Sustainable growth, exhaustible resource and stochastic discovery process

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Abstract

This paper aims to show how a continuous effort in exploration and development activities can relax the availability constraint of a non-renewable resource. We also investigate the possibility for the economy to reach, in average, a sustainable growth path if the impatience of society is balanced by the positive effects of new deposit discoveries. Formally, a necessary condition of sustainability requires the social discount rate to be small enough compared to the marginal probability of discovery.

JEL classification : Q30, Q32, C61, O41.

Keywords : Exhaustible resource; Exploration and development; Hotelling Rule; Sustainable growth; Non stationary Poisson process.

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1 Introduction

When a non renewable natural resource is used as essential input into production, this raises, in fine, a scarcity constraint on final consumption decisions since initial reserves are finite. The hardness of such a constraint depends on the initial levels of resource stocks and also on the conditions of the resource substitution with other inputs. This paper proposes to show how this availability constraint can be relaxed with a permanent investment into exploration and development activities. We also investigate the conditions for an optimal growth path to be sustainable. Under a strict economic point of view, any optimal growth path is qualified as sustainable when utility is at least non decreasing over time. For more details, Chichilnisky, Heal and Vercelli (1998) provide a good inventory of sustainable criterion in the economic literature.

According to Pindyck (1980), the outlets of exploration activities are twofold. First, exploration provides information on the distribution of reserves when this distribution is not perfectly known. In that case, exploration works as a "Bayesian" learning process (see for instance Clark et Mangel, 1986). Second, the exploration of new areas results in the discovery of new deposits so that existing reserves can be extended. However, the yield of exploration is unknown by definition since the time which is necessary to obtain a success is turned out to be random. Furthermore, each new exploration phase is characterized or not by the discovery of a new deposit and the trajectory of reserves thus becomes piecewise discontinuous.

This paper focuses on the second function of exploration more particularly. Discoveries are generated by a non stationary Poisson process whose arrival rate depends both on time and on exploration investment. On that account, an increase in the exploration effort causes the probability of discovery to be higher. Since in standard search models of natural resource economics the size of additional stocks traditionally depends on the exploration effort (see for instance Arrow and Chang, 1982, Devarajan and Fisher, 1982 or Quyen, 1991), this other way to make exploration process endogenous represents the originality of the paper.

Under a more general point of view, such an exploration activity can be interpreted as a R&D process which randomly generates innovations. In that case, any discovery lead to the accumulation of a stock of knowledge and the resource productivity or its degree of substitution with other inputs directly depends on this knowledge. If the R&D process yields an infinite number of innovations, the resource becomes potentially renewable. Consider for instance the implications of the nuclear fusion in terms of uranium use. Such a technological change may have the same consequences on the resource scarcity than if additional uranium deposits were permanently discovered.

The model is exposed in section 2. In section 3, we examine the optimal conditions and we derive the dynamic evolution rule of the resource rent. We thus get a modified Hotelling rule, extended to the case where new deposits can be discovered. Analytical optimal solutions are discussed in section 4. In section 5, we study the optimal trajectories of extraction and consumption and in particular, we determine the necessary conditions for those optimal growth paths to be sustainable. Finally, we briefly conclude in section 6.

2 The model

We consider an economy in which the population is constant over time. Labor supply is inelastic and equal to 1 at each point of time. A final consumption good is produced from labor and from a non renewable resource according to the following constant return to scale technology:

$$F\left(R_t, L_t\right) = R_t^{\theta} L_t^{1-\theta}, \tag{1}$$

where R_t is the resource extraction rate at date t, L_t, the quantity of labor and θ, $\theta \in (0, 1)$, a parameter of substituability between factors. The initial stock of resource, S_0, is finite and non reproducible. On that account, the only possibility to delay the exhaustion of S_0 consists in exploring new areas in order to discover new deposits, or to invest into R&D activities in order to increase the resource productivity. Hence, labor can be devoted either to production activities, either to exploration and development activities, which implies the following constraints:

$$
\begin{aligned}
1-L_t-N_t &\geq 0, &(2)\\
L_t &\geq 0, &(3)\\
N_t &\geq 0, &(4)
\end{aligned}
$$

where N_t denotes the fraction of labor that is devoted to exploration. Costs of extraction, exploration and development are assumed to be zero, excepted those coming from the labor affectation to both sectors.

Reserves of resource fluctuate stochastically over time according to the following process:

$$dS_t = -R_t dt + xS_t d\widetilde{q}_t. \tag{5}$$

Parameter x denotes the expected percentage rate of growth of reserves due to exploration and development. As in Pindyck (1987), we implicitly divorce exploration decisions from production decisions since ongoing exploration and development activity is resulting in changes in reserves, but those changes can be view as exogenous to the production decision.

In (5), $\widetilde{q}_t$ denotes the random cumulated number of discoveries or innovations since the initial date. Its variations $d\widetilde{q}_t$ follow a non stationary Poisson process with arrival rate $\lambda(N_t)\,dt$:

$$P(\widetilde{q}_t-\widetilde{q}_s=k)=\frac{[\lambda(N_t)\,t-\lambda(N_s)\,s]^k}{k!}e^{-\lambda(t-s)},\ \text{for } 0\leq s\leq t. \tag{6}$$

Instantaneous probability function $\lambda(.)$ is assumed to be increasing and concave in N_t, and such that $\lambda(N_t)\in[0,1]\,,\ \forall N_t\in[0,1]$. Hence, current probability of success increases with the effort devoted to exploration and development activities. In addition, $\lambda(0)=0$ so that the probability of success is zero as long as none effort is allocated to exploration. At every moment, either a new deposit or an innovation is discovered and $d\widetilde{q}_t=1$ with probability $\lambda(N_t)\,dt$, either exploration is not successful and $d\widetilde{q}_t=0$ with probability $[1-\lambda(N_t)\,dt]$.

As long as there is no success or during any phase of exploration and development, $d\widetilde{q}_t=0$ and equation (5) reads:

$$\dot{S}_t=-R_t. \tag{7}$$

Since the resource is not renewable, the available stock at date t decreases at the extraction rhythm. When a new deposit or an innovation is found, $\widetilde{q}_t$ is instantaneously increased by one unit and $dt = 0$ if we assume that consequences of discoveries are instantaneous. Then, current reserves instantaneously grow and their trajectory makes an upward jump whose size is given by:

$$\Delta S_t = S\left(t, \widetilde{q}_t + 1\right) - S\left(t, \widetilde{q}_t\right) = xS\left(t, \widetilde{q}_t\right), \tag{8}$$

where $S\left(t, \widetilde{q}_t + 1\right)$, noted $\breve{S}_t$, measures the level of current reserves following on an additional success. As in Pindyck (1987), those discrete changes in reserves are assumed to be proportional to the size of the resource stock, reflecting the idea that exploration activity is proportional to the size of the resource industry.

Along an infinite time horizon, the stock trajectory is thus piecewise discontinuous since it makes a discrete upward jump at each new success of the exploration and development process. In average, the positive effect of discoveries on reserves may balance the negative impact of extraction if the probability of success is, at least, higher than the extracted-discovered quantities ratio[1].

Finally, the representative consumer is assumed to be risk adverse. The instantaneous utility function is characterized by:

$$u\left(C_t\right) = \frac{\left(C_t\right)^{1-\gamma}}{1-\gamma}, \ \gamma > 0 \text{ and } \gamma \neq 1, \tag{9}$$

where C_t denotes the consumption quantity of the final good at date t and $1/\gamma$ the elasticity of intertemporal substitution of consumption.

3 The modified Hotelling Rule

The objective of the central planner is to maximize the expected present value of life-time utility of the representative consumer over an infinite time horizon, subject to constraints (1)-(5). Future utility flows are discounted at rate δ, $\delta > 0$. Clearly, (2) will be binding at the optimum: the labor that is not

[1] Expected variations of reserves are $\frac{E_t dS_t}{dt} = -R_t + \lambda\left(N_t\right) xS_t$ and are non negative for $\lambda\left(N_t\right) \geq \frac{R_t}{xS_t}$. Since $\lambda\left(N_t\right)$ is a probability function, a necessary condition for an expected growth rate of reserves to be non negative is $xS_t \geq R_t$. Hence, the rate of growth of reserves due to exploration should be higher than the rate of extraction in order to infinitely delay resource exhaustion.

devoted to production will be allocated to the exploration and development sector. We can then rule out a control variable by setting $L_t = 1 - N_t$, $\forall t \geq 0$. Let (P) be the program of the social planner:

$$(P) \qquad \max_{\{R_t, N_t, t \geq 0\}} E \int_0^\infty u\left(C_t\right) e^{-\delta t} dt$$

$$\text{s.t.} \left| \begin{array}{l} C_t = F\left[R_t, \left(1 - N_t\right)\right] \\ dS_t = -R_t dt + x S_t d\widetilde{q}_t, \; S_0 \text{ given} \\ R_t, N_t, \left(1 - N_t\right) \text{ and } S_t \geq 0, \; \forall t \geq 0. \end{array} \right.$$

Since resource and labor are necessary inputs to production, inequality constraints $(1 - N_t) \geq 0$ and $R_t \geq 0$ will never be binding, excepted asymptotically. We solve (P) using dynamic programming methods as detailed in Merton (1990). Defining $J\left(S_t\right)$ as the value function of (P), the associated Hamilton-Bellman-Jacobi equation writes:

$$\delta J\left(S_t\right) = \max_{R_t, N_t} \left\{ u\left[F\left(R_t, 1 - N_t\right)\right] + \frac{1}{dt} E_t dJ\left(S_t\right) + \mu_t N_t \right\}. \tag{10}$$

Expanding the stochastic differential $dJ\left(S_t\right)$, (10) can be rewritten as:

$$\delta J\left(S_t\right) = \max_{R_t, N_t} \left\{ u\left[F\left(R_t, 1 - N_t\right)\right] - R_t J'\left(S_t\right) + \lambda\left(N_t\right) \Delta J\left(S_t\right) + \mu_t N_t \right\}, \tag{11}$$

where $\Delta J\left(S_t\right)$ denotes the instantaneous increase in social welfare due to a success of exploration, i.e. $\Delta J\left(S_t\right) = J\left(\breve{S}_t\right) - J\left(S_t\right) = J\left[\left(1 + x\right) S_t\right] - J\left(S_t\right)$. We examine the case where the economy is not constraint by $N_t \geq 0$. First order conditions of (P) are then:

$$u'\left(C_t\right) F'_R = J'\left(S_t\right), \tag{12}$$

$$u'\left(C_t\right) F'_L = \lambda'\left(N_t\right) \Delta J\left(S_t\right). \tag{13}$$

The interpretation of (12) is standard: along any optimal trajectory, the marginal benefit from extraction in terms of instantaneous utility (LHS) must permanently be equal to the resource rent $J'\left(S_t\right)$. This term, formally the derivative of the objective function with respect to the dynamic constraint, corresponds to the shadow price of the resource, i.e. the price of a marginal unit of the in situ stock if there where a competitive market for the resource. It can also reflect the scarcity of the resource when being defined as the so-called Hotelling rent.

Similarly, condition (13) means that along any optimal path, the marginal cost of exploration and development in terms of instantaneous utility (LHS) must be equal to the expected gain in terms of utility due to a success in the exploration process (RHS), $\lambda'(N_t)$ being the marginal probability of success and $\Delta J(S_t)$ the instantaneous increment of the social welfare.

Replacing functions $U(.)$ and $F(.,.)$ by their analytical forms, and dividing expression (12) by (13), it comes:

$$\frac{\theta(1-N_t)}{(1-\theta)R_t} = \frac{J'(S_t)}{\lambda'(N_t)\Delta J(S_t)}. \tag{14}$$

In (14), $\lambda'(N_t)\Delta J$ reads as the expected marginal value of exploration and development in terms of potential increment of the resource stock. Hence, condition (14) means that along any optimal growth path, the marginal rate of transformation must be equal to the ratio of input prices.

In standard extractive resource models, the resource price (or at least its marginal value) grows at the same rhythm than the interest rate. This result, that had been established by Hotelling in 1931, illustrates the permanent intertemporal trade-off of the resource holder: he must be indifferent between extracting one unit of resource, selling it on the resource market and investing gains on a financial market, or keeping it non exploited. The following property extends the standard Hotelling rule to our context of exploration and development activities.

Proposition 1 *: If the evolution of reserves obeys to stochastic process (5), the resource rent grows, in average, at the following rate:*

$$\frac{\frac{1}{dt}EdJ'(S_t)}{J'(S_t)} = \delta - \lambda(N_t)\,x\frac{J'\left(\breve{S}_t\right)}{J'(S_t)}, \tag{15}$$

where $J'\left(\breve{S}_t\right) = J'[(1+x)S]$ *is the value of the resource rent following on an additional success in the exploration and development process.*

Proof: Partially differentiating (11) with respect to S yields:

$$\delta J'(S) = -RJ''(S) + \lambda(N)\Delta J'(S) + x\lambda(N)J'\left(\breve{S}\right), \tag{16}$$

where $\Delta J'(S) = J'\left(\breve{S}\right) - J'(S) = J'[(1+x)S] - J'(S)$ denotes the instantaneous variation of the resource rent if a new deposit is found with an exploratory effort N. The expected growth rate of $J'(S)$ can be obtained by computing its marginal increments along discrete intervals of time. The value of the resource rent at date $t+dt$ is defined as:

$$J'(S)\mid_{t+dt} = J'(S)\mid_t + J''(S)\,dS,$$

which clearly depends on the result of exploration:

$$J'(S)\mid_{t+dt} = \begin{cases} J'(S)\mid_t - RJ''(S)\,dt & \text{, with probability } 1-\lambda(N)\,dt \\ J'\left(\breve{S}\right) & \text{, with probability } \lambda(N)\,dt. \end{cases}$$

This implies:

$$\frac{J'(S)\mid_{t+dt} - J'(S)\mid_t}{dt} = \begin{cases} -RJ''(S) & \text{, with probability } 1-\lambda(N)\,dt \\ \frac{1}{dt}\Delta J'(S) & \text{, with probability } \lambda(N)\,dt. \end{cases}$$

Expected variation over time interval dt is:

$$E\left(\frac{J'(S)\mid_{t+dt} - J'(S)\mid_t}{dt}\right) = -\left[1-\lambda(N)\,dt\right]RJ''(S) + \lambda(N)\,\Delta J'(S).$$

From (16),

$$\delta J'(S) - x\lambda(N)\,J'\left(\breve{S}\right) = -RJ''(S) + \lambda(N)\,\Delta J'(S)$$

and then:

$$E\left(\frac{J'(S)\mid_{t+dt} - J'(S)\mid_t}{dt}\right) = \delta J'(S)\mid_t - x\lambda(N)\,J'\left(\breve{S}\right) + \lambda(N)\,RJ''(S)\,dt. \tag{17}$$

Dividing each side of (17) by $J'(S)\mid_t$ and making dt tend to zero, this concludes the proof. ■

In average, the rate of growth of the resource rent equals to the social discount rate less a partial correction term. Then, it is lower than the rate advocated by the Hotelling Rule. The reason is the following. At the time an additional deposit or innovation is discovered, the physical scarcity of the resource is instantaneously reduced. As a result, uncertainty on the return of the exploration and development process may speed up extraction.

The partial correction term in (15) reads as a regeneration rate of the resource. The growth rate of the shadow price of a renewable resource with regeneration function $H(S_t)$ is defined by the difference between the social discount rate and the marginal rate of regeneration, i.e. $[\delta - H'(S_t)]$. Under this point of view, (15) can be view as an Hotelling Rule extended to a renewable resource, but with a regeneration law which is both random and endogenous.

Finally, (15) allows to verify the transversality condition of (P). In fact, a necessary condition for the transversality condition to hold is:

$$\frac{1}{dt}E\left[\frac{d\left(e^{-\delta t}J'(S)\right)}{J'(S)}\right] < 0.$$

And from (15), it comes:

$$\frac{1}{dt}E\left[\frac{d\left(e^{-\delta t}J'(S)\right)}{J'(S)}\right] = e^{-\delta t}\left[-\delta + \frac{\frac{1}{dt}EdJ'(S)}{J'(S)}\right] = -e^{-\delta t}\left[\lambda(N_t)\,x\frac{\breve{J}'_S}{J'_S}\right] < 0.$$

4 Optimal policy functions

For algebraic simplicity, we set $\lambda(N_t) = \lambda N_t$, where $\lambda \in [0,1]$ denotes the marginal probability of success. Then, an increase in the R&D effort leads to an higher probability of discovery, but it let the marginal probability unchanged. Though it seems more realistic to introduce concavity in this probability form in order to mention the "breathlessness" of the exploration and development process, linear probability is the unique functional form that allows an analytical resolution of (P). The optimal policy functions of extraction and exploration effort are defined in proposition 2 below.

Proposition 2 *: The optimal path of exploration and development effort is unique and regular. The associated optimal extraction rule is proportional to current reserves. Those optimal solutions are:*

$$1 - N^* = L^* = \frac{(1-\gamma)(1-\theta)\mu}{\lambda\left[(1+x)^{\theta(1-\gamma)} - 1\right]}, \tag{18}$$

$$\widehat{R}(S_t) = \mu S_t,\ \textit{where}\ \mu = \left\{\frac{\delta - \lambda\left[(1+x)^{\theta(1-\gamma)} - 1\right]}{\gamma}\right\}. \tag{19}$$

Proof: We begin by computing optimal solutions under feedback forms. Solutions of system (12)-(13) are:

$$\begin{aligned} R(S) &= \left[\frac{1-\theta}{\lambda\Delta J(S)}\right]^{\frac{(1-\theta)(1-\gamma)}{\gamma}} \left[\frac{\theta}{J'(S)}\right]^{\frac{1-(1-\theta)(1-\gamma)}{\gamma}}, && (20) \\ L(S) &= 1-N(S) = \left[\frac{1-\theta}{\lambda\Delta J(S)}\right]^{\frac{1-\theta(1-\gamma)}{\gamma}} \left[\frac{\theta}{J'(S)}\right]^{\frac{\theta(1-\gamma)}{\gamma}}. && (21) \end{aligned}$$

Substituting $R(S)$ and $L(S)$ into (11) and after rearrangements, we get the following ordinary differential equation in $J(S)$:

$$\delta J(S) = \left(\frac{\gamma}{1-\gamma}\right)\left[\frac{1-\theta}{\lambda\Delta J(S)}\right]^{\frac{(1-\theta)(1-\gamma)}{\gamma}} \left[\frac{\theta}{J'(S)}\right]^{\frac{\theta(1-\gamma)}{\gamma}} + \lambda\Delta J(S). \qquad (22)$$

Solution of (22) is:

$$\begin{aligned} J(S) &= \left\{\frac{\gamma(1-\gamma)^{-\frac{\gamma+\theta(1-\gamma)}{\gamma}}\left[\frac{1-\theta}{\lambda\left[(1+x)^{\theta(1-\gamma)}-1\right]}\right]^{\frac{(1-\theta)(1-\gamma)}{\gamma}}}{\delta-\lambda\left[(1+x)^{\theta(1-\gamma)}-1\right]}\right\}^{\gamma} S^{\theta(1-\gamma)} \\ &= \Phi S^{\theta(1-\gamma)}. && (23) \end{aligned}$$

Then, the resource rent and the instantaneous increase in social welfare when exploration is successful are, respectively:

$$\begin{aligned} J'(S) &= \theta(1-\gamma)\Phi S^{\theta(1-\gamma)-1}, \\ \Delta J(S) &= \Phi\left[(1+x)^{\theta(1-\gamma)}-1\right] S^{\theta(1-\gamma)}, \end{aligned}$$

which, reinjected into (20) and (21) concludes the proof. ■

To simplify notations, define $\eta = \left[(1+x)^{\theta(1-\gamma)}-1\right]$. This term is positive if $\gamma < 1$, negative otherwise. Substituting R and S by their optimal expressions (18) and (19) into (1), we get the optimal consumption policy (i.e. optimal consumption as a function of current reserves):

$$\widehat{C}(S_t) = \mu\left[\frac{(1-\gamma)(1-\theta)}{\lambda\eta}\right]^{1-\theta} S_t^{\theta}. \qquad (24)$$

Solutions under feedback form of program (P) are characterized by expressions (18), (19) and (24). From (18), the optimal allocation of effort between productive and explorative sectors is proved to be constant over time, which illustrates a standard result of the economic growth theory when production and utility functions are chosen such that elasticities of substitution are constant. In addition, a permanent effort is affected to exploration and development activities along any optimal growth path.

Since N^* and L^* are assumed to be included into $(0,1)$, we must determine the necessary conditions of existence of solutions (18)-(19). The feasible set of parameter can be defined with respect to ratio (δ/λ). An interior optimal solution exists if and only if:

$$\left(\frac{\delta}{\lambda}\right) \in \left\{\left(\frac{\delta}{\lambda}\right)^{\inf}, \left(\frac{\delta}{\lambda}\right)^{\sup}\right\}$$

where:

$$\begin{aligned}(\delta/\lambda)^{\inf} &= \eta \text{ and } (\delta/\lambda)^{\sup} = \frac{[1-\theta(1-\gamma)]\,\eta}{(1-\gamma)(1-\theta)}, \text{ if } \gamma < 1 \\ (\delta/\lambda)^{\inf} &= 0 \text{ and } (\delta/\lambda)^{\sup} = \frac{[1-\theta(1-\gamma)]\,\eta}{(1-\gamma)(1-\theta)}, \text{ if } \gamma > 1.\end{aligned}$$

Hence, two cases have to be considered according to the intertemporal elasticity of substitution of consumption. First, the optimal extraction rule and the productive labor are positive (which directly implies $N^* < 1$) if $(\delta/\lambda) > \eta$ for $\gamma < 1$, and if $(\delta/\lambda) > 0$ for $\gamma > 1$. Second, whatever the value of γ, the social discount rate (i.e. the index of preferences for the present) must be small enough compared with the marginal probability of success in order to guarantee a positive optimal effort allocated to exploration and development activities.

Table 1 below summarizes the effects of parameter on the optimal policies. However, only optimal solutions under feedback form are concerned so that these results do not take into account direct effects on S_t.

	S_t	δ	x	λ
N^*	#	$-$	$+$	$+$
L^*	#	$+$	$-$	$-$
$\widehat{R}(S_t)$	$+$	$+$	$-$ if $\gamma < 1$ $+$ if $\gamma > 1$	$-$ if $\gamma < 1$ $+$ if $\gamma > 1$
$\widehat{C}(S_t)$	$+$	$+$	$-$ if $\gamma < 1$ $+$ if $\gamma > 1^{(*)}$	$-$ if $\gamma < 1$ $+$ if $\gamma > 1^{(*)}$

Table 1: Static behavior of optimal policies.

Intuitively, $\widehat{R}(S)$ and $\widehat{C}(S)$ are increasing functions of current reserves. An increase in the social discount rate causes the inputs use to increase, the effort of exploration and development to diminish and the consumption to increase. If the discovery rate or the marginal probability of success increases, productive labor will be reduced and R&D investment will be simultaneously augmented; that also increases current extraction if and only if society favors its present consumption capacity, i.e. if $(1/\gamma) < 1$. In that case, the global effect on $\widehat{C}(S)$ is turned out to be undetermined a priori since L^* decreases as x and λ increase. The positive effect on extraction overrides the negative effect on labor if and only if $(\delta/\lambda) \leq -\theta\eta/(1-\theta)$ – this situation which corresponds to $(*)$ in Table 1. Nevertheless, an increase in x or λ always implies that optimal policies of extraction and consumption diminishes if $\gamma < 1$.

5 Optimal trajectory analysis

This section aims to characterize the exact optimal trajectories and to determine their asymptotic behavior. From solutions (18) and (19), stochastic differential equation (5) becomes:

$$dS_t = -\mu S_t dt + x S_t d\widetilde{q}_t. \tag{25}$$

As long as there is no discovery or innovation, $d\widetilde{q}_t = 0$ and equation (25) has the following solution:

$$S(t, 0) = S_0 e^{-\mu t}. \tag{26}$$

Hence, reserves follow the "exponential decline curve" that is often used by petroleum engineers (see Lerche and McKay, 1993). When a new deposit or a new innovation is found, $d\widetilde{q}_t = 1$ and current reserves are instantaneously increased by $x\%$. The level of stock measured just after the discovery is thus $S(t, \widetilde{q}_t + 1) = (1+x) S(t, \widetilde{q}_t)$. Since the random experience that consists in obtaining a success with probability $\lambda N^* dt$ and a failure with probability $(1 - \lambda N^* dt)$ is infinitely repeated along an infinite time horizon, we thus get a geometric series with reason $(1+x)$. Solution of this series is:

$$S(t, \widetilde{q}_t) = (1+x)^{\widetilde{q}_t} S(t, 0), \tag{27}$$

where $S(t,0)$ is defined by (26). As a result, reserves, extraction and consumption follow optimal trajectories whose equations are, respectively:

$$\begin{aligned} S^*(t,\widetilde{q}_t) &= (1+x)^{\widetilde{q}_t} S_0 e^{-\mu t}, && (28)\\ R^*(t,\widetilde{q}_t) &= \mu S^*(t,\widetilde{q}_t), && (29)\\ C^*(t,\widetilde{q}_t) &= (1+x)^{\theta\widetilde{q}_t}(\mu S_0)^\theta (1-N^*)^{1-\theta} e^{-\theta\mu t}. && (30) \end{aligned}$$

At each point of time t, those equations determine the position of the economic variables as a function of the number $\widetilde{q}_t$ of success already incurred since the initial date until particular date t. Trajectories (28), (29) and (30) are piecewise discontinuous since they make an upward jump at the instant the discovery or the innovation takes place. In addition their asymptotic behavior is undetermined. Since along an infinite time horizon, the Poisson process will never be stopped, $\widetilde{q}_t$ tends to infinity in probability as $t\to\infty$ whereas $S(t,0)$ declines to zero. Clearly in expression (28), the term $(1+x)^{\widetilde{q}_t}$ due to the exploration and development process can asymptotically balance the exponential term $\left(e^{-\mu t}\right)$ caused by depletion. On that account, resource exhaustion may be avoided, or at least infinitely delayed, if the extraction intensity is balanced by exploration and development activities.

Since exact trajectories are piecewise discontinuous and their asymptotic properties, undetermined, we compute the smoothed trajectories. In particular, we determine the necessary conditions for those path to be qualified as sustainable. By sustainable growth path, we name any optimal consumption trajectory which is technically feasible and is at least non decreasing over time. Given (6)[2], the resource stock, the extraction and consumption levels go along, in average, the following optimal paths:

$$\begin{aligned} \overline{S}_t &= S_0 e^{(\lambda x N^*-\mu)t}, && (31)\\ \overline{R}_t &= \mu\overline{S}_t, && (32)\\ \overline{C}_t &= (\mu S_0)^\theta (1-N^*)^{1-\theta} e^{\left[\lambda N^*(1+x)^\theta-\lambda N^*-\theta\mu\right]t}, && (33) \end{aligned}$$

[2] In particular, $E\left[(1+x)^{\widetilde{q}_t}\right] = \sum_{k=0}^{\infty}(1+x)^k \frac{(\lambda N^* t)^k}{k!}e^{-\lambda N^* t} = e^{x\lambda N^* t}$, and $E\left[(1+x)^{\theta\widetilde{q}_t}\right] = \exp\left\{\lambda N^* t\left[(1+x)^\theta-1\right]\right\}$.

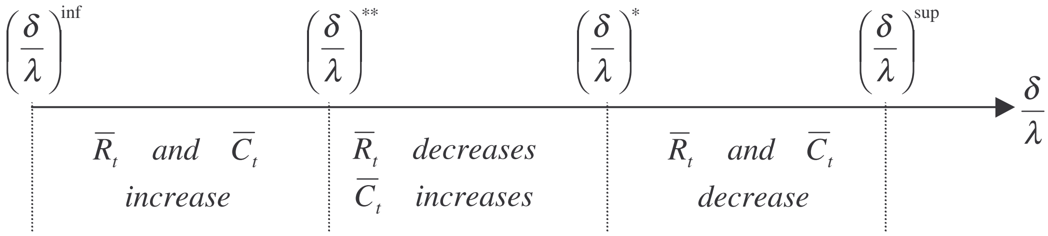

Figure 1: Necessary conditions for sustainabality.

The growth rates of those trajectories are constant over time and are defined as follows:

$$\begin{aligned} g_{\overline{S}} &= g_{\overline{R}} = \frac{\lambda}{\gamma}\left\{\left[x\left(1-\gamma\right)\left(1-\theta\right)+\eta\right]\left[1-\frac{\delta}{\lambda\eta}\right]+x\gamma\right\}, \\ g_{\overline{C}} &= \frac{\lambda}{\gamma}\left\{\left[\left(1+x\right)^{\theta}-1\right]\left[1-\theta\left(1-\gamma\right)-\frac{\delta\left(1-\gamma\right)\left(1-\theta\right)}{\lambda\eta}\right]+\theta\eta\left(1-\frac{\delta}{\lambda\eta}\right)\right\}. \end{aligned}$$

Signs of those rates depend first on the intertemporal elasticity of substitution of consumption and second, on ratio (δ/λ) :

$$\begin{aligned} g_{\overline{R}} &\geq 0 \Leftrightarrow \left(\frac{\delta}{\lambda}\right) \leq \left(\frac{\delta}{\lambda}\right)^{*}, \\ g_{\overline{C}} &\geq 0 \Leftrightarrow \left(\frac{\delta}{\lambda}\right) \leq \left(\frac{\delta}{\lambda}\right)^{**}, \end{aligned}$$

where

$$\begin{aligned} \left(\frac{\delta}{\lambda}\right)^{*} &= \left[\left(\frac{\delta}{\lambda}\right)^{\sup}-K\right], \\ \left(\frac{\delta}{\lambda}\right)^{**} &= \left[\left(\frac{\delta}{\lambda}\right)^{\sup}-K'\right], \\ K &= \frac{\gamma\eta^2}{\left(1-\gamma\right)\left(1-\theta\right)\left[x\left(1-\gamma\right)\left(1-\theta\right)+\eta\right]}, \\ K' &= \frac{\gamma\theta\eta^2}{\left(1-\gamma\right)\left(1-\theta\right)\left\{\left(1-\gamma\right)\left(1-\theta\right)\left[\left(1+x\right)^{\theta}-1\right]+\theta\eta\right\}}. \end{aligned}$$

The reader can numerically verify that thresholds $(\delta/\lambda)^*$ and $(\delta/\lambda)^{**}$ are positive for any $\gamma > 0$, $\gamma \neq 1$, $\theta \in (0,1)$ and $x \in (0,1)$. In addition, it is easily proved that $(\delta/\lambda)^* > (\delta/\lambda)^{**}$. Results are depicted in Figure 1.

Figure 1 can be viewed as a regeneration scale of the resource. In fact, even if it is initially exhaustible, the resource becomes potentially renewable as the exploration and innovation process becomes more and more successful. If the marginal probability of discovery is high enough compared with the degree of impatience of society, i.e. (δ/λ) small enough, the average positive effect of exploration and development activities overrides the average negative effect of depletion so that the smoothed trajectory of stock increases over time. Given the optimal labor affectation to each sector, average consumption is an increasing function of time if and only if society is patient enough compared with the probability of finding new deposits. The left hand side of Figure 1 thus deals with a renewable resource whose regeneration is both random and endogenous. As ratio (δ/λ) increases, the resource scarcity gradually overrides the regeneration properties. If social preferences for the present are too important compared with the marginal probability of extending reserves, resource stock will be asymptotically exhausted and average consumption will decline to zero. The economy thus behaves as if the resource was non renewable.

We note that the condition on λ is less restricting if we want an increasing smoothed trajectory of consumption than if we want an increasing smoothed trajectory of extraction. In other words, if $(\delta/\lambda) \in [(\delta/\lambda)^{**}, (\delta/\lambda)^{*}]$, average consumption can be increasing over time whereas average extraction is decreasing. Such a result is due to Jensen' inequality since production function is concave in R. Consequently, the growth rate of average consumption can be higher than the growth rate of average extraction if the resource productivity is high enough compared with the labor productivity.

6 Conclusion

We have shown how the availability constraint of a non-renewable resource can be relaxed by a continuous investment into exploration and development. In particular, the evolution of the resource rent over time obeys to a modified Hotelling rule which advocates a lower rate of growth than the standard rule. By doing that, we implicitly consider the scarcity diminution as the exploration process is successful. We have also investigated the possibility for the economy to reach, in average, a sustainable growth path if the impatience of society is balanced by the positive effects of new deposit discoveries, i.e. for a small enough discount rate compared to the marginal probability of success.

The results presented here are based on a model that contains a number of simplifying assumptions, some of which may be quite limiting. Perhaps more important, ongoing exploration and development activities was resulting in changes in reserves which were assumed to be proportional to the current stock level. A more realistic process may be $dS_t = -R_t dt + \widetilde{X} d\widetilde{q}_t$, where $\widetilde{X}$ is a time independent random variable that denotes the size of new discovered deposits.

Finally, we have assumed that agents perfectly know the distribution of reserves so that they can adapt continuously to reserves change. One aspect that seems to be particularly relevant nowadays is to consider a more sophisticated stochastic framework by assuming that the resource holder does not perfectly know the size of reserves which will be available in the future. This case is similar to a "cake-eating" problem but with both the total size of the cake and how it increases in size if more flour and eggs are added, unknown. This more general consideration of the environment is not only more realistic, but it also leads exploration activity to exhibit its second function, the possibility to provide some information about the available reserves.

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