## **Transitive Existential Paths**

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The concept of a transitive existential path  $\exists (f \to g)$  formalizes what it means to have "information" and "symmetry" stored in some space and performing information-preserving transformations. It does so without referring to the specific data structure used to model the information. In addition, a modular transitive existential path  $\exists (f \% g)$  gives solutions of infinite spaces  $\exists (f \% g) = 1$  and groups  $\exists (f \% g) = \infty$ .

A transitive existential path of `f` by `g` with inverse `g-¹` is defined as the following:

$$\exists (f \rightarrow g) := \(x : [\exists g] \text{ true}) = (\exists f)(g^{-1}(x))$$
 $g : A \rightarrow A$ 
 $g^{-1} <=> id[id \rightarrow g]$ 

The transitive existential path of `f` by `id` is equal to the existential path:

$$\exists (f \rightarrow id) \leq \exists f$$

Repeated transformations is written:

$$\exists (f \rightarrow g^n) := \ (x : [\exists g^n] \text{ true}) = (\exists f)(g^{-n}(x))$$
  
n : nat

It is easy to see that the transitive existential path can be defined in terms of any "previously defined" transitive existential path:

$$\exists (f \rightarrow g^{n+m}) := \backslash (x : [\exists g^{n+m}] \text{ true}) = (\exists (f \rightarrow g^n))(g^{-m}(x))$$

$$n : \text{nat}$$

$$m : \text{nat}$$

$$g^0 <=> \text{id}$$

Some examples:

$$\begin{split} &\exists (f \to g^1) := \backslash (x : [\exists g^1] \text{ true}) = (\exists f)(g^{-1}(x)) \\ &\exists (f \to g^2) := \backslash (x) = (\exists (f \to g^1))(g^{-1}(x)) \\ &\exists (f \to g^3) := \backslash (x) = (\exists (f \to g^2))(g^{-1}(x)) \\ &\exists (f \to g^3) := \backslash (x) = (\exists (f \to g^1))(g^{-2}(x)) \end{split}$$

As a closely related concept, a modular transitive existential path is a function returning `true` for `n` reducing to the existential path.

$$\exists (f \% g) : nat \rightarrow bool$$

The function captures a characteristic variable `n` such that the following condition is satisfied:

$$\forall x : \text{nat } \{ \exists (f \rightarrow g^{x \cdot n}) \leq \exists f \}$$

Notice that any natural number `m` multiplied with `n` is also a characteristic variable, so this forms a subset relationship between the two series of modular transitive existential paths:

$$\exists (f \% g^{n \cdot m}) \subseteq \exists (f \% g^n)$$

For infinite spaces, a modular transitive existential path is defined as:

$$\exists (f \% g) := \(x : nat) = x == 0$$
  
 $\exists (f \% g) <=> (= 0)$   
 $|\exists (f \% g)| == 1$ 

For groups, a modular transitive existential path is defined as:

$$\exists (f \% g) := \(x : nat) = (x \% n) == 0$$
  
 $|\exists (f \% g)| == \infty$ 

A group is naturally unbounded, so you can change `nat` to `int`, if you want to.

This can also be done for infinite spaces, if there exists an inverse for all input. Remember that a transitive existential path got a domain constraint on the argument:

$$\exists (f \rightarrow g) := \ (x : [\exists g] \text{ true}) = (\exists f)(g^{-1}(x))$$

The domain constraint `[ $\exists g$ ] true` prevents `x` from taking on values that `g-¹` does not map to. If you wonder whether `g-¹` is a partial function, you are both right and wrong, because `g-¹ <=> id[id  $\rightarrow g$ ]` defines a logical equivalence between the path sets and supports partial functions of `g-¹` when this is the case.

For example, for real numbers a function `g` might return all values:

$$\exists g := (\_ : real) = true$$

This means you can drop the constraint:

$$\exists (f \rightarrow g) := \langle (x : real) = (\exists f)(g^{-1}(x)) \rangle$$

A simple example of an infinite space is the `add(k)` function:

add(k): nat 
$$\rightarrow$$
 nat  
add:=\(k: nat  $\land$  (>0)) = \(x: nat) = x + k

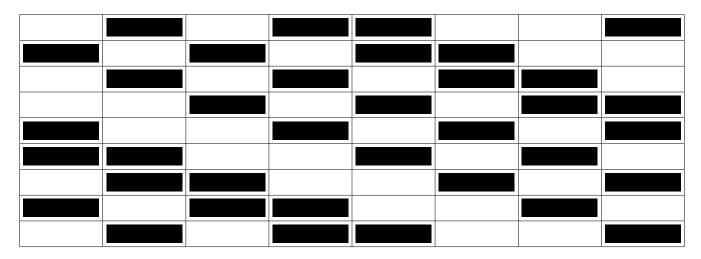
Let us say `f` is `even\_sequence` and `g` is `add(k)`, it is just a matter of substitute and solve:

The modular transitive existential path has only one solution, because if you move even numbers by adding with k > 0, you lose 0 from the transitive existential path.

A simple example of a group is one that increments numbers and maps back using modulus:

$$g := (x : nat) = (x + 1) \% 8$$

Think of this as a rotating pattern where f := [1, 3, 4, 7]:



After 8 rounds, we end up with the same pattern as we started with. This is because  $g^8 <=> g$ . So, why not use  $g^2$  Why do we have to worry about  $f^2$  The reason is that some patterns repeats themselves more frequently, so to express this precisely we need to write  $G(f \otimes g)$ .