doi:10.1111/j.1467-9892.2008.00598.x

QUASI-MAXIMUM LIKELIHOOD ESTIMATION OF PERIODIC GARCH AND PERIODIC ARMA-GARCH PROCESSES

By Abdelhakim Aknouche and Abdelouahab Bibi

University of Sciences and Technology Houari Boumediene and University Mentouri

First Version received August 2007

Abstract. This article establishes the strong consistency and asymptotic normality (CAN) of the quasi-maximum likelihood estimator (QMLE) for generalized autoregressive conditionally heteroscedastic (GARCH) and autoregressive moving-average (ARMA)-GARCH processes with periodically time-varying parameters. We first give a necessary and sufficient condition for the existence of a strictly periodically stationary solution of the periodic GARCH (PGARCH) equation. As a result, it is shown that the moment of some positive order of the PGARCH solution is finite, under which we prove the strong consistency and asymptotic normality of the QMLE for a PGARCH process without any condition on its moments and for a periodic ARMA-GARCH (PARMA-PGARCH) under mild conditions.

Keywords. Periodic GARCH processes; periodic ARMA-GARCH models; strict periodic stationarity; periodic ergodicity; strong consistency; asymptotic normality.

AMS Subject Classification (2000). Primary: 62F12. Secondary: 62M10, 91B84.

1. INTRODUCTION

Periodic generalized autoregressive conditionally heteroscedastic (PGARCH) processes introduced by Bollerslev and Ghysels (1996) have proved useful and appropriate for modelling many time series encountered in practice, which are characterized by a stochastic conditional variance with periodic dynamics (see also Franses and Paap, 2000, 2004; Ghysels and Osborn, 2001; Taylor, 2004, 2006; Martens *et al.*, 2002, and the references therein). As for the standard GARCH model, the PGARCH process may be seen as a non-Gaussian white noise (or more precisely, as a martingale difference) but whose conditional variance follows a linear periodic autoregressive moving-average (PARMA) dynamics in terms of the squared process, implying a periodic structure for the underlying process itself. In contrast to PARMA models, which may be written as an equivalent-vector ARMA form (Tiao and Grupe, 1980), there is no direct correspondence between the PGARCH model and the multivariate GARCH model. In fact, any PGARCH process can be written as only a weak multivariate GARCH model (see Drost and Nijman, 1993, for the definition of weak GARCH and Hafner and Rombouts, 2007, for further applications), meaning that the study of PGARCH may not be trivially deduced from the existing multivariate GARCH theory and thus it constitutes a useful and interesting task.

Since their introduction, PGARCH models have been considered much fairly in the literature. Bollerslev and Ghysels (1996) have studied the second-order periodic stationarity of the PGARCH model and derived the corresponding quasi-maximum likelihood estimator (QMLE) but without studying its asymptotic properties. They have successfully applied PGARCH modelling to certain financial data. Franses and Paap (2000) have considered the PGARCH process as an innovation of a more general periodic ARMA-GARCH (PARMA-PGARCH) model. In fact, empirical applications (see also Taylor, 2006) show that there is also evidence for a PARMA structure in time-series returns, which are better fitted by PARMA-PGARCH models. Important applications of such processes may be found in Ghysels and Osborn (2001), Franses and Paap (2004) and Taylor (2006). On the other hand, some probabilistic properties of the PGARCH model such as strict periodic stationarity, existence of higher-order moment and geometric ergodicity have been studied recently by Bibi and Aknouche (2007). Apart from the above mentioned studies, it seems that there are no other studies concerning the asymptotic inference of such models. However, this problem has been exhaustively studied recently for the standard GARCH case. Indeed, a considerable amount of research has been executed for studying the asymptotic properties of the QMLE for GARCH processes. This research, including contributions by authors such as Lee and Hansen (1994) , Lumsdaine (1996), Boussama (2000), Berkes *et al*. (2003), Ling and McAleer (2003), Berkes and Horvath (2004), Jensen and Rahbek (2004a,b), Straumann and Mikosch (2006), and Francq and Zakoïan (2004, 2007), is aimed at establishing consistency and asymptotic normality of the QMLE for GARCH processes with weak conditions on the moment, as this makes an undesirable restriction on the parameter space. Francq and Zakoïan's (2004) result seems to be the weaker one obtained at present.

In the spirit of Francq and Zakoïan's study, the main goal of this article is to establish the strong consistency and asymptotic normality of the QML estimators for the PGARCH and for the PARMA-PGARCH. We first show that the sufficient condition for strict periodic stationarity of the PGARCH process given by Bibi and Aknouche (2007) is also necessary. As a direct consequence, it will be shown that the moment of some positive order of the periodic stationary solution is finite. This result will be exploited in order to obtain strong consistency and asymptotic normality for the QMLE of PGARCH parameters irrespective of any moment requirements. For the PARMA-PGARCH case, strong consistency of the corresponding QMLE is also established without any moment condition but asymptotic normality is only obtained, as for the standard ARMA-GARCH case (cf. Francq and Zakoïan, 2004), under the existence of fourth moment for the PGARCH component and second moment of the PARMA component.

The rest of this article is organized as follows. Section 2 gives a necessary and sufficient condition for strict periodic stationarity for the PGARCH process and some important consequences thereof. The main results concerning strong consistency and asymptotic normality of the QMLE for the pure PGARCH process are found in section 3 while those for the PARMA-PGARCH are given in section 4. The proofs of the main results are left in the Appendix.

Through the rest of this paper we use the following notations: the symbols '$\rightsquigarrow$' and 'a.s.' stand for 'convergence in law' and 'almost sure convergence', respectively. A matrix A of order $m \times n$ is denoted by $A_{m\times n}$ and the $k \times k$ identity matrix by $I_{k\times k}$. The spectral radius, i.e. the maximum modulus of the eigenvalues of squared matrix A is denoted by $\rho(A)$, A' stands for the transpose of A and the vectorial relation $A > B$ means that each element of A is greater than the corresponding element of B.

2. STRICT PERIODIC STATIONARITY OF THE PGARCH AND SOME OF ITS IMPORTANT CONSEQUENCES

This article partly studies the periodic GARCH process $(y_t,\ t \in \mathbb{Z})$ of orders p and q and period $S \geq 1$, that is a solution to the stochastic difference equation

$$\begin{aligned} y_t &= \sqrt{h_t}\eta_t \\ h_t &= \omega_t + \sum_{i=1}^{q} \alpha_{t,i} y_{t-i}^2 + \sum_{j=1}^{p} \beta_{t,j}\, h_{t-j}, \quad t \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}, \end{aligned} \tag{1}$$

where $(\eta_t,\ t \in \mathbb{Z})$ is a sequence of independent and identically distributed (i.i.d.) random variables defined on a probability space $(\Omega,\mathcal{A},P)$, such that $E(\eta_t) = 0$, $E(\eta_t^2) = 1$ and η_k is independent of y_t for all $k > t$. The parameters ω_t, $\alpha_{t,i}$ and $\beta_{t,j}$ are periodic in t with period S (i.e. $\omega_{t+kS} = \omega_t$, $\alpha_{t+kS,i} = \alpha_{t,i}$ and $\beta_{t+kS,\,j} = \beta_{t,j}$) such that $\omega_t > 0$, $\alpha_{t,i} \geq 0$ and $\beta_{t,j} \geq 0$ for all k, $t \in \mathbb{Z}$, $1 \leq i \leq q$ and $1 \leq j \leq p$. So, by setting $t = v + nS$, model (1) may be equivalently written as

$$\begin{aligned} y_{v+nS} &= \sqrt{h_{v+nS}}\eta_{v+nS} \\ h_{v+nS} &= \omega_v + \sum_{i=1}^{q} \alpha_{v,i} y_{v+nS-i}^2 + \sum_{j=1}^{p} \beta_{v,j}\, h_{v+nS-j}, \quad n \in \mathbb{Z},\ \ 1 \leq v \leq S, \end{aligned} \tag{2}$$

highlighting the periodicity in the model. In the difference eqn (2) y_{v+nS} may refer to (y_t) during the vth 'season' of the year n and $\omega_v,\alpha_{v,1},\ldots,\alpha_{v,q}$ together with $\beta_{v,1},\ldots,\beta_{v,p}$ are the model coefficients at season v. The orders p and q can also be considered periodic in time, say p_t and q_t, respectively, in order to reduce the number of parameters to be estimated. However, in this case, a constrained estimation procedure, which is not the objective of this article, would be needed.

A standard approach for studying the structure of an ARCH-type model is to write it as a random coefficient representation. Indeed, it is easy to write model (1) in terms of the squared process as follows

$$\begin{aligned} y_t^2 &= \omega_t \eta_t^2 + \sum_{i=1}^{q} \alpha_{t,i} \eta_t^2 y_{t-i}^2 + \sum_{j=1}^{p} \beta_{t,j} \eta_t^2 h_{t-j} \\ h_t &= \omega_t + \sum_{i=1}^{q} \alpha_{t,i} y_{t-i}^2 + \sum_{j=1}^{p} \beta_{t,j} h_{t-j}, \quad t \in \mathbb{Z}, \end{aligned} \tag{3}$$

which is ready to be cast in a stochastic recurrence equation with random coefficients.

For defining the $(p+q)$-random vectors $Y_t = (y_t^2, \ldots, y_{t-q+1}^2, h_t, \ldots, h_{t-p+1})'$ and $B_t = (\omega_t \eta_t^2, 0'_{(q-1)\times 1}, \omega_t, 0'_{(p-1)\times 1})'$, together with the $(p+q) \times (p+q)$-random matrix A_t given by

$$A_t = \begin{pmatrix} \alpha_{t,1}\eta_t^2 \alpha_{t,2}\eta_t^2 \ldots \alpha_{t,q-1}\eta_t^2 & \alpha_{t,q}\eta_t^2 & \beta_{t,1}\eta_t^2 \beta_{t,2}\eta_t^2 \ldots \beta_{t,p-1}\eta_t^2 & \beta_{t,p}\eta_t^2 \\ I_{(q-1)\times(q-1)} & 0_{(q-1)\times 1} & 0_{(q-1)\times p} & \\ \alpha_{t,1}\alpha_{t,2} \ldots \alpha_{t,q-1} & \alpha_{t,q} & \beta_{t,1}\beta_{t,2} \ldots \beta_{t,p-1} & \beta_{t,p} \\ 0_{(p-1)\times q} & & I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} \end{pmatrix},$$

($0_{m\times n}$ and $I_{n\times n}$ stand for the $m \times n$-null matrix and the $n \times n$-identity matrix, respectively) one can rewrite model (3) in the following generalized AR model

$$Y_t = A_t Y_{t-1} + B_t, \quad t \in \mathbb{Z}, \tag{4}$$

which differs from the standard formulation studied by Bougerol and Picard (1992a, b) in that the coefficients (A_t, B_t) are rather independent and periodically distributed (i.p.d.). Moreover, their expectations $E\{A_t\}$ and $E\{B_t\}$ defined element-wise are S-periodic in time.

Now, we focus on the stochastic recurrence eqn (4) and the existence of a strictly periodically stationary (s.p.s.) solution thereof. Recall that the process $(Y_t, t \in \mathbb{Z})$ is said to be s.p.s. (with period $S \geq 1$) if its finite-dimensional distributions are invariant under a multiple shift of S , that is the distribution of $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})'$ is the same as that of $(Y_{t_1+hS}, Y_{t_2+hS}, \ldots, Y_{t_n+hS})'$ for all $n \geq 1$ and h, $t_1, t_2, \ldots, t_n \in \mathbb{Z}$. Furthermore, it is called periodically ergodic (cycloergodic; cf. Boyles and Gardner, 1983) if for all integer m and all Borel set B of $\mathbb{R}^{m+1}$

$$\frac{1}{n} \sum_{t=1}^{n} \mathbb{I}_B(Y_{v+St}, Y_{v+1+St}, \ldots, Y_{v+m+St}) \longrightarrow P((Y_v, Y_{v+1}, \ldots, Y_{v+m}) \in B), \text{ a.s.} \quad \text{as } n \to \infty,$$

for all $1 \leq v \leq S$, where $\mathbb{I}_B(\cdot)$ denotes the indicator function of the set B. It follows immediately that a standard ergodic process is periodically ergodic with period $S = 1$. As for the stationary case (Billingsley, 1995, Thm 36.4), periodic ergodicity is closed under certain transformations. In particular, if $(\varepsilon_t,\ t \in \mathbb{Z})$ is s.p.s. and periodically ergodic and if $(Y_t,\ t \in \mathbb{Z})$ is given by $Y_t = f(\ldots, \varepsilon_t, \varepsilon_{t+1}, \ldots)$, where f is a measurable function from $\mathbb{R}^\infty$ to $\mathbb{R}$, then $(Y_t,\ t \in \mathbb{Z})$ is also s.p.s. and periodically ergodic (A. Aknouche and H. Guerbyenne, unpublished manuscript). A periodic analogue of the ergodic theorem for stationary sequences can be stated as follows. If $(Y_t,\ t \in \mathbb{Z})$ is s.p.s. and periodically ergodic and if f is a measurable function from $\mathbb{R}^\infty$ to $\mathbb{R}$ such that $E\{f(\ldots,Y_{t-1},Y_t,Y_{t+1},\ldots)\} < \infty$ then for all $1 \leq v \leq S$,

$$\frac{1}{n}\sum_{t=1}^{n} f(\ldots,Y_{v+(t-1)S},Y_{v+tS},Y_{v+(t+1)S},\ldots) \to E\{f(\ldots,Y_{v-S},Y_v,Y_{v+S},\ldots)\}, \text{a.s.} \quad \text{as } n \to \infty.$$

It is well known that the periodic stationarity theory can be translated immediately into a stationarity theory by an appropriate transformation. Since the seminal paper by Gladyshev (1961), we know that a periodically stationary process $(Y_t, t \in \mathbb{Z})$ is equivalent to a vector-valued stationary process $(\underline{Y}_n, n \in \mathbb{Z})$, where

$$\underline{Y}_n = (Y'_{1+nS}, Y'_{2+nS}, \ldots, Y'_{S+nS})',$$

meaning that any property about periodic processes translates at once into a corresponding result about stationary processes. In our case, the process $(\underline{Y}_n, n \in \mathbb{Z})$ satisfies nonuniquely the stochastic difference equation

$$\underline{Y}_n = \underline{A}_n \underline{Y}_{n-1} + \underline{B}_n, \quad n \in \mathbb{Z}, \tag{5}$$

where $\underline{A}_n$ and $\underline{B}_n$ are defined by blocks as

$$\underline{A}_n = \begin{pmatrix} 0_{r\times r} & \ldots & 0_{r\times r} & A_{1+nS} \\ 0_{r\times r} & \ldots & 0_{r\times r} & A_{2+nS}A_{1+nS} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{r\times r} & \ldots & 0_{r\times r} & \prod_{v=0}^{S-1} A_{nS+S-v} \end{pmatrix}_{rS\times rS},$$

$$\underline{B}_n = \begin{pmatrix} B_{1+nS} \\ A_{2+nS}B_{1+nS} + B_{2+nS} \\ \vdots \\ \sum_{k=1}^{S}\left\{\prod_{v=0}^{S-k-1} A_{S-v+nS}\right\}B_{k+nS} \end{pmatrix}_{rS\times 1},$$

with $r = p + q$ and empty products are set equal to one. Therefore, the process, solution of eqn (4) is s.p.s. if and only if the corresponding solution of eqn (5) is strictly stationary. Similarly, the process given by eqn (4) is periodically ergodic if and only if the corresponding process solution of eqn (5) is ergodic. So, we can substitute the study of the properties of model (4) with those of model (5) if they are much easy to obtain. In the sequel, we use only model (4) as it seems to be simpler.

As for strict stationarity, the primordial tool in studying strict periodic stationarity is the top Lyapunov exponent for i.p.d. random matrices, properties of which can be found in A. Aknouche and H. Guerbyenne (unpublished manuscript). Let $||\cdot||$ be an arbitrary operator norm in $\mathcal{M}^r$, the space of real square matrices of dimension r. The top Lyapunov exponent associated with the i.p.d. sequence of matrices $A\colon = (A_t, t \in \mathbb{Z})$ is defined, when

$$\sum_{v=1}^{S} E(\log^+ \|A_v\|) < \infty,$$

by

$$\gamma^S(A) = \inf_{n\in\mathbb{N}^*} \frac{1}{n} E\{\log \|A_{nS}A_{nS-1}\ldots A_1\|\}, \tag{6}$$

where $\log^+(x) = \max(\log(x), 0)$ and $\mathbb{N}^*$ denotes the set of positive integers. For $S = 1$, eqn (6) reduces to the definition of the top Lyapunov exponent for i.i.d. matrices (e.g. Bougerol and Picard, 1992a). It is clear that $\gamma^S(\cdot)$ inherits the properties of the standard top Lyapunov exponent. In particular, the following inequality

$$\gamma^S(A) \leq \sum_{v=1}^{S} E(\log \|A_v\|)$$

holds with equality for the scalar case $r = 1$. Furthermore, when the $(A_t, t \in \mathbb{Z})$ is a sequence of S-periodic nonrandom matrices, then from eqn (6)

$$\gamma^S(A) = \log \lim_{n\to\infty} \left(\left\| \left(\prod_{v=0}^{S-1} A_{S-v} \right)^n \right\| \right)^{1/n} \stackrel{\text{def}}{=} \log \rho \left(\prod_{v=0}^{S-1} A_{S-v} \right).$$

On the other hand, by the i.p.d. property of $(A_t, t \in \mathbb{Z})$, the sequence $(\prod_{v=0}^{S-1} A_{nS-v}, n \in \mathbb{Z})$ is i.i.d.; applying the subadditive ergodic theorem (cf, Kingman 1973) or Theorem 2 of Furstenberg and Kesten (1960) to the sequence, it follows that a.s.

$$\gamma^S(A) = \lim_{n\to\infty} \frac{1}{n} \log \|A_{nS}A_{nS-1}\cdots A_1\|. \tag{7}$$

As emphasized above, properties of model (4) can be obtained from those of model (5), which is a generalized AR with non-negative i.i.d. coefficients for which a fairly exhaustive theory exists (see Bougerol and Picard, 1992b, and the references therein). Since $(\eta_t, t \in \mathbb{Z})$ is i.i.d. with finite variance and the model parameters are S-periodic time-varying, the sequence $((A_t, B_t), t \in \mathbb{Z})$ is s.p.s. and periodically ergodic and the random pair $((\underline{A}_t, \underline{B}_t), t \in \mathbb{Z})$ is stationary and ergodic. Moreover, because

$$\sum_{v=1}^{S} E(\log^+ \|A_v\|) \leq \sum_{v=1}^{S} E(\|A_v\|) \quad \text{and} \sum_{v=1}^{S} E(\log^+ \|B_v\|) \leq \sum_{v=1}^{S} E(\|B_v\|),$$

the expectations $\sum_{v=1}^{S} E(\log^+ \|A_v\|)$, $\sum_{v=1}^{S} E(\log^+ \|B_v\|)$, $E(\log^+\|\underline{A}_0\|)$ and $E(\log^+\|\underline{B}_0\|)$ are finite. Thus, from Theorem 2.3 of Bougerol and Picard (1992b), a necessary and sufficient condition for model (5) to possess a *nonanticipative* (future-independent) strictly stationary solution is that the top Lyapunov exponent associated with $\underline{A} = (\underline{A}_t, t \in \mathbb{Z})$, that is

$$\gamma(\underline{A}) := \inf_{t \in \mathbb{N}^*} \frac{1}{t} E\{(\log \|\underline{A}_t \underline{A}_{t-1} \cdots \underline{A}_1\|)\}$$

is strictly negative.

It must be noted that the latter result is not suitable for our purposes since it gives a condition about the transformed non-periodic model (5), not the original periodic one (4), which is the objective of this section. Fortunately, from the relation between $(\underline{A}_t, t \in \mathbb{Z})$ and $(A_t, t \in \mathbb{Z})$, it is easily seen, taking a multiplicative norm, that $\gamma(\underline{A}) \leq \gamma^S(A)$. So, a sufficient condition for model (4) to possess a nonanticipative s.p.s. solution is that $\gamma^S(A) < 0$, which is Theorem 2 of Bibi and Aknouche (2007). Nevertheless, using model (5) it seems to be difficult to show that the condition $\gamma^S(A) < 0$ is also necessary. Using model (4) directly, Theorem 1 shows that this condition is also necessary.

Theorem 1. *Model* (4) *admits a nonanticipative s.p.s. solution given by*

$$Y_t = \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} A_{t-j} B_{t-k} + B_t, \quad t \in \mathbb{Z}, \tag{8}$$

if and only if the top Lyapunov exponent γ^S(A) given by eqn (6) *is strictly negative, where the above series converges a.s. for all $t \in \mathbb{Z}$. Moreover, the solution is unique and periodically ergodic.*

We now give a necessary condition for the strict stationarity, which coincides for the case $S = 1$ (non-periodic) to Corollary 2.3 of Bougerol and Picard (1992b) and which will be needed in studying asymptotics for the QMLE.

Corollary 1. *If the PGARCH model* (1) *possesses an s.p.s. solution*, *then*

$$\rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v}\right) < 1,$$

where $\boldsymbol{\beta}_t$ is the submatrix of A_t, defined by

$$\boldsymbol{\beta}_t = \begin{pmatrix} \beta_{t,1}\beta_{t,2}\cdots\beta_{t,p-1} & \beta_{t,p} \\ I_{(p-1)\times(p-1)} & 0_{(p-1)\times 1} \end{pmatrix}.$$

We now turn to an important consequence of strict periodic stationarity and the corresponding log-moment condition of Theorem 1, that is the existence of a moment of some positive order. A similar result for the non-periodic GARCH case was proved by Nelson (1990) for the GARCH(1,1) and Berkes *et al.* (2003; Lemma 2.3) for the general *GARCH* case.

Theorem 2. *If $\gamma^S(A) < 0$ then there is $\delta > 0$ such that for all t*

$$E(h_t^{\delta}) < \infty \quad \textit{and} \quad E(y_t^{2\delta}) < \infty. \tag{9}$$

3. ASYMPTOTIC PROPERTIES OF THE QMLE FOR PGARCH MODELS

Let $\theta = (\theta_1', \theta_2', \ldots, \theta_S')'$, where $\theta_v = (\omega_v, \alpha_{v,1}, \ldots, \alpha_{v,q}, \beta_{v,1}, \ldots, \beta_{v,p})'$ for $1 \leq v \leq S$, be the parameter vector which is supposed to belong to a compact parameter space $\Theta \subset (]0, +\infty[\times [0, +\infty[^{(p+q)})^S$. The true parameter value is unknown and is denoted by $\theta^0 = (\theta_1^{0\prime}, \theta_2^{0\prime}, \ldots, \theta_S^{0\prime})'$ with $\theta_v^0 = (\omega_v^0, \alpha_{v,1}^0, \ldots, \alpha_{v,q}^0, \beta_{v,1}^0, \ldots, \beta_{v,p}^0)'$. Consider a time series $\{y_1, y_2, \ldots, y_T, T = NS\}$ generated from model (1) with true parameter $\theta^0 \in \Theta$, i.e. from the model

$$\begin{aligned} y_{v+nS} &= \sqrt{h_{v+nS}}\eta_{v+nS} \\ h_{v+nS} &= \omega_v^0 + \sum_{i=1}^{q} \alpha_{v,i}^0 y_{v+nS-i}^2 + \sum_{j=1}^{p} \beta_{v,j}^0 h_{v+nS-j} \quad n \in \mathbb{Z},\ 1 \leq v \leq S, \end{aligned} \tag{10}$$

where the sample size T is supposed, without loss of generality, to be a multiple of the period S. The Gaussian log-likelihood function of $\theta \in \Theta$ conditional on initial values $y_0, \ldots, y_{1-q}, \tilde{h}_0, \ldots, \tilde{h}_{1-p}$ is given (ignoring a constant) by

$$\tilde{L}_{NS}(\theta) = -\frac{1}{NS} \sum_{k=0}^{N-1} \sum_{v=1}^{S} \tilde{l}_{v+kS}(\theta), \tag{11}$$

with

$$\tilde{l}_t(\theta) = \frac{y_t^2}{\tilde{h}_t} + \log \tilde{h}_t, \quad t \geq 1, \tag{12}$$

where $\tilde{h}_t = \tilde{h}_t(\theta)$ is obtained for $t \geq 1$ from the conditional model

$$\begin{aligned} y_{v+nS} &= \sqrt{\tilde{h}_{v+nS}}\eta_{v+nS} \\ \tilde{h}_{v+nS} &= \omega_v + \sum_{i=1}^{q} \alpha_{v,i} y_{v+nS-i}^2 + \sum_{j=1}^{p} \beta_{v,j} \tilde{h}_{v+nS-j}, \quad n \in \mathbb{N},\ 1 \leq v \leq S, \end{aligned} \tag{13}$$

given initial values $y_0, \ldots, y_{1-q}, \tilde{h}_0, \ldots, \tilde{h}_{1-p}$ ($\mathbb{N}$ being the set of non-negative integers). These values may be chosen, taking into account the S-periodicity of the distribution of $(y_t, t \in \mathbb{Z})$, as an approximation of the unconditional variance. For instance, for the first-order PGARCH(1,1) model, the unconditional variance is given by

$$E(y_t^2) = E(h_t) = \frac{\omega_t + \sum_{j=1}^{S-1} \prod_{i=0}^{j-1} (\alpha_{t-i,1} + \beta_{t-i,1}) \omega_{t-j}}{1 - \prod_{i=0}^{S-1} (\alpha_{t-i,1} + \beta_{t-i,1})},$$

which may be negative for some parameter values from the strict periodic stationarity domain. Thus, taking $\alpha_{t,1} = \beta_{t,1} = 0$, we have the initial values

$$y_0^2 = \tilde{h}_0 = \omega_0, \quad y_{-1}^2 = \tilde{h}_{-1} = \omega_{-1}^2, \cdots, y_{1-d}^2 = \tilde{h}_{1-d} = \omega_{-d}^2, \tag{14a}$$

which can be considered the same for the general PGARCH model. Another choice of $y_0, \ldots, y_{1-q}, \tilde{h}_0, \ldots, \tilde{h}_{1-p}$ is

$$y_0^2 = \tilde{h}_0 = y_S^2, \quad y_{-1}^2 = \tilde{h}_{-1} = y_{[S-1]}^2, \ldots, y_{1-d}^2 = \tilde{h}_{1-d} = y_{[S-d]}^2, \tag{14b}$$

where $d = \max(p, q)$ and $[k] = k$ if $k \geq 1$ and $[k] = lS + k$ otherwise, l being the lower integer such that $lS + k \geq 1$. Obviously, these choices have only a practical value and do not affect the asymptotic properties of the QMLE.

The QMLE of θ^0, denoted by $\hat{\theta}_{NS}$, is the maximizer of $\tilde{L}_{NS}(\theta)$ on Θ and then the minimizer of

$$\frac{1}{NS}\sum_{k=0}^{N-1}\sum_{v=1}^{S}\tilde{l}_{v+kS}(\theta).$$

Let $\gamma^S(A^0)$ denote the top Lyapunov exponent associated with $(A_t^0, t \in \mathbb{Z})$ where A_t^0 is just A_t defined in eqn (4) with θ^0 in place of θ. To study the strong consistency of $\hat{\theta}_{NS}$, consider the following assumptions.

Assumption A1. $\gamma^S(A^0) < 0$ *and* $\forall\theta \in \Theta$,

$$\rho\left(\prod_{v=0}^{S-1}\boldsymbol{\beta}_{S-v}\right) < 1.$$

Assumption A2. *The polynomials*

$$\alpha_v^0(z) = \sum_{j=1}^{q}\alpha_{v,j}^0 z^j \quad \textit{and} \quad \beta_v^0(z) = 1 - \sum_{j=1}^{p}\beta_{v,j}^0\, z^j$$

have no common root, $\alpha_v^0(1) \neq 0$ *and* $\alpha_{v,q}^0 + \beta_{v,p}^0 \neq 0$ *for all* $1 \leq v \leq S$.

Assumption A3. $(\eta_{\mathrm{t}}^2, \mathrm{t} \in \mathbb{Z})$ *is non-degenerate.*

As seen in Theorem 2, Assumption A1 ensures the existence of a finite moment, for the solution of eqn (10), which is the key for proving consistency and asymptotic normality irrespective of any moment condition. The condition

$$\rho\left(\prod_{v=0}^{S-1}\boldsymbol{\beta}_{S-v}\right) < 1$$

is imposed for any $\theta \in \Theta$ in order to obtain $h_t(\theta)$ (to be defined in eqn (16)) as a causal solution of $(y_t, y_{t-1}, \ldots)$, while Assumption A2 is made to guarantee the identifiability of the parameter. Similar conditions for the GARCH case have been considered by Francq and Zakoïan (2004).

Given the realization $\{y_1, \ldots, y_{NS}\}$, $(\tilde{l}_t(\theta))_t$ can be approximated for $1 \leq t \leq NS$ by an s.p.s. and the periodically ergodic sequence $(l_t(\theta))_t$ given by

$$l_t(\theta) = \frac{y_t^2}{h_t(\theta)} + \log h_t(\theta), \tag{15}$$

where $(h_t(\theta), t \in \mathbb{Z})$ is the unique nonanticipative s.p.s. and periodically ergodic solution of

$$h_{v+nS}(\theta) = \omega_v + \sum_{i=1}^{q} \alpha_{v,i} y_{v+nS-i}^2 + \sum_{j=1}^{p} \beta_{v,j} h_{v+nS-j}(\theta), \quad n \in \mathbb{Z}, \ 1 \leq v \leq S, \tag{16}$$

for $\theta \in \Theta$ with $h_t = h_t(\theta^0)$. Let

$$L_{NS}(\theta) = -\frac{1}{NS} \sum_{k=0}^{N-1} \sum_{v=1}^{S} l_{v+kS}(\theta). \tag{17}$$

The following result establishes the strong consistency of $\hat{\theta}_{NS}$.

Theorem 3. *Under Assumptions* A1–A3, $\hat{\theta}_{NS}$ *is strongly consistent in the sense that* $\hat{\theta}_{NS} \rightarrow \theta^0$ *a.s. as* $N \rightarrow \infty$.

Remark 1. From Assumptions A1–A3, it is clear that the above result remains true for the particular periodic ARCH (PARCH) case, i.e. when $\beta_{t,j} = 0$.

Remark 2. In order that Assumption A1 holds for, for example, the particular PGARCH(1,1), we may choose the parameter space Θ as a compact of the form $\Theta = ([\epsilon,1/\epsilon] \times]0,1/\epsilon] \times [0,1-\epsilon])^S$, where $\epsilon > 0$ is so small that the parameter $\theta^0 = (\omega_1^0, \alpha_{1,1}^0, \beta_{1,1}^0, \omega_2^0, \alpha_{2,1}^0, \beta_{2,1}^0, \ldots, \omega_S^0, \alpha_{S,1}^0, \beta_{S,1}^0)'$ belongs to Θ. The fact that $\beta_{v,1}^0$ must be lower than 1 for all v is so restrictive and can be weakened so that

$$\prod_{v=1}^{S} \beta_{v,1}^0 < 1$$

which is just Assumption A1 for the particular PGARCH(1,1).

Now we turn to the asymptotic normality of $\hat{\theta}_{NS}$. Consider the following additional assumptions.

Assumption A4. θ^0 *is in the interior of* Θ.

Assumption A5. $E(\eta_t^4) < \infty$.

Assumption A4 is standard and allows to validate the first-order condition on the maximizer of the log-likelihood while Assumption A5 is necessary for the existence of the limiting covariance matrix of the QMLE.

The following result establishes the asymptotic normality of $\hat{\theta}_{NS}$.

THEOREM 4. *Under Assumptions A1–A5 we have*

$$\sqrt{NS}\left(\hat{\theta}_{NS} - \theta^0\right) \rightsquigarrow N(0, (E(\eta_t^4) - 1)J^{-1}), \quad as\ N \to \infty, \tag{18}$$

where the matrix J given by

$$J := \left[\sum_{v=1}^{S} E_{\theta^0}\left(\frac{\partial^2 l_v(\theta^0)}{\partial\theta\partial\theta'}\right)\right] = \sum_{v=1}^{S} E_{\theta^0}\left(\frac{1}{h_v^2(\theta^0)}\frac{\partial h_v(\theta^0)}{\partial\theta}\frac{\partial h_v(\theta^0)}{\partial\theta'}\right), \tag{19}$$

is block-diagonal.

REMARK 3. Jensen and Rahbek (2004a,b) established consistency and asym ptotic normality of the QMLE for nonstationary time-invariant GARCH models where nonstationarity stems from the fact that the parameters do not belong to the strict stationarity domain (given by $\gamma^S(A) < 0$ for $S = 1$). Thus, it would be fruitful to study asymptotics of the QMLE for the PGARCH model when the parameters are outside the strict periodic stationarity domain, generalizing Jensen and Rahbek's (2004a,b) results. It is also interesting to study asymptotic normality of QMLE when the PGARCH parameter θ is on the boundary of Θ, generalizing the results of Francq and Zakoïan (2007).

REMARK 4. Straumann and Mikosch (2006) considered stationarity and asym ptotic properties of the QMLE of a general class of time-invariant GARCH models in which the conditional variance is defined through a smoothed parametric function of the squared process. However, their general class does not cover our periodic time-varying case.

REMARK 5. It is worth noting that the obtained asymptotic results for the QMLE are still valid for the particular periodic integrated GARCH model obtained from the PGARCH model when the parameters are on the boundary of the second-order periodic stationarity domain. This is because the latter domain is strictly included in the strict periodic stationarity domain, in which it has been proved that these asymptotic results hold.

REMARK 6. When considering the PGARCH model (1), it has been imposed that $\omega_v > 0$ for all $1 \leq v \leq S$ in order to ensure positive conditional (and unconditional) variance. However, as suggested by an anonymous referee, this condition may be restrictive since as showed by Franses and Paap (2000, Appendix), it is possible to have positive conditional variance even when some of the ω_v are negative. This is also true for the unconditional variance (see the expression of $E(y_t^2) = E(h_t)$ in (14)). Franses and Paap (2000) have also estimated negative values for some of the ω_v in a PGARCH(1,1) specification. Hence, instead of imposing $\omega_v > 0$ for $1 \leq v \leq S$, it may be possible to consider the domain $D = \{\omega_v : h_{v+nS} > 0, 1 \leq v \leq S, n \in \mathbb{Z}\}$ which is strictly larger. On this domain, it suffices to consider the Assumptions A1–A5, under which Theorems 3 and 4 remain true.

Let us now apply the foregoing results to the first-order PARCH process given by

$$y_{v+nS} = \eta_{v+nS}\sqrt{\omega_v^0 + \alpha_v^0 y_{v+nS-1}^2}, \quad n \in \mathbb{Z}, \ 1 \le v \le S,$$

where $\omega_v^0 > 0$ and $\alpha_v^0 > 0$. It is easily seen that the strict periodic stationarity condition for the PARCH(1) reduces to

$$0 \le \prod_{v=1}^{S} \alpha_v^0 < \exp\{-E(\log(\eta_t^2))\} := a,$$

under which, supposing that $\theta^0 = (\omega_1^0, \alpha_{1,1}^0, \omega_2^0, \alpha_{2,1}^0, \ldots, \omega_S^0, \alpha_{S,1}^0)'$ belongs to a compact Θ of the form

$$\Theta = ([\epsilon, 1/\epsilon] \times [0, a^{1/S} - \epsilon])^S, \tag{20}$$

($\epsilon > 0$ is defined as in Remark 2) the QMLE is then, by Theorem 3, strongly consistent.

Moreover, if we assume that θ^0 is in the interior of Θ given by eqn (20), then from Theorem 4 the QMLE is also asymptotically Gaussian and its limiting distribution given by eqns (18) and (19) reduces in the PGARCH(1,1) case to

$$J = \begin{pmatrix} J_1 & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ 0_{2\times 2} & J_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0_{2\times 2} \\ 0_{2\times 2} & \cdots & 0_{2\times 2} & J_S \end{pmatrix},$$

$$\text{with } J_v = E\begin{pmatrix} \frac{1}{\left(\omega_v + \alpha_{v,1} y_{v-1}^2\right)^2} & \frac{y_{v-1}^2}{\left(\omega_v + \alpha_{v,1} y_{v-1}^2\right)^2} \\ \frac{y_{v-1}^2}{\left(\omega_v + \alpha_{v,1} y_{v-1}^2\right)^2} & \frac{y_{v-1}^4}{\left(\omega_v + \alpha_{v,1} y_{v-1}^2\right)^2} \end{pmatrix}, \quad 1 \le v \le S.$$

4. ASYMPTOTIC PROPERTIES OF THE QMLE FOR PARMA MODELS WITH PGARCH INNOVATIONS

This section extends the previous asymptotic results to the case where the underlying PGARCH process is no longer directly observed but is an innovation of an observed PARMA model. The proposed results extend on the one hand, Basawa and Lund's asymptotics for PARMA models and, on the other hand QMLE asymptotic properties obtained by Francq and Zakoïan (2004) for ARMA-GARCH models. Besides further tedious but simple technical manipulations, the extension from ARMA-GARCH to its periodic counterpart does not entail any substantial difficulty. As for the ARMA-GARCH case (cf.

Francq and Zakoïan, 2004), consistency of the QMLE will be proved without any additional moment restriction but for asymptotic normality, we need a fourth moment condition on the PGARCH component.

Let $\{X_1, X_2, \ldots, X_T, T = NS\}$ be a time series generated from a nonanticipative s.p.s. solution, $(X_t, t \in \mathbb{Z})$, to the PARMA(P,Q)-PGARCH(p,q) stochastic difference equation

$$X_t - c_t^0 = \sum_{i=1}^{P} \phi_{t,i}^0 (X_{t-i} - c_{t-i}^0) + \varepsilon_t - \sum_{j=1}^{Q} \psi_{t,j}^0 \varepsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{21a}$$

$$\varepsilon_t = \sqrt{h_t}\eta_t, \quad h_t = \omega_t^0 + \sum_{i=1}^{q} \alpha_{t,i}^0 \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_{t,j}^0 h_{t-j}, \tag{21b}$$

where $(\eta_t, t \in \mathbb{Z})$, $\alpha_{t,i}^0$ and $\beta_{t,j}^0$ are defined as in eqn (1) and where c_t^0, $\phi_{t,i}^0$ and $\psi_{t,j}^0$ are periodic in t with period S.

For $\varphi = (\varphi_1', \varphi_2', \ldots, \varphi_S')'$ with $\varphi_v = (c_v, \phi_{v,1}, \ldots, \phi_{v,P}, \psi_{v,1}, \ldots, \psi_{v,Q})'$, $1 \le v \le S$, denote by $\pi = (\varphi', \theta')'$ the model parameter which is supposed to belong to a compact parameter space $\Pi \subset (\mathbb{R}^{P+Q+1})^S \times (]0, +\infty[\times[0, +\infty[^{(p+q)})^S$. The true parameter value denoted by π^0 is still unknown and is supposed to belong to Π. Given initial values $X_0, \ldots, X_{1-q+Q}, \tilde{\varepsilon}_{-q+Q}, \ldots, \tilde{\varepsilon}_{1-Q}, \tilde{h}_0, \ldots, \tilde{h}_{1-p}$ for $q \ge Q$ and $X_0, \ldots, X_{1-q+Q}$, $\tilde{\varepsilon}_0, \ldots, \tilde{\varepsilon}_{1-Q}, \tilde{h}_0, \ldots, \tilde{h}_{1-p}$ otherwise, the Gaussian conditional log-likelihood $\tilde{L}_{NS}(\pi)$ of $\pi \in \Pi$ is now given by

$$\tilde{L}_{NS}(\pi) = -\frac{1}{NS} \sum_{k=0}^{N-1} \sum_{v=1}^{S} \tilde{l}_{v+kS}(\pi), \quad \tilde{l}_t(\pi) = \frac{\tilde{\varepsilon}_t^2}{\tilde{h}_t} + \log \tilde{h}_t,$$

where $\tilde{h}_t = \tilde{h}_t(\pi)$ and $\tilde{\varepsilon}_t = \tilde{\varepsilon}_t(\varphi)$ are obtained for $t \ge 1$ from the conditional PARMA-PGARCH model

$$X_{v+nS} - c_v = \sum_{i=1}^{P} \phi_{v,i} (X_{v+nS-i} - c_{v-i}) + \tilde{\varepsilon}_{v+nS} - \sum_{j=1}^{Q} \psi_{v,j} \tilde{\varepsilon}_{v+nS-j}$$

$$\varepsilon_{v+nS} = \sqrt{\tilde{h}_{v+nS}}\eta_{v+nS}, \tilde{h}_{v+nS} = \omega_v + \sum_{i=1}^{q} \alpha_{v,i}^2 \tilde{\varepsilon}_{v+nS-i}$$

$$+ \sum_{j=1}^{p} \beta_{v,j} \tilde{h}_{v+nS-j}, \quad n \in \mathbb{N}, \ 1 \le v \le S,$$

for the given series $\{X_1, X_2, \ldots, X_{NS}\}$, given $\pi \in \Pi$ and conditional on the aforementioned initial values.

The QMLE of π^0 denoted by $\hat{\pi}_{NS}$ is the maximizer of $\tilde{L}_{NS}(\pi)$ on Π. For $1 \le v \le S$, consider the polynomials

$$\phi_v^0(z) = 1 - \sum_{i=1}^{P} \phi_{v,i}^0 z^i \quad \text{and} \quad \psi_v^0(z) = 1 - \sum_{j=1}^{Q} \psi_{v,j}^0 z^j$$

and the matrices

$$\Phi_v = \begin{pmatrix} \phi_{v,1} \cdots \phi_{v,P-1} & \phi_{v,P} \\ I_{(P-1)\times(P-1)} & 0_{(P-1)\times 1} \end{pmatrix} \quad \text{and} \quad \Psi_v = \begin{pmatrix} \psi_{v,1} \cdots \psi_{v,Q-1} & \psi_{v,Q} \\ I_{(Q-1)\times(Q-1)} & 0_{(Q-1)\times 1} \end{pmatrix}.$$

For the consistency of $\hat{\pi}_{NS}$, we make the following additional assumptions.

Assumption A6. *For all $\pi \in \Pi$,*

$$\rho\left(\prod_{v=0}^{S-1} \Phi_{S-v}\right) < 1 \quad \textit{and} \quad \rho\left(\prod_{v=0}^{S-1} \Psi_{S-v}\right) < 1.$$

Assumption A7. *The polynomials $\phi_v^0(\mathrm{z})$ and $\psi_v^0(\mathrm{z})$ have no common root and $\phi_{v,\mathrm{P}}^0 \neq 0$ or $\psi_{v,\mathrm{Q}}^0 \neq 0$ for all $1 \leq v \leq S$.*

Assumption A6 is made in order to ensure causality and invertibility of the PARMA component given by (21a) (cf. Aknouche, 2007). Equivalent conditions may be found in Tiao and Grupe (1980) and Bentarzi and Hallin (1994). Assumption A7 is an identifiability condition but its first part is also a necessary and sufficient condition for the limiting Fisher information matrix of the pure PARMA component to be invertible (see Theorem 5.1 of Bentarzi and Aknouche, 2005). Under Assumptions A1 and A6, $(X_t, t \in \mathbb{Z})$ is the unique nonanticipative s.p.s. solution to eqn (21a) with finite second moment. Set

$$\varepsilon_{v+nS}(\varphi) = \psi_v^{-1}(L)\phi_v(L)(X_{v+nS} - c_v),$$

(L being the backward shift operator) and let $(h_t(\pi), t \in \mathbb{Z})$ be the unique nonanticipative s.p.s. and periodically ergodic solution to the equation

$$h_{v+nS}(\pi) = \omega_v + \sum_{i=1}^{q} \alpha_{v,i}\varepsilon_{v+nS-i}^2(\varphi) + \sum_{j=1}^{p} \beta_{v,j} h_{v+nS-j}(\pi), \quad n \in \mathbb{Z}, \ 1 \leq v \leq S,$$

from which we have of course $\varepsilon_t(\varphi^0) = \varepsilon_t$ and $h_t(\pi^0) = h_t$. As in the PGARCH case, we approximate $(\tilde{l}_t(\pi))_{t\geq 1}$ by the s.p.s. sequence $(l_t(\pi))_{t \in \mathbb{Z}}$ where

$$l_t(\pi) = \frac{\varepsilon_t^2(\varphi)}{h_t(\pi)} + \log h_t(\pi) \quad \text{for} \quad t \in \mathbb{Z}.$$

Strong consistency of $\hat{\pi}_{NS}$ is established by the following result.

Theorem 5. *Under Assumptions* A1–A3 *and* A6–A7 *we have $\hat{\pi}_{NS} \to \pi^0$ a.s. as $N \to \infty$.*

As for the standard PGARCH case, the process $(\varepsilon_t, t \in \mathbb{Z})$ and hence $(X_t, t \in \mathbb{Z})$ need not have a finite second moment. For the case of PARMA models with i.p.d. innovations (that is, $\varepsilon_t = \eta_t$) our result is a complement to the asymptotic inferences for PARMA models given by Basawa and Lund (2001) which have not established strong consistency of the Gaussian MLE.

While strong consistency of $\hat{\pi}_{NS}$ follows irrespective of any moment requirements, this is not the case for the asymptotic normality. As the standard ARMA-GARCH case (cf. Francq and Zakoïan, 2004), we will prove asymptotic normality of $\hat{\pi}_{NS}$ under the fourth moment condition on the $(\varepsilon_t, t \in \mathbb{Z})$. From Theorem 4 of Bibi and Aknouche (2007), such an assumption is expressed by:

$$\rho\left(\prod_{v=0}^{S-1} E(A_{S-v} \otimes A_{S-v})\right) < 1$$

where '$\otimes$' stands for the Kronecker product. Thus, we make the following assumptions.

Assumption A8.

$$\rho(\prod_{\mathrm{v}=0}^{\mathrm{S}-1} \mathrm{E}(\mathrm{A}_{\mathrm{S}-\mathrm{v}} \otimes \mathrm{A}_{\mathrm{S}-\mathrm{v}})) < 1$$

and for all $\theta \in \Theta$,

$$\rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v}\right) < 1.$$

Assumption A9. π^0 *is in the interior of* Π.

Assumption A10. *There exists no set* Λ *of cardinal* 2 *containing* η_t *with probability* 1.

Assumption A8 clearly implies that $E(\eta_t^4) < \infty$ and $\gamma^S(A) < 0$ for all $\theta \in \Theta$, thereby making Assumption A1 redundant. Assumption A9 is an adaptation of A5 to the PARMA-PGARCH case while Assumption A10, an identifiability assumption, is just Assumption A12 of Francq and Zakoïan (2004) and which is required for the proof of the existence and invertibility of the limiting variance of the QMLE. Theorem establishes the $\sqrt{NS}$-consistency of $\hat{\pi}_{NS}$.

Theorem 6. *Under Assumptions* A2–A3 *and Assumptions* A6–A10 *we have*

$$\sqrt{NS}(\hat{\pi}_{NS} - \pi^0) \rightsquigarrow N(0, J^{-1}IJ^{-1}), \quad \text{as } N \to \infty,$$

where the matrices I and J given by

$$I = \left[\sum_{v=1}^{S} E_{\pi^0}\left(\frac{\partial l_v(\pi^0)}{\partial \pi}\frac{\partial l_v(\pi^0)}{\partial \pi'}\right)\right], \quad J = \left[\sum_{v=1}^{S} E_{\pi^0}\left(\frac{\partial^2 l_v(\pi^0)}{\partial \pi \partial \pi'}\right)\right],$$

may be partitioned as

$$I = \begin{pmatrix} I_{\varphi\varphi} & I_{\varphi\theta} \\ I_{\theta\varphi} & I_{\theta\theta} \end{pmatrix}, \quad J = \begin{pmatrix} J_{\varphi\varphi} & J_{\varphi\theta} \\ J_{\theta\varphi} & J_{\theta\theta} \end{pmatrix},$$

such that the sub-matrices $I_{\varphi\varphi}$, $I_{\varphi\theta}$, $I_{\theta\theta}$, $J_{\varphi\varphi}$, $J_{\varphi\theta}$ and $J_{\theta\theta}$ are S-block-diagonal. Moreover, if the distribution of η_t is symmetric, then

$$I = \begin{pmatrix} I_{\varphi\varphi} & 0 \\ 0 & I_{\theta\theta} \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} J_{\varphi\varphi} & 0 \\ 0 & J_{\theta\theta} \end{pmatrix}$$

where

$$\begin{aligned}
I_{\varphi\varphi} &= (E(\eta_t^4) - 1)\sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v^2}\frac{\partial h_v}{\partial \varphi}\frac{\partial h_v}{\partial \varphi'}(\pi^0)\right) + 4\sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v}\frac{\partial \varepsilon_v}{\partial \varphi}\frac{\partial \varepsilon_v}{\partial \varphi'}(\pi^0)\right),\\
I_{\theta\theta} &= (E(\eta_t^4) - 1)\sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v^2}\frac{\partial h_v}{\partial \theta}\frac{\partial h_v}{\partial \theta'}(\pi^0)\right),\\
J_{\varphi\varphi} &= \sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v^2}\frac{\partial h_v}{\partial \varphi}\frac{\partial h_v}{\partial \varphi'}(\pi^0)\right) + 2\sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v}\frac{\partial \varepsilon_v}{\partial \varphi}\frac{\partial \varepsilon_v}{\partial \varphi'}(\pi^0)\right),\\
J_{\theta\theta} &= \sum_{v=1}^{S} E_{\pi^0}\left(\frac{1}{h_v^2}\frac{\partial h_v}{\partial \theta}\frac{\partial h_v}{\partial \theta'}(\pi^0)\right).
\end{aligned}$$

Remark 7. The fact that the submatrices $I_{\varphi\varphi}$, $I_{\varphi\theta}$, $I_{\theta\theta}$, $J_{\varphi\varphi}$, $J_{\varphi\theta}$ and $J_{\theta\theta}$ are S-block diagonal implies the asymptotic independence of the estimates for each season $1 \leq v \leq S$ which is not a surprising result in periodic time-varying models. Moreover, the asymptotic independence also appears for the estimates of the PARMA component and those of the PGARCH component when the distribution of η_t is symmetric.

Remark 8. It is clear from Theorem 6 that when $\varepsilon_t = \eta_t$ for all t, our asymptotic results coincide with those for the pure PARMA model with i.i.d. innovations (cf. Basawa and Lund, 2001).

APPENDIX

Appendix A. Proofs of Section 2s results

Proof of Theorem 1. We only give a proof of the necessity part of the theorem as the sufficiency part was established earlier by Bibi and Aknouche (2007). Suppose that model

(4) admits a nonanticipative s.p.s. solution $(Y_t, t \in \mathbb{Z})$. Then iterating eqn (4) m times for some $m > 1$ and some $v \in \{1, \ldots, S\}$ gives

$$Y_v = \sum_{j=0}^{m} \prod_{i=0}^{j-1} A_{v-i} B_{v-j} + \prod_{j=0}^{m+1} A_{v-j} Y_{v-m-1}.$$

Exploiting the non-negativity of the coefficients of A_t and those of Y_v it follows that for all $m > 1$,

$$Y_v \geq \sum_{j=0}^{m} \prod_{i=0}^{j-1} A_{v-i} B_{v-j}, \quad \text{a.s.},$$

implying that the series

$$\sum_{j=0}^{\infty} \prod_{i=0}^{j-1} A_{v-i} B_{v-j}$$

converges a.s. Therefore,

$$\prod_{i=0}^{j-1} A_{v-i} B_{v-j} \to 0, \quad \text{a.s.} \quad \text{as } j \to \infty, \tag{A.1}$$

from which we want to show that

$$\prod_{i=0}^{j-1} A_{v-i} \to 0, \quad \text{a.s.} \quad \text{as } j \to \infty. \tag{A.2}$$

This holds whenever

$$\lim_{j \to \infty} \prod_{i=0}^{j-1} A_{v-i} e_k = 0, \text{ a.s.} \quad \text{for all } 1 \leq k \leq r, \tag{A.3}$$

where $r = p + q$ and $(e_k)_{1 \leq k \leq r}$ is the canonical basis of $\mathbb{R}^r$. Observing that $B_{v-j} = \omega_{v-j} \eta_{v-j}^2 e_1 + \omega_{v-j} e_{q+1}$, (A.1) implies that

$$\omega_{v-j} \eta_{v-j}^2 \prod_{i=0}^{j-1} A_{v-i} e_1 \to 0, \text{ a.s. as } j \to \infty \quad \text{and} \quad \omega_{v-j} \prod_{i=0}^{j-1} A_{v-i} e_{q+1} \to 0, \text{a.s.} \quad \text{as } j \to \infty, \tag{A.4}$$

and since $\omega_{v-j} > 0$, this shows that eqn (\hbox{A}.3) for $k = q + 1$. Moreover, since for all $k = 1, \ldots, p$

$$A_{v-j+1} e_{q+k} = \beta_{v-j+1,k} \eta_{v-j+1}^2 e_1 + \beta_{v-j+1,k} e_{q+1} + e_{q+k+1},$$

then for $k = 1$ we have a.s.

$$0 = \lim_{j \to \infty} \prod_{i=0}^{j-1} A_{v-i} e_{q+1} \geq \lim_{j \to \infty} \prod_{i=0}^{j-1} A_{v-i} e_{q+2} \geq 0,$$

showing eqn (A.3) for $k = q + 2$ and then recursively for $k = q + j$, $1 \leq j \leq p$. On the other hand, since

$$A_{v-j+1}e_q = \alpha_{v-j+1,q}\eta^2_{v-j+1}e_1 + \alpha_{v-j+1,q}e_{q+1},$$

then from eqn (A.4), follows (A.3) for $k = q$. Finally, from the identity

$$A_{v-j+1}e_k = \alpha_{v-j+1,k}\eta^2_{v-j+1}e_1 + \alpha_{v-j+1,k}e_{q+1} + e_{k+1}, \quad k = 1, \ldots, q-1,$$

eqn (A.3) follows for the rest values of k using a backward recursion. This shows eqn (A.2), which implies that any subsequence of the sequence

$$(U_j)_j := \left(\prod_{i=0}^{j-1} A_{v-i}\right)_j$$

converges a.s. to zero as $j \to \infty$. In particular, for the subsequence $(U_{jS})_j$ we have

$$U_{jS} = \prod_{i=0}^{Sj-1} A_{v-i} = \prod_{i=0}^{j-1}\left(\prod_{k=0}^{S-1} A_{v-iS-k}\right) \to 0, \text{ a.s.} \quad \text{as } j \to \infty. \tag{A.5}$$

Now, since the sequence

$$\left(\prod_{k=0}^{S-1} A_{v-iS-k}, i \in \mathbb{Z}\right)$$

is i.i.d. for all v, then from eqn (A.5) and Lemma 2.1 of Bougerol and Picard (1992b) it follows that the top Lyapunov exponent associated with the i.i.d. sequence

$$\left(\prod_{k=0}^{S-1} A_{v-iS-k}, i \in \mathbb{Z}\right),$$

which is exactly $\gamma^S(A)$, is strictly negative. This completes the proof. □

Proof of Corollary 1. As the $(A_t, t \in \mathbb{Z})$ is non-negative, the corresponding top Lyapunov exponent $\gamma^S(A)$ is greater that the top Lyapunov exponent corresponding to the nonrandom sequence of matrices, *say* $\boldsymbol{\beta} = (\boldsymbol{\beta}_t, t \in \mathbb{Z})$, obtained when one sets the elements of the first q lines and those of the first q columns to be equal to zero. In other words,

$$\gamma^S(A) \geq \gamma^S(\boldsymbol{\beta}) := \log \rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v}\right).$$

Thus, if the model has an s.p.s. solution, which is equivalent to the condition $\gamma^S(A) < 0$, the conclusion of the corollary is true. □

Proof of Theorem 2. The proof is similar to that of Lemma 2.3 of Berkes *et al.* (2003). First, we have to show that if $\gamma^S(A) < 0$ then there is $\delta > 0$ and n_0 such that

$$E(\|A_{n_0S}A_{n_0S-1}\ldots A_1\|^\delta) < 1. \tag{A.6}$$

Since

$$\gamma^S(A) = \inf_{n \in \mathbb{N}^*}\left\{\frac{1}{n}E(\log\|A_{nS}A_{nS-1}\ldots A_1\|)\right\}$$

is strictly negative, there is a positive integer n_0 such that

$$E(\log \|A_{n_0S}A_{n_0S-1}\ldots A_1\|) < 0.$$

On the other hand, working with a multiplicative norm and by the i.p.d. property of the sequence $(A_t, t \in \mathbb{Z})$ we have

$$\begin{aligned} E(\|A_{n_0S}A_{n_0S-1}\ldots A_1\|) &= \|E(A_{n_0S}A_{n_0S-1}\ldots A_1)\| \\ &= \|E(A_SA_{S-1}\ldots A_1)^{n_0}\| \\ &\leq \|E(A_SA_{S-1}\ldots A_1)\|^{n_0} < \infty. \end{aligned}$$

Let

$$f(t) = E(\|A_{n_0S}A_{n_0S-1}\ldots A_1\|^t).$$

Since

$$f'(0) = E(\log \|A_{n_0S}A_{n_0S-1}\ldots A_1\|) < 0,$$

$f(t)$ decrease in a neighbourhood of 0 and since $f(0) = 1$, it follows that there exists $0 < \delta < 1$ such that eqn (A.6) holds.

Now from Theorem 1 we have for some $v \in \{1, \ldots, S\}$

$$\|Y_v\| \leq \sum_{k=1}^{\infty} \left\| \prod_{j=0}^{k-1} A_{v-j} \right\| \|B_{v-k}\| + \|B_v\|.$$

Because $0 < \delta < 1$, then

$$\|Y_v\|^{\delta} \leq \sum_{k=1}^{\infty} \left\| \prod_{j=0}^{k-1} A_{v-j} \right\|^{\delta} \|B_{v-k}\|^{\delta} + \|B_v\|^{\delta},$$

which, by the independence of A_{v-j} and B_{v-k} for $j < k$, implies that

$$\begin{aligned} E\|Y_v\|^{\delta} &\leq \sum_{k=1}^{\infty} E\left(\left\| \prod_{j=0}^{k-1} A_{v-j} \right\|^{\delta} \right) E(\|B_{v-k}\|^{\delta}) + E(\|B_v\|^{\delta}) \\ &\leq B(\delta) \sum_{k=1}^{\infty} E\left(\left\| \prod_{j=0}^{k-1} A_{v-j} \right\|^{\delta} \right) + E(\|B_v\|^{\delta}), \end{aligned}$$

where

$$B(\delta) = \max_{0 \leq v \leq S-1} E\left(\|B_{v-k}\|^{\delta} \right).$$

Using eqn (A.6) there exist $a_v > 0$ and $0 < b_v < 1$ such that

$$E\left(\left\| \prod_{j=0}^{k-1} A_{v-j} \right\|^{\delta} \right) \leq a_v b_v^k \leq ab^k,$$

where

$$ab^k = \max_{0 \leq v \leq S-1} \{a_v b_v^k\}.$$

This proves that $E||Y_v||^{\delta} < \infty$ and hence eqn (9). □

Appendix B. Proofs of Theorems 3 and 4

The proofs of Theorems 3 and 4 are by now standard and follow from similar arguments used in showing the strong consistency and asymptotic normality of the QMLE for standard GARCH models (Francq and Zakoïan, 2004). The main aim here is to reveal the basic assumptions and to quantify the asymptotic distribution of the QMLE for the PGARCH. Since there are several similarities between the standard GARCH and the PGARCH , certain steps of the proof for the PGARCH case are similar in spirit to that of the standard GARCH one. Thus, we give proofs only when it seems pertinent to us and refer to Francq and Zakoïan (2004) for further details.

Proof of Theorem 3. Theorem 3 will be proved by showing several lemmas. Lemma B.1 establishes the uniform asymptotic forgetting of initial values, Lemma B.2 ensures identifiability of the parameter, Lemma B.3 shows the finiteness of the limiting criterion $\sum_{v=1}^{S} E_{\theta^0}(\mathrm{l}_v(\theta))$ and that this one is uniquely minimized at θ^0, while Lemma B.4 uses the compactness of Θ and a periodically ergodic argument to conclude the strong convergence. In the sequel, $M > 0$ and $\rho \in]0,1[$ will denote any constants non necessarily the same when appearing in different terms.

Lemma B.1. *Under Assumption* A1 *we have*

$$\sup_{\theta\in\Theta}\left|L_{NS}(\theta) - \tilde{L}_{NS}(\theta)\right| \to 0, a.s. \quad as\ N \to \infty,$$

Proof. Rewriting eqn (16) in a vectorial form as follows

$$\underline{h}_t = \boldsymbol{\beta}_t \underline{h}_{t-1} + \underline{\boldsymbol{\alpha}}_t, \quad t \in \mathbb{Z}, \tag{A.7}$$

where

$$\underline{h}_t = (h_t(\theta), h_{t-1}(\theta), \ldots, h_{t-p+1}(\theta))' \quad \text{and} \quad \underline{\boldsymbol{\alpha}}_t = \left(\omega_t + \sum_{i=1}^{q} \alpha_{t,i} y_{t-i}^2, 0, \ldots, 0\right)'_{1\times p}.$$

From Corollary 1, the assumption

$$\rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v}\right) < 1$$

of Assumption A1 and the compactness of Θ we have

$$\sup_{\theta\in\Theta} \rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v}\right) < 1, \tag{A.8}$$

which implies, by iterating eqn (A.7), that

$$\underline{h}_t = \sum_{k=0}^{t-1} \prod_{i=0}^{k-1} \boldsymbol{\beta}_{t-i} \underline{\boldsymbol{\alpha}}_{t-k} + \prod_{i=0}^{t} \boldsymbol{\beta}_{t-i} \underline{h}_0, \quad t \in \mathbb{Z}.$$

If we denote by $\underline{\tilde{h}}_t$ and $\underline{\tilde{\boldsymbol{\alpha}}}_t$ the vectors obtained from $\underline{h}_t$ and $\underline{\boldsymbol{\alpha}}_t$, respectively, while replacing $h_{t-j}(\theta)$ by $\tilde{h}_{t-j}$ with initial values given by eqn (14), we have

$$\underline{\tilde{h}}_t = \sum_{k=0}^{t-q-1} \prod_{i=0}^{k-1} \boldsymbol{\beta}_{t-i} \underline{\boldsymbol{\alpha}}_{t-k} + \sum_{k=t-q}^{t-1} \prod_{i=0}^{k-1} \boldsymbol{\beta}_{t-i} \underline{\tilde{\boldsymbol{\alpha}}}_{t-k} + \prod_{i=0}^{t} \boldsymbol{\beta}_{t-i} \underline{\tilde{h}}_0. \tag{A.9}$$

From eqn (A.8) it follows that

$$\begin{aligned}\sup_{\theta\in\Theta} \|\underline{h}_t - \underline{\tilde{h}}_t\| &= \sup_{\theta\in\Theta} \left\| \sum_{k=t-q}^{t-1} \prod_{i=0}^{k-1} \boldsymbol{\beta}_{t-i} (\underline{\boldsymbol{\alpha}}_{t-k} - \underline{\tilde{\alpha}}_{t-k}) + \prod_{i=0}^{t-1} \boldsymbol{\beta}_{t-i} (\underline{h}_0 - \underline{\tilde{h}}_0) \right\| \\ &\leq M\rho^t,\end{aligned}$$

from which we get

$$\begin{aligned}\sup_{\theta\in\Theta} \left| L_{NS}(\theta) - \tilde{L}_{NS}(\theta) \right| &\leq \frac{1}{N} \sum_{t=1}^{NS} \sup_{\theta\in\Theta} \left[\left| \frac{\tilde{h}_t - h_t(\theta)}{\tilde{h}_t h_t(\theta)} \right| y_t^2 + \left| \log\left(\frac{h_t(\theta)}{\tilde{h}_t} \right) \right| \right] \\ &\leq \left(\max_{1\leq v\leq S} \sup_{\theta\in\Theta} (\omega_v^{-2}) \right) \frac{M}{N} \sum_{t=1}^{NS} \rho^t y_t^2 + \left(\max_{1\leq v\leq S} \sup_{\theta\in\Theta} (\omega_v^{-1}) \right) \frac{M}{N} \sum_{t=1}^{NS} \rho^t,\end{aligned}$$

where the inequality

$$\left| \log \frac{y}{x} \right| \leq \frac{|y - x|}{\min(y, x)}$$

for positive x and y has been used. The existence of a moment for y_t (cf, Corollary 1) implies, by the Borel-Cantelli lemma, that $\rho^t y_t^2 \to 0$ a.s. so that the conclusion of the lemma follows from the Toeplitz lemma. □

Lemma B.2. *Under Assumptions* A1–A3 *there is $t \in \mathbb{Z}$ such that if $h_t(\theta) = h_t(\theta^0)$ a.s. then $\theta = \theta^0$.*

Proof. From the assumption

$$\rho\left(\prod_{v=0}^{S-1} \boldsymbol{\beta}_{S-v} \right) < 1$$

of Assumption A1 the polynomials $(\beta_v^0(L))_{1\leq v\leq S}$ are invertible. Suppose that $h_t(\theta) = h_t(\theta^0)$ a.s. for some t, then using (3.7) we have

$$\left(\frac{\alpha_v(L)}{\beta_v(L)} - \frac{\alpha_v^0(L)}{\beta_v^0(L)} \right) y_{v+nS}^2 = \left(\frac{\omega_v^0}{\beta_v^0(L)} - \frac{\omega_v}{\beta_v(L)} \right), \quad \text{for all } 1 \leq v \leq S.$$

If

$$\frac{\alpha_v(L)}{\beta_v(L)} \neq \frac{\alpha_v^0(L)}{\beta_v^0(L)} \quad \text{for some } 1 \leq v \leq S$$

then there exists a deterministic periodic time-varying combination of $y_{v+St-j}^2, j \geq 1$. This contradicts the fact that

$$y_{v+St}^2 = E(y_{v+St}^2 / y_{v+St-1}^2, \ldots) + h_{v+St}(\eta_{v+St}^2 - 1),$$

since by Assumption A3 $(\eta_t, t \in \mathbb{Z})$ is non-degenerate. Therefore,

$$\frac{\alpha_v(z)}{\beta_v(z)} = \frac{\alpha_v^0(z)}{\beta_v^0(z)} \forall |z| \leq 1 \quad \text{and} \quad \frac{\omega_v^0}{\beta_v^0(L)} - \frac{\omega_v}{\beta_v(L)}, \text{ for all } 1 \leq v \leq S,$$

which by the assumption A2 of absence of common root implies that $\alpha_v(z) = \alpha_v^0(z), \beta_v(z) = \beta_v^0(z)$ and $\omega_v = \omega_v^0$ for all $1 \leq v \leq S$, proving the lemma. □

Lemma B.3. *Under Assumption* A1

$$\sum_{v=1}^{S} E_{\theta^0}(l_v(\theta^0)) < \infty,$$

and $\sum_{v=1}^{S} E_{\theta^0}(l_v(\theta))$ *is minimum at* $\theta = \theta^0$.

Proof. By Theorem 2 we have

$$\begin{aligned}\sum_{v=1}^{S} E_{\theta^0}(\log h_v(\theta^0)) &= \sum_{v=1}^{S} E_{\theta^0} \frac{1}{\delta}(\log h_v(\theta^0)^{\delta}) \\ &\leq \frac{1}{\delta}\sum_{v=1}^{S} \log E_{\theta^0}(h_v(\theta^0)^{\delta}) < \infty,\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_{v=1}^{S} E_{\theta^0}(l_v(\theta^0)) &= \sum_{v=1}^{S} E_{\theta^0}\left[\frac{h_v(\theta^0)\eta_v}{h_v(\theta^0)} + \log h_v(\theta^0)\right] \\ &= S + \sum_{v=1}^{S} E_{\theta^0}(\log h_v(\theta^0)) < \infty.\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{v=1}^{S} E_{\theta^0}(l_v(\theta)) - \sum_{v=1}^{S} E_{\theta^0}(l_v(\theta^0)) &= \sum_{v=1}^{S} E_{\theta^0}\left[\log\frac{h_v(\theta)}{h_v(\theta^0)} + \frac{h_v(\theta^0)}{h_v(\theta)} - 1\right] \\ &\geq \sum_{v=1}^{S} E_{\theta^0}\left[\log\frac{h_v(\theta)}{h_v(\theta^0)} + \log\frac{h_v(\theta^0)}{h_v(\theta)}\right] = 0,\end{aligned} \tag{A.10}$$

showing that the limit criterion is minimized at θ^0. □

Lemma B.4. *Under* A1, *for all* $\theta \neq \theta^0$ *there is a neighbourhood* $\mathcal{V}(\theta)$ *such that*

$$\lim_{N\to\infty} \inf \inf_{\tilde{\theta}\in\mathcal{V}(\theta)} \left(-\frac{1}{N}\tilde{L}_{NS}\left(\tilde{\theta}\right)\right) > \sum_{v=1}^{S} E_{\theta^0}(l_v(\theta^0)).$$

Proof. For all $\theta \in \Theta$ and all integer k, let $\mathcal{V}_k(\theta)$ be an open sphere of centre θ and radius $1/k$. Using Lemma B.1 we have

$$
\begin{aligned}
\lim_{N\to\infty} \inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} \inf\left(-\frac{1}{N}\tilde L_{NS}\left(\tilde\theta\right)\right) &\geq \lim_{N\to\infty} \inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} \inf\left(-\frac{1}{N}L_{NS}\left(\tilde\theta\right)\right)\\
&\quad - \lim_{N\to\infty} \sup \frac{1}{N}\sup_{\theta\in\Theta}\left|\tilde L_{NS}(\theta)-L_{NS}(\theta)\right|\\
&\geq \lim_{N\to\infty} \inf \frac{1}{N}\sum_{n=0}^{N-1}\sum_{v=1}^{S}\inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} l_{v+nS}\left(\tilde\theta\right).
\end{aligned}
$$

Applying the ergodic theorem for the i.i.d. sequence

$$
\left(\sum_{v=1}^{S} l_{v+nS}\left(\tilde\theta\right)\right)_n \quad \text{with } E\left(\sum_{v=1}^{S} l_{v+nS}\left(\tilde\theta\right)\right)\in\mathbb{R}\cup\{\infty\}
$$

(cf. Billingsley 1995, p. 284, 495) it follows that

$$
\lim_{N\to\infty} \inf \frac{1}{N}\sum_{n=0}^{N-1}\sum_{v=1}^{S}\inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} l_{v+nS}\left(\tilde\theta\right) = \sum_{v=1}^{S} E_{\theta^0}\left(\inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} l_{v+nS}(\tilde\theta)\right),
$$

and by the Beppo-Levi theorem (e.g. Billingsley, 1995, p. 219), we have

$$
\sum_{v=1}^{S} E_{\theta^0}\left(\inf_{\tilde\theta\in\mathcal{V}_k(\theta)\cap\Theta} l_{v+nS}(\tilde\theta)\right) \to \sum_{v=1}^{S} E_{\theta^0}(l_v(\theta)) \quad \text{as } k\to\infty,
$$

which by eqn (A.10) proves the lemma. □

Proof of Theorem 3. In view of Lemmas B.1–B.4, the proof of the theorem is completed by using an argument of compactness of Θ. First, for all neighbourhood $\mathcal{V}(\theta^0)$ *of* θ^0 we have

$$
\begin{aligned}
\lim_{N\to\infty} \sup \inf_{\tilde\theta\in\mathcal{V}(\theta^0)}\left(-\frac{1}{N}\tilde L_{NS}\left(\tilde\theta\right)\right) &\leq \lim_{N\to\infty}\left(-\frac{1}{N}\tilde L_{NS}(\theta^0)\right) = \lim_{N\to\infty}\left(-\frac{1}{N}L_{NS}(\theta^0)\right)\\
&= \sum_{v=1}^{S} E_{\theta^0}\left(l_v(\theta^0)\right). \qquad \text{(A.11)}
\end{aligned}
$$

The compact Θ is recovered by a union of a neighbourhood $\mathcal{V}(\theta^0)$ of θ^0 and the set of neighbourhoods $\mathcal{V}(\theta)$,$\theta\in\Theta\backslash\mathcal{V}(\theta^0)$, where $\mathcal{V}(\theta)$ fulfills Lemma B.4. Therefore, there exists a finite sub-covering of Θ by $\mathcal{V}(\theta_0)$,$\mathcal{V}(\theta_1)$,…,$\mathcal{V}(\theta_k)$ such that

$$
\inf_{\theta\in\Theta}\left(-\frac{1}{N}\tilde L_{NS}\left(\tilde\theta\right)\right) = \min_{i\in\{1,2,\ldots,k\}} \inf_{\tilde\theta\in\Theta\cap\mathcal{V}(\theta_i)}\left(-\frac{1}{N}\tilde L_{NS}\left(\tilde\theta\right)\right).
$$

From eqn (A.11) and Lemma B.4, the latter relation shows that $\hat\theta_{NS}\in\mathcal{V}(\theta^0)$ for N sufficiently large, which complete the proof of the theorem. □

Proof of Theorem 4. The proof of Theorem 4 is based, along the lines of Francq and Zakoïan (2004), on a Taylor expansion of $\partial\tilde L_{NS}(\theta)/\partial\theta$ at θ^0 which is given by

$$
0=(N)^{-1/2}\sum_{t=1}^{NS}\frac{\partial\tilde l_t}{\partial\theta}\left(\hat\theta_{NS}\right) = (N)^{-1/2}\sum_{t=1}^{NS}\frac{\partial\tilde l_t}{\partial\theta}(\theta^0) + \left((N)^{-1}\sum_{t=1}^{NS}\frac{\partial^2\tilde l_t}{\partial\theta\partial\theta'}\left(\tilde\theta\right)\right)(N)^{1/2}\left(\hat\theta_{NS}-\theta^0\right),
$$

where the coordinates of $\tilde{\theta}$ are between the corresponding entrees of $\hat{\theta}_{NS}$ and those of θ^0.

The proof is also based on several lemmas, Lemmas B.5–B.9, which are aimed at establishing: the integrability of the first derivatives of the limiting criterion at θ^0, the invertibility of J and its relation with the first derivatives of the limiting criterion, the uniform integrability of the third-order derivatives of the limiting criterion, the asymptotic forgetting of starting values for the derivatives, and a central limit theorem for martingale difference together with a periodically ergodic theorem for the second derivatives of the criterion.

Lemma B.5. *We have*

$$\sum_{v=1}^{S} E_{\theta^0}\left\|\frac{\partial l_v(\theta^0)}{\partial\theta}\frac{\partial l_v(\theta^0)}{\partial\theta'}\right\| < \infty \quad \textit{and} \quad \sum_{v=1}^{S} E_{\theta^0}\left\|\frac{\partial^2 l_v(\theta^0)}{\partial\theta\partial\theta'}\right\| < \infty.$$

Lemma B.6. *Under Assumptions* A1–A5, *J is invertible and*

$$\sum_{v=1}^{S} E_{\theta^0}\left(\frac{\partial l_v(\theta^0)}{\partial\theta}\frac{\partial l_v(\theta^0)}{\partial\theta'}\right) = (E(\eta_t^4) - 1)J.$$

Lemma B.7. *The following limit relations*

$$N^{-\frac{1}{2}}\left\|\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial l_{v+kS}(\theta^0)}{\partial\theta} - \frac{\partial \tilde{l}_{v+kS}(\theta^0)}{\partial\theta}\right\| = o_p(1),$$

$$\sup_{\theta\in\mathcal{V}(\theta^0)} N^{-1}\left\|\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial^2 l_{v+kS}(\theta^0)}{\partial\theta\partial\theta'} - \frac{\partial^2 \tilde{l}_{v+kS}(\theta^0)}{\partial\theta\partial\theta'}\right\| = o_p(1),$$

hold.

Lemma B.8. *There is a neighbourhood* $\mathcal{V}(\theta^0)$ *of* θ^0 *such that for all* $i, j, k \in \{1, \ldots, S(p+q+1)\}$

$$\sum_{v=1}^{S} E_{\theta^0} \sup_{\theta\in\mathcal{V}(\theta^0)}\left|\frac{\partial^3 l_v(\theta)}{\partial\theta_i\partial\theta_j\partial\theta_k}\right| < \infty.$$

Lemma B.9. *The following limit results are true*

$$N^{-\frac{1}{2}}\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial l_{v+kS}}{\partial\theta}(\theta^0) \rightsquigarrow N(0, (E(\eta_t^4) - 1)J) \quad \textit{and} \quad N^{-1}\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial^2 l_{v+kS}}{\partial\theta\partial\theta'}(\tilde{\theta}) \to J, \ \ \textit{a.s.}$$

The proofs are very similar to those of Francq and Zakoïan (2004). It suffices to replace the stationarity and ergodicity arguments by the periodic stationarity and periodic ergodicity ones, respectively. For this we omit the proofs which are quite lengthy. □

Appendix C. Proofs of Theorems 5 and 6

The proofs of Theorems 5 and 6 are simple adaptations of those of Theorem 3 and 4

Proof of Theorem. Along the lines of Theorem 4's proof, Theorem 5 will be proved whenever establishing the following lemmas.

Lemma C.1. *Under Assumptions* A1 *and* A6 *we have*

$$\sup_{\pi\in\Pi}\left|L_{NS}(\pi)-\tilde{L}_{NS}(\pi)\right| \to 0, \quad a.s. \quad as\ N\to\infty,$$

Proof. Relation (A.9) is now replaced by

$$\underline{\tilde{h}}_t=\sum_{k=0}^{t-1}\prod_{i=0}^{k-1}\boldsymbol{\beta}_{t-i}\underline{\tilde{\boldsymbol{\alpha}}}_{t-k}+\prod_{i=0}^{t}\boldsymbol{\beta}_{t-i}\underline{\tilde{h}}_0. \tag{A.12}$$

where

$$\underline{\tilde{\boldsymbol{\alpha}}}_t=\left(\omega_t+\sum_{i=1}^{q}\alpha_{t,i}\tilde{\varepsilon}_{t-i}^2,0,\ \ldots,\ 0\right)'_{1\times p}.$$

From Assumption A6, the coefficients of the PAR(∞) representation of X_t decrease geometrically to zero. Using the compactness assumption on Π, it follows that for $k\geq 1$ and $1\leq i\leq q$,

$$\sup_{\pi\in\Pi}\left|\varepsilon_{k-i}^2(\varphi)-\tilde{\varepsilon}_{k-i}^2\right|\leq M\rho^k,$$

so that

$$\begin{aligned}\|\underline{\boldsymbol{\alpha}}_{t-k}-\underline{\tilde{\boldsymbol{\alpha}}}_{t-k}\| &\leq \sum_{i=1}^{q}|\alpha_{t-k,i}|\left|\varepsilon_{k-i}^2(\varphi)-\tilde{\varepsilon}_{k-i}^2\right| \\ &\leq M\rho^k\left(\sum_{i=1}^{q}1+|\varepsilon_{k-i}^2(\varphi)|\right).\end{aligned}$$

Thus, from eqn (A.12) we have

$$\begin{aligned}\sup_{\theta\in\Theta}\|\underline{h}_t-\underline{\tilde{h}}_t\| &= \sup_{\theta\in\Theta}\left\|\sum_{k=t-q}^{t-1}\prod_{i=0}^{k-1}\boldsymbol{\beta}_{t-i}(\underline{\boldsymbol{\alpha}}_{t-k}-\underline{\tilde{\boldsymbol{\alpha}}}_{t-k})+\prod_{i=0}^{t-1}\boldsymbol{\beta}_{t-i}\left(\underline{h}_0-\underline{\tilde{h}}_0\right)\right\| \\ &\leq M\rho^t\left(\sum_{k=-q}^{t}1+|\varepsilon_k^2(\varphi)|\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\theta\in\Theta}\left|L_{NS}(\theta)-\tilde{L}_{NS}(\theta)\right| &\leq \frac{1}{N}\sum_{t=1}^{NS}\sup_{\theta\in\Theta}\left[\left|\frac{\tilde{h}_t-h_t(\theta)}{\tilde{h}_t h_t(\theta)}\right|\varepsilon_t^2+\left|\log\left(\frac{h_t(\theta)}{\tilde{h}_t}\right)\right|+\frac{|\varepsilon_t^2(\varphi)-\tilde{\varepsilon}_t^2|}{\tilde{h}_t}\right] \\ &\leq \left(\max_{1\leq v\leq S}\sup_{\theta\in\Theta}\left(\omega_v^{-2},\omega_v^{-1}\right)\right)MN^{-1}\sum_{t=1}^{NS}\rho^t(\varepsilon_t^2+1)\sum_{k=-q}^{t}1+|\varepsilon_k^2(\varphi)|,\end{aligned}$$

and the conclusion of the lemma follows from the same argument used in proving Lemma B.1. □

Lemma C.2. *Under Assumptions* A1–A3 *and* A6–A7 *there is* $t \in \mathbb{Z}$ *such that if* $h_t(\pi) = h_t(\pi^0)$ *and* $\varepsilon_v(\varphi) = \varepsilon_v(\varphi)$ *a.s. then* $\pi = \pi^0$.

Proof. If $\varphi \neq \varphi^0$ then the equality $\varepsilon_t(\varphi) = \varepsilon_t(\varphi^0)$ implies the existence of a constant combination of X_{t-j}, $j > 0$. However, the linear innovation of X_t which is equal to $X_t - E(X_t/\mathcal{F}_{t-1}) = h_t(\pi^0)\eta_t$, cannot be null *a.s.* because $E(\eta_t^2) = 1$ and $h_t(\pi^0) > 0$. Thus $\varphi = \varphi^0$. The fact that $\theta = \theta^0$ follows from the same argument used in proving Lemma B.2. □

Lemma C.3. *If* $\pi \neq \pi^0$ *then*

$$\sum_{v=1}^{S} E_{\pi^0}(l_v(\pi)) > \sum_{v=1}^{S} E_{\pi^0}(l_v(\pi^0)).$$

Proof. Using the argument used in proving Lemma B.4, we have

$$\begin{aligned}\sum_{v=1}^{S} E_{\pi^0}(l_v(\pi)) - E_{\pi^0}(l_v(\pi^0)) &= \sum_{v=1}^{S} E_{\pi^0}\left(\log\frac{h_v(\pi)}{h_v(\pi^0)} + \frac{h_v(\pi^0)}{h_v(\pi)} - 1\right) + E_{\pi^0}\left(\frac{\varepsilon_v^2(\varphi) - \varepsilon_v^2(\varphi^0)}{h_v(\pi)}\right)\\ &= \sum_{v=1}^{S} E_{\pi^0}\left(\log\frac{h_v(\pi)}{h_v(\pi^0)} + \frac{h_v(\pi^0)}{h_v(\pi)} - 1\right) + E_{\pi^0}\left(\frac{(\varepsilon_v(\varphi) - \varepsilon_v(\varphi^0))^2}{h_v(\pi)}\right).\end{aligned}$$

Therefore,

$$\sum_{v=1}^{S} E_{\pi^0}(l_v(\pi)) \geq \sum_{v=1}^{S} E_{\pi^0}(l_v(\pi^0))$$

with equality if and only if $\varepsilon_v(\varphi) = \varepsilon_v(\varphi^0)$ and $h_t(\pi) = h_t(\pi^0)$ and hence $\pi = \pi^0$. □

Lemma C.4. *For all* $\pi \neq \pi^0$ *there is a neighbourhood* $\mathcal{V}(\pi)$ *such that*

$$\lim_{N\to\infty}\inf \inf_{\tilde{\pi}\in\mathcal{V}(\pi)}\left(-\frac{1}{N}\tilde{L}_{NS}(\tilde{\pi})\right) > \sum_{v=1}^{S} E_{\pi^0}(l_v(\pi^0)).$$

Proof. The proof is similar to that of Lemma B.4 and hence it will be omitted. □

Proof of Theorem 6. Theorem 6 will be proved via the following lemmas while using Lemma 4.1 of Francq and Zakoïan (2004), which remains true in our periodic case.

Lemma C.5. *We have*

$$\sum_{v=1}^{S} E_{\pi^0}\left\|\frac{\partial l_v(\pi^0)}{\partial \pi}\frac{\partial l_v(\pi^0)}{\partial \pi'}\right\| < \infty \quad \textit{and} \quad \sum_{v=1}^{S} E_{\pi^0}\left\|\frac{\partial^2 l_v(\pi^0)}{\partial \pi \partial \pi'}\right\| < \infty.$$

Lemma C.6. *Under Assumptions* A1–A10, *I and J are invertible and are block-diagonal when the distribution of* η_t *is symmetric.*

LEMMA C.7. *The following limit relations hold*

$$N^{-\frac{1}{2}}\left\|\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial l_{v+kS}(\pi^0)}{\partial\pi}-\frac{\partial\tilde{l}_{v+kS}(\pi^0)}{\partial\pi}\right\|=o_p(1),$$

$$\sup_{\pi\in\mathcal{V}(\pi^0)}N^{-1}\left\|\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial^2 l_{v+kS}(\pi^0)}{\partial\pi\partial\pi'}-\frac{\partial^2\tilde{l}_{v+kS}(\pi^0)}{\partial\pi\partial\pi'}\right\|=o_p(1).$$

LEMMA C.8. *The following limit results are true*

$$N^{-\frac{1}{2}}\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial l_{v+kS}}{\partial\pi}(\pi^0)\rightsquigarrow N(0,(E(\eta_t^4)-1)J)\ \text{and}\ N^{-1}\sum_{k=1}^{N}\sum_{v=1}^{S}\frac{\partial^2 l_{v+kS}}{\partial\pi\partial\pi'}(\tilde{\pi})\to J,\ a.s.$$

for any $\tilde{\pi}$ *between* $\hat{\pi}_{NS}$ *and* π^0. □

ACKNOWLEDGEMENTS

We are deeply grateful to Prof. M. Priestley, the Editor-in-Chief, and two anonymous referees for their constructive remarks and suggestions that led to a substantial improvement in the form and the content of the paper, in particular for suggesting to us to consider asymptotics for the general Periodic ARMA-GARCH model.

NOTES

Corresponding authors: Abdelhakim Aknouche, Faculty of Mathematics, University of Sciences and Technology Houari Boumediene (USTHB), Algiers, Algeria. E-mail: aknouche_ab@yahoo.com. Abdelouahab Bibi, Department of Mathematics, University Mentouri, Constantine, Algeria. E-mail: abd.bibi@gmail.com

REFERENCES

AKNOUCHE, A. (2007) Causality conditions and autocovariance calculations in PVAR models. *Journal of Statistical Computation and Simulation* 77, 769–80.

BASAWA, I. V. and LUND, R. (2001) Large sample properties of parameters estimates for periodic ARMA models. *Journal of Time Series Analysis* 22, 1–13.

BENTARZI, M. and AKNOUCHE, A. (2005) Calculation of the Fisher information matrix for periodic ARMA models. *Communications in Statistics – Theory and Methods* 34, 891–903.

BENTARZI, M. and HALLIN, M. (1994) On the invertibility of periodic moving average models. *Journal of Time Series Analysis* 15, 263–8.

BERKES, I. and HORVATH, L. (2004) The efficiency of the estimators of the parameters in GARCH processes. *Annals of statistics* 32, 633–55.

Berkes, I., Horvath, L. and Kokoskza, P. (2003) GARCH processes: structure and estimation. *Bernoulli* 9, 201–27.

Bibi, A. and Aknouche, A. (2007) On some probabilistic properties of periodic GARCH processes. arXiv: 0709.2983v1. Available at: http://arxiv.org/abs/0709.2983

Billingsley, P. (1995) *Probability and Measure*, 3rd ed. New York: Wiley-Interscience.

Bollerslev, T. and Ghysels, E. (1996) Periodic autoregressive conditional heteroskedasticity. *Journal of Business and Economic Statistics* 14, 139–52.

Bougerol, P. and Picard, N. (1992a) Strict stationarity of generalized autoregressive processes. *Annals of Probability* 20, 1714–30.

Bougerol, P. and Picard, N. (1992b) Stationarity of GARCH processes and some nonnegative time series. *Journal of Econometrics* 52, 115–27.

Boussama, F. (2000) Normalité asymptotique de l'estimateur du pseudo-maximum de vraisemblance d'un modèle GARCH. *Comptes Rendus de l'Académie des Sciences, Paris* 331, 81–4.

Boyles, R. A. and Gardner, W. A. (1983) Cycloergodic properties of discrete-parameter nonstationary stochastic processes. *IEEE Transactions on Information Theory* 29, 105–14.

Drost, F. C. and Nijman, T. E. (1993) Temporal aggregation of GARCH processes. *Econometrica* 61, 909–27.

Francq, C. and Zakoïan, J. M. (2004) Maximum likelihood estimation of pure GARCH and ARMA-GARCH processes. *Bernoulli* 10, 605–37.

Francq, C. and Zakoïan, J. M. (2007) Quasi-maximum likelihood estimation in GARCH processes when some coefficients are equal to zero. *Stochastic Processes and their Applications* 117, 9, 1265–84.

Franses, P. and Paap, R. (2000) Modeling day-of-the-week seasonality in the S&P 500 Index. *Applied Financial Economics* 10, 483–8.

Franses, P. and Paap, R. (2004) *Periodic Time Series Models*. New York: Oxford University Press.

Furstenberg, H. and Kesten, H. (1960) Product of random matrices. *Annals of Mathematical Statistics* 31, 457–69.

Ghysels, E. and Osborn, D. (2001) *The Econometric Analysis of Seasonal Time Series*. UK: Cambridge University Press.

Gladyshev, E. G. (1961) Periodically correlated random sequences. *Soviet Mathematics* 2, 385–88.

Hafner, C. M. and Rombouts, J. V. K. (2007) Estimation of temporally aggregated multivariate GARCH models. *Journal of Statistical Computation and Simulation* 77, 629–50.

Jensen, S. T. and Rahbek, A. (2004a) Asymptotic normality of the QML estimator of ARCH in the nonstationary case. *Econometrica* 72, 641–6.

Jensen, S. T. and Rahbek, A. (2004b) Asymptotic inference for nonstationary GARCH. *Econometric Theory* 20, 1203–26.

Kingman, J. F. C. (1973) Subadditive ergodic theory. *Annals of Probability* 1, 883–904.

Lee, S. W. and Hansen, B. E. (1994). Asymptotic theory for the GARCH(1,1) Quasi-maximum likelihood estimator. *Econometric Theory* 10, 29–52.

Ling, S. and Mcaleer, M. (2003) Asymptotic theory for a new vector ARMA-GARCH model. *Econometric Theory* 19, 280–310.

Lumsdaine, R. L. (1996) Consistency and asymptotic normality of the quasi-maximum likelihood estimator in IGARCH(1,1) and covariance stationary GARCH(1,1) models. *Econometrica* 64, 575–96.

Martens, M., Chang, Y. and Taylor, S. J. (2002) A comparison of seasonal adjustment methods when forecasting intraday volatility. *Journal of Financial Research* 25, 283–99.

Nelson, D. B. (1990) Stationarity and persistence in the GARCH(1,1) model. *Econometric Theory* 6, 318–34.

Straumann, D. and Mikosch, T. (2006). Quasi-maximum likelihood estimation in conditionally heteroscedastic time series: A stochastic recurrence equation approach. *The Annals of Statistics* 34, 2449–95.

Taylor, J. W. (2006) Density forecasting for the efficient balancing of the generation and consumption of electricity. *International Journal of Forecasting* 22, 707–24.

Taylor, N. (2004) Modeling discontinuous periodic conditional volatility: Evidence from the commodity futures market. *Journal of Futures Markets* 24, 805–34.

Tiao, G. C. and Grupe, M. R. (1980) Hidden periodic autoregressive-moving average models in time series data. *Biometrika* 67, 365–73.