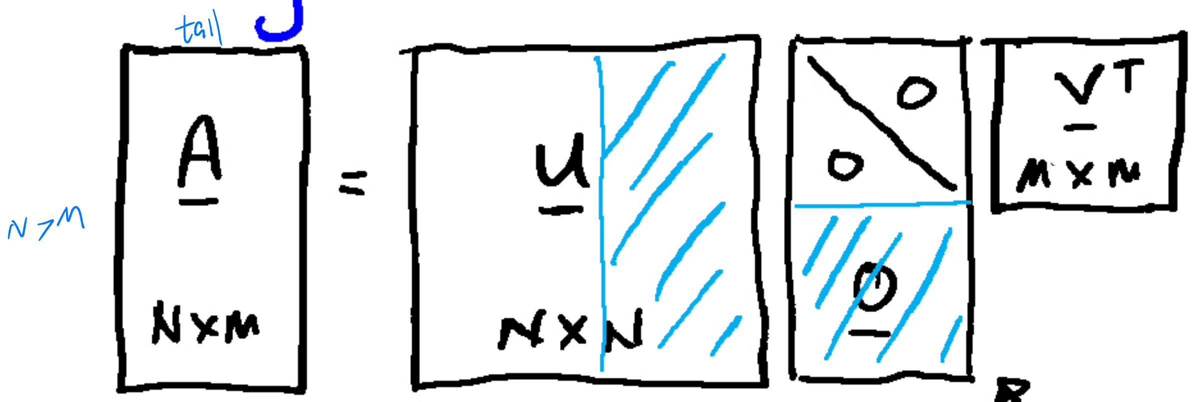
Properties of Singular Value Decomposition

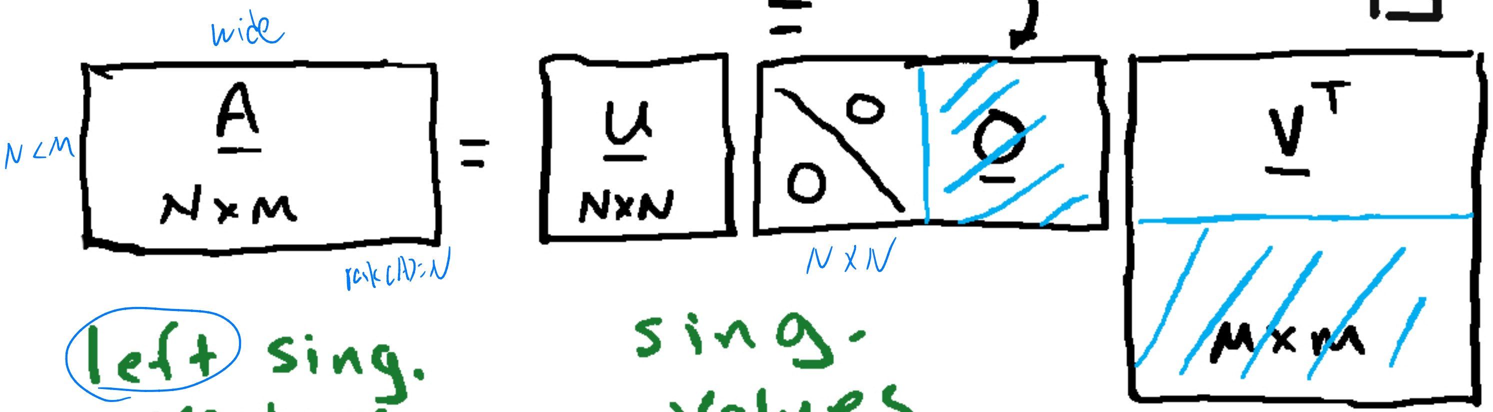
Objectives

- review orthormality of singular vectors
- review rank and singular values
- explore singular vectors as bases
- connect SVD and matrix inversion

Singular Value Decomposition



$\underline{\Sigma}$: N×M



left sing. vectors

$$\underline{U} = \begin{bmatrix} | & | & & | \\ \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_N \\ | & | & & | \end{bmatrix}$$

sing. values

$\sigma_1 \geq \sigma_2 \geq \cdots$
$\geq \sigma_{\min\{N,M\}} \geq 0$

right sing. vectors

$$\underline{V} = \begin{bmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_M \\ | & | & & | \end{bmatrix}$$

Orthonormality

U/V are defined to have orthonormal columns in it

$\underline{U}^T\underline{U} = \underline{I}$; $\underline{V}^T\underline{V} = \underline{I}$ full econ

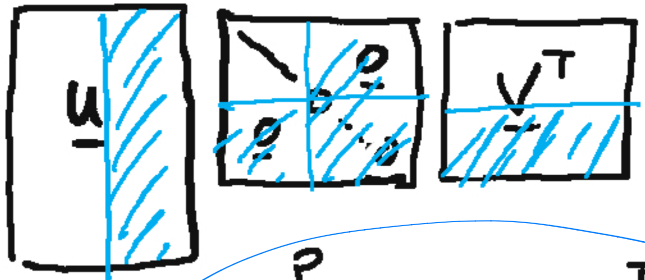
$\underline{U}\underline{U}^T = \underline{I}_N$; $\underline{V}\underline{V}^T = \underline{I}_M$ full only

Rank

Rank deficient matrix

$\text{rank}(\underline{A}) = p \Leftrightarrow$

$\sigma_1 \geq \cdots \geq \sigma_p > \sigma_{p+1} = \cdots = \sigma_{\min(N,M)} = 0$



$\sigma_{p+2} = \cdots = \sigma_m = 0$

$$\underline{A} = \sum_{i=1}^{p} \sigma_i \underline{u}_i \underline{v}_i^T$$

☆ (The rank of a matrix = num of σ)

Why in skinny SVD scenario: UU^T and VV^T is not Identity matrix?

A is $N \times M$

① $N > M$, $U = N \times M$ / $U^T = M \times N$

$UU^T = N \times N$ (but rank is $M \neq N$, not I)

② $N < M$, $V = M \times N$ / $V^T = N \times M$

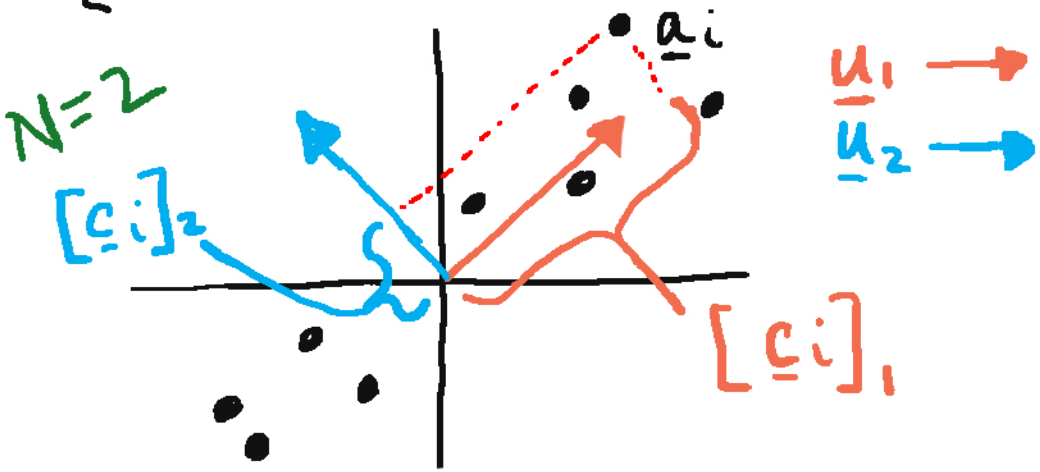
$\therefore VV^T = M \times M$ (but rank is $N \neq M$, not I)

Singular vectors are o/n bases for rows/columns

$$\underbrace{\begin{bmatrix} | & | & & | \\ \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_M \\ | & | & & | \end{bmatrix}}_{N\times M} = \underbrace{\begin{bmatrix} | & | & & | \\ \underline{u}_1 & \underline{u}_2 & \cdots & \underline{u}_P \\ | & | & & | \end{bmatrix}}_{N\times P} \underbrace{\begin{bmatrix} | & | & & | \\ \underline{c}_1 & \underline{c}_2 & \cdots & \underline{c}_M \\ | & | & & | \end{bmatrix}}_{\underline{C} = \underline{\Sigma}\,\underline{V}^T \;\; (P\times M)} \Rightarrow \underline{a}_i = \sum_{j=1}^{P} \underbrace{\underline{u}_j}_{\text{cols}} \underbrace{[\underline{c}_i]_j}_{\text{rows of } C_i}$$

coords of $\underline{a}_i$ in basis $\underline{U}$

(Span the subspace of A)



left sing. vec. $\underline{U}$: o/n basis cols $\underline{A}$

j^{th} coord $\sim \sigma_j$ $\quad [\underline{c}_i]_j = \sigma_j \underbrace{[\underline{V}^T]_{j,i}}_{\max = 1}$

Due to orthonormal

$$\underbrace{\begin{bmatrix} - & \underline{x}_1^T & - \\ - & \underline{x}_2^T & - \\ & \vdots & \\ - & \underline{x}_N^T & - \end{bmatrix}}_{N\times M} = \underbrace{\begin{bmatrix} - & \underline{d}_1^T & - \\ - & \underline{d}_2^T & - \\ & \vdots & \\ - & \underline{d}_N^T & - \end{bmatrix}}_{\underline{D} = \underline{U}\,\underline{\Sigma}} \begin{bmatrix} - & \underline{v}_1^T & - \\ - & \underline{v}_2^T & - \\ & \vdots & \\ - & \underline{v}_P^T & - \end{bmatrix} \Rightarrow \underline{x}_i^T = \sum_{j=1}^{P} \underline{v}_j^T [\underline{d}_i]_j$$

right sing. vec. $\underline{V}$: o/n basis for rows of $\underline{A}$

Same as above

j^{th} coord $\sim \sigma_j$ $\quad [\underline{d}_i]_j = \sigma_j [\underline{U}]_{i,j}$

SVD gives inverse of square matrices

$N = M \quad \underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T \qquad \underline{U}, \underline{\Sigma}, \underline{V} : N \times N$

Noninvertible (Singular): $\text{rank}(\underline{A}) < N$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > \sigma_{p+1} = \cdots = \sigma_N = 0$$

Invertible: $\text{rank}(\underline{A}) = N \qquad \underline{A}^{-1} = \underline{V}\,\underline{\Sigma}^{-T}\underline{U}^T$

$$\underline{A}\cdot\underline{A}^{-1} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T\underline{V}\,\underline{\Sigma}^{-T}\underline{U}^T = \underline{U}\,\underline{\Sigma}\,\underline{\Sigma}^{-T}\underline{U}^T = \underline{U}\,\underline{\Sigma}\,\underline{\Sigma}^{-1}\underline{U}^T$$

$$= \underline{U}\,\underline{U}^T = \underline{I}$$ (no econ SVD for full rank square)

$$\underline{A} = \sum_{i=1}^{N} \sigma_i \underline{u}_i \underline{v}_i^T, \quad \underline{A}^{-1} = \sum_{i=1}^{N} \frac{1}{\sigma_i} \underline{v}_i \underline{u}_i^T$$

SVD of $\underline{A}$ gives SVD of $\underline{A}^{-1}$