

Approximate Solutions, Norms, and the Least-Squares Problem

Objectives

- state need for approximate solutions
- introduce norms
- define properties of norms
- introduce the least-squares problem

Solving Systems of Linear Equations is Important 2

$$\underline{A}\underline{w} = \underline{d}$$

- classification
- modeling

Exact solution: \underline{d} must lie in the subspace spanned by the columns of \underline{A}

$$\sum_i \underline{a}_i w_i = \underline{d}$$

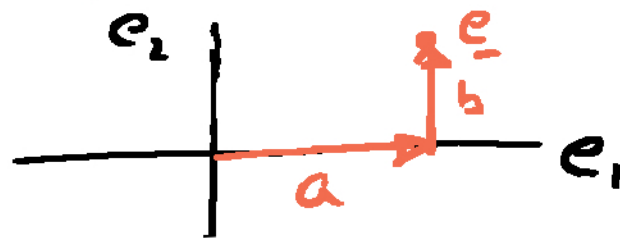
Rarely satisfied in real problems due to noise, model limitations, nonidealities, etc

Can we find \underline{w} so $\underline{A}\underline{w} \approx \underline{d}$? → close to \underline{d} (Approximate)

★ Error $\underline{e} = \underline{A}\underline{w} - \underline{d}$, want \underline{e} small

A vector norm measures the "size" of a vector. 3

- $\|\underline{e}\|_1 = \sum_i |e_i|$
 $= a + b$



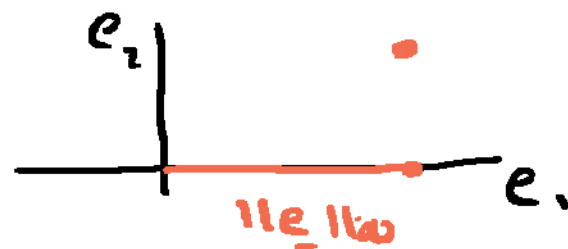
Manhattan norm
 Taxi-cab norm l_1

- $\|\underline{e}\|_2 = \left(\sum_i |e_i|^2 \right)^{1/2}$



Euclidean norm l_2

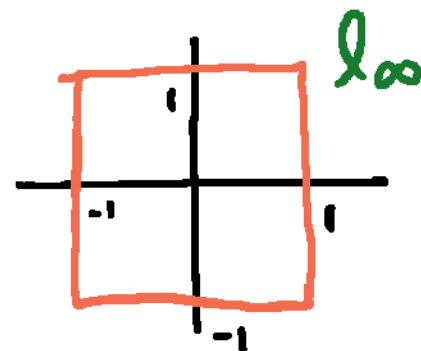
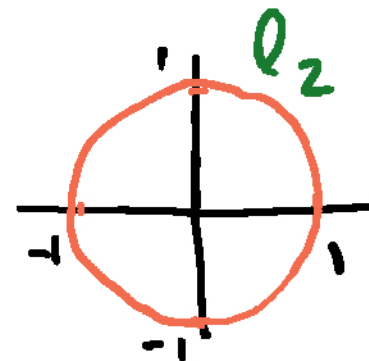
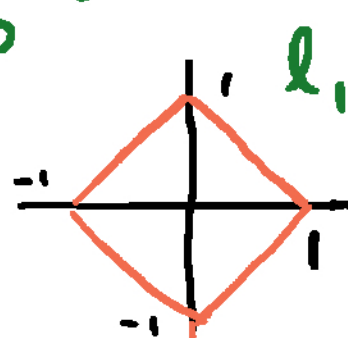
- $\|\underline{e}\|_\infty = \max_i |e_i|$



Sup or max norm l_∞

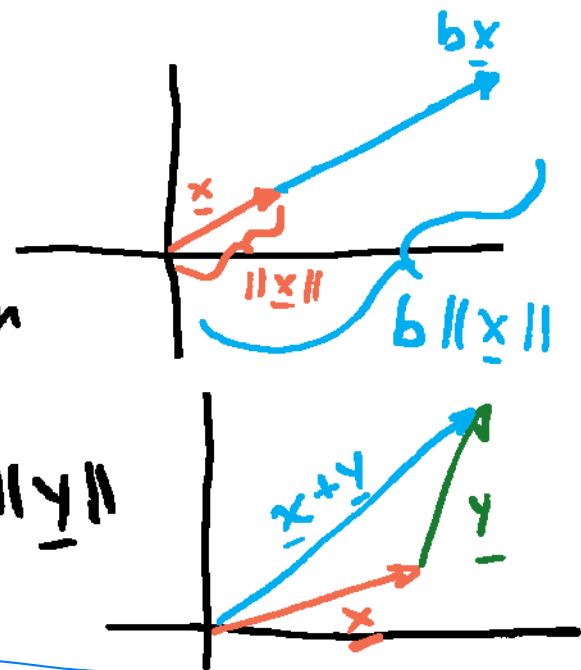
- $\|\underline{e}\|_q = \left(\sum_i |e_i|^q \right)^{1/q}$ l_q norm

Unit ball: $\{\underline{x} : \|\underline{x}\| = 1\}$



A vector norm $\|\cdot\|$ maps from $\mathbb{R}^n \rightarrow \mathbb{R}$ and satisfies⁴ the following properties:

- 1) $\|\underline{x}\| \geq 0$ for all \underline{x}
- 2) $\|\underline{x}\| = 0$ if and only if $\underline{x} = \underline{0}$
- 3) $\|b\underline{x}\| = |b|\|\underline{x}\|$ for all $b \in \mathbb{R}, \underline{x} \in \mathbb{R}^n$
- 4) Triangle inequality $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$



Example: $\|\underline{x}\|_1 = \sum_i |x_i|$

- 1) $\|\underline{x}\|_1 \geq 0$
- 2) if $\underline{x} = \underline{0}$, then $\|\underline{x}\|_1 = 0$
if $\|\underline{x}\|_1 = \sum_i |x_i| = 0$, then $|x_i| = 0$
- 3) $\|b\underline{x}\|_1 = \sum_i |bx_i| = |b| \sum_i |x_i| = |b| \|\underline{x}\|_1$
- 4) $\|\underline{x} + \underline{y}\|_1 = \sum_i |x_i + y_i| \leq \sum_i |x_i| + \sum_i |y_i| = \|\underline{x}\|_1 + \|\underline{y}\|_1$



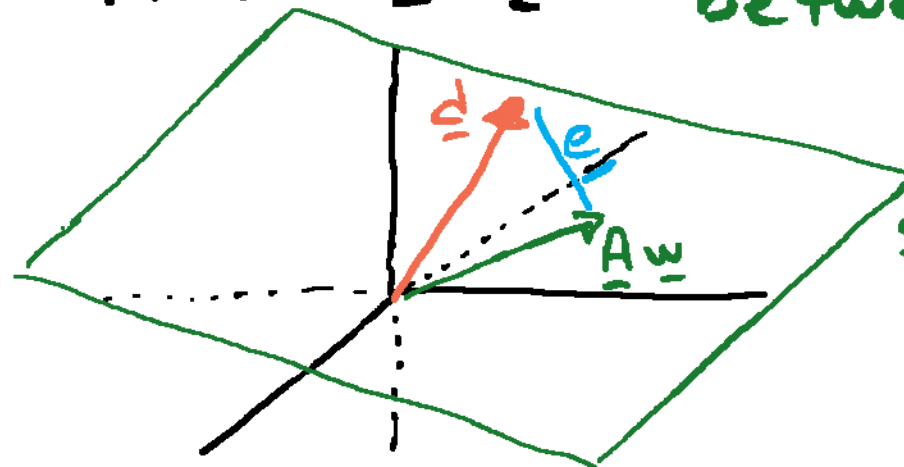
Inclusion
(ℓ_p norms): $1 \leq p \leq q \leq \infty$
 $\|\underline{x}\|_q \leq \|\underline{x}\|_p$
Ex: $\max_i |x_i| \leq \sum_i |x_i|$
 $\|\underline{x}\|_\infty \leq \|\underline{x}\|_1$

Different norms are used to encourage different 5 attributes

- $\|\underline{x}\|_1$ sparse solutions
- $\|\underline{x}\|_\infty$ constant magnitude solutions
- $\|\underline{x}\|_2$ minimize squared error

Least-Squares Problem

$\min_{\underline{w}} \|\underline{A}\underline{w} - \underline{d}\|_2^2$ minimize squared error between $\underline{A}\underline{w}$ and \underline{d}



$$\underline{e} = \underline{A}\underline{w} - \underline{d}$$

↓ We want to have the smallest euclidean norm

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