Practice Problem 1 for Probabilistic Paths

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Probabilistic paths can be very difficult to wrap your head around, so here is a problem that you can practice on. The `inc` function is a very simple function that just increases a number:

inc := \(x : nat) = x + 1

Prove that the probabilistic symmetric path of `inc[even]` is equal to:

$\mathrm{inc[even]}_p$:= \([e_0, e_1] : [bool] ∧ [len] 2, [p_0] : [real] ∧ [len] 1) =
if e_1 == e_0 { 0 } else { 1 } · ($2p_0$ – 1) +
0.5 · (1 – ($2p_0$ – 1))

Testing:

$\mathrm{inc[even]}_p$([true, true], [1]) = 0 + 0 = 0
$\mathrm{inc[even]}_p$([true, true], [0]) = 0 + 1 = 1
$\mathrm{inc[even]}_p$([true, false], [1]) = 1 + 0 = 1
$\mathrm{inc[even]}_p$([true, false], [0]) = -1 + 1 = 0
$\mathrm{inc[even]}_p$([false, false], [0.5]) = 0 + 0.5 = 0.5

Pro tip: Break the problem into smaller sub-problems and combine together the solutions.

1. Find `$\exists_p$(even · inc)`.
2. Find `$\exists_p$(even · inc){[even] e}` by combining two cases.
    1. Find `$\exists_p$(even · inc){[even] true}`.
    2. Find `$\exists_p$(even · inc){[even] false}`.
3. Find `$\exists_p$even`.
4. Simplify `($p_0$ - ($\exists_p$even)($e_0$)) / (1 - ($\exists_p$even)($e_0$))` by using `$\exists_p$even`.
5. Combine all the above by putting them into the definition of a probabilistic path.

Now, try something harder. This requires more creativity, e.g. making tables:

$\mathrm{inc\{(< 10)\}[even \rightarrow (> 3)]}_p$:= \([b_0, b_1] : [bool] ∧ [len] 2, [p_0] : [real] ∧ [len] 1) =
0.7 · (1 – ($2p_0$ – 1)) +
if b_0 { if b_1 { 0.6 } else { 0.4 } } else { if b_1 { 0.8 } else { 0.2 } } · ($2p_0$ – 1)

Testing:

$\mathrm{inc\{(< 10)\}[even \rightarrow (> 3)]}_p$([true, true], [1]) = 0 + 0.6 = 0.6
$\mathrm{inc\{(< 10)\}[even \rightarrow (> 3)]}_p$([true, true], [0]) = 1.4 + -0.6 = 0.8

Pro tip: Notice the constraint `{(< 10)}` is static. As before, break it down and combine the solutions.