiplicative-seasonal component via seasonal filter $S_{n\times m}$ for detrended series achieved by implementing Henderson filter.

2.2.3 The Pilipovic two-factor model for energy

Dragana Pilipovic in 1997 wrote the first book in Energy Risk [2], where she presented a two-factor model for energy taking into consideration in the complex spot-futures dynamics, the futures curve with respect to the spot price, which long-run mean is implied with the futures curve. She presented the following two-factor model

$$The\ first\ factor: \quad d\widetilde{S_t} = \alpha(L_t - S_t)dt + \sigma S_t d\widetilde{W_t}^1 \qquad (21)$$

$$The\ second\ factor: \quad dL_t = \mu L_t dt + \gamma L_t dW_t^2 \qquad (22)$$

where $\widetilde{W_t}^1$ and W_t^2 are uncorrelated standard Brownian motions, spot price process $\widetilde{S_t}$ as a mean-reverted risk-adjusted process and the long-run mean L_t as a geometric Brownian motion (GBM). If W_t^1 is a Wiener process, which drives the spot process S_t, then $d\widetilde{W_t}^1 = dW_t^1 - \lambda_t dt$, becomes a risk-adjusted Wiener process which drives $\widetilde{S_t}$.

Notice that by and application of Girsanov's theorem, the dependency of Wiener process $W_t^1, 0 \le t \le T$, on the market price of risk, $\lambda_t, 0 \le t \le T$ defined on the same probability space (Ω, F, P) , can be absorbed into an equivalent martingale measure. The Wiener process $d\widetilde{W_t}^1$ under the equivalent martingale Q are given by

$$\widetilde{W_t}^1 = W_t^1 - \int_0^t \lambda_s \, ds \qquad (23)$$

so that $d\widetilde{W_t}^1 = dW_t^1 - \lambda_t dt$. A risk-neutral (adjusted) measure Q is any probability measure, equivalent to the market measure P, which makes all discounted bond prices martingales.

In energy market with seasonality, the underlying price refers to the spot price with the seasonality factors taken away:

$$S_t = S_t^{UND} + seasonality\ factors \qquad (24)$$

where S_t^{UND} is the underlying spot price at time *t*.

Sometimes the processes of the spot prices are stripped of seasonality because we need to strip the effect of seasonality out of price data in order to analyze the underlying price behaviour. The removing of the seasonality from price data allows one to model the seasonality separately from modelling the underlying price processes, however, seasonality should be modelled as a stochastic process. Note that there are three seasonal factors: summer seasonality, winter seasonality, third seasonality in order to capture any additional repetitive annual event behaviour, such as an additional peaking or hump behaviour in the summer or winter, for example.

Thus, the seasonality terms will be defined in the same way for all models. However, spot price removed from the seasonality effects, will be defined uniquely by each model being tested. The calibration of the model parameters and the seasonality parameters will be performed simultaneously. For each model, calibrating will end up with specifying the model parameters, and all the seasonality parameters. Sometimes, there is seasonality of seasonality, that is, not only are there annual summer and winter seasonality patterns, but also 10-year and even 100-year cycles.

Ultimately, if there is seasonality in the futures curve, as in natural gas, the futures curve should be modelled by first stripping off the seasonality such as seasonal hump in the fall, i.e.,

$$F(t,T) = F^{UND}(t,T) + seasonality\ contribution. \qquad (25)$$

where F^{UND} is underlying (UND) futures prices.

The spot process of which is an exponential affine form has also an explicit form for its futures price as shown by Pilipovic [2, 3]:

$$F(t,T) = (S_t - L_t)e^{-(\alpha+\lambda\gamma)(T-t)} + L_t e^{(\mu-\lambda\gamma)(T-t)} \qquad (26)$$

2.2.4 Xu's generalization of Pilipovic's model

Xu [4] added a seasonality term $f(t)$ to Pilipovic's model, i.e. equations (21) and (22). He exclusively studied the natural gas, and the following model was proposed by Xu:

$$S_t = f(t) + X_t \qquad (27)$$

$$The\ first\ factor: \quad dX_t = \alpha(L_t - X_t)dt + \sigma(t)X_t dW_t^{\ 1} \qquad (28)$$

$$The\ second\ factor: \quad dL_t = \mu(\gamma - L_t)dt + \tau L_t dW_t^{\ 2} \qquad (29)$$

$$The\ third\ factor: \quad \sigma(t) = e^{\left(c+\sum_{i=1}^{i=2}[\lambda_i \cos 2\pi jt + \omega_j \sin 2\pi jt]\right)} \qquad (30)$$

$$Seasonality: \quad f(t) = bt + \sum_{i=1}^{i=2}[\beta_i \cos 2\pi jt + \eta_j \sin 2\pi jt] \qquad (31)$$

For $\gamma = 0$ and constant $\sigma(t)$, the Xu’s model would be equivalent to Pilipovic's model. He considered the above model when L_t is constant, i.e., one-factor model, and studied the model with and without seasonality. He concluded that the models final seasonality terms performed the best.

2.2.5 Hikspoors and Jaimungal's model

In 2007, Hikspoors and Jaimungal [7] proposed a class of models with long-run mean and a seasonal component g_t. They presented the following two-factor model

$$S_t = e^{(g_t + X_t)} \ , \qquad (32)$$

$$The\ first\ factor: \qquad dX_t = \beta(Y_t - X_t)dt + \sigma_X dW_t \qquad (33)$$

$$The\ second\ factor: \qquad dY_t = \alpha(\varphi - Y_t)dt + \sigma_Y dZ_t \qquad (34)$$

$$d[W, Z]_t = \rho dt \ , \qquad (35)$$

where X_t is a stochastic process satisfied in the observed equation

$$X_t = log(S_t) - g_t \quad . \qquad (37)$$

Both of X_t and Y_t are OU processes with the advantage of being able to estimate the conditional probabilities. Hikspoors and Jaimungal [7] also developed a three-factor model with additional stochastic volatility, with the spot price process given by:

$$S_t = e^{g_t + X_t} \ ,$$

$$The\ first\ factor: \quad dX_t = \beta(Y_t - X_t)dt + \sigma_X(Z_t)dW_t^1 \qquad (38)$$

$$The\ second\ factor: \quad dY_t = \alpha(\varphi - Y_t)dt + \sigma_Y dW_t^2 \qquad (39)$$

$$The\ third\ factor: \quad dZ_t = \eta(\mu - Z_t)dt + \sigma_Z dW_t^3 \qquad (40)$$

$$d[W^1, W^2]_t = \rho_{xy} dt\ ,\ d[W^1, W^3]_t = \rho_{xz} dt\ ,\ d[W^2, W^3]_t = \rho_{yz} dt. \qquad (41)$$

They extended their two and three-factor models to include jumps, such that

$$S_t = e^{g_t + X_t + J_t} \ , \qquad (42)$$

where the jump component J_t satisfies

$$dJ_t = -\kappa J_{t^-} dt + dQ_t \ , \qquad (43)$$

where Q_t is a compound Poisson process.

Instead of inserting the jump term in X_t, the jump term J_t is included directly to the spot price dynamics of S_t. That means, a jump is randomly added to commodity prices rather than a jump exerts the whole prices in altering with its changes, and then the price returns back to its original state. This description is an advantage of the model, since the typical behaviour of commodity prices exhibit that of spikes in prices and typically returning to its regular level. Interestingly, Quan [15] in 2006 also studied exclusively on one and two-factor models with affine jump-diffusion, with and without seasonality.

2.2.6 Eydeland and Wolyniec's model

In Eydeland and Wolyniec [16], a model is introduced to capture the entire forward curve. The forward equation is determined by the Schwartz model in a form of an HJM (Heath-Jarrow-Morton) model, which is induced as an underlying forward curve without seasonality to follow an interest rate type model. The forward process generally has the following multifactor form:

$$dF(t,T) = \mu\big(t,T,F(t,T)\big)dt + \sum_j \sigma_j\big(t,T,F(t,T)\big)dW_t^j \tag{44}$$

where W_t^j are correlated Wiener processes.

For commodities, the forward curve model can be simplified to

$$dF(t,T) = F(t,T)\sum_j \sigma_j(t,T)dW_t^j \quad . \tag{45}$$

The above equations propose a very different approach without worrying about the actual spot price process. Eydeland and Wolyniec [16] try to capture the dynamics of the futures curve which actually has a very erratic behaviour in gas due to its dependence on long and short terms supply and demand.

2.2.7 Nedunthally's models.

Nedunthally [17] introduces Levy processes into spot modeling of gas spot prices, and claims that the one-factor model with OU processes, together with the normal inverse Gaussian (NIG) process is the most effective in the class of one-factor models. Since there are few jumps in natural gas spot prices, and so it is suitable to use a stable process to model the spot price. The advantage of the alpha-stable Levy process is the ability to model the skewness and kurtosis. Nedunthally [17] also presents a new two factor model that assigns an affine structure for its seasonality term, and the model being an extension to Pilipovic's two-factor model.

Nedunthally [17] also shows that the normal inverse Gaussian (NIG) process is a computationally feasible case of the generalized hyperbolic (GH) process, which manages to retain properties of asset returns such as semi-heavy tails. He discusses about the important criteria for modeling natural gas spot prices and how expected value of the spot price process at different times in the future must be consistent with that of the futures curve. He also introduces two different one-factor models based on an alpha-stable process and NIG process, and take advantage of the fact that OU processes are based on stable or NIG processes having explicit solutions. The parameters of the seasonality term are obtained from a combination of spot and futures prices, which is used in Levy based OU and Cox-Ingersoll-Ross (CIR) processes to match the futures price. To calibrate the two-factor models, Nedunthally [17] uses linear regression to strip the underlying futures curve and then uses maximum likelihood to estimate the parameters.

Nedunthally [17] work deals with one-factor Levy-based stochastic models of the following form

$$dX_t = \lambda(b - X_t)dt + {X_t}^r dL_t \qquad (46)$$

$$S_t = f(t) + e^{X_t} \quad , \qquad (47)$$

where L_t is a Levy process, $f(t)$ is the seasonality parameter and S_t is the natural gas price process.

If $r = 0$, it turns out to be an OU process and for $r = 0.5$ it becomes a CIR type process.

To determine the spot price processes satisfying the following boundary condition:

$$F(t, T, s) = E_Q(S(T)|S(t) = s) \qquad (48)$$

where $F(s, t, T)$ is the price of the contract at time *t* with the maturity date being *T*, and when the spot price at time *t* is given by *s*. The value of a futures contract at maturity *T* tells about the markets expectation of the spot price at time *T* under the risk-neutral measure *Q*,

$$F(t, T, S_t) = F_{UND}(t, T, S_t) + f(t, T, S_t) \, , \qquad (49)$$

where $f(t, T)$ is a seasonality term whose parameters are calibrated by

$$f(t, T) = \sum_{i=1}^{i=2} \left[u_i sin\left(2\pi rfc\left((T-t) - t_i^C\right)\right) + v_i cos\left(2\pi rfc\left((T-t) - t_i^C\right)\right)\right]. \quad (50)$$

Observations show that this seasonality is not constant, but an affine form for the coefficient allows capturing the amplitude of the seasonality,

that is

$$f(t, T, S_t) = m(t, T, S_t) f(t, T) \;\; , \qquad (51)$$

where $m(t, T, S_t) = a + bS_t$, i.e. it has an affine structure, or

$$f(t, T, S_t) = m\big(t, T, F(t, T)\big) f(t, T) \qquad (52)$$

where

$$m\big(t, T, F(t, T)\big) = a + b\left(max\big(F(t, [T_M \ldots T_{M-12}])\big) - min\big(F(t, [T_M \ldots T_{M-12}])\big)\right). \quad (53)$$

2.2.8 Parsons's Models

Parsons [18] develops a two-factor tree model which consistently captures optionality in large amounts on both fast and slow-cycle leases, based on assuming the strong mean-reverting in United States (U.S.) natural gas spot prices. Also he drives the discrete-time spot price process in order to be applied in tree method, and in order to model natural gas storage value, which is outside of the scope of this work. Parsons [18] shows that according to the principal components analysis, and presuming two factors of risk would explain approximately 95% of movements in U.S. natural gas forward prices. Since forward prices are merely expected spot prices in risk-neutral measure, he claims a two-factor prices model as a natural starting point for the spot price process. His model is very similar to the one in Pilipovic [2] with a mean-reverting spot price and a geometric Brownian motion long-run mean. In Parsons pricing model [18], a two-component long-run mean is assumed with one component as a mean-reverting process, and another component as a deterministic process, and the model assumes as follows:

$$\frac{d\,S_t}{S_t} = a\big(ln(L_t) + \mu_t - ln(S_t)\big)dt + \sigma_{S,t}dZ_t \qquad (54)$$

$$\frac{d\,L_t}{L_t} = b\big(ln(L) - ln(L_t)\big)dt + \sigma_{L,t}dW_t\ , \qquad (55)$$

where

S_t = Gas-daily (spot) price at time *t*

L_t = Stochastic component of the long-run mean at time *t*

μ_t = Deterministic component at time *t*

L = Long-run mean of the L_t process

a = Mean-reversion speed of the S_t process

b = Mean-reversion speed of the L_t process

$\sigma_{S,t}$ = Deterministic volatility of the S_t process at time *t*

$\sigma_{L,t}$ = Deterministic volatility of the L_t process at time *t*

Z_t = Independent Brownian motion of the S_t process

W_t = Independent Brownian motion of the L_t process

The model introduces the stochastic component to the long-run mean, so as to overcome the shortfalls of the one-factor price model and to be more realistic from the analysis of mean-reversion. Moreover, the forward curve can bend and shift parallels in the model and as a consequence resulting in less correlation between spot and forward prices.

Furthermore, with introducing the deterministic component to the long-run mean, the model allows seasonality to be incorporated into the spot prices as well as to facilitate calibration. Deterministic volatilities in both S_t and L_t processes exert to capture seasonality as well.

In order to find the solution for $ln(L_t)$, the Ito's lemma is applied to equation (55) and then inserting this into (54) to solve (S_t), and this spot price process for $T > t$ is obtained as below,

$$S_T = S_t^{e^{-a(T-t)}} L_t^{\frac{a}{(b-a)}(e^{-b(T-t)}-e^{-a(T-t)})} .A.B.C.D \quad (56)$$

where

S_T = The gas-daily (spot) price at time $T > t$

$$A = e^{ln(L)\left(1-e^{-a(T-t)}-\frac{a}{(b-a)}(e^{-b(T-t)}-e^{-a(T-t)})\right)} \quad (57)$$

$$B = e^{\int_t^T ae^{-a(T-\tau)}(\mu_\tau - \int_t^\tau e^{-b(\tau-u)}\sigma_{L,u}{}^2/du)d\tau} \quad (58)$$

$$C = e^{\int_t^T ae^{-a(T-\tau)}(\int_t^\tau e^{-b(\tau-u)}\sigma_{L,u}dW_u)d\tau + \int_t^T e^{-a(T-\tau)}\sigma_{S,u}dZ_u} \quad (59)$$

$$D = e^{-\int_t^T e^{-b(T-\tau)}\sigma_{S,\tau}{}^2/d\tau} \quad . \quad (60)$$

2.2.9 One-Factor Regime Switching Model

Although one-factor mean-reverting models typically are used in the literature of natural gas storage evaluation, these do not give us enough efficiency in practice. On the other hand, multi-factor models are computationally expensive. In 2007, Chan and Forsyth [5] introduced a one-factor regime switching model to evaluate the gas storage facilities. This model seems to work almost as well as two-factor models with respect to fitting forward curves. The model to be proposed has two regimes, i.e. mean-reverting process (MR) and geometric Brownian motion process (GBM), and switches between a combination of MR and GBM processes. Therefore, this produce several model resulted from variations of regimes, i.e. MRMR, MRGBM, GBMMR, and GBMGBM. We are interested in the three variations: MRMR, MRGBM and GBMGBM. In MRMR model, the processes in both regimes are mean-reverting. An MRGBM shows the mean-reverting process in one regime and GBM with positive drift in another regime. In GBMGBM variation, the process in one regime follows GBM with a positive drift while that in another regime is GBM with negative drift. Moreover, they used futures curves and options on futures to obtain the models parameter, and in particular used the options on futures to find the volatility parameter.

It is observed that when regimes switch between MR and MR's equilibrium price takes place, the dynamics of Xu [4] is reproduced. This includes seasonality and mean reverting long-run mean as in section 2.2.4, incorporating the equations (27) through (31). However, when regimes switch between GBM and GBM (with different signs of drifts) happens, Gibson-Schwartz [6] model is reproduced instead and which extends the typical mean-reverting OU model with additional stochastic factor of convenience yield. This is similar as in section 2.2.2, comprising of the equations (10) and (11).

We then examined the three variations; MRMR, MRGBM and GBMGBM as well as other models for calibration spot-forward dynamics. Their results showed that the MRMR and MRGBM variations of the regime-switching model are capable of fitting the market gas forward curves more accurately than the MR model. And, the GBMGBM does not appear to be consistent with market data.

The switch between two regimes can be modelled by a two-state continuous-time Markov chain $m(t)$, and taking two values 0 or 1. The value of $m(t)$ indicates the regime in which the risk-adjusted gas spot price resides at time t. Letting $\lambda^{0\to1}dt$ denotes probability of shifting from regime 0 to regime 1 over a small time interval dt, and lettng $\lambda^{1\to0}dt$ be the probability of switching from regime 1 to regime 0 over dt, then $m(t)$ can be represented by

$$dm(t) = (1 - m(t^-))dq^{0\to1} - m(t^-)dq^{1\to0} \tag{61}$$

where $t-$ is the infinitesimal time before t, and $q^{0\to1}$ and $q^{1\to0}$ are the independent Poisson processes with intensity $\lambda^{0\to1}$ and $\lambda^{1\to0}$, respectively. In the regime-switching model, the risk-adjusted natural gas spot price is modelled by an SDE given by

$$dP = \alpha^{m(t^-)}\left(K_0^{m(t^-)} - P\right)dt + \sigma^{m(t^-)}PdZ + S^{m(t^-)}(t)Pdt \tag{62}$$

$$S^{m(t^-)}(t) = \beta_A^{m(t^-)}sin\big(2\pi(t - t_0 + C_A(t_0))\big) + \beta_{SA}^{m(t^-)}sin\big(4\pi(t - t_0 + C_{SA}(t_0))\big). \tag{63}$$

As indicated in equations (62-63), within a regime $k \equiv m(t^-),$ the gas spot price follows the process (64-65) with parameters $\alpha^k, K_0^k, S^k(t), \sigma^k$ (but the signs of α^k and K_0^k are not constrained).

$$dP = \alpha(K_0 - P)dt + \sigma PdZ + S(t)Pdt \tag{64}$$

$$S(t) = \beta_A sin\big(2\pi(t - t_0 + C_A(t_0))\big) + \beta_{SA} sin\big(4\pi(t - t_0 + C_{SA}(t_0))\big) \tag{65}$$

where
$\alpha > 0$ is the mean-reverting rate,
$K_0 > 0$ is the long-term equilibrium price,
$\sigma > 0$ is the volatility,
dZ is an increment of the standard Gauss-Wiener process,
$S(t)$ is a time-dependent term so that $S(t)Pdt$ is the price change at time t contributed by the seasonality effect. Note that multiplying $S(t)$ with P guarantees the price of natural gas always stays positive,
β_A is the annual seasonality parameter,
t_0 is a reference time satisfying $t_0 < t$.
$C_A(t_0)$ is the annual seasonality centering parameter for t_0.

We define

$$C_A(t_0) = A_0 + D(t_0) \tag{66}$$

where A_0 is a constant time adjustment parameter obtained through calibration; $D(t_0)$ is the distance between the reference time t_0 and the first date in January in the year of t_0. Thus, by calibrating the value of A_0, we are able to determine the evolution of the annual seasonality effect over time. Similarly, for

β_{SA} is the semiannual seasonality parameter,
$C_{SA}(t_0)$ is the semiannual seasonality centering parameter for t_0.

Similar to the definition of $C_A(t_0)$, we define

$$C_{SA}(t_0) = SA_0 + D(t_0) \tag{67}$$

where the constant time adjustment parameter SA_0 is obtained from a calibration process.

Meanwhile, the stochastic factors for the two regimes are perfectly correlated. Note that we assume that the centring parameters $C_A(t_0)$ and $C_{SA}(t_0)$, as given in equations (66-67), respectively, are identical for the two regimes in order to reduce the number of calibrated parameters.

This simple model is considered by several authors ([2] and [4]), although the seasonality feature is handled in a slightly different manner.

Remark 2.2 (Effect of the seasonality term on gas price dynamics).

We can rewrite equation

$$dP = \alpha K_0 dt + (S(t) - \alpha)Pdt + \sigma PdZ \ , \tag{68}$$

since $-(|\beta_A| + |\beta_{SA}|) \leq S(t) \leq (|\beta_A| + |\beta_{SA}|)$ according to equation (65), if

$$|\beta_A| + |\beta_{SA}| > \alpha \tag{69}$$

then there exists certain periods of time within which $S(t) - \alpha > 0$. In this case, if P is large and $(S(t) - \alpha)Pdt \gg \alpha K_0 dt$ in equation (68), then the process (64) becomes a GBM process with positive drift rate due to the strong seasonality effect. At other times, the process is mean-reverting.
Note that the deseasoned process (i.e., setting $S(t) = 0$ in SDE (64)) is a mean-reverting process.

Remark 2.3 (Mean-reverting or GBM-like process).

From the model (62-63), the deseasoned spot price in regime $m(t^-)$ can follow either a mean-reverting process or a GBM-like process by setting parameter values. If we choose $\alpha^{m(t^-)} > 0$ and $K_0^{m(t^-)} > 0$, then the deseasoned gas price (obtained from setting the seasonality term $S^{m(t^-)}(t) = 0$ in SDE (62)) follows a mean-reverting process

$$dP = \alpha^{m(t^-)}\left(K_0^{m(t^-)} - P\right)dt + \sigma^{m(t^-)}PdZ \tag{70}$$

with equilibrium level $K_0^{m(t^-)}$ and mean-reversion rate $\alpha^{m(t^-)}$.
If we set $K_0^{m(t^-)} = 0$ in equation (62), then the deseasoned gas price SDE becomes

$$dP = -\alpha^{m(t^-)}Pdt + \sigma^{m(t^-)}PdZ \tag{71}$$

This is a GBM-like process. Specifically, if the drift coefficient $-\alpha^{m(t^-)} > 0$, then SDE (71) is a standard GBM process, i.e., gas price P will drift up at a rate $\left|\alpha^{m(t^-)}\right|$ at time *t*; if $-\alpha^{m(t^-)} < 0$,then the gas price will drift down at a rate $\left|\alpha^{m(t^-)}\right|$.

2.2.10 Variations of the regime-switching model

As indicated in Remark 2.3, the deseasoned spot price in each regime can follow either a mean-reverting process or a GBM-like process. Consequently, there exist many possible variations of the regime-switching model by choosing different combinations of the stochastic processes in two regimes. We are interested in the following three variations, i.e. MRMR, MRGBM, GBMGBM variations which are described in the following:

MRMR variation

The processes in both regimes are mean-reverting with different equilibrium levels, i.e., $K_0^k > 0, \alpha^k > 0, k \in \{0, 1\}$ in SDE (62). In this variation, the equilibrium level of the gas spot price switches between two constants, K_0^k, K_1^k, which thus creates a sort of mean-reverting effect on the equilibrium level. This simulates the behaviour of the equilibrium price in the two-factor model proposed by Xu [4], where the gas spot price P follows a one-factor mean-reverting process and its equilibrium price evolves over time according to the other one-factor mean-reverting process.

MRGBM variation

The process in one regime is mean-reverting while the other regime is a GBM process with a positive drift, i.e., $K_0^0 > 0, K_0^1 = 0, \alpha^0 > 0, \alpha^1 < 0$ in SDE (62). The mean-reverting regime represents the normal price dynamics, and the GBM regime can be regarded as the sudden drifting up of the gas price driven by exogenous events.

GBMGBM variation

The processes in both regimes are GBM processes with a positive drift in one regime and a negative drift in the other, i.e., $K_0^0 = K_0^1 = 0, \alpha^0 < 0, \alpha^1 > 0$ in SDE (62). This simulates the behaviour of the two-factor model in Gibbon-Schwartz [6], where the risk adjusted commodity spot price process is modelled by a GBM-like process given by

$$P = (r - \delta)Pdt + \sigma PdZ \qquad (64) \ ,$$

where r is the constant riskless interest rate; δ is the instantaneous convenience yield, following an Ornstein-Uhlenbeck mean-reverting process. The drift coefficient $r - \delta$ can switch between positive and negative values during a time interval since the value of δ is stochastic and may change signs during the interval. Thus the gas price P will either drift up or drift down at any time depending on the sign of $r - \delta$. According to (70), the GBMGBM variation can produce behaviour similar to the SDE (71).

3.0 SUMMARY AND CONCLUSION

The article accounts for an overview on natural gas spot modelling without diving into calibration, spot-futures and spot-forward dynamics. In this work, based on mean-reversion property of spot price in energy markets, the spot modelling is divided into two categories:

mean-reversion models and regime-switching models. The historical extensions or new techniques based on these two categories are explained, where some emphasizing on fitting the forward curve, as examples [18] and [16], and some others attempt to capture spot-futures dynamics such as [16,10,14,5,6,11,17 and 6] and others capture both the forward and futures fitting, as [14,12, 2 and 3]. This article presented different spot prices models for natural gas in two types: mean-reverting models and regime-switching models.

According to the literature on gas spot price modeling, there are some historical ways and ideas to extend these two type models. For mean-reverting models, there are some strategies to extend as in the following:

1. Modify the first factor or spot price equation [15,11,17,18,15 and 13]
2. Manipulate the second factors
    a. Change the second factor: [2, 3 and 13]
    b. Split into some components:[18]
3. Add the seasonality terms to spot price model: [4] and [7].
    a. Sinusoidal term
    b. Co-sinusoidal term
    c. Linear combination of sinusoidal and co-sinusoidal functions
    d. Exponential term
4. Use sinusoidal or co-sinusoidal with different periods
5. Add some factor:[14,12, 6 and 4]
6. Add jump process: [15]
7. Use the Levy process instead of Wiener diffusion, [17].

For the second type model, we have Chen and Forsyth [5], and due to its novelty there seems to be no modification or extension as yet; possibly therefore there would be further active progress later in some future work. Since natural gas is the most volatile markets, it would be expected that newer models would have to be developed in order to respond to the complexity in market behaviours.

In conclusion, this article is a survey on some recent literature in natural gas spot modelling without plunging into calibration, spot-futures and spot-forward dynamics. This work, based on the fact that the crucial property of spot price in energy markets, as a commodity market, is mean-reverting. We observe that the natural gas spot modelling can be divided into essential categories, i.e. mean-reversion models and regime-switching models. We were able to examine their historical extensions in the form of new techniques. In the former models, one-factor, two-factor and three-factor spot mean-reverting models as well as their extensions are resulted from splitting the long-run mean into two stochastic and deterministic components. The consideration also led us to the Levy diffusion based on alpha-stable process and normal inverse Gaussian process as well as affine structure for seasonality term. As the latter category, one-factor regime-switching model is considered, which consists of two regimes either mean-reverting process or geometric Brownian motion with positive/negative drift.

References

[1] Schwartz, S. The Stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedging. *The Journal of Finance,* 1997. 52 (3): 923–973.

[2] Pilipovic, D. *Energy Risk: Valuing and Managing Energy Derivatives*. New York: McGraw-Hill, 1997.

[3] Pilipovic, D. *Energy Risk: Valuing and Managing Energy derivatives*. Second edition, New York: McGraw-Hill, 2007.

[4] Xu, Z. *Stochastic Models for Gas Prices*. University of Calgary: Master's thesis. 2004.

[5] Chen, Z. and Forsyth, P. Implication of Regime-Switching Model on Natural Gas Storage Valuation and Optimal Operation. *Quantitative Finance,* 2010. 10(2): 159-176.

[6] Gibson, R. and Schwartz E. S. Stochastic Convenience Yield and The Pricing of Oil Contingent Claims. *Journal of Finance*, 1990. 45(3):959-976.

[7] Hikspoors, S. and Jaimungal, S. *Energy Spot Price Models and Spread Options Pricing*. COMMODITIES 2007, Birkbeck College, University of London, Jan 17-19 2007.

[8] Boogert, A. and De Jong, C. *Gas Storage Valuation Using a Monte Carlo Method*. Birkbeck Working Papers in Economics and Finance BWPEF0704, Birkbeck College, School of Economics, Mathematics and Statistics, University of London. Dec 2006.

[9] Thompson, M., Davison, M., and Rasmussen, H. *Natural Gas Storage Valuation and Optimization: A Real Options Application.* Technical report, University of Western Ontario. 2003.

[10] Bringedal, B. *Valuation of Gas Storage: A Real Options Approach*. Department of Industrial Economics and Technology Management, Section for Managerial Economics and Operations Research, at the Norwegian University of Science and Technology: Master Thesis. 2003.

[11] Li,Y. *Natural Gas Storage Evaluation*. Georgia Institute of Technology: Master's thesis. 2007.

[12] Carmona, R. and Ludkovski, M. Spot Convenience Yield Models for The Energy Markets. In G. Yin and Y. Zhang, editors. *AMS Mathematics of Finance, Contemporary Mathematics,* 2004. 351: 65-80.

[13] Runggaldier, W.J. *Estimation via Stochastic Filtering in Financial Market Models*. Preprint. 2003.

[14] Carmona, R. and Ludkovski, M. Gas Storage and Supply Guarantees: An Optimal Switching Approach. *Quantitative Finance* 10(4): 359-374, 2010.

[15] Quan, G. *Stochastic Models for Natural Gas and Electricity Prices*. University of Calgary: Master's thesis. 2006.

[16] Eydeland, A. and Wolyniec, K. *Energy and Power Risk Management: New Developments in Modeling, Pricing and Hedging*, Wiley Finance, New York, Dec 20, 2002.

[17] Nedunthally, T. Alpha-Stable, Normal Inverse Gaussian and Multi-Factor Models for Spot and Futures Modeling in Natural Gas, University of Calgary: Master's thesis. 2009.

[18] Parsons, C.A. Quantifying Natural Gas Storage Optionality: A Two-Factor Tree Model, Preprint. 2011.