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The Exact Maximum Likelihood-Based Test for Fractional Cointegration: Critical Values, Power and Size

EMMANUEL DUBOIS[1], SANDRINE LARDIC[2] and VALÉRIE MIGNON[3]
[1]*PESOR, University of Paris XI, Paris, France;* [2]*MODEM-CNRS, University of Paris X, Paris, France;* [3]*THEMA-CNRS, University of Paris X, 200 Avenue de la République, 92001 Nanterre Cedex, France; E-mail: Valerie.Mignon@u-paris10.fr*

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Abstract. The exact maximum likelihood (EML) procedure can be used as a residual-based test of the hypothesis of no cointegration against the alternative of fractional cointegration. Since the corresponding asymptotic properties have not yet been established, this paper provides simulated critical values, power and size relating to the EML-based test for fractional cointegration. Monte Carlo simulations indicate that the simulated density of the EML-based test is shifted to the left compared to the standard normal distribution and exhibits a strong excess of kurtosis in the absence of autoregressive components in the regression residuals. The power and size comparison indicates that the EML-based test is more powerful than other fractional cointegration tests (Lo, Lobato-Robinson and Geweke and Porter-Hudak) in small and medium sample sizes. Moreover, by simulating integrated time series with AR(1), and respectively MA(1), disturbances, it is shown that, whatever the sample size, the EML-based test exhibits the lowest size distortions for positive AR(1) and negative MA(1) coefficients, respectively.

Key words: exact maximum likelihood procedure, fractional cointegration, Monte Carlo experiment

JEL classifications: C15; C22

1. Introduction

The concept of fractional cointegration introduced by Granger (1986) has become an important topic in time series analysis. It is linked to fractional integration which, in turn, is linked to the long-memory property of time series (see Granger and Joyeux (1980) and Hosking (1981)). Under these conditions, two integrated series are called fractionally integrated if there exists a linear combination that is fractionally integrated. In other words, the integration order of the error in the equilibrium relation is not necessarily 0 or 1, it can also be a real number between 0 and 1: the error term may be fractionally integrated. This allows us to obtain more various mean reverting behaviors (see Chou and Shih (1997) among others). More specifically, a fractionally integrated error in the equilibrium relation implies the existence of an equilibrium long-term relationship between the considered variables. Thus, the error in the equilibrium relation needs not to be $I(0)$.

The growing empirical literature on fractional cointegration highlights the importance of powerful tests of the null hypothesis of no cointegration against fractionally cointegrated alternatives. The fractional cointegration tests are based on the long-memory tests and *ARFIMA* (autoregressive fractionally integrated moving average) estimation procedures (see Baillie (1996) for a review). In the case of fractional cointegration, these tests are applied to the residuals of the long-term relationship between the considered variables. It is worth noting that the properties of these tests, such as the asymptotic distribution under the null hypothesis, are known only if true equilibrium errors are observable. However, this is not the case here since fractional cointegration tests are applied to estimated *OLS* residuals. As noted by Dittmann (2000), since the *OLS* regression method tends to reduce too much of the residual's variance, the regression residuals are likely to be biased towards stationarity in finite samples. One should thus use other critical values than those calculated on the basis of the truly observed series. Dittmann (2000) has tabulated critical values for some fractional cointegration tests (unit root tests, Lo (1991) test, Lobato and Robinson (1998) test and Geweke and Porter-Hudak (1983) test). We propose here to tabulate critical values and to study the power and size associated with the exact maximum likelihood procedure (EML). This topic is relevant since, according to Dahlhaus (1988) and Sowell (1992), EML is the most efficient estimation procedure for *ARFIMA* processes (see also Cheung and Diebold (1994)). It permits us to tackle the short memory contamination problem and the small-sample bias associated notably with the popular Geweke and Porter-Hudak two-step method. However, despite these advantages, this technique has received little attention in the literature, compared to the Geweke and Porter-Hudak procedure, because the EML estimation of a long-range dependent process is computationally intensive, due to the time required to compute the elements of the covariance matrix of the process and its inverse (see, for example, Bollerslev and Jubinski (1999)).[1] This procedure has thus been criticized as too computationally demanding, while the Geweke and Porter-Hudak method has been criticized as inaccurate for finite samples (see e.g. Sowell (1992)). Under these conditions, it seems interesting to proceed to an extensive Monte Carlo experiment in order to study the distribution of the EML-based test under the null hypothesis of no cointegration. Moreover, in order to determine which residual-based test (among Lo, Lobato-Robinson, Geweke and Porter-Hudak and EML tests) of the null hypothesis of no cointegration is most powerful against fractionally cointegrated alternatives, we simulate also the power and size of the EML-based test. To our knowledge, the EML-based test has not been used in fractional cointegration analysis either, though it seems a promising candidate. Thus, no result is available regarding critical values, power and size for the EML-based test for fractional cointegration.

The paper is organized as follows. Section 2 briefly describes the concept of fractional cointegration and the EML-based test. Section 3 reports Monte Carlo simulations concerning the distribution of the EML-based test under the null hypothesis. Section 4 gives Monte Carlo results relating to the power and size of EML. It provides also a power and size comparison to other fractional cointegration tests. Section 5 concludes.

2. EML-Based Test for Fractional Cointegration

Consider two series, x_t and y_t, each of which being integrated of order 1. x_t and y_t are fractionally cointegrated if there exists a cointegration relationship:

$$y_t = \alpha + \beta x_t + z_t \tag{1}$$

where z_t is a long-memory process, such as an *ARFIMA*(p, d, q) process[2]:

$$\Phi(L)(1 - L)^d z_t = \Theta(L)\varepsilon_t \tag{2}$$

where $\Phi(L)$ and $\Theta(L)$ are autoregressive and moving average polynomials, respectively. ε_t is a white noise, L is the lag operator and:

$$(1 - L)^d = 1 - dL - \frac{d(1 - d)}{2!}L^2 - \frac{d(1 - d)(2 - d)}{3!}L^3 - \cdots \tag{3}$$

Residual-based tests for fractional cointegration are based on the null hypothesis:

H_0 : x_t and y_t are not cointegrated, *i.e.* z_t is $I(1)$, for all $\alpha, \beta \in \Re$,

against the alternative:

H_1 : x_t and y_t are cointegrated, *i.e.* z_t is $I(d)$, with $d < 1$

These tests are applied to the residuals $\hat{z}_t$ estimated from an *OLS* regression of y_t on x_t (see (1)).

The parameters of the *ARFIMA*(p, d, q) process in (2) can be jointly estimated by the EML method (see Sowell (1992) for details). Let $z_t, t = 1, \ldots, T$, be a fractionally integrated stationary Gaussian time series. z_t follows a normal law with mean zero and covariance matrix Σ. Its density function is given by:

$$f(z_t, \Sigma) = (2\pi)^{-T/2}|\Sigma|^{-1/2} \quad \exp\left(-\frac{1}{2}z_t'\Sigma^{-1}z_t\right) \tag{4}$$

Due to the stationarity property, the covariance matrix has a Toeplitz form: $\Sigma = [\gamma_{i-j}]$ with $i, j = 1, 2, \ldots, T$. Estimation of *ARFIMA* processes by EML requires writing the spectral density function of z_t, denoted as $f_z(\lambda)$, in terms of the parameters of the model and then evaluating the autocovariance function γ_s at lag s by:

$$\gamma_s = \frac{1}{2\pi} \int_0^{2\pi} f_z(\lambda) e^{i\lambda s} d\lambda \tag{5}$$

If the true equilibrium errors z_t are observable, the EML estimator of d has an asymptotic normal distribution (see Dahlhaus (1989)). However, the EML procedure is here applied to regression residuals, and asymptotic results for this case have not yet been established. Thus, critical values for $t-$statistics of d have to be simulated.

The pertinence of this method lies in its using all information concerning the short and long-term behavior of the series since it estimates all parameters of the $ARFIMA(p, d, q)$ representation simultaneously. Moreover, the application of this procedure to residual series allows us to test the null hypothesis of a unit root ($d = 1$) against the alternative of fractional integration ($d < 1$). This is equivalent to a test of the null $d' = 0$ against $d' < 0$, with $d' = d - 1$ where d is the fractional difference parameter of the series in levels and d' the fractional difference parameter of the series in first differences, i.e. Δz_t.

3. Distribution Under the Null Hypothesis

We generate 16 $ARFIMA(p, d, q)$ processes, from $(0, d, 0)$ to $(3, d, 3)$.[3] For the simulation of the critical values, 200,000 iterations of each process were conducted. Following Dittmann (2000), for each iteration, we generate two random walks of appropriate length as follows. We consider two uncorrelated standard normal variables x'_t and y'_t. The two random walks are constucted by calcultating the partial sums: $x_t = \sum_{k=1}^{t} x'_k$ and $y_t = \sum_{k=1}^{t} y'_k$. Then, we run the regression $y_t = \alpha + \beta y_t + z_t$ in order to derive the estimated residuals $\hat{z}_t$. The null distribution is simulated for five different sample sizes: $T = 50$, 100, 150, 200 and 500.

Tables I and II report the critical values for the usual significance levels (1, 5 and 10%). Figures 1–4 show the simulated density function of the EML-based test for $T = 500$ together with the standard normal density function for various values of p and q. The standard normal distribution is the asymptotic distribution of the EML-based test when calculated from the true equilibrium error process rather than from the regression residuals.

The figures indicate that the simulated density is undoubtedly distinct from the standard normal. From Figure 1, one can remark that the simulated density presents too many observations in the "middle" in relation to the normal distribution. This indicates a strong excess of kurtosis when there is no AR component. This excess decreases with the introduction of MA components and disappears when AR components are present (see Figures 2 and 4 for example). This illustrates that

Table I. Critical values of the EML-based test for fractional cointegration.

ARFIMA	$T = 50$			$T = 100$			$T = 150$		
(p, d, q)	1%	5%	10%	1%	5%	10%	1%	5%	10%
$(0, d, 0)$	−4.65	−2.81	−2.42	−3.41	−2.68	−2.02	−3.22	−2.33	−1.93
$(0, d, 1)$	−4.43	−3.29	−2.75	−3.98	−2.99	−2.50	−3.76	−2.77	−2.27
$(0, d, 2)$	−5.20	−3.72	−3.08	−4.52	−3.35	−2.78	−4.24	−3.11	−2.57
$(0, d, 3)$	−6.01	−4.15	−3.37	−4.93	−3.61	−2.97	−4.54	−3.35	−2.76
$(1, d, 0)$	−5.49	−4.16	−3.61	−7.14	−5.46	−4.44	−8.11	−5.81	−3.39
$(1, d, 1)$	−5.14	−3.50	−2.92	−5.86	−4.03	−3.30	−6.45	−4.48	−3.49
$(1, d, 2)$	−5.81	−3.68	−2.98	−6.23	−4.01	−3.23	−6.80	−4.31	−3.38
$(1, d, 3)$	−7.23	−3.99	−3.14	−6.69	−4.10	−3.27	−7.10	−4.34	−3.41
$(2, d, 0)$	−12.1	−5.97	−3.14	−9.77	−3.84	−3.22	−7.09	−4.33	−3.63
$(2, d, 1)$	−16.5	−5.31	−3.39	−14.1	−5.16	−3.61	−11.4	−5.36	−3.58
$(2, d, 2)$	−31.5	−6.60	−3.75	−23.6	−5.42	−3.64	−24.3	−5.35	−3.63
$(2, d, 3)$	−54.1	−8.27	−4.09	−45.9	−6.34	−3.88	−42.7	−6.16	−3.95
$(3, d, 0)$	−18.2	−6.64	−3.63	−15.5	−4.22	−2.88	−11.7	−3.80	−3.12
$(3, d, 1)$	−30.5	−6.96	−3.22	−22.7	−4.34	−2.89	−15.1	−4.06	−3.09
$(3, d, 2)$	−72.9	−11.3	−5.95	−94.7	−11.4	−7.29	−97.5	−12.1	−8.35
$(3, d, 3)$	−128.3	−12.6	−5.09	−171.6	−11.9	−5.38	−206.0	−12.2	−5.91

Table II. Critical values of the EML-based test for fractional cointegration.

ARFIMA	$T = 200$			$T = 500$		
(p, d, q)	1%	5%	10%	1%	5%	10%
$(0, d, 0)$	−2.89	−1.77	−1.47	−2.68	−1.33	−0.61
$(0, d, 1)$	−3.45	−2.53	−2.04	−2.90	−1.97	−1.48
$(0, d, 2)$	−4.02	−2.96	−2.42	−3.33	−2.41	−1.92
$(0, d, 3)$	−4.31	−3.17	−2.62	−3.68	−2.66	−2.15
$(1, d, 0)$	−8.64	−4.28	−2.51	−4.25	−2.90	−2.07
$(1, d, 1)$	−6.85	−4.75	−3.51	−7.97	−2.78	−2.21
$(1, d, 2)$	−7.09	−4.48	−3.45	−7.95	−4.47	−2.66
$(1, d, 3)$	−7.18	−4.47	−3.45	−7.94	−4.74	−3.22
$(2, d, 0)$	−7.18	−4.70	−3.76	−6.91	−3.26	−2.37
$(2, d, 1)$	−10.4	−5.28	−3.44	−10.3	−3.96	−2.54
$(2, d, 2)$	−18.7	−5.05	−3.62	−12.2	−5.31	−3.04
$(2, d, 3)$	−33.4	−5.79	−3.84	−21.4	−5.78	−3.54
$(3, d, 0)$	−7.75	−3.99	−3.35	−6.52	−4.27	−2.71
$(3, d, 1)$	−9.88	−4.26	−3.26	−8.09	−4.46	−2.69
$(3, d, 2)$	−67.7	−12.1	−8.62	−41.2	−11.8	−3.78
$(3, d, 3)$	−133.4	−10.3	−5.79	−115.1	−10.0	−5.20

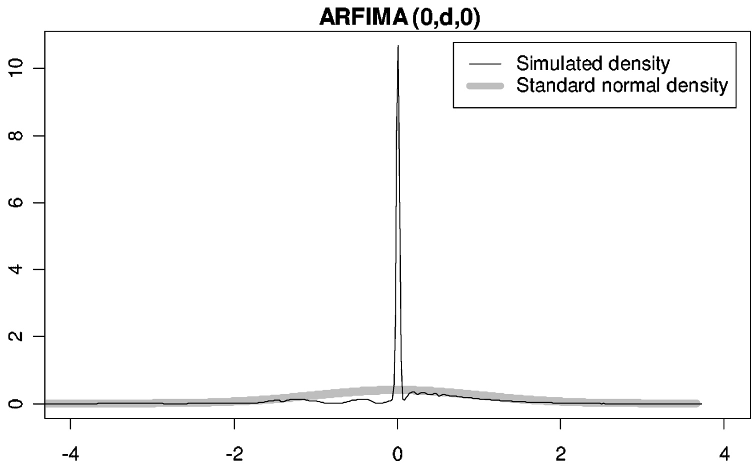

Figure 1. Simulated density function of the EML-based test under the null hypothesis and density function of the standard normal distribution.

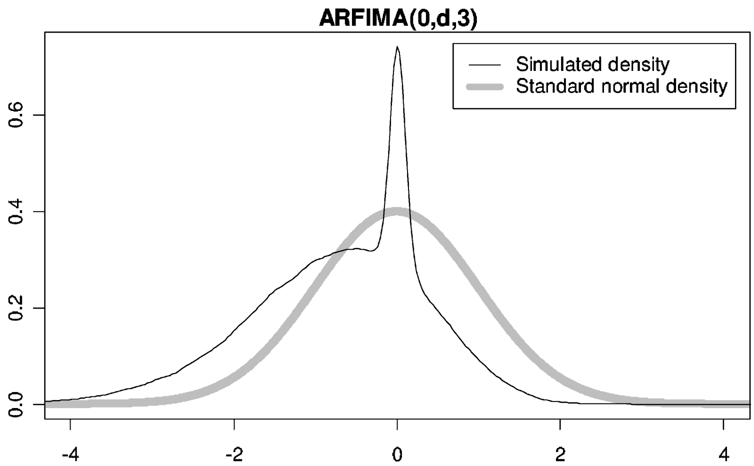

Figure 2. Simulated density function of the EML-based test under the null hypothesis and density function of the standard normal distribution.

the distribution of the EML-based test is strongly dependent from the presence or not of short-term components.

Moreover, as it can be seen from Figures 2–4, the density of the EML-based test is shifted to the left compared to the standard normal distribution: the simulated density has a clear mode left of zero. In other words, the test finds less evidence for long memory in the regression residuals than it would find in the true equilibrium errors.

Note that these results are, to some extent, close to those obtained by Hauser (1999), though not in the case of cointegration. Indeed, by means of simulations, Hauser (1999) studies the properties of four maximum likelihood estimators,

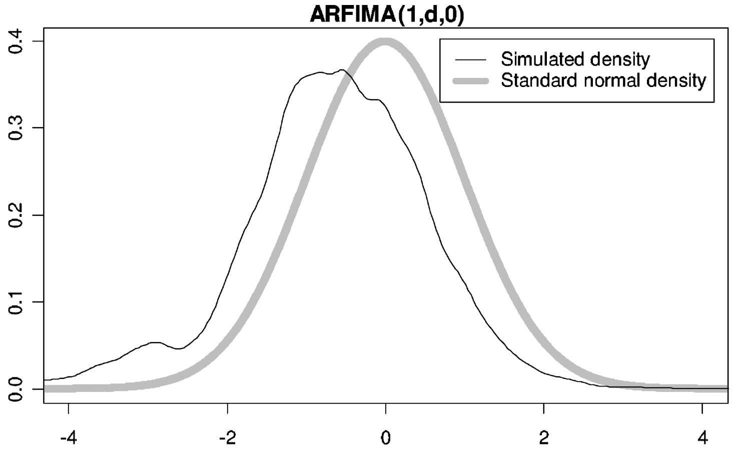

Figure 3. Simulated density function of the EML-based test under the null hypothesis and density function of the standard normal distribution.

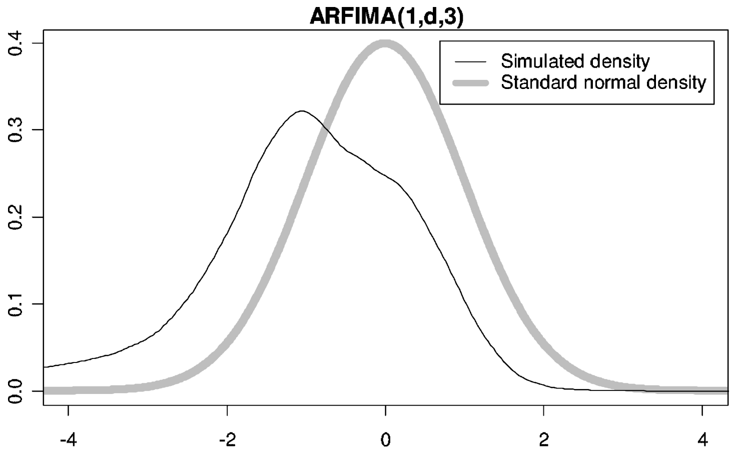

Figure 4. Simulated density function of the EML-based test under the null hypothesis and density function of the standard normal distribution.

including EML, for fitting ARFIMA processes. He shows, like Li and McLeod (1986), that the EML estimate tends to give a negatively biased parameter for pure fractionally integrated processes. Moreover, this negative bias in the fractional integration parameter estimate is enlarged by the inclusion of the AR and MA parameters (for an explanation of this negative bias, see Hauser (1999)).

4. Power and Size

We now consider the power and size of the EML-based test for fractional cointegration. We suppose that, under the alternative hypothesis, z_t follows an $ARFIMA(0, d, 0)$ process. In other words, z_t is a pure fractionally integrated process.

This allows us to compare the power and size of EML with those of other fractional cointegration tests.

For the power and size simulations, 10, 000 iterations were used. Following Dittmann (2000), we consider the system $\{x_t, y_t\}$ defined as:

$$\begin{cases} y_t = 2u_t - v_t \\ x_t = v_t - u_t \end{cases} \tag{6}$$

where u_t is a random walk. In the case of power simulations, v_t is a fractionally integrated series. In the case of size simulations, v_t is a simulated $ARIMA(1, 1, 0)$ or $ARIMA(0, 1, 1)$ process (see below). The length of u_t and v_t is $T + 50$, where T is the sample size. After having computed the system $\{x_t, y_t\}$ according to (6), the first 50 observations were discarded to reduce the effect of initial values. In order to compute the power and size of the EML-based test, the previously simulated critical values are used (see Tables I and II).

4.1. POWER

4.1.1. *Power Simulations*

Table III reports the simulated power of the EML-based test under the alternative of fractional cointegration. We consider ten long-memory parameters $d = 0, 0.1, 0.2, \ldots, 0.9$ and five sample sizes $T = 50, 100, 150, 250$ and 500. Recall that the case $d = 0$ denotes the alternative "classical cointegration", while $d = 1$

Table III. Power of the EML-based test for fractional cointegration.

		d									
T	(%)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
50	1	96.12	98.3	98.92	99.38	99.41	99.62	99.56	99.64	99.66	99.69
	5	99.9	99.92	99.95	99.96	99.96	99.95	99.92	99.93	99.92	99.96
	10	99.94	99.93	99.97	99.97	99.96	99.97	99.95	99.95	99.95	99.97
100	1	100	99.99	99.99	99.98	99.94	99.89	99.92	99.83	99.85	99.82
	5	100	100	100	100	99.98	99.94	99.93	99.89	99.93	99.9
	10	100	100	100	100	99.99	99.96	99.96	99.95	99.96	99.95
150	1	99.96	99.93	99.93	99.92	99.95	99.89	99.86	99.88	99.86	99.92
	5	99.97	99.98	99.98	99.96	99.99	99.94	99.92	99.93	99.93	99.94
	10	99.97	99.98	99.99	99.96	99.99	99.96	99.94	99.94	99.95	99.97
200	1	99.9	99.99	99.91	99.96	99.99	99.97	99.95	99.95	99.96	99.98
	5	99.97	100	99.98	100	100	100	100	99.98	99.98	99.99
	10	99.97	100	99.99	100	100	100	100	99.99	99.98	99.99
500	1	92.76	92.2	91.92	90.48	87.9	82.95	78.73	71.38	64.26	60.46
	5	92.78	92.21	91.92	90.48	87.9	82.95	78.74	71.38	64.27	60.46
	10	92.79	92.22	91.93	90.48	87.91	82.95	78.74	71.38	64.27	60.46

corresponds to the null hypothesis of no cointegration. In between, we find the fractionally cointegrated alternatives.

The results in Table III indicate that the EML-based test is very powerful against fractionally cointegrated alternatives, especially for $T < 500$. Moreover the power is stable, whatever the value of d. Indeed, the power of the 5% test is always greater than 99.9%. For $T = 500$, the power tends to decrease as d increases. The power remains however reasonable, since it is around 92% for low d values and 65% for high d values. Thus, these power simulations indicate that the EML-based test appears to be very powerful in small and medium sample sizes.

4.1.2. *Power Comparison*

We propose to compare the power of the EML-based test to the power of other tests for fractional cointegration studied by Dittmann (2000): the modified rescaled range statistic derived by Lo, the LM-type test developed by Lobato and Robinson (LR) and the semi-nonparametric procedure proposed by Geweke and Porter-Hudak (GPH).

Figure 5 shows the rejection percentages of the EML, Lo, LR and GPH tests for $T = 100$ plotted against the long-memory parameter d. Figure 6 is relating to the case $T = 500$. According to Figure 5, EML is much more powerful than the other tests for all values of d. Moreover, it is worth noting that, contrary to GPH and LR, the power of EML does not decrease as d increases. The Lo test is clearly the least powerful test among the four tests. However, as Figure 6 suggests, the power of the EML-based test decreases as the number of observations increases. Indeed, for $T = 500$, the two most powerful tests are LR and GPH. Note that the power of the four tests decreases as d increases. Finally, one can remark that EML is more powerful than LR and GPH only for $d = 0.9$ and than Lo test for $d \geq 0.7$.

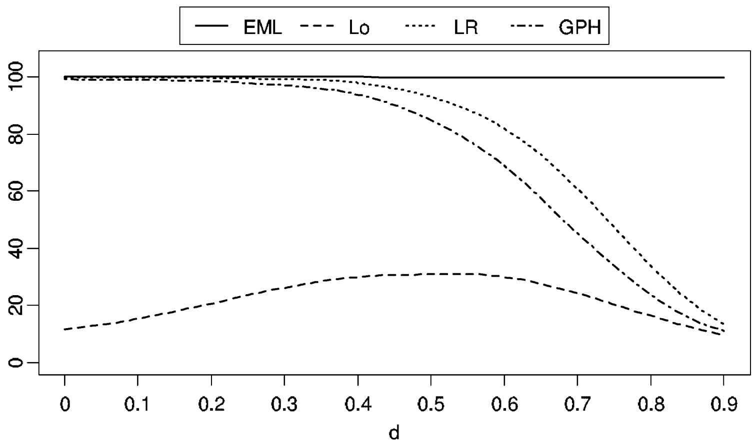

Figure 5. Power of EML, Lo, LR and GPH against fractionally cointegrated alternatives with $T = 100$ and significance level 5%.

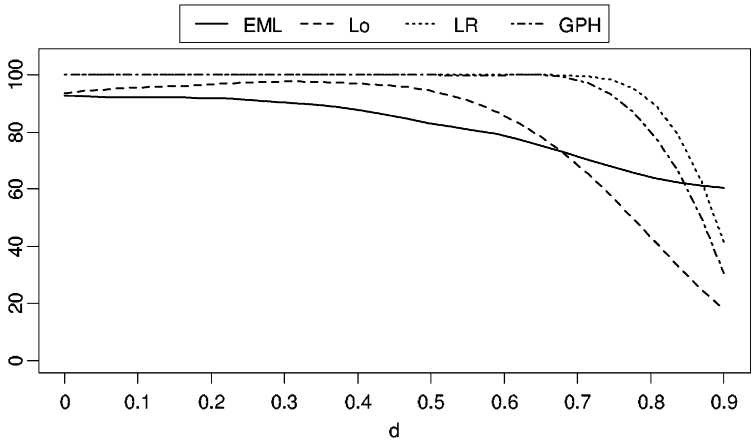

Figure 6. Power of EML, Lo, LR and GPH against fractionally cointegrated alternatives with $T = 500$ and significance level 5%.

Note that, as shown in Table III, there is a jump in terms of power when T goes from 200 to 500. This finding may be the result of a loss of accuracy during the estimation process. Indeed, recall that EML needs the inversion of the $T \times T$ covariance matrix, which is rather difficult for high T. The loss of accuracy can occur at this stage and explain our results, but further developments need to be done on this point.

To sum up, EML is the most powerful test among the four considered tests in small and medium sample sizes ($T < 500$). For $T \geq 500$, LR provides better results in terms of power, except for near-unity values of d where EML outperforms the other tests.

4.2. SIZE

4.2.1. *Size Simulations*

As for the power simulations, five sample sizes are considered: $T = 50$, 100, 150, 200 and 500. In order to proceed to the size simulations, we simulate two integrated – but not cointegrated – time series with $AR(1)$ (and, respectively, MA(1)) disturbances. Thus, in (6), v_t is either an *ARIMA*(1, 1, 0) process:

$$\Delta v_t = \phi \Delta v_{t-1} + \eta_t \tag{7}$$

where η_t is a white noise, or an *ARIMA*(0, 1, 1) process:

$$\Delta v_t = \mu_t - \theta \mu_{t-1} \tag{8}$$

where μ_t is a white noise.

The parameters ϕ and θ vary between -0.9 and 0.9, with a step of 0.2. Tables IV and V report the size of EML in the case of an *ARIMA*(1, 1, 0) and an *ARIMA*(0, 1, 1) process, respectively.

Table IV. Size of the EML-based test for fractional cointegration. Case of an *ARIMA*(1, 1, 0) process.

		ϕ									
T	(%)	−0.9	−0.7	−0.5	−0.3	−0.1	0.1	0.3	0.5	0.7	0.9
50	1	87.84	60.09	28.26	8.46	1.87	0.54	0.43	0.48	0.47	0.45
	5	96.79	86.02	60.32	29.15	9.76	4.15	2.74	2.67	2.81	3.15
	10	97.97	90.64	71.2	41.58	16.94	7.89	5.61	5.59	6.25	6.63
100	1	98.44	89.87	66.73	26.94	3.59	0.7	0.45	0.46	0.64	0.67
	5	99	93.69	78.29	42.95	11.06	2.64	2.1	2.37	3.1	3.33
	10	99.37	96.3	85.56	58.25	21.16	6.48	4.72	5.26	6.18	7.3
150	1	99	93.8	74.98	31.41	3.9	0.51	0.35	0.51	0.61	0.74
	5	99.23	95.75	80.12	44.51	13.08	2.74	2.08	2.52	3.02	3.45
	10	99.31	96.46	83.38	53.84	22.46	5.94	4.31	5.05	5.85	7.6
200	1	98.76	92.9	66.86	23.88	5.34	0.63	0.46	0.51	0.71	0.92
	5	98.97	94.5	73.02	45.07	23.98	5.07	2.95	3.18	3.95	4.74
	10	99.01	94.7	74.09	47.79	31.17	8.46	5.47	6	6.65	8.85
500	1	74.72	62.26	18.07	36.85	3.57	0.25	0.25	0.44	0.48	0.58
	5	74.73	62.6	19.54	42.83	21.38	3.84	2.37	3.32	3.38	4.11
	10	74.75	62.67	19.83	44.31	27.77	6.38	4.07	5.43	6.15	8.04

Table V. Size of the EML-based test for fractional cointegration. Case of an *ARIMA*(0, 1, 1) process.

		θ									
T	(%)	−0.9	−0.7	−0.5	−0.3	−0.1	0.1	0.3	0.5	0.7	0.9
50	1	0.46	0.43	0.46	0.43	0.53	1.65	10.19	37.66	76.03	95.29
	5	2.41	2.55	2.66	2.69	3.72	9.6	33.34	74.41	97.04	99.88
	10	5.23	4.91	4.99	5.38	7.52	17.35	46.67	83.65	98.59	99.94
100	1	0.38	0.52	0.35	0.54	0.56	3.95	32.82	82.91	99.52	100
	5	2.08	2.31	1.79	2.18	2.92	11.52	51.35	90.52	99.82	100
	10	4.68	4.7	3.93	4.7	6.52	22.22	65.74	95.19	99.91	100
150	1	0.4	0.39	0.48	0.44	0.51	4.02	37.48	89.62	99.77	99.97
	5	2.6	2.11	2.12	2.03	2.87	14.01	51.16	92.37	99.91	99.99
	10	4.95	4.27	4.2	4.2	5.63	23.65	59.72	93.65	99.91	99.99
200	1	0.56	0.66	0.49	0.44	0.67	5.47	28.66	81.3	99.52	99.87
	5	3.27	3.15	2.87	3.05	5.39	24.99	46.71	83.49	99.6	99.96
	10	5.99	5.69	5.23	5.62	8.92	32.2	49.27	83.66	99.62	99.97
500	1	0.34	0.36	0.25	0.33	0.34	4.17	41.26	15.03	55.52	86.69
	5	2.55	2.79	2.25	2.44	3.88	22.09	45.73	15.4	55.52	86.71
	10	4.67	4.59	4.25	4.34	6.21	27.57	46.47	15.44	55.52	86.71

Results in Tables IV and V indicate that the EML-based test performs badly in terms of size for negative values of ϕ and positive values of θ. The empirical size is very high and the test rejects too frequently the null hypothesis. Moreover, size distortions increase as ϕ tends to -0.9 and θ to 0.9.

In contrast, for positive values of ϕ and negative values of θ, the EML-based test performs very well in terms of size, whatever the number of observations: the empirical size is always below the nominal size. For example, for $T = 100$ and $\theta = -0.5$, the empirical size of the 5% test is 1.79%.

4.2.2. *Size Comparison*

Using the simulations made by Dittmann (2000) concerning the other fractional cointegration tests, one can remark that, for $\phi < 0$ and $\theta > 0$, EML leads to the worst results among Lo, LR and GPH. However, as mentioned above, the size of EML is very low for $\phi > 0$ and $\theta < 0$ (see Tables IV and V). Moreover, according to the results obtained by Dittmann (2000) concerning Lo, LR and GPH, the test which exhibits the lowest size distortions is the modified rescaled range statistic developed by Lo (1991). It thus seems relevant to compare the sizes of EML and Lo, respectively. Since Dittmann's simulations concern negative values of ϕ and positive values of θ, we extend the simulations concerning the size of the Lo test to $\phi > 0$ and $\theta < 0$. As in the EML case, 10,000 iterations were used.

Figures 7–10 report the obtained results. Figures 7 and 8 concern the AR(1) size, and Figures 9 and 10 are relating to the MA(1) size. According to these figures, whatever the sample size and the values of ϕ and θ, the size distortions of EML are much smaller than those of Lo.

To sum up, for $\phi > 0$ and $\theta < 0$, EML performs much more better than the other tests in terms of size for all sample sizes. However, it is worth noting that

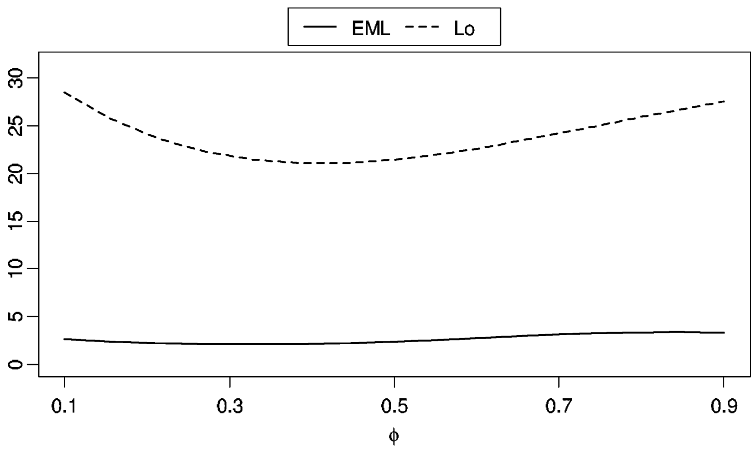

Figure 7. Size of EML and Lo with $T = 100$ and significance level 5%. Case of an *ARIMA*(1, 1, 0) process.

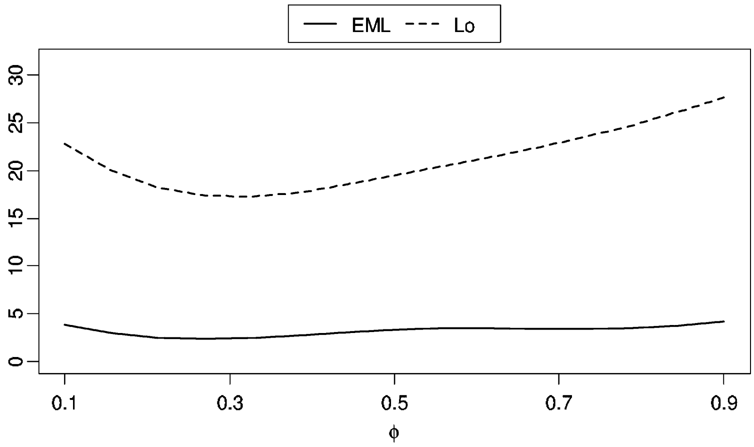

Figure 8. Size of EML and Lo with $T = 500$ and significance level 5%. Case of an *ARIMA*(1, 1, 0) process.

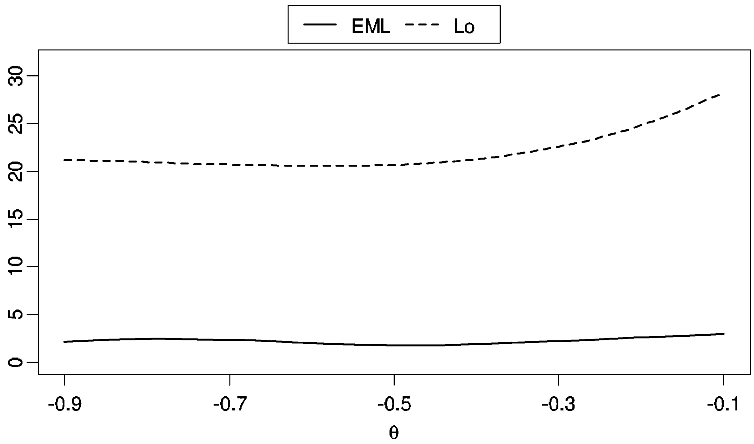

Figure 9. Size of EML and Lo with $T = 100$ and significance level 5%. Case of an *ARIMA*(0, 1, 1) process.

EML is sensitive to the signs of ϕ and θ since it leads to the worst results for $\phi < 0$ and $\theta > 0$.

4.3. SOME COMPUTATIONAL ASPECTS

The estimation of *ARFIMA* processes by EML involves two complicating factors: computation of the autocorrelation function, and evaluation of the likelihood which requires the inversion of the $T \times T$ covariance matrix.

Concerning the autocovariances, Sowell (1992) suggests the use of hypergeometric functions. Assuming the unicity of the roots ρ_j^{-1}, $j = 1, \ldots, p$, of the autoregressive polynomial $\Phi(z) = 0$, each autocovariance γ_k requires the

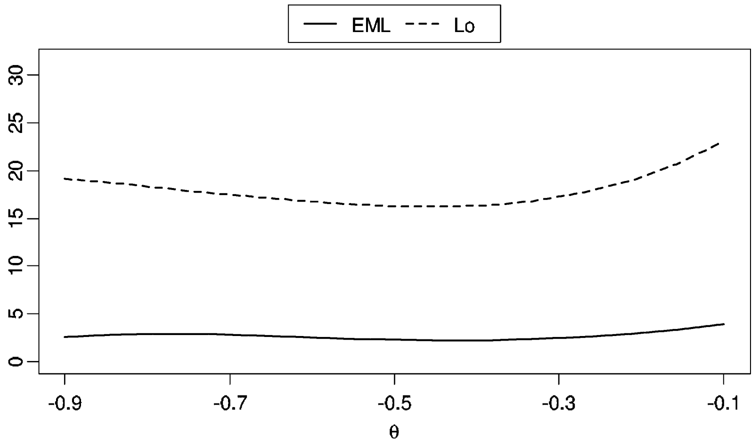

Figure 10. Size of EML and Lo with $T = 500$ and significance level 5%. Case of an *ARIMA*(0, 1, 1) process.

computation of at least p hypergeometric functions $F(a, 1, c, \rho)$ where a and c are functions of d and j:

$$F(a, 1, c, \rho) = \sum_{i=1}^{\infty} \frac{a(a+1)\cdots(a+i-1)(i-1)!\rho^i}{c(c+1)\cdots(c+i-1)i!} \tag{9}$$

The computation of $F(a, 1, c, \rho)$ is done recursively:

$$F(a, 1, c, \rho) = \frac{c-1}{\rho(a-1)}[F(a-1, 1, c-1, \rho) - 1] \tag{10}$$

As argued by Ooms and Doornik (1999), this allows Sowell (1992) to achieve a major speed-up of the algorithm: only the hypergeometric functions for γ_T have to be calculated separately, the remaining $\gamma_{T-1}, \ldots$ being derived recursively. Note that the Sowell algorithm requires that no roots are zero, while Doornik and Ooms (1999) have adapted the algorithm to make it feasible for $\rho_j = 0$. Finally, it is worth noting that the calculation of the autocovariance function by the Sowell procedure is fast and accurate (see, e.g., the Table I of Ooms and Doornik (1999)).

Let us now turn to the evaluation of the likelihood. In order to compute the log-likelihood, one needs to evaluate the $T \times T$ Toeplitz matrix of the autocorrelations, denoted as Ω. Indeed, written (4) in logarithms gives:

$$l = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\log|\Sigma| - \frac{1}{2}z_t'\Sigma^{-1}z_t \tag{11}$$

By concentrating the error variance σ_ε^2 out of the likelihood and writing $\Sigma = \Omega\sigma_\varepsilon^2$, the function used in the maximization procedure can be written (see Sowell (1992) and Ooms and Doornik (1999) for details):

$$-\frac{1}{2}\left[T^{-1}\log|\Omega| + \log\sigma_\varepsilon^2\right] \tag{12}$$

Table VI. Estimation of 16 *ARFIMA* processes. Timings in seconds.

$T = 50$	$T = 100$	$T = 150$	$T = 200$	$T = 500$	$T = 1000$
1.78	3.22	4.80	6.76	48.46	215.87

In order to evaluate (12), Sowell (1992) uses an efficient Choleski decomposition. This method avoids the storage of Ω, but requires the storage of the $\frac{1}{2}T(T+1)$ Choleski factors. Note that this storage is time-consuming for large T. For this reason, Doornik and Ooms (1999) use the Levinson algorithm in order to avoid the storage of the $\frac{1}{2}T(T+1)$ Choleski factors (see also Hauser (1999)). The timings provided by Doornik and Ooms (1999) indicate that, for $T < 200$, the two methods lead to similar results in terms of speed. However, above 2000 observations, the $\frac{1}{2}T(T+1)$ matrix starts to push the program into virtual memory, with a great impact on speed (see Doornik and Ooms (1999), p. 28).

Finally, note that the time required to obtain the set of maximum likelihood parameter estimates depends on the number of observations, the number of parameters being estimated, the numerical optimization algorithm, the starting values of the parameters used in the algorithm, etc. As argued by Sowell (1992), “in extreme situations (e.g., large number of parameters, large number of observations, roots near the unit circle) the time can be quite lengthy”. Moreover, the time needed to evaluate an ARFIMA process by EML is similar to the time required for the estimation of a non fractional process. In order to quantify this time and to actualize the results of Doornik and Ooms (1999), we compute the time needed to estimate the 16 *ARFIMA*(p, d, q) processes, from $(0, d, 0)$ to $(3, d, 3)$ using the Sowell program for our five considered observation numbers, and for $T = 1000$. These timings are given in Table VI and are obtained on a 640 MB Pentium II 450 Mhz, linux box. They indicate that the Sowell routine is not very time-consuming and can be quickly implemented for the sizes considered here.

5. Conclusion

In this paper, we have proceed to a Monte Carlo experiment in order to derive the critical values, power and size of the EML-based test for fractional cointegration. From the simulation of the density function of the EML-based test under the null hypothesis, three main innovative results emerge. First, the distribution strongly depends on the presence or the absence of short-term components (i.e. AR and MA components) in the regression residuals. Second, in the absence of AR components, the simulated density deviates from the standard normal density in kurtosis. This excess of kurtosis decreases with the introduction of MA components. Third, the density of the EML-based test is strongly shifted to the left, which means that this test finds less evidence for long memory in the regression residuals than it would find in the true equilibrium errors.

Concerning the power and size, our results indicate that the EML-based test is very powerful in small and medium sample sizes. Indeed, for a number of observations below 500, it strongly outperforms the modified rescaled range statistic, the LM-type test of Lobato and Robinson and the semi-nonparametric procedure developed by Geweke and Porter-Hudak. For higher sample sizes, the power of the EML-based test tends to decrease as the fractional integration parameter d increases and the LM-type test provides better results (except for near-unity values of d). Finally, by considering two integrated—but not cointegrated—series with AR(1) and, respectively, MA(1) disturbances, our size simulations demonstrate that, whatever the sample size, the EML-based test exhibits very low size distortions for positive AR(1) and negative MA(1) components, respectively.

Notes

[1]For a recent discussion on this subject, see Doornik and Ooms (2003). See also Section 4.3.

[2]For a presentation of *ARFIMA* processes, see Granger and Joyeux (1980) and Hosking (1981).

[3]In this paper, we use the Fortran program GQSTFRAC written by Sowell (1992). The Fortran library of numerical optimization algorithms GQOPT developed by Richard Quandt has also been used. Note that other implementations of the EML procedure exist, like the Ox routine developed by Doornik and Ooms (1999).

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