Long-Range Dependence in Mortgage Termination

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Abstract. A popular modeling technique in mortgage banking involves regarding mortgage terminations observed in a continuously replenished system (such as a servicing portfolio) as a Markov process of finite order k > 0 with values in a finite state space. In this article, we consider the ramifications of such an assumption. In particular,we conduct hypothesis tests of long-range dependence on a widely available data source. Finally, we consider the effects of the fractional cointegration explanatory variables such as housing price appreciation and the term structure of interest rate on the mortage termination process.

Keywords: residential mortgage, mortgage termination, Markov process, long-range dendepence, fractional integration

1. Introduction

As described in (Duffie and Singleton, 2003) and (Bluhm et. al., 2003), a popular technique for modeling the term structure of corporate default is to consider the problem from the point-of-view of a finite state space Markov process, or Markov chain. Beyond estimating default probability, such models provide a forecast of probability that a given corporate debt transitions to/from a degraded credit state, potentially sharpening understanding of credit risk. Motivated by the desire to increase understanding of the mortgage termination process, the banking industry has begun to adopt Markov models (see (Betancourt, 1999) and (Duffy et. al., 2005)) as a framework for understanding the evolution of mortgages and revolving credit lines from origination to termination.

In contrast to the Markov setting, some phenomenon exhibit long-range dependence, and are characterized by the property that observations separated by large time intervals can be highly correlated. Beginning with the hydrological studies of (Hurst, 1951), and the agriculture studies of (Whittle, 1956), the seminal investigations of long range dependent processes were carried out in (Mandelbrot and Wal-

lis, 1968), (Mandelbrot and Van Ness, 1968), (Mandelbrot and Wallis, 1969) and (Mandelbrot, 1969).

Long-range dependent processes have been used successfully to study naturally occurring phenomena. The analysis in (Granger, 1966) suggests that the 'typical' economic variable has unbounded spectral density as frequency tends towards zero (the defining characteristic of long-range dependence for stationary processes). In a ground-breaking study of business cycles (Kydland and Prescott, 1982), it is argued that housing construction and other aggregate economic variables can be better explained with shocks endowed with long-range dependence. Fractionally differenced time series were employed in (Backus and Zin, 1993), and evidence was provided suggesting that long-range dependence models of short-term interest rates, inflation and the growth of the money supply have superior performance over short-range dependent alternatives. The study of (Leland et. al., 1993) used fractional Brownian motion to explain certain network traffic phenomena such as time to next packet request. Comprehensive surveys of applications of long-range dependent processes to the fields of finance, hydrology and network traffic can be found in (Doukhan et. al., 2003).

Our aim is to study the mortgage termination process from the point of view of long-range dependence. The paper is organized as follows. In Section 2, we review the basic concepts and results from the theory of long range dependent processes. We set up the mortgage system in Section 3, which includes the mortgage servicing framework, along with observed terminations of an actual portfolio of mortgages. We also give two detailed heuristic arguments in support of long-range dependence for the mortgage termination. The data used for this analysis is reviewed in Section 4.We detail and review our statistical methods and results in Section 5. We devote Section 6 to a discussion of fractional cointegraion as a means of better understanding the mortgage system. We conclude by discussing open questions and future research directions in Section 7.

2. Foundations of Long-Range Dependence

In this section, we provide a brief review of the fundamental concepts from the theory of long-range dependent processes For comprehensive coverage, the reader is directed to (Beran, 2003) and (Doukhan et. al., 2003).

2.1. Long-Range Dependence in Stationary Processes

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with filtration $\{\mathcal{F}_t\}$, indexed by $t \in [0, T]$, where $T \leq \infty$. We assume a given stochastic process X_t, adapted to the filtration $\mathcal{F}_t$, where $\mathcal{F}_t$ satisfies the usual conditions regarding completeness and right-continuity.

The analysis for such a process X_t begins under the assumption of stationarity. Strictly speaking, a stationary process is one all of whose finite-dimensional distributions are invariant under a parallel shift of the time index t. For the purposes of this analysis, we will relax these requirements and consider a process to be stationary in the wide sense.

DEFINITION 1. *(Stationary Process) Given $(\Omega, \mathcal{F}, \mathbb{P})$,$\{\mathcal{F}_t\}$ and X_t as above, we say that X_t is (wide-sense) stationary provided that there is a finite constant μ such that*

$$\mu = \mathbb{E}[X_t], \forall t \tag{1}$$

and for $s \leq t$, the autocovariance function,$\gamma(s, t)$, given by

$$\gamma(s, t) = \mathbb{E}[(X_s - \mu)(X_t - \mu)] \tag{2}$$

is a function of $t - s$, i.e.

$$\gamma(s, t) = \gamma(0, t - s) = \gamma(t - s) \tag{3}$$

Given a process which is stationary as in Definition (1) above, we may then define the autocorrelation function,

$$\rho(t) = \frac{\gamma(t)}{\gamma(0)} \tag{4}$$

It is with the autocorrelation function of a stationary process that the analysis of long-range dependence usually commences.

When we consider the autocorrelation function of a Markov process, we discover that, asymptotically, the autocorrelation function decays to zero geometrically, i.e., like α^t, as t becomes unbounded. Intuitively then, a long-range dependent process should therefore be one whose autocorrelation decays much more slowly, say hyperbolically instead of geometrically. Motivated by this analysis, we arrive at the following definition.

DEFINITION 2. *(Long-Range Dependence) Let X_t be a stationary process. ThenX_t is said to be long-range dependent provided $\exists$ positive constants α and C,with $\alpha < 1$, such that*

$$\frac{\rho(t)}{Ct^\alpha} \to 1 \ as \ t \to \infty \tag{5}$$

As demonstrated in (Priestly, 1981), we may uncover significant properties of the autocorrelation function of a stationary process as a consequence of studying its spectral density function.

DEFINITION 3. *(Spectral Density Function) Let X_t be a stationary process. The spectral density function, $f(\lambda)$ is defined to be the Fourier transformation of the autocovariance function of the centered process $Y_t = X_t - \mu$*

$$f(\lambda) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-i\lambda t}\gamma(t)dt \tag{6}$$

We may now reformulate Definition (2) and consider long-range dependence from the point of view of the spectral density function.

THEOREM 4. *(Zygmund,1953) Let X_t be a stationary process. Then X_t is long-range dependent $\Longleftrightarrow \exists$ positive constants β, D, with $\beta < 1$ such that*

$$\frac{f(\lambda)}{D\,|\lambda|^{\beta}} \to 1 \ as \ \lambda \to 0 \tag{7}$$

Reviewing Theorem 4, we see that long-range dependence is equivalent to the existence of a certain integrable pole at $\lambda = 0$ of the spectral density function. This equivalent formulation of the long-range dependence property is leveraged by several statistical tests of long-range dependence, as we will discuss further detail in following sections.

2.2. Long Range Dependence in Non-stationary Processes

A fairly complete and satisfactory theory is now in place for the analysis of long-range dependence in stationary processes. However, no such comprehensive theory, as yet, has emerged for the analysis of non-stationary processes, which are inherently much more complex than their stationary analogues. See (Burke and Hunter, 2005) for a good overview of general techniques for non-stationary processes, both short-range and long-range dependent.

One important theoretical tool for studying long-range dependence in non-stationary processes has been the development (see (Granger and Joyeux, 1980) and (Hosking, 1981))of the fractionally integrated, autoregressive moving average, or $ARFIMA(p, n + d, q)$ process To introduce these processes, we start with a review of the traditional $ARIMA(p, n, q)$ process.

DEFINITION 5. *(ARIMA process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with filtration $\{\mathcal{F}_n\}$, where $n = 1, 2, \ldots$. Let X_n be a process*

adapted to $\mathcal{F}_n$. The X_n is said to be an $ARIMA(p, d, q)$ process provided $\exists$ non-negative integers $p, n,$ and q, such that X_n has the representation

$$\phi(B)\nabla^n X_n = \theta(B)\varepsilon_n \tag{8}$$

where B is the backshift operator, $\phi(z)$ is a degree p polynomial of the form

$$\phi(z) = a_0 z^p + a_1 z^{p-1} + \cdots + a_p \tag{9}$$

$\theta(z)$ is a degree q polynomial of the form

$$\theta(z) = b_0 z^q + b_1 z^{q-1} + \cdots + b_q \tag{10}$$

$\nabla = 1 - B$ is the self-differencing operator, and we assume that $\{\varepsilon_n\}$ is a white-noise process adapted to $\mathcal{F}_n$, i.e. $\{\varepsilon_n\}$ is a sequence of mean zero, mutually independent and identically distributed random variables.

As described in (Hosking, 1981), we may extend the traditional $ARIMA(p, d, q)$ process by allowing the degree of self-differencing d to take values in $(-\frac{1}{2}, \frac{1}{2})$. In order to accomplish this, consider a generalized binomial expansion of $\nabla^d = (1 - B)^d$. For $d \in (-\frac{1}{2}, \frac{1}{2})$, define the fractional difference operator

$$\nabla^d = (1 - B)^d = \sum_{k=0}^{\infty} \binom{d}{k} B^k \tag{11}$$

Here, the combinatorial term is given by

$$\binom{d}{k} = \frac{\Gamma(k - d)}{\Gamma(-d)\Gamma(k + 1)} \tag{12}$$

where $\Gamma(x)$ is the gamma function.

Introduced in (Hosking, 1981), we define the $ARFIMA(0, d, 0)$ process.

DEFINITION 6. *($ARFIMA(0, d, 0)$ process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with filtration $\{\mathcal{F}_n\}$, where $n = 1, 2, \ldots$. Let X_n be a process adapted to $\mathcal{F}_n$. The X_n is said to be an $ARFIMA(0, d, 0)$ process provided $\exists\ d \in (-\frac{1}{2}, \frac{1}{2})$ such that X_n has the representation*

$$\nabla^d X_n = \varepsilon_n \tag{13}$$

where $\{e_i\}$ is a white-noise process adapted to $\mathcal{F}_n$

As established in (Hosking, 1981, page 167), every $ARFIMA(0, d, 0)$ process with $d \in (0, \frac{1}{2})$ is a stationary, invertible long range dependent process. Using an $ARFIMA(0, d, 0)$ process as a building block, it is therefore possible to build a family of processes (some of which are non-stationary), and members of this family are called an $ARFIMA(p, n + d, q)$ process.

DEFINITION 7. *(ARFIMA$(p, n+d, q)$ process) Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space endowed with filtration $\{\mathcal{F}_n\}$, where $n = 1, 2, \ldots$. Let X_n be a process adapted to $\mathcal{F}_n$. The X_n is said to be an ARIMA$(p, n+, d, q)$ process provided $\exists$ non-negative integers p, n, and q, and real number $d \in (-\frac{1}{2}, \frac{1}{2})$ such that X_n has the representation*

$$\phi(B)\nabla^n(\nabla^d X_n) = \theta(B)\varepsilon_n \tag{14}$$

where $\phi(B), \theta(B)$, and $\{\varepsilon_n\}$ are as in Definition (5).

Given an $ARFIMA(p, n+d, q)$ process X_n with $d \neq 0$, assume that $\theta(B)$ is invertible. Then we may recover the stationary long-range dependent component of X_n, Y_n as

$$Y_n = \theta^{-1}(B)\phi(B)\nabla^n X_n \tag{15}$$

3. The Mortgage System

3.1. Definition of the Servicing Framework

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with filtration $\{\mathcal{F}_t\}$. Our main object of interest is a continually replenished portfolio of mortgage obligations and we model this as a nonlinear system

$$\lambda = F(\pi, \rho, \delta, t) \tag{16}$$

where λ is the $\mathcal{F}_t$−adapted process representing the percentage of active loans just prior to time t which terminate at time t (also known as the hazard process). The variables π, ρ, and δ represent real-valued $\mathcal{F}_t$−adapted processes, where π is the aggregate housing price process, and δ represents the aggregate demand for mortgages. We further assume that $\rho(t, T)$ represents the term structure of mortgage interest rates and is a progressively measurable function on the space $\Omega \times [0, \infty) \times [0, \infty)$. We begin our analysis by assuming that

$$F : \mathbb{R}^4 \longrightarrow [0, 1] \tag{17}$$

is Borel measurable, although often additional regularity assumptions are imposed on F.

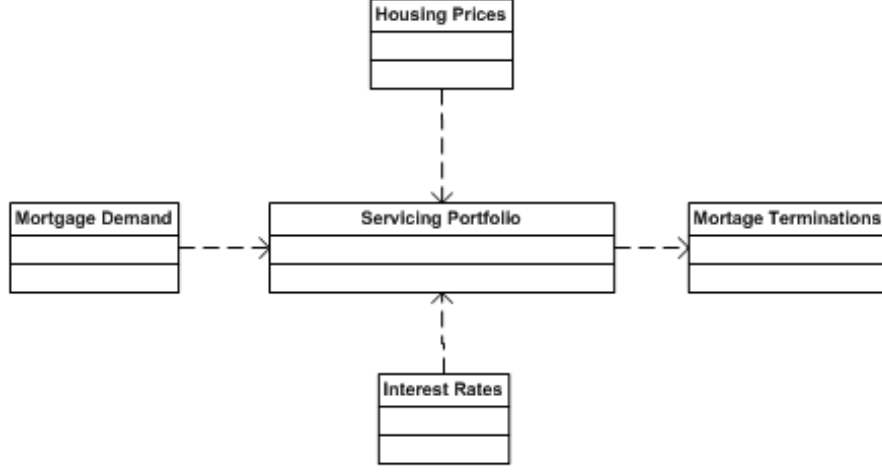The model is depicted graphically in the Figure 3.1 below.

Figure 3.1: The Mortgage System

3.2. Heuristic Argument for Long-Range Dependence in λ : Aggregation

It is known that, in certain situations, long-range dependence can be introduced into time series via aggregation. One of the first results to uncover this phenomenon is due to (Granger, 1980). Other investigations have been conducted on the nature of aggregates, including (Zaffaroni, 2004).

THEOREM 8. *(Granger 1980) Consider the aggregate model*

$$X_t = \sum_{j=1}^{\infty} X_t^i \tag{18}$$

where $\{X_t^i\}$ is a sequence of independent, $AR(1)$ processes of the form

$$X_t^i = \alpha_j X_{t-1}^i + \varepsilon_t^j \tag{19}$$

with $\alpha_j \in (-1, 1)$ and whose innovations $\{\varepsilon_t^j\}$ form a sequence of independent, identically distributed, mean zero process with variance σ_j. Assume the parameters $\{\alpha_j\}, \{\sigma_j\}$ are sequences of independent random variables, drawn randomly from populations F_α with mean α, and F_σ with means σ_ε^2. If F_α is an admissible beta distribution, then X_t is a long-range dependent process.

For details, including the definition of admissible beta distribution, the reader is referred to (Granger, 1980).

The mortgage servicing system as given by eq.(16) is inherently an aggregate model whose overall behavior is governed by the behavior of the individual mortgages in the servicing pool. Moreover, it is reasonable the many of the assumptions of Theorem 8 might hold for the mortgage servicing system.

3.3. Heuristic Argument for Long-Range Dependence in λ : A System of Fractional Diffusions

There is evidence in the literature (e.g. (Backus and Zin, 1993) and (Kydland and Prescott, 1982)) to suggest that at least some of the input parameters, particularly interest rates, ρ, and housing prices, π, are best modeled via long range dependence. To introduce another heuristic argument in favor of long-range dependence in the hazard process λ, we begin with the definition of fractional Brownian Motion, as can be found in (Doukhan et. al., 2003).

DEFINITION 9. *(Fractional Brownian Motion) Given a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$, a continuous time process B_t^H is said to be a standard one-dimensional fractional Brownian motion with Hurst parameter $H, H \in (0,1)$, provided*

1. *B_t^H is a Gaussian process with mean 0, and $B_0^H = 0$*

2. *$\exists \, \sigma > 0$ so that $E[(B_t^H)^2] = \sigma^2 \, |t|^{2H}$*

3. *The increments of B_t^H are stationary, i.e. for each real s,the process given by $\{B_{t+s}^H - B_s^H\}$ is distributionally equivalent to the process $\{B_t^H - B_0^H\}$*

As discussed in (Doukhan et. al., 2003) and the upcoming (Biagini et. al., 2006), an analogue of classical Itô stochastic calculus can be developed for fractional Brownian motion. In particular, the notion of a process having dynamics governed by a fractional diffusion equation can be well-defined and explored.

Suppose further that the interest rate process ρ, the housing price process π, and the mortgage demand process δ each have dynamics governed by fractional stochastic differential equations of the form:

$$
\begin{aligned}
d\rho &= \mu_\rho \, dt + \sigma_\rho \, dB_{H_\rho} \\
d\pi &= \mu_\pi \, dt + \sigma_\pi \, dB_{H_\pi} \\
d\delta &= \mu_\delta \, dt + \sigma_\delta \, dB_{H_\delta}
\end{aligned}
\tag{20}
$$

where $B_{H_\rho}, B_{H_\pi}, B_{H_\delta}$ are standard one-dimensional fractional Brownian motions processes with Hurst parameter H_ρ, H_δ and H_π respectively. Assuming sufficient regularity, we can determine the dynamics of $\lambda = F(\pi, \rho, \delta, t)$ via an application of the fractional Itô's rule. Under reasonable hypotheses on this dynamical system, one might expect $d\lambda = dF(\pi, \rho, \delta, t)$ to also exhibit long-range dependence. An investigation into long-range and short-range dependence of systems given by eq.(20) above can be found in (Sinek, Sun and Thurston, (in preparation))

4. The Mortgage Performance Data

4.1. OVERVIEW OF THE DATA

The data used in this analysis consist of mortgage performance data for subprime, 30 year loans as made available in the ABS master database from the Loan Performance Corporation.The data investigated consist entirely of first lien loans, and include both adjustable rate and fixed rate products

4.2. SUBPRIME MORTGAGE CPR PERFORMANCE DATA

In Figure 4.2 below, we display the overall CPR curve (in dark blue) on the primary axis, together with the loan count (in red) on the secondary axis.
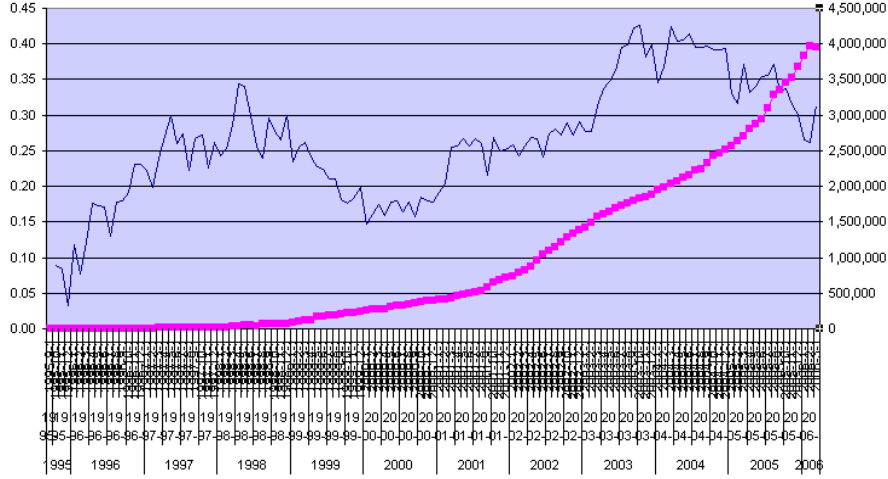

Figure 4.2 Subprime Overall CPR (source: Loan Performance Corp.)

This monthly data spans the time interval from the second quarter of 1995 through the first quarter of 2006. We observe that the majority of the loans in the dataset were originated since 2002.

4.3. SUBPRIME MORTGAGE CPR 1ST SELF-DIFFERENCE

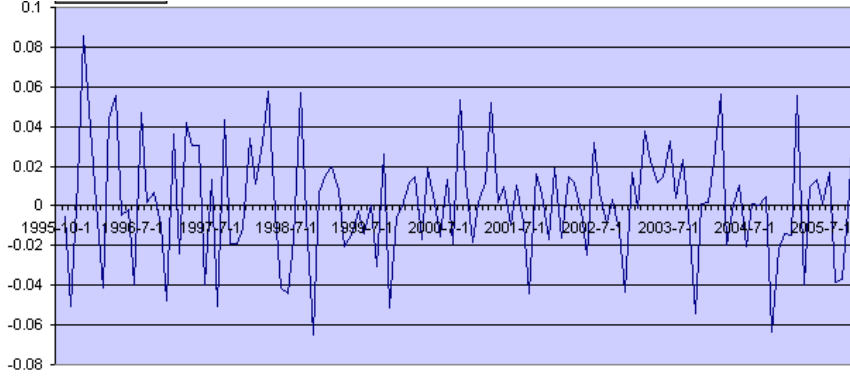In Figure 4.3 below, the first self-difference of the subprime CPR data is shown (in dark blue).

Figure 4.3 First Order Self-Difference (source: Loan Performance Corp.)

The self-differenced data appears reasonably well-behaved and we are led to conjecture that this series is both stationary and invertible.

5. Statistical Methods and Results

5.1. THE RESULTS

We drew the following conclusions based on our statistical analysis of the subprime termination data:

− The undifferenced CPR series X_n is $I(1)$, in the sense that the series of first differences ∇X_n is stationary and invertible.

− The series of first differences ∇X_n is a Gaussian series and, moreover, is (statistically) short-range dependent, with Hurst parameter estimate $\widehat{H} \doteq 0.49$ and self-differencing parameter

$$\widehat{d} \doteq -0.01 \qquad (21)$$

− The 95% confidence interval for the self-differencing parameter is

$$d \in [-0.146284124350426, 0.128529386219655]. \qquad (22)$$

In the remainder of the section, we provide our analysis and statistical results in support of these conclusions.

5.2. I(1) Analysis: Stationarity/Invertibility of the Self-Differenced Series

In order to test the hypothesis that the subprime data is drawn from an I(1) process, we are required to test whether the series of first self-differences is stationary and invertible. Our approach involves an $ARIMA$ analysis of the first differenced series, added by several statistical hypothesis tests along the way.

5.2.1. *Minimum Information Criterion*

In Figure 5.2.1 below, we display the minimum information criterion for the $ARIMA(p, d, q)$ model.

Minimum Information Criterion						
Lags	MA 0	MA 1	MA 2	MA 3	MA 4	MA 5
AR 0	-7.32501	-7.31081	-7.27682	-7.25939	-7.25497	-7.23363
AR 1	-7.32827	-7.2959	-7.28238	-7.27351	-7.25327	-7.21627
AR 2	-7.30688	-7.30715	-7.27669	-7.29265	-7.28109	-7.24726
AR 3	-7.2907	-7.29974	-7.27901	-7.25626	-7.24977	-7.21424
AR 4	-7.28792	-7.28857	-7.28664	-7.24926	-7.21332	-7.18545
AR 5	-7.25769	-7.25271	-7.25793	-7.22005	-7.18308	-7.14703

Figure 5.2.1: ARIMA Minimum Information Criteria

The minimum information score of -7.32827 is achieved at AR 1, MA 0, therefore we choose these values as the order of the $ARIMA(p, 0, q)$ model to fit to our series of first self-differences.

Based on the minimum information criteria results, we selected an $ARIMA(1, 0, 0)$ model and estimated the autoregressive parameter using conditional maximum likelihood estimation.

5.2.2. *Conditional Least Squares Estimation*

The estimation results of the conditional least squares estimation of the $ARIMA(1, 0, 0)$ are given in Figure 5.2.2 below.

Conditional Least Squares Estimation					
Parameter	**Estimate**	**Standard Error**	**t Value**	**Approx Pr > |t|**	**Lag**
MU	0.0017176	0.0019926	0.86	0.3904	0
AR1,1	-0.25581	0.08828	-2.90	0.0045	1

Figure 5.2.2: ARIMA Regressions Results

As the loan root of the AR term lies outside of the unit circle, we deduce that this $ARIMA(1, 0, 0)$ is stationary and invertible.

5.2.3. *Goodness-of-Fit*

In order to check goodness-of-fit of our selected model, we show the results of the residuals analysis in Figure 5.2.3 below.

Autocorrelation Check of Residuals									
To Lag	**Chi-Square**	**DF**	**Pr > ChiSq**	**Autocorrelations**					
6	5.72	5	0.3343	-0.027	-0.084	0.029	-0.161	0.096	0.009
12	26.14	11	0.0062	0.111	-0.010	0.017	-0.128	0.071	0.335
18	31.48	17	0.0175	-0.093	-0.006	-0.063	-0.130	0.086	0.005
24	47.19	23	0.0021	-0.116	0.040	-0.174	-0.148	0.138	0.126

Figure 5.2.3: Residuals Analysis

In general, the quality of the residuals are high, as evidenced by the autocorrelation data. Under null hypothesis of no residual serial correlation, we would fail to reject this hypothesis for reasonably high levels of significance.

5.3. TESTS FOR GAUSSIAN DATA

As described in (Beran, 2003, Chapter 6). given a stationary time series,which is assumed to be drawn from an $ARFIMA(p, d, q)$ process, we may conduct a maximum likelihood investigation in order to estimate the degree of fractional differencing, d. This analysis relies heavily on methods from the spectral analysis of Gaussian processes.

The histogram of the series of first differences is diplayed in Figure 5.3.1 below.
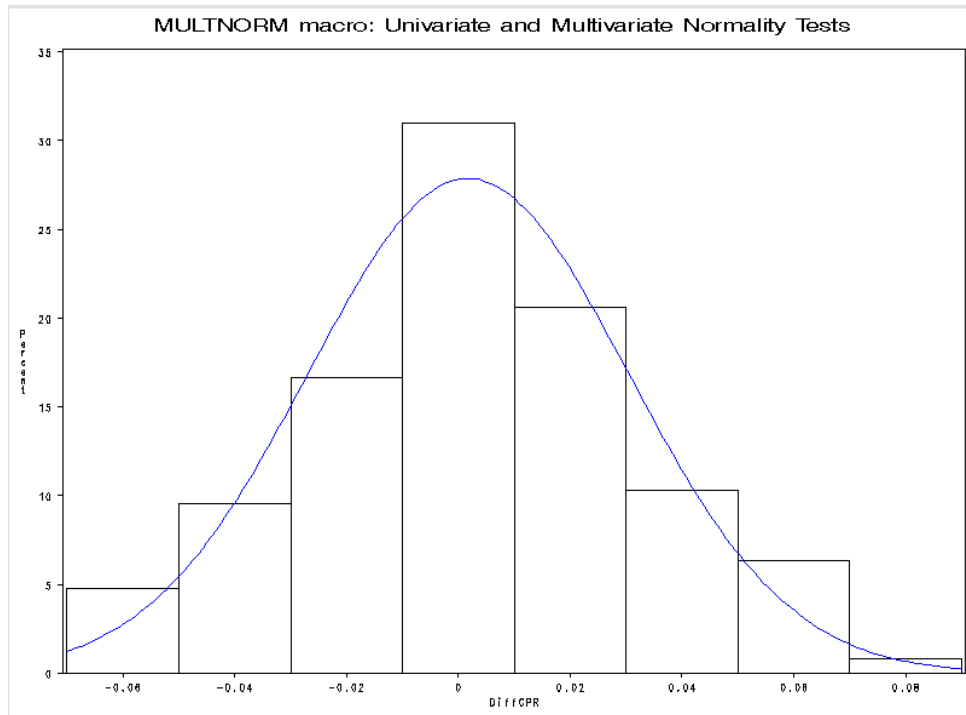
Figure 5.3.1: Histogram for the Series of First Differences

In support of the applicability of Whittle's method, we apply a normality tests to the series of first differences. The results of these test are displayed in Figure 5.3.2 below.

The UNIVARIATE Procedure
Fitted Distribution for DiffCPR

Goodness-of-Fit Tests for Normal Distribution					
Test	Statistic		p Value		
Kolmogorov-Smirnov	D	0.06734338	Pr > D	>0.150
Cramer-von Mises	W-Sq	0.10837373	Pr > W-Sq	0.089
Anderson-Darling	A-Sq	0.66246792	Pr > A-Sq	0.085

Figure 5.3.2: Normality Tests on 1st Differenced Series

The normality test results indicate normality in the data, with reasonably large p-values from all tests. Given these p-value thresholds, we fail to reject the null hypothesis that the data was drawn from a Gaussian process.

5.4. Whittle's Estimate of the fractional differencing parameter d

In this section, we provide an overview of Whittle maximum likelihood estimate of the fractional differencing parameter d. The reader will find a detailed treatment of Whittle's approximation in (Beran, 2003, Chapter 5).

Assume $\{X_n\}$ is a stationary, invertible time series with mean μ and variance σ^2. Assume that X_n is long-range dependent, with parameter $\alpha \in (0, 1)$ as in Definition 2. The Hurst parameter H is such that

$$\alpha = 2 - 2H \tag{23}$$

and we assume that $H \in (\frac{1}{2}, 1)$. To exploit the well-know properteis of stationary Gaussian processes (e.g. such a process is fully characterized by its mean and covariances /spectral density), we further assume that X_n is Gaussian.

5.4.1. The Exact Gaussian MLE

Following (Beran, 2003, page 104.), the joint distribution function of $X = \{X_1, X_2, \ldots, X_n\}$ is (in the case that X is also a causal, linear process), is given by

$$h(x, \theta) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \|\Sigma(\theta)\|}} \exp(-\frac{1}{2} x^T \Sigma^{-1}(\theta) x) \tag{24}$$

In this expression, $x \in \mathbb{R}^n, \theta = (\sigma^2, H, \theta_3, \ldots, \theta_m)$ is an unknown m-dimensional vector of parameters, and $\Sigma(\theta)$ is the covariance matrix of X.

The Exact Gaussian maximum likelihood estimate for θ is obtained by maximizing $\log h(x, \theta)$ with respect to the parameter θ. This estimate is known to have standard asymptotic normality properties. and the variance of the estimate can be approximated by the inversion Hessian of the log likelihood, evaluted at the parameter estimate, $\widehat{\theta}$.

5.5. Whittle's Approximate MLE

In order to reduce the numerical complexity of the problem of optimizing the exact Gaussian likelihood given in eq.(24), (Whittle, 1953)

proposed the following approximation to the exact log likelihood function

$$L_W(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda; \theta) d\lambda + \frac{x^T A(\theta) x}{n} \tag{25}$$

where $f(\lambda; \theta)$ is the spectral density of the process, and $A(\theta)$ is a numerically stable estimator of $\Sigma^{-1}(\theta^0)$. The asymptotic statistical properties of this estimate are the same as those for the exact maximum likelihood estimate.

We note that eq.(25) may be further approximated by appealing to standard discrete time estimates (e.g., Riemann sums for Riemann integrals, periodogram for spectral density).

5.6. ESTIMATION AND HYPOTHESIS TESTING

In Section 5.2 above, classical time series analysis indicated that the self-differenced CPR series follows $ARIMA(1,0,0)$ process. We now further assume that the self-differenced CPR series follows a fractional $ARFIMA(1,d,0)$ process, and we wish to obtain an estimate for the fractional self-differencing parameter d To this end, we form the unknown parameter

$$\theta = (\sigma^2, H) \tag{26}$$

where $H = d + \frac{1}{2}$, and appeal to the explicit form of the spectral density of the $ARFIMA(1,d,0)$ in the Whittle MLE calculation.See (Beran, 2003, page 63.) for details.

Equipped with the Whittle estimate $\widehat{d}$ of the self-differencing parameter d, we consider the asymptotic distribution in order to test the null hypothesis

$$H_0 : d = 0 \tag{27}$$

The Hurst parameter $H = d + \frac{1}{2}$, so that $d = H - \frac{1}{2}$. We may then reformulate the hypothesis test as

$$H_0 : H = \frac{1}{2} \tag{28}$$

Appealing to (Beran, 2003, Theorem 5.1, page 105), we may appeal to the asymptotic normality property of the estimate. Therefore given our approximate $\widehat{H}$, we may build a normal $N(0,1)$ distributed test statistic via

$$z = \sqrt{n} \frac{(\widehat{H} - \frac{1}{2})}{\sigma} \tag{29}$$

where σ^2 is the estimate of the variance obtained from the information matrix in the usual manner.

Optimizing the likelihood, we obtain an estimate for H, achieved with a convergence criterion of 10^{-8}. In the table below we give these estimates and parameter values:

$$
\begin{aligned}
\widehat{H} &= 0.491122630934614 \\
\frac{\sigma}{\sqrt{n}} &= 0.0701054873903268 \\
n &= 125
\end{aligned}
\tag{30}
$$

Computing the confidence interval for H at the $\alpha = .05$ significance level, we have

$$
[\widehat{H} + \frac{(-z_{\frac{\alpha}{2}})(\sigma)}{\sqrt{125}}, \widehat{H} + \frac{(z_{\frac{\alpha}{2}})(\sigma)}{\sqrt{125}}] = [0.353715875649574, 0.628529386219655]
\tag{31}
$$

At the $\alpha = .05$ significance level, two-sided critical values $z_{\frac{\alpha}{2}} = \pm 1.96$ for the hypothesis test lead us to fail to reject the null hypothesis that series of first differences is short-range dependent. We conclude that, with self-differencing parameter estimate $\widehat{d} \doteq -0.01$ and Hurst parameter estimate $\widehat{H} \doteq 0.49$, that the mortgage termination data exhibits short-range dependence.

6. Complexity Reduction via Fractional Cointegration

Reviewing our heuristic arguments in favor in long-range dependence for mortgage termination from Section 3, we note that while both arguments predict long-range dependence,our results show no statistical evidence for long-range dependence, and our estimate of the Hurst parameter is close to $\frac{1}{2}$. One possible explanation for the absence of the long range dependence in the hazard process could reside in the notion of fractional cointegration as a complexity reducing mechanism. Following (Burke and Hunter, 2005, page 171), we have the following definition.

DEFINITION 10. *(Fractional Cointegration) Let X_t, Y_t be a pair of non-stationary time series. Assume there is a real $d \in (-\frac{1}{2}, \frac{1}{2})$ so that*

$$
\nabla^d X_t, \nabla^d Y_t \in I_s(0)
\tag{32}
$$

Then X_t, and Y_t are said to be fractionally cointegrated provided there are real, constants β_X, β_Y (not both zero) such that

$$
\beta_X X_t + \beta_Y Y_t \in I_s(0)
\tag{33}
$$

here $I_s(0)$ denotes the collection of stationary, invertible, short-range dependent time series.

We note that more general definitions of fractional cointegration exist in which the order of fractional differencing is merely reduced, not eliminated.

It is possible that the absence of the long-range dependence of the hazard process λ could be, in part, due to fractional cointegration among some or all of the non-stationary, long-range dependent explanatory processes. We note that investigations into fractional cointegration and fractional error correction is proving to be a fruitful area of investigation, as evidenced by the number of recent research publications along these lines.

7. Directions for Further Research

We have identified several directions which might warrant additional investigation.

- Additional Mortgage Products - The investigation here has been focused on subprime 30 year first lien mortgages.Other types of mortgages, including home equity loans and second lien mortgages could be investigated.

- Other Credit Related Products - The investigation could also be extended to cover credit cards, student loans, automotive & marine loans, and other credit related products.

- Voluntary Prepayment versus Default - The termination process includes terminations due to both voluntary prepayments and involuntary defaults. This investigation could be applied to each of these component of mortgage termination separately.

- Fractional Cointegration - A new investigation could be pursued to uncover to what extent the explanatory variables of the termination process are fractionally cointegrated, as well as what role fractional cointegration plays in the behavior of the termination process.

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