Isomorphic and Homotopy Paths

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An isomorphic path is a pair of two functions and two maps that reflect elements of each other.

$(f, h)[g_{i\to n}, e_{i\to n}]_{iso}$

$f <=> e_n \cdot h \cdot g_i$
$h <=> g_n \cdot f \cdot e_i$

$f[g_{i\to n}] <=> h$
$h[e_{i\to n}] <=> f$

$e_{in} \cdot g^{in} <=> id_{(A, B)}$
$g_{in} \cdot e^{in} <=> id_{(C, D)}$

$f : A \to B$
$g_{i\to n} : (A \to C, B \to D)$
$h : C \to D$
$e_{i\to n} : (C \to A, D \to B)$

Reflexivity (This is called the identity isomorphic path):

$(f, f)[id, id]_{iso} <=> (f, f)[id, id]_{iso}$

Symmetry:

$(f, h)[g_{i\to n}, e_{i\to n}]_{iso} <=> (h, f)[e_{i\to n}, g_{i\to n}]_{iso}$

Transitivity:

$(f_0, f_1)[g_{01}, g_{10}]_{iso} \land (f_1, f_2)[g_{12}, g_{21}]_{iso} <=> (f_0, f_2)[g_{12} \cdot g_{01}, g_{10} \cdot g_{21}]_{iso} <=> (f_0, f_2)[g_{02}, g_{20}]_{iso}$

If any path exists for any function that has a collection of isomorphic paths, then all isomorphic paths are contractible by a family of paths.

$\exists\ x \{ f_x[g_{xh}] <=> h \} <=> \forall\ x \{ f_x[g_{xh}] <=> h \}$

$(f_0, f_1)[g_{01}, g_{10}]_{iso}[g_{0h}, g_{1h}]$
$(f_0, f_1)[g_{0h}, g_{1h}][g_{01}[g_{0h} \to g_{1h}], g_{10}[g_{1h} \to g_{0h}]]_{iso}$
$(f_0[g_{0h}], f_1[g_{1h}])[id, id]_{iso}$
$(h, h)[id, id]_{iso}$

In general, an isomorphic path satisfies the following:

$$[x_0, x_1]_{iso}[y_0, y_1] \Longleftrightarrow [y_0, y_1][x_0[y_0 \rightarrow y_1], x_1[y_1 \rightarrow y_0]]_{iso}$$

This way of propagating a copy of the path to the left gives rise to the idea of a "homotopy path", a higher N-dimensional path between isomorphic paths. It serves a syntactic purpose to propagate a copy to the left and "erase" the leftovers from the inner isomorphic paths.

$$[x_i]_{hom}[y_i] \Longleftrightarrow [y_i][x_i[y^i]]_{hom}$$

A homotopy path allows one to separate the isomorphic structure of functions and their mappings. One can also put a homotopy path bracket inside a homotopy path bracket.

For example, this is how to prove that two different isomorphic paths are contractible to the same path:

$$((f_0, f_1), (f_2, f_3))[[g_{01}, g_{10}]_{iso}, [g_{23}, g_{32}]_{iso}]_{hom}[[g_{0h}, g_{1h}], [g_{2h}, g_{3h}]]$$
$$((f_0, f_1), (f_2, f_3))[[g_{0h}, g_{1h}], [g_{2h}, g_{3h}]][[g_{01}, g_{10}]_{iso}[g_{0h}, g_{1h}], [g_{23}, g_{32}]_{iso}[g_{2h}, g_{3h}]]_{hom}$$
$$((f_0, f_1)[g_{0h}, g_{1h}], (f_2, f_3)[g_{2h}, g_{3h}])[g_{01}[g_{0h} \rightarrow g_{1h}], g_{10}[g_{1h} \rightarrow g_{0h}]_{iso}, [g_{23}[g_{2h} \rightarrow g_{3h}], g_{32}[g_{3h} \rightarrow g_{2h}]]_{iso}]_{hom}$$
$$((f_0[g_{0h}], f_1[g_{1h}]), (f_2[g_{2h}], f_3[g_{3h}]))[[id, id]_{iso}, [id, id]_{iso}]_{hom}$$
$$((h, h), (h, h))[[id, id]_{iso}, [id, id]_{iso}]_{hom}$$

Here is another example, using unknowns to extract equations:

$$((f_0, f_1), (f_0, f_2))[[g_{01}, g_{10}]_{iso}, [g_{02}, g_{20}]_{iso}]_{hom}[[x_0, x_1], [x_2, x_3]]$$
$$((f_0, f_1), (f_0, f_2))[[x_0, x_1], [x_2, x_3]][[g_{01}, g_{10}]_{iso}[x_0, x_1], [g_{02}, g_{20}]_{iso}[x_2, x_3]]_{hom}$$
$$((f_0, f_1)[x_0, x_1], (f_0, f_2)[x_2, x_3])[[g_{01}[x_0 \rightarrow x_1], g_{10}[x_1 \rightarrow x_0]]_{iso}, [g_{02}[x_2 \rightarrow x_3], g_{20}[x_3 \rightarrow x_2]]_{iso}]_{hom}$$
$$((f_0[x_0], f_1[x_1]), (f_0[x_2], f_2[x_3]))[[g_{01}[x_0 \rightarrow x_1], g_{10}[x_1 \rightarrow x_0]]_{iso}, [g_{02}[x_2 \rightarrow x_3], g_{20}[x_3 \rightarrow x_2]]_{iso}]_{hom}$$

The equations to be solved are extracted from the homotopy path bracket:

$$g_{01}[x_0 \rightarrow x_1] \Longleftrightarrow g_{02}[x_2 \rightarrow x_3]$$
$$g_{10}[x_1 \rightarrow x_0] \Longleftrightarrow g_{20}[x_3 \rightarrow x_2]$$

$$x_0 \Longleftrightarrow id \qquad x_1 \Longleftrightarrow g_{12} \qquad x_2 \Longleftrightarrow id \qquad x_3 \Longleftrightarrow id$$

$$g_{01}[id \rightarrow g_{12}] \Longleftrightarrow g_{02}[id \rightarrow id] \Longleftrightarrow g_{02}$$
$$g_{10}[g_{12} \rightarrow id] \Longleftrightarrow g_{20}[id \rightarrow id] \Longleftrightarrow g_{20}$$

Inserting:

$$((f_0[id], f_1[g_{12}]), (f_0[id], f_2[id]))[[g_{01}[id \rightarrow g_{12}], g_{10}[g_{12} \rightarrow id]]_{iso}, [g_{02}[id \rightarrow id], g_{20}[id \rightarrow id]]_{iso}]_{hom}$$
$$((f_0, f_1[g_{12}]), (f_0, f_2))[[g_{01}[id \rightarrow g_{12}], g_{10}[g_{12} \rightarrow id]]_{iso}, [g_{02}, g_{20}]_{iso}]_{hom}$$
$$((f_0, f_1[g_{12}]), (f_0, f_2))[[g_{02}, g_{20}]_{iso}, [g_{02}, g_{20}]_{iso}]_{hom}$$

Solution:

$$f_1[g_{12}] \Longleftrightarrow f_2$$