Abrupt Change in Complex Systems by Small Changes in Micro Behavior

by Sven Nilsen, 2017

Here I represent an inference technique to study what probability a micro behavior need to have in order to change the overall system drastically, but otherwise keeps the system in one of two states. It is perhaps the simplest possible model where abrupt changes can occur due to small changes in micro behavior, which makes the technique applicable to a wide variety of applications.

Alice designs a spaceship and tries to figure what level of safety is required. She needs to decide between two designs, one where 5 things need to go wrong and one where 20 things need to go wrong for an accident to occur.

In general, such problems can be modeled as a list of events `$x_i$` that all happens simultaneously. The safety aspect can be thought of as the existence of a such list among other ones, where each list represents a single use of the system. Assuming the events are independent, this corresponds to the probability assigned to the following logical expression:

$$P(\exists\ i\ \{\ \forall\ j\ \{\ x_{ij}\ \}\ \})$$

Deriving the formula for the probability:

$$P(\exists\ i\ \{\ \forall\ j\ \{\ x_{ij}\ \}\ \})$$
$$1 - \prod\ i\ \{\ 1 - P(\forall\ j\ \{\ x_{ij}\ \})\ \}$$
$$1 - \prod\ i\ \{\ 1 - \prod\ j\ \{\ x_{ij}\ \}\ \}$$

By setting a uniform probability for each event, one can obtain a simpler formula that makes it easier to study the sensitivity:

$$1 - \prod\ i\ \{\ 1 - \prod\ j\ \{\ x_{ij}\ \}\ \}$$
$$1 - \prod\ i\ \{\ 1 - p^a\ \}$$
$$1 - (1 - p^a)^b$$

Solving this for `p` given some confidence of the safety of the whole system:

$$1 - c = 1 - (1 - p^a)^b$$
$$c = (1 - p^a)^b$$
$$\ln(c) = b * \ln(1 - p^a)$$
$$\ln(c) / b = \ln(1 - p^a)$$
$$\exp(\ln(c) / b) = 1 - p^a$$
$$1 - \exp(\ln(c) / b) = p^a$$
$$\ln(1 - \exp(\ln(c) / b)) = a * \ln(p)$$
$$\ln(1 - \exp(\ln(c) / b)) / a = \ln(p)$$
$$p = \exp(\ln(1 - \exp(\ln(c) / b)) / a)$$

Writing this as a function `p`:

p := \(a, b, c) = exp(ln(1 - exp(ln(c) / b)) / a)

Now, Alice can compute some confidence intervals, using `b = 1000` since she does not want to use the spaceship unless she is 90% confident she would survive a thousand times over:

Things that can go wrong	**c = 90%**	**c = 50%**	**c = 10%**
a = 5	p ~= 16%	p ~= 23%	p ~= 30%
a = 20	p ~= 63%	P ~= 70%	p ~= 74%

Notice that when 20 things are required to go wrong at the same time to produce an accident, you need a very high probability for a single event to go wrong. It does not matter much whether this is 1% or 40% as long you keep it below 63%.

$1 – (1 – 0.01^{20})^{1000}$ ~= 0 (Alice never see an accident occur)
$1 – (1 – 0.4^{20})^{1000}$ ~= 0.00001 (Alice never see an accident occur)

Once it starts to go above 63% the chance that an accident occurs raises very rapidly.

$1 – (1 – 0.99^{20})^{1000}$ ~= 1 (Alice will see at least one accident among 1000 uses)
$1 – (1 – 0.8^{20})^{1000}$ ~= 0.99999 (Alice will see at least one accident among 1000 uses)

This is the surprising thing about such systems: They concentrate the change around a very sensitive spot for a single event to go wrong. Even with a large sample of repeated use, for example if all the people in the world using Alice’s spaceship, the observations made tends to fall into two categories:

- Either an accident always occur all the time...
- ...or an accident never happens any time!

This system analysis demonstrates that with enough redundancy, which is not necessarily a very high number, a good design for safety should actually *never* have accidents in practice!