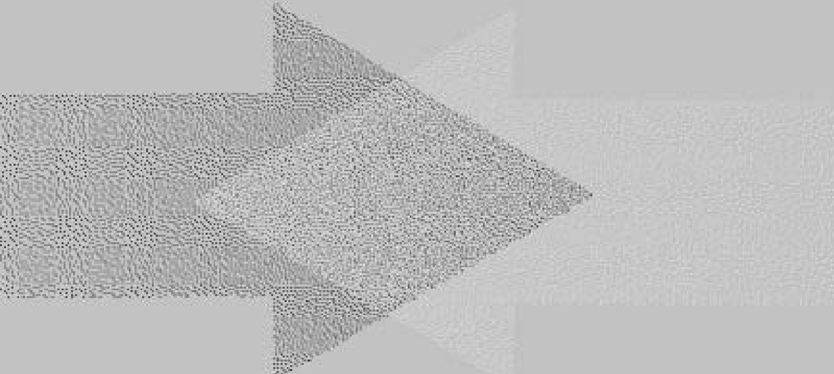
Modeling Aggregate Behavior and Fluctuations in Economics

STOCHASTIC VIEWS OF INTERACTING AGENTS



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Stochastic Views of Interacting Agents

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published in printed format 2001

ISBN 0-511-03074-6 eBook (Adobe Reader)
ISBN 0-521-78126-4 hardback

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CHAPTER 7

Solving nonstationary master equations

Often, we need to analyze nonstationary probability distributions to investigate, for example, how the distributions behave as time progresses. For instance, we may be interested in knowing how the distributions of market shares of firms behave in some sector as the sector or industry matures.

If we can't solve master equations directly in the time domain, we may try to solve them by the method of probability generating functions. In cases where that approach does not work, we may try solving ordinary differential equations for the first few moments of the distributions by the method of cumulant generating functions; see Cox and Miller (1965, p. 159). Alternatively, we may be content with deriving probabilities such as $P_0(t)$, this being the probability for extinction of certain types (of their sizes being reduced to zero).

This section describes the probability- and cumulant-generating-function methods for solving the master equations. In those cases where the transition rates are more general nonlinear functions of state variables than polynomials, we can try Taylor series expansions of transition rates to solve the master equations approximately.

In this chapter, we illustrate some procedures to obtain nonstationary probability distributions on some elementary models. (See also the method of Langevin equations, which is discussed in Section 8.7.) This leads to (approximate) solutions of Fokker–Planck equations.

7.1 Example: Open models with two types of agents

The next example is an open market or sector model with two types of agents. By reinterpreting the number of agents as measured in some basic units, we may translate the results in terms of the number into results in terms of sizes of agents such as firms. We solve this problem both for stationary and for nonstationary distributions. For stationary solutions we use the detailed-balance conditions. For nonstationary solutions, we use the generating-function methods. We first dispose of the stationary case, since it is straightforward.

7.1.1 Equilibrium distribution

Assume that all transition rates are time-homogeneous. Suppose that the entry probability intensity is given by

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) = \nu_i$$

for $n_i \geq 0$, and the exit transition rate is specified by

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i) = \mu_i n_i$$

for $n_i \geq 1$, $i = 1, 2$. In addition, agents change from type 1 to type 2 with probability intensity

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e_1} + \mathbf{e_2}) = \lambda_{12} n_1,$$

and likewise from type 2 to type 1 with the coefficient λ_{21}.

The detailed-balance conditions hold, and it is easy to see that the equilibrium probability distribution is given by

$$\pi(\mathbf{n}) = \pi_1(n_1)\pi_2(n_2),$$

with

$$\pi_i(n_i) = c_i \frac{\alpha_i^{n_i}}{n_i!},$$

$i = 1, 2$, where c_i is the normalizing constant and the constants are $\alpha_i = \nu_i/\mu_i$, and provided $\lambda_{12}\alpha_1 = \lambda_{21}\alpha_2$.

Exercise 1. Verify that with the expression for the equilibrium distributions, the detailed-balance conditions are satisfied.

Exercise 2. Change the entry transition rate specification to $w(\mathbf{n}, \mathbf{n} + \mathbf{e}_i) = \nu_i n_i$, and the type switching transition rates to $\lambda_{12} n_1 n_2$ and $\lambda_{21} n_2 n_1$, respectively. Assume that α's are less than one in magnitude. Show that π_i changes to $c_i \alpha_i^{n_i}/n_i$.

7.1.2 Probability-generating-function method

The master equation is given by

$$\begin{aligned}\frac{dP(n_1, n_2)}{dt} &= \mu_1(n_1 + 1)P(n_1 + 1, n_2) + \mu_2(n_2 + 1)P(n_1, n_2 + 1)\\ &\quad + \nu_1 P(n_1 - 1, n_2) + \nu_2 P(n_1, n_2 - 1)\\ &\quad + \lambda_{12}(n_1 + 1)P(n_1 + 1, n_2 - 1)\\ &\quad + \lambda_{21}(n_1 - 1)P(n_1 - 1, n_2 + 1)\\ &\quad - \{\mu_1 n_1 + \mu_2 n_2 + \nu_1 + \nu_2 + \lambda_{12} + \lambda_{21} n_1\}P(n_1, n_2),\end{aligned}$$

where we supress the time argument of n_1 and n_2, and where z_1 and z_2 below are dummy or auxiliary variables of the generating function to extract appropriate probabilities from the function. See Feller (1968) for example.

Define the probability generating function by

$$G(z_1, z_2; t) = \sum_{n_1} \sum_{n_2} z_1^{n_1} z_2^{n_2} P(n_1, n_2).$$

Multiply the master equation by $z_1^{n_1} z_2^{n_2}$, and sum over n_1 and n_2. Using relations such as $\sum_{n_1} \sum_{n_2} n_1 z_1^{n_1} z_2^{n_2} P(n_1, n_2) = z_1 \frac{\partial G}{\partial z_1}$, we deduce the partial differential equation for the probability generating function G:

$$\begin{aligned}\frac{\partial G}{\partial t} &= [\nu_1(-1 + z_1) + \nu_2(-1 + z_2)]G \\ &\quad + [\mu_1(1 - z_1) + \lambda_1(z_2 - z_1)]\frac{\partial G}{\partial z_1} \\ &\quad + [\mu_2(1 - z_2) + \lambda_2(z_1 - z_2)]\frac{\partial G}{\partial z_2},\end{aligned}$$

where we abbreviate λ_{12} as λ_1, and likewise for λ_2.

7.1.3 *Cumulant generating functions*

Often, we are interested in the time response patterns of the first few moments such as the mean, variance, and skewness. Now, by using the device in Cox and Miller (1965, p. 159), we derive the ordinary differential equations for the mean and variance.

Given a random variable $X(t)$, its probability generating function is changed into the moment generating function by setting $z = e^{-\theta}$ and defining

$$H(\theta, t) = E\left(e^{-\theta X(t)}\right).$$

Next introduce the cumulant generating function by

$$K(\theta, t) = \ln H(\theta, t).$$

Denoting the rth cumulant by κ_r, we extract it from $K(\theta, t)$ as the coefficient of $(-\theta)^r/r!$.

Now, instead of solving the partial differential equation for G, we derive equations to determine the mean and the second moments. Letting $z_i = e^{-\theta_i}$, $i = 1, 2$, and $K(\theta_1, \theta_2, t) = \ln G(z_1, z_2, t)$, we obtain, by noting that $\partial/\partial z_i = -e^{\theta_i}\ \partial/\partial\theta_i$, $i = 1, 2$,

$$K(\theta_1, \theta_2; t) = \nu_1(e^{-\theta_1} - 1) + \nu_2(e^{-\theta_2} - 1) + [\mu_1(1 - e^{-\theta_1}) + \lambda_1(1 - e^{\theta_1-\theta_2})]\frac{\partial K}{\partial\theta_1} + [\mu_2(1 - e^{\theta_2}) + \lambda_2(1 - e^{\theta_2-\theta_1})]\frac{\partial K}{\partial\theta_2}.$$

Expand the function K in a power series of θ_i as

$$K = -\kappa_{10}\theta_1 - \kappa_{01}\theta_2 + \frac{1}{2}\left(\kappa_{20}\theta_1^2 + 2\kappa_{11}\theta_1\theta_2 + \kappa_{02}\theta_2^2\right) + \cdots.$$

By collecting expressions for the coefficients of θ_i, θ_i^2, $i = 1, 2$, and that of $\theta_1\theta_2$, we derive

$$\frac{d\kappa_{10}}{dt} = \nu_1 - (\mu_1 + \lambda_1)\kappa_{10} + \lambda_2\kappa_{10},$$

$$\frac{\kappa_{01}}{dt} = \nu_2 - (\mu_2 + \lambda_2)\kappa_{01} + \lambda_1\kappa_{01},$$

$$\frac{d\kappa_{11}}{dt} = -(-\mu_1 + \mu_2 + \lambda_1 + \lambda_2)\kappa_{11} - \lambda_1(\kappa_{10} + \kappa_{20}) - \lambda_2\kappa_{01},$$

$$\frac{d\kappa_{20}}{dt} = \nu_1 + (\lambda_1 + \mu_1\kappa_{10} - (\lambda_1 - 2\mu_1)\kappa_{20} + 2\lambda_2\kappa_{11},$$

and at last the equation for κ_{02}, which is obtained from that for κ_{20} by substituting 2 for 1 in the subscripts.

7.2 Example: A birth–death-with-immigration process

We may reinterpret this model as a representation of a process that governs the growth of a firm by interpreting a birth as the addition of a basic unit to the firm size, death as a reduction in size, again in some basic unit, and immigration as an innovation that increases the size of a firm by a unit. This last term may also represent the feedback effects of the average (field) effect. We let μ denote the rate of size reduction and λ the rate of size increase by one unit, and α the innovation rate. Accordingly, the transition rates are specified by

$$l_k = \mu k$$

for size reduction by a unit, and

$$r_k = \alpha + \lambda k$$

for size increase by a unit, which comes either as random innovation or as random proportional growth. We assume that $0 < \lambda \leq \mu$.

The master equation is

$$dP_k/dt = (k+1)\mu P_{k+1} + [\alpha + \lambda(k-1)]P_{k-1} - [\alpha + (\lambda + \mu)k]P_k,$$

with the initial condition that $k(0) = k_0$. A model with $\alpha = 0$ is discussed in Cox and Miller (1965, p. 165).

7.2.1 *Stationary probability distribution*

Setting the left-hand side of the master equation equal to zero, and applying the detailed-balance conditions, which hold for this model because of the tree graph structure, we derive the stationary probability for k, π_k, as a negative binomial distribution

$$\pi_k (1-\gamma)^\theta \binom{\theta + k - 1}{n} \gamma^k,$$

where $\gamma = \lambda/\mu$ and $\theta = \alpha/\lambda$.

The maximum of the probability occurs at k^*, which satisfies

$$1 \le \frac{\alpha - \mu}{\mu - \lambda} \le k^* \le \frac{\alpha - \lambda}{\mu - \lambda},$$

provided that $\alpha > \mu$ and $\mu < (\alpha + \lambda)/2$.

Next, we verify that the time-dependent solution of the master equation indeed approaches this steady-state solution.

7.2.2 *Generating function*

Let the probability generating function be defined by

$$G(z, t) = \sum_{k \ge 0} z^k P_k(t).$$

Note that $\sum_{k\ge 0} z^k \; P_{k-1}(t) = z\,G(z,t)$, $\sum_{k\ge 1} (k+1) z^k \; P_{k+1}(t) = \partial G/\partial z$, $\sum_{k\ge 1} k z^k P_k(t) = z\,\partial G/\partial z$, and $\sum_{k\ge 1} (k-1) z^k P_{k-1}(t) = z^2\,\partial G/\partial z$.

Using these relations, the master equation is transformed into a partial differential equation for the generating function:

$$\frac{\partial G}{\partial t} = (\lambda z - \mu)(z-1)\frac{\partial G}{\partial z} + \alpha(z-1)G.$$

This equation is solved by the method of characteristics. See Hildebrand (1976, Chap. 8) for example. A brief outline of the method is in the Section A.1.

The characteristic curves are defined by[1]

$$\frac{dt}{1} = \frac{dz}{-(\lambda z - \mu)(z-1)} = \frac{dG}{\alpha(z-1)G}.$$

From the first pair we derive

$$\frac{z-1}{\lambda z - \mu} = C(z,t)e^{\beta t},$$

with $\beta = \mu - \lambda$, and from the second pair we derive

$$G(z,t) = D(z,t)(\lambda z - \mu)^{-\alpha/\lambda},$$

where C and D are the constants of integration. From the initial condition, we note that

$$G(z,0) = z^{n_0} = D(z,0)(\lambda z - \mu)^{-\alpha/\lambda},$$

and we replace z by

$$z = \frac{C(z,0)\mu - 1}{C(z,0)\lambda - 1}$$

to obtain the relation between $C(z,0)$ and $D(z,0)$, which is

$$D(z,0) = z^{n_0}(\lambda z - \mu)^{-\alpha/\lambda},$$

in which z is substituted out by the expression above.

Now to obtain the expression for $G(z,t)$, we substitute into $G(z,t) = (\lambda z - \mu)^{-\alpha/\lambda}D(z,t)$ the relation between $D(z,t)$ and $C(z,t)$ by replacing 0 with t. The result is

$$G(z,t) = (\lambda z - \mu)^{-\alpha/\lambda}\left\{\frac{\mu C(z,t) - 1}{\lambda C(z,t) - 1}\right\}^{n_0}\left\{\frac{\mu - \lambda}{\lambda C(z,t) - 1}\right\}^{\alpha/\lambda},$$

which, after substituting $C(z,t)$ out, becomes

$$G(z,t) = \left\{\frac{\mu - \lambda}{\lambda(z-1)e^{-\beta t} - (\lambda z - \mu)}\right\}^{\alpha/\lambda}\left\{\frac{\mu(z-1)e^{-\beta t} - (\lambda z - \mu)}{\lambda(z-1)e^{-\beta t} - (\lambda z - \mu)}\right\}^{n_0}.$$

The effect of the initial condition disappears as t becomes large, because the second factor above approaches one. The mean, which is calculated as $\partial G/\partial z|_{z=1}$, can be shown to approach a constant α/β at the rate $e^{-\beta t}$. With

[1] In general, when a term dependent on G is absent, the equation becomes $dt/1 = dz/h(z) = dG/0$. This equation has $G = a$, with some constant a, as one of the two independent solutions. The other solution is obtained from the first equality, $\phi(z,t) = b$, say. See Cox and Miller (1965, p. 158) for the necessary relation between the two constants.

$\alpha = 0$ and the initial condition $n_0 = 1$, the mean is one: $\bar{n}_t = 1$. The probability $P_0(t)$ approaches $(1 - \lambda/\mu)^{\alpha/\lambda}$, which is less than 1, as time goes to infinity. With $\alpha = 0$ it approaches as time goes to infinity.

Exercise 3.

(a) A process with $r_n = \nu n$, and $l_n = 0$ is governed by

$$\frac{\partial P_n(t)}{\partial t} = -\nu n P_n(t) + \nu(n-1)P_{n-1}(t).$$

Suppose that the initial condition is $P_n(0) = \delta_{n-n_0}$ for some positive integer n_0. Show that its moment generating function is given by

$$G(z,t) = e^{-\nu n_0 t} z^{n_0}[1 - (1 - e^{-\nu t})z]^{-n_0}.$$

By extracting the coefficient of z^n obtain the expression

$$P_n(t) = C_{n-1,n-n_0} e^{-\nu n_0 t}(1 - e^{-\nu t})^{n-n_0}$$

for $n \geq n_0$. Use the identity that $C_{-n,k} = C_{n+k-1}(-1)^k$. Suppose that the parameter ν above is replaced with ν/n, i.e., the birth rate is monotonically descreasing in n. Then, $P_n(t)$ has the Poisson distribution.

(b) Derive the moment generating function for the stationary and nonstationary probability distributions with the initial condition $P_n(0) = \delta(n - n_0)$, and the transition rates $l_n = n\mu$ and $r_n = (N - n)\nu$ where $\nu > \mu$. It is given by

$$G(z,t) = \left(\frac{\nu}{\nu+\mu}\right)^N \left\{\frac{z + (\mu/\nu) + (\mu/\nu)(z-1)e^{\theta t}}{z + (\mu/\nu) - (\mu/\nu)(z-1)e^{-\theta t}}\right\}^{n_0} H^N,$$

where

$$H = \left\{\frac{z + (\mu/\nu) - (z-1)e^{-\theta t}}{z + (\mu/\nu) - (\mu/\nu)(z-1)e^{-\theta t}}\right\}^N, \quad \text{with} \quad \theta = \nu + \mu.$$

(c) Suppose that each agent is characterized by a set of K attributes, each of which takes on the value of 1 or -1. The resulting K-dimensional vector is his state vector. The distance between the states of two agents is measured by the Hamming distance, which is the number of attributes on which the two agents are different. Let $P_k(t)$ be the probability at time t that the Hamming distance is k between two specified agents. Assume that attributes change with time in such a way that μ is the rate of change, that is, in a short time span dt, the Hamming distance changes by one with probability $\mu\, dt + o(dt)$. The probability is governed by the master equation

$$dP_k(t)/dt = \mu\{(k+1)P_{k+1}(t) + (K-k+1)P_{k-1}(t) - kP_k(t)\}.$$

Show that the solution of this equation is

$$P_k(t) = \frac{K!}{2^K k!(K-k)!}(1 - e^{-2\mu t})^k (1 + e^{-2\mu t})^{K-k}.$$

Verify that the average Hamming distance is $(K/2)(1 - e^{-2\mu t})$.

7.2.3 *Time-inhomogeneous transition rates*

We follow Kendall (1948a) in discussing solutions of master equations with time-varying transition rates. Consider

$$\frac{\partial G}{\partial t} = f(z)\frac{\partial G}{\partial z},$$

where $f(z) = (\lambda z - \mu)(z - 1)$. The associated differential equation $dz/dt = f(z)$ is a known Riccati type. The general solution is of the form (Watson 1952, Sec. 8.4)

$$z = \frac{f_1 + Cf_2}{f_3 + cf_4},$$

where the fs are functions of time. Solving for the constant we have

$$C = z\frac{f_3 - f_1}{f_2 - zf_4}.$$

The generating function with the initial condition $P_0(0) = 1$, $P_n(0) = 0$, $n \neq 1$, is

$$G(z, t) = \frac{g_1 + zg_2}{g_3 + zg_4}$$

for some gs. Expanding G in power series in z, we obtain

$$P_0(t) = g_1(t)/g_3(t) := \xi(t),$$

and

$$P_n(t) = \eta P_{n-1}(t),$$

with $\eta(t) = -g_4(t)/g_3(t)$. Normalizing the probabilities to sum to one, we have

$$P_n(t) = (1 - P_0(t))(1 - \eta(t))\eta(t)^{n-1},$$

$n \geq 1$.

To determine the functions ξ and η, substitute the generating function thus determined, $G(z, t) = \{\xi + (1 - \xi - \eta)z\}/(1 - \eta z)$, into the partial differential equation. From it we derive

$$\xi'\eta - \xi\eta' + \eta' = \lambda(1 - \xi)(1 - \eta),$$

and

$$\xi' = \mu(1-\xi)(1-\eta),$$

where the prime denotes differentiation with respect to time. Next, change variables to $U = 1 - \xi$ and $V = 1 - \eta$. The differential equation for V becomes

$$V' = (\mu - \lambda)V - \mu V^2.$$

On further changing the variable to $W = 1/V$, this becomes

$$W' = -(\mu - \lambda)W + \mu,$$

which is integrated to give

$$W = e^{-\rho}\left[1 + \int_0^t e^{\rho(u)}\mu(u)\,du\right]$$

with $\rho(t) = \int_0^t [\mu(u) - \lambda(u)]\,du$.

The differential equation for U becomes

$$U'/U = -\mu/W = -W'/W - (\mu - \lambda).$$

From these we derive

$$\xi(t) = 1 - e^{-\rho(t)}/W,$$

and

$$\eta(t) = 1 - 1/W.$$

These can be used to show that $P_0(t)$ goes to one with time going to infinity if and only if $I = \int_0^\infty e^{\rho(u)}\mu(u)\,du$ is infinite.

7.2.4 The cumulant-generating-function

Now, instead of solving the partial differential equation for G as we have done above, we derive ordinary differential equations to determine the means, the covariance, and the variances. For this purpose we use the cumulant generating function. We set $z = e^{-\theta}$ in the probability generating function to convert it to the moment generating function, and take the logarithm of it to obtain the cumulant generating function

$$K(\theta; t) = \ln G(e^{-\theta}; t).$$

Noting that

$$\frac{\partial}{\partial z} = -e^{\theta}\frac{\partial K}{\partial \theta},$$

we rewrite the partial differential equation for G of Section 7.2.2 as

$$\frac{\partial K}{\partial t} = -(\lambda e^{-\theta} - \mu)(1 - e^{\theta})\frac{\partial K}{\partial \theta} + \alpha(e^{-\theta} - 1).$$

Let $K(\theta, t) = -\kappa_1(t)\theta + \kappa_2\theta^2/2 + \cdots$, where κ_1 is the mean and κ_2 is the variance of the state variable k. We derive the ordinary differential equations

$$d\kappa_1/dt = -\beta\kappa_1 + \alpha,$$

and

$$d\kappa_2/dt = -2\beta\kappa_2 + (\lambda + \mu)\kappa_1 + \alpha/2.$$

Solving these, we derive the mean and variance. We see that the variance approaches its steady-state value at least as fast as $e^{-2\beta t}$.

We use next the cumulant generating function to discuss market shares of firms.

7.3 Models for market shares by imitation or innovation

We compare deterministic and master-equation treatments of imitation and innovation processes by two types of firms. We expect the deterministic solutions to be identical or analogous to those of the mean dynamic equations provided the variances about the mean asymptotically vanish, although we should be aware of the possibility of disappointment, as Feller's example in Chapter 5 shows. The expectation is indeed satisfied for the examples in this section except for the last one.

We provide additional illustration of these generating-function techniques in the context of market shares with two types of firms, one technically advanced and the other technically less advanced. The exact natures of the two classes of firms are not important so long as firms of one class can become firms of the second class.

To be concrete, we consider a collection of n firms. This assumption allows us to use one state variable rather than two, by taking the number of firms of the advanced type to be the state variable, $k = 0, 1, 2, \dots, n$, where n is fixed. The disadvantage of this approach is that we must provide separate boundary conditions for $k = 0$ and $k = n$, which look different from the master equations valid for $0 < k < n$. Since we are merely illustrating the use of generating functions, we do not bother with these boundary conditions here. In Chapter 5 we show how to deal with models with two state variables without assuming that the total number of firms is exogenously fixed. Elsewhere, we have dealt with a model with two types of agents. There, the transition rates $w(k, k+1) = \lambda n(1 - k/n)\eta_1(k/n)$ and $w(k, k-1) = \mu n(k/n)\eta_2(k/n)$ have been used. In other words, the birth and death rates are not constant but state-dependent. Then, the functional form of η_1 is specified as $e^{\beta h(k/n)}/\{e^{\beta h(k/n)} + e^{-\beta h(k/n)}\}$, where β is a parameter to incorporate uncertainty or imprecise information about alternative choices, and $h(\cdot)$ is a function equal to the difference of means of the alternative discounted present values associated with the alternative choices. When this function h is expanded in Taylor series, we see that we obtain both the terms $x(1-x)$ and $x^2(1-x)$, which are singled out in the imitation process of this chapter, and effects of congestion. For the latter, see Hirsch and Smale (1974, Chap. 12) for example.

7.3.1 Deterministic innovation process

Suppose we divide the firms into two groups: group A of k firms with superior technologies to the firms in group B, consisting of the remainder, i.e., $n-k$ firms. The identities of the firms belonging to the groups are not important. What matters is the number – or the value of the fraction, $x = k/n$ – of firms belonging to group A.

Suppose, for the sake of simplicity, that an innovation, when it occurs to firms of group B, turns them into members of group A. The fraction is then often modeled by

$$\frac{dx}{dt} = \lambda(1-x). \tag{7.1}$$

The solution is

$$x(t) = 1 - [1 - x(0)]e^{-\lambda t}.$$

This shows that eventually all firms belong to group A, that is, the fraction converges to 1. The reason, of course, is that no firm leaves the market; hence eventually all firms belong to group A. Our purpose is not to implement more realistic assumptions, but rather to justify the ordinary differential equation above, which is often used without much justification.

We now reformulate this process as the birth process with transition rate $w(k, k+1) = \lambda(n-k)$. This specifies that firms of the less advanced class can have individual probability rates λ of advancing to the superior class. The master equation is

$$dP_k(t)/dt = P_{k-1}(t)w(k-1, k) - P_k(t)w(k, k+1),$$

except for the boundary conditions for $P_0(t)$ and $P_n(t)$, with which we do not bother here.

Next, convert this equation into the partial differential equation for the probability generating function,

$$\frac{\partial G(z,t)}{\partial t} = \lambda n(z-1)G - \lambda z(z-1)\frac{\partial G}{\partial z}.$$

We can solve this equation by the characteristic-curve method as shown in Section 7.2. When we do this, $G(z, t)$ approaches z^n as t goes to infinity for all initial numbers of firms in group B. This shows that all firms become technically advanced as time progresses.

Our purpose is to point out that the posited ordinary differential equation can be derived from the cumulant generating function, to show the relations between it and the partial differential equations, and to show that it can be used by itself, since no coupling exists between the mean and variance dynamics for this simple process.

Let $K(\theta, t) = \ln G(e^{-\theta}, t)$. The partial differential equation now becomes

$$\frac{\partial K}{\partial t} = \lambda n(e^{-\theta} - 1) + \lambda(e^{-\theta} - 1)\frac{\partial K}{\partial \theta}.$$

Expand $K(\theta, t) = -\kappa_1\theta + \kappa_2\theta^2/2 + \cdots$, from which we derive

$$d\kappa_1/dt = -\lambda\kappa_1 + \lambda n,$$

with $\kappa_1(0) = k_0$, and

$$d\kappa_2 = -2\lambda\kappa_2 - \lambda(n - \kappa_1),$$

with $\kappa_2(0) = 0$. Note that κ_3 does not appear in this set of equations.

In this example there is no dynamic coupling from the variance to the mean. The solutions are

$$\kappa_1(t) = ne^{-\lambda t} - (n - k_0)e^{-\lambda t};$$

hence κ_1/n may be interpreted as the deterministic counterpart of the fraction x defined by $x = \kappa_1/n$, and

$$\kappa_2(t) = (n - k_0)(e^{-\lambda t} - e^{-2\lambda t}).$$

This shows that the variance asymptotically vanishes. Hence, the above interpretation is justified.

7.3.2 Deterministic imitation process

We next introduce interactions between firms of different classes. We follow Iwai (1984a,b, 1996) and assume that firms in group B individually imitate firms in group A and succeed in becoming members of group B at the rate μ. Using the deterministic approach, this is expressed by[2]

$$\frac{dx}{dt} = \mu x(1 - x).$$

[2] If only firms with backward technology die, we add $-d(1 - x)$ to the next equation. This does not introduce anything new. Change the variable from x to $y = x - d$. Then, for $y \geq 0$ the equation becomes $\mu y(1 - d - y)$. In other words, y approaches $1 - d$ as time goes to infinity.

Writing this as $dx/x(1-x) = \mu\, dt$ and integrating it, we obtain its solution as

$$x(t) = \frac{x(0)}{x(0) + [1 - x(0)]e^{-\mu t}}.$$

Again, we see that all firms succeed in becoming members of the advanced class.

In terms of the master equation, we respecify the transition rate to be $w(k, k+1) = \mu k(n-k)$, in

$$dP_k(t)/dt = P_{k-1}(t)w(k-1, k) - P_k(t)w(k, k+1).$$

Now, the partial differential equation for the probability generating function is slightly more complicated:

$$\frac{\partial G}{\partial t} = \mu N z(z-1)\frac{\partial G}{\partial z} - \mu z(z-1)\left[\frac{\partial G}{\partial z} + z\frac{\partial^2 G}{\partial z^2}\right].$$

Going directly to the partial differential equation for the new cumulant generating function, we note that $z[\partial^2 G/\partial z^2 + \partial G/\partial z]$ equals $Ge^{2\theta}[(\partial K/\partial\theta)^2 + \partial^2 K/\partial\theta^2]$. We derive

$$\frac{\partial K}{\partial t} = \mu(1 - e^{-\theta})\left[n\frac{\partial K}{\partial\theta} + \left(\frac{\partial K}{\partial\theta}\right)^2 + \frac{\partial^2 K}{\partial\theta^2}\right].$$

The ordinary differential equations for the mean and variance are now coupled:

$$d\kappa_1/dt = \mu(n - \kappa_1)\kappa_1 - \mu\kappa_2,$$

and

$$d\kappa_2/dt = \mu 2n\kappa_2 + \mu(n - \kappa_1)\kappa_1 - 4\mu\kappa_1\kappa_2 - 2\mu\kappa_3.$$

However, as κ_1 approaches n, the effect of the variance on the mean vanishes. Thus dropping the κ_2 term in the differential equation for κ_1 leads to the correct limiting value.

The reader may have noticed the absence of the phenomenon of firms going out of business – the death process in the birth-and-death process models. When bankruptcy is modeled, we must drop the assumption that n is fixed exogenously. See Chapter 10 for discussions of processes with both birth and death effects on firm market shares.

7.3.3 A joint deterministic process

We now combine these two effects into a single equation:

$$\frac{dx}{dt} = \mu x(1-x) + \lambda(1-x). \tag{7.2}$$

We can solve this directly by noting that

$$\frac{1}{\mu x(1-x)+\lambda(1-x)} = \frac{1}{\mu+\lambda}\left[\frac{\mu}{\mu x+\lambda} + \frac{1}{1-x}\right].$$

The solution is

$$x(t) = \frac{C - \lambda e^{-\gamma t}}{C + \mu e^{-\gamma t}},$$

where $\gamma = \lambda + \mu$, and where $C = (\lambda + \mu x_0)/(1 - x_0)$. Unsurprisingly, $x(t)$ goes to one as time progresses.

The reason for this limiting behavior is clear. No firms leave the market, and eventually all firms belong to group A. To remedy this, we must allow some firms to leave the market or go bankrupt. We need to abandon the assumption of a fixed number of firms unless we artificially allow entry to keep the number fixed. See Chapter 10 for this alternative model.

7.3.4 *A stochastic dynamic model*

Next, we set up the master equation for the joint processes to model market shares of group A. We assume that n is fixed. We take the transition rate of the number of firms in group A from k to $k+1$ to be

$$r_k = \mu\frac{k}{n}\left(1-\frac{k}{n}\right) + \lambda\left(1-\frac{k}{n}\right).$$

Denote the probability $P_k(t) = \Pr(k_A(t) = k)$, where k_A is the number of firms of group A. It is governed by

$$\frac{dP_k(t)}{dt} = r_{k-1}P_{k-1}(t) - r_k P_k(t)$$

for $k = 1, \ldots, n$. There is an obvious boudary equation for $k = 1$.

It is convenient to regroup the right-hand side as $[r_{k-1} - r_k]P_k(t) + r_{k-1}[P_{k-1} - P_k(t)]$, change the variable by

$$\frac{k}{n} = \phi + \frac{\xi}{\sqrt{n}},$$

and denote

$$\Pi(\xi, t) := P_k(t).$$

Then, the left-hand side of the master equation becomes

$$\frac{\partial \Pi}{\partial t} - \sqrt{n}\frac{\partial \Pi}{\partial \xi}\frac{d\phi}{dt}.$$

This change of variable was introduced earlier in Section 5.2. See also Aoki (1996a, p. 123) or Aoki (1995). As it turns out, to match orders of magnitude of terms on both sides, we need to change the time scale as well by

$$t = n\tau.$$

We rewrite the time derivative as $d\phi/dt = n^{-1}\, d\phi/d\tau$, and do likewise to rewrite $\partial\Pi/\partial t$.

We note that on the right-hand side

$$r_{k-1} - r_k = -\frac{\mu}{n}\left(1 - 2\phi - 2\frac{\xi}{\sqrt{n}}\right) + \frac{\lambda}{n} + o(n^{-1}),$$

and

$$P_{k-1} - P_k = -\frac{1}{\sqrt{n}}\frac{\partial\Pi}{\partial\xi} + \frac{1}{2n}\frac{\partial^2\Pi}{\partial\xi^2}.$$

Now equate terms of the same order. The highest-order ones are of the order n. We obtain

$$\frac{d\phi}{d\tau} = \mu\phi(1-\phi) + \lambda(1-\phi).$$

The next highest-order terms in n produce what is known as the (linear) Fokker–Planck equation. We look for the stationary solution and set $\partial\Pi/\partial\tau$ equal to zero:

$$0 = [\lambda - \mu(1-2\phi)]\frac{\partial(\xi\Pi)}{\partial\xi} + \frac{\mu\phi(1-\phi) + \lambda(1-\phi)}{2}\frac{\partial^2\Pi}{\partial\xi^2}.$$

The deterministic equation has the logistic curve as its solution. The Fokker–Planck equation has the Gaussian distribution with mean zero and variance $[\mu\phi(1-\phi) + \lambda(1-\phi)]/[\lambda - \mu(1-2\phi)]$ when the denominator is positive. By setting μ to zero, we recover the aggregate dynamics (7.1) and the associated Fokker–Planck equation.

7.4 A stochastic model with innovators and imitators

We examine an open model with two types of firms, called innovators and imitators, to discover the market shares of the two types. We incorporate asymmetric interactions between the two types of firms.

Let $\mathbf{n} = (n_1, n_2)$ be the state vector. We assume that the transition rates are such that

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_1) = c_1(n_1 + h_1) = c_1 n_1 + f_1,$$

where type 1 firms grow at the innovation rate f_1 plus a proportional growth rate of $c_1 n_1$. On the other hand type 2 firms, which are less technically advanced that those of type 1, grow at the rate

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_2) = f_2,$$

where $f_2 < f_1$.

The two types go out of business at the rates

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = d_j n_j,$$

$j = 1, 2$. We assume that type 2 firms fail more often than type 1 firms: $d_2 \geq d_1$. By imitating type 1 firms, a type 2 firm may become type 1 at the rate

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2) = \lambda_{21} d_2 c_1 n_2 (n_1 + h_1),$$

while a type 1 firm may slip back to type 2 at the rate

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_2 - \mathbf{e}_1) = \lambda_{12} f_2 d_1 n_1.$$

With these transition rates, it is easily verified that the steady-state probability distribution exists by imposing the detailed-balance conditions, provided $\lambda_{12} = \lambda_{21}$. Let $g_1 = c_1/d_1$, and $g_2 = f_2/d_2$. We define $\mu := \lambda d_1 d_2$. Then we can write the transition rates more succinctly by noting that $\lambda_{21} d_2 c_1 = \mu g_1$ and $\lambda_{12} f_2 d_1 = \mu g_2$:

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2) = \mu g_1 n_2 (n_1 + h_1),$$

while a type 1 firm may slip back to type 2 at the rate

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_2 - \mathbf{e}_1) = \mu g_2 n_1 n_1.$$

The number of type 1 firms is governed by a negative binomial distribution, while the number of type 2 firms has a Poisson distribution,

$$\pi(n_1, n_2) = \pi_1(n_1)\pi_2(n_2),$$

with

$$\pi_1 = (1 - g)^{-h_1} \binom{n_1 + h_1 - 1}{n_1} g_1^{n_1},$$

and

$$\pi(n_2) = e^{-g_2} \frac{g^{n_2}}{n_2!}.$$

We next examine the nonstationary solution of the master equation

$$\partial P(\mathbf{n})/\partial t = I(\mathbf{n}, t) - O(\mathbf{n}, t),$$

where the probability influx denoted by the first term is

$$\begin{aligned}I(\mathbf{n},t) = {} & P_{n_1+1,n_2}(t)d_1(n_1+1) + P_{n_1,n_2+1}(t)d_2(n_2+1)\\ & + P_{n_1-1,n_2}(t)c_1(n_1+h_1-1) + P_{n_1,n_2-1}(t)f_2\\ & + P_{n_1+1,n_2-1}(t)\mu g_1(n_1+1)\\ & + P_{n_1-1,n_2+1}(t)\mu g_2(n_1+h_1-1)(n_2+1),\end{aligned}$$

and the outflux is given by

$$\begin{aligned}O(\mathbf{n},t) = P_{n_1,n_2}(t)[& c_1(n_1+h_1) + f_2 + d_1 n_1 + d_2 n_2\\ & + \mu g_1 n_2(n_1+f_1) + \mu g_2 n_1].\end{aligned}$$

We rewrite it in terms of the probability generating function

$$G(z_1,z_2,t) = \sum_{n_1,n_2} z_1^{n_1} z_2^{n_2} P(n_1,n_2).$$

The partial differential equation for the generating function is

$$\begin{aligned}\frac{\partial G}{\partial t} = {} & [d_1(1-z_1) + c_1 z_1(z_1-1) + \mu g_2(z_2-z_1)]\,\frac{\partial G}{\partial z_1} + \ d_2(1-z_2)\\ & + \mu g_1 h_1(z_1-z_2)\frac{\partial G}{\partial z_2} + \mu g_1 z_1(z_1-z_2)\frac{\partial^2 G}{\partial z_1 \partial z_2}\\ & + [f_1(z_1-1) + f_2(z_2-1)]\,G.\end{aligned}$$

Since this equation is rather complicated, we change it to one for the cumulant generating function defined by

$$K(\theta_1,\theta_2,t) = \ln G(e^{-\theta_1}, e^{-\theta_2}, t).$$

In doing so, we note that $\partial G/\partial t = G\,\partial K/\partial t$, $\partial G/\partial z_i = -Ge^{\theta_i}\partial K/\partial\theta_i$, $i = 1, 2$, and $\partial^2 G/\partial z_1 \partial z_2 = Ge^{\theta_1+\theta_2}[(\partial K/\partial\theta_1)(\partial K/\partial\theta_2) - \partial^2 K/\partial\theta_1\partial\theta_2)]$. We derive

$$\frac{\partial K}{\partial t} = -\phi_1\frac{\partial K}{\partial\theta_1} - \phi_2\frac{\partial K}{\partial\theta_2} + \phi_3\left[\frac{\partial K}{\partial\theta_1}\frac{\partial K}{\partial\theta_2} + \frac{\partial^2}{\partial\theta_1\partial K}\theta_2\right] + \phi_4,$$

where

$$\phi_1 = d_1\left(\theta_1 + \frac{\theta_1^2}{2}\right) + c_1\left(-\theta_1 + \frac{\theta_1^2}{2}\right) + \mu g_2\left(\theta_1 - \theta_2 + \frac{(\theta_1-\theta_2)^2}{2}\right),$$

$$\phi_2 = d_2\left(\theta_2 + \frac{\theta_2^2}{2}\right) + \mu g_1 h_1\left(\theta_2 - \theta_1 + \frac{(\theta_2-\theta_1)^2}{2}\right),$$

$$\phi_3 = \mu g_1(\theta_2 - \theta_1 + (\theta_1-\theta_2)^2),$$

and

$$\phi_4 = f_1\left(-\theta_1 + \theta_1^2/2\right) + f_2\left(-\theta_2 + \theta_2^2/2\right).$$

Then, expanding K as $K = -\kappa_1\theta_1 - \kappa_2\theta_2 + (1/2)(\kappa_{11}\theta_1^2 + \kappa_{12}\theta_1\theta_2 + \kappa_{22}\theta_2^2) + \cdots$, we derive coupled ordinary differential equations for the first and second moments. Unfortunately, the ordinary equations are not solvable in closed form.

We mention some special cases by imposing some conditions on the parameters in the transiton rates.

7.4.1 *Case of a finite total number of firms*

First, assume that $g_2 = 0$, that $d_2 = d_1 - c_1$, and that h_1 is much smaller than these parameters. The first assumption means that no type 1 firms become type 2 firms. Once technically advanced, firms remain technically advanced. The second assumption means that the net dropout rate of type 1 firm is the same as that of type 2 firms. The third assumption means that c_1 is much larger than f_1, that is, the rate of growth of type 1 firms comes primarily from existing firms generating new firms of the same type and not from new entries. Then,

$$\frac{d(\kappa_1 + \kappa_2)}{dt} = -d_2(\kappa_1 + \kappa_2) + f_1 + f_2;$$

hence the sum of the numbers of firms of both types asymptotically approaches $(f_1 + f_2)/d_2$.

Another consequence of these parameter values is that κ_{12} is constant. Suppose that it is zero at time 0. Then, it remains zero for all times. With $g_2 = 0$ and $h_1 = 0$, we can drop the assumption that $d_2 = d_1 - c_1$ and solve for κ_1 and κ_2 separately, because

$$d\kappa_1/dt = -(d_1 - c_1)\kappa_1 + f_1 + \mu g_1(\kappa_1\kappa_2),$$

and

$$d\kappa_2/dt = -d_2\kappa_2 f_2 - \mu g_1(\kappa_1\kappa_2).$$

We can solve algebraically for the limiting values of κ_1 and κ_2 from these. Compared with the case of $\mu g_1 = 0$, interactions between firms of the two groups cause $\kappa_1(\infty)$ to be larger with $\mu g_1 \neq 0$, and $\kappa_2(\infty)$ less, as expected.

Next, suppose that type 2 firms never become type 1. Once technically behind, they remain behind. This is expressed by assuming that $g_1 = 0$. In this case, the dynamics for κ_1 is solved first, then substituted into the dynamics for κ_2. We find that $\kappa_1(\infty) = f_1/(d_1 - c_1)$ and $\kappa_2 = f_2/d_2 + \mu g_2 f_1/d_2(d_1 - c_1)$. There are more type 2 firms, while the number of type 1 firms remains the same.

In examining the joint dynamics for κ_1 and κ_2, we see that the covariance κ_{12} acts on the time derivatives of κ_1 and κ_2 with opposite signs and the same magnitude.

7.5 Symmetric interactions

7.5.1 Stationary state distribution

Suppose we now specify entry transition rates for the two groups in the same form, that is, we replace f_2 in the previous section by

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_2) = c_2(n_2 + h_2) = c_2 n_1 + f_2,$$

and respecify the transition rate from type 1 to type 2 as

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_1 + \mathbf{e}_2) = \mu g_2 n_1 (n_2 + h_2).$$

Then both types have negative binomial distributions as their equilibrium distributions:

$$\pi_i(n_i) B_i g_i^{n_i} \binom{n_i + h_i - 1}{n_i},$$

$i = 1, 2.$

7.5.2 Nonstationary distributions

We derive the differential equation for the cumulant generating function as in the previous section. The detailed expressions for ϕ_1 through ϕ_4 are slightly different, but the general procedure of analysis remains the same. We do not bother with the detailed results.

Rather, we later examine in Chapter 11, after discussing some growth and business-cycle models in between in Chapters 8 and 10, what happens when the number of groups becomes large, as well as the total number of firms. We find a perhaps surprising connection with the Ewens distribution.

CHAPTER 8

Growth and fluctuations

This chapter is loosely grouped into four parts. Part one is composed of Sections 8.1 through 8.5. Part two consists of a single long Section 8.6. Part three is made up of Sections 8.7 and 8.8, and part four is Section 8.9. The first part of this chapter collects a number of bare-bones models and topics that are loosely tied to the notion of growth, market shares, and fluctuations.

The bare-bones models in this part may be used, singly or in some combinations, to construct more fully specified models of growth or fluctuation. For example, Aoki and Yoshikawa (2001) describe a model that uses some of the bare-bones models as components to show how demand saturation limits growth. A second example is described in Section 8.6. The third example is discussed in Chapter 9.

We begin this chapter by discussing two mini-models, called Poisson and urn models, for explaining how economies grow by inventing new goods or creating new industries. These models provide different explanations of growth from those in the literature on endogenous growth models.

The two models in Section 8.1 provide two explanations of economic growth that are different from standard ones based on technical progress, that is, total factor productivity models or endogenous growth models. The flavor of the difference may be captured by saying that in the endogenous growth models an economy grows by improving the quality of existing goods, whereas in our models it grows by introducing new goods.

In the first model, firms or sectors independently invent new goods or improve on the existing ones. The numbers of new goods are then functions of the number of the firms, or the size of the economy. The growth rate of this model eventually converges to the rate at which new goods are being introduced to the economy. In the second model, the numbers of goods that are introduced are independent of the numbers of the existing goods, and the rate of introduction of new goods decreases as the number of goods grows. The growth rate eventually reduces to zero. This is not a totally absurd idea. Actually, to quote Kuznets (1953), "The industries that have matured technologically account for a progressively increasing ratio of the total production of the economy. Their maturity does imply that economic effects of further improvements will necessarily be more limited than in the past."

These two linear models are followed by another set of two related models. In the first one the exit rate of goods is nonlinear, to quantify the idea that older goods disappear from the markets more quickly than newer goods. In the second, demand saturates as time goes on.

We then turn to discuss a simple stochastic business-cycle model, after taking a quick look at a deterministic version suggested by Iwai (1984a, b, 1996). His model considers an economy composed of two sectors of firms with a fixed total number of firms. The rates of change of shares of the market change in response to gaps between demands and supplies of the two sectors. The share converges monotonically to one. This generates no business-cycle-like fluctuations of the total output.

A stochastic version of this model, which keeps the central idea that shares change in response to the gaps, is next discussed for comparison. A simplified two-sector version of this model has been described in Section 4.1. The model generalizes this to a K-sector model in which sectors respond to gaps between supplies and demands of individual sectors. It is an open business-cycle model. Unlike the deterministic ones, it generates fluctuations. The expected output of the economy responds to changing patterns of demand shares. It increases as more demands are shifted to more productive sectors of the economy.

We accomplish two things by this model. First, we illustrate a possibility that fluctuations of the aggregate economy arise as an outcome of interactions of many agents/sectors in a simple model. Second, we demonstrate that the *level* of the *aggregate* economic activity depends on the structure of demand. In the standard neoclassical equilibrium, where the marginal products of production factors such as labor are equal in all activities and sectors, demand determines only the composition of goods and services to be produced, not the level of the aggregate economic activity.

There are two ways for demand to affect the aggregate level of economic activities. One is externality associated with demand, which might produce multiple equilibria as in Diamond (1982). We return to his model in Chapter 9. The other is differences in productivity across sectors/activities. Recent works by Murphy et al. (1989) and Matsuyama (1995), for example, emphasize the importance of increasing returns in order to demonstrate the role of demand in determining the level of the aggregate production. They, in effect, allow differences in productivity across sectors to draw their conclusions. In this chapter, we keep Iwai's idea and assume that productivities differ across sectors in the economy. Here, we just show how the output of the economy is maximized for a suitable choice of the demand shares by assuming that the productivity coefficients of the two sectors are not equal.

8.1 Two simple models for the emergence of new goods

Two models are introduced to explain how new goods appear. In the first model, called the *Poisson model*, firms or sectors independently invent new goods or improve on the existing ones. The numbers of new goods are then functions of the number of firms, or the size of the economy. The growth rate of this model eventually converges to that of the rate at which new goods are being introduced into the economy. In the second, called the *urn model*, the number of goods that are introduced is independent of the numbers of the existing goods, and the rate of introduction of new goods decreases as the number of goods increases.

8.1.1 Poisson growth model

Let $Q_k(t)$ be the probability that there are k goods or sectors in the economy at time t. Each firm (sector) independently has probability $\lambda\ \Delta t$ in a small time interval Δt of introducing one new good (sector). Thus the total numbers of the final goods (sectors) go from k to $k+1$ in time interval of Δt with probability

$$\Pr\left(N(t+\Delta t) = k+1 | N(t) = k\right) = \lambda k Q_k(t)\ \Delta t + o(\Delta t),$$

where $N(t)$ is the number of goods being produced at time t, and λk is the overall rate of new goods being introduced, on the assumption that sectors act independently. The probability $Q_k(t)$ is governed by the differential equation

$$\frac{dQ_k(t)}{dt} = -\lambda k Q_k(t) + \lambda(k-1)Q_{k-1}(t),$$

with the initial condition $Q_k(0) = \delta(k - k_0)$. We assume goods once introduced are not withdrawn from the market. A different model is later discussed, which has different effects on goods to be introduced to the markets in the future. For simplicity take k_0 to be one.

Solving this differential equation, the probability is given by

$$Q_k(t) = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}.$$

Suppose that output at time t of a good that was introduced at time s, $s \leq t$, is given by

$$y(t; s) = \frac{\mu}{\nu + (\mu - \nu)e^{-\mu(t-s)}}.$$

For definiteness assume that $\mu > \nu$. This expression implies that the output changes monotonically from 1 at the time of introduction of the good, and incorporates the assumption that the output eventually levels off at μ/ν as time goes to infinity.

The total output of this economy is then given by

$$Y_t = \sum_{k \geq 1} \lambda k \int_0^t Q_k(s) y(t, ; s)\, ds + y(t; 0).$$

The second term is the output of the original firm that exists in the economy at time zero. Using the generating function to express the sum, it is straightforward to show that the rate of growth of the total output converges to λ, that is, the rate of entry of new goods (sectors) as time goes to infinity:

$$g_t := \frac{d \ln Y_t}{dt} = \lambda + \frac{y(t, 0)}{Y_t} \frac{d \ln y(t, 0)}{dt} \rightarrow \lambda,$$

since $d \ln y(t, 0)/dt \rightarrow 0$.

In this subsection, we have used a constant λ. More realistically, λ may be a decreasing function of N. For example, by interpreting N as proxy for the stock of R&D, diminishing returns to R&D due to congestion in research, increasing difficulties, and such may cause the growth rate to approach zero, as in Jones (1995), Jones and Williams (1998), Segerstrom (1998), and Young (1998) among others. In the next model, we present a different take on this aspect.

8.1.2 *An urn model for growth*

The model of this subsection incorporates the idea that goods/sectors that will emerge are not directly linked to R&D, so that their rate of emergence is not tied to the birth rate of the Poisson process, but is strongly conditioned on some opportunities for innovations, which are independent of stock or flows of R&D, such as advances in basic scientific knowledge. We assume that a new good or sector is introduced at time t with probability

$$P(t) = \frac{\omega}{\omega + t},$$

for some positive ω, and $t = 1, 2, \ldots$. We use a discrete-time description for brevity.

Here, the rate of innovation is simply a descreasing function of time. This probability may be regarded as a probability of drawing a black ball from an urn that initially contains ω black balls. After each drawing, the drawn ball is returned, and one white ball is added to the urn. At time t, then, the urn contains ω black balls and t white balls. Urns to which one or more balls of different colors are added belong to a class of urn models called *Pólya urns*. Such models are extensively used in population genetics models. See Hoppe (1984, 1987), or Appendix A.2.

Pólya urns are used also in the standard R&D-based total factor productivity models such as Jones (1995), Jones and Williams (1998), Segerstrom (1998), and Young (1998). Their rate of innovation is a decreasing function of the R&D capital stock. Here, the rate of innovation is simply a decreasing function of time.

The probability of k goods being available in the market at time t is denoted as before by $Q_k(t)$, but is now governed by the difference equation

$$Q_k(t+1) = [1 - P(t)]Q_k(t) + P(t)Q_{k-1}(t),$$

with the boundary conditions

$$Q(1,t) = \frac{1}{\omega+1}\frac{2}{\omega+2}\cdots\frac{t-1}{\omega+t-1}$$

and

$$Q(t,t) = \frac{\omega^t}{(\omega+1)(\omega+2)\cdots(\omega+t)}.$$

The first is the probability that no new goods are introduced up to time t, and the second is the probability that new goods are introduced at each and every period up to time t.

We introduce a notation for the ascending factorial

$$\omega^{[t]} := \omega(\omega+1)\cdots(\omega+t-1).$$

The difference equation has the solution

$$Q_k(t) = \frac{c(k,t)\omega^k}{\omega^{[t]}}.$$

Here, we have introduced also an important number, the (unsigned) Stirling number of the first kind. It satisfies the recursion relation

$$c(k,t+1) = tc(k,t) + c(k-1,t).$$

This number is also defined by

$$x^{[m]} = \sum_{j=0}^{m} c(m,j)x^j,$$

for some positive integer m, i.e., the coefficient of x^j in the expansion of the ascending factorial $x^{[m]}$. See Appendix A.5 on Stirling numbers, as well as Abramovitz and Stegun (1968, p. 825). Aoki (1997, p. 279) has some nongenetic applications.

The total output is now given by

$$Y_t = \sum_{l=1}^{y}\sum_{j=1}^{l} \frac{c(l-1,j-1)\omega^{j-1}}{\omega^{[l-1]}}\frac{\omega}{\omega+l}y(t-l),$$

where $y(t-l)$ is the production at time t of the final good that emerged at time $l, l < t$.

For simplicity take ω to be a positive integer. Then

$$\sum_{l=1}^{t} \frac{\omega}{\omega + l} = \omega \left(\sum_{k=1}^{\omega+l} \frac{1}{k} - \sum_{k=1}^{\omega} \frac{1}{k} \right) \approx \ln \frac{\omega + t}{\omega}.$$

Thus, approximately,

$$Y_t = \ln(\omega + t).$$

The growth rate goes asymptotically to zero.

Alternatively, suppose that $P(t)$ depends on the number of existing goods. We define the probability that the kth good is invented during period t by

$$P_k(t) = \frac{\omega(k-1)}{\omega + k}.$$

In this case, the rate of growth is given by

$$\frac{d \ln Y_t}{dt} = \frac{1}{t} + \frac{1}{t \ln t}.$$

When opportunities for innovation declines, sustained growth is not possible. Solow (1994) makes a similar point about endogenous innovations. He points out that if R&D does not produce a proportional increase in the (Hicks-neutral) technical progress factor A in the production function $AF(K, L)$, but only an absolute increase in A, then greater allocation of resources to R&D buys a one-time jump in productivity, but no faster productivity growth. The model in the first Subsection 8.1.1 corresponds to proportionate growth in A, and the second to an absolute increase in A.

8.2 Disappearance of goods from markets

In the standard economic literature, diminishing returns to capital stocks essentially restrain economic growth. The model of this section is constructed to have its growth impeded by demand saturation, and is led by new goods, which randomly appear on the markets.[1] We examine a process of invention of goods and disappearance of goods as a nonlinear birth–death process.

Put simply, we assume that the stochastic process for demand changes is a birth-and-death process with birth rate λ and death rate μn_t. The only nonstandard feature is a nonlinear death rate in order to embody an idea that older

[1] This section is based in part on Aoki and Yoshikawa (2000).

products have higher probability of dying out. This aspect is somewhat reminiscent of the old-age effect of Arley referred to in Kendall (1948b). Arley use μt with constant μ to indicate that older particles die faster, probabilistically, than newly formed particles in his study of cosmic showers. We can handle an alternative senario of constant death rate and diminishing birth rate by replacing λ by λ/n_t. Basically the same qualitative conclusion follows from this alternative. We do not pursue this alternative further.

8.2.1 Model

Write the probability that output is n (in a suitable unit) as $P_n(t)$. The master equation for this growth process is

$$\frac{\partial P_n(t)}{\partial t} = \lambda(n-1)P_{n-1}(t) + \mu(n+1)^2 P_{n+1}(t) - (\lambda + \mu n)nP_n(t),$$

$n > 1$. The boundary condition is $\partial P_0(t)/\partial t = \mu P_1(t)$. We assume that $\lambda > \mu$. In this model the birth rate is a constant λ, but the death rate is taken to be μ times the number n. This effect may be congestion effects or old-age effect.

We can solve the master equation for a steady-state (stationary) distribution by setting the left-hand side equal to zero and replacing $P_n(t)$ by π_n. Try the detailed-balance condition

$$\lambda(n-1)\pi_{n-1} = \mu n^2 \pi_n,$$

$n > 1$. This has the solution

$$\pi_n = \frac{1}{Z}\frac{(\lambda/\mu)^n}{nn!},$$

where $n \geq 1$, and where $Z = \sum_n (\lambda/\mu)^n/nn! < \infty$.

All stationary moments are finite. For example, $\bar{n} = \sum_n n\pi_n = e^{\lambda/\mu}/Z$, $\overline{n^2} = (\lambda/\mu)\bar{n}$, $\overline{n^3} = (\lambda/\mu)(\lambda/\mu + 1)/Z$, and so on.

To obtain some information on nonstationary behavior of the growth process, we next derive the probability generating function $G(z,t) = \sum_n z^n P_n(t)$. Since this does not seem to have a closed-form solution, we convert it to the cumulant generating function and derive the ordinary differential equations for the first two cumulants.

The probability generating function is

$$\frac{\partial G}{\partial t} = \lambda z(z-1)\frac{\partial G}{\partial z} + \mu(1-z)\left\{\left(\frac{z\partial^2 G}{\partial z^2} + \frac{\partial G}{\partial z}\right)\right\},$$

and the cumulant generating function is

$$\frac{\partial K}{\partial t} = -\lambda(e^{-\theta} - 1)\frac{\partial K}{\partial \theta} + \mu(e^{\theta} - 1)\left[\left(\frac{\partial K}{\partial \theta}\right)^2 + \frac{\partial^2 K}{\partial \theta^2}\right],$$

where $K(\theta, t) = G(e^{-\theta}, t)$.

We expand K as $-\kappa_1\theta + \kappa_2\theta^2/2 - \kappa_3\theta^3/3! + \cdots$. Then, equating the expressions of the same powers in θ on both sides, we obtain the differential equations for the first two cumulants, κ_1 and κ_2:

$$d\kappa_1/dt = \lambda\kappa_1 - \mu\big(\kappa_1^2 + \kappa_2\big),$$

and

$$d\kappa_2/dt = (2\lambda + \mu)\kappa_2 + \lambda\kappa_1 + \mu\kappa_1^2 - 4\mu\kappa_1\kappa_2 - 2\mu\kappa_3.$$

Unfortunately, the equations for the cumulants do not terminate at any finite moments. By assuming that the steady state of κ_3 is a small bounded number, we can solve for the steady-state values of the first two moments by ignoring this term. We then examine if the linearized differential equations are asymptotically stable. If the answer is yes, then the stationary values are such that the third cumulants are zero. For certain ranges of the ratio λ/μ, there are two steady-state variance values for a positive stationary value of the mean.

8.2.2 *Stability analysis*

We drop the time argument for simplicity. Assume that $\mu < \lambda \le 2\mu$. With this assumption, the linearized equations for x and v about x_∞ and v_∞ are asymptotically stable, as we show later, and the stationary values are obtained by setting the left-hand side of the differential equation equal to zero, and assuming that the third central moment remains bounded for all time and has a stationary value as well. The stationary values are given by

$$\lambda x_\infty = \mu\big(x_\infty^2 + v_\infty\big),$$

and

$$(2\lambda + \mu)v_\infty + \lambda x_\infty + \mu x_\infty^2 - 4\mu x_\infty v_\infty = 2\mu\kappa_3(\infty).$$

In terms of x_∞, the stationary values are related by

$$v_\infty = \left(\frac{\lambda}{\mu} - x_\infty\right)x_\infty,$$

and

$$\kappa_\infty = x_\infty\frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu} - 2x_\infty\right)\left(\frac{\lambda}{\mu} - 2x_\infty\right)x_\infty.$$

When the third central moment does not have zero steady-state value, we need to bound the effects of this nonzero value by the Gronwall inequality. We drop these inessential complications here.

The steady-state variance needs to be smaller than $(\mu/2\lambda)^2$ to have positive x_∞; we assume that, and focus on the mean of n, or equivalently on the mean of the demand, which becomes

$$\bar{A}(t) = \frac{\lambda A_0}{\mu A_0 + (\lambda - \mu A_0)e^{-\lambda t}}.$$

The expected demand follows the logistic curve. Suppose that the initial value A_0 is smaller than λ/μ. Then, the mean demand initially grows almost exponentially, but eventually the growth diminishes to zero and the mean demand approaches its ceiling value λ/μ.

Subtract the steady-state values obtained by setting the derivatives equal to zero, and let $\xi = x - x_\infty$ and $\zeta = v - v_\infty$. Retaining only linear terms in the differential equations for these newly introduced variables, we have

$$\frac{d\xi}{dt} = \lambda\xi - \mu\zeta - 2\mu x_\infty \xi,$$

and

$$\frac{d\zeta}{dt} = (\lambda - 4\mu v_\infty + 2\mu x_\infty)\xi + (2\lambda + \mu - 4\mu x_\infty)\zeta - 2\mu(\kappa - \kappa_\infty).$$

The eigenvalues of this set of two equations have negative real part if the trace is negative and the determinant is positive of the matrix of the two equations above. At $x_\infty = \lambda/\mu$, for example, v_∞ is zero, and the two eigenvalues of this linearized equation both have negative real part, under the condition $\mu \le \lambda \le 2\mu$.

8.3 Shares of dated final goods among households

This section applies the techniques and elementary building blocks for models, discussed in Chapter 7, to analyze a model in which new final goods become available randomly over time, and they are being adopted or purchased gradually by some fraction of n households. To simplify presentation we work with expected values of stochastic variables. These new goods sustain growth of the economy.[2]

The mechanism of growth of the model is basically that of the Ramsey model. Unlike the latter, which relies on the shift of preference of a representative household for the engine of growth, growth in our model is due to diffusion or spread of consumption of the newly available goods for purchase by the households. Spread of consumption of new goods among households creates demand, which

[2] This section is based in part on Aoki and Yoshikawa (2000).

in turn induces capital investment and growth. Higher growth rate creates higher income among households, which induces more households to consume.

Because the amount of final goods purchased by households is bounded, growth of production of goods necessarily decelerates. Creation of new goods is the ultimate engine to sustain growth in the model.

8.3.1 Model

For simplicity, assume that there are n households, where n is exogenously fixed. Household i either buys or does not buy good j at time t. See Aoki and Yoshikawa (2000) for full specification of the model.

The purchase pattern is denoted by $q_{i,j}(t,\tau)$, which is 1 if household i purchases good j at time or period t and zero otherwise. Actually this depends also on the epoch τ at which good j has appeared on the market. To shorten notation, we sometimes drop this argument. We should and can treat $q_{i,j}$ as stochastic, but for simplicity we stay with the deterministic version.

The total number of households that buy good i is given by

$$\sum_{i=1}^{n} q_{i,j}(t,\tau) := d_j(t,\tau),$$

where $d_j(t,\tau)$ is the number of households that buy good j, which has existed since time τ.

We model the spread of purchase shares among the households by the birth–death process discussed in Chapter 7. As discussed there, we incorporate a nonlinear death rate in the model to reflect the assumption that demands for older goods decline with time.

The (expected) value of the mean of the demand for good j, then, has the S-shaped time profile

$$m_j(t,\tau) = \frac{\mu}{\nu + (\mu - \lambda)e^{-\mu(t-\tau)}}.$$

This expression is obtained by solving a nonlinear master equation, having μ as the birth rate and $j\nu$ as the nonlinear exit or disappearance rate. Recall our discussion of a model with a nonlinear exit rate in Chapter 4.

Next, assume that each household has the saving rate s, which is assumed to be the same for all households. Households purchase $1 - s$ units of any final goods that they consume.

The budget constraints for household i is

$$I_i(t) = \sum_{j=2}^{\infty} \int_0^t \lambda n e^{-\lambda\tau}(1-e^{-\lambda\tau})^{n-1}(1-s)q_{i,j}(t,\tau)d\tau + (1-s)m_{i,1} + sI_i(t),$$

where $I_i(t)$ is the income of household i, and $m_{i,1}$ is the purchase share of good 1, which is the initially available good at time 0. Here we assume that the number of goods initially available is 1. This budget constraint simplifies to

$$I_i(t) = \sum_{j=2}^{\infty} \int_0^t \lambda n e^{-\lambda\tau}(1 - e^{-\lambda\tau})^{n-1} q_{i,j}(t,\tau)\, d\tau + m_{i,1}.$$

Recall our discussion of the solution of the master equation

$$\frac{dQ(n,t)}{dt} = -\lambda n Q(n,t) + \lambda(n-1)Q(n-1,t),$$

where $Q(n,t)$ is the probability that the number of final goods available to the households is n at time t. We have seen that

$$Q(n,t) = e^{-\lambda t}(1 - e^{-\lambda t})^{n-1}.$$

Summing the incomes of all the households, we arrive at

$$\begin{aligned}\sum_i I_i(t) = \sum_j \int_0^t \lambda n e^{-\lambda\tau}(1 - e^{-\lambda\tau})^{n-1} \frac{\mu}{\lambda + (\mu - \lambda)e^{-\mu(t-\tau)}}\, d\tau \\ + \frac{\mu}{\lambda + (\mu - \lambda)e^{-\mu t}} \\ = Y(t),\end{aligned}$$

where $Y(t)$ is the GDP of this economy.

The equilibrium condition of the goods market is $s(\rho(t), \psi(t)) = \phi(g)$, where g is the growth rate of the economy, $\phi(\cdot)K$ is the investment in the economy with capital stock K, and ψ is a shift parameter of the saving rate. In the stationary state

$$s(\rho^*, \psi^*) = \phi(\lambda),$$

where $*$ indicate logarithmic derivative ($\phi^* := d\phi/dt$, for example), and ρ is the instantaneous rate of discount.

8.4 Deterministic share dynamics

We next switch to sectoral models. Denote the market shares of the n firms by s_i, with $s_i > 0$, for all i and $\sum_1^n s_i = 1$. These firms belong to two subgroups or clusters, denoted by A and B. We define

$$S_A := \sum_{j \in A} s_j.$$

For shorter notation we denote the sum as $\sum_A$. Similarly for $S_B = 1 - S_A$.

The rate of changes of the share of group A is

$$\frac{dS_A/dt}{S_A} = \sum_A \frac{ds_j/dt}{s_j}\frac{s_j}{S_A}.$$

We assume that individual firms' rates of change of shares are proportional to their deviations from the share-weighted average over all firms of some variable denoted as x_j, $j = 1, 2, \ldots, n$, such as the price charged by firm j or unit cost of firm j:

$$\frac{ds_j/dt}{s_j} = \gamma(\bar{x} - x_j),$$

where

$$\bar{x} = \sum_A s_j x_j + \sum_B s_j x_j.$$

The difference of the share-weighted averages between the two subgroups is

$$\delta = \frac{\sum_B s_j x_j}{S_B} - \frac{\sum_A s_j x_j}{S_A}.$$

The growth of the share of group A can be written as

$$\frac{dS_A}{dt} = \gamma\delta S_A(1 - S_A),$$

by expressing the sum over group B in terms of that over A and δ, and substituting it back into the original expression for the rate of change.

This last equation shows that the difference between the subgroups, δ, drives the dynamics for the group share, S_A. $S_A(t)$ converges to 1 if δ remains positive, and to 0 if δ remains negative. It is only with $\delta = 0$ that the shares of groups stabilize.

8.5 Stochastic business-cycle model

In this section, we consider a stochastic model in which the gap between demands and supplies of firms in subgroups drives the dynamics.

Let N denote the total number of firms, assumed to be fixed. The number of firms of group A is denoted by n. There are $N - n$ firms in group B, which is assumed to be less productive than those of firms in group A. Thus the total output of the economy is

$$y = c_1 n + c_2(N - n) = c_2 N + (c_1 - c_2)n,$$

where $c_1 \geq c_2 > 0$, and time arguments are suppressed from y and n. We express this in terms of the fraction

$$x = \frac{n}{N}.$$

Denote the share of demand for group As goods by s. That for group B is then $1 - s$. Both efficiency of production and shares could be functions of x in our analysis.

The gap between demand and supply for group A is

$$g = sy - c_1 n = N[sc_2 - \{c_2 s + c_1(1-s)\}x].$$

In a special case in which $c_1 = c_2 := c$, we have

$$g = cN(s - x).$$

In a deterministic model, one might postulate some adjustment mechanism that increases x if this gap is positive, and decreases x if it is negative, with $x = sc_2/[c_2 s + c_1(1-s)]$ being the equilibrium share of group A. Instead, we use the framework of birth–death processes, similar to the one in Aoki (1996a, Chap. 5), and postulate the transition rate for the number of firms in group A as

$$r_n := w(n, n+1) = N(1-x)\eta_1(x),$$

and

$$l_n := w(n, n-1) = Nx\eta_2(x),$$

with

$$\eta_1(x) = \frac{e^{\beta h(g)}}{e^{\beta h(g)} + e^{-\beta h(g)}},$$

where $h(g)$ is an increasing function of the gap g. It may include some adjustment or moving-cost component as well. The parameter β plays a crucial role. As in our earlier applications, β incorporates the effects of uncertainty, incomplete information, ignorance, and so on. In the simple case of $c_1 = c_2 = c$, g is positive when $s > x$ and η_1 is bigger than $1/2$, and is less than $1/2$ when $s < x$.

If we treat s, c_1, and c_2 as fixed parameters, then g is linear in x. If we assume that h is linear in g, then h is linear in x. With h nonlinear in g, or g made nonlinear by assuming that share the s, or efficiency of production, is a function of x, we could have situations in which h is cubic in x. In this case we know from models discussed in Aoki (1996a, Sec. 5.10) that there may be three critical points to the aggregate dynamics, two of which may be locally stable. All depends on the value of β introduced in the transition rates to embody uncertainty or lack of information on the future streams of profits that firms face. We show next how to use the master equation, the aggregate dynamics, and the Fokker–Planck equation to gain information on the fluctuations.

The master equation is

$$\partial P(n,t)/\partial t = (z-1)P(n,t)w(n,n-1) + (z^{-1}-1)P(n,t)w(n,n+1),$$

where we use the operator notation

$$zP(n,t)w(n,n-1) := P(n+1,t)w(n+1,n),$$

and

$$z^{-1}P(n,t)w(n,n+1) = P(n-1,t)w(n-1,n).$$

This is analogous to lead and lag operators in econometrics, and z-transforms in system theory. In physics this notation is used by van Kampen for example. Next, we change variables by

$$\frac{n}{N} = x = \phi + N^{-1/2}\xi,$$

and set

$$P(n,t) = \Pi(\xi,t).$$

With this change of the variable, we note that

$$z - 1 = N^{-1/2}\frac{\partial}{\partial\xi} + \frac{1}{2}N^{-1}\frac{\partial^2}{\partial\xi^2} + \cdots,$$

and

$$z^{-1} - 1 = -N^{-1/2}\frac{\partial}{\partial\xi} + \frac{1}{2}N^{-1}\frac{\partial^2}{\partial\xi^2} + \cdots.$$

Change time to $\tau = t/N$. Then, the left-hand side of the master equation becomes

$$\frac{\partial P}{\partial t} = N^{-1}\frac{\partial\Pi}{\partial\tau} - N^{1/2}\frac{d\phi}{d\tau}\frac{\partial\Pi}{\partial\xi}.$$

Equating the terms on the two sides of the largest order in N, we arrive at the aggregate dynamic equation

$$\frac{d\phi}{d\tau} = \alpha(x),$$

with

$$\alpha = r_n - l_n.$$

Stationary solutions are obtained as the zeros of the function $\alpha(\phi) = 0$; by substituing the expressions for the transition rates, we find that the zeros are the solutions of

$$\ln\frac{\phi}{1-\phi} = 2\beta h(g(\phi)).$$

As we remarked earlier, there are at most three solutions when h is cubic in ϕ. Depending on the magnitude of β, a unique locally stable ϕ, two locally stable ϕ's, or a single unstable ϕ is found in the range $0 < \phi < 1$. See Aoki (1996a, Chap. 5; 1998a). We also know that these critical points correspond to those of a double-well potential

$$U(\phi) = -\int^{\phi} h(g(x))\,dx - \frac{1}{2\beta}H(\phi),$$

with $H(\phi) = -\phi \ln \phi - (1-\phi)\ln(1-\phi)$ the entropy.

The remainder of the master equation yields

$$\frac{\partial \Pi}{\partial \tau} = \frac{1}{2}(r_n + l_n)\frac{\partial^2 \Pi}{\partial \xi^2} - \alpha' \frac{\partial}{\partial \xi}(\xi \Pi),$$

where r_n and l_n are evaluated at the equilibrium value ϕ^e of the aggregate equation, when we set the left-hand side equal to zero to obtain the stationary Fokker–Planck equation. We obtain a Gaussian distribution for ξ, with mean zero and variance $2(r_n - l + n)'/(r_n + l_n)'$, where the prime indicates differentiation with respect to ϕ, and evaluated at locally stable ϕ-values.

8.6 A new model of fluctuations and growth: Case with underutilized factor of production

We now return to the two-sector model of Section 4.1, and develop it more fully.

As we mentioned there, fluctuations of aggregate economic activities or business cycles have long attracted attention of economists. Here, we discuss dynamics for the two-sector model introduced in Section 4.1, and generalize the model to consist of K sectors.

Resources are stochastically allocated to sectors in response to excess demands or supplied of the sectors. We show that the total outputs of such an economy fluctuate, and that the average level of aggregate production (or GDP) depends on the patterns of demand.[3] Because we assume zero adjustment cost for the sizes of sectors, our model is a model of an economy with underutilized factors of production, such as hours of work of employees.

[3] In this sense, the model may be thought of a particular kind of quantity adjustment model. Leijonhufvud (1974, 1993) described a Marshallian quantity adjustment model. He envisioned a representative firm that adjusts outputs to narrow the gap between the supply price and demand prices of the good produced by the firm. Since the demand price schedule is unknown to the firm, the market-clearing price is substituted for it. Aoki analyzed this model in Aoki (1976, pp. 193 ff., 319 ff.) In this paper, sectors are not the same, and sectors adopt stochastic rules of response to gaps between demands and supplies. Sectors are subject to aggregate externalities, as we discuss in the text. See also Leijonhufvud (1993).

In the literature, economic fluctuations are usually explained as a direct outcome of the individual agents behavior. The focus is thus on individual agents. Often, elaborate microeconomic models of optimization or rational expectations are the starting points. The more strongly one wishes to interpret aggregate fluctuations as something "rational" or "optimal," the more one is led to this essentially microeconomic approach.

The model of this section proposes a different approach to explain economic fluctuations. The focus is not on individual agents, nor on elaborate microeconomic optimization modeling. Rather, the focus is on the manner in which a large number of agents interact.

Although studies of macroeconomies with many possibly heterogeneous agents are not new, the dynamic behavior of economies in disequilibrium is not satisfactorily analyzed. Clower (1965) and Leijonhufvud (1968) pointed out that quantity adjustment might be actually more important than price adjustment in economic fluctuations. Although this insight spawned a vast literature of the so-called "non-Walrasian" or "disequilibrium" analysis, this approach suffers from the basically static or deterministic nature of the analysis. See, for example, Malinvaud (1977) or Dréze (1991).

8.6.1 The model

Our model is a simple quantity adjustment model composed of a large number of sectors or agents. Resources are stochastically allocated to sectors in response to excess demands or supplies of the sectors. We show that the total output of such an economy fluctuates, and that the average level of aggregate production (or GDP) depends on the patterns of demand. Because we assume zero adjustment cost for the sizes of sectors, our model is a model of an economy with underutilized factors of production, such as hours of work of employees. For empirical studies of such economies, see Davis et al. (1996).

We assume that sector i has productivity coefficient c_i, which is exogenously given and fixed. Assume, for convenience, that the sectors are arranged in decreasing order of productivity. Sector i employs N_i units of the factor of production. It is a nonnegative integer-valued random variable. We call its value the **size** of the sector. When $N_i(t) = n_i$, $i = 1, 2, \ldots, K$, the output of sector i is $c_i n_i$, and the total output (GDP) of this economy is

$$Y(t) := \sum_{i=1}^{K} c_i n_i(t). \tag{8.1}$$

Demand for the output of sector i is denoted by $s_i Y(t)$, where $s_i > 0$ is the share of sector i, and $\sum_i s_i = 1$. The shares are also assumed to be exogenously given and fixed.

We denote the excess demand for goods of sector i by

$$f_i(t) := s_i Y(t) - c_i n_i(t), \tag{8.2}$$

$i = 1, 2, \ldots, K$. We keep the cs and demand shares fixed exogenously. Denote the set of sectors with positive excess demands by

$$I_+ = \{i; f_i > 0\},$$

and similarly for the set of sectors with negative demands by

$$I_- = \{j; f_j \leq 0\}.$$

To shorten notation, summations over these subsets are denoted as $\sum_+$ and $\sum_-$. Denote by n_+ the number of n's in the set I_+, that is,

$$n_+ := \sum_+ n_i,$$

where the subscript + is a shorthand for the set I_+, and similarly

$$n_- := \sum_- n_j,$$

for the sum over the sectors with negative excess demands. Let $n = n_+ + n_-$.

Sectors with nonzero excess demands attempt to reduce the sizes of excess demands by adjusting their sizes, up or down, depending on the signs of the excess demands. Section 8.6.3 makes this precise.

8.6.2 Transition-rate specifications

The transition probabilities are such that

$$\Pr(N_i(t+h) = n_i + 1 | N_i(t) = n_i) = \gamma_i h + o(h)$$

for $i \in I_+$, and

$$\Pr(N_i(t+h) = n_i - 1 | N_i(t) = n_i) = \rho_i h + o(h)$$

for $i \in I_-$, where the transition rates, γ and ρ, of the jump Markov process are specified later.

We assume that the γ's and ρ's depend on the total number of sizes and the current size of the sector that adjusts:

$$\gamma_i = \gamma_i(n_i, n),$$

and

$$\rho_j = \rho(n_j, n).$$

This is an example of applying W. E. Johnson's sufficientness postulate. We have discussed specifications of entry and exit probabilities in Aoki (2000b). See also Costantini and Garibaldi (1979, 1989), who give clear discussions on reasons for these specifications. As explained fully by Zabell (1992), there is a long history of statisticians who have discussed this type of problems. There are good reasons for γ_i to depend only on n_i and n, and similarly for ρ_i. See Zabell for further references on the statistical reasons for this specification.

We specify the entry rate, that is, the rate of size increase, by

$$\gamma_a(n_a, n) = \frac{\alpha + n_a}{K\alpha + n},$$

and that of the exit rate, namely, the rate of size decrease, by

$$\rho_a(n_a, n) = \frac{n_a}{n}.$$

If α is much smaller than K, then $\gamma_i \approx n_i/(\theta + n)$, where we set $\theta := K\alpha$, that is,[4]

$$\gamma_i(n_i, n) \approx \frac{n}{\theta + n}\frac{n_i}{n}.$$

So long as θ is kept constant, the above expression implies that the choices of K and α do not matter, provided α is much smaller than K. It is also clear that γ_i is nearly the same as the fraction n_i/n, which is the probability for exit. Then, the time histories of n_i are nearly those of fair coin tosses. We have K such coin tosses available at each jump. The sector that jumps determines which coin toss is selected from these K coins.

We set $\alpha = 0$ to discuss economies with fixed numbers of sectors, and set it to a positive number to allow for new sectors to emerge. In the latter case, a new sector emerges with probability $\theta/(\theta + n)$, while the size of sector i increases by one when the sector has positive excess demand with probability $(\alpha + n_i)/(\theta + n)$. See Ewens (1972).

8.6.3 Holding times

We assume that the time it takes for sector i to adjust its size by one unit (up or down), T_i, is exponentially distributed:

$$\Pr(T_i > t) = \exp(-b_i t),$$

where b_i is either γ_i or ρ_i, depending on the sign of the excess demand. This time is called the sojourn time or holding time in the probability literature. We

[4] There is an obvious interpretation of this approximate expression in terms of the Ewens sampling formula (Ewens 1972).

assume that the random variables T_i of the sectors with nonzero excess demand are independent.

The sector that adjusts first is the sector with the shortest holding time. Let T^* be the minimum of all the holding times of the sectors with nonzero excess demands. Lawler calculates that for $a \in I_+$

$$\Pr(T_a = T^*) = \frac{\gamma_a}{\gamma_+ + \rho_-},$$

where $\gamma_+ = \sum_+ \gamma_i$ and $\rho_- = \sum_- \rho_j$, and if $a \in I_-$, then the probability of a jump in sector a is given by

$$\rho/(\gamma_+ + \rho_-),$$

and similarly for the γ's. See Lawler (1995, p. 56), Appendix A.4, or Aoki (1996a, Sec. 4.2)

8.6.4 Aggregate outputs and demands

After a change in the size of a sector, the total output of the economy changes to

$$Y(t+h) = Y(t) + \mathrm{sgn}\{f_a(t)\}\, c_a,$$

where a is the sector that jumped first by the time $t + h$.[5]

After the jump, this sector's excess demand changes to

$$f_a(t+h) = f_a(t) - c_a(1 - s_a)\,\mathrm{sgn}\{f_a(t)\}. \tag{8.3}$$

Other, nonjumping sectors have the excess demand changed to

$$f_i(t+h) = f_i(t) + \mathrm{sgn}\{f_a(t)\}\, s_i c_a \tag{8.4}$$

for $i \neq a$.

These two equations show the effects of an increase of size in one sector. An increase by c_a of output increases the GDP by the same amount. However, sector a experiences an increase of its demand by only a fraction s_a of it, while all other sectors experience increase of their demands by $s_i c_a$, $i \neq a$. Equation (8.3) shows a source of externality for this model that affects the model behavior significantly. The index sets I_+ and I_- also change in general.

Defining $\Delta Y(t) := Y(t+h) - Y(t)$ and $\Delta f_i(t) := f_i(t+h) - f_i(t)$, rewrite (8.1) through (8.4) as

$$\Delta Y(t) = \mathrm{sgn}\{f_a(t)\}\, c_a,$$
$$\Delta f_a(t) = -(1 - s_a)\, \Delta Y(t),$$

[5] For the sake of simplicity, we may think of the skeleton Markov chain, in which the directions of jump are chosen appropriately but the holding times themselves are replaced by a fixed unit time interval. The limiting behavior of the original and that of the skeletal version are known to be the same under certain technical conditions, which hold for this example. See Çinlar (1975).

and

$$\Delta f_i(t) = s_i \, \Delta Y(t)$$

for $i \neq a$.

8.6.5 Equilibrium sizes of the sectors (Excess demand conditions)

When the excess demands of all sectors are zero, no section changes its output. We solve K equations of zero excess demands $f_i = 0$, $i = 1, 2, \ldots, K$, and obtain the equilibrium sizes, denoted by superscript e, of the fractions of sector sizes, n_i^e/n^e for $i = 1, \ldots, K$, and the ratio of the total output to the total number of units, Y^e/n^e.

Define K-dimensional column vectors $\mathbf{c} := (c_1, c_2, \ldots, c_K)'$ and $\mathbf{s} := (s_1, s_2, \ldots, s_K)'$. A diagonal $K \times K$ matrix $C := \mathrm{diag}(c_1, c_2, \ldots, c_K)$ is introduced to simplify our discussion.

The output is $Y = \langle \mathbf{c}, \mathbf{n} \rangle$, and the set of zero-excess-demand conditions is expressed by $s_i Y = c_i n_i$, $i = 1, 2, \ldots, K$, which is rewritten compactly as

$$C\mathbf{n} = \mathbf{s}Y = \mathbf{s}\mathbf{c}'\mathbf{n},$$

or

$$\Phi\mathbf{n} = \mathbf{0},$$

with $\Phi = C - \mathbf{s}\mathbf{c}'$.

Noting that the shares sum to one, the matrix Φ does not have full rank, because $|\Phi| = |C|(1 - \mathbf{c}C^{-1}\mathbf{s}) = 0$. It has rank $K - 1$, and its null space has dimension one and is spanned by the solutions we give next.

The solution is

$$\mathbf{n} = C^{-1}\mathbf{s}Y, \tag{8.5}$$

or

$$\frac{n_i^e}{n^e} = \frac{s_i/c_i}{\sum s_i/c_i}, \qquad i = 1, \ldots, K. \tag{8.6}$$

That is, the fraction is uniquely determined. Multiply it by c_i and sum over i to obtain the relation between Y and n as

$$Y^e = \frac{n^e}{\sum_i s_i/c_i}.$$

Note that $Y_i^e = s_i Y^e$, for all i, as it should.

We later give examples to show that simulation results support the analytical calculations remarkably well. Simulations show that, after the initial transient periods, the model moves around the equilibrium or near-equilibrium values of the sector sizes and outputs, in other words, we have equilibrium cycles when $\alpha = 0$, and growth with cycles with positive α.

8.6.6 Behavior out of equilibrium: Two-sector model

The expected level of total output, which is the equilibrium level in a deterministic model of the kind given in the previous subsection, is indeterminate. Here, we explore the behavior of the economy out of equilibrium in a stochastic model.

After a jump by a sector, the patterns of signs of the excess demands in sectors change. The changes are rather complicated to analyze in generality. To gain some insight, we analyze a simple two-sector model, and comment on how the results of that model may generalize. Note that $s_2 = 1 - s_1$ in the two-sector model. This model is characterized then by two parameters s_1 and c_2/c_1. (If you wish, c_1 may be set to one with a suitable choice of unit to measure n_1.)

Equation (8.5) shows that $n_1^e/n_2^e = (s_1/c_1)/(s_2/c_2)$, that is, the sign of $(1 - s_1)/s_1 - c_2/c_1$ determines the relative sizes of the two sectors at equilibrium. Hence, it does matter in the details of stochastic evolution whether n_1^e is larger than n_2^e or not. We describe the model behavior assuming that this sign is positive, that is $n_2^e \geq n_1^e$. The other case may be examined by switching the subscripts.

We suppress time arguments. The nonnegative quadrant of the plane for n_1 and n_2, with the horizontal axis labeled by n_1 and the vertical by n_2, is divided into six regions, denoted by R_k, $k = 1, 2, \dots, 6$. They are bounded by $n_i \geq 0$, $i = 1, 2$, and by five other straight lines with a common slope $\beta := (c_1/c_2)(1 - s_1)/s_1$. This slope is larger than one for our choice of the parameter values. The intercepts of the five lines are β, 1, 0, -1, and $-\beta$. These five lines are denoted by $L_1, L_2, \dots, L_5$, respectively. Line 3 cuts the n_1 axis at 0, line 4 at 1, and line 5 at –1, and $-\beta$.

In different regions, either the signs of the excess demands, or those after size changes by sector 1 or 2, are different, as detailed below. We note first that the two-sector model is special in that $f_1 + f_2 = 0$. Further, denoting by $f_i{}'(\pm)$ the value of the excess demand of f_i after a change of n_1 by ± 1, and similarly by $f_i{}''(\pm)$ the excess demand of f_i after a change in n_2 by ± 1, we note that $f_1'(\pm) + f_2'(\pm) = 0$ and similarly $f_1''(\pm) + f_2''(\pm) = 0$ from (8.1) and (8.2). Recall that only an increase in n_1 is possible when $f_1 > 0$ in sector 1. Similarly, with $f_2 < 0$, $f_1''(+)$ does not happen. To see these facts we note, for example, that in R_1, R_2, and R_3, which are above L_3, $f_1 > 0$. Hence, $f_2 < 0$ in these regions.

Table 8.1. *Excess demand: signs and sign changes.*

	Sign of excess demand				
Region	f_1	$f_1'(+)$	$f_1'(-)$	$f_1''(+)$	$f_1''(-)$
R_1	$+$	$+$	$*$	$*$	$+$
R_2	$+$	$-$	$*$	$*$	$+$
R_3	$+$	$-$	$*$	$*$	$-$
R_4	$-$	$*$	$+$	$+$	$*$
R_5	$-$	$*$	$+$	$-$	$*$
R_6	$-$	$*$	$-$	$-$	$*$

$*$ marks non-applicable or theoretically not possible combinations irrelevant situation.

After a change in n_1 by ± 1

$$f_1'(\pm) = s_1 c_2[n_2 - \beta(n_1 + 1)] > 0$$

above L_1, and so on.

The signs of the excess demands and how the sign changes by a change in size in sector 1 and 2 are summarized in Table 8.1.

The symbol $*$ marks entries that do not apply. Note that signs of f_1 are reversed for regions 4 to 6.

The probability of a size increase in sector 2 is larger than that of a size decrease in sector 2 when

$$\gamma_1(n_1, n_2) \leq \rho_2(n_1, n_2).$$

With $\alpha = 0$, this inequality holds when $n_1 < n_2$.

From a state (n_1, n_2) in R_1, consecutive jumps in sector 1 will bring the state to the boundary L_1 by increasing n_1; then the model state enters R_2, and the nature of the dynamics changes. This is so because f_1 continues to be positive after jumps in sector 1. Similarly, consecutive jumps in sector 2 from a state in R_1 also eventually bring the state to the same boundary by descreasing n_2. In general, we can calculate the various combinations of jumps in sector 1 and sector 2 to bring the state to the boundary, L_1. We thus see that the state leaves R_1 with probability 1. From a state in R_2 consecutive decreases in n_2 are possible until the state enters R_3. From a state on L_2 a jump in sector 2 brings the states to L_3.

The sector that jumps first is determined by the sector with the shortest holding time. Given that sector i changes its size, if $f_i(t)$ is positive, then we assume that n_i will increase by one. If the excess-demand expression is negative, we assume that n_i will decrease by one. Since no adjustment cost is included in the model, the sizes n_i may be interpreted as some measure of the capacity utilization factor in situations where the capacity constraint is not binding. With fixed numbers of employees in each sector, hours worked per period are an example of units of production factor entering and leaving production processes.

We show by simulation that cycles are possible in this model, and that the average level of output responds to demand patterns, that is, larger demand shares for more productive sector outputs tend to produce higher average output of the economy as a whole than smaller demand shares.

8.6.7 *Stationary probability distribution: the two-sector model*

Here, we derive the stationary probability distribution for the sizes of the two-sector model.

A general discussion of dynamics is conducted via the master (Chapman–Kolmogorov) equation. Here, we report on the derivation of the stationary probability distribution near the equilibrium states represented by L_3.

Table 8.1 in Section 8.6.6 indicates that patterns of signs of excess demands in regions 3 and 4 are such that the state of the model alternates between these two regions, thus exhibiting oscillations in n_i, $i = 1, 2$, and consequently in GDP. The line that defines the region $n_2 \geq n_1$ lies below L_3 and cuts across L_4 and L_5 from left to right. In the regions above this line sector 2 is more likely to jump than sector 1 in R_4. With $\beta > 1$, the lines L_1, L_2, and L_3 are above the line $n_1 = n_2$. Among states below L_3, most will be $n_1 < n_2$.

We show that the sign of the derivative of the expected value of Y with respect to s_1 to be positive near the equilibrium shown in (8.5) in our two-sector model (Sec. 8.6.6). For simpler presentation we just treat the case with $\alpha = 0$.

Suppose that the system enters R_3. We have shown that the state will oscillate in the region $R_3 \cup R_4$. Take the initial state b, which is on or just below L_3. Let $n(b) = (n_1(b), n_2(b))$ be the state, and define two adjacent positions e and c by $n_1(e) = n_1(b) + 1, n_2(e) = n_2(b) + 1, n_1(c) = n_1(b) - 1$, and $n_2(c) = n_2(b) - 1$.

By the detailed-balance conditions between states e and b, and those between b and c, we derive the relations for the stationary probabilities:

$$\frac{\pi(e)}{\pi(b)} = \frac{n_1(b)}{n_1(b) + 1} \frac{n_2(b)}{n_2(b) + 1} \frac{n + 2}{n},$$

where $n := n_1(b) + n_2(b)$, and

$$\frac{\pi(c)}{\pi(b)} = \frac{n_1(b)}{n_1(b) - 1} \frac{n_2(b)}{n_2(b) - 1} \frac{n - 2}{n}.$$

By repeating the process of expressing the ratios of probabilities, we obtain

$$\frac{\pi(b+(k,k))}{\pi(b)} = \left(\frac{n_1(b)}{n_1(b)+k}\right)^2 \frac{n+2k}{n}$$

for $k = 1, 2, \dots,$ where we use $n_2 = \beta n_1$ on or near L_3. Similarly,

$$\frac{\pi(b-(l,l))}{\pi(b)} = \left(\frac{n_1(b)}{n_1(b)-l}\right)^2 \frac{n-2l}{n}$$

for $l = 1, 2, \dots, \bar{l}-1$, where $\bar{l}$ is the largest positive integer such that $n - 2\bar{l} \geq 0$. Without loss of generality we treat it as an integer. Noting that $n = (1+\beta)n_1$, we write these ratios as

$$\frac{\pi(b+(k,k))}{\pi(b)} = \gamma^{-\mu k},$$

and

$$\frac{\pi(b-(l,l))}{\pi(b)} = \gamma^{\mu l},$$

with $\gamma = \exp(2/n_1(b))$ and $\mu = \beta/(1+\beta)$.

From now on we write b for $n_1(b)$ since there is no ambiguity.

Now, $E(Y) = (c_1 + c_2\beta)E(n_1)$, where

$$E(n_1) = A\left[\sum_{k\geq 1}(b+k)\gamma^{-\mu k} + \sum_{0}^{\bar{l}-1}(b-l)\gamma^{\mu l}\right]$$

for $l = 1, 2, \dots, \bar{l}-1$, where A is the normalizing constant $A^{-1} = \sum_0^{\bar{l}-1}\gamma^{\mu l} + \sum_{k\geq 1}\gamma^{-\mu k} = \gamma^{\mu\bar{l}}/[\gamma^{\mu}-1]$. We calculate this sum by means of the generating function,

$$E(n_1) = b + AG'(1),$$

with

$$G(z) = \sum_k (\gamma^{-\mu}z)^k - \sum_l (\gamma^{\mu}z)^l,$$

with the obvious upper limits of summation, which we drop for simplicity. Note that

$$G(z) = \frac{z}{\gamma^{\mu}-z} - \frac{\gamma^{\mu}z - (\gamma^{\mu}z)^{\bar{l}}}{1-(\gamma^{\mu}z)^{\bar{l}}}.$$

Substituting $(1+\beta)b/2$ for $\bar{l}$, we obtain, after some algebra,

$$E(n_1) = \frac{1}{1-\gamma^{-\mu}} + b - \bar{l} = \frac{1}{1-\gamma^{-\mu}} + \frac{b}{2}(\beta - 1).$$

For $\beta = 1$, this is clearly positive. For $\beta > 1$, it is positive if $1/(1-\gamma^{-\mu}) > (b/2)(\beta-1)$, which is satisfied for $\beta > \beta^*$ for some β^*. We assume that this condition is satisfied. Then,

$$EY = (c_1 + c_2\beta)E(n_1).$$

Its derivative with respect to s is

$$\frac{dE(Y)}{ds} = \frac{dE(Y)}{d\beta}\frac{d\beta}{ds},$$

where we note that $d\beta/ds = -(c_1/c_2)(1/s^2) \le 0$, and that

$$\frac{dE(Y)}{d\beta} = [-H(\beta)c_1 - G(\beta)c_2](\gamma^{\mu}-1)^{-2},$$

with

$$H(\beta) = \frac{2}{b}\frac{1+\beta^2}{(1+\beta)^2} + o(1/b),$$

and

$$G(\beta) = \frac{2}{b}\frac{2\beta^2(\beta-1)}{(1+\beta)^2} + o(1/b).$$

Consequently, we have

$$\frac{dE(Y)}{d\beta} \le 0$$

for all $\beta \ge 1$. This establishes

Proposition. *The expected value of Y will increase as the demand for sector 1 is increased in the range of $\beta > 1$.*

Hence, $E(Y)$ increases with a small inrease in s if and only if $dE(Y)/d\beta < 0$. When this inequality holds, we conclude that $d^2E(Y)/ds^2) \le 0$ if $d^2E(Y)/d\beta^2 < 0$, because of the relation $d^2E(y)/ds^2 = (d^2E(y)/d\beta^2)(d\beta/ds)^2 + (dE(y)/d\beta)d^2\beta/ds^2$, which is negative when $dE(Y)/d\beta$ is.

We can also show that $E(Y)$ is concave in s, that is, the increase in $E(Y)$ decreases as s becomes larger. To see this, note that $d^2E(Y)/d\beta^2 = -c_1H'(\beta) - c_2G'(\beta)$, where $c_1H'(\beta) + c_2G'(\beta) \ge 0$, with $H' = (4/b)(\beta-1)/(1+\beta)^3$ and $G' = (4\beta/b)\beta(\beta^2+3\beta-2)/(1+\beta)^3$, both expressions are true up to $o(1/b)$.

Hence we have the inequality

$$d^2E(Y)/d\beta^2 \le 0.$$

Writing $c_2\beta$ as $c_1 z$, with $z = (1 - s)/s$, and rewriting β as κz where $\kappa = c_1/c_2$, we see that the second derivative of $E(Y)$ with respect to β is negative in the range of z where

$$f(z) := \kappa^2 z^3 + 3\kappa z^2 + (\kappa - 2)z - 1 > 0.$$

For example, with $c_1 = c_2$, $f(z) > 0$ for $z > z^*$ with z^* somewhere between 0.6 and 0.7. This means that for $s \leq 0.5$ so that $\beta \geq 1$, the sign of this second derivative is negative. We can combine this result with that of the first derivative and conclude the following fact.

Fact. *$E(Y)$ is a convex increasing function of s in the range $0 < s < c_1/(c_1 + c_2 z^*)$. When $z^* \leq 1$, $\beta \geq 1$ in this range.*

Analogous proposition may be established for the range $\beta < 1$ in similar manner.

8.6.8 Emergence of new sectors

Next, suppose that new sectors appear at a rate proportional to $\theta/(\theta + n_+)$. This transition rate may be justified as a limiting case in which the parameter α goes to zero, while $K\alpha$ approaches a positive value θ. More in detail, we assume that either one of the sectors with positive excess demand increases size by one with probability $(\alpha' + n_j)/(K_+\alpha + n_+)$, where K_+ denotes the number of sectors with positive excess demand, and n_+ is the total size of such sectors, or a new sector emerges with rate proportional to $(K_+ - 1)\alpha/(K_+\alpha + n_+)$. In the limit of letting α go to zero, and assuming that $K_+\alpha$ approaches a common positive value for the sake of simplicity, we have a model in which either one of the existing sectors with positive excess demand increases size by one, or a new sector emerges.[6] That is, (8.1) is now modified to read that the conditional change in $Y(t + h)$ given $Y(t)$ consists of two terms, the first conditional on the event of the new sector appearing, which occurs with probability $\theta/(\theta + n_+)$, and the second conditional on the event that no new sector appears.

We assume that a new sector, when it emerges, inherits the characteristics – that is, c and s – of one of the existing sectors with equal probability. That is, if there are L sectors, then with probability $1/L$, the value of c and s of randomly selected sector is inherited. The s's are then renormalized so that they sum to one, including the newborn sector. This is merely for convenience. Other schemes may also be tried.

[6] We could assume that $K_+\alpha$ converges to θ_+, which may change each epoch. This would lead to a slight modification of the Ewens sampling formula.

8.6.9 *Simulation runs for multi-sector model*

This section summarizes our findings of the model's behavior by simulation.

What is most striking is the fact that the production levels, that is, the sizes of the different sectors of the model, are such that high-productivity sectors are constrained by demands. By starting the simulation with the initial condition of equal sizes for all sectors, we see that inflows of production factors into high-productivity sectors are clearly constrained, and the sizes of more productive sectors actually shrink in simulation. This is consistent with the views of Yoshikawa expressed in some of his writings (Yoshikawa 1995, 2000). We keep the total number of sectors at $K = 10$. We have done simulations with $K = 15$ and 20, but do not report them, since we have not observed any substantive differences. As our discussion above indicates, for small value of θ, which ranges from 0.2 to 0.6 in our experiments, there is not much loss of generality in keeping the value of K fixed. We also keep fixed the order of the productivities from $c_1 = 1$ to $c_K = 1/K$ at equal intervals. We start the simulation runs with the initial condition $n_i = 10$ for all sectors, $i = 1, 2, \ldots, 10$. In the graphs below, we skip the first 150 or 200 periods to avoid transient responses.

We vary the demand patterns for the outputs of the sectors as follows. We try five patterns, $P_i, i = 1, \ldots, 5$:

Pattern P_1 has $\mathbf{s} = (5, 5, 4, 4, 3, 1, 1, 1, 1, 1)/26$;
Pattern P_2 has $\mathbf{s} = (5, 3, 2, 1, 1, 1, 1, 1, 1, 1)/17$;
Pattern P_3 has $\mathbf{s} = (2, 2, 2, 2, 2, 1, 1, 1, 1, 1)/15$;
Pattern P_4 has $\mathbf{s} = (1, 1, 1, 3, 3, 3, 3, 1, 1, 1)/18$;
Pattern P_5 has $\mathbf{s} = (2, 2, 2, 1, 1, 1, 1, 1, 1, 1)/13$.

The sum of the shares of the top five sectors are 0.8, 0.7, 0.66, 0.5, and 0.61 respectively. Our analysis of the two-sector model may be adapted to these five patterns by lumping the top five sectors and the bottom five sectors separately to produce a two-sector model. The simulation confirms what the anaysis predicts, that is, the output is the largest for P_1, followed by those of P_2, P_3, and so on.

All patterns were run 200 times for 500 periods with $\theta = 0.6$ except as we note below. Pattern P_2 has also been run with $\theta = 0.2$. Pattern P_5 was also run for 1000 periods 400 times.

Runs of 200 are small for Monte Carlo experiments. Our interest here, however, is not in accurate estimates of any statistical properties of output variations, but rather in exhibiting possibilities of cyclical behavior in this simple model, and showing that output levels respond to demand pattern shifts.

One of the clear effects of different patterns is the dependence of average output levels on the patterns. Mean outputs are approximately in the order of

P_1 to P_5. By putting larger demand shares in higher-productivity sectors, the averge output shifts up. Because the standard deviations of outputs are still large due to the small numbers of runs, effects of different demand patterns on the statistical features of cycles are not so clear cut. Peak-to-peak swings are about 2 percent of the mean levels of outputs.

Figure 8.1 shows outputs, averaged over 200 Monte Carlo runs, with demand shares P_1 through P_5, each for the case of $\theta = 0.6$. Figure 8.2 is the plot of P_2 outputs averaged over 200 runs with $\theta = 0.2$. Figure 8.3 shows 1000 time periods of outputs averaged over 400 runs, with $\theta = 0.6$. Figure 8.4 shows per-unit output Y/n averaged over 200 runs. The equilibrium value is $y^e/n^e = 0.4196$. This value is independent of θ. Figure 8.5 shows the outputs for $\theta = 0.1$ and $\theta = 1$ with P_3 demand share pattern. These two figures are included to give some feel for the effects of the magnitude of θ on the outputs. As is pointed out by Feller (1968, Chap. 3), the random walks generated by fair coin tosses show much counterintuitive behavior. The numbers of periods and runs are not large enough to draw any precise conclusions. These simulation experiments serve to show the existence of equilibrium cycles even in this extremely simple quantity adjustment model.

The four panels of Fig. 8.6 show a sample of how the number of sectors increases, together with the total number of sizes, total output, and output per unit size, for the demand pattern P_3 with $\theta = 0.3$. As the value of θ is increased, the number of new sectors increases more quickly. For small values such as $\theta = 0.01$, new sectors come in much more slowly.

8.6.10 Discussion

Instead of assuming that resources are instantaneously reallocated to equalize productivities in all sectors, the model of this section assumes that (re)allocation of resources takes time. The model calculates the holding times of all sectors, which determine the probability of the sector that actually increases output in response to positive excess demands for goods of sectors of the economy. As parts of this calculation, the probability of a new sector emerging is also determined. The model solves the conceptual problem, in the usual agent-based simulation models, of which agent moves first and by how much.

Without building microeconomic structures into models, this model shows that cyclical fluctuations and growth with fluctuations are possible. What is most striking is the fact that production levels, that is, the sizes of the different sectors of the model, are such that high-productivity sectors are constrained by demands. By starting the simulation with the initial condition of equal sizes for all sectors, we see that inflows of production factors into high-productivity sectors are clearly constrained and sizes of more productive sectors actually

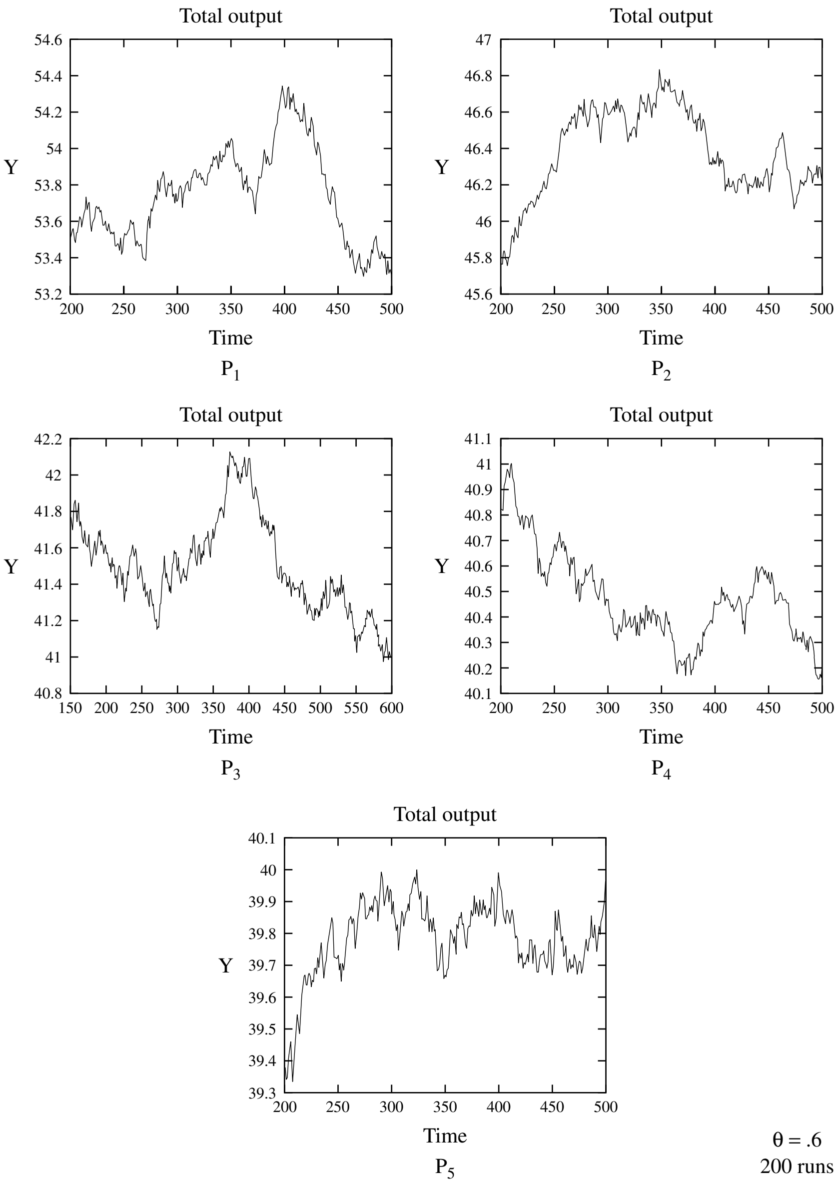

Fig. 8.1. Total outputs of five demand patterns, P_1 through P_5, all for 500 time periods, average of 200 Monte Carlo runs, and $\theta = 0.6$.

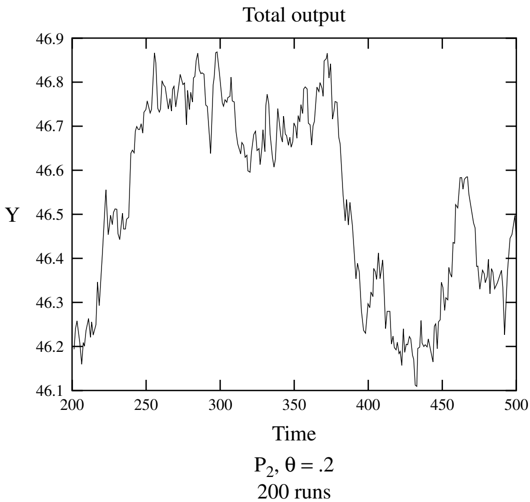

Fig. 8.2. Total output with P_2 pattern with 500 times periods, average of 200 runs, $\theta = 0.2$.

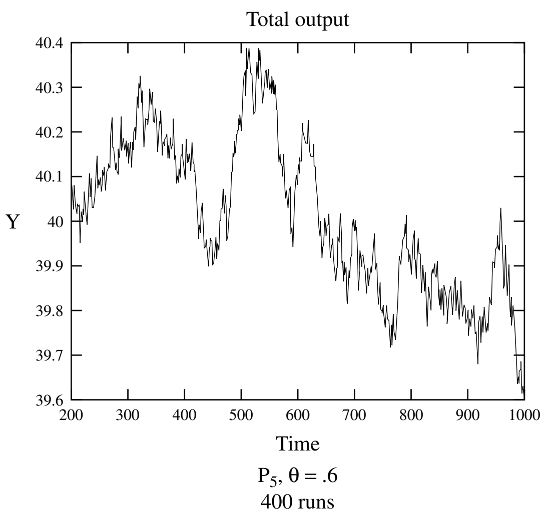

Fig. 8.3. Total output with P_5 pattern with 1000 times periods, average of 400 runs, $\theta = 0.6$.

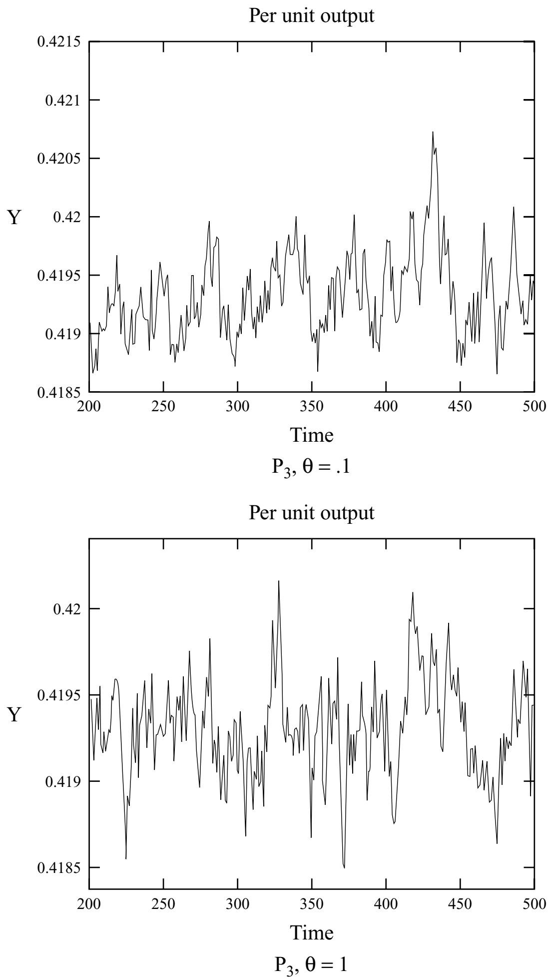

Fig. 8.4. Per unit output, Y/n, with pattern P_3. Upper panel for $\theta = 0.1$ Lower panel with $\theta = 1$.

shrink in simulation. This is consistent with the views in Yoshikawa (2000, 1995). We may also call the reader's attention to Davis et al. (1996, p. 83), which seems to lend support to the kind of modeling described in this section. They complain about downplay between cycles and the restructuring of industries and jobs in the traditional economics literature, and call for going beyond the stress placed on the role of aggregate shocks in business cycles.

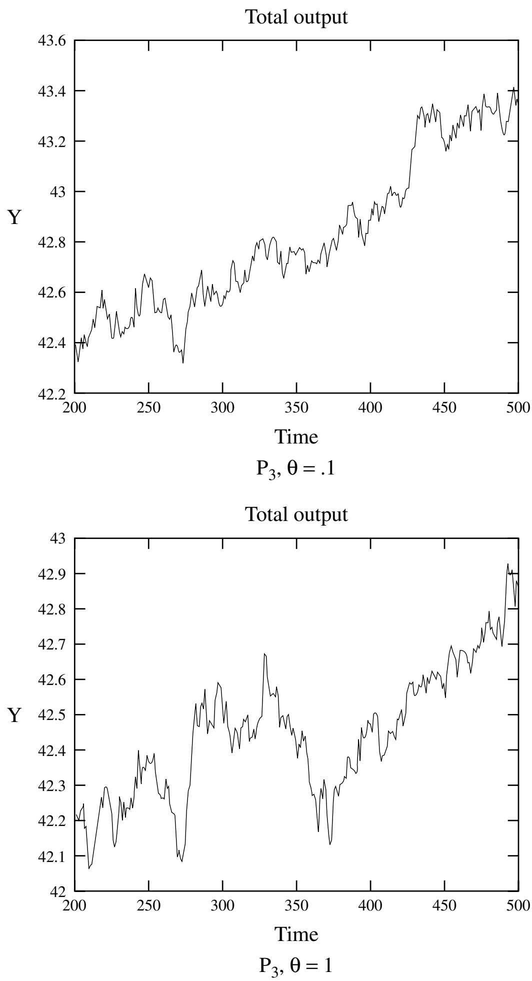

Fig. 8.5. Total output with pattern P_3 . Upper panel $\theta = 0.1$. Lower panel $\theta = 1$.

We have taken the entry and exit probabilities to depend on the sizes of the sectors. An alternative specification will specify them to depend on the excess demands themselves. This possibility is definitely worth pursuing.

Also, we note that changing the outputs from linear ones in (8.1) to concave ones $c_i n_i^\gamma$, with $0 < \gamma \leq 1$, does not change the patterns of the sign changes of excess demands in response to changes in n_i in the two-sector model if we

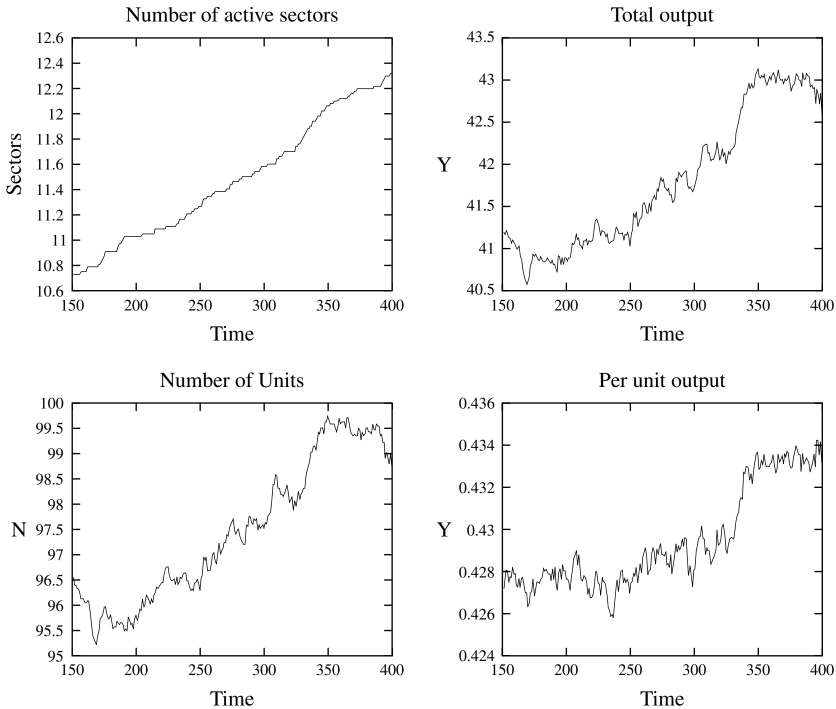

Fig. 8.6. New sectors with $\theta = 0.3$.

replace β with $\beta^{1/\gamma}$. For example, the inequality $n_2 > \beta(n_1 + 1)$ is replaced with $(n_2)^\gamma > \beta(n_1 + 1)^\gamma$, that is, with $n_2 > \beta^{1/\gamma}(n_1 + 1)$.

The regions R_1 through R_6 are analogously defined by lines L_1 through L_5 with slope $\beta^{1/\gamma}$. Arguments to derive the stationary distribution go through with β replaced by $\beta^{1/\gamma}$. Since the Proposition in Section 8.6.7 holds for all values of β, it also holds for economies with $c_i n_i^\gamma$, $i = 1, 2, \ldots, K$.

8.7 Langevin-equation approach

The dynamics of conventional economic models is specified by an n-dimensional deterministic dynamical equation

$$\frac{dx}{dt} = h(x),$$

where x is an n-dimensional state vector. Its stochastic version, especially in econometric models, is often proposed by tacking noises, usually additive, onto deterministic equations. Instead, we consider

$$\frac{dX}{dt} = g(X, \epsilon\xi(t))$$

as its stochastic version, with $g(\cdot,\cdot)$ some smooth function of the two arguments, where $X(t)$ is a stochastic state vector, and $\epsilon\xi(t)$ a vector-valued white noise process. Here, ϵ is a small positive constant, and $\xi(t)$ is standardized to have variance 1. We drop the time argument for simplicity.

Its linearization is called the Langevin equation:

$$\frac{dX}{dt} = b(X) + \epsilon\sigma(X)\xi(t),$$

with $b(X) = g(X, 0)$, and $\epsilon\sigma_{i,j} = \partial g_i(X, 0)/\partial\xi_j$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where m is the dimension of the vector ξ. This equation is usually presented as an Ito stochastic differential equation

$$dX = b(X)\,dt + \epsilon\sigma(X)\,dW(t),$$

where $dW_i(t) = W_i(t + dt) - W_i(t)$ is the Wiener-process increment. See Todorovic (1992) for a brief account of Langevin's approach. Cox and Miller (1965, p. 298) make a brief comment on it as well. Soize (1994) has more details.

Consider a scalar stochastic process

$$dx = b(x)\,dt + \epsilon\sigma(x)\,dW(t)$$

for a twice continously differentiable f on a compact set in the real line, with

$$df(x) = f'(x)\,dx + \frac{1}{2}f''(x)(dx)^2.$$

Substituting dx out, we have

$$df = f'(x)\{b(x)\,dt + \epsilon\sigma(x)\,dW(t)\} + \frac{\epsilon^2 f''(x)}{2}\sigma^2(x)\,dt.$$

Suppose that a probability density function $p(x, t)$ exists. Taking the expectation of the above with this density function, and interchanging the order of differentiation and integration, we evaluate the partial derivative with respect to t of the expectation of $f(x)$ by integration by parts. Noting that f and its partial derivatives have compact support, the resulting expression evaluated at the limits of integrations is zero, e.g., $f(x)b(x)p(x,t)|_{-\infty}^{\infty} = 0$, and so on. The resulting expression is

$$\int f(x)\frac{\partial p}{\partial t} = \int f(x)\left\{-\frac{\partial}{\partial x}[b(x)p] + \frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)p]\right\} dx,$$

so, because f is arbitrary, we derive

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}[b(x)p] + \frac{\epsilon^2}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)p].$$

This is a second-order linear parabolic partial differential equation. To be more precise on the technical conditions, see Soize (1994, Chap. VI), for example. This equation is called Fokker–Planck equation. In addition to the sources cited above, see Cox and Miller (1965, Chap. 5).

We sometimes rewrite the Fokker–Planck equation as

$$\frac{\partial p}{\partial x} = -\frac{\partial J}{\partial x},$$

where

$$J(x) = b(x)p - \frac{1}{2}\frac{\partial p}{\partial x}$$

is defined as the probability current through the point x.

8.7.1 *Stationary density function*

Setting the left-hand side of the Kolmogorov equation equal to zero, the stationary probability density is obtained. See Soize (1994, Sec. VI.5) for example.

The density must have probability mass 1. In the case of scalar equations defined on the real line and where the positive and negative regions are distinct, and there is no probability flow from one side to the other at the origin, $x = 0$; then the probability current is zero at $x = 0$.

In the steady-state case, the current is constant; hence it is zero throughout if it is zero at any point, such as the origin. Given that $J = 0$ at all x, it follows that

$$b(x)p^e(x) = \frac{1}{2}\frac{\partial p^e(x)}{\partial x},$$

where p^s denotes the stationary probability density. Integrating this equation, we obtain

$$p^e(x) = C \exp -\phi(x),$$

with

$$\phi(x) = \int_x^0 b(u)\, du.$$

8.7.2 *The exponential distribution of the growth rates of firms*

Let $S(t)$ be a stochastic process of the size of a firm. Here the word "size" is to be interpreted broadly as meaning some quantity related to the scale of firm's activities, such as the number of employees, sales in dollars, or plant and equipment or capitalization in dollar terms. The parameter values of the distributions vary somewhat depending on the S being used, but the functional form of the distribution remains the same. See Amaral et al. (1997) for detail.

Here, we postulate that $s(t) = \ln S(t)$ grows by the rule

$$s(t+\Delta) = s(t) - \mathrm{sgn}[s(t) - s^*]\,(\ln k) + \sigma_\epsilon \xi(t),$$

where σ_ϵ is a positive number, $\xi(t)$ is a mean-zero and variance-one normal random variable, k is some constant greater than 1, and s^* is a parameter assumed to be fixed for now. This equation simply means that a firm has a desired value for its size, S^*, and $S(t)$ grows by the factor $ke^{\sigma_\epsilon \xi(t)}$ if $S(t) < S^*$, and shrinks by the factor $(1/k)e^{\sigma_\epsilon \xi(t)}$ when the reverse inequality holds. With $S(t) = S^*$, it undergoes a random change $e^{\sigma_\epsilon \xi(t)}$.

Later, we describe how it may vary. The expression sgn means a sign function: $\mathrm{sgn}(u) = 1$ for a positive variable u, and $= -1$ if u is negative.

Approximating $s(t+\Delta) - s(t)$ by $\frac{ds}{dt}\Delta$, this dynamics has the Ito representation

$$ds = b(s)\,dt + \sigma\,dW(t),$$

where $W(t)$ is a standard Wiener process, with $b(s) = -\mathrm{sgn}(s - s^*)\,(\ln k)/\Delta$ and $\sigma = \sigma_\epsilon/\Delta$. Translated into the Fokker–Planck equation for the probability density $p(s,t)$, the equation is

$$\frac{\partial p}{\partial t} = \mathrm{sgn}(s - s^*)\,\frac{\ln k}{\Delta}\frac{\partial p}{\partial s} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial s^2}.$$

We next change the variables to put it into a standard form. Define

$$x = \frac{s - s^*}{s_0},$$

with $s_0 = \Delta\sigma^2/\ln k$, and

$$\tau = t/t_0,$$

with $t_0 = (\Delta\sigma)^2/(\ln k)^2$. The Fokker–Planck equation in the new variables is

$$\frac{\partial p}{\partial \tau} = \mathrm{sgn}(x)\,\frac{\partial p}{\partial x} + \frac{1}{2}\frac{\partial^2 p}{\partial x^2}. \tag{8.7}$$

The solution of this equation gives the time-dependent probability density for s or its normalized version x.

The steady-state probability density in the original variable, denoted by $p^e(s)$, is

$$p^e = \frac{1}{s_0}\exp\left(-\frac{|s - s^*|}{2s_o}\right).$$

8.8 Time-Dependent density and heat equation

In nonstationary case, we can further simplify the diffusion equation by eliminating the term $\partial p/\partial x$. Set

$$p(x, \tau) = e^{\alpha x+\beta\tau} V(x, \tau),$$

and choose $\alpha = -\text{sgn}(x)$ and $\beta = -1/2$. Then (8.7) becomes the standard heat-equation form

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\frac{\partial^2 V}{\partial x^2}.$$

This is a specially simple and well-known parabolic partial differential equation. It is called the heat equation because it arose as a model of the temperature distribution in one-dimensional heat-conducting media in steady heat flow (conduction or diffusion). The book by Sommerfeld (1949) discusses this and other physics examples. As mentioned in the introductory section, the option-pricing equation by Black and Scholes is a slightly more complicated example of this equation, to which it may be reduced by suitable transformation as in Willmot et al. (1993, Sec. 5.4).

The solution of the heat equation is seen to depend on x and t in a special combination $\xi = x/\sqrt{t}$. This is so because the equation and the initial and boundary conditions are invariant under scaling of x by a factor λ, and t by λ^2, for any real number λ. We have

$$\left.\frac{\partial \ln p}{\partial x}\right|_{0+} = -2.$$

We set $V(x, \tau) = \tau^{-1/2}U(\xi)$, as suggested by Willmot et al. (1993, p. 73), to have the integral of U finite. Then, the heat equation for V becomes

$$U'' = -(\xi U)'.$$

The solution is of the form $ce^{-\xi^2/2}$. From the conservation of the probability mass we impose

$$\frac{1}{2} = \int_0^{\infty} p(x, \tau)\, dx = \int_{-\infty}^{0} p(x, \tau)\, dx.$$

The constant c is given by

$$c^{-1} = \sqrt{2\pi}\ [1 - \text{erfc}(\sqrt{\tau}\,)],$$

where

$$\text{erfc}(w) = \sqrt{2}\pi \int_w^{\infty} e^{-u^2/2}\, du.$$

For large τ the erfc expression is approximately equal to $\tau^{-1/2}e^{-\tau/2}$, with $w = \sqrt{\tau}$. Thus, we recover the steady-state solution discussed above.

8.9 Size distribution for old and new goods

We have described distributions of cluster sizes based on Dirichlet distributions. These are stationary distributions from the viewpoint of random combinatorial analysis. Here, we examine the same subject from a different perspective. We obtain some information on nonstationary distributions this way.

8.9.1 Diffusion-equation approximation

We calculate the probability that there are k clusters formed by n agents at time t starting from n singletons, that is, n individuals initially. We follow the analyses by Derrida and Peliti (1991).

Suppose that there are n agents of either type 1 or 2. Suppose that a Markov chain $X(t)$ is defined by

$$\Pr(X(t+1) = j \mid X(t) = i)) = \frac{n!}{(n-j)!j!}(i/n)^j(1-i/n)^{n-j},$$

where $X(t)$ is the number of agents of type 1. This process is known to be approximated by a diffusion process with the forward Kolmogorov equation for the density with mean zero and variance $\sigma(x)^2$:

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2(\sigma^2(x)f)}{\partial x^2},$$

with $\sigma(x)^2 = x(1-x)/n$. This is the equation for the density of the fraction x of the diffusion approximation with a system of n agents. We define $\tau = t/n$ and rewrite the time derivative in terms of τ to remove the $1/n$ in the variance equation. See Ewens (1979, p. 140).

This diffusion equation was first solved by Kimura (1955). We discuss the solution of this equation with the initial condition $\delta(x-p)$.

We posit $f(x, \tau; p) = T(\tau)X(x; p)$ to try the separation of the variables to solve the equation. The equation separates into

$$\frac{T'(\tau)}{T(\tau)} = \frac{1}{2}\frac{\{x(1-x)X(x;p)\}''}{X(x;p)} = \kappa,$$

where κ is a constant.

The function $T(\tau) = T(0)e^{\kappa\tau}$ is immediate, and we change variable from x to $z = 1 - 2x$ in X. It satisfies

$$(1-z^2)X'' - 4zX' - (2\kappa + 2) = 0,$$

where $'$ denotes differentiation with respect to z now. This differential equation has a solution with $\kappa = -i(i+1)/2$ for any positive integer i. The function that solves the differential equation

$$(1-z^2)\psi'' - 2(\beta+1)z\psi' + \alpha(\alpha+2\beta+1)\psi = 0$$

is known as the Gegenbauer function, a type of hypergeometric function. See Morse and Feshbach (1953, pp. 547, 731) on the Gegenbauer functions. We see that the case with $\beta = 1$ and $\alpha = i-1$ is for our function X with $\kappa = -i(i+1)/2$. These values are the eigenvalues, and the corresponding ψ are eigenfunctions. To be explicit, we have

$$X(z) = T_{i-1}^1(z).$$

The class of Gegenbauer polynomials is known to be a system of complete orthogonal polynomials with weight $1-z^2$ on the interval $[-1, 1]$: For any positive integers m and n,

$$\int_{-1}^{1} (1-z^2) T_m^1(z) T_n^1(z)\, dz = a_n \delta_{m,n}.$$

Hence the solution of the diffusion equation can be expressed as

$$f(x,\tau;p) = \sum_{i\geq 1} C_i T_{i-1}^1(z) T_{i-1}^1(p) e^{-i(i+1)\tau/2},$$

where $C_i = (2i+1)(1-r^2)/[i(i+1)]$, where $r = 1-2p$.

The recursion relations are given by

$$(i+1)T_{i+1}^1(z) = (2i+3)zT_i^1 - (i+2)T_{i-1}^1,$$

with $T_0^1 = 1$ and $T_1^1 = 3z$. This is obtained from the generating function

$$\sum_{n\geq 0} t^n T_n^1(z) = (1-2tz+t^2)^{-3/2}.$$

We note that $T_n^1(1) = (n+1)(n+2)/2$ and $T_n^1(-1) = (-1)^n(n+1)(n+2)/2$.

8.9.2 Lines of product developments and inventions

We derive joint distributions for shares of old and new goods in a sense we now explain. Pick some past time instant t. Some of the goods or products currently available on the markets were in existence at least t time units or periods ago, that is, when we go back in time t periods, these products were already invented or being produced. Call these goods or products **old** goods or products. The remainder of goods or products that are currently available but were not available t periods in the past have been either invented or improved upon since that time. Call these **new** goods or products.

In this section, we use the theory of coalescents, which was invented by Kingman (1982) and has been applied extensively in the genetics literature, to explain shares of old and new goods (species). See, among others, Watterson (1984), Donnelly and Tavaré (1987), Ewens (1990), and Hoppe (1987) on coalescents and related topics. We describe the distributions of the numbers of old goods and new goods, and the probability density of the shares or fractions of these goods.

At present time, we take a sample of n products or goods, and we examine their histories of developments, going back in time. Some goods can trace their developmental history to an invention or innovation that took place some time ago. Others may have branched out from common prototypes some time in the past.

Pick a time t units in the past, and fix a sample of size n goods out of all goods that are available now in the markets. When the histories of development or improvements, or mere existence in the markets, are traced back in time for these n goods, they can be put in equivalence relations using the notion of **defining events** in the terminology of Ewens (1990, Sec. 7). A defining event is either the emergence of two products from a common prototype, or the invention of a new good or product some time ago. It is assumed that the overall rate at which the former takes place in an interval of length h is $[k(k-1)/2]h + o(h)$ when there are k goods, and the overall rate of invention is specified by $[k\theta/2]h + o(h)$. The rate of arrival of defining events is therefore $k(k+\theta-1)/2$ when there are k products. The mean time of arrival is $2/\{k(k+\theta-1)\}$.

Using the notation in Watterson (1984), at time t ago there were D_t old goods, and their equivalence classes are denoted by $\xi_i, i = 1, 2, \ldots, D_t$. Their sizes are $\lambda_i = |\xi_i|$. New goods are also put into equivalence classes, $\eta_j, j = 1, 2, \ldots, F_t$ (where F_t is the number of the equivalence classes), having the sizes $\mu_j = |\eta_j|$. See the figures in Ewens (1990).

Watterson (1984, (2.9)) has shown

$$\begin{aligned}&\Pr(l; \lambda_1, \lambda_2, \ldots \lambda_k; \mu_1, \mu_2, \ldots, \mu_l | k)\\ &\quad = \frac{(n-k)!k!}{n!} \frac{\theta^l}{(k+\theta)^{[n-k]}} \prod_i \lambda_i! \prod_j (\mu_j - 1)!,\end{aligned}$$

where k is the number of old goods and l the number of new goods. As t recedes into remote past, k will become zero, because all goods coalesce to a single prototype good, and the above simplifies to

$$\Pr(l; \mu_1, \ldots, \mu_l) = \frac{\theta^l}{\theta^{[n]}} \prod_{j=1}^{l} (\mu_j - 1)!.$$

This distribution is related to the Ewens sampling formula. Because there are

$$\frac{n!}{\mu_1!\mu_2!\cdots\mu_l!b_1!b_2!\cdots b_n!}$$

distinguishable ways that the sample can have $D_t = 0$ and $F_t = l$, and new class sizes μ_i, where b_i is the number of classes with size i, we recover the Ewens formula by multiplying the above two expressions.

This can be generalized to the case with nonzero old goods. Let a_i denote the number of old classes of size i. Then, we have

$$\begin{aligned}&\Pr(F_t = l; \lambda_1, \ldots, \lambda_k; \mu_1, \ldots, \mu_l | D_t = k)\\ &\quad = \frac{(n-k)!k!\theta^l}{(k+\theta)^{[n-k]}} \frac{1}{\prod_j \mu_j \prod_j b_j! \prod_i a_i!}.\end{aligned}$$

Noting that $\sum_{i=1}^n a_i = k$, $\sum_{j=1}^n b_j = l$, $\sum_{i=1}^n i(a_i + b_i) = n$, and $\prod_j (\theta/j)^{b_i} = \theta^j/\mu_1\mu_2\cdots\mu_l$, we have

$$\Pr(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n | D_t = k) = \frac{(n-k)!k!}{(k+\theta)^{[n-k]}} \prod_i \frac{1}{a_i!} \prod_j \frac{(\theta/j)^{b_j}}{b_j!}.$$

See Watterson (1984) for proof. Summing this over all l and bs, we derive

$$\Pr(a_1, a_2, \ldots, a_n | D_t = k) = \frac{k!}{a_1! \cdots a_n!} C_{\theta+n-z-1,n-z}/C_{\theta+n-1,n-k},$$

with $z = \sum_i i a_i$.

The process $\{D_t, t \geq 0\}$ is a continuous-time Markov (pure death) process, with $D_0 = n$. Tavaré (1984, (5.5)) has derived its probability,

$$\Pr(D_t = k) = \sum_{j=k}^{\infty} e^{-j(j+\theta-1)t/2} \frac{(-1)^{j-k}(2j+\theta-1)\Gamma(k+\theta+j-1)}{k!(j-k)!\Gamma(k+\theta)},$$

for $k \geq 1$.

Hoppe (1987, Sec. 8) provides related discussions and offers an urn-model interpretation.

Donnelly and Tavaré (1987) characterize the fractions of old goods, x_1, $x_2, \ldots, x_k$, and those of the new goods, $x_{k+1}, x_{k+2}, \ldots$, as follows: the sum $v_k := x_1 + x_2 + \cdots + x_k$ has the density function of Beta(k, θ):

$$f_k(v) = \frac{\Gamma(k+\theta)}{\Gamma(k)\Gamma(\theta)} v^{k-1}(1-v)^{\theta-1},$$

and

$$x_i = v_k u_i,$$

$i = 1, 2, \ldots, k$, where $u_1, \ldots, u_k$ are uniformly distributed on [0, 1] and sum to one, and $x_{k+1} = (1 - v_k)z_1$, $x_{k+2} = (1 - v_k)z_2(1 - z_1)$, and so on, where the z's are i.i.d. with the density of Beta$(1, \theta)$,

$$f(z) = \theta(1 - z)^{\theta - 1},$$

$0 \leq z \leq 1$. The factor $1 - v_k$ is residually allocated to the shares of new goods, in the order of appearance, that is, in the order of ages of the new goods. See also Ewens (1990).