Universally Optimal Compression

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In logic, a lot of statements can be proved that are not explicitly stated, simply because they follow from the assumptions. This property can be exploited to construct a universally optimal compression algorithm. This is how you do it.

Both the sender and the receiver has a boolean expression which the N bits transferred must satisfy. The bits are then ordered by sender and receiver such that when the first bit is sent, there is a probability distribution P of it being set to 1 and a solution distribution S which counts the number of satisfied states. This results in an expected number of solutions for the first bit.

$$E(x_0) = P(x_0) \cdot S(x_0) + P(\neg x_0) \cdot S(\neg x_0)$$

This is the same as:

$$E(x_0) = P(x_0) \cdot S(x_0) + (1 - P(x_0)) \cdot (S - S(x_0))$$

For the first bit, there are no other bits that got lower expected number of solutions. This ensures the first bit is most significant.

The next bit has an expected number of solutions conditioned on the state of the first bit:

$$\begin{aligned} E(x_1|x_0) = \quad & P(x_1|x_0) \cdot S(x_1|x_0) + \\ & P(\neg x_1|x_0) \cdot S(\neg x_1|x_0) + \\ & P(x_1|\neg x_0) \cdot S(x_1|\neg x_0) + \\ & P(\neg x_1|\neg x_0) \cdot S(\neg x_1|\neg x_0) \end{aligned}$$

This process is repeated, such that for e.g. 4 bits, the product of expected solutions is minimized:

$$C(x_3|x_2|x_1|x_0) = E(x_0) \cdot E(x_1|x_0) \cdot E(x_2|x_1|x_0) \cdot E(x_3|x_2|x_1|x_0)$$

A such compression algorithm is optimal for any boolean expression relative to the order of bits.

A universally optimal compression algorithm for a solution `s ` finds an optimal representation `$f_{opt}$`, by enumerating all boolean functions of type `$f : bool^N \rightarrow bool^M$` which can be inverted back to the solution domain, finding the minimum `$C_{\exists f}$` and using the logical compression for encoding:

$$f_{opt} : (\exists f^{-1}\{\exists f(s)\} \Rightarrow s) \wedge (\forall f : (\exists f^{-1}\{\exists f(s)\} \Rightarrow s) \{ C_{\exists f\_opt} \leq C_{\exists f} \})$$

$$s : bool^N \rightarrow bool$$
$$f_{opt} : bool^N \rightarrow bool^M$$
$$f^{-1} \Leftrightarrow id[f \rightarrow id]$$

Since there is no better representation and no better order of bits, the communication is optimal.