Nilsen Cartesian Product Notation

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This notation is useful when doing asymmetric path semantics in a way that can easily be reduced to symmetric path semantics by erasing indices.

A single lowered index annotates a cartesian product:

$$a_i = (a_0, a_1, \dots, a_{n-1})$$
$$a_j = (a_0, a_1, \dots, a_{m-1})$$

`i` and `j` has paired cardinality with `n` and `m` respectively. When `n` and `m` are used as lowered indices, they can be thought of as an extra item at the end:

$$a_{in} = (a_i, a_n) = (a_0, a_1, \dots, a_{n-1}, a_n)$$
$$a_{jm} = (a_j, a_m) = (a_0, a_1, \dots, a_{m-1}, a_m)$$

A common technique in path semantics is to create partial function products, e.g.:

$$a_{in} = (x_i, f(x_i))$$

A lowered index paired with a raised index annotates function application per element:

$$f_i(x^i) = (f_0(x_0), f_1(x_1), \dots, f_{n-1}(x_{n-1}))$$

This makes it possible to apply a product of functions to a partial function product:

$$g_{in}(a^{in}) = g_{in}(x^i, f(x_i)) = (g_0(x_0), g_1(x_1), \dots, g_{n-1}(x_{n-1}), g_n(f(x_0, x_1, \dots, x_{n-1})))$$

For all partial function products of the same function:

$$(a_i = b_i) => (a_n = b_n)$$

A path function product `$g_{i \to n}$` splits the arguments from return value at `n`:

$$[g_{i \to n}] = [g_i \to g_n] = [g_0 \times g_1 \times \dots g_{n-1} \to g_n]$$

This means an asymmetric path can be turned into a symmetric path by erasing indices:

$f[g_{i \to n}]$	asymmetric path
$f[g]$	symmetric path

Equations often compare products:

$$g_i(x^i) = g_i(y^i)$$
$$(g_0(x_0), g_1(x_1), \dots, g_{n-1}(x_{n-1})) = (g_0(y_0), g_1(y_1), \dots, g_{n-1}(y_{n-1}))$$