# Lecture 5: Course allocation

Thilo Klein

ZEW Mannheim

# Introduction

"Traditional" assignments problems are either:

- one-to-one (e.g., assigning bedrooms)
- or many-to-one (e.g., assigning students to schools)

For those problems, we can identify well-behaved mechanisms (strategyproof, efficient or stable).

Many-to-many assignment problems are more complex. A typical example is course allocation:

- Each student has to enroll in several courses;
- Each course can admit many students.

# Preliminaries

A course allocation problem is given by:

- A (finite) set of students;
- A (finite) set of courses, and each course has a maximal capacity.
- Students have to be assigned a schedule: a collection of courses.

A course allocation problem is a special case of a combinatorial assignment problem.

In a course allocation problem students have preferences over schedules.

For many-to-one assignment problems we usually assume that objects' priorities are responsive. This is a strong assumption for course allocation problems:

- Some courses are complements
- Some courses are substitute.

Responsive preferences do not allow us to capture such realistic constraints.

We will study what can be done with responsive preferences, but eventually we will have to drop that assumption.

Another complication is about the choice of the assignment mechanism.

Unlike for one-to-one or many-to-one problems, we do not have much choice. Before that we need the following concept:

### Definition

A mechanism $\varphi$ is non-bossy if for any profile of submitted preferences $P$, and for any individual $i$, and preference ordering $P'_i$,

$$\underbrace{\varphi(P'_i, P_{-i})(i)}_{i\text{'s assignment}} = \varphi(P_i, P_{-i})(i) \quad \Rightarrow \quad \underbrace{\varphi(P'_i, P_{-i})}_{\text{the whole assignment}} = \varphi(P_i, P_{-i})$$

The mechanisms is non-bossy nobody can change other individuals' assignment without her own assignment.

### Theorem (Pápai)

*A many-to-many assignment mechanism is strategyproof, non-bossy and Pareto efficient if, and only if, it is a Serial Dictatorship mechanism.*

Note that here courses do not have priorities over students. So Serial Dictatorship and Top Trading Cycles (with public endowments) are the same.

Another way to read Pápai's result: there is no way to run TTC, respecting all possible priorities that courses have.

# Bidding for courses

A common assignment mechanisms in many business schools consists of running an auction (Columbia Business School, Haas Business School, etc.).

The basic structure of those auctions is:

1. Each student is endowed with a budget (of coins, tokens, points, etc.).
2. Students submits bids for each of the course they want to enroll in.
3. Bids are used to produce an ordering of students. A Serial Dictatorship mechanism is run, using this ordering.

Bids are only submitted once, so it is a first-price, sealed-bid auction.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 3 | 2 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 2 | 2 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X$ | | | |

Alice's bid for $X$ is the highest, so she is assigned that course.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 1 | 2 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X$ | $X$ | | |

Next highest bid is Bob's bid for $X$, so he is assigned that course.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 2 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X$ | $X$ | $X$ | |

Next highest bid is Carol's bid for $X$, so she is assigned that course. Course $X$ is now full.

Subsequent bids for course $X$ will be ignored.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 1 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X,Y$ | $X$ | $X$ | |

Next highest bid is Alice's for course $Y$. So she gets it. She is no longer participating, she has 2 courses.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 0 | 4 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X, Y$ | $X$ | $X$ | $Y$ |

Next highest bid is Erik's for course $Y$. So he gets it. Course $Y$ is now full.

Subsequent bids for course $Y$ will be ignored.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 0 | 3 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X, Y$ | $X, Z$ | $X$ | $Y$ |

Next highest bid is Bob's for course $Z$. So he gets it. He is no longer participating, he has 2 courses.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 0 | 2 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X,Y$ | $X,Z$ | $X,Z$ | $Y$ |

Next highest bid is Carol's for course $Z$. So she gets it. She is no longer participating, she has 2 courses.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 0 | 1 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X,Y$ | $X,Z$ | $X,Z$ | $Y,Z$ |

Erik is the last one without a second course. Only $Z$ is available, so he gets it.

# Example

Alice, Bob, Carol and Erik need to take two courses among courses $X$, $Y$ and $Z$. The capacities are 3, 2 and 4, respectively. The bids are:

| **capacity** | 0 | 0 | 1 |
|---|---|---|---|
| | $X$ | $Y$ | $Z$ |
| Alice | 60 | 38 | 2 |
| Bob | 48 | 22 | 30 |
| Carol | 47 | 28 | 25 |
| Erik | 45 | 35 | 20 |

| Alice | Bob | Carol | Erik |
|---|---|---|---|
| $X,Y$ | $X,Z$ | $X,Z$ | $Y,Z$ |

The auction is over.

# Issues

In principle, with a course bidding mechanism we could interpret bids as preferences with intensities.

But it is easy to see that it cannot be strategyproof:

My favorite course is the least demanded.

$\Rightarrow$ I submit the lowest possible bid for that course.

$\Rightarrow$ If truthful this would be my highest bid.

Also, courses prices in the bidding mechanism cannot be considered as market-clearing prices.

If an auction is efficient it should produce competitive prices. It is not the case here.

**Example**:

- Liza bids 1 for course $X$ (her favorite), 50 for course $Y$ and 50 for course $Z$.
- She knows $X$ is underdemanded: she'll be enrolled for sure.
- Suppose she ends up being enrolled in courses $Y$ and $Z$, and nobody is enrolled in course $X$.
- The market clearing price for course $X$ should be 0.
- But Liza bids above the competitive price for $X$.

  If the assignment is competitive she should be enrolled in $X$.

# Deferred Acceptance with bids

Sönmez and Ünver argue that we need to separate:

- Inferring student's preference orderings over courses;
- Determining which student has a bigger claim (priority) over each course.

The propose a modified mechanism that separates these two problems.

For the presentation of the mechanism it is assumed that each student has to enroll in $k$ courses.

# Gale-Shapley Pareto-Dominant Market Mechanism

**Step 1**
Students are randomly ordered. This order will be used to break ties between students, if needed.

**Step 2**
Each student submits her preferences over courses (not over schedules).

**Step 3**
Each student submits a bid for each course.

**Step 4** Run the Deferred Acceptance algorithm where:

- students first propose to their $k$ most preferred courses.
- When a student is rejected by $p$ courses, she propose to her next $p$ most preferred courses.
- Courses accept and reject students using the bids to prioritize students (e.g., the student with the highest bid for a course has the highest priority).
- If two students have identical bids for the same course their relative priority is given by the random ordering of Step 1.

In this mechanism, revealing one's true preferences over courses (Step 2) is a dominant strategy.

However, choosing how much to bid is still a strategic decision. We need to consider equilibria of the bidding game (Step 3).

### Proposition

*With the Gale-Shapley Pareto-Dominant Market Mechanism, if students choose bids that maximize their expected payoffs then the course assignment (and the prices) correspond to a market equilibrium.*

### Proposition

*The course assignment obtained using the Gale-Shapley Pareto-Dominant Market Mechanism Pareto dominates the assignment of any competitive equilibrium whenever the bids used constitute an equilibrium.*

*Also, it cannot be Pareto dominated by the assignment obtained with the course bidding mechanism (using the same bids).*

# The Harvard Business School method

The Harvard Business School (HBS) takes a different approach:

Budish and Cantillon (2012) studied it, they found that:

- The mechanism is not strategyproof or efficient;
- It produces outcomes that can be more efficient than a strategyproof mechanism!

# The HBS draft mechanism

The mechanism used at HBS is a modification of the Random Serial Dictatorship

**Step 1**:
Each student submits a preference list over courses.

**Step 2**: Students are assigned a random number (no two students have the same number).

$\mathbf{Step k, k \geq 3, k\ odd}$
Each student who still needs a course is assigned her most preferred course among the courses that are still available, starting with the student with the highest random number.

$\mathbf{Step k, k \geq 3, k\ even}$
Each student who still needs a course is assigned her most preferred course among the courses that are still available, starting with the student with the lowest random number.

# Example

Suppose that the random order is

Alice, Bob, Carol, Denis

The HBS Draft mechanism works as follows (once the students submitted their preferences):

1. Run the Serial Dictatorship with this order:

   Alice, Bob, Carol, Denis

2. Run the Serial Dictatorship with this order:

   Denis, Carol, Bob, Allice

3. Run the Serial Dictatorship with this order:

   Alice, Bob, Carol, Denis

4. And so on.

# Strategic behavior

The HBS Draft mechanisms is not strategyproof: there is a conflict between

- students' preferences, and
- courses popularity.

Being truthful may not be longer an option if the most preferred course is not very popular.

In this case a student can be better off declaring:

- The very popular course as top choice;
- The most preferred course (and not popular) lower in the submitted preferences.

# Example

Each student has to take 2 courses, each course has a capacity of 2 students.

| $P_{\text{Alice}}$ | $P_{\text{Bob}}$ | $P_{\text{Carol}}$ |
|---|---|---|
| $c_1$ | $c_2$ | $c_1$ |
| $c_2$ | $c_1$ | $c_3$ |
| $c_3$ | $c_3$ | $c_4$ |
| $c_4$ | $c_4$ | $c_2$ |

Suppose all students are truthful. Then:

- For any ordering of the students Bob is sure to get $c_2$.
- At the end of the 1st round $c_1$ is no longer available: taken by Alice and Carol.
- Top choices at beginning of 2nd round:
  $c_2$ (Alice), $c_3$ (Bob), $c_3$ (Carol).
  So Bob will get $c_3$ for sure.

Suppose Bob deviates, and we have

| $P_{\text{Alice}}$ | $P'_{\text{Bob}}$ | $P_{\text{Carol}}$ |
|---|---|---|
| $c_1$ | $c_1$ | $c_1$ |
| $c_2$ | $c_2$ | $c_3$ |
| $c_3$ | $c_3$ | $c_4$ |
| $c_4$ | $c_4$ | $c_2$ |

- Bob gets $c_1$ if he's 1st or 2nd. So he gets $c_1$ with probability $\frac{2}{3}$.
- If Alice is last in the queue: one seat left for $c_2$. The top choices in the 2nd step are: $c_3$ (Alice), $c_2$ (Bob), $c_3$ (Carol). So Bob gets $c_2$.
- If Carol is last in the queue: one seat left for $c3$. The top choices in the 2nd step are: $c_2$ (Alice), $c_2$ (Bob), $c_4$ (Carol). So Bob gets $c_2$.
- If Bob is last in the queue, he picks $c_2$ in the first step and then $c_3$ in the 2nd step.

To sum up.

- If Bob submits $P_{\text{Bob}} = c_2, c_1, c_3, c_4$
  He gets
  - $\{c_2, c_3\}$
- If Bob submits $P'_{\text{Bob}} = c_1, c_2, c_3, c_4$
  He gets
  - $\{c_2, c_3\}$ with probability $\frac{1}{3}$ (he is last in the order)
  - $\{c_1, c_2\}$ with probability $\frac{2}{3}$ (he is not the last in the order)

$\Rightarrow$ Submitting $P'_{\text{Bob}}$ is a better option for Bob.

Proposition (Budish and Cantillon)

*(a) Students should not reverse the relative ranking of two courses in their submitted preference lists (with respect to their true preferences) if by doing so they do not obtain the preferred course for sure.*

*(b) Students should reverse the relative ranking of two courses if this does not come at the cost of not obtaining the more preferred course.*

Proposition (Budish and Cantillon)

*The outcome of the Harvard Draft mechanism under equilibrium play can be ex-post inefficient.*

# Empirical evidence

Budish and Cantillon analyzed a series of data consisting of:

- a poll conducted in May, 2005, asking students their true preferences over courses;
- preference lists submitted for a trial run in May, 2005; and
- preference lists submitted for the real run.

They observe that:

- preferences submitted in the trial run or the real run differ significantly from the true preferences.
- Changes between the true preferences and the submitted preferences change according to their theoretical findings (proposition in previous slide).

Using the true preferences obtained in the first poll, Budish and Cantillon ran simulations.

- On average, 64% of the students would benefit by trading at least one of the courses obtained with the HBS Draft mechanism.
- Under the HBS Draft mechanism, strategic plays lowers students' welfare compared to truthful play. It reduces the number of students obtaining their top choices (from 82% to 63%), and on average students obtained lower ranked courses.

- Under the HBS Draft mechanism, strategic plays increases students welfare compared to a strategyproof mechanism (Serial Dictatorship in 1 round where students picks their entire schedule).

One key feature that can explain the popularity of the HBS Draft mechanis is that strategic misrepresentations are relatively easy to undertake.

# Course assignment at Wharton

The bidding mechanism, the Gale-Shapley Pareto-Dominant Market mechanism or the HBS draft mechanisms implicitly assume that students' preferences over courses are responsive.

But students may well see some courses as substitutable or complement.

Ideally, we would like to have a combinatorial assignment mechanism. There are two problems, though:

- In general, it is not possible to have a nice and robust mechanism unless we adopt a Serial Dictatorship mechanism.
- Asking students preferences over scheduel is, in practice, not feasible.

# Approximate Competitive Equilibrium from Equal Incomes

Budish (2011) proposed a solution, using a "traditional" economics approach, the competitive equilibrium, with a twist: instead of an exact equilibrium he seeks only an approximate equilibrium.

The approach is like for the bidding mechanism: endow students with a budget of "tokens".

In a competitive equilibrium we need to find prices for each course such that:

$$\sharp \text{ student asking for a course} \leq \text{course capacity},$$

and each for each student, given the courses prices, her demand is her most preferred, affordable schedule.

# Example

Budget is 1,000 tokens and I need to enroll in 2 courses among the following 3 courses:

- Marketing;
- Corporate Finance;
- Accounting.

Let $p_M$, $p_C$ and $p_A$ be the prices for thoses courses.
My demand could be

| $p_M$ | $p_C$ | $p_A$ | Demand |
|---|---|---|---|
| 10 | 800 | 200 | Accounting, Corporate Finance |
| 100 | 800 | 500 | Marketing, Corporate Finance |
| 200 | 900 | 500 | Marketing, Accounting |

Budish's solution is the Approximate Competitive Equilibrium with Equal Incomes (ACEEI).

- An approximate equilibrium: prices do not necessarily clear the markets.

  For instance, the prices could be that there some courses are oversubscribed.

  For HBS, he calculates that with 900 students chosing 5 courses (among 50 possible courses), there are only 11 instances of oversubscription.

- Students are not all endowed with the same budget. But differences can be (very) small.

  At Wharton the average budget is 5,000 tokens. The maximum difference between 2 budgets is 80 tokens (1.6%).

Budish' approximate competitive equilibrium works well on theory. But in practice, it may be very difficult to implement.

In 2011–2012 the Wharton School at the University of Pennsylvania considered changed is course assignment mechanism (until then, a bidding mechanism).

Asking students their demands is impossible: the number of possible schedules is over 100 million.

# How to report complex demands?

To see whether the ACEEI can be implemented Budish and Kessler set up an experiment.

A typical experimental setup would be:

- Endow subjects with some preferences;
- Incentivize subjects to express their demands.

This approach is not suited for the present problem:

> the purpose is to elicit students' preferences over schedules.
>
> Endowing them with artifical preferences does not help to see how to elicit their real preferences.

# The experimental mechanism

Budish and Kessler propose the following mechanism:

1. Students give a score between 1 and 100 for any course they are interested in.
   (1 for the least desired courses, 100 for the most desired courses).

2. Students can report adjustments for any pair of courses:

   - A positive adjustment for courses $X$ and $Y$: signals they are viewed as complements.
   - A negative adjustment for courses $X$ and $Y$: signals they are viewed as substitutes.

3. For each schedule: add the score of each courses (and ajdustements, if any). The preferences over schedules is given by the ranking of their total scores.

# Example

A student reports the following scores:

| Course | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| Score | 90 | 50 | 40 | 20 |

This means the student prefers $A$ and $D$ $(90+20=110)$ to $B$ and $C$ $(50+40=90)$.

If the student puts an adjustment of $+25$ to $B$ and $D$, then

$$\underbrace{B \text{ and } D}_{50+20+25=95} \quad \text{preferred to} \quad \underbrace{B \text{ and } C}_{50+40=90}$$

Subjects were then given what the computer found as being their 10 most preferred schedules.

$\rightarrow$ it allows subjects to fine tune their preferences.

Once the scoring of each course (and the adjustments) is finalized, an heavy computation is made to calculate the ACEEI.

# Results

The experiment was a success! Wharton adopted the mechanism in fall 2013.

In the experiments, subjects were also asked to play with the former bidding mechanism, and then compare the schedules they would obtain for both mechanisms.

Note that:

- Reporting preferences over combinations of courses is difficult
- Compare two schedules and say which one is the preferred one is easy.

Students were also asked to compare their schedules with that of other students.

8 sessions were run.

- For 6 sessions, the majority of the students preferred the schedule obtained with ACEEI. For 2 sessions there was a tie.
- Envy is almost impossible to eliminate in problems like course allocation.

  ACEEI reduced envyness by about 30%.
- The scoring methodology allows to report preferences over schedules accurately:

  When asking students to compare schedules, 85% of the comparisons were consistent with the predicted preference (i.e., using the scores).

# Take-away

- Course allocation is a many-to-many problem:
  - Each student can enroll in many courses;
  - Many students enroll in the same courses.
- Many-to-many problems are more complex than many-to-one problems. The only strategyproof, non-bossy and Pareto efficient mechanism is the Serial Dictatorship.
- Auctions are often used to allocate courses.
  - Students are endowed with a budget (tokens, points, etc.)
  - Students bid for courses.
  - Bids are ordered and a Serial Dictatorship is run.

  This mechanism is not strategyproof, and bids cannot be considered as market-clearing prices.

- A solution is to use a mix of bids (to determine students' priority rankings for each course) and the Deferred Acceptance algorithm (with students submitting preferences over courses).

  This mechanism is efficient mechanism and bids correspond to market prices.

- Harvard Business School Draft: A series of Serial Dictatorship where students choose one course, with the order of dictators reversed for each run.

- HBS mechanism is not strategyproof, not efficient.

- A simple strategy is to ask first for popular courses, not necessarily the most preferred.

  Empirical evidence shows that students use such strategies.

  But strategic plays increases welfare compared to a strategyproof mechanism!

- Wharton put in place a new mechanism that allows students to express substitutabilities and complementarities.
- Students' preferences are elicited, then heavy computation determines students' demand (with a budget constraint) and calculate a competitive market equilibrium.
- A key feature: the equilibrium is approximate and not students have unequal budgets (but differences are small).
- Outcomes under the Wharton mechanism is preferred by students to the traditional bidding mechanism.
- Budish and Kessler developed a new methodology to elicit students' preferences.