Law of Sub-Type Reduction by Transitivity Variable Elimination

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Sometimes a variable can be eliminated from a proof, making it much easier to check. This is possible when there is some function `f` that has the following property by material implication:

$$\forall a, b, c : A \{ f(b, c) \rightarrow (f(a, b) \rightarrow f(a, c)) \}$$

$$f : A \times A \rightarrow bool$$

This property is called “transitivity” and is equivalent to the traditional definition:

$$\forall a, b, c : A \{ f(a, b) \land f(b, c) \rightarrow f(a, c) \}$$

One can easier see why a variable can be eliminated by writing it another way:

$$\forall b, c : A \{ f(b, c) \rightarrow \forall a : A \{ f(a, b) \rightarrow f(a, c) \} \}$$

With other words, proving `f(b, c)` is sufficient to prove `f(a, b) → f(a, c)` for any `a`.

For example, in path semantics one might want to reduce the following sub-type to `x : (> 4)`:

$$x : (> 2) \land (> 4)$$

This is the same as two inequalities:

$$(x > 2) \land (x > 4)$$

It is easy to see that if `x` is greater than 4, it must also be greater than `2`, so one implies the other.

$$\forall a, b \{ ((a \rightarrow b) \land a) \rightarrow a \land b \}$$

From the law above one can deduce that `( x > 2 ) ∧ ( x > 4 )` must be true if `x > 4` and:

$$(x > 4) \rightarrow (x > 2)$$

The `>` operator is transitive, so the above is proved because the following is true:

$$4 > 2$$

Therefore, what remains is `x > 4` and the sub-type can be reduced to:

$$x : (> 4)$$