

Tensors

- Formal definition: context dependent
- Definition in this class: A linear transformation within or between vector spaces

- Another definition:
$$\mathbf{A}\mathbf{x} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N A_{1,i} x_i \\ \vdots \\ \sum_{i=1}^N A_{N,i} x_i \end{bmatrix}$$

- Matrix product (general form of op above):
$$\mathbf{A}\mathbf{B} = \sum_{k=0}^{N-1} A_{i,k} B_{k,j}$$

- Hadamard (a.k.a. element-wise, a.k.a. Schur) product:
$$(\mathbf{A} \circ \mathbf{B})_{i,j} = (A_{i,j})(B_{i,j})$$

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} A_{1,1} & \dots & A_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1} & \dots & A_{N,N} \end{bmatrix} \cdot \begin{bmatrix} B_{1,1} & \dots & B_{1,N} \\ \vdots & \ddots & \vdots \\ B_{N,1} & \dots & B_{N,N} \end{bmatrix} = \begin{bmatrix} A_{1,1}B_{1,1} & \dots & A_{1,N}B_{1,N} \\ \vdots & \ddots & \vdots \\ A_{N,1}B_{N,1} & \dots & A_{N,N}B_{N,N} \end{bmatrix}$$

Vector and tensor norms

- Measure of the size, or magnitude of a vector or tensor
- Definition: The following criteria qualify $f(\cdot)$ as a norm:
 - Positive definite: $f(\mathbf{x}) = 0 \quad \text{iff} \quad \mathbf{x} = \mathbf{0}$
 - Triangle inequality: $f(\mathbf{x} + \mathbf{y}) \leq f(\mathbf{x}) + f(\mathbf{y})$
 - Homogeneity: $\forall \alpha \in \mathbb{R} : f(\alpha \mathbf{x}) = |\alpha| f(\mathbf{x})$
- The ones we care about are:

- Lp norm:
$$\|\mathbf{x}\|_p = \left(\sum_i |x_i|^p \right)^{1/p}$$

- Frobenius norm:
$$\|\mathbf{A}\|_F = \sqrt{\sum_{i,j} A_{i,j}^2}$$