Chapter 5

Stochastic One-Sector and Two-Sector Growth Models in Continuous Time

Bjarne S. Jensen
University of Southern Denmark and Copenhagen Business School

Martin Richter
Department of Economics, Copenhagen Business School
Danske Research, Danske Bank, Copenhagen

5.1 Introduction

To set the stage for the main content, exposition, and organisation of our subject matter - stochastic neoclassical models of capital accumulation in continuous time with steady-state or persistent growth per capita - we will review some fundamental issues of "stochastic processes" [parameterized collections (sequences) of stochastic variables] that were first raised in the seminal papers of Mirman (1972, 1973), Brock and Mirman (1972, 1973), Merton (1975), Bourguignon (1974).

Since the concept of a steady state equilibrium has played an important role in both positive and optimal theory of economic growth, Mirman asked whether the same questions can be posed in *random growth models* as in deterministic growth models:

> "In what sense should one even discuss the *random evolution* of the system ? How does one define a concept in the random case analogous to the deterministic steady state? Added to these questions are the usual questions of existence, uniqueness, and stability for the random analogue of the steady state", Mirman (1973, p. 220).

Then he redefined the concept of a steady state in a stochastic sense:

> "This is done by using the distribution function of possible capital labor ratios generated by the stochastic growth process. Having *defined the steady state* in terms of a *distribution* function, we then show that, for each admissible policy, the corresponding stochastic system has a unique steady state distribution, which is a degenerate distribution in the deterministic theory.
>
> Moreover, it is shown that this unique steady state distribution is stable in the sense that the set of possible states of the system converges over time to a well-defined set, the analogue of the deterministic steady state, which supports the unique steady state distribution.
>
> Finally, it is shown that the *sequence of distributions converge* to a *unique* steady state distribution ", Mirman (1973, p. 220).

In implementing this research program and in the mathematical analysis, Mirman (1972, pp. 224) first assumed that his *random variables* A(t) - "*technology shocks*" - were independent, identically distributed in *discrete time*, and at any time independent of the capital labor ratios, k(t); further, it was assumed that the *shocks* A(t) were always strictly *positive and finite*, (bounded away from both zero and infinity). To further simplify the analysis and avoid the possibility of a steady state at either zero or infinity, the technology (production function) was assumed to satisfy the simple derivative conditions of Inada (satisfied by the CD technology). With these assumptions, Mirman used mathematical techniques similar to those in the theory of Markov chains to establish the stochastic generalization of the Solow growth model by showing the existence, uniqueness, and stability of stationary probability measures. With his assumptions, he proved that a stationary measure will always exist, and the stationary measure will be unique if the recurrent states all communicate and admit no cyclically moving subsets. Stability meant that iterates (sequences) of the transition probability tend to a unique asymptotic (time invariant) probability measure (distribution). The tools of the Markov processes were used to demonstrate such stability (convergence) to the unique stationary (steady-state) distribution of the capital labor ratio. In short, particular *neoclassical* assumptions and

> "techniques from positive *deterministic* growth theory were combined with the tools of *Markov processes* to achieve a positive theory of *stochastic* economic growth", Mirman (1973, p. 230).

Mirman (1973) had assumed that his *random variables* A(t), (technology shocks), which influence the production process, are *strictly positive* and *finite*. "More precisely, for any capital stock, output can neither be arbitrarily large nor arbitrarily small (even with arbitrarily small probability)", but "it was not clear where the bounds of possible random effects should be set", Mirman (1972, p. 271). Next, he addressed this problem with arbitrarily large/small outputs. Still, the existence of a stationary measure (steady-state distribution) was demonstrated with a fixed-point argument, Mirman (1972, p. 279):

> "However, it is possible that there *exists* a *positive probability* of *extinction* or *positive probability* of an *infinite capital stock* in the stochastically generalized notion of a steady state".

Mirman (1972, p. 271) studied *conditions* "for the existence of stationary measures having *zero probability* at *zero and infinity"*, - which turned out as *analogous* to imposing the *Inada conditions* on the production process. For recent advances in the field of stochastic neoclassical growth models in discrete time, see Schenk-Hoppe (2002), Lau (2002).

The first and important *extension* of the stochastic study by Mirman and Brock of the *discrete-time*, neoclassical one-sector growth model to *continuous-time stochastic processes* was done by Merton (1975) and Bourguignon (1974). Besides existence and uniqueness, much more *specific* (parametric) structures of the steady-state (asymptotic, limit) *distributions* for the capital-labor ratio and other variables could now be examined. Thus, Merton (1975) derived *density* functions and *first and second moments* that would be obtained in a steady-state, and a *comparison* of the results and biases (in expected value) between *deterministic* (certainty) and *stochastic* modelling were rigorously derived for a stochastic growth model with a CD production function. The source of *uncertainty* was not positive technology shocks, but the uncertainty in Merton (1975) affected the *evolution* of the *labor* (population) stock, which he assumed followed a *geometric Wiener process*. With the latter, the *boundary problems* at zero and infinity for the capital-labor ratio were essentially *absent* from the first *stochastic* Solow growth model in *continuous time*.

In the neoclassical one-sector growth model, the sources of *uncertainty* were extended by Bourguignon (1974) to *saving* and *depreciation rates*, and the more general CES technology was adopted. Hence, the *boundary problem* for the *capital-labor ratio* naturally came into focus, and the result was that

> *"uncertainty can make the neoclassical model closer to the Harrod-Domar-type models of growth in introducing the possibility of a collapse of the economy"*, Bourguignon (1974, p. 142).

In particular, uncertainty in the saving rate posed (without further parameters restrictions) a serious problem of hitting the lower boundary (absorption, $k = 0$). Since 1975, the consequences of such critical boundary problems have somehow been that this field of research in *continuous-time stochastic growth models* has not matured and in fact mostly *disappeared* from the economic literature.

A *new start* is needed and is here attempted, partly by first *resolving* the older methodological *problems* with *absorbing boundaries* and steady state and partly by *extending* the *stochastic neoclassical* framework to *one-sector and two-sector* models with parameters generating *endogenous* (persistent) *per capita growth*. Simulations of *sample paths* and asymptotic *density functions* will illustrate our Theorems and the properties of the parametric stochastic processes.

The study of *deterministic general equilibrium dynamics* in two-sector and multi-sector growth models has been reviewed and extended in Jensen [2003], Jensen and Larsen [2005] - with emphasis on factor *allocation*, output *composition,* and the dualities for *commodity* and *factor prices*. Sample paths of the *stochastic two-sector analogue* are here discussed and demonstrated.

For our purposes and as benchmarks, *production functions* of the CD and CES form are used in both the stochastic one-sector and two-sector growth models. These technologies must therefore first be introduced and adequately described.

5.2 Neoclassical technologies and CES forms

The sector *technologies* are - in stochastic one-sector and two-sector dynamics - described by nonnegative smooth concave homogeneous production functions, $F_i(L_i, K_i), i = 1, 2$, with *constant returns* to scale in labor and capital,

$$Y_i = F_i(L_i, K_i) = L_i F_i(1, k_i) \equiv L_i f_i(k_i) \equiv L_i y_i, L_i \neq 0; F_i(0, 0) = 0 \quad (1)$$

where the function $f_i(k_i)$ is strictly concave and monotonically increasing in the capital-labor ratio $k_i \in [0, \infty)$, i.e.

$$\forall k_i > 0\colon \;\; f_i'(k_i) = df_i(k_i)/dk_i > 0, \quad f_i''(k_i) = d^2 f_i(k_i)/dk_i^2 < 0 \qquad (2)$$

The *sectorial output elasticities*, $\epsilon_{L_i}, \epsilon_{K_i}$, ϵ_i - with respect to marginal and proportional factor variation - are, cf. (1),

$$\begin{aligned}
\epsilon_{L_i} &\equiv E(Y_i, L_i) \equiv \frac{\partial Y_i}{\partial L_i}\frac{L_i}{Y_i} = \frac{MP_{L_i}}{AP_{L_i}} = 1 - \frac{k_i f_i'(k_i)}{f_i(k_i)} > 0,\; k_i \neq 0 &(3)\\
\epsilon_{K_i} &\equiv E(Y_i, K_i) \equiv \frac{\partial Y_i}{\partial K_i}\frac{K_i}{Y_i} = \frac{MP_{K_i}}{AP_{K_i}} = \frac{k_i f_i'(k_i)}{f_i(k_i)} = E(y_i, k_i) > 0 &(4)\\
\epsilon_i &\equiv \epsilon_{L_i} + \epsilon_{K_i} = 1. &(5)
\end{aligned}$$

At any point on the isoquants, the *marginal rates* of *technical substitution,* $\omega_i(k_i)$ are, by (2), positive *monotonic* functions,

$$\omega_i(k_i) = \frac{MP_{L_i}}{MP_{K_i}} = \frac{f_i(k_i)}{f_i'(k_i)} - k_i = \frac{\epsilon_{L_i}}{\epsilon_{K_i}} k_i > 0, \quad \forall k_i > 0. \qquad (6)$$

CES Production Functions

General CES forms of $F_i(L_i, K_i)$, (1),$\gamma_i > 0$, $0 < a_i < 1$, $\sigma_i > 0$ are

$$Y_i = F_i(L_i, K_i) = \gamma_i L_i^{1-a_i} K_i^{a_i} = L_i \gamma_i k_i^{a_i} \equiv L_i f_i(k_i) \qquad (7)$$

$$\begin{aligned}
Y_i &= F_i(L_i, K_i) = \gamma_i \left[(1-a_i) L_i^{\frac{\sigma_i - 1}{\sigma_i}} + a_i K_i^{\frac{\sigma_i - 1}{\sigma_i}}\right]^{\frac{\sigma_i}{\sigma_i - 1}} &(8)\\
&= L_i \gamma_i \left[(1-a_i) + a_i k_i^{(\sigma_i - 1)/\sigma_i}\right]^{\sigma_i/(\sigma_i - 1)} \equiv L_i f_i(k_i) &(9)\\
f_i'(k_i) &= \gamma_i a_i k_i^{a_i - 1},\; f_i'(k_i) = \gamma_i a_i \left[a_i + (1-a_i) k_i^{-(\sigma_i - 1)/\sigma_i}\right]^{1/(\sigma_i - 1)} &(10)
\end{aligned}$$

The *limits* of $f_i(k_i)$ and $f_i'(k_i)$ become, ($\forall i : \sigma_i \gtrless 1 \Rightarrow a_i^{\sigma_i/(\sigma_i - 1)} \lesseqgtr 1$),

$$\sigma_i < 1 : \begin{cases} \lim\limits_{k_i \to 0} f_i(k_i) = 0, & \lim\limits_{k_i \to \infty} f_i(k_i) = \gamma_i (1-a_i)^{\frac{\sigma_i}{\sigma_i - 1}} \\ \lim\limits_{k_i \to 0} f_i'(k_i) = \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}}, & \lim\limits_{k_i \to \infty} f_i'(k_i) = 0 \end{cases} \qquad (11)$$

$$\sigma_i = 1 : \begin{cases} \lim\limits_{k_i \to 0} f_i(k_i) = 0, & \lim\limits_{k_i \to \infty} f_i(k_i) = \infty \\ \lim\limits_{k_i \to 0} f_i'(k_i) = \infty, & \lim\limits_{k_i \to \infty} f_i'(k_i) = 0 \end{cases} \qquad (12)$$

$$\sigma_i > 1 \colon \begin{cases} \lim\limits_{k_i \to 0} f_i(k_i) = \gamma_i (1-a_i)^{\frac{\sigma_i}{\sigma_i - 1}}, & \lim\limits_{k_i \to \infty} f_i(k_i) = \infty \\ \lim\limits_{k_i \to 0} f_i'(k_i) = \infty, & \lim\limits_{k_i \to \infty} f_i'(k_i) = \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}} \end{cases} \qquad (13)$$

For the CES technologies, the *monotonic* relations between marginal rates of substitution, factor proportions, and output elasticities are, cf. (8-10),

$$\omega_i = \frac{1-a_i}{a_i}\, k_i^{1/\sigma_i}, \qquad k_i = \frac{1}{c_i}\,[\omega_i]^{\sigma_i}, \quad c_i = \left[\frac{1-a_i}{a_i}\right]^{\sigma_i} \quad i = 1, 2. \tag{14}$$

$$\epsilon_{K_i} = \left[1 + \frac{1-a_i}{a_i} k_i^{\frac{1-\sigma_i}{\sigma_i}}\right]^{-1} = \frac{1}{1 + c_i \omega_i^{1-\sigma_i}}, \quad \epsilon_{L_i} = \frac{c_i \omega_i^{1-\sigma_i}}{1 + c_i \omega_i^{1-\sigma_i}} \tag{15}$$

With two-sector models and CES technologies, it is apparent from (14) that sectorial *factor* ratio ("intensity") *reversals* can only be avoided if and only if $\sigma_1 = \sigma_2$ and $a_1 \neq a_2$. Hence, with $\sigma_1 \neq \sigma_2$, there will be a *reversal* point, $(k_i, \omega_i) = (\bar{k}, \bar{\omega})$:

$$\bar{k} = \left[\frac{a_1(1-a_2)}{a_2(1-a_1)}\right]^{\frac{\sigma_1\sigma_2}{\sigma_2-\sigma_1}} = \left[\frac{c_2^{\sigma_1}}{c_1^{\sigma_2}}\right]^{\frac{1}{\sigma_2-\sigma_1}}, \quad \bar{\omega} = \left[\frac{c_2}{c_1}\right]^{\frac{1}{\sigma_2-\sigma_1}} \tag{16}$$

5.3 Stochastic one-sector growth models

5.3.1 Introduction of stochastic elements

The standard deterministic neoclassical one-sector growth model is described by the *ordinary differential equations* (ODE), cf. (1)

$$dL/dt \equiv \dot{L} = Ln \tag{17}$$

$$dK/dt \equiv \dot{K} = Lsf(k) - \delta K \tag{18}$$

$$dk/dt \equiv \dot{k} = sf(k) - (n+\delta)k \tag{19}$$

This general model becomes, with *uncertainty* (stochastic elements ϵ_i) in the growth rate of *labor* n, the gross *saving* rate s and the capital *depreciation* rate δ,

$$\dot{L} = L(n + \beta_1\, \epsilon_1) \tag{20}$$

$$\dot{K} = L(s + \phi_3(k)\, \epsilon_3) f(k) - (\delta + \beta_2\, \epsilon_2) K \tag{21}$$

where $\beta_i \geq 0$, and $(\epsilon_1, \epsilon_2, \epsilon_3)$ are "white noise" (stochastic process with a constant spectral density function), related to Wiener processes (w_1, w_2, w_3) with the correlation structure

$$d\langle w_i, w_j\rangle = \rho_{ij}\, dt, \quad \rho_{ii} = 1, \quad i, j = 1, 2, 3 \tag{22}$$

and $\langle w_i, w_j\rangle$ is the quadratic variation process for the components of the Wiener process, Karatzas and Shreve (1991), Øksendal (2003). For the formal connection between Wiener processes and "white noise", see Holden et. al (1996, chap.3).

The function ϕ_3 is, as later explained, here conveniently chosen (to avoid boundary problems at zero) as

$$\phi_3(k) = \beta_3 \tanh(\lambda_3 k), \quad k \in [0, \infty), \quad \phi_3(0) = 0, \quad \phi_3(\infty) = \beta_3 \tag{23}$$

For the labor and capital stock, the associated *stochastic differential equations* (SDE) to (20–21) are given by

$$\begin{aligned} dL &= Ln\, dt + L\beta_1 dw_1 && (24) \\ dK &= (sLf(k) - \delta K)\, dt - \beta_2 K\, dw_2 + Lf(k)\phi_3(k)\, dw_3 && (25) \end{aligned}$$

The *drift* and *diffusion* coefficients of the stochastic dynamic system (24–25) are *homogeneous* functions of *degree one* in the state variables L and K. The homogeneity of degree one allows us to reduce the two-dimensional stochastic system (24–25) to one-dimensional stochastic dynamics of the capital-labor ratio.

As an alternative to the uncertainty in the saving rate, we also consider *uncertainty* (stochastic element ϵ_4) in *technology*, more precisely, uncertainty (ϵ_4) in the *total productivity* parameter (γ) of the production function $f(k)$, i.e.

$$\dot{K} = Ls(\gamma + \phi_4(k)\, \epsilon_4)\, [f(k)/\gamma] - (\delta + \beta_2\, \epsilon_2)K \tag{26}$$

where ϕ_4 is similar to ϕ_3 defined in equation (23). Hence, with (26), the stochastic differential equation (25) is replaced by

$$dK = (sLf(k) - \delta K)\, dt - \beta_2 K\, dw_2 + Ls\, [f(k)/\gamma]\, \phi_4(k)\, dw_4. \tag{27}$$

The stochastic differential equations (24-25) or (24), (27) represent a *two-dimensional stochastic system*, driven by a *three-dimensional* Wiener process. For the purposes of *simulations*, i.e., computing the sample paths $L(t, \omega)$ and $K(t, \omega)$ or the ratio, $k(t, \omega) = K(t, \omega)/L(t, \omega)$, it is sufficient to use the equations (24), (25) or (24),(27). In fact, the SDE (24) is the well-known geometric Wiener proces. There is in general no closed form expression for the solutions (sample paths) for $K(t, \omega)$ or $k(t, \omega)$.

However, to precisely examine the *absorbing boundary* conditions and *stationarity* conditions for the diffusion process, $k(t)$, it is necessary to obtain an analytical expression for $k(t)$ as given by a particular one-dimensional Wiener proces. Fortunately, as both drift and diffusion coefficients are homogenous functions of degree one in K and L, this allows us to *analytically* describe $k(t)$ as a *one-dimensional* SDE, driven by a *one-dimensional* Wiener process, where the relevant drift coefficient and diffusion coefficient now need to be exactly determined, cf. Jensen and Wang (1999, Lemma 1).

5.3.2 The SDE of the capital-labor ratio

Theorem 1. *The stochastic neoclassical dynamics for the capital-labor ratio $k(t)$ of (24-25) is a diffusion process given by the SDE,*

$$dk = -(K/L^2)\,dL + (1/L)\,dK + (K/L^3)\,dL^2 - (1/L^2)\,dLdK \tag{28}$$

$$= \left\{[s - \rho_{13}\beta_1\phi_3(k)]f(k) - \left[n + \delta - (\beta_1^2 + \rho_{12}\beta_1\beta_2)\right]k\right\}\,dt \tag{29}$$
$$- \beta_1 k\,dw_1 - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3$$

The SDE (29) can in its domain be given in the compact form,

$$dk = a(k)\,dt + b(k)\,dw, \qquad k(t) \in (0, \infty) \tag{30}$$

with the drift coefficient,

$$a(k) = \bar{s}(k)f(k) - \Theta k \tag{31}$$

$$\bar{s}(k) = s - \rho_{13}\beta_1\phi_3(k) \qquad \Theta = n + \delta - (\beta_1^2 + \rho_{12}\beta_1\beta_2) \tag{32}$$

and the diffusion coefficient,

$$b^2(k) = \beta^2 k^2 + \phi_3(k)^2 f(k)^2 - \rho\phi_3(k)f(k)k \tag{33}$$

$$\beta^2 = \beta_1^2 + \beta_2^2 + 2\rho_{12}\beta_1\beta_2 \qquad \rho = 2(\rho_{13}\beta_1 + \rho_{23}\beta_2). \tag{34}$$

PROOF: Ito's Lemma: Let $X(t) \in \mathbb{R}^n$ be a general diffusion process, and if $F(X)$ is an arbitrary C^2 map from $\mathbb{R}^n \to \mathbb{R}$, then

$$dF(X) = F_x^T dX + 1/2 dX^T F_{xx} dX \tag{35}$$

i.e. $F(X)$, determined by diffusion process $X(t)$, is again a diffusion process, where F_x represent the partial derivatives with respect to x of the function $F(x)$, and F_{xx} represents the Hessian matrix of the function $F(x)$, and where, $(dw_i)^2 = dt\ \forall i$; $dw_i \cdot dw_j = 0$ for $i \neq j$; $(dt)^2 = 0$; $dt \cdot dw_i = 0$. Hence with,

$$X = (L, K)^T, \quad dX = (dL, dK)^T, \qquad F(X) = K/L \equiv k, \tag{36}$$

$$F_X = \begin{pmatrix} \frac{\partial F}{\partial L} \\ \frac{\partial F}{\partial K} \end{pmatrix} = \begin{pmatrix} -\frac{K}{L^2} \\ \frac{1}{L} \end{pmatrix}, \; F_{XX} = \begin{pmatrix} \frac{\partial^2 F}{\partial L^2} & \frac{\partial^2 F}{\partial L \partial K} \\ \frac{\partial^2 F}{\partial K \partial L} & \frac{\partial^2 F}{\partial K^2} \end{pmatrix} = \begin{pmatrix} \frac{2K}{L^2} & \frac{-1}{L^2} \\ \frac{-1}{L^2} & 0 \end{pmatrix} \tag{37}$$

we get, cf. (35–37),

$$dk = -\frac{K}{L^2}\,dL + \frac{1}{L}\,dK + \frac{1}{2}\left(\frac{2K}{L}\,dL^2 - 2\frac{1}{L^2}\,dLdK + 0\,dK^2\right) \tag{38}$$

$$= -\frac{K}{L^2}\,dL + \frac{1}{L}\,dK + \frac{K}{L^3}\,dL^2 - \frac{1}{L^2}\,dLdK \tag{39}$$

which is (28).
Inserting (24) and (25) into (39) gives,

$$\begin{aligned}
dk &= -\frac{K}{L^2}L(n\,dt + \beta_1\,dw_1) + \frac{1}{L}L[\{sf(k) - \delta k\}\ dt - \beta_2 k\,dw_2 \\
&\quad + \phi_3(k)f(k)\,dw_3] + \frac{K}{L^3}L^2(n\,dt + \beta_1\,dw_1)^2 \\
&\quad - \frac{1}{L^2}\,L(n\,dt + \beta_1\,dw_1)L[\{sf(k) - \delta k\}\ dt \\
&\quad - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3] \\
&= -k(n\,dt + \beta_1\,dw_1) + \{sf(k) - \delta k\}\ dt - \beta_2 k\,dw_2 \\
&\quad + \phi_3(k)f(k)\,dw_3 + k(n\,dt + \beta_1\,dw_1)^2 - (n\,dt + \beta_1\,dw_1) \\
&\quad \times [\{sf(k) - \delta k\}\ dt - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3] \\
&= -nk\,dt - \beta_1 k\,dw_1 + sf(k)\,dt - \delta k\,dt - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3 \\
&\quad + k\beta_1^2\,dt - \beta_1\,dw_1\,[-\beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3] \\
&= -nk\,dt - \beta_1 k\,dw_1 + sf(k)\,dt - \delta k\,dt - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3 \\
&\quad + k\beta_1^2\,dt + \rho_{12}\beta_1\beta_2 k\,dt - \rho_{13}\beta_1\phi_3(k)f(k)\,dt \\
&= \left\{sf(k) - \rho_{13}\beta_1\phi_3(k)f(k) - nk - \delta k + \beta_1^2 k + \rho_{12}\beta_1\beta_2)k\right\}\ dt \\
&\quad - \beta_1 k\,dw_1 - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3
\end{aligned} \tag{40}$$

which establishes (29).

Finally, using Levy's characterization, the local martingale term, $-\beta_1 k\,dw_1 - \beta_2 k\,dw_2 + \phi_3(k)f(k)\,dw_3$, can be simplified to $b(k)\,dw$, where w is a new one-dimensional Wiener process and the diffusion coefficient $b(k)$ can be calculated by determining the quadratic variation of : $-\beta_1 k\, dw_1 - \beta_2 k\, dw_2 + \phi_3(k) f(k)\, dw_3$. Hence, we get

$$\begin{aligned}
b(k)\, dw &\equiv -\beta_1 k\, dw_1 - \beta_2 k\, dw_2 + \phi_3(k) f(k)\, dw_3 \\
b^2(k)\, dt &\equiv [-\beta_1 k\, dw_1 - \beta_2 k\, dw_2 + \phi_3(k) f(k)\, dw_3]^2 \\
b^2(k)\, dt &= \beta_1^2 k^2\, dt + \beta_2^2 k^2\, dt + \phi_3(k)^2 f(k)^2\, dt + 2\beta_1\beta_2 k^2\, [dw_1, dw_2] \\
&\quad - 2\beta_1 k \phi_3(k) f(k)\, [dw_1, dw_3] - 2\beta_2 k \phi_3(k) f(k)\, [dw_2, dw_3] \\
b^2(k)\, dt &= \beta_1^2 k^2\, dt + \beta_2^2 k^2\, dt + \phi_3(k)^2 f(k)^2\, dt + 2\rho_{12}\beta_1\beta_2 k^2\, dt \\
&\quad - \rho_{13} 2\beta_1 k \phi_3(k) f(k)\, dt - 2\rho_{23}\beta_2 k \phi_3(k) f(k)\, dt \\
b^2(k) &= (\beta_1^2 + \beta_2^2 + 2\rho_{12}\beta_1\beta_2) k^2 + \phi_3(k)^2 f(k)^2 \qquad (41) \\
&\quad - 2(\rho_{13}\beta_1 + \rho_{23}\beta_2)\phi_3(k) f(k) k
\end{aligned}$$

which is succinctly summarized in (33) together with (34). □

The *sample path* (trajectory) of the process $k(t)$, (30–34), is formally given by

$$k(t;\omega) = k(0) + \int_0^t a(k[u;\omega])\, du + \int_0^t b(k[u;\omega])\, dw_u(\omega) \qquad (42)$$

where $k(0)$ is a fixed initial condition and ω symbolizes a particular realization of the Wiener process. The sample path (42) is in this paper approximated by the Euler scheme, Kloeden and Platen (1995).

Remark 1. Note that correlation ρ_{23} does not enter the drift coefficient in (31–32), because the coefficient of dK^2 is zero, cf. (37–38). Furthermore, note that our introduction of uncertainties (24–25) implies that the *deterministic* accumulation *parameters*, s, n, δ, only appear in the *drift coefficient* , (31–32), but not in the diffusion coefficient, (33–34). Some diffusion parameters β_i, ρ_{ij},however, may appear in the drift coefficient, (31–32). ▽

From the derived diffusion process of the capital-labor ratio $k(t)$ in **Theorem 1**, we can now analyze the *evolution* of the one-sector economy, with emphasis on long-run behavior (asymptotic properties). The drift and diffusion coefficients, however, *govern* the evolution only at *interior* points of the state space. To fully define a diffusion process, the behavior at any *boundary* points requires *separate* specification. For our purposes of studying the long-run evolution of nontrivial states, careful examination is needed of the *conditions* that will make the *boundaries*, $k = 0$ or $k = \infty$, *inaccessible* for any *finite* time ($t < \infty$). If $a(0) = 0$ and $b(0) = 0$ – as is often seen, cf. (30–34) and (11–13) – then $k(t) = 0$ is an *absorbing* boundary, i.e., the sample paths $k(t)$ remain at the zero position, once it is attained. Even if $a(0) \neq 0$, or $b(0) \neq 0$, and hence $k = 0$ is not an absorbing state, we cannot admit negative state values of $k(t)$, i.e., a *viable* (working) *economic* diffusion model must require that the *inaccessibility* of the boundary state $k = 0$ is ensured by imposing sufficient *parameter restrictions* on the actual drift and diffusion coefficients.

As the incremental Wiener processes $dw_i(t) \in (-\infty, \infty)$ may occasionally take on very large negative values, the drift and diffusion coefficients must indeed be carefully studied to prevent the random variable $k(t)$ from hitting the lower boundary, $k = 0$.

5.4 Boundaries, steady-state, and convergence

5.4.1 Terminology, concepts and definitions

Let the transition probability in case of a *one-dimensional* stochastic process $X(t)$ be denoted

$$P(x, t; x_0, t_0) = \Pr[X(t) \leq x \mid X(t_0) = x_0] \tag{43}$$

where $X(t)$ is the state of the process at instant t. The transition probability distribution $P(x, t; x_0, t_0)$ is assumed to have a probability density function $p(x, t; x_0, t_0)$, defined everywhere.

Boundary conditions. In terms of notation in (30), we define in the one-dimensional case the following indefinite integrals (functions),

$$J(x) = \int_{x_0}^{x} \frac{a(u)}{b^2(u)} du; \quad \mathfrak{s}(x) = \exp\{-2J(x)\},\ \mathcal{S}(x) = \int_{x_0}^{x} \mathfrak{s}(u) du \tag{44}$$

$$\mathfrak{m}(x) = \frac{\exp\{2J(x)\}}{b^2(x)} = \frac{1}{b^2(x)\mathfrak{s}(x)},\ \mathcal{M}(x) = \int_{x_0}^{x} \mathfrak{m}(u) du \tag{45}$$

The functions $\mathfrak{s}(x)$, $\mathcal{S}(x)$, $\mathfrak{m}(x)$, and $\mathcal{M}(x)$ are called, respectively, the *scale density* function, the *scale* function, the *speed density* function, and the *speed measure* of the stochastic processes $X(t)$; cf. Karlin and Taylor (1981, p. 194–96, p. 229).

Inaccessible boundaries. Let the diffusion process $X(t)$ have two boundaries $r_1 < r_2$. *Sufficient* conditions: The boundaries r_1 and r_2 are inaccessible, if

$$\begin{aligned}&\forall x_0 \in [r_1, r_2], \mathcal{S}(r_1) = -\infty; \quad \mathcal{S}(r_2) = +\infty; \quad \text{equivalently,}\\ &\mathfrak{s}(x) \text{ is not integrable on the closed interval } [r_i, x_0]\end{aligned} \tag{46}$$

or if

$$\mathcal{S}(r_i) = \lim_{x \to r_i} \mathcal{S}(x) = \lim_{x \to r_i} \int_{x_0}^{x} \mathfrak{s}(x)dx = \mp\infty; \quad i = 1, 2 \tag{47}$$

The *necessary* and *sufficient* condition is

$$\Sigma(r_i) \equiv \int_{x_0}^{r_i} [\mathcal{S}(r_i) - \mathcal{S}(x)]\mathfrak{m}(x)dx = +\infty; \quad i = 1, 2 \tag{48}$$

Existence of steady-state distribution. A *time-invariant* distribution function $P(x)$ exists if and only if

$$\mathcal{S}(r_i) = \mp\infty \text{ and } \mathcal{M}(x) \text{ is finite at } r_i, \text{ i.e. } |\mathcal{M}(r_i)| < \infty; \ i = 1, 2 \tag{49}$$

The existence of steady-state distribution $P(x)$ – implying inaccessible boundaries – also implies the *convergence* of the nonstationary distribution functions $P(x, t)$ towards $P(x)$ as $t \to \infty$.

Existence of steady-state density function. A *time-invariant* probability density function $p(x)$ exists if and only if the *speed* density $\mathfrak{m}(x)$ satisfies

$$\textstyle\int_{r_1}^{r_2} \mathfrak{m}(x)dx < \infty, \quad p(x) = m\mathfrak{m}(x), \quad \int_{r_1}^{r_2} p(x)dx = 1 \tag{50}$$

where m is the normalizing constant.

The conditions and formulas above can be applied directly when we have the *same* stochastic differential equation (drift and diffusion coefficients) for the *whole interval* of x. For more details, see Karlin and Taylor (1981) and Mandl (1968).

Remark 2. It is well-known, (Karlin and Taylor, 1981, p. 359), that the solution (Ito-integral) to the stochastic differential equation (24) is the geometric Wiener process with the *continuous sample paths* (stochastic trajectories, realizations),

$$L(t) = L_0 \exp\{(n - \frac{1}{2}\beta_1^2)t + \beta_1 w_1(t)\}; \ 2n \gtrless \beta_1^2 \colon \lim_{t \to \infty} L(t) = \begin{cases} \infty \\ 0 \end{cases} \tag{51}$$

Moreover, the *boundaries*, zero or infinity, are *inaccessible*, as the sample paths (51) *cannot* attain any of the two boundaries in *finite* time. It is instructive to prove the latter statement as a prelude to the general procedure of proving inaccessibility. From (24) and (44), the *scale density* $\mathfrak{s}(L)$ and the *scale function* $\mathcal{S}(L)$ become, cf. (44),

$$\mathfrak{s}(L) = \exp\{-2 \int_{L_0}^{L} nL/(\beta_1^2 L^2)dL\} = (L/L_0)^{-2n/\beta_1^2} \tag{52}$$

$$\mathcal{S}(L) = \int_{L_0}^{L} \mathfrak{s}(L)dL = L_0/(1 - 2n/\beta_1^2)[(L/L_0)^{1-2n/\beta_1^2} - 1] \tag{53}$$

As $\mathcal{S}(0) = -\infty$ for $2n \geq \beta_1^2$, the latter is a *sufficient* parameter condition for the *inaccessibility* of the boundary: $k = 0$. But despite the finite $\mathcal{S}(0) = -L_0/(1 - 2n/\beta_1^2)$ for $2n < \beta_1^2$, the boundary $k = 0$ may still not be attainable.

The *speed density* $\mathfrak{m}(L)$ is, cf. (45),

$$\mathfrak{m}(L) = 2L^{2n/\beta_1^2}/(\beta_1^2 L^2) = (L_0^{2n/\beta_1^2}/\beta_1^2)L^{2n/\beta_1^2-2} \tag{54}$$

and we must now with (53)–(54) and (48) evaluate

$$\Sigma(0) = \int_0^{L_0} [\mathcal{S}(L) - \mathcal{S}(0)]\mathfrak{m}(L)dL = \frac{L_0^{4n/\beta_1^2}}{\beta_1^2(1 - 2n/\beta_1^2)} \int_0^{L_0} \frac{dL}{L} = \infty \tag{55}$$

Thus, $\Sigma(0) = +\infty$ says that it takes *infinite* time to reach zero boundary from any interior state, i.e., $L = 0$ is after all *inaccessible* for $2n < \beta_1^2$, as it *cannot* be attained in *finite* time. The same analysis can be applied to boundary $L = \infty$, which is neither attainable in finite time. From this examination of the *labor diffusion* process (24) – which has no steady-state distribution – it is clear that boundary problems for the capital-labor ratio diffusion are essentially due to boundary problems associated with the capital stock diffusion process. $\bigtriangledown$

5.4.2 Boundary conditions – neoclassical growth models

Labor growth and capital depreciation rates are uncertain

From (30–34) with $\beta_3 = 0$, and the CES, $f = f_i$, (7–9), we have,

$$\sigma = 1, \quad dk = \{s\gamma k^a - \Theta k\}dt - \beta k dw, \tag{56}$$

$$\sigma \neq 1, \quad dk = \{s\gamma[(1-a) + ak^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} - \Theta k\}\, dt - \beta k\, dw. \tag{57}$$

Theorem 2. *The sufficient conditions for the diffusion processes of one-sector neoclassical growth models to have inaccessible boundaries – with CES technologies, (56–57), and uncertainties in both labor growth and capital depreciation – are:*

$$\sigma < 1\colon \begin{cases} k = 0 \ : \ 2(n + \delta - s\gamma a^{\sigma/(\sigma-1)}) \leq \beta_1^2 - \beta_2^2 \\ k = \infty\colon \ 2(n+\delta) \geq \beta_1^2 - \beta_2^2 \end{cases} \tag{58}$$

$$\sigma = 1\colon \begin{cases} k = 0 \ : \ \textit{always inaccessible} \\ k = \infty\colon \ 2\Theta + \beta^2 \geq 0 \Leftrightarrow 2(n+\delta) \geq \beta_1^2 - \beta_2^2 \end{cases} \tag{59}$$

$$\sigma > 1\colon \begin{cases} k = 0 \ : \ \textit{always inaccessible} \\ k = \infty\colon \ 2(n + \delta - s\gamma a^{\sigma/(\sigma-1)}) \geq \beta_1^2 - \beta_2^2 \end{cases} \tag{60}$$

The parametric conditions, (58)–(60), with strict inequalities, ensure the existence and the long-run convergence of the stochastic capital-labor ratio $k(t)$ to a time-invariant (steady-state) probability distribution $P(k)$.

PROOF: $\sigma = 1$: For an arbitrary $k_0 \in (0, \infty)$, the scale density function is given by, cf. (44), (56),

$$\mathfrak{s}(k) = \exp\{-2\int_{k_0}^{k} \frac{s\gamma k^a - \Theta k}{\beta^2 k^2} dk\}, \quad 0 < k \leq \infty \tag{61}$$

and, by simple integration, we get

$$\mathfrak{s}(k) = (k/k_0)^{2(\Theta/\beta^2)} \exp\{\frac{2s\gamma}{(1-a)\beta^2}(k^{-(1-a)} - k_0^{-(1-a)})\} \tag{62}$$

From (62), we have,

$$\mathcal{S}(0) = \int_{k_0}^{0} \mathfrak{s}(k)dk \equiv m_0 \int_{k_0}^{0} k^{2(\Theta/\beta^2)} \exp\{\frac{2s\gamma}{(1-a)\beta^2} k^{-(1-a)}\}\, dk \tag{63}$$

Since $1-a > 0$ and $2s\gamma/[(1-a)\beta^2] > 0$, cf. (7), the exponential term in (63) will dominate and explode for $k \to 0$, and hence $\mathcal{S}(0)$ diverges, i.e., $\mathcal{S}(0) = -\infty$. Thus, the *lower boundary* $k = 0$ is always *inaccessible*, *irrespective* of the size of the drift and diffusion parameters.

$$\mathcal{S}(\infty) = \int_{k_0}^{\infty} \mathfrak{s}(k)dk \equiv m_0 \int_{k_0}^{\infty} k^{2(\Theta/\beta^2)} \exp\{\frac{2s\gamma}{(1-a)\beta^2} k^{-(1-a)}\}dk \tag{64}$$

Since $1-a > 0$, cf. (7), the divergence of $\mathcal{S}(\infty)$ here only depends on the polynomial term in (64) with the exponent $2(\Theta/\beta^2)$. Hence divergence of $\mathcal{S}(\infty)$ requires that $2(\Theta/\beta^2) \geq -1$, or, equivalently, $2(n+\delta) \geq \beta_1^2 - \beta_2^2$. Hence, the *upper* boundary $k = \infty$ is *inaccessible* by imposing the parameter restriction stated in (59).

$\sigma \neq 1$: From (57), we have the expressions, cf. (61),

$$\mathfrak{s}(k) = \exp\{-2\int_{k_0}^{k} \frac{s\gamma[(1-a) + ak^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} - \Theta k}{\beta^2 k^2}\, dk\} \tag{65}$$

$\sigma < 1$: From (65) and (11), we have,

$$\mathcal{S}(0) \;=\; \lim_{k\to 0}\int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{0} (k/k_0)^{-2[s\gamma a^{\sigma/(\sigma-1)} - \Theta]/\beta^2}\, dk \tag{66}$$

Hence, it follows from (66), that the divergence of $\mathcal{S}(0)$ to $-\infty$ requires that the exponent must be less than or equal to -1, or equivalently, $2s\gamma a^{\sigma/(\sigma-1)} \geq 2\Theta + \beta^2$, which is the lower boundary condition in (58). From (65) and (11), we have,

$$\mathcal{S}(\infty) = \lim_{k\to\infty} \int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{\infty} (\frac{k}{k_0})^{\frac{2\Theta}{\beta^2}} \exp\{\frac{2s\gamma(1-a)^{\frac{\sigma}{\sigma-1}}}{\beta^2}(\frac{1}{k} - \frac{1}{k_0})\}\, dk \tag{67}$$

The polynomial term in (67) decides the divergence of $\mathcal{S}(\infty)$; it diverges to $+\infty$ if the exponent $2\Theta/\beta^2 \geq -1$, which gives the upper inaccessibility condition in (58).

$\sigma > 1$: From (65) and (13), we have,

$$\mathcal{S}(0) = \lim_{k\to 0} \int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{0} (\frac{k}{k_0})^{\frac{2\Theta}{\beta^2}} \exp\{\frac{2s\gamma(1-a)^{\frac{\sigma}{\sigma-1}}}{\beta^2}(\frac{1}{k} - \frac{1}{k_0})\}\, dk \tag{68}$$

The exponential term in (68) will always explode for $k \to 0$. Hence, $\mathcal{S}(0)$ is diverging, and accordingly, $k = 0$ is *inaccessible*, *irrespective* of parameters restrictions.

From (65) and (13), we have,

$$\mathcal{S}(\infty) = \lim_{k\to\infty} \int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{\infty} (k/k_0)^{-2[s\gamma a^{\sigma/(\sigma-1)} - \Theta]/\beta^2}\, dk \tag{69}$$

$\mathcal{S}(\infty)$ is divergent, if and only if the *exponent* of the polynomial in (69) is larger than or equal to -1, which gives the parametric *inaccessibility* restriction as stated in (60). □

Remark 3. Corresponding to the CD case, (59), Bourguignon (1974 , pp. 153–54), gave the upper-boundary inaccessibility condition as

$$2b/c \geq -1 \quad \Leftrightarrow \quad 2(n+\delta) \geq \beta_1^2 - \beta_2^2 - \beta_1\beta_2\rho_{12} \tag{70}$$

which differs from our simpler expression in (59). The result (70) is due a "misprint" in his formula for dk (p. 146), equivalent to our (38–39). Thus, we observe that, in contrast to his result, (70), the *correlation* ρ_{12} has *no implication* for the *inaccessibility* of the upper-boundary. Incidentally, note that ρ_{12} does not enter the boundary condition for the CES cases, (58), (60).

By the way, LHS of the boundary conditions (58–60) represent with $\beta_1 = \beta_2 = 0$, the necessary and sufficient conditions for the existence of a non-trivial (non-zero and finite) deterministic steady state. Evidently, $\beta_1 > 0$ makes it easier to avoid the trivial boundary $k = 0$, but more likely to explode. Note that $\beta_2 > 0$ makes it easier to avoid explosion, but more likely to hit the boundary $k = 0$. The economic-mathematical intuition of such $\beta_i > 0$ effects is left to the reader. $\bigtriangledown$

The saving rate is uncertain

It was seen in (30–34) that the *uncertainty* in *saving* behaviour will always introduce *nonlinear* terms in the *diffusion* coefficient. Inaccessible boundaries here raise conditions that clash with common, deterministic, dynamic regularity properties. By (30–34) with $\beta_1 = \beta_2 = 0$, $\lambda_3 = \infty$, $\phi_3(k) = \beta_3$, cf. (23), we get

$$\sigma = 1, \quad dk = \{s\gamma k^a - (n+\delta)k\}dt + \beta_3 \gamma k^a dw, \tag{71}$$

$$\sigma \neq 1, \quad dk = \{sf(k) - (n+\delta)k\}\, dt - \beta_3 f(k)\, dw, \;\; f(k) : (9) \tag{72}$$

Theorem 3. *The diffusion process (30–34) with CD or CES functions, (7), (9), and uncertainty only in the saving rate -* $\beta_1 = \beta_2 = 0$, $\beta_3 \neq 0$ *- will have boundary properties and sufficient inaccessibility conditions as follows,*

$$\sigma < 1 \colon \begin{cases} k = 0 \;\; : \;\; 2[s\gamma a^{\sigma/(\sigma-1)} - (n+\delta)] \geq [\beta_3 \gamma a^{\sigma/(\sigma-1)}]^2 \\ k = \infty \colon \;\; \textit{always inaccessible} \end{cases} \tag{73}$$

$$\sigma = 1 \colon \begin{cases} k = 0 \;\; : \;\; \textit{inaccessible, if } a > \frac{1}{2};\ \textit{poss.access. if } a < \frac{1}{2} \\ k = \infty \colon \;\; \textit{always inaccessible} \end{cases} \tag{74}$$

$$\sigma > 1 \colon \begin{cases} k = 0 \;\; : \;\; \textit{possibly accessible} \\ k = \infty \colon \;\; 2[s\gamma a^{\sigma/(\sigma-1)} - (n+\delta)] \leq [\beta_3 \gamma a^{\sigma/(\sigma-1)}]^2 \end{cases} \tag{75}$$

Proof: **CD case.** Applying (44) to (71) gives,

$$\mathfrak{s}(k) = \exp\{\frac{(n+\delta)(k^{2(1-a)} + k_0^{2(1-a)}) - 2s\gamma(k^{1-a} - k_0^{1-a})}{\beta_3^2 \gamma^2 (1-a)}\} \tag{76}$$

From (44) and (76), we have,

$$\mathcal{S}(0) = \int_{k_0}^{0} \mathfrak{s}(k)dk \equiv m_0 \int_{k_0}^{0} \exp\{\frac{(n+\delta)k^{2(1-a)} - 2s\gamma k^{1-a}}{\beta_3^2 \gamma^2 (1-a)}\}dk \tag{77}$$

Since $1 - a > 0$, $\mathcal{S}(0)$ will *converge*, and, accordingly, $k = 0$ may possibly be accessible.

To decide whether $k = 0$ is in fact inaccessible, we must calculate $\Sigma(0)$. In the CD case, we have, for the lower boundary $k = 0$, cf. (48),

$$\Sigma(0) = \int_{k_0}^{0} [\mathcal{S}(0) - \mathcal{S}(k)]\mathfrak{m}(k)dk \tag{78}$$

The limit of the integrand $\mathcal{S}(k)\mathfrak{m}(k)$ in (78) is, cf. (44)–(45), (77),

$$\begin{aligned}&\lim_{k\to 0} \mathcal{S}(k)\mathfrak{m}(k) = \\ &\lim_{k\to 0} \frac{m_0 \int_{k_0}^{k} \exp\{[\beta_3^2\gamma^2(1-a)]^{-1}[(n+\delta)k^{2(1-a)} - 2s\gamma k^{1-a}]\}dk}{\beta_3^2\gamma^2 k^{2a} \exp\{[\beta_3^2\gamma^2(1-a)]^{-1}[(n+\delta)k^{2(1-a)} - 2s\gamma k^{1-a}]\}}\end{aligned} \tag{79}$$

Since $[\beta_3^2\gamma^2(1-a)]^{-1} > 0$, the limit (79) *converges* iff $2(1-a) > 1$, i.e., $a < \frac{1}{2}$. Hence, with $a > \frac{1}{2}$, $\Sigma(0)$, (78) will be *divergent*, and thus the lower boundary is inaccessible.
From (44) and (76), we have,

$$\mathcal{S}(\infty) = \int_{k_0}^{\infty} \mathfrak{s}(k)dk \equiv m_0 \int_{k_0}^{\infty} \exp\{\frac{(n+\delta)k^{2(1-a)} - 2s\gamma k^{1-a}}{\beta_3^2\gamma^2(1-a)}\}dk \tag{80}$$

Since $1 - a > 0$, and $k^{2(1-a)}$ is the dominating term, $\mathcal{S}(\infty)$ will always diverge, i.e., $k = \infty$ is *inaccessible*.

CES case. Applying (44) to (72) gives

$$\mathfrak{s}(k) = \exp\{-2\int_{k_0}^{k} \frac{s\gamma[(1-a) + ak^{(\sigma-1)/\sigma}]^{\sigma/(\sigma-1)} - (n+\delta)k}{\beta_3^2\gamma^2[(1-a) + ak^{(\sigma-1)/\sigma}]^{2\sigma/(\sigma-1)}}dk\} \tag{81}$$

$\sigma < 1$: From (81), we have, for small k, cf. (11),

$$\begin{aligned}\mathcal{S}(0) &= \lim_{k\to 0} \int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{0} \exp\{-2\int_{k_0}^{k} \frac{s\gamma a^{\sigma/(\sigma-1)} - (n+\delta)}{\beta_3^2\gamma^2 a^{2\sigma/(\sigma-1)}} k^{-1}dk\}dk \\ &= \int_{k_0}^{0} (k/k_0)^{\frac{-2[s\gamma a^{\sigma/(\sigma-1)} - (n+\delta)]}{\beta_3^2\gamma^2 a^{2\sigma/(\sigma-1)}}} dk\end{aligned} \tag{82}$$

$\mathcal{S}(0)$ diverges if the exponent of k/k_0 is less than or equal to -1, which immediately gives the condition (73).
From (81), we have, for large k, cf. (11),

$$\begin{aligned}\mathcal{S}(\infty) &= \lim_{k\to\infty} \int_{k_0}^{k} \mathfrak{s}(k)dk = \int_{k_0}^{\infty} \exp\{-2\int_{k_0}^{k} \frac{s\gamma(1-a)^{\sigma/(\sigma-1)} - (n+\delta)k}{\beta_3^2\gamma^2(1-a)^{2\sigma/(\sigma-1)}}dk\}dk \\ &= \int_{k_0}^{\infty} \exp\{\frac{(n+\delta)(k^2 - k_0^2) - 2s\gamma(1-a)^{\sigma/(\sigma-1)}(k-k_0)]}{\beta_3^2\gamma^2(1-a)^{2\sigma/(\sigma-1)}}\}dk \\ &\equiv m_0^3 \int_{k_0}^{\infty} \exp\{\frac{n+\delta}{\beta_3^2\gamma^2(1-a)^{2\sigma/(\sigma-1)}}k^2 - \frac{2s}{\beta_3^2\gamma(1-a)^{\sigma/(\sigma-1)}}k\}dk\end{aligned} \tag{83}$$

As $1 - a > 0$, the constant denominators in (83) are positive, and since the k^2 term is the dominating term in the exponential expression, $\mathcal{S}(\infty)$ will always be divergent; hence, the upper boundary is always *inaccessible.*

$\sigma > 1$: From (81), we have, for small k, cf. (13),

$$\begin{aligned}\mathcal{S}(0) &= \lim_{k\to 0}\int_{k_0}^{k}\mathfrak{s}(k)dk = \int_{k_0}^{0}\exp\{-2\int_{k_0}^{k}\tfrac{s\gamma(1-a)^{\sigma/(\sigma-1)}-(n+\delta)k}{\beta_3^2\gamma^2(1-a)^{2\sigma/(\sigma-1)}}dk\}dk\\ &\equiv m_1^0\int_{k_0}^{0}\exp\{\tfrac{n+\delta}{\beta_3^2\gamma^2(1-a)^{2\sigma/(\sigma-1)}}k^2 - \tfrac{2s}{\beta_3^2\gamma(1-a)^{\sigma/(\sigma-1)}}k\}dk \qquad (84)\end{aligned}$$

Since $1 - a > 0$, $\mathcal{S}(0)$ will always converge, and hence $k = 0$ may possibly be accessible. Whether in fact $k = 0$ is attainable in finite time requires similar evaluations as shown above, cf. (78)–(79), (55). From (81), we have, for large k, cf. (13),

$$\begin{aligned}\mathcal{S}(\infty) &= \lim_{k\to\infty}\int_{k_0}^{k}\mathfrak{s}(k)dk = \int_{k_0}^{\infty}\exp\{-2\int_{k_0}^{k}\tfrac{s\gamma a^{\sigma/(\sigma-1)}-(n+\delta)}{\beta_3^2\gamma^2a^{2\sigma/(\sigma-1)}}k^{-1}dk\}\\ &= \int_{k_0}^{\infty}(k/k_0)^{\frac{-2[s\gamma a^{\sigma/(\sigma-1)}-(n+\delta)]}{\beta_3^2\gamma^2a^{2\sigma/(\sigma-1)}}}dk \qquad (85)\end{aligned}$$

$\mathcal{S}(\infty)$ diverges if the exponent of k/k_0 is larger than or equal to -1, which is equivalent to the upper inaccessibilty condition in (75). □

With uncertainty in the saving rate, the *drift* and *diffusion* coefficients in (20)–(21) now have *similar nonlinear elements*, that, if dominating, will prevent us from satisfying a sufficient lower inaccessibility condition, as the scale function $\mathcal{S}(k)$ at $k = 0$ is now finite, *whenever* $\sigma \geq 1$.

The factor accumulation process is likely to be much more severely affected (large volatility) by *uncertainty* about the *saving rate* than by uncertainties in labor growth and depreciation rates. The *lack* of any parametric *restrictions* preventing the *accessibility* of the *absorbing boundary* $k = 0$ (implosion,"economic collapse"), cf. (74), (75), represents a *critical* stochastic dynamic model complication for the system (30–34) and a mathematical issue to be adequately resolved below.

5.4.3 General parameter uncertainty and inaccessible boundaries

To dampen the impact of the Wiener process dw_3, near $k = 0$ in our (30–34), the random element ϵ_3 in the saving parameter must be *state-dependent*, and to preserve (24)–(25) as a *homogenous* stochastic *dynamic* system, the function $\phi_3(k)$ was chosen, cf. (23).

From economic reasons, the actual *shape* of $\phi_3(k)$ on the *domain* $k \in [0, \infty)$ is chosen as a monotonically *increasing* curve, but this curve should also – to avoid creating excessive saving parameter volatility – be *bounded* above by a *horizontal* asymptote. With these two stipulations upon relevant selections of $\phi_3(k)$, one choice might be the logistic (S-shaped) curve described by well-known exponential expression. But among the exponentials, a relevant and convenient choice of $\phi_3(k)$, with proper domain and range for our purposes, is (23).

Theorem 4. *The sufficient conditions for the general diffusion process (30–34) – with CES technologies and uncertainties in labor growth, capital depreciation, and saving rates – to have inaccessible lower and upper boundaries are:*

$$\begin{aligned}
&k = 0\,; \ \sigma < 1 \ : \quad 2(n+\delta - s\gamma a^{\sigma/(\sigma-1)}) \le \beta_1^2 - \beta_2^2 && (86)\\
&k = 0\,; \ \sigma \ge 1 \ : \quad \textit{Always inaccessible} && (87)\\
&k = \infty; \ \sigma \le 1 \ : \quad 2(n+\delta) \ge \beta_1^2 - \beta_2^2 && (88)\\
&k = \infty; \ \sigma > 1 \ : \quad 2(n+\delta - s\gamma a^{\sigma/(\sigma-1)}) \ge \beta_1^2 - \beta_2^2 - \Delta && (89)\\
&\qquad\qquad\qquad\quad \Delta \equiv [\beta_3\gamma a^{\sigma/(\sigma-1)}]^2 - 2\rho_{23}\beta_2\beta_3\gamma) a^{\sigma/(\sigma-1)}.
\end{aligned}$$

PROOF: The hyperbolic function $\phi_3(k) = \beta_3 \tanh(\lambda_3 k)$, (23), is

$$\phi_3(k) = \beta_3 \tanh(\lambda_3 k) = \beta_3(e^{\lambda_3 k} - e^{-\lambda_3 k})/(e^{\lambda_3 k} + e^{-\lambda_3 k}), \quad k \ge 0 \qquad (90)$$

It is well-known and easily verified from (90) that for

$$\begin{aligned}
&\text{small } k \ : \quad \phi_3(k) \sim \beta_3\lambda_3 k, \ \Leftrightarrow \ \phi_3(k)/\beta_3\lambda_3 k \to 1 \text{ as } k \to 0 && (91)\\
&\text{large } k \ : \quad \phi_3(k) \sim \beta_3, \qquad \Leftrightarrow \ \phi_3(k)/\beta_3 \to 1 \text{ as } k \to \infty && (92)
\end{aligned}$$

Lower boundary. $\sigma = 1$: With the CD production function (7), the scale density function $\mathfrak{s}(k)$ now becomes, cf. (44), (90),

$$\mathfrak{s}(k) = \exp\{-2\int_{k_0}^{k} \frac{[s - \rho_{13}\beta_1\phi_3(k)]\gamma k^a - \Theta k}{\beta^2 k^2 + \phi_3^2(k)\gamma^2 k^{2a} - \rho\phi_3(k)\gamma k^{(1+a)}}\, dk\} \qquad (93)$$

Since $a < 1$, the *dominating* term in the numerator and the denominator of (93) becomes, for small k, cf. (91),

$$\mathfrak{s}(k) \sim \exp\{-2\int_{k_0}^{k} \frac{s\gamma k^a}{\beta^2 k^2} dk\} = \exp\{\frac{2s\gamma}{(1-a)\beta^2}(k^{-(1-a)} - k_0^{-(1-a)})\} \quad (94)$$

Since $1-a > 0$, it is seen, from (94), that $\mathcal{S}(0) = \int_{k_0}^{0} \mathfrak{s}(k)dk$ is diverging at $k = 0$, cf. (63), and hence the lower boundary is inaccessible.

$\sigma < 1$: With the CES function (9), the scale function with the dominating terms becomes, cf. (9), (11), (91), (94),

$$\mathcal{S}(0) = \lim_{k \to 0} \int_{k_0}^{k} \mathfrak{s}(k) dk = \int_{k_0}^{0} (k/k_0)^{2[\Theta - s\gamma a^{\sigma/(\sigma-1)}]/\beta^2} dk \tag{95}$$

Hence, it follows from (95), that the divergence of $\mathcal{S}_1(0)$ requires that the exponent $2[\Theta - s\gamma a^{\sigma/(\sigma-1)}]/\beta^2 \leq -1$, which is the lower boundary condition in (86), cf. (66).

$\sigma > 1$: With the CES function (7), the scale function with the dominating terms becomes, cf. (13),

$$\mathcal{S}(0) = \lim_{k \to 0} \int_{k_0}^{k} \mathfrak{s}(k) dk = \lim_{k \to 0} \int_{k_0}^{0} \exp\{\frac{2s\gamma(1-a)^{\frac{\sigma}{\sigma-1}}}{\bar{b}^2}(k^{-1} - k_0^{-1})\} dk \tag{96}$$

where $\bar{b}^2 \equiv \beta^2 + \beta_3^2 \lambda_3^2 \gamma^2 (1-a)^{2\sigma/(\sigma-1)} - \rho\beta_3\lambda_3\gamma(1-a)^{\sigma/(\sigma-1)} > 0$. Since the parameter $2s\gamma(1-a)^{\sigma/(\sigma-1)}/\bar{b}^2$ in the exponential term is always positive, it seen by (96) that $\mathcal{S}(0)$ is always diverging; hence, $k = 0$ is inaccessible, irrespective of parameter restrictions, cf. (68).

Upper boundary. With the CD function (7), the scale density $\mathfrak{s}(k)$ becomes, cf. (44), (90),

$$\mathfrak{s}(k) = \exp\{-2 \int_{k_0}^{k} \frac{[s - \rho_{13}\beta_1\phi_3(k)]\gamma k^a - \Theta k}{\beta^2 k^2 + \phi_3^2(k)\gamma^2 k^{2a} - \rho\phi_3(k)\gamma k^{(1+a)}} dk\} \tag{97}$$

Since $a < 1$, the *dominating* term in the numerator and denominator of (97) becomes, for large k, cf. (92),

$$\mathfrak{s}(k) \sim \exp\{-2 \int_{k_0}^{k} \frac{-\Theta k}{\beta^2 k^2} dk\} = (k/k_0)^{2\Theta/\beta^2} \tag{98}$$

The divergence of $\mathcal{S}(\infty)$ from (98) is analogous to the result in (64), (59); hence, we have (88) for $\sigma = 1$.

$\sigma < 1$: With the CES function (9), the scale density $\mathfrak{s}(k)$ becomes, keeping the dominant terms for large k, cf. (97), (92),

$$\mathfrak{s}(k) \sim \exp\{-2 \int_{k_0}^{k} \frac{-\Theta k}{\beta^2 k^2} dk\} = (k/k_0)^{2\Theta/\beta^2} \tag{99}$$

which is the same as (98), and the divergence of $\mathcal{S}(\infty)$ is analogous to (67), (58).

$\sigma > 1$: From (97), (9), (13), (92), we have, for large k,

$$\begin{aligned}\mathfrak{s}(k) &\sim \exp\{-2\int_{k_0}^{k}\frac{(s-\rho_{13}\beta_1\beta_3)\gamma a^{\sigma/(\sigma-1)}k-\Theta k}{\beta^2k^2+\beta_3^2\gamma^2a^{2\sigma/(\sigma-1)}k^2-\rho\beta_3\gamma a^{\sigma/(\sigma-1)}k^2}dk\} \\ &= \exp\{-2\int_{k_0}^{k}(\bar{a}/\bar{b}^2)k^{-1}dk\} = (k/k_0)^{\bar{a}/\bar{b}^2} \qquad (100)\end{aligned}$$

$\bar{a} \equiv (s-\rho_{13}\beta_1\beta_3)\gamma a^{\frac{\sigma}{\sigma-1}} - \Theta$, $\bar{b}^2 \equiv \beta^2 + \beta_3^2\gamma^2 a^{\frac{\sigma}{\sigma-1}} - \rho\beta_3\gamma a^{\frac{\sigma}{\sigma-1}} > 0$. Now $\mathcal{S}(\infty)$ from (100) *diverges*, cf. the analogue (85) and (75), if the exponent: $\bar{a}/\bar{b}^2 \geq -1$. Rewriting the latter, using (32), (34), gives our condition (89), where $\Delta \equiv \bar{b}^2 - \beta^2 + 2\rho_{13}\beta_1\beta_3\gamma a^{\sigma/(\sigma-1)} = [\beta_3\gamma a^{\sigma/(\sigma-1)}]^2 - 2\rho_{23}\beta_2\beta_3\gamma a^{\sigma/(\sigma-1)}$. □

The *assumptions* about $\phi_3(k)$, (23) have *removed* the uncertainty in saving rates (21) entirely from the *lower boundary* problems with, $\sigma \geq 1$, cf. (74) and (75), because with (23), we now have that (87) holds, irrespective of the size of *any drift* and *diffusion* parameters. With $\sigma < 1$, proper parameter restrictions (86) can safeguard against attaining $k = 0$.

Thus, for *any substitution elasticity* of the CES technology, the *stochastic neoclassical growth model* of **Theorem 1**, (29–34), is made fully *workable without any boundary problems* (extinction, explosion).

5.4.4 Neoclassical SDE and asymptotic non-stationarity

The relaxation of the *sufficient* inaccessibility condition $\mathcal{S}(\infty) = \infty$ does not itself in the long run imply an explosion. Still, to avoid any risk of implosion, we want to keep the sufficient *condition* $\mathcal{S}(0) = -\infty$. But, together with a *finite* $\mathcal{S}(\infty)$, we have the following well-known implications (with probability one),

$$[\mathcal{S}(0) = -\infty \wedge \mathcal{S}(\infty) < \infty] \Rightarrow \lim_{t\to\infty} k(t) = \infty \Rightarrow \lim_{t\to\infty} \mathsf{E}[k(t)] = \infty \qquad (101)$$

Within our stochastic neoclassical growth model, a finite $\mathcal{S}(\infty)$ is simply equivalent to reversing the inequality in (88)–(89). For $\sigma \leq 1$, the reverse of (88) is $2(n+\delta) \leq \beta_1^2 - \beta_2^2$. The latter implies that $k(t) \to \infty$ as $t \to \infty$, but it is a pathological case, as the reversal of (88) also implies that $L(t) \to 0$ (although never reached in finite time). In short, no relevant stochastic endogenous growth is possible with $\sigma \leq 1$. Hence, as in the deterministic case, stochastic endogenous economic growth requires that the marginal product of capital is bounded below, i.e., $\sigma > 1$, cf. (13).

By reversing (89), the sufficient condition of growth becomes, cf. (101), (87)

$$\sigma > 1\colon \mathcal{S}(\infty) < \infty \Leftrightarrow s\gamma a^{\sigma/(\sigma-1)} \geq n + \delta + 1/2(-\beta_1^2 + \beta_2^2 + \Delta) \tag{102}$$

which is the *stochastic analogue* to the *deterministic* condition (with only $n+\delta$ on RHS) of endogenous (persistent) growth; see Jensen and Wang (1997, p. 93), Jensen and Larsen (1987).

We note from (102) that it is generally *more difficult* (higher *saving rates* are required) to *achieve persistent economic growth* per capita in the face of *uncertainty* – as $n - \frac{1}{2}\beta_1^2 > 0$ is now taken for granted in (102), and Δ is always positive when $\rho_{23} = 0$, cf. (89). Uncertainties in the *accumulation of capital*, (23), (25), ($\beta_2 \neq 0$, $\beta_3 \neq 0$) make the stochastic analogue (102) harder to satisfy.

The *rapidity* of stochastic growth is not directly seen by (102). However, with $\mathcal{S}(\infty) < \infty$, the stochastic differential equation (30) is, asymptotically,

$$\begin{aligned} dk \sim \bar{a}kdt + \bar{b}kdw \equiv\ & \{(s - \rho_{13}\beta_1\beta_3)\gamma a^{\sigma/(\sigma-1)} - \Theta\}k\,dt\ + \\ & [\beta^2 + \beta_3^2\gamma_2^2 a^{2/(\sigma-1)/\sigma} - \rho\beta_3\gamma a^{\sigma/(\sigma-1)}]^{1/2} k dw \end{aligned} \tag{103}$$

i.e., *geometric* Wiener *processes* with sample paths and expectations:

$$k(t) \sim k_0 \exp\{(\bar{a} - \bar{b}^2/2)t + \bar{b}w(t)\},\ 2\bar{a} > \bar{b}^2;\ \mathsf{E}[k(t)] \sim k_0 \exp\{\bar{a}t\} \tag{104}$$

It is easily verified that the exponential growth condition, $\bar{a} - \frac{1}{2}\bar{b}^2 > 0$ in (104), is equivalent to (102). Thus, the *stochastic condition* (102) is indeed the *analogue of deterministic* exponential per capita growth in the *neoclassical* growth model.

5.5 Explicit steady-state distribution with CD technologies

Having obtained the conditions for the *existence* of and convergence to a *steady-state* (time-invariant) *distribution*, cf. (59), we also want to obtain as a *benchmark* – with CD sector technologies – a *closed form* expression for the *time invariant probability density* function $p(k)$ and the *distribution function* $P(k)$ of the diffusion process (56).

It turns out that the benchmark *distribution* function $P(k)$ for the CD economy can be expressed by *gamma* $\Gamma(\alpha)$ and *incomplete gamma* functions $\Gamma(\alpha, x_0)$, which are generally defined, respectively, by the improper integrals,

$$\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1}e^{-x}dx, \quad \Gamma(\alpha, x_0) \equiv \int_{x_0}^\infty x^{\alpha-1}e^{-x}dx,\ \alpha > 0 \tag{105}$$

Theorem 5. *The time invariant (steady-state) distribution $P(k)$ for the stochastic process (56), cf. Theorem 1, will have a density function $p(k)$ if and only if:*

$$2\Theta + \beta^2 > 0 \quad \Leftrightarrow \quad 2(n+\delta) > \beta_1^2 - \beta_2^2 \tag{106}$$

With (106), the time invariant probability density function $p(k)$ is in closed form,

$$p(k) = c_0 k^{-2[1+\Theta/\beta^2]} \exp\{-ck^{-(1-a)}\}, \qquad 0 < k < \infty \tag{107}$$

where the constants c and c_0 are given by,

$$c = \frac{2s\gamma}{(1-a)\beta^2}, \quad c_0 = \frac{(1-a)c^\alpha}{\Gamma(\alpha)}, \quad \alpha = \frac{2\Theta + \beta^2}{(1-a)\beta^2} \tag{108}$$

With the stationary density function $p(k)$, (107), the stationary distribution function

$$P(k) = \int_0^k p(u)du$$

can be given by the incomplete gamma functions as,

$$P(k) = \frac{1}{\Gamma(\alpha)}\Gamma(\alpha, ck^{-(1-a)}) \tag{109}$$

The distribution $P(k)$, (109), will have finite first-order and second-order moments, $\mathsf{E}(k)$ and $\mathsf{E}(k^2)$ if and only if, respectively,

$$\Theta > 0 \quad \Leftrightarrow \quad n + \delta > \beta_1^2 + \rho_{12}\beta_1\beta_2 \tag{110}$$
$$2\Theta - \beta^2 > 0 \quad \Leftrightarrow \quad 2(n+\delta) > 3\beta_1^2 + 4\rho_{12}\beta_1\beta_2 + \beta_2^2 \tag{111}$$

With (110)–(111), the steady-state distribution $P(k)$ will have first-order and second-order moments given by,

$$\mathsf{E}(k) = \frac{\Gamma(\alpha^*)}{\Gamma(\alpha)} c^{(1-a)^{-1}}, \mathsf{E}(k^2) = \frac{\Gamma(\alpha^{**})}{\Gamma(\alpha)} c^{2(1-a)^{-1}}, \sigma^2 = \mathsf{E}(k^2) - [\mathsf{E}(k)]^2 \tag{112}$$

where

$$\alpha^* = \alpha - (1-a)^{-1} \qquad \textit{and} \qquad \alpha^{**} = \alpha - 2(1-a)^{-1} \tag{113}$$

and α was given by (108).

PROOF: For the stochastic system (56), the speed densities are, cf. (44)–(45)

$$\mathfrak{m}(k) = \frac{1}{\beta^2 k^2} \exp\{-2 \int_{\tilde{k}}^{k} \frac{s\gamma u^a - \Theta u}{\beta^2 u^2} du\}, \quad 0 < k < \infty \tag{114}$$

The constant $\tilde{k} > 0$ can be chosen arbitrarily, but for numerical purposes it is convenient to let $\tilde{k}$ be close to the mode of the distribution. After simple integration, we obtain,

$$m(k) = c_0 k^{-2[1+\Theta/\beta^2]} \exp\{-\frac{2s\gamma}{(1-a)\beta^2} k^{-(1-a)}\}. \tag{115}$$

Under the condition (106), the expression (115) is the density $p(k)$ given in (107), with the constants c and c_0 as stated in (108). With the condition (106), the integral in (115) can be normalized to a density. The normalizing constant c_0 is obtained by identifying that k^{a-1} is a Γ-distribution with position parameter c and scala parameter $1/\alpha$ given by (108).

The first-order and second-order moments are obtained from the density function as,

$$\mathsf{E}(k) \equiv \int_0^\infty kp(k)dk, \qquad \mathsf{E}(k^2) \equiv \int_0^\infty k^2 p(k)dk.$$

After several substitutions of variables and lengthy rearrangements, we get the expression for $\mathsf{E}(k)$ and $\mathsf{E}(k^2)$.
Finally, we observe that

$$\alpha^* > 0 \quad \Leftrightarrow \quad \Theta > 0, \qquad \alpha^{**} > 0 \quad \Leftrightarrow \quad 2\Theta > \beta^2 \tag{116}$$

which gives the moment existence restrictions (110–111), and Theorem 5 is established. □

5.6 Sample paths and asymptotic densities with CD and CES technologies

The steady value (κ) of the capital-labor ratio in a deterministic growth model (17–19) is given by

$$\dot{k} = 0 \quad \Leftrightarrow \quad k(t) = \kappa \quad \Leftrightarrow \quad f(\kappa)/\kappa = (n+\delta)/s \tag{117}$$

Table 1. Steady-state values (certainty equivalents) of one-sector growth models

Variables	CD ($\sigma = 1$)	CES ($\sigma \neq 1$)
capital-labor ratio: $K/L = \kappa$	$\left(\frac{\gamma s}{n+\delta}\right)^{\frac{1}{1-a}}$	$\left(\frac{1-a}{a}\right)^{\frac{\sigma}{1-\sigma}} \left[\frac{1}{a}\left(\frac{\gamma s}{n+\delta}\right)^{\frac{1-\sigma}{\sigma}} - 1\right]^{\frac{\sigma}{1-\sigma}}$
output labor ratio: $AP_L = f(\kappa)$	$\left(\frac{\gamma s}{n+\delta}\right)^{\frac{a}{1-a}}$	$\gamma(1-a)^{\frac{\sigma}{\sigma-1}} \left[1 - a\left(\frac{\gamma s}{n+\delta}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{1-\sigma}}$
marginal productivity: $MP_L(\kappa)$	$(1-a)\left(\frac{\gamma s}{n+\delta}\right)^{\frac{a}{1-a}}$	$\gamma(1-a)^{\frac{\sigma}{\sigma-1}} \left[1 - a\left(\frac{\gamma s}{n+\delta}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{1-\sigma}}$
output–capital ratio: $AP_K(\kappa),\ f(\kappa)/\kappa$	$\frac{n+\delta}{s}$	$\frac{n+\delta}{s}$
capital–output ratio: $K/Y,\ \kappa/f(\kappa)$	$\frac{s}{n+\delta}$	$\frac{s}{n+\delta}$
real interest rate: $MP_K(\kappa) = f'(\kappa)$	$\frac{a(n+\delta)}{s}$	$\gamma a \left(\frac{n+\delta}{\gamma s}\right)^{\frac{1}{\sigma}}$
wage–rental ratio: $\omega(\kappa)$	$\frac{1}{\gamma}\frac{1-a}{a}\left(\frac{\gamma s}{n+\delta}\right)^{\frac{1}{1-a}}$	$\left(\frac{1-a}{a}\right)^{\frac{\sigma}{1-\sigma}} \left[\frac{1}{a}\left(\frac{\gamma s}{n+\delta}\right)^{\frac{1-\sigma}{\sigma}} - 1\right]^{\frac{1}{1-\sigma}}$
capital share: $\epsilon_K(\kappa) = 1 - \epsilon_L(\kappa)$	a	$a\left(\frac{\gamma s}{n+\delta}\right)^{\frac{\sigma-1}{\sigma}}$
per capita consumption: $c_L(\kappa)$	$(1-s)f(\kappa)$	$(1-s)f(\kappa)$
per capita saving: $s_L(\kappa)$	$sf(\kappa)$	$sf(\kappa)$

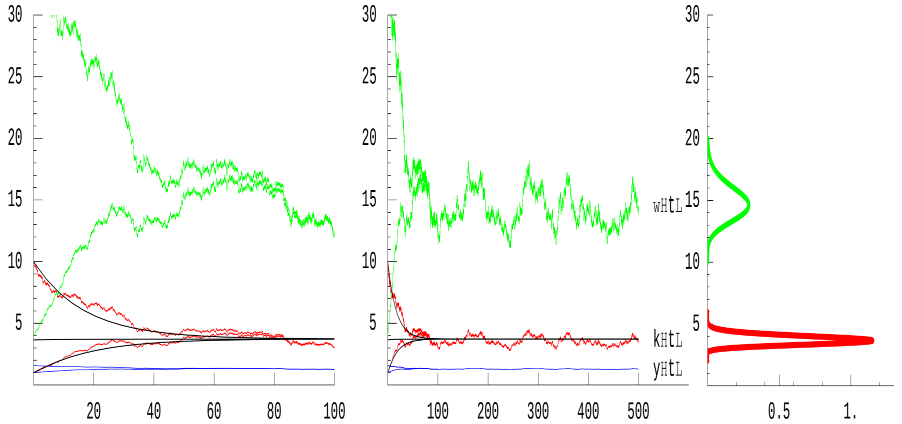

Figure 5.1: CD1

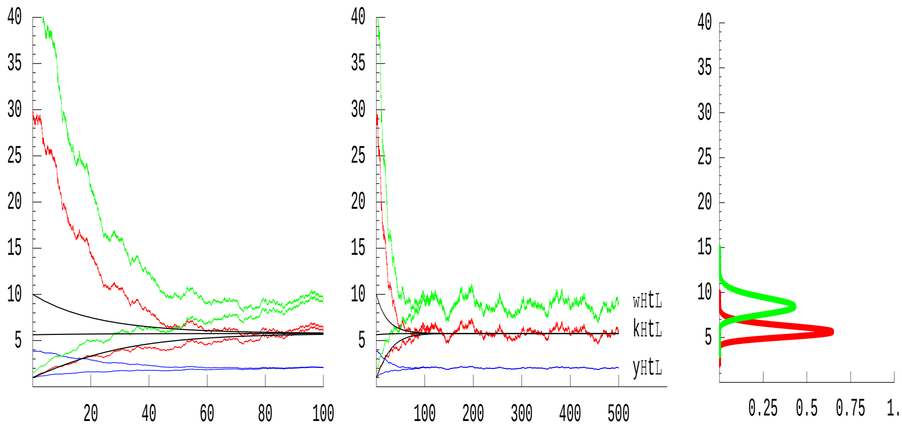

Figure 5.2: CD3a

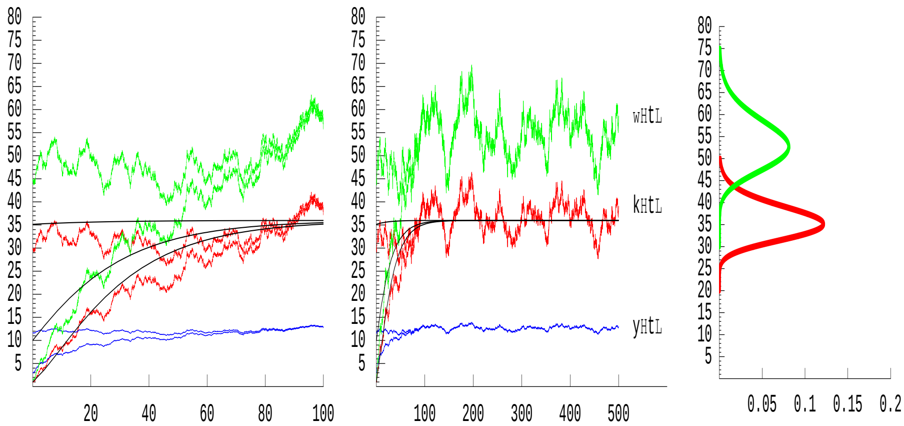

Figure 5.3: CD3b

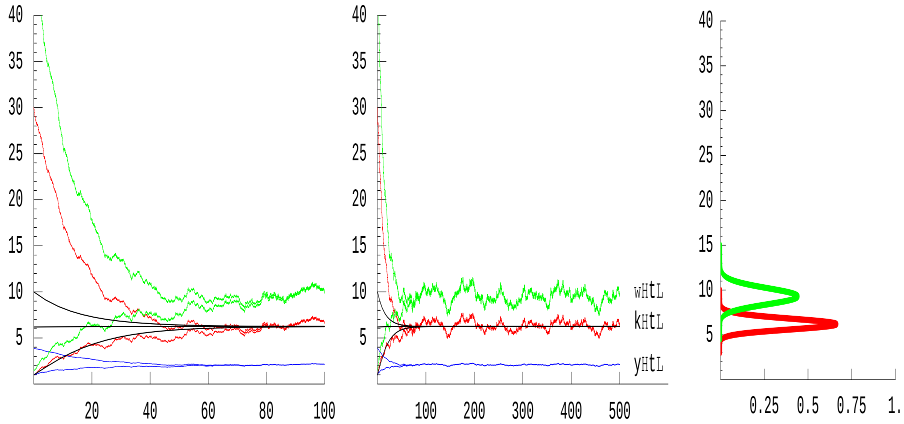

Figure 5.4: CD7

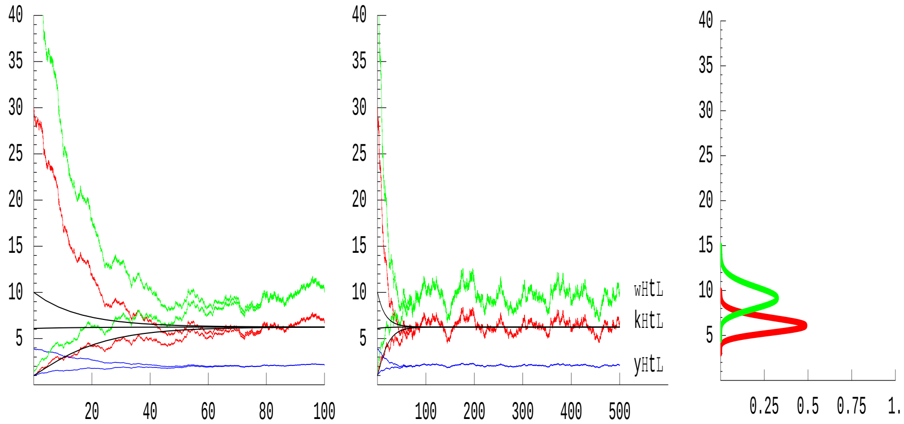

Figure 5.5: CD8

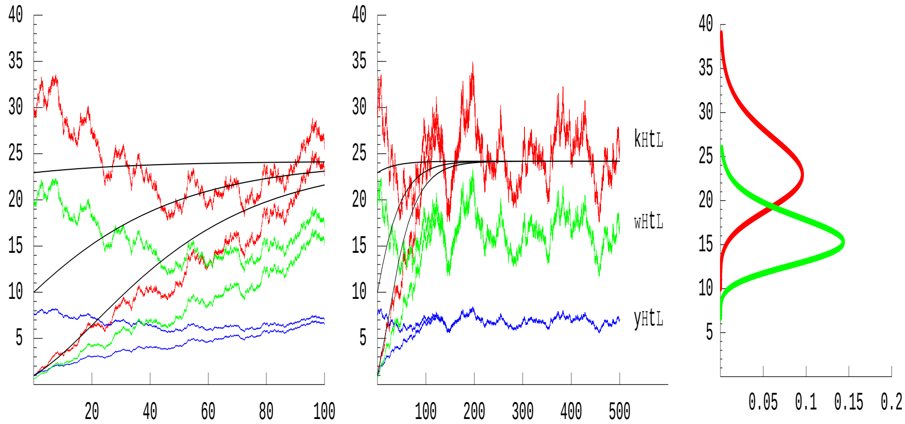

Figure 5.6: CD13

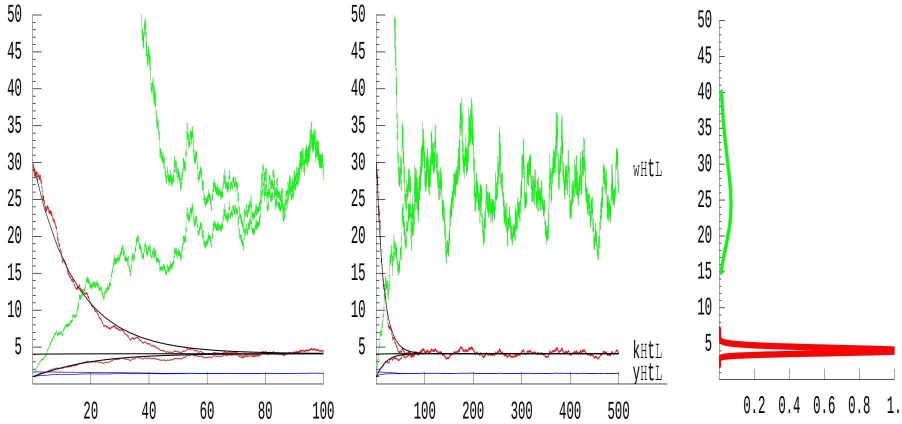

Figure 5.7: CES I2

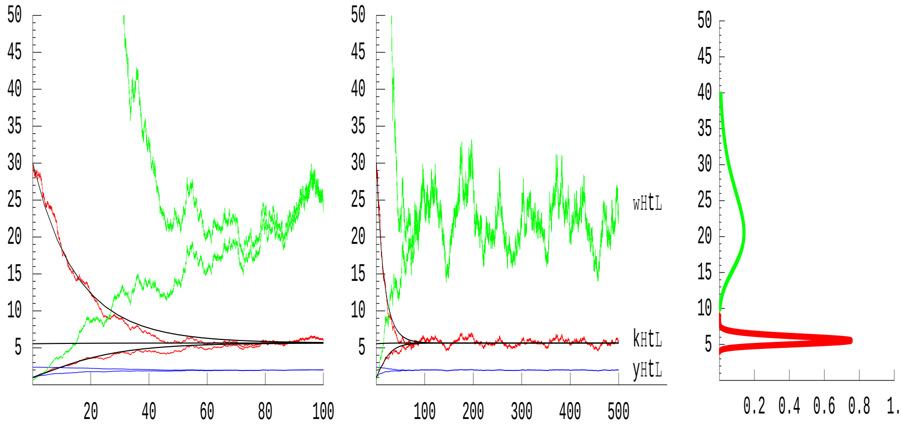

Figure 5.8: CES 3a

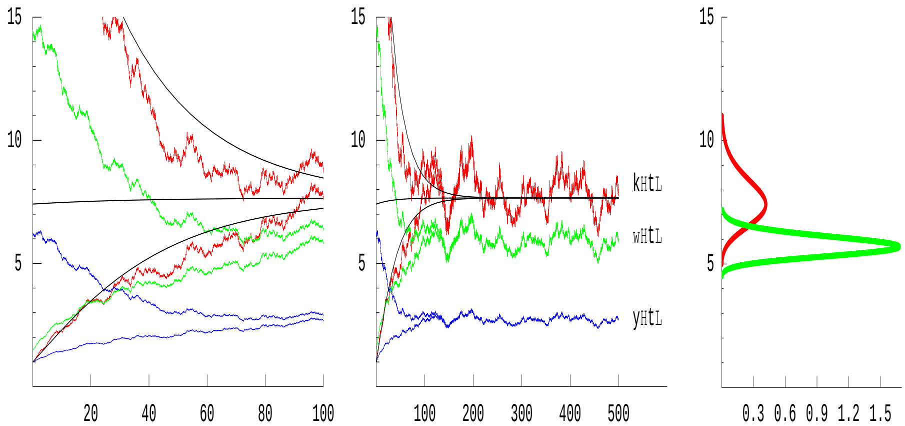

Figure 5.9: CES4

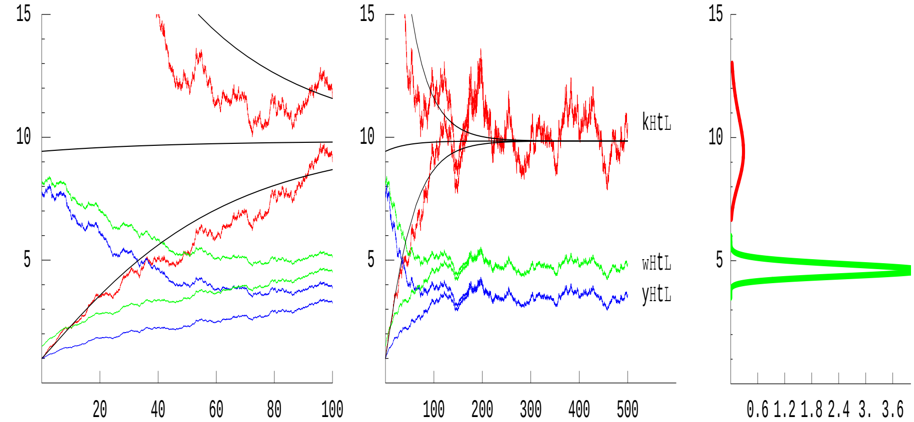

Figure 5.10: CES 5

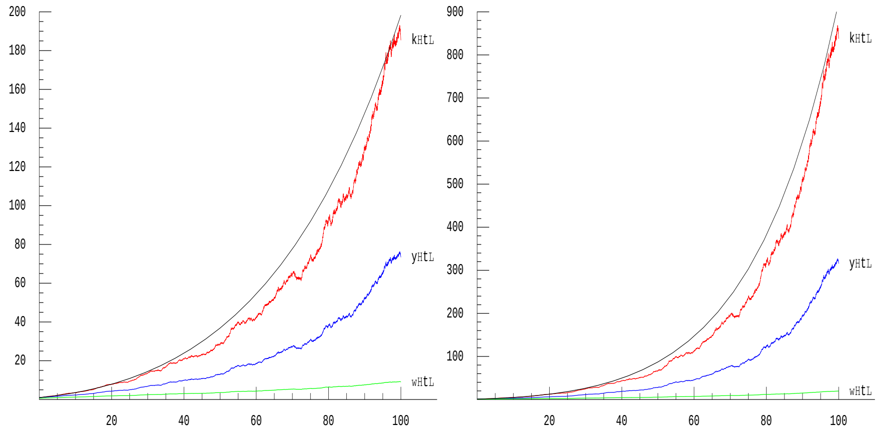

Figure 5.11: CES 11 and CES 12

Table 2. Examples of key numbers for one sector CD and CES models

Parameter values - stationary examples												Model Characteristics										
s	n	δ	γ	a	σ	β_1	β_2	β_3	β_4	λ_3	λ_4	κ	$E(k)$	mode(k)	$\sigma(k)$	$\bar{\omega}$	$f(\kappa)$	$f(\kappa)/\kappa$	$f'(\kappa)$	ϵ_κ	K/Y	$\frac{n+\delta}{s}$
0.20	0.02	0.05	1.0	0.20	1.0	0.01	0.03	0.00	0.0	0	0	3.715	3.718	3.656	0.353	14.858	1.323	0.356	0.070	0.20	2.857	0.350
0.20	0.02	0.05	1.0	0.25	1.0	0.01	0.03	0.00	0.0	0	0	4.054	4.057	3.986	0.398	12.163	1.388	0.342	0.088	0.25	2.857	0.350
0.20	0.02	0.05	1.0	0.40	1.0	0.01	0.03	0.00	0.0	0	0	5.753	5.753	5.632	0.631	8.629	2.014	0.350	0.140	0.40	2.857	0.350
0.20	0.02	0.05	3.0	0.40	1.0	0.01	0.03	0.00	0.0	0	0	35.900	35.900	35.143	3.936	53.849	12.565	0.350	0.140	0.40	2.857	0.350
0.20	0.02	0.05	1.0	0.40	1.0	0.00	0.00	0.10	0.0	1	0	5.753	5.736	5.685	0.694	8.629	2.014	0.350	0.140	0.40	1.000	1.000
0.25	0.02	0.05	1.0	0.40	1.0	0.00	0.00	0.05	0.0	1	0	8.345	8.341	8.329	0.403	12.517	2.336	0.280	0.112	0.40	3.571	0.280
0.25	0.02	0.05	1.0	0.40	1.0	0.00	0.00	0.10	0.0	1	0	8.345	8.329	8.282	0.806	12.517	2.336	0.280	0.112	0.40	3.571	0.280
0.30	0.02	0.08	1.0	0.40	1.0	0.00	0.00	0.10	0.0	1	0	6.240	6.22862	6.194	0.600	9.360	2.080	0.333	0.133	0.40	3.000	0.333
0.30	0.02	0.08	1.0	0.40	1.0	0.01	0.03	0.10	0.0	1	0	6.240	6.22858	6.102	0.830	9.360	2.080	0.333	0.133	0.40	3.000	0.333
0.20	0.02	0.08	1.0	0.40	1.0	0.01	0.03	0.10	0.0	1	0	3.175	3.162	3.072	0.541	4.762	1.587	0.500	0.200	0.40	2.000	0.500
0.20	0.01	0.08	1.0	0.40	1.0	0.00	0.00	0.10	0.0	1	0	3.784	3.770	3.726	0.517	5.676	1.703	0.450	0.180	0.40	2.222	0.450
0.20	0.01	0.08	2.0	0.40	1.0	0.00	0.00	0.10	0.0	1	0	12.014	11.969	11.833	1.644	18.021	5.406	0.450	0.180	0.40	2.222	0.450
0.20	0.01	0.08	0.3	0.40	1.0	0.00	0.00	0.10	0.0	1	0	0.509	0.508	0.504	0.033	0.763	0.229	0.450	0.180	0.40	2.222	0.450
0.30	0.01	0.08	0.2	0.40	1.0	0.00	0.00	0.00	0.1	0	1	0.509	0.509	0.504	0.033	0.763	0.153	0.300	0.120	0.40	3.333	0.300
0.33	0.01	0.08	1.0	0.40	1.0	0.00	0.00	0.00	0.1	0	1	8.719	2.375	8.719	0.079	13.078	2.378	0.273	0.109	0.40	3.667	0.273
0.25	0.02	0.05	1.0	0.60	1.0	0.01	0.03	0.10	0.0	1	0	24.105	23.960	22.949	4.312	16.070	6.749	0.280	0.168	0.60	3.571	0.280
0.20	0.02	0.05	1.0	0.25	0.5	0.01	0.03	0.0	0.0	0	0	3.476	3.479	3.428	0.309	36.252	1.217	0.031	0.350	0.088	2.857	0.350
0.20	0.02	0.05	1.0	0.40	0.5	0.01	0.03	0.0	0.0	0	0	4.095	4.097	4.035	0.375	25.157	1.433	0.049	0.350	0.140	2.857	0.350
0.20	0.02	0.05	1.0	0.60	0.5	0.01	0.03	0.0	0.0	0	0	5.643	5.642	5.552	0.539	21.228	1.975	0.074	0.350	0.210	2.857	0.350
0.20	0.02	0.05	3.0	0.60	0.5	0.01	0.03	0.0	0.0	0	0	19.929	19.949	19.657	1.756	264.766	6.975	0.024	0.350	0.070	2.857	0.350
0.20	0.02	0.05	1.0	0.40	1.5	0.01	0.03	0.0	0.0	0	0	7.633	7.635	7.412	0.988	5.815	2.672	0.199	0.350	0.568	2.857	0.350
0.25	0.02	0.05	1.0	0.40	1.5	0.01	0.03	0.0	0.0	0	0	13.148	13.147	12.724	1.795	8.356	3.681	0.171	0.280	0.611	3.571	0.280
0.25	0.02	0.05	1.0	0.60	1.5	0.01	0.03	0.0	0.0	0	0	401.664	398.071	345.493	120.230	36.293	112.466	0.257	0.280	0.917	3.571	0.280
0.20	0.02	0.05	1.0	0.60	1.5	0.01	0.03	0.0	0.0	0	0	55.715	55.493	51.178	12.365	9.725	19.500	0.298	0.350	0.851	2.857	0.350
0.20	0.01	0.08	0.3	0.40	1.5	0.00	0.00	0.1	0.0	1	0	0.523	0.523	0.518	0.033	0.974	0.235	0.157	0.450	0.349	2.222	0.450
0.30	0.01	0.08	0.2	0.40	1.5	0.00	0.00	0.0	0.1	0	1	0.523	0.523	0.518	0.033	0.974	0.157	0.105	0.300	0.349	3.333	0.300
0.20	0.02	0.05	1.0	0.40	2.0	0.01	0.03	0.0	0.0	0	0	9.806	9.812	9.429	1.471	4.697	3.432	0.237	0.350	0.676	2.857	0.350
0.20	0.02	0.05	1.0	0.40	3.0	0.01	0.03	0.0	0.0	0	0	15.469	15.507	14.502	3.021	3.737	5.414	0.282	0.350	0.805	2.857	0.350
0.20	0.02	0.05	1.0	0.40	7.0	0.01	0.03	0.0	0.0	0	0	191.698	203.598	105.010	182.373	3.178	67.094	0.344	0.350	0.984	2.857	0.350
0.25	0.02	0.05	1.0	0.40	3.0	0.01	0.03	0.0	0.0	0	0	99.222	99.721	82.518	34.630	6.944	27.782	0.262	0.280	0.935	3.571	0.280
Parameters - Endogenous growth examples																	Limits for $k \to \infty$					
0.25	0.02	0.05	1.0	0.40	4.0	0.01	0.03	0	0	0	0						0.2947	0.2947	∞			
0.20	0.02	0.05	1.0	0.60	2.0	0.01	0.03	0	0	0	0						0.3600	0.3600	∞			
0.25	0.02	0.05	1.0	0.60	2.0	0.01	0.03	0	0	0	0						0.3600	0.3600	∞			
0.30	0.02	0.05	1.0	0.60	2.0	0.01	0.03	0	0	0	0						0.3600	0.3600	∞			

5.7 General equilibria of two-sector economies

Great emphasis was naturally first given to labor and capital accumulation in *aggregate* (one-sector) *growth* models. An extensive literature on two-sector growth models, however, began in the 1960s. The *seminal* work on *two-sector* growth models with flexible sector technologies was done by Uzawa (1961-62, 1963), Solow (1961-62), Inada (1963), Drandakis (1963). The main expositions and references to the early two-sector growth literature are: Stiglitz and Uzawa (1969), Burmeister and Dobell (1970), Wan Jr. (1971), Gandolfo (1980).

The study of *general equilibrium dynamics* in two-sector and multisector growth models has been reviewed and extended in Jensen (2003), Jensen and Larsen (2005) - with emphasis on factor *allocation*, output *composition,* and the dualities for *commodity* and *factor prices*.

5.7.1 Factor Endowment Allocation and Prices

We now consider an economy consisting of a *capital* good industry (sector) and a *consumer* good industry, labeled 1 and 2, respectively. The *factor endowments,* total labor force (L) and the total capital stock (K), are inelastically supplied and are *fully employed* (utilized):

$$\begin{aligned}
L &= L_1 + L_2, \quad L_1/L + L_2/L \equiv \lambda_{L_1} + \lambda_{L_2} \equiv 1, && (118)\\
K &= K_1 + K_2, \quad K_1/K + K_2/K \equiv \lambda_{K_1} + \lambda_{K_2} \equiv 1, && (119)\\
k &\equiv K/L \equiv \lambda_{L_1} k_1 + \lambda_{L_2} k_2 \equiv k_2 + (k_1 - k_2)\lambda_{L_1}, && (120)\\
\lambda_{L_1} &\equiv (k - k_2)\,/\,(k_1 - k_2)\,, \quad \lambda_{K_i} \equiv (k_i/k)\,\lambda_{L_i},\ k_1 \neq k_2, && (121)
\end{aligned}$$

where the factor *allocation fractions* are denoted λ_{L_i}, λ_{K_i}, (118-119). Free *factor mobility* between the two industries and *efficient* factor *allocation* impose the *common* MRS condition, cf. (6),

$$\omega = \omega_1(k_1) = \omega_2(k_2), \tag{122}$$

For the variables k_1 and k_2 to satisfy (122), it is, beyond (2) and (6), further required that the *intersection* of the sectorial *range* for $\omega_1(k_1)$ and $\omega_2(k_2)$ is not empty,

$$\omega_i(k_i) \in \Omega_i = [\underline{\omega}_i, \overline{\omega}_i] \subseteq \mathbf{R}_+,\ \omega \in \Omega \equiv \Omega_1 \cap \Omega_2 = [\underline{\omega}, \overline{\omega}] \neq \emptyset, \tag{123}$$

The two industries are assumed to operate under *perfect competition* (zero excess profit); absolute (money) *input* (factor) *prices* (w, r) are the same in both industries; and absolute (money) *output* (product, commodity) *prices* (P_1, P_2) represent unit cost. Hence, we have the *competitive producer equilibrium* equations,

$$\begin{aligned} w &= P_i \cdot MP_{L_i}, \quad r = P_i \cdot MP_{K_i}; \quad \omega = w/r, \qquad P_i \neq 0. & (124)\\ Y_i &= Ly_i l_i,\ P_i Y_i = wL_i + rK_i,\ \epsilon_{L_i} = wL_i/P_iY_i,\ \epsilon_{K_i} + \epsilon_{L_i} = 1 & (125)\\ p &\equiv \frac{P_1}{P_2} = \frac{MP_{K_2}}{MP_{K_1}} = \frac{f_2'(k_2)}{f_1'(k_1)} = \frac{f_2(k_2) - k_2 f_2'(k_2)}{f_1(k_1) - k_1 f_1'(k_1)} = \frac{MP_{L_2}}{MP_{L_1}} & (126) \end{aligned}$$

Gross domestic product, Y, is the monetary value of sector outputs,

$$Y \equiv P_1Y_1 + P_2Y_2 = L(P_1y_1l_1 + P_2y_2l_2) \equiv Ly \tag{127}$$

and is, with (124-126), equal to the total *factor incomes*:

$$Y = wL + rK = L(w + rk) = L(\omega + k)P_i f_i'(k_i) = Ly, \tag{128}$$

Hence, the factor income *distribution shares,* $\delta_K + \delta_L = 1$, become,

$$\delta_K \equiv rK/Y \equiv rk/y, \delta_L \equiv wL/Y; \delta_K \equiv k/(\omega + k), \delta_K/\delta_L \equiv k/\omega \tag{129}$$

The macro equivalence of total revenues and total expenditures gives the *decomposition* of GDP (127) into *expenditure shares,* s_i, as

$$s_i = P_iY_i/Y, \quad \sum_{i=1}^{2} s_i \equiv \sum_{i=1}^{2} P_iY_i/Y = 1 \tag{130}$$

Lemma 1. *The macro factor income shares δ_L, δ_K, (129), are expenditure-weighted combinations of sectorial factor (cost) shares,*

$$\delta_L = \sum_{i=1}^{2} s_i\epsilon_{L_i}, \qquad \delta_K = \sum_{i=1}^{2} s_i\epsilon_{K_i}, \quad \delta_K + \delta_L = 1 \tag{131}$$

The factor allocation fractions (118-119) are obtained by,

$$L_i/L = \lambda_{L_i} = s_i\epsilon_{L_i}/\delta_L \qquad K_i/K = \lambda_{K_i} = s_i\epsilon_{K_i}/\delta_K \tag{132}$$

The total factor endowment ratio, (120), satisfy the identity, cf. (129):

$$K/L = k = \frac{\omega\delta_K}{\delta_L} = \omega \sum_{i=1}^{2} s_i\epsilon_{K_i} \Big/ \sum_{i=1}^{2} s_i\epsilon_{L_i} \tag{133}$$

which is a representation of the Walras's law.

PROOF: By definition we have,

$$\delta_L = wL/Y = [wL_1 + wL_2]/Y, \delta_K = rK/Y = [rK_1 + rK_2]/Y \quad (134)$$

From (125)and (130), we get

$$wL_i = \epsilon_{L_i} P_i Y_i = s_i \epsilon_{L_i} Y, \qquad rK_i = \epsilon_{K_i} P_i Y_i = s_i \epsilon_{K_i} Y \quad (135)$$

Hence, by (134) and (135) we obtain (131). Next, we obtain

$$\lambda_{L_i} = \frac{L_i}{L} = \frac{wL_i}{wL} = \frac{s_i \epsilon_{L_i} Y}{\delta_L Y} = \frac{s_i \epsilon_{L_i}}{\delta_L} \quad (136)$$

$$\lambda_{K_i} = \frac{K_i}{K} = \frac{rK_i}{rK} = \frac{s_i \epsilon_{K_i} Y}{\delta_K Y} = \frac{s_i \epsilon_{K_i}}{\delta_K} \quad (137)$$

as stated in (132). □

5.7.2 Commodity prices and factor prices with CES

The *connection* between relative *factor* (service) prices and relative *commodity* prices follows from (6, 122, 124, 126),

$$p(\omega) = \frac{P_1}{P_2}(\omega) = \frac{MP_{K_2}[k_2(\omega)]}{MP_{K_1}[k_1(\omega)]} = \frac{f_2'[k_2(\omega)]}{f_1'[k_1(\omega)]}, \quad \omega = w/r. \quad (138)$$

The exact form of the function (138) needs particular attention. With (10) and (14), the *relative commodity* prices (comparative costs) (138) become, with $\sigma_i = 1$, $\sigma_i \neq 1$, and $\sigma_1 = \sigma_2 = \sigma$, respectively,

$$p(\omega) = \frac{f_2'[k_2(\omega)]}{f_1'[k_1(\omega)]} = \frac{\gamma_2 a_2 k_2(\omega)^{a_2-1}}{\gamma_1 a_1 k_1(\omega)^{a_1-1}} = \frac{\gamma_2 a_2^{a_2}(1-a_2)^{1-a_2}}{\gamma_1 a_1^{a_1}(1-a_1)^{1-a_1}} \omega^{(a_2-a_1)} \quad (139)$$

$$\begin{aligned} p(\omega) &= \frac{f_2'[k_2(\omega)]}{f_1'[k_1(\omega)]} = \frac{\gamma_2 a_2 \left[a_2 + (1-a_2)k_2(\omega)^{-(\sigma_2-1)/\sigma_2}\right]^{1/(\sigma_2-1)}}{\gamma_1 a_1 \left[a_1 + (1-a_1)k_1(\omega)^{-(\sigma_1-1)/\sigma_1}\right]^{1/(\sigma_1-1)}} \\ &= \frac{\gamma_2 a_2^{\sigma_2/(\sigma_2-1)} \left(1 + c_2 \omega^{1-\sigma_2}\right)^{1/(\sigma_2-1)}}{\gamma_1 a_1^{\sigma_1/(\sigma_1-1)} \left(1 + c_1 \omega^{1-\sigma_1}\right)^{1/(\sigma_1-1)}} \quad (140) \\ p(\omega) &= \frac{\gamma_2}{\gamma_1} \left[\left[\frac{a_2}{a_1}\right]^{\sigma} \frac{1 + c_2\omega^{1-\sigma}}{1 + c_1\omega^{1-\sigma}}\right]^{1/(\sigma-1)}_{,} \quad c_i = \left[\frac{1-a_i}{a_i}\right]^{\sigma} \quad (141) \end{aligned}$$

5.7.3 Walrasian general equilibrium and CES

As to the demand (expenditure share) decomposition between consumption and investment (saving), we shall employ the "neoclassical" saving assumption, which has been the standard in much of the growth literature. It is immaterial for our purposes whether investment is controlled by owners or managers.

Hence, we use the aggregate monetary saving function:

$$S = sY, \quad 0 < s < 1; \tag{142}$$

or alternatively stated by the expenditure condition for the commodities, (newly produced capital goods) and (consumer goods), cf. (127)

$$P_1 Y_1 \ = \ sY; \ s = P_1 Y_1 / Y; \ 1 - s = P_2 Y_2 / Y. \tag{143}$$

The *competitive general equilibrium* (CGE) states (Walrasian equilibria), with market clearing prices on the commodity/factor markets and Pareto efficient endowments allocation, are obtained as follows.

From (143), (131), (132) we have,

$$\lambda_{L_1} = \frac{s\epsilon_{L_1}}{s\epsilon_{L_1} + (1-s)\epsilon_{L_2}}, \quad \lambda_{K_1} = \frac{s\epsilon_{K_1}}{s\epsilon_{K_1} + (1-s)\epsilon_{K_2}} \tag{144}$$

$$\delta_L = s\epsilon_{L_1} + (1-s)\epsilon_{L_2}, \qquad \delta_K = s\epsilon_{K_1} + (1-s)\epsilon_{K_2}. \tag{145}$$

Finally, by (133) and (145), we obtain, $\forall k \in R_+, \forall \omega \in \Omega$, (123)

$$k = \frac{\omega \delta_K}{\delta_L} = \frac{\omega[s\epsilon_{K_1} + (1-s)\epsilon_{K_2}]}{s\epsilon_{L_1} + (1-s)\epsilon_{L_2}} = \Psi(\omega). \tag{146}$$

The formula of $\Psi(\omega)$, (146), is a "reduced form" expression derived from the "structural equations" of the model, which relate the *"endogenous"* two-sector *general equilibrium* solutions of the wage-rental ratio $\omega = w/r$ to the *"exogenous"* factor endowments ratio, $k = K/L$. Then, having obtained ω from (146), we can go back through (144), (126) to get the associated general equilibrium values of all other endogenous variables - sector outputs, allocation fractions of inputs, income shares, relative commodity prices.

The *competitive general equilibrium,* (CGE) function, $\Psi(\omega)$ and its *graph*, as loci of timeless general equilibrium values and as trajectories of motion – are of paramount importance for inquiring into the statics, comparative statics, and dynamics of two-sector economies,

and they will be called the *factor endowment-factor price* (FEFP) *correspondence*. Alternatively, a CGE function Ψ may be dubbed the *Walrasian kernel*, since it selects in the Edgeworth box diagram the relevant *Walrasian equilibrium allocation* as a specific allocation (*point*) within the *core* of the *contract* curve, cf. Mas-Colell et al. (1995, p. 654). The *locus* of the *Walrasian kernel*, $k = \Psi(\omega)$, effectively *links* the *Pareto optimal allocations* [changing contract curves of expanding Edgeworth boxes] to the relevant *Walrasian equilibrium price vector*, (p, ω) [on the locus of the Factor Price-Commodity Price (FPCP) correspondence $p(\omega)$, (138-141)].
CES case. From (144-146), (15), we get

$$\lambda_{L_1} = \frac{1 + c_2\omega^{1-\sigma_2}}{1 + \frac{1-s}{s}\frac{c_2}{c_1}\omega^{\sigma_1-\sigma_2} + \frac{c_2}{s}\omega^{1-\sigma_2}}, \quad \lambda_{K_1} = \frac{1 + c_2\omega^{1-\sigma_2}}{\frac{1}{s} + \frac{1-s}{s}c_1\omega^{1-\sigma_1} + c_2\omega^{1-\sigma_2}} \tag{147}$$

$$\begin{aligned} \delta_K &= s\left(1 + c_1\omega^{1-\sigma_1}\right)^{-1} + (1-s)\left(1 + c_2\omega^{1-\sigma_2}\right)^{-1} \\ &= \frac{1 + (1-s)c_1\omega^{1-\sigma_1} + sc_2\omega^{1-\sigma_2}}{1 + c_1\omega^{1-\sigma_1} + c_2\omega^{1-\sigma_2} + c_1c_2\omega^{2-\sigma_1-\sigma_2}}, \quad \delta_L = 1 - \delta_K \end{aligned} \tag{148}$$

and hence finally by (148)

$$k = \Psi(\omega) = \frac{\omega\delta_K}{\delta_L} = \frac{\omega^{\sigma_1+\sigma_2} + sc_2\omega^{1+\sigma_1} + (1-s)c_1\omega^{1+\sigma_2}}{sc_1\omega^{\sigma_2} + (1-s)c_2\omega^{\sigma_1} + c_1c_2\omega} \tag{149}$$

The function $\Psi(\omega)$, (149), is monotonic, cf. Jensen (2003, p. 65), and the inverse $\omega(k) = \Psi^{-1}(k)$ exists (not necessarily in closed form) for every $k \in [0, \infty)$.

5.8 Dynamics of two-sector economies

5.8.1 Deterministic dynamics of two-sector growth models

The equations of *factor accumulation* for neoclassical two-sector growth models with flexible sector technologies are formally given, Uzawa (1963, p. 106), by (δ is the depreciation rate of capital),

$$\frac{dL}{dt} \equiv \dot{L} = nL, \tag{150}$$

$$\frac{dK}{dt} \equiv \dot{K} = Y_1 - \delta K = Ly_1\lambda_{L_1} - \delta K = L\left\{f_1(k_1)\lambda_{L_1} - \delta k\right\}. \tag{151}$$

In the general equilibrium models of two-sector economies, k_1 and λ_{L_1} are through ω (122), *uniquely* determined by the *factor* endowments *ratio* k, cf. (146).

Hence, the accumulation equations (150-151) become genuine *autonomous* (time invariant) *differential equations* in the state variables L and K and represent a standard *homogeneous dynamic system*,

$$\begin{aligned} \dot{L} &= Ln \equiv L\mathfrak{f}(k), & (152)\\ \dot{K} &= L\left\{f_1(k_1[\Psi^{-1}(k)])l_1[\Psi^{-1}(k)] - \delta k\right\} \equiv L\mathfrak{g}(k). & (153)\end{aligned}$$

As $\mathfrak{g}(k)$, (153), are intricate functions of k, we rewrite $\mathfrak{g}(k)$ in alternative forms by (151), (128-129), cf. Burmeister and Dobell (1970, p. 111), Wan (1971, pp. 119), Gandolfo (1980, pp. 490),

$$\dot{K} = sY/P_1 - \delta K = L(s(y/P_1) - \delta k) \tag{154}$$

$$= Ls(\omega + k)f_1'(k_1) - \delta K = Lk\{\frac{sf_1'(k_1)}{\delta_K} - \delta\} = L\mathfrak{g}(k) \tag{155}$$

From the *governing* functions $\mathfrak{g}(k)$, (154), the *director* functions, $h(k) \equiv \mathfrak{g}(k) - k\mathfrak{f}(k)$, of (152-153) that control $dk/dt \equiv \dot{k}$ become,

$$\dot{k} = h(k) = k\left[\frac{sf_1'(k_1[\omega(k)])}{\delta_K[\omega(k)]} - (n+\delta)\right]; \quad \omega(k) = \Psi^{-1}(k) \tag{156}$$

The neoclassical demand side (savings) was not affected by factor income distribution, cf. (142). However, to succinctly express and decompose the governing functions of capital accumulation (154), the bounded variable $\delta_K(k)$ is mainly a formal auxiliary term helpful in evaluating concrete cases.
CES case. The *qualitative* properties of the family of *solutions* $k(t)$ in the *Walrasian general equilibrium* growth models with CES sector technologies are summarized in:

Proposition 1. *For the two-sector growth models (150-153) with CES sector technologies, the sufficient conditions for the existence of at least one positive steady-state solution* [*no positive, attractive, steady-state solution* κ *exists with the RHS inequalities of (157-158) reversed*] *are:*

$$\sigma_1 < 1,\ \sigma_2 < 1: \quad \bar{\beta}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)/s \tag{157}$$

$$\sigma_1 < 1,\ \sigma_2 > 1: \quad \bar{\beta}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta) \tag{158}$$

$\sigma_1 \leq 1:$ *(sufficient condition), persistent growth of* $k(t)$ *is impossible.*
$\sigma_1 > 1:$ *necessary and sufficient conditions for* $\lim_{t\to\infty} k(t) = \infty$ *are:*

$$\sigma_1 > 1,\ \sigma_2 > 1: \quad \underline{\beta}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)/s \tag{159}$$

$$\sigma_1 > 1,\ \sigma_2 < 1: \quad \underline{\beta}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta) \tag{160}$$

except that (160) is occasionally not sufficient for small initial values.

PROOF: See Jensen (2003). □

Proposition 1 shows explicitly that the *global existence issues* of any *steady state* or *persistent* growth depend on the *size* of the key parameters: σ_i, a_1, γ_1, s, s_K, n, δ. While the *accumulation* parameters (s, s_K, n, δ) play some roles, the *fundamental* role of the *technology* parameters in the *capital* good sector (σ_1, γ_1, a_1) for deciding the types of the long-run *evolution* in the CGE growth models *complies* with *observation* and economic *intuition*, as well as confirms the strategic importance ascribed to capital good industries by economic historians and the general public, cf. Mahalanobis (1955), Rosenberg (1963).

The most important parameter in **Proposition 1** is the *substitution elasticity* in the capital good sector, σ_1 – with the CD technology as the critical bifurcation value. The "*total productivity*" parameter γ_1 in the capital good sector matters in all the stated conditions (157-160), and they can all be violated by giving γ_1 any value between 0 and ∞. If we restrict $\gamma_1 = 1$ and if $\sigma_1 \simeq 2$, then (159) will usually be satisfied for other relevant parameters, in particular with high saving rates. Evidently, the critical role in **Proposition 1** is played by σ_1 rather than σ_2. The conclusions in this proposition contrast sharply with the standard literature on two-sector growth models.

5.8.2 Neoclassical SDE of the capital-labor ratio

The deterministic model (150-151) becomes, with *uncertainties* (stochastic elements ϵ_i) in the growth rate of *labor* n, the gross *saving* rate s, and the capital *depreciation* rate δ:

$$\dot{L} = L(n + \beta_1 \epsilon_1) \tag{161}$$

$$\dot{K} = L(s + \phi_3(k)\,\epsilon_3)Y/P_1 - (\delta + \beta_2\,\epsilon_2)K \tag{162}$$

For the labor and capital stock, the associated *stochastic differential equations* (SDE) to (161–162), are given by

$$dL = Ln\,dt + L\beta_1 dw_1 \tag{163}$$

$$dK = (sLY/P_1) - \delta K)\,dt - \beta_2 K\,dw_2 + LY/P_1\phi_3(k)\,dw_3 \tag{164}$$

Theorem 6. *The stochastic neoclassical dynamics for the capital-labor ratio of the two-sector model (163-164) is a diffusion process, given by the following SDE:*

$$dk = a(k)\,dt + b(k)\,dw, \qquad k \in (0, \infty) \tag{165}$$

where the drift coefficient $a(k)$ and diffusion coefficient $b(k)$ are,

$$\begin{aligned}
a(k) &= \bar{s}(k)(y/P_1) - \Theta k \\
\bar{s}(k) &= s - \rho_{13}\beta_1\phi_3(k) \\
\Theta &= n + \delta - (\beta_1^2 + \rho_{12}\beta_1\beta_2) \\
b^2(k) &= \beta^2 k^2 + \phi_3(k)^2 (y/P_1)^2 - \rho\phi_3(k)(y/P_1)k \\
\beta^2 &= \beta_1^2 + \beta_2^2 + 2\rho_{12}\beta_1\beta_2 \\
\rho &= 2(\rho_{13}\beta_1 + \rho_{23}\beta_2)
\end{aligned} \tag{166}$$

where y/P_1 is given by, cf. (128), (149),

$$y/P_1 = (\omega + \Psi(\omega)) f_1'(k_1(\omega)) = (\Psi^{-1}(k) + k) f_1'[k_1(\Psi^{-1}(k))] \tag{167}$$

PROOF: The proof of **Theorem 6** is a replication of the proof of **Theorem 1**, with $f(k)$ replaced by $y/P_1(k)$. □

5.9 Sample paths of two-sector models and CES

The stochastic dynamics of the capital-labor ratio in the two-sector growth model is formally equivalent to that of the diffusion process of the capital-labor in the aggregate (one-sector) growth model. We will only explore the consequences of uncertainties in the labor growth parameter n and the depreciation parameter δ under various assumptions about CES sector technologies. Our illustrations cover both convergence to steady state and sample path with persistent (long-run) stochastic growth.

We include simulations of both one-sector and two-sector stochastic growth models with some variation of the parameter to illustrate the critical parameter values in long-run stationary processes and alternatively with infinity as attractor, stochastic endogenous growth.

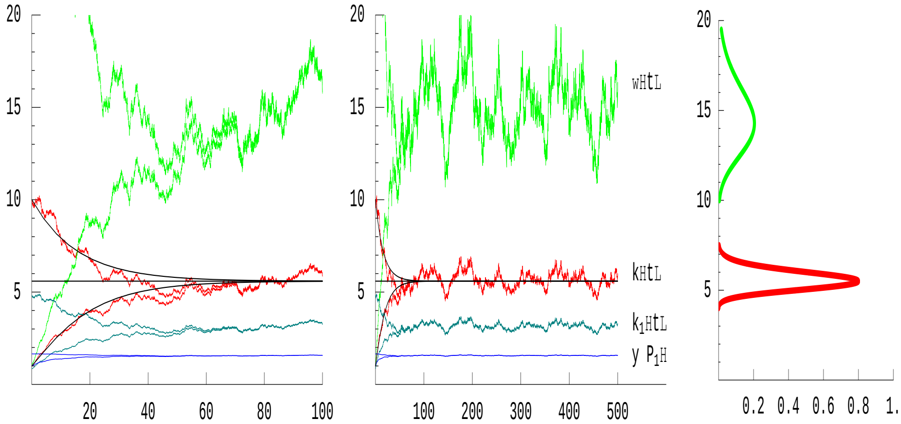

Figure 5.12: CESII 1

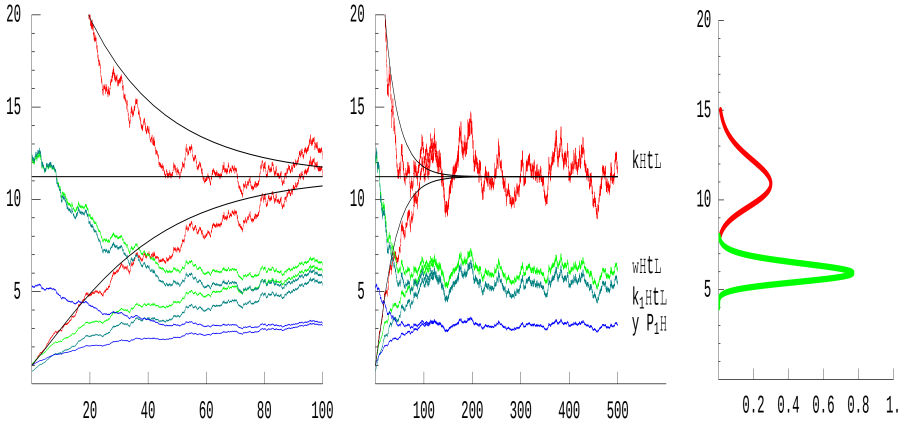

Figure 5.13: CESII 3

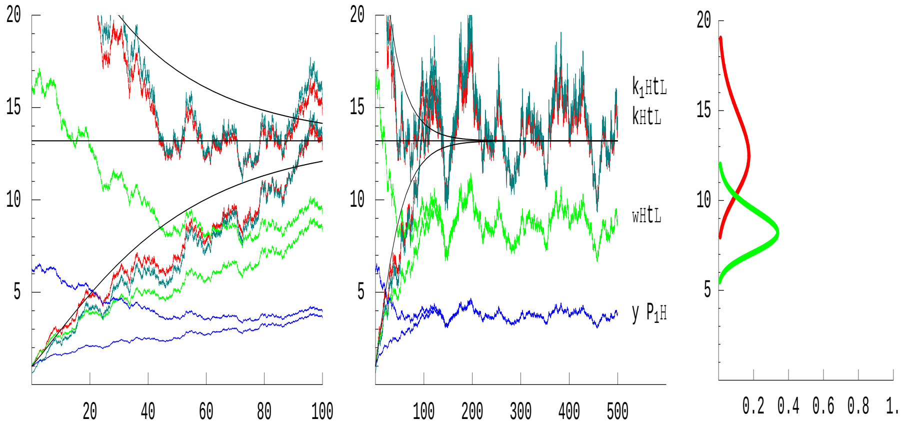

Figure 5.14: CESII 6

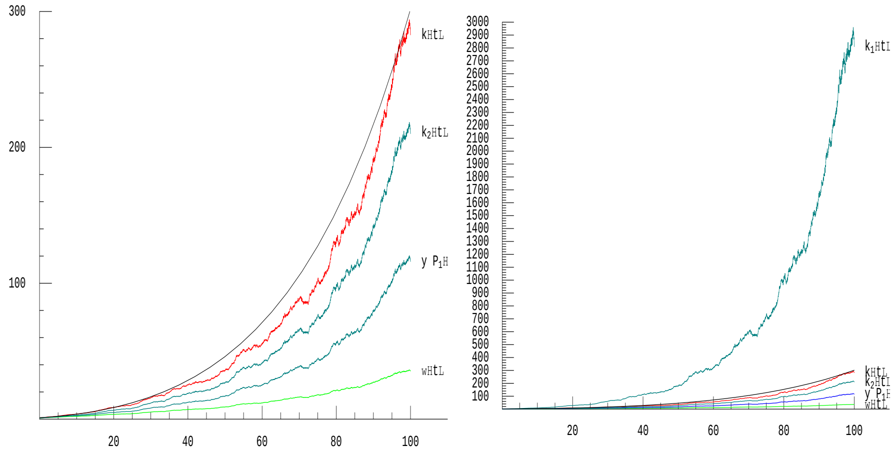

Figure 5.15: CESII 7

Table 3. Examples of key numbers for two sector CD and CES models

Parameters - stationary examples												Model Characteristics									
s	n	δ	a_1	a_2	σ_1	σ_2	β_1	β_2	β_3	λ_3	κ	$E(k)$	mode(k)	$\sigma(k)$	$\bar{\omega}$	$f_1(\kappa_1(\bar{\omega}))$	$\frac{f_1(\kappa_1(\bar{\omega}))}{\kappa_1(\bar{\omega})}$	$f'(\kappa_1(\bar{\omega}))$	ϵ_{κ_1}	$\delta_K(\bar{\omega})$
0.20	0.02	0.05	0.2	0.5	1.0	1.0	0.01	0.03	0.0	0	4.357	4.361	4.288	0.414	5.546	1.068	0.770	0.154	0.200	0.440
0.20	0.02	0.05	0.4	0.5	1.0	1.0	0.01	0.03	0.0	0	5.878	5.878	5.754	0.644	6.368	1.783	0.420	0.168	0.400	0.480
0.25	0.02	0.05	0.4	0.5	0.5	0.7	0.01	0.03	0.0	0	5.583	5.586	5.502	0.507	14.684	1.374	0.439	0.077	0.176	0.275
0.25	0.02	0.05	0.6	0.5	0.5	0.7	0.01	0.03	0.1	1	7.586	7.564	7.435	0.950	20.606	1.969	0.354	0.075	0.212	0.269
0.25	0.02	0.05	0.4	0.5	1.2	1.5	0.01	0.03	0.0	0	11.196	11.192	10.902	1.369	6.029	2.063	0.389	0.182	0.468	0.650
0.25	0.02	0.05	0.4	0.5	1.2	1.5	0.01	0.03	0.1	1	11.196	11.146	10.769	1.797	6.029	2.063	0.389	0.182	0.468	0.650
0.25	0.02	0.05	0.4	0.5	1.5	1.2	0.01	0.03	0.0	0	13.198	13.150	12.725	1.797	8.499	3.739	0.277	0.170	0.613	0.607
0.25	0.02	0.05	0.4	0.5	1.5	1.2	0.01	0.03	0.1	1	13.198	13.113	12.501	2.398	8.499	3.739	0.277	0.170	0.613	0.607
Parameters - Endogenous growth examples																Limits for $k \to \infty$				
0.25	0.02	0.05	0.6	0.5	2.0	1.5	0.01	0.03	0.0	0						0.360	0.360	∞		
0.25	0.02	0.05	0.6	0.5	2.0	1.5	0.01	0.03	0.1	1						0.360	0.360	∞		
0.30	0.02	0.05	0.6	0.5	2.0	1.5	0.01	0.03	0.1	1						0.360	0.360	∞		

In Merton's paper with one-sector CD-technology and only uncertainty on labor it follows from footnote 1 on page 383 that the certainty steady-state value for k, κ, is the same as the mode of the steady-state distribution for k but not the same as mean for that distribution. With uncertainty on labor and depreciation the mode and κ do no longer need to agree. The mode, m, can be found by solving the equation $b'(m)b(m) = a(m)$, which yield $m = (\gamma s/(\Theta + \beta^2))^{1/(1-\alpha)}$. The certainty steady-state value κ equals $\kappa = (\gamma s/(n+\delta))^{1/(1-\alpha)}$ and hence $m = \kappa$ if and only if $\beta_2 = -\rho_{12}\beta_1$. As an example this equality can only be satisfied if $\beta_1 \geq \beta_2$ and with $\rho_{12} = 0$ and $\beta_2 > 0$ we find that $m < \kappa$.

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