MAXIMUM-LIKELIHOOD ESTIMATION OF FRACTIONAL COINTEGRATION WITH AN APPLICATION TO U.S. AND CANADIAN BOND RATES

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Abstract—We estimate a multivariate ARFIMA model to illustrate a cointegration testing methodology based on joint estimates of the fractional orders of integration of a cointegrating vector and its parent series Previous cointegration tests relied on a two-step testing procedure and maintained the assumption in the second step that the parent series were known to have a unit root. In our empirical example of fractional cointegration, we illustrate how uncertainty regarding the order of integration of the parent series can be even more important than uncertainty regarding the order of integration of the cointegrating vector when testing for cointegration

I. Introduction

IN THE PAST two decades economists have developed a number of tools to examine whether economic variables trend together in ways predicted by theory, most notably cointegration tests. The multivariate testing procedure of Johansen (1988, 1991) has become a popular method of testing for cointegration of the $I(1)/I(0)$ variety, where $I(1)$ and $I(0)$ stand for integration of orders 1 and 0, respectively. In the Johansen methodology, series are pretested for unit roots; series that appear to have unit roots are put into a vector autoregression from which one can test for the existence of one or more $I(0)$ linear combinations.

A broader notion of cointegration, however, simply requires that cointegrating linear combinations have lower orders of integration than their parent series (Granger (1986)). Stemming from the literature on fractional integration introduced by Granger and Joyeux (1980) and Hosking (1981), where continuous orders of integration from the real line are considered, the case where there exists an $I(d - b)$ linear combination of two or more $I(d)$ series has become known as fractional cointegration. Fractional cointegration refers to cases where the reduction in the order of integration from the cointegrated parent series to the cointegrating residuals can take fractional values. A continuous measure of the reduction in order from cointegration, $b \geq 0$, provides more information than the $I(1)/I(0)$ framework regarding the extent to which series share a common stochastic trend. Moreover, the methods discussed here are also useful when testing for unit cointegration, because in many cases one is hesitant to claim to have found cointegration due to uncertainty regarding the order of integration of the original series, that is, whether they really have unit roots.

When testing for cointegration, especially fractional cointegration and possibly small values of b, it is desirable not to rely on an assumed value, $d = d_0$ (usually $d_0 = 1$), when concluding that there is significant cointegration. Previous cointegration tests, however, entail a two-step testing procedure, which is contrasted in table 1 with the test advocated here based on joint estimates of d and $d' = d - b$.

Typically the first step of the two-step procedure is a low-powered test for a unit root in the parent series. Despite the low power of the test, the unit root (or any value of d_0) is assumed certain to be the relevant reference point for the test in the second step. The cointegration test based on joint estimates, in contrast, takes into account uncertainty regarding d in its inference, which makes it more difficult to reject the null hypothesis of no cointegration. Hence, even though the null hypothesis is no cointegration in both procedures, the test using joint estimates is a more rigorous way to establish cointegration.

In cases of fractional cointegration with a small value of b, the two-step procedure may not prove convincing. Recent empirical investigations using the two-step procedure to test for fractional cointegration are published in Cheung and Lai (1993) and Baillie and Bollerslev (1994a,b). In these examples, standard $I(1)/I(0)$ tests reject cointegration, but the two-step procedure suggests the existence of long-run relationships, where departures from the long-run relationship are fractionally integrated. Baillie and Bollerslev (1994a) perform tests that fail to reject unit roots in the nominal exchange rates they study. They estimate a cointegrating vector via ordinary least squares (OLS) and then estimate the fractional order of integration of the cointegrating residuals, arriving at an estimate of $d' = d - b = 0.89$. Conditional on the maintained hypothesis that $d = 1$, the reduction in order from cointegration is presumed significant, since d' is significantly less than 1. Baillie and Bollerslev (1994b) share the same hypothesis testing approach: forward exchange rates, such as the dollar/deutschemark rate, are presumed to be linked by a long-run cointegrating relationship to the spot rate, because univariate analysis shows the order of integration of the forward premium to have a fractional order of integration significantly less than 1. In an investigation of fractional cointegration between exchange rates and relative price levels (purchasing/power parity), Cheung and Lai (1993) also test only the hypothesis that d' is less than unity, conditioning on the maintained assumption that $d = 1$. Such inferences are tenuous, however, considering how close $b = d - d'$ is to zero, especially given that Cheung (1993) finds evidence in favor of the hypothesis $d < 1$ among nominal exchange rates. Clearly it would be better to have joint estimates of d and d' from which to test directly the cointegration hypothesis $b > 0$.

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TABLE 1 —COINTEGRATION TEST PROCEDURES: TWO-STEP VERSUS JOINT ESTIMATES

	First Step: Levels	Second Step: Residuals	Single Step: Both	Inference
Two-step	Test H_0: $d = d_0$	Test H_0: $d' = d_0$		Cointegration if first step not rejected, second step rejected
Joint test			Estimate jointly d, d'	Cointegration if H_0: $b = d - d' = 0$ is rejected

In this paper we present a testing methodology that allows direct tests for fractional cointegration from joint estimates of d and d'. We use the multivariate autoregressive fractionally integrated moving-average (ARFIMA) model and draw heavily on Sowell's (1989, 1992a) work on calculating ARFIMA autocovariances. To the best of our knowledge, no one has implemented Sowell's procedure for exact maximum-likelihood estimation of multivariate ARFIMA models, although univariate applications exist, such as Sowell (1992b). Sowell (1989) discusses using the multivariate ARFIMA model to estimate fractional cointegration, but does not implement the procedure he suggests. We explain here the relationship between Sowell's specification of a multivariate model with fractional cointegration and an error correction specification for fractional cointegration, found in Granger (1986) and cited in Cheung and Lai (1993) and Baillie (1996).

In the next section, we discuss the multivariate ARFIMA model as a way to obtain joint estimates of d and d'. The third section presents an illustration using data on 10-year government bond rates from the United States and Canada. With these data, the standard Johansen (1988) testing procedure rejects $I(1)/I(0)$ cointegration. From our multivariate estimates, a two-step testing method for fractionally integrated cointegrating residuals strongly favors the cointegration hypothesis by strongly rejecting a unit root in the cointegrating residuals. A joint hypothesis test from the same estimates provides an intermediate result by rejecting with roughly equal significance levels the hypotheses of unit reduction and no reduction in the order of integration due to cointegration. Only the latter test considers uncertainty regarding the order of integration of the parent series before drawing inferences about cointegration. In fact, the standard errors on d strongly influence the test statistic for the null hypothesis of no cointegration, which serves to caution against two-step tests that assume $d = 1$ and only perform univariate tests of the order of integration of the cointegrating residuals.

II. Multivariate ARFIMA Models with Cointegration

The standard ARFIMA(p, d, q) process for a univariate, mean-zero time-series y_t can be written

$$\Phi(L)(1 - L)^d y_t = \Theta(L)\epsilon_t \tag{1}$$

where ϵ_t is a serially uncorrelated, mean-zero disturbance, $\Phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ is a stationary autoregressive process, and $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$ is an invertible moving-average process. For estimation, the series must be differenced enough times so that $d < 0.5$; series with $d > 0.5$ are not covariance stationary.[1]

Without loss of generality, we consider bivariate ARFIMA models with cointegration as presented by Sowell (1989),

$$\Phi(L)\begin{pmatrix}(1-L)^d & 0\\ 0 & (1-L)^{d-b}\end{pmatrix}\begin{pmatrix}1 & 0\\ -\beta & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \Theta(L)\epsilon. \tag{2}$$

Sowell (1989) discusses the estimation of multivariate ARFIMA models and derives the autocovariances needed for exact maximum-likelihood estimation. The multivariate ARFIMA autocovariances are calculated by the direct method of evaluating the integral of the weighted spectral density,

$$E[y_t y'_{t-s}] = \Sigma(s) = \Sigma(-s)' = \frac{1}{2\pi}\int_0^{2\pi} f_y(\lambda)e^{i\lambda s}\,d\lambda \tag{3}$$

where f_y is the spectral density of y.

The spectral density of an ARFIMA process is (with tildes suppressed)

$$\begin{aligned} f_y(\lambda) = {} & D(\omega)^{-1}[\Phi(\omega)^{-1}\Theta(\omega)] \\ & \Sigma(0)[\Theta(\omega^{-1})\Phi(\omega^{-1})^{-1}]D(\omega^{-1})^{-1} \end{aligned} \tag{4}$$

where $\omega = e^{-i\lambda}$,

$$\Sigma(0) = E[y_t y'_t] = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}$$

and

$$D(L) = \text{diag}\,[(1 - L)^d, (1 - L)^{d-b}].$$

Sowell (1989) shows that the (i, j) element of the bivariate ARFIMA autocovariance at lag s is

$$\begin{aligned}\Sigma_{i,j}(s) = {} & \sum_{n=1}^{2}\sum_{r=1}^{2}\sigma_{nr}\sum_{m=0}^{M}\sum_{l=0}^{M}B_{i,n}(m)B_{j,r}(l)\sum_{h=1}^{2p} \\ & \times\ \zeta_h C(d_n, d_r, 2p + s + m - l, \rho_h)\end{aligned} \tag{5}$$

where $(d_1, d_2) = (d, d - b)$,

[1] See Baillie (1996) for an overview of long-memory, fractionally integrated processes.

$$\zeta_h = \left[\rho_h \prod_{i=1}^{2p} (1 - \rho_i\rho_h) \prod_{k=1, k\neq h}^{2p} (\rho_h - \rho_k)\right]^{-1}$$

and

$$C(d, d-b, f, \rho_h) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\rho_h^{4p}}{1-\rho_h\omega} - \frac{1}{1-\rho_h^{-1}\omega}\right) \times (1-\omega)^{-d} (1-\omega^{-1})^{b-d} \omega^f \, d\lambda.$$

Sowell (1989) discusses how $C(d, d-b, 2p+s+m-l, \rho_h)$ can be written as the product of a gamma function and the sum of two hypergeometric functions. The hypergeometric functions can be evaluated numerically to any desired accuracy by summing enough terms,

$$\begin{aligned} C(d_n, d_r, f, \rho) = {} & \Gamma(1-d_n-d_r)\rho^{4p} \sum_{k=0}^{\infty} \\ & \times \frac{\rho^k(-1)^{f+k}}{\Gamma(1-d_n+f+k)\Gamma(1-d_r-f-k)} \\ & + \Gamma(1-d_n-d_r) \sum_{j=1}^{\infty} \\ & \times \frac{p^j(-1)^{f-j}}{\Gamma(1-d_n+f-j)\Gamma(1-d_r-f+j)}. \end{aligned} \tag{6}$$

The multivariate normal log-likelihood function for mean-zero (or demeaned) series y up to a constant is

$$-0.5 \ln|\Sigma| - 0.5y' \Sigma^{-1} y \tag{7}$$

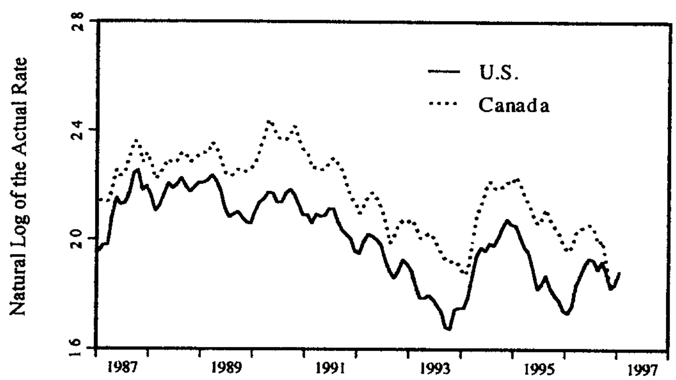
and the dimension of the covariance matrix is $(2T \times 2T)$.

III. Fractional Cointegration Among U.S. and Canadian Bond Rates: An Empirical Illustration

To illustrate the issue of hypothesis testing for cointegration, we have taken 121 monthly observations from January 1987 to February 1997 on 10-year government bond rates from the United States and Canada (the entire history on Canada from the Haver Analytics database). We take the logs of the rates to impose nonnegativity constraints on forecasted values, denoting the U.S. and Canadian rates by R_{US} and R_{CAN}, respectively.

We begin with "integer" tests for integration and cointegration. For each interest rate, we calculated augmented Dickey–Fuller statistics with four lags to be -2.72 for R_{US} and -2.43 for R_{CAN}. The 10% critical value is -3.149, so in neither case would we reject a unit root. We then looked for a cointegrating relationship between R_{US} and R_{CAN}. Johansen cointegration tests do not reject the null hypotheses of zero cointegrating vectors (with and without trends) with a likelihood ratio test statistic of 10.5 (with trends), which is less than the 5% critical value of 25.3.[2] Thus with traditional analysis, one would reject the notion that U.S. and Canadian long-term interest rates share a long-run relationship. Visual inspection of the data in figure 1, however, would suggest that the two rates tend to move together in the long run. This is the sort of commonsense observation made by proponents of fractional cointegration: not all interesting cointegrating relationships are necessarily of the $I(1)/I(0)$ variety. In this paper we present a comparison of the two-step and joint-estimate procedures for testing the cointegration hypothesis, using no cointegration as the null hypothesis and allowing for fractional cointegration as the alternative.

FIGURE 1 —U.S. AND CANADIAN LONG-TERM INTEREST RATES



A. Bivariate ARFIMA Estimates and Cointegration Tests

Here we utilize the bivariate ARFIMA model of equation (2) as a way to derive joint estimates of d and $d' - b$ that are necessary to test the cointegration hypothesis $b > 0$. Our empirical examination is meant to be illustrative, and for that reason we do not conduct an extensive model selection procedure for the bivariate ARFIMA lag lengths.[3] Our analysis of the results from the ARFIMA model with cointegration consists of hypothesis tests, plots of model-implied versus sample autocorrelations, and forecast performance, relative to a corresponding autoregressive integrated moving-average (ARIMA) model with cointegration.

The specific model we estimate has a second-order autoregressive (AR) process (8 AR parameters), a first-order moving-average (MA) process (4 MA coefficients), a cointegrating vector, and two orders of integration, one for the differenced series and one for the cointegrating vector, that is, an ARFIMA $(2, d, d-b, 1)$ model.[4] Because a formal model selection procedure would be time consuming and because underparameterized AR processes can be confused

[2] These results were obtained using four lags in the AR specification, but the findings are robust to other lag lengths

[3] Sowell (1992b) discusses use of the Akaike and Schwartz information criteria for selecting an ARFIMA model, but notes that not much is yet known about choosing among long-memory models

[4] Note that the bond rates do not appear to be covariance stationary in levels, so they must be differenced, whereas the cointegrating vector need not be differenced

TABLE 2.—BIVARIATE ARFIMA (2, d, $d-b$, 1) MODEL OF FRACTIONAL COINTEGRATION BETWEEN U.S AND CANADIAN 10-YEAR BOND RATES

Variable	Parameter	Parameter Value	Standard Error
Log-likelihood		−556.36	
Fractional root	d	−0.326	0 250
	$d' = 1 + d - b$	0.200	0.098
Cointegration parameter	β	1.14	0.050
	ϕ_{11}	0.332	0.317
	ϕ_{12}	0 130	0.096
	ϕ_{13}	0.037	0.013
	ϕ_{14}	−0 207	0.114
	ϕ_{21}	−0.001	0.009
	ϕ_{22}	−0.289	0 490
	ϕ_{23}	0.750	0.105
	ϕ_{24}	0.227	0.336
	θ_{11}	0.544	0.159
	θ_{12}	−0.093	0.197
	θ_{21}	1.05	0.050
	θ_{22}	0.265	0.328
U.S. bond rate	σ_1^2	3.05	0.197
Cointegrating residual	σ_2^2	1.93	0.127
	σ_{12}	−1.03	1.54

Note In levels the series are $I(1 + d)$, the cointegrating residuals are $I(d')$

with fractionally integrated long-memory processes, we are conservative in including a generously parameterized AR process.[5] We use Wald tests to indicate the significance of the included AR lags. The individual coefficients are labeled as follows:

$$\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}\phi_{11} & \phi_{12}\\\phi_{21} & \phi_{22}\end{pmatrix}L - \begin{pmatrix}\phi_{13} & \phi_{14}\\\phi_{23} & \phi_{24}\end{pmatrix}L^2\right] \times \begin{pmatrix}(1-L)^{1+d} & 0\\0 & (1-L)^{d'}\end{pmatrix}\begin{pmatrix}1 & 0\\-\beta & 1\end{pmatrix}\begin{pmatrix}R_{US}\\R_{CAN}\end{pmatrix} = \left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}\theta_{11} & \theta_{12}\\\theta_{21} & \theta_{22}\end{pmatrix}L\right]\begin{pmatrix}\varepsilon_1\\\varepsilon_2\end{pmatrix} \tag{8}$$

where $(\varepsilon_1, \varepsilon_2)$ are assumed to be mean-zero Gaussian errors that are uncorrelated across time, but may have a contemporaneous cross correlation.

Table 2 provides the parameter estimates for the bivariate ARFIMA model with fractional cointegration (equation (8)). The estimated order of integration of the first-differenced parent series ($d = -0.326$) has a large standard error, which raises the standard error on $b = 1 + d - d'$. In accord with the results from the Johansen estimator, unit $I(1)/I(0)$ cointegration (H_0: $d = 0$, $d' = 0$) is rejected with a Wald test statistic of 10.09, which has a probability value of 0.007. Note, however, that if we followed a two-step procedure testing for fractional cointegration in the residuals, we would observe that the order of integration of the parent series d is not significantly less than 1 and the order of integration of the cointegrating residuals d' is very significantly less than 1, leading to a supposedly strong claim that a fractionally cointegrating long-run relationship exists between the two series under the assumption that $1 + d = 1$. In fact, the significance of the reduction in the order of integration brought by any long-run cointegrating relationship depends on the significance of the test statistic for $b = 0$, not the significance of $d' = 1$. In our example, the standard error on $d - d'$ is 0.2247. Thus, we can reject H_0: $b = 1 + d - d' = 0$ with a t-statistic of 2.11 as well as H_0: $b = 1 + d - d' = 1$ with a t-statistic of 2.34. Based on the joint test, the reduction in the order of integration due to cointegration is significant, but the significance levels of rejections of $b = 1$ and $b = 0$ are quite balanced, whereas the significance levels of rejections of $d' = 0$ and $d' = 1$ are not. The balanced significance levels result from symmetric treatment in the joint test of uncertainty in both orders of integration, d and d'. In this way, the joint estimates provide cointegration test results that are not distorted by the maintained unit-root assumption. If a unit root is assumed to be known, the resulting test statistics can be deceptively decisive, suggesting that either the null hypothesis of no cointegration (as in the two-step test described above) or the null hypothesis of unit cointegration (as in the Johansen test) is almost certainly false.

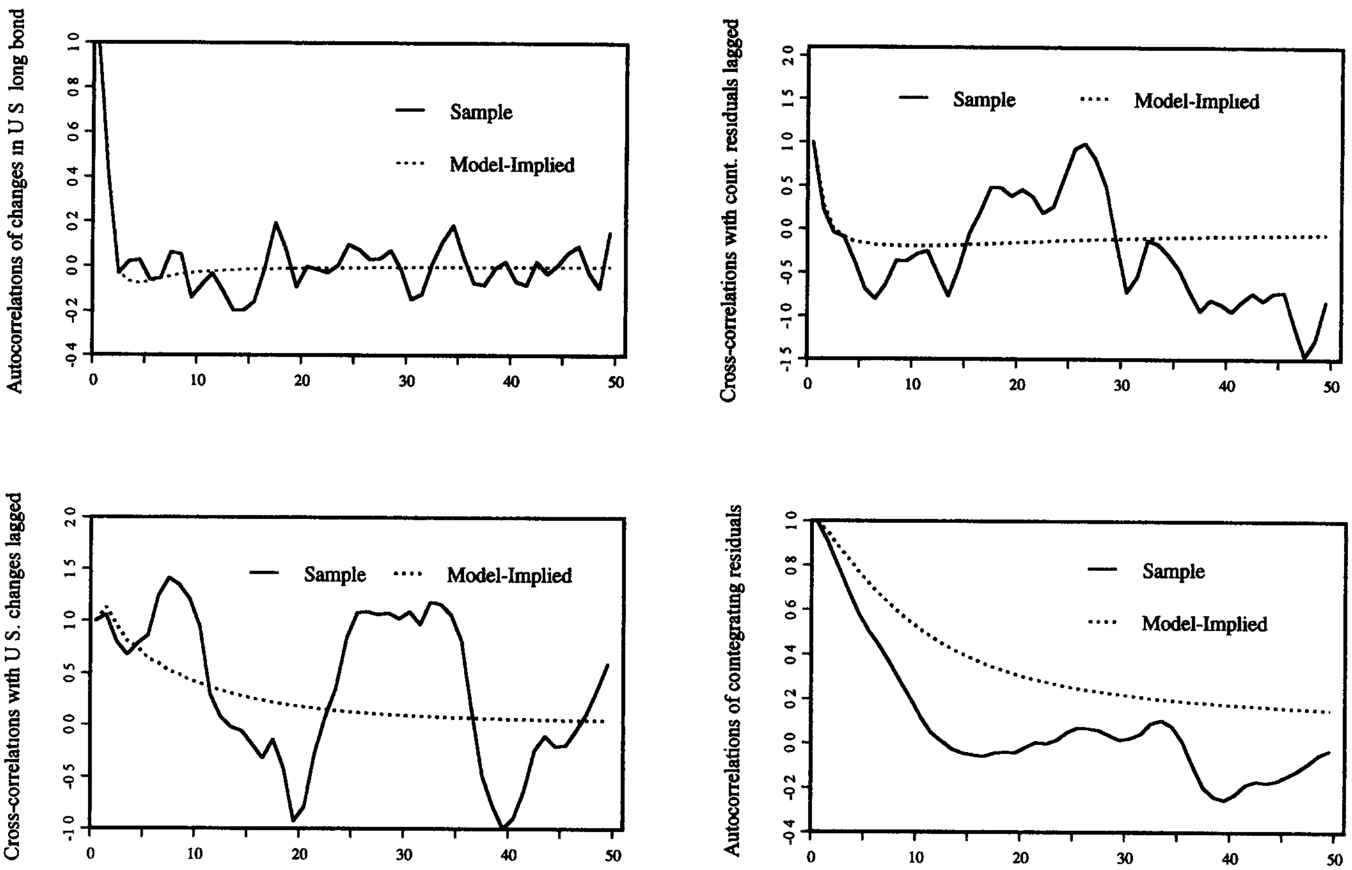
We also checked whether the second lag of the AR process was significant, using a Wald test. The Wald test statistic is 10.5, which has a probability value of 0.033. Combining these four restrictions with the restriction $b = 0$ leads to a Wald test statistic of 14.2, which has a probability value of only 0.0144. Thus it does not appear that it is desirable to reduce the order of the AR process.

Figure 2 plots the sample and model-implied autocorrelations for the ARFIMA $(2, d, d - b, 1)$ model reported in table 2. In general the model-implied autocorrelations match the sample autocorrelations in shape and magnitude. The sample cross correlations are subject to wide swings, but the model-implied cross correlations tend to cut them through the middle. Figure 3 shows mean-squared forecast errors (in sample) for the changes in the U.S. bond rate and the cointegrating vector, where forecasts are from the OLS projection formula. The forecasts range from 1 to 15 periods ahead. The estimated model from table 2 is compared with an ARIMA(2,1) model in which $I(1)/I(0)$ cointegration has been imposed. The bivariate model with fractional cointegration uniformly achieves lower mean-squared forecast errors.

Another diagnostic test for the ARFIMA model, relative to the usual ARIMA model, is a test for serial correlation in the one-step-ahead prediction errors, denoted by $\hat{e}$. Hosking (1980) derives the multivariate analogue to the Box–Pierce portmanteau test statistic for autocorrelation. The multivari-

[5] This particular model takes 35 minutes on a 200-MHz PC per iteration of exact maximum likelihood. Because demonstrating the properties of alternative, but computationally simpler estimators that yield joint estimates of both orders of integration is beyond the scope of this paper, we present exact ML estimates.

FIGURE 2 —AUTOCORRELATIONS FROM A COINTEGRATED BIVARIATE ARFIMA MODEL OF U.S AND CANADIAN BOND RATES



ate (k-variable) statistic using v lags is

$$P = T \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{r=1}^{v} (\hat{S}_{ijr})^2 \tag{9}$$

where

$$\hat{S}_r = L' \hat{C}_r L$$

$$LL' = \hat{C}_0^{-1}$$

$$\hat{C}_r = \frac{1}{T} \sum_t \hat{e}_t \hat{e}'_{t-r}.$$

The multivariate test includes the cross correlations of the forecast errors to check whether the model is exhausting the possibilities for using lagged forecast errors in one variable to forecast the other variable. The portmanteau test statistic is distributed chi-square with a number of degrees of freedom equal to the number of lags v minus the number of coefficients used to model the lag structure, which is 14 in equation (8). In table 3 we present results for the multivariate test statistic and univariate Box–Pierce test statistics that sum only each error's own squared autocorrelations.

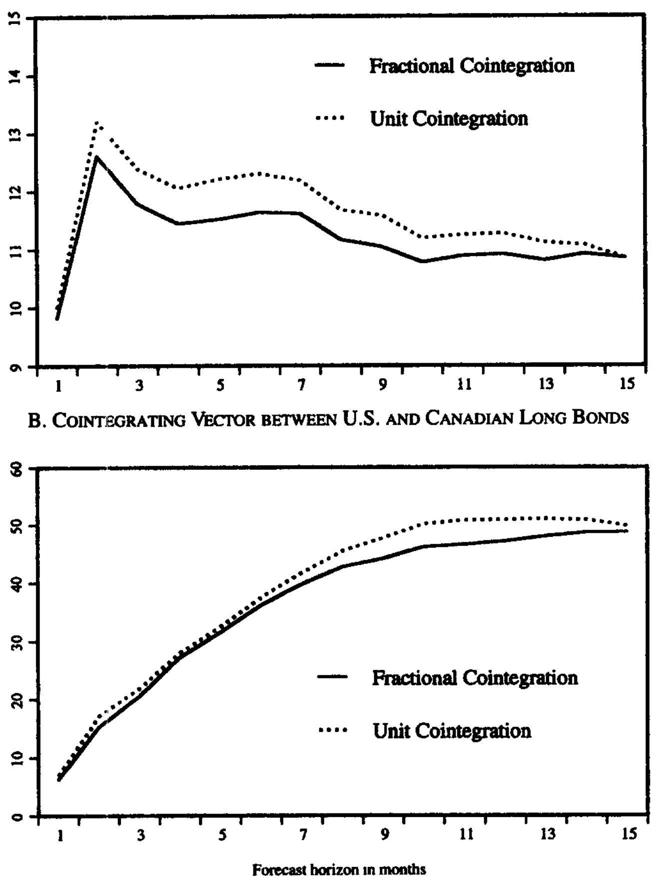
Table 3 shows, from the univariate statistics, that the ARFIMA model does not have significant correlations between a forecast error and its *own* lags. The ARIMA model, in contrast, does not eliminate significant correlation between errors in predicting departures from the long-run cointegrating relationship and its own past values. Both multivariate statistics easily reject, however, indicating that both models leave significant cross correlations of forecast errors with lags of errors in the other variable. Seemingly unexploited possibilities for better predictions stemming from cross correlations have also been documented in size-sorted stock portfolios (Lo and MacKinlay (1990)), so the failure of our multivariate portmanteau statistic is not unique. As with the ARFIMA forecast errors presented here, Lo and MacKinlay (1990) find that stock returns are uncorrelated with their own past, but have significant cross correlations with past returns of other stocks.

B. *Semiparametric Frequency-Domain Evidence*

In the bivariate ARFIMA model discussed above, cointegration is rejected in favor of fractional cointegration. Obviously if unit cointegration does not hold, then either the parent series do not have a unit root or the cointegrating

FIGURE 3.—MEAN-SQUARED FORECAST ERRORS WITH AND WITHOUT FRACTIONAL COINTEGRATION

A. CHANGES IN U.S. 10-YEAR BOND RATE FROM BIVARIATE MODELS



residuals are not $I(0)$. The point estimates suggest that neither condition for $I(1)/I(0)$ cointegration is met, but only the cointegrating residuals have a long-memory parameter that is significantly different from zero. The significance of the apparent long memory in the cointegrating residuals is not overwhelming, however, and is not guaranteed to hold in all bivariate ARFIMA specifications or across the range of uncertainty surrounding the estimator of the cointegrating parameter β. As discussed above, we do not attempt to conduct a formal model selection procedure across possible ARFIMA lag lengths, so we look for corroborating evidence of long memory in the cointegrating residuals from semiparametric frequency-domain estimates.

Semiparametric estimators of long memory in the frequency domain use a limited number of periodogram ordinates near zero to avoid influence from the short-run cycles in the data. Lobato and Robinson (1996) derive the limiting distribution of Robinson's (1994a) univariate averaged periodogram estimator of long memory.[6] The estimator of the fractional order of integration of a series is

$$\hat{d}_q = \frac{1}{2} - \frac{1}{2\ln q}\ln\left[\frac{\hat{F}(q\lambda_m)}{\hat{F}(\lambda_m)}\right] \tag{10}$$

where

$$\hat{F}(\lambda) = \frac{2\pi}{n}\sum_{j=1}^{\lambda n/2\pi} I(\lambda_j)$$

$I(\lambda_j)$ being the periodogram ordinates at the Fourier frequencies and n the number of observations in the data set. Lobato and Robinson (1996) tabulate optimal values for q for various values of the long-memory parameter d. Based on the bivariate ARFIMA estimate of $d = 0.2$, the optimal value of q is 0.4. A formula for the optimal bandwidth m is taken from Robinson (1994b). For the cointegrating residuals in question, the suggested bandwidth is $m = 6$. Because this number seems small, we present in table 4 results for bandwidths from 5 to 15. We also calculate the estimator of the long-memory parameter for both the estimated long-run relationship with $\beta = 1.135$ and a unit cointegrating vector.

For all bandwidths less than 10 the Lobato–Robinson (1996) estimator of the long-memory parameter is close to the estimate of 0.2 from the bivariate ARFIMA model in table 2. Moreover, for bandwidths greater than 10, the estimated value jumps even higher, although one would suspect that the short-run cycles in the data contaminate the estimates that include these higher frequency ordinates.

At $m = 7$, 8, and 9, $\hat{d}$ (using the unit cointegrating vector) is less than 0.25, in which case Lobato and Robinson (1996) show that the limiting distribution of $\hat{d}$ is normal. The variance of the limiting distribution of $\hat{d}$ from Lobato and Robinson (1996) implies standard errors on $\hat{d}$ ranging from 0.108 to 0.096. In these cases, the estimator for the long-memory parameter is significantly greater than zero and remarkably similar in magnitude and precision to the estimator for d' for the cointegrating residuals from the bivariate ARFIMA model in table 2.

IV. Conclusions

The concept of fractional cointegration, especially the prospect of claims that one has found a significant long-run relationship between two series because the cointegrating vector has a fractional differencing parameter 0.1 or 0.2 less than 1, has been met with some skepticism. In this paper we demonstrate the importance of testing jointly the orders of integration of the parent series and the cointegrating vector to have a true test for a *reduction* in order brought by cointegration. The order of integration of the original series ought not be assumed to be known when testing for cointegration. For this reason, we demonstrate with a multivariate ARFIMA model the first cointegration tests

[6] Note that these estimators are written for the Hurst coefficient, which equals the coefficient for a fractional order of integration d plus one-half

TABLE 3.—PORTMANTEAU TESTS WITH 50 LAGS FOR SERIAL CORRELATION IN FORECAST ERRORS

Model	Variable	Test Statistic	Degrees of Freedom	p-Value
ARFIMA (2, d, 1)	Univariate			
	Δ U.S. rate	30.19	43	0.93
	U.S.–Can cointegration vector	50.56	43	0.20
ARFIMA (2, d, $d-b$, 1)	Bivariate	163.3	36	0.000
ARIMA (2, 1)	Univariate			
	Δ U.S. rate	31.37	44	0.92
	U.S.–Can cointegration vector	67.18	44	0.01
	Bivariate	251.8	38	0.000

TABLE 4.—FREQUENCY-DOMAIN ESTIMATES OF LONG-MEMORY PARAMETER $\hat{d}$ IN COINTEGRATING RESIDUALS BETWEEN U.S. AND CANADIAN BOND RATES

	Cointegrating Vector	
Bandwidth	$\beta = 1$	$\beta = 1.135$
$m = 5$	0.264	0.285
$m = 6$	0.260	0.281
$m = 7$	0.219	0.261
$m = 8$	0.211	0.273
$m = 9$	0.209	0.272
$m = 10$	0.380	0.399
$m = 11$	0.359	0.382
$m = 12$	0.355	0.378
$m = 13$	0.408	0.436
$m = 14$	0.392	0.424
$m = 15$	0.388	0.421

based on joint estimates of the orders of integration of the cointegrating vector and its parent series. In our empirical illustration of cointegration between long-term government bond rates of the United States and Canada, uncertainty regarding the order of integration of the parent series accounts for more than half the standard error on the estimated reduction in the order of integration due to cointegration, making rejection of the null hypothesis of no cointegration less likely.

Thus we argue that appropriate testing methodology (joint estimates of both orders of integration) ought to give tests for cointegration better power and size properties, relative to the usual two-step procedure. If, in the two-step procedure, the null hypothesis is that the cointegrating residuals are integrated of order zero, the test will likely suffer from spurious rejections of the null hypothesis of no cointegration against the alternative of fractional cointegration. Similarly, if the null hypothesis is that the cointegrating residuals are $I(1)$ in a two-step test, spurious rejections of cointegration are likely to occur. Given a joint test with better size properties, we could then take more seriously findings of significant instances of fractional cointegration, where deviations from the long-run relationship display long memory. Future research can quantify through Monte-Carlo simulation the size properties of the joint cointegration test illustrated here, relative to two-step tests.

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