# Douglas-Rachford method and ADMM

Acknowledgement: this slides is based on Prof. Lieven Vandenberghes lecture notes

# Outline

1. Douglas-Rachford splitting method
2. examples
3. alternating direction method of multipliers
4. image deblurring example
5. convergence

# Douglas-Rachford splitting algorithm

Consider

$$\min \quad f(x) = g(x) + h(x)$$

$g$ and $h$ are closed convex functions

**Douglas-Rachford iteration:** starting at any $z^{(0)}$, repeat

$$\begin{aligned}
x^{(k)} &= \operatorname{prox}_{th}(z^{(k-1)}) \\
y^{(k)} &= \operatorname{prox}_{tg}(2x^{(k)} - z^{(k-1)}) \\
z^{(k)} &= z^{(k-1)} + y^{(k)} - x^{(k)}
\end{aligned}$$

- $t$ is a positive constant (simply scales the objective)
- useful when $g$ and $h$ have inexpensive prox-operators
- under weak conditions (existence of a minimizer), $x^{(k)}$ converges

# Equivalent form

- start iteration at $y$-update

$$y^+ = \mathrm{prox}_{tg}(2x - z); \quad z^+ = z + y^+ - x; \quad x^+ = \mathrm{prox}_{th}(z^+)$$

- switch $z$- and $x$-updates

$$y^+ = \mathrm{prox}_{tg}(2x - z); \quad x^+ = \mathrm{prox}_{th}(z + y^+ - x); \quad z^+ = z + y^+ - x$$

- make change of variables $w = z - x$

**alternate form of DR iteration:** start at $x^{(0)} \in \mathbf{dom}\, h, w^{(0)} \in t\partial h(x^{(0)})$

$$\begin{aligned} y^+ &= \mathrm{prox}_{tg}(x - w) \\ x^+ &= \mathrm{prox}_{th}(y^+ + w) \\ w^+ &= w + y^+ - x^+ \end{aligned}$$

# Interpretation as fixed-point iteration

Douglas-Rachford iteration can be written as

$$z^{(k)} = F(z^{(k-1)})$$

where $F(z) = z + \mathrm{prox}_{tg}(2\mathrm{prox}_{th}(z) - z) - \mathrm{prox}_{th}(z)$

**fixed points of $F$ and minimizers of $g + h$**

- if $z$ is a fixed point, then $x = \mathrm{prox}_{th}(z)$ is a minimizer:

$$\begin{aligned} z = F(z), \quad x = prox_{th}(z) &\Rightarrow \mathrm{prox}_{tg}(2x - z) = x = \mathrm{prox}_{th}(z) \\ &\Rightarrow -x + z \in t\partial g(x); z - x \in t\partial h(x) \\ &\Rightarrow 0 \in t\partial g(x) + t\partial h(x) \end{aligned}$$

- if $x$ is a minimizer and $u \in t\partial g(x) \cap -t\partial h(x)$, then $x - u = F(x - u)$

# Douglas-Rachford iteration with relaxation

**fixed-point iteration with relaxation**

$$z^+ = z + \rho(F(z) - z)$$

$1 < \rho < 2$ is overrelaxation, $0 < \rho < 1$ is underrelaxation

**first version of DR method**

$$\begin{aligned} x^+ &= \mathrm{prox}_{th}(z) \\ y^+ &= \mathrm{prox}_{tg}(2x^+ - z) \\ z^+ &= z + \rho(y^+ - x^+) \end{aligned}$$

**alternate version**

$$\begin{aligned} y^+ &= \mathrm{prox}_{tg}(x - w) \\ x^+ &= \mathrm{prox}_{th}((1-\rho)x + \rho y^+ + w) \\ w^+ &= w + \rho y^+ + (1-\rho)x - x^+ \end{aligned}$$

# Outline

# Sparse inverse covariance selection

$$\min \quad \mathbf{tr}(CX) - \log\det X + \rho \sum_{i>j} |X_{ij}|$$

variable is $X \in \mathbf{S}^n$; parameters $C \in \mathbf{S}^n_{++}$ and $\rho > 0$ are given

**Douglas-Rachford splitting**

$$g(X) = \mathbf{tr}(CX) - \log\det X, h(x) = \rho \sum_{i>j} |X_{ij}|$$

- $X = \mathrm{prox}_{tg}(\hat{X})$ is positive solution of $C - X^{-1} + (1/t)(X - \hat{X}) = 0$
  easily solved via eigenvalue decomposition of $\hat{X} - tC$
- $X = \mathrm{prox}_{th}(\hat{X})$ is soft-thresholding

# Spingarn's method of partial inverses

**equality constrained convex problem**

$$\begin{array}{ll} \min & h(x) \\ \text{s.t.} & x \in V \end{array}$$

$h$ a closed convex function; $V$ a subspace

**Douglas-Rachford splitting:** take $g = I_V$ (indicator of $V$ )

$$\begin{aligned} x^+ &= \mathrm{prox}_{th}(z) \\ y^+ &= P_V(2x^+ - z) \\ z^+ &= z + y^+ - x^+ \end{aligned}$$

# Application to composite optimization problem

$$\min f_1(x) + f_2(Ax)$$

$f_1$ and $f_2$ have simple prox-operators

- equivalent to minimizing $h(x_1, x_2)$ over subspace $V$ where

$$h(x_1, x_2) = f_1(x_1) + f_2(x_2), \quad V = \{(x_1, x_2) \mid x_2 = Ax_1\}$$

- $\mathrm{prox}_{th}$ is separable: $\mathrm{prox}_{th}(x_1, x_2) = (\mathrm{prox}_{tf_1}(x_1), \mathrm{prox}_{tf_2}(x_2))$
- projection of $(x_1, x_2)$ on $V$ reduces to linear equation:

$$\begin{aligned} P_V(x_1, x_2) &= \begin{pmatrix} I \\ A \end{pmatrix} (I + A^T A)^{-1}(x_1 + A^T x_2) \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} A^T \\ -I \end{pmatrix} (I + A^T A)^{-1}(x_2 - Ax_1) \end{aligned}$$

# Decomposition of separable problems

$$\min \quad \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(A_{i1}x_1 + \cdots + A_{in}x_n)$$

- same problem as the lecture on "dual proximal gradient method", but without strong convexity assumption
- we assume the functions $f_j$ and $g_i$ have inexpensive prox-operators

**equivalent formulation**

$$\begin{array}{ll} \min & \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(y_{i1} + \cdots + y_{in}) \\ \text{s.t.} & y_{ij} = A_{ij}x_j, \quad i = 1, \cdots, m; \quad j = 1, \cdots, n \end{array}$$

- prox-operator of cost involves uncoupled prox-evaluations for $f_j, g_i$
- projection on constraint set reduces to $n$ independent linear equations

# Decomposition of separable problems

**second equivalent formulation** with extra splitting variables $x_{ij}$:

$$\begin{array}{ll} \min & \sum_{j=1}^{n} f_j(x_j) + \sum_{i=1}^{m} g_i(y_{i1} + \cdots + y_{in}) \\ \text{s.t.} & x_{ij} = x_j, \quad i = 1, \cdots, m; \quad j = 1, \cdots, n \\ & y_{ij} = A_{ij} x_{ij}, \quad i = 1, \cdots, m; \quad j = 1, \cdots, n \end{array}$$

- make first set of constraints part of domain of $f_j$:

  $$\tilde{f}_j(x_j, x_{1j}, \cdots, x_{mj}) = \begin{cases} f_j(x_j) & x_{ij} = x_j, \quad i = 1, \cdots, m \\ +\infty & \text{otherwise} \end{cases}$$

  prox-operator of $\tilde{f}_j$ reduces to prox-operator of $f_j$
- projection on other constraints involves $mn$ independent linear equations

# Outline

# Dual application of Douglas-Rachford method

**separable convex problem**

$$\begin{array}{ll} \min & f_1(x_1) + f_2(x_2) \\ \text{s.t.} & A_1 x_1 + A_2 x_2 = b \end{array}$$

**dual problem**

$$\max \quad -b^T z - f_1^*(-A_1^T z) - f_2^*(-A_2^T z)$$

we apply the Douglas-Rachford method (page 3) to minimize

$$\underbrace{b^T z + f_1^*(-A_1^T z)}_{g(z)} + \underbrace{f_2^*(-A_2^T z)}_{h(z)}$$

# Douglas Rachford on the dual

$$y^+ = \mathrm{prox}_{tg}(z - w), \quad z^+ = \mathrm{prox}_{th}(y^+ + w), \quad w^+ = w + y^+ - z^+$$

**first line:** use result in "lect-dualProxGrad.pdf" to compute $y^+ = \mathrm{prox}_{tg}(z - w)$

$$\begin{aligned} \hat{x_1} &= \operatorname*{argmin}_{x_1}(f_1(x_1) + z^T(A_1x_1 - b) + \frac{t}{2}\|A_1x_1 - b - w/t\|_2^2) \\ y^+ &= z - w + t(A_1\hat{x_1} - b) \end{aligned}$$

**second line:** similarly, compute $z^+ = \mathrm{prox}_{th}(z + t(A_1\hat{x_1} - b))$

$$\begin{aligned} \hat{x_2} &= \operatorname*{argmin}_{x_1}(f_1(x_2) + z^T A_2x_2 + \frac{t}{2}\|A_1\hat{x_1} + A_2x_2 - b\|_2^2) \\ z^+ &= z + t(A_1\hat{x_1} + A_2\hat{x_2} - b) \end{aligned}$$

**third line** reduces to $w^+ = -tA_2\hat{x_2}$

# Alternating direction method of multipliers

1. minimize augmented Lagrangian over $x_1$

$$x_1^{(k)} = \operatorname*{argmin}_{x_1} \left( f_1(x_1) + (z^{(k-1)})^T A_1 x_1 + \frac{t}{2} \|A_1 x_1 + A_2 x_2^{(k-1)} - b\|_2^2 \right)$$

2. minimize augmented Lagrangian over $x_2$

$$x_2^{(k)} = \operatorname*{argmin}_{x_2} \left( f_2(x_2) + (z^{(k-1)})^T A_2 x_2 + \frac{t}{2} \|A_1 x_1^{(k)} + A_2 x_2 - b\|_2^2 \right)$$

3. dual update

$$z^{(k)} = z^{(k-1)} + t(A_1 x_1^{(k)} + A_2 x_2^{(k)} - b)$$

also known as split Bregman method

# Comparison with other multiplier methods

**alternating minimization method** with $g(y) = I_{\{b\}}(y)$

- same dual update, same update for $x_2$
- $x_1$-update in alternating minimization method is simpler:

$$x_1^{(k)} = \underset{x_1}{\operatorname{argmin}} \left( f_1(x_1) + (z^{(k-1)})^T A_1 x_1 \right)$$

- ADMM does not require strong convexity of $f_1$

**augmented Lagrangian method** with $g(y) = I_{\{b\}}(y)$

- dual update is the same
- AL method requires joint minimization of the augmented Lagrangian

$$\min_{x_1, x_2} f_1(x_1) + f_2(x_2) + (z^{(k-1)})^T (A_1 x_1 + A_2 x_2) + \frac{t}{2} \|A_1 x_1 + A_2 x_2 - b\|_2^2$$

# Application to composite optimization (method 1)

$$\min \quad f_1(x) + f_2(Ax)$$

apply ADMM to

$$\begin{array}{ll} \min & f_1(x_1) + f_2(x_2) \\ \text{s.t.} & Ax_1 = x_2 \end{array}$$

- augmented Lagrangian is

$$f_1(x_1) + f_2(x_2) + \frac{t}{2}\|Ax_1 - x_2 + z/t\|_2^2$$

- $x_1$-update requires minimization of $f_1(x_1) + \frac{t}{2}\|Ax_1 - x_2 + z/t\|_2^2$
- $x_2$-update is evaluation of $\mathrm{prox}_{t^{-1}f_2}$

# Application to composite optimization (method 2)

introduce extra 'splitting' or 'dummy' variable $x_3$

$$\begin{array}{ll} \min & f_1(x_3) + f_2(x_2) \\ \text{s.t.} & \begin{pmatrix} A \\ I \end{pmatrix} x_1 = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \end{array}$$

- alternate minimization of augmented Lagrangian over $x_1$ and $(x_2, x_3)$

$$f_1(x_3) + f_2(x_2) + \frac{t}{2}\left(\|Ax_1 - x_2 + z_1/k\|_2^2 + \|x_1 - x_3 + z_2/k\|_2^2\right)$$

- $x_1$-update: linear equation with coefficient $I + A^T A$
- $(x_2, x_3)$-update: decoupled evaluations of $\text{prox}_{t^{-1}f_1}$ and $\text{prox}_{t^{-1}f_2}$

# Outline

# Image blurring model

$$b = Kx_t + w$$

- $x_t$ is unknown image
- $b$ is observed (blurred and noisy) image; $w$ is noise
- $N \times N$-images are stored in column-major order as vectors of length $N^2$

**blurring matrix $K$**

- represents 2D convolution with space-invariant point spread function
- with periodic boundary conditions, block-circulant with circulant blocks
- can be diagonalized by multiplication with unitary 2D DFT matrix $W$:

$$K = W^H \mathbf{diag}(\lambda) W$$

  equations with coefficient $I + K^T K$ can be solved in $O(N^2 \log N)$ time

# Total variation deblurring with 1-norm

$$\begin{array}{ll} \min & \|Kx - b\|_1 + \gamma \|Dx\|_{tv} \\ \text{s.t.} & 0 \le x \le 1 \end{array}$$

second term in objective is **total variation penalty**

- $Dx$ is discretized first derivative in vertical and horizontal direction

$$\left( \begin{array}{c} I \otimes D_1 \\ D_1 \otimes I \end{array} \right), \qquad \left( \begin{array}{ccccccc} -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{array} \right)$$

- $\| \cdot \|_{tv}$ is a sum of Euclidean norms: $\|(u, v)\|_{tv} = \sum_{i=1}^{n} \sqrt{u_i^2 + v_i^2}$

# Solution via Douglas-Rachford method

an example of a composite optimization problem

$$\min \quad f_1(x) + f_2(Ax)$$

with $f_1$ the indicator of $[0,1]^n$ and $A = \begin{pmatrix} K \\ D \end{pmatrix}$ ,$f_2(u,v) = \|u\|_1 + \gamma\|v\|_{tv}$

$$\min \ \|u\|_1 + \gamma\|v\|_{tv}, \ \text{ s.t. } u = Kx - b, \ v = Dx, y = x, 0 \le y \le 1$$

**primal DR method** and **ADMM** require:

- decoupled prox-evaluations of $\|u\|_1$ and $\|v\|_{tv}$, and projections on $C$
- solution of linear equations with coefficient matrix

$$I + K^T K + D^T D$$

solvable in $O(N^2 \log N)$ time

# Example

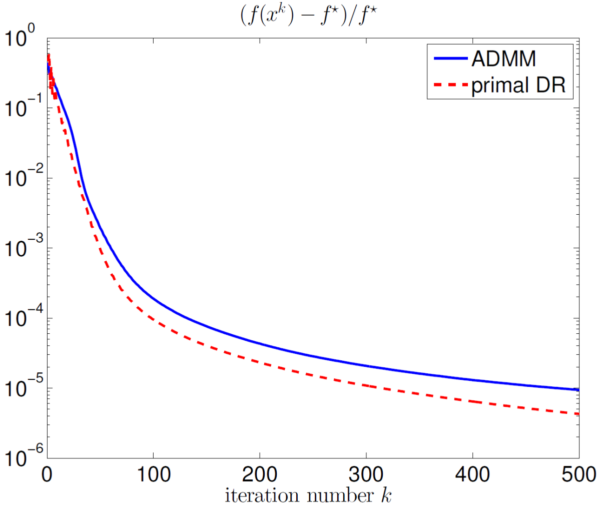
- $1024 \times 1024$ image, periodic boundary conditions
- Gaussian blur
- salt-and-pepper noise (50% pixels randomly changed to 0/1)

original

noisy/blurred

restored

# Convergence




cost per iteration is dominated by 2D FFTs

# Outline

# Nonexpansiveness

if $u = \text{prox}_h(x), v = \text{prox}_h(y)$, then

$$(u-v)^\top(x-y) \geq \|u-v\|2$$

$\text{prox}_h$ is *firmly nonexpansive*, or *co-coercive* with constant 1

- follows from characterization of proximal mapping and monotonicity

$$x-u \in \partial h(u), y-v \in \partial h(v) \quad \Rightarrow \quad (x-u-y+v)^\top(u-v) \geq 0$$

- implies (from Cauchy-Schwarz inequality)

$$\|\text{prox}_h(x) - prox_h(y)\|_2 \leq \|x-y\|_2$$

  $\text{prox}_h$ is *nonexpansive*, or *Lipschitz continuous* with constant 1

# Douglas-Rachford iteration mappings

define iteration map $F$ and negative step $G$

$$\begin{aligned} F(z) &= z + \mathrm{prox}_{th}(2\mathrm{prox}_{th}(z) - z) - \mathrm{prox}_{th}(z) \\ G(z) &= z - F(z) \\ &= \mathrm{prox}_{th}(z) - \mathrm{prox}_{tg}(2\mathrm{prox}_{th}(z) - z) \end{aligned}$$

- $F$ is firmly nonexpansive (co-coercive with parameter 1)

$$(F(z) - F(\hat{z}))^T(z - \hat{z}) \geq \|F(z) - F(\hat{z})\|_2^2 \quad \forall z, \hat{z}$$

- implies that G is firmly nonexpansive:

$$\begin{aligned} & (G(z) - G(\hat{z}))^T(z - \hat{z}) \\ = \quad & \|G(z) - G(\hat{z})\|_2^2 + (F(z) - F(\hat{z}))^T(z - \hat{z}) - \|F(z) - F(\hat{z})\|_2^2 \\ \geq \quad & \|G(z) - G(\hat{z})\|_2^2 \end{aligned}$$

## Proof.

firm nonexpansiveness of $F$

- define $x = \text{prox}_{th}(z), \hat{x} = \text{prox}_{th}(\hat{z})$, and

$$y = \text{prox}_{tg}(2x - z), \quad \hat{y} = \text{prox}_{tg}(2\hat{x} - \hat{z})$$

- substitute expressions $F(z) = z + y - x$ and $F(\hat{z}) = \hat{z} + \hat{y} - \hat{x}$:

$$\begin{aligned}
&(F(z) - F(\hat{z}))^T(z - \hat{z}) \\
&\geq (z + y - x - \hat{z} - \hat{y} + \hat{x})^T(z - \hat{z}) - (x - \hat{z})^T(z - \hat{z}) + \|x - \hat{x}\|_2^2 \\
&= (y - \hat{y})^T(z - \hat{z}) + \|z - x - \hat{z} + \hat{x}\|_2^2 \\
&= (y - \hat{y})^T(2x - z - 2\hat{x} + \hat{z}) - \|y - \hat{y}\|_2^2 + \|F(z) - F(\hat{z})\|_2^2 \\
&\geq \|F(z) - F(\hat{z})\|_2^2
\end{aligned}$$

inequalities use firm nonexpansiveness of $\text{prox}_{th}$ and $\text{prox}_{tg}$

$$(x - \hat{x})^T(z - \hat{z}) \geq \|x - \hat{x}\|_2^2, \quad (2x - z - 2\hat{x} + \hat{z})^T(y - \hat{y}) \geq \|y - \hat{y}\|_2^2$$

$\square$

# Convergence result

$$\begin{aligned} z^{(k)} &= (1-\rho_k)z^{(k-1)} + \rho_k F(z^{(k-1)}) \\ &= z^{(k-1)} - \rho_k G(z^{(k-1)}) \end{aligned}$$

**assumptions**

- optimal value $f^* = \inf_x (g(x) + h(x))$ is finite and attained
- $\rho_k \in [\rho_{\min}, \rho_{\max}]$ with $0 < \rho_{\min} < \rho_{\max} < 2$

**result**

- $z^{(k)}$ converges to a fixed point $z^*$ of $F$
- $x^{(k)} = \mathrm{prox}_{th}(z^{(k-1)})$ converges to a minimizer $x^* = \mathrm{prox}_{th}(z^*)$ (follows from continuity of $\mathrm{prox}_{th}$)

## Proof.

Let $z^*$ be any fixed point of $F(z)$ (zero of $G(z)$). Consider iteration $k$ (with $z = z^{(k-1)}, \rho = \rho_k, z^+ = z^{(k)}$):

$$\begin{aligned} \|z^+ - z^*\|_2^2 - \|z - z^*\|_2^2 &= 2(z^+ - z)^T(z - z^*) + \|z^+ - z\|_2^2 \\ &= -2\rho G(z)^T(z - z^*) + \rho^2\|G(z)\|_2^2 \\ &\leq -\rho(2-\rho)\|G(z)\|_2^2 \\ &\leq -M\|G(z)\|_2^2 \end{aligned} \tag{1}$$

where $M = \rho_{\min}(2 - \rho_{\max})$ (line 3 is firm nonexpansiveness of $G$)

- (1) implies that

$$M\sum_{k=0}^{\infty}\|G(z^{(k)})\|_2^2 \leq \|z^{(0)} - z^*\|_2^2, \quad \|G(z^{(k)})\|_2 \to 0$$

- (1) implies that $\|z^{(k)} - z^*\|_2$ is nonincreasing; $z^{(k)}$ bounded
- since $\|z^{(k)} - z^*\|_2$ is nonincreasing, the limit $\lim_{k\to\infty}\|z^{(k)} - z^*\|_2$ exists

## continued.

- since the sequence $z^{(k)}$ is bounded, it has a convergent subsequence
- let $\bar{z_k}$ be a convergent subsequence with limit $\bar{z}$; by continuity of $G$,

$$0 = \lim_{k\to\infty} G(\bar{z_k}) = G(\bar{z})$$

  hence, $\bar{z}$ is a zero of $G$ and the limit $\lim_{k\to\infty} \|z^{(k)} - \bar{z}\|2$ exists
- let $\bar{z_1}$ and $\bar{z_2}$ be two limit points; the limits

$$\lim_{k\to\infty} \|z^{(k)} - \bar{z_1}\|_2, \quad \lim_{k\to\infty} \|z^{(k)} - \bar{z_2}\|_2$$

  exist, and subsequences of $z^{(k)}$ converge to $\bar{z_1}$, resp. $\bar{z_2}$; therefore

$$\|\bar{z_2} - \bar{z_1}\|_2 = \lim_{k\to\infty} \|z^{(k)} - \bar{z_1}\|_2 = \lim_{k\to\infty} \|z^{(k)} - \bar{z_2}\|_2 = 0$$

$\square$

# References


**Douglas-Rachford method, ADMM, Spingarn's method**

- J. E. Spingarn, *Applications of the method of partial inverses to convex programming: decomposition*, Mathematical Programming (1985)
- J. Eckstein and D. Bertsekas, *On the Douglas-Rachford splitting method and the proximal algorithm for maximal monotone operators*, Mathematical Programming (1992)
- P.L. Combettes and J.-C. Pesquet, *A Douglas-Rachford splitting approach to nonsmooth convex variational signal recovery*, IEEE Journal of Selected Topics in Signal Procesing (2007)
- S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein, *Distributed optimization and statistical learning via the alternating direction method of multipliers* (2010)
- N. Parikh, S. Boyd, *Block splitting for distributed optimization* (2013)

**image deblurring:** the example is taken from
D. O'Connor and L. Vandenberghe, *Primal-dual decomposition by operator splitting and applications to image deblurring* (2014)