Finding Normal Paths Using Zero Order Existential Paths

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The zero order existential path `even_sequence` which first order existential path is `even` is a function mapping every natural number to an even number:

even_sequence := \(x: nat) = 2·x

∃even_sequence <=> even

The inverse zero order existential path is `odd_sequence`:

odd_sequence := \(x: nat) = 2·x + 1

∃odd_sequence <=> ¬∃even_sequence

In path semantics, you can search for a normal path by transforming input and output of the function, but this requires the assumption that there are no 2 objects which the transformed output gives different results:

$\exists\ x_i, y_i\ \{\ g_i(x_i) = g_i(y_i) \land g_n(f(x_i))\ \neg{=}\ g_n(f(y_i))\ \}\ \Rightarrow \neg\exists\ f[g_{i \to n}]$

Notice that `∃ f[$g_{i\to n}$]` means “existence of path” while `∃f` means “existential path”. The space is significant.

One can break down the operations in an expressions, here with the first order existential path, to find a normal path. For example:

f(a, b) = a + b + a·b

add[even] <=> eq
add[odd] <=> xor
mul[even] <=> or
mul[odd] <=> and

even(a + b + a·b)
even(a + b) == even(a·b)
(even(a) == even(b)) == (even(a) || even(b))
even(a) && even(b)
f[even] <=> and

Once `f[even]` is found one can easily find `f[odd]`:

f[odd] <=> f[odd][not] <=> and[not] <=> or

There is another way that uses the zero order existential path. Instead of transforming the expression, one inserts the expressions of `even_sequence` and `odd_sequence` and determines whether the solved equation is a natural number. Since there is a map from any natural number to the sequence of even or odd numbers, it means if the solution yields a non-natural number, the value is returned by the inverse zero order existential path:

$$a + b + a \cdot b$$

$$(2 \cdot n) + (2 \cdot m) + (2 \cdot n) \cdot (2 \cdot m) = (2 \cdot r)$$
$$(2 \cdot n) + (2 \cdot m + 1) + (2 \cdot n) \cdot (2 \cdot m + 1) = (2 \cdot r)$$
$$(2 \cdot n + 1) + (2 \cdot m) + (2 \cdot n + 1) \cdot (2 \cdot m) = (2 \cdot r)$$
$$(2 \cdot n + 1) + (2 \cdot m + 1) + (2 \cdot n + 1) \cdot (2 \cdot m + 1) = (2 \cdot r)$$

$n + m + 2 \cdot n \cdot m = r$ (natural number, returned by `even`)
$n + m + \frac{1}{2} + n + 2 \cdot n \cdot m = r$ (rational number, returned by `odd`)
$n + \frac{1}{2} + m + 2 \cdot n \cdot m + m = r$ (rational number, returned by `odd`)
$n + \frac{1}{2} + m + \frac{1}{2} + 2 \cdot n \cdot m + n + m + \frac{1}{2} = r$ (rational number, returned by `odd`)

f[even] <=> and

This method works by using `even_sequence` and `odd_sequence` as boolean values to find the truth table. The proof relies on a non-trivial way to detect whether an expression is a natural number or a rational number. For example, you can add `½ + ½  = 1` which is a natural number, but if you have 3 of them then you get a rational number.

The simplest way of proving it is through transformation with path semantics:

even(a + b + a·b)
even(a) && even(b)
f[even] <=> and

One reason this is simpler is because normal paths are total functions. When we solve the equation for `r` we only check a partial function, so we need to solve 2^N number of equations where N is number of inputs. finding normal paths using zero order existential paths is more work than using the first order existential path directly.