Probabilistic Path of If-Expressions

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Assume one constructs a function using the higher order `if` function and two elements:

$$\mathrm{if}(a_0, a_1) : \mathrm{bool} \rightarrow A$$

$$a_0 : A$$
$$a_1 : A$$

When there is some property of these elements:

$$g : A \rightarrow B$$

The probabilistic path is:

$$\begin{aligned}&\mathrm{if}(a_0, a_1)[\mathrm{id} \rightarrow g]_p := \backslash([b_0, b_1] : [\mathrm{bool}, B] \wedge [\mathrm{len}]\ 2, [p_0] : [\mathrm{real}] \wedge [\mathrm{len}]\ 1) = \\ &\quad (\mathrm{if}\ g(a_0) == b_1\ \{ 0.5 \}\ \mathrm{else}\ \{ 0 \} + \mathrm{if}\ g(a_1) == b_1\ \{ 0.5 \}\ \mathrm{else}\ \{ 0 \}) \cdot (1 - (2p_0\text{-}1)) + \\ &\quad (\mathrm{if}\ b_0\ \{ \mathrm{if}\ g(a_0) == b_1\ \{ 1 \}\ \mathrm{else}\ \{ 0 \} \}\ \mathrm{else}\ \{ \mathrm{if}\ g(a_1) == b_1\ \{ 1 \}\ \mathrm{else}\ \{ 0 \} \}) \cdot (2p_0\text{-}1)\end{aligned}$$

Rest of this paper is about proving this from the definition of a probabilistic path.

A probabilistic path is defined as following:

$$\begin{aligned}&f[g_{i \rightarrow n}]_p := \backslash(b_{i \rightarrow n} : [] \wedge [\mathrm{len}]\ |g_{i \rightarrow n}|, p : [\mathrm{real}] \wedge [\mathrm{len}]\ |g_i|) = \textstyle\sum j\ 2\text{^}|g_i|\ \{ \\ &\quad (\exists_p(g_n \cdot f)\{2\text{^}[g_i]\ b_i\}(\beta_j))(b_n) \cdot \textstyle\prod k\ |g_i|\ \{ \beta_{jk}((p_k\text{-}(\exists_p g_{ik})(b_k))/(1\text{-}(\exists_p g_{ik})(b_k))) \} \\ &\}\end{aligned}$$

Inserting by substituting `f => if(a₀, a₁)` and `g_{i→n} => id → g` and unrolling the sum loop:

$$\begin{aligned}&\mathrm{if}(a_0, a_1)[\mathrm{id} \rightarrow g]_p := \backslash([b_0, b_1] : [\mathrm{bool}, B] \wedge [\mathrm{len}]\ 2, [p_0] : [\mathrm{real}] \wedge [\mathrm{len}]\ 1) = \\ &\quad (\exists_p(g \cdot \mathrm{if}(a_0, a_1)))(b_1) \cdot (1 - (p_0\text{-}(\exists_p \mathrm{id})(b_0))/(1\text{-}(\exists_p \mathrm{id})(b_0))) + \\ &\quad (\exists_p(g \cdot \mathrm{if}(a_0, a_1))\{[\mathrm{id}]\ b_0\})(b_1) \cdot (p_0\text{-}(\exists_p \mathrm{id})(b_0))/(1\text{-}(\exists_p \mathrm{id})(b_0))\end{aligned}$$

Solving `$\exists_p$id`:

$$\exists_p \mathrm{id} := \backslash(\_ : \mathrm{bool}) = 0.5$$

Reducing sub-expression:

$$(p_0\text{-}(\exists_p \mathrm{id})(b_0))/(1\text{-}(\exists_p \mathrm{id})(b_0))$$
$$(p_0\text{-}0.5)/(1\text{-}0.5)$$
$$(p_0\text{-}0.5)/0.5$$
$$2p_0\text{-}1$$

Inserting:

$$\begin{aligned}&\text{if}(a_0, a_1)[\text{id} \to g]_p := \backslash([b_0, b_1] : [\text{bool}, B] \wedge [\text{len}]\ 2, [p_0] : [\text{real}] \wedge [\text{len}]\ 1) = \\ &\qquad (\exists_p(g \cdot \text{if}(a_0, a_1)))(b_1) \cdot (1 - (2p_0\text{-}1)) + \\ &\qquad (\exists_p(g \cdot \text{if}(a_0, a_1))\{[\text{id}]\ b_0\})(b_1) \cdot (2p_0\text{-}1)\end{aligned}$$

Next step is to work out the probabilistic existential path of the function composition.

The unconstrained term is found by averaging over the probabilities:

$$\begin{aligned}&\exists_p(g \cdot \text{if}(a_0, a_1)) := \backslash(x : B) = \\ &\qquad 0.5 \cdot \text{if } g(a_0) == x\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \} + \\ &\qquad 0.5 \cdot \text{if } g(a_1) == x\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \} \\ &\exists_p(g \cdot \text{if}(a_0, a_1)) := \backslash(x : B) = \\ &\qquad \text{if } g(a_0) == x\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \} + \\ &\qquad \text{if } g(a_1) == x\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \}\end{aligned}$$

$$\begin{aligned}&(\exists_p(g \cdot \text{if}(a_0, a_1)))(b_1) \\ &\text{if } g(a_0) == b_1\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \} + \text{if } g(a_1) == b_1\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \}\end{aligned}$$

The constrained term passes on `$a_0$` or `$a_1$` to `g`, depending on `$b_0$`:

$$\begin{aligned}&(\exists_p(g \cdot \text{if}(a_0, a_1))\{[\text{id}]\ b_0\})(b_1) \\ &(\text{if } b_0\ \{\ \exists_p g\{(= a_0)\}\ \}\ \text{else}\ \{\ \exists_p g\{(= a_1)\}\ \})(b_1) \\ &\text{if } b_0\ \{\ \text{if } g(a_0) == b_1\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \}\ \}\ \text{else}\ \{\ \text{if } g(a_1) == b_1\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \}\ \}\end{aligned}$$

The probability of `g` returning `$b_1$` constrained by a specific input value is, of course, either 100% or 0%, which can be checked by using the function `g` and compare the result.

Inserting:

$$\begin{aligned}&\text{if}(a_0, a_1)[\text{id} \to g]_p := \backslash([b_0, b_1] : [\text{bool}, B] \wedge [\text{len}]\ 2, [p_0] : [\text{real}] \wedge [\text{len}]\ 1) = \\ &\qquad (\text{if } g(a_0) == b_1\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \} + \text{if } g(a_1) == b_1\ \{\ 0.5\ \}\ \text{else}\ \{\ 0\ \}) \cdot (1 - (2p_0\text{-}1)) + \\ &\qquad (\text{if } b_0\ \{\ \text{if } g(a_0) == b_1\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \}\ \}\ \text{else}\ \{\ \text{if } g(a_1) == b_1\ \{\ 1\ \}\ \text{else}\ \{\ 0\ \}\ \}) \cdot (2p_0\text{-}1)\end{aligned}$$

Testing that probability of believing the condition is `true` is equal to the inverse probability of believing the condition is `false`:

$$\begin{aligned}&\text{if}(a_0, a_1)[\text{id} \to g]_p([\text{true}, x], [p]) = \\ &\qquad \text{if } g(a_0) == x\ \{\ p\ \}\ \text{else}\ \{\ 0\ \} + \text{if } g(a_1) == x\ \{\ 1 - p\ \}\ \text{else}\ \{\ 0\ \}\end{aligned}$$

$$\begin{aligned}&\text{if}(a_0, a_1)[\text{id} \to g]_p([\text{false}, x], [1\text{-}p]) = \\ &\qquad \text{if } g(a_0) == x\ \{\ p\ \}\ \text{else}\ \{\ 0\ \} + \text{if } g(a_1) == x\ \{\ 1 - p\ \}\ \text{else}\ \{\ 0\ \}\end{aligned}$$