Proving non-existence of asymmetric paths

by Sven Nilsen, 2017

To prove the non-existence of asymmetric paths, the following formula can be used:

$$
\begin{aligned}
&\exists\ (x_0, x_1, \dots, x_{n-1}), (y_0, y_1, \dots, y_{n-1})\ \{ \\
&\qquad (\ (g_0(x_0), g_1(x_1), \dots, g_{n-1}(x_{n-1})) = (g_0(y_0), g_1(y_1), \dots, g_{n-1}(y_{n-1}))\ ) \land \\
&\qquad (\ g_n(f(x_0, x_1, \dots, x_{n-1})) \neg= g_n(f(y_0, y_1, \dots, y_{n-1}))\ ) \\
&\} \Rightarrow \neg\exists\ f[g_0 \times g_1 \times \dots \times g_{n-1} \rightarrow g_n]
\end{aligned}
$$

In Nilsen cartesian product notation:

$$\exists\ x_i, y_i\ \{\ g_i(x^i) = g_i(y^i) \land g_n(f(x_i)) \neg= g_n(f(y_i))\ \} \Rightarrow \neg\exists\ f[g_{i\rightarrow n}]$$

This can be proved the following way, by constructing two partial function pairs:

$$
\begin{aligned}
&a_{in} = g_{in}(x^i, f(x_i)) \\
&b_{in} = g_{in}(y^i, f(y_i))
\end{aligned}
$$

Assume that the partial function pairs `$a_{in}$` and `$b_{in}$` are of the same function. If the arguments are equal, then the function must return same value, so the return values must be equal:

$$(\ a_i = b_i\ ) \Rightarrow (\ a_n = b_n\ )$$

If the return values are not equal, then the partial function pairs do not belong to the same function, therefore the path `$f[g_{i\rightarrow n}]$` does not exist.