

# Approximate Solutions, Norms, and the Least-Squares Problem

# Objectives

1

- state need for approximate solutions
- introduce norms
- define properties of norms
- introduce the least-squares problem

# Solving Systems of Linear Equations is Important 2

$$\underline{A}\underline{w} = \underline{d}$$

- classification
- modeling

Exact solution:  $\underline{d}$  must lie in the subspace spanned by the columns of  $\underline{A}$

$$\sum_i \underline{a}_i \underline{w}_i = \underline{d}$$

Rarely satisfied in real problems due to noise, model limitations, nonidealities, etc

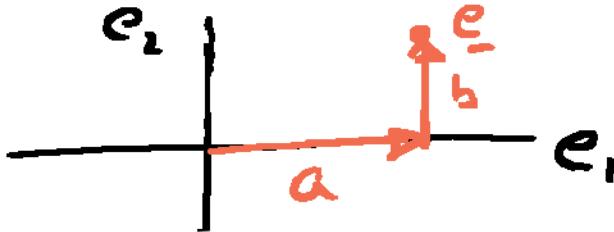
Can we find  $\underline{w}$  so  $\underline{A}\underline{w} \approx \underline{d}$ ?

close to  $\underline{d}$  (Approximate)

★ Error  $\underline{e} = \underline{A}\underline{w} - \underline{d}$ , want  $\underline{e}$  small

A vector norm measures the "size" of a vector. 3

- $\|\underline{e}\|_1 = \sum_i |e_i|$   
 $= a + b$



Manhattan norm

Taxi-cabs  
norm

$\| \cdot \|_1$

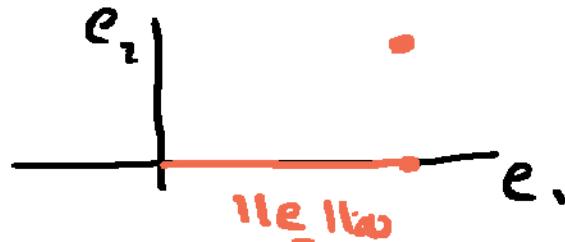
- $\|\underline{e}\|_2 = \left( \sum_i |e_i|^2 \right)^{1/2}$



Euclidean  
norm

$\| \cdot \|_2$

- $\|\underline{e}\|_\infty = \max_i |e_i|$

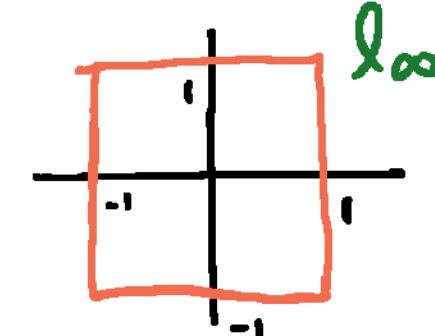
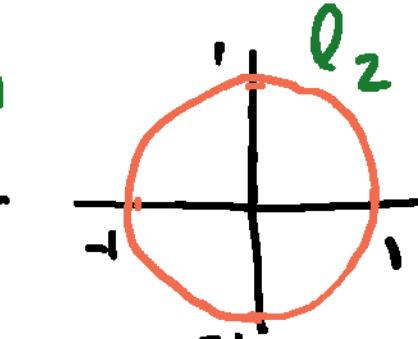
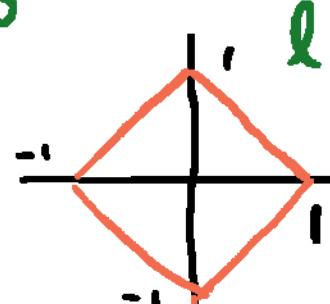


Sup or max  
norm

- $\|\underline{e}\|_q = \left( \sum_i |e_i|^q \right)^{1/q}$

$l_q$  norm

Unit ball:  $\{\underline{x} : \|\underline{x}\|=1\}$



A vector norm  $\|\cdot\|$  maps from  $\mathbb{R}^n \rightarrow \mathbb{R}$  and satisfies<sup>4</sup> the following properties:

- 1)  $\|\underline{x}\| \geq 0$  for all  $\underline{x}$
- 2)  $\|\underline{x}\| = 0$  if and only if  $\underline{x} = \underline{0}$
- 3)  $\|b\underline{x}\| = |b|\|\underline{x}\|$  for all  $b \in \mathbb{R}$ ,  $\underline{x} \in \mathbb{R}^n$
- 4) Triangle inequality  $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$

Example:  $\|\underline{x}\|_1 = \sum_i |\underline{x}_i|$

- 1)  $\|\underline{x}\|_1 \geq 0$
- 2) if  $\underline{x} = \underline{0}$ , then  $\|\underline{x}\|_1 = 0$   
if  $\|\underline{x}\|_1 = \sum_i |\underline{x}_i| = 0$ , then  $|\underline{x}_i| = 0$
- 3)  $\|b\underline{x}\|_1 = \sum_i |b\underline{x}_i|$   
 $= |b| \sum_i |\underline{x}_i|$
- 4)  $\|\underline{x} + \underline{y}\|_1 = \sum_i |\underline{x}_i + \underline{y}_i|$   
 $\leq \sum_i |\underline{x}_i| + |\underline{y}_i| = \|\underline{x}\|_1 + \|\underline{y}\|_1$

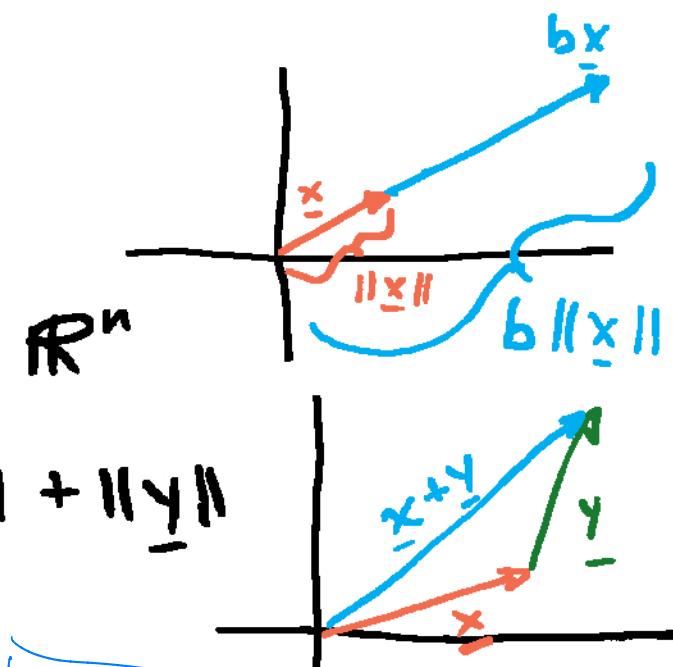


Inclusion  
( $l_p$  norms):  $1 \leq p \leq q \leq \infty$

$$\|\underline{x}\|_q \leq \|\underline{x}\|_p$$

Ex:  $\max_i |\underline{x}_i| \leq \sum_i |\underline{x}_i|$

$$\|\underline{x}\|_\infty \leq \|\underline{x}\|_1$$

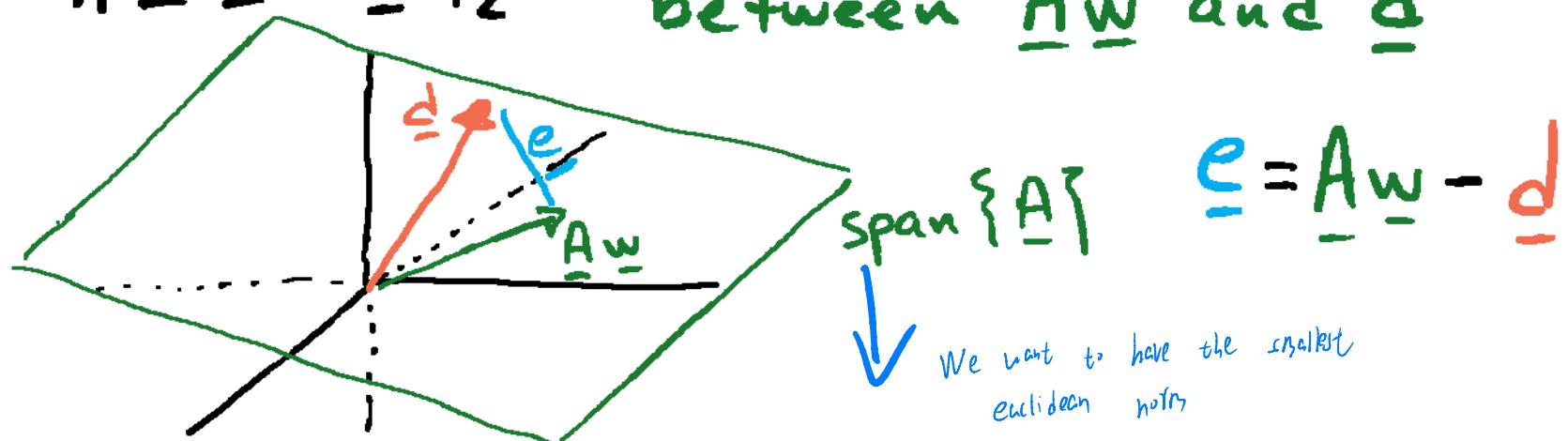


Different norms are used to encourage different attributes 5

- $\|\underline{x}\|_1$  sparse solutions
- $\|\underline{x}\|_\infty$  constant magnitude solutions
- $\|\underline{x}\|_2$  minimize squared error

## Least-Squares Problem

$$\min_{\underline{w}} \|\underline{A}\underline{w} - \underline{d}\|_2^2 \quad \begin{array}{l} \text{minimize squared error} \\ \text{between } \underline{A}\underline{w} \text{ and } \underline{d} \end{array}$$



Copyright 2019  
Barry Van Veen