Q2. (a). $f(t) = Ae^{-at^2}$

$\Rightarrow \hat{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt$

$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Ae^{-at^2}\cdot e^{-2\pi i\omega t}\,dt$

$= \dfrac{Ae^{-\omega^2/4a}}{2\sqrt{a\pi}}$

(b). $f(t) = \sin(2\pi\omega t) + \cos(2\pi\omega t)$

$\Rightarrow \hat{f}(s) = \frac{1}{2}i[\delta(s+\omega)+\delta(s-\omega)] + \frac{1}{2}[\delta(s+\omega)+\delta(s-\omega)]$

(c). $\delta(t) = \int_{-\infty}^{\infty} f(t)f(t+t')\,dt'$, this is by definition the convolution of f

$\Rightarrow F(\delta(t)) = F^2(f)$

(d). $f(t) = \begin{cases} \frac{1}{a} & \text{for } -a/2 < t < a/2 \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow F(f(t)) = F(f(t)\cdot 1) = \frac{1}{a}\operatorname{sinc}(a\pi\omega)$

Q4. for ARMA(2,2), $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t$

$\Rightarrow \gamma(h) = E(x_t x_{t+h}) = \phi_1 E(x_{t-1}x_{t-h}) + \phi_2 E(x_{t-2}x_{t-h}) + \theta_1 E(w_{t-1}x_{t-h}) + \theta_2 E(w_{t-2}x_{t-h}) + E(w_t x_{t-h})$

① $\gamma(0) = E(x_t x_t)$

$= \phi_1 E(x_{t-1}x_t) + \phi_2 E(x_{t-2}x_t) + \theta_1 E(w_{t-1}x_t) + \theta_2 E(w_{t-2}x_t) + E(w_t x_t)$

$= \phi_1\gamma(1) + \phi_2\gamma(2) + \theta_1 w_{t-1}\cdot E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t)$

$+ \theta_2 w_{t-2}\cdot E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t) + w_t E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t)$

↳ $\theta_1 w_{t-1} E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t)$

$= \theta_1\phi_1 E(w_{t-1}x_{t-1}) + \theta_1\phi_2 E(w_{t-2}x_{t-2}) + \theta_1^2\sigma_w^2$

also as $E(w_{t-1}x_{t-1}) = \sigma_w^2$ $\Rightarrow$ this expression is $(\theta_1\phi_1 + \theta_1^2)\sigma_w^2$

↳ $\theta_2 w_{t-2}\cdot E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t)$

$= \theta_2\phi_1 E(w_{t-2}x_{t-1}) + \phi_2\theta_2 E(w_{t-2}x_{t-1}) + \theta_2^2\sigma_w^2$

similarly, this expression is $(\theta_2\phi_2 + \theta_2^2)\sigma_w^2$

↳ $w_t E(\phi_1 x_{t-1} + \phi_2 x_{t-2} + \theta_1 w_{t-1} + \theta_2 w_{t-2} + w_t) = \sigma_w^2$

Therefore $\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + (\theta_1\phi_1 + \theta_1^2)\sigma_w^2 + (\theta_2\phi_2 + \theta_2^2)\sigma_w^2 + \sigma_w^2$

$$\begin{aligned}\gamma(1) &= \phi_1\gamma(0) + \phi_2\gamma(1) + \theta_1 E(w_{t-1}x_{t-1}) + \theta_2 E(w_{t-2}w_{t-1}) + E(w_t x_{t-1}) \\ &= \phi_1\gamma(0) + \phi_2\gamma(1) + \theta_1 w_{t-1} E(\phi_1 x_{t-2} + \phi_2 x_{t-3} + \theta_1 w_{t-2} + \theta_2 w_{t-3} + w_{t-1}) + \theta_2 w_{t-2} E(\phi_1 x_{t-2} + \\ &\quad \phi_2 x_{t-3} + \theta_1 w_{t-2} + \theta_2 w_{t-3} + w_{t-1}) + w_t E(\phi_1 x_{t-2} + \phi_2 x_{t-3} + \theta_1 w_{t-2} + \theta_2 w_{t-3} + w_{t-1}) \\ &= \phi_1\gamma(0) + \phi_2\gamma(1) + (\theta_1\theta_2 + \theta_2\phi_1 + \theta_1)\sigma_w^2\end{aligned}$$

$$\begin{aligned}\gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \theta_1 E(w_{t-1}x_{t-2}) + \theta_2 E(w_{t-2}x_{t-2}) + E(w_t x_{t-2}) \\ &= \phi_1\gamma(1) + \phi_2\gamma(0) + \theta_1 w_{t-1} E(\phi_1 x_{t-3} + \phi_2 x_{t-4} + \theta_1 w_{t-3} + \theta_2 w_{t-4} + w_{t-2}) + \theta_2 w_{t-2} E(\phi_1 x_{t-3} + \\ &\quad \phi_2 x_{t-4} + \theta_1 w_{t-3} + \theta_2 w_{t-4} + w_{t-2}) \\ &= \phi_1\gamma(1) + \phi_2\gamma(0) + \theta_2\sigma_w^2\end{aligned}$$

Therefore, we have:
$$\begin{cases}\gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + (\theta_1\phi_1 + \theta_1^2 + \theta_2\phi^2 + \theta_2^2 + 1)\sigma_w^2 \\ \gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1) + (\theta_1\theta_2 + \theta_2\phi_1 + \theta)\sigma_w^2 \\ \gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0) + \theta_2\sigma_w^2 \\ \gamma(h) = \phi_1\gamma(h-1) + \phi_2\gamma(h-2) \qquad (h>3)\end{cases}$$

Q4.2. (a). $P(z_1, z_2) = \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2} = \frac{1}{2\pi} \cdot e^{-(z_1^2+z_2^2)/2}$

$\Rightarrow \begin{cases} z_1 = A\cos\phi \\ z_2 = A\sin\phi \end{cases}$ for $A = \sqrt{z_1^2+z_2^2}$ and $\phi = \tan^{-1}(\frac{z_2}{z_1})$

$$\Rightarrow P(A^2, \phi) = \begin{vmatrix} \partial z_1/\partial A^2 & \partial z_1/\partial\phi \\ \partial z_2/A^2 & \partial z_2/\partial\phi \end{vmatrix} P(z_1, z_2) = \begin{vmatrix} \cos\theta/2A & -A\sin\theta \\ \sin\theta/2A & A\cos\theta \end{vmatrix} P(z_1, z_2)$$

(bc $\frac{\partial z_1}{\partial A} = \cos\phi$, $\frac{\partial z_1}{\partial A^2} = \frac{\cos\phi}{2A}$)

$$\Rightarrow P(A^2, \phi) = (\tfrac{1}{2}\cos^2\theta + \tfrac{1}{2}\sin^2\theta) P(z_1, z_2) = \tfrac{1}{2} P(z_1, z_2) = \frac{1}{4\pi} e^{-A^2/2}$$

$$\Rightarrow f_{A^2}(A^2) = \int P(A^2, \phi)\,d\phi = \int \frac{1}{4\pi} e^{-A^2/2}\,d\phi = \tfrac{1}{2} e^{-A^2/2} \Rightarrow A^2 \sim chi(2)$$

$$f_\phi(\phi) = \int P(A^2, \phi)\,dA^2 = \int \frac{1}{4\pi} e^{-A^2/2}\,dA^2 = \frac{1}{4\pi}\int e^{-A^2/2}\,dA^2 = \frac{1}{2\pi} \Rightarrow \phi \sim U(-\pi.$$

and also A^2 and ϕ are independent variables.

(b). $P(A^2, \phi) = P(A^2)\,P(\phi) = \frac{1}{4\pi} e^{\cdots}$

$\Rightarrow P(z_1, z_2) = P(A^2, \phi) / J\left(\frac{z_1 z_2}{A^2 \phi}\right) = \frac{1}{2\pi} e^{-A^2/2} = \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2}$

$\Rightarrow f_{z_1}(z_1) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2}\, dz_2 = \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} \Rightarrow z_1 \sim N(0,1)$

similarly: $f_{z_2}(z_2) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-(z_1^2+z_2^2)/2}\, dz_1 = \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2} \Rightarrow z_2 \sim N(0,1)$

Also, as $f_{z_1}(z_1) \cdot f_{z_2}(z_2) = P(z_1, z_2)$, z_1 and z_2 are independent variables.