Pol Sci 630: Problem Set 2 Solutions - Properties of Random Variables

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Due Date for Grading: Friday, September 11, 2015, 10 AM
(Beginning of Class)

1. Expected Value and Its Properties

a.

(1/4 point) (DeGroot, p. 216) Suppose that one word is to be selected at random from the sentence 'the girl put on her beautiful red hat'. If X denotes the number of letters in the word that is selected, what is the value of E(X)?

Solution

As the number of letters in a word, X can take on following values: $x \in \{2, 3, 4, 9\}$, with probability as follows:

$$P(X = 2) = \frac{1}{8} \qquad \text{(1 word ("on") out of 8 words in the sentence)} \tag{1}$$

$$P(X = 3) = \frac{5}{8} \tag{2}$$

$$P(X = 4) = \frac{1}{8} \tag{3}$$

$$P(X = 9) = \frac{1}{8} \tag{4}$$

Therefore,

$$E(X) = \sum_{all x_i} x_i P(X = x_i) = 3.75$$

b.

(2/4 point) (Degroot p. 216) Suppose that one letter is to be selected at random from the 30 letters in the sentence given in Exercise 4. If Y denotes the number of letters in the word in which the selected letter appears, what is the value of E(Y)?

Hint: 1a) and 1b) force you to think carefully about the definition of expectation value. For each problem, think about what is your random variable (X), which values it takes on ($x \in \{?, ?, ...\}$) and with what probability ($P(X = x) =?$)

c.

(1/4 point) (Degroot, p. 224) Suppose that three random variables X_1, X_2, X_3 are uniformly distributed on the interval [0, 1]. They are also independent. Determine the value of $E[(X_1 2X_2 + X_3)^2]$.

2. Variance and its properties

For this problem, you can use the properties of expected value.

a.

(1/4 point) Prove that $Var(aX + b) = a^2 Var(X)$.

b.

(2/4 point) Prove that if two random variables are independent, the variance of the sum is the sum of the variance. In other words, if X_1, X_2 are independent, then

$$Var(X_1 + X_2) = Var(X_1) + Var(X_2)$$

c.

(1/4 point) (Degroot, p. 232) Suppose that one word is selected at random from the sentence ‘the girl put on her beautiful red hat‘. If X denotes the number of letters in the word that is selected, what is the value of $Var(X)$?

3. Binomial distribution

(Credit to Jan) This problem is taken from Pitman (1993) Probability

Suppose a fair coin is tossed n times. Find a simple formula in terms of n and k for the following probability: $Pr(k\ heads | k-1\ heads\ or\ k\ heads)$. Please pay close attention to the formula, particularly what event is conditioned on what events. (Ch. 2.1, Problem 10 b) (p. 91)

Hint 1: Use the binomial distribution to model this.

Hint 2: Because those events are mutually exclusive, calculate the following:
$$\frac{Pr(k\ heads)}{Pr(k\ heads) + Pr(k-1\ heads)}$$

This is true because: $Pr(A|B) = \dfrac{Pr(A \cap B)}{Pr(B)}$

The intersection of events A and B in this case, $Pr(k\ heads \cap (k\ heads \cup k-1\ heads))$, reduces to $Pr(k\ heads)$ because the two events are mutually exclusive.

4. Plotting distribution

For this problem, you'll need to Google some R techniques (e.g. side-by-side / overlapping plot). Also, label the axes and the plots accordingly.

a.

(1/4 point) Download a variable you are interested in, using `WDI`. Plot the histogram, density plot, boxplot, and normal quantile plot.

b.

(1/4 point) Plot the histogram of that variable for Europe and Asia, 1) side by side (Hint: `par(mfrow=c(?, ?))`), and 2) overlapping in the same plot.

c.

(1/4 point) Draw the scatterplot of that variable against another variable.

d.

(1/4 point) Label the point that represents your country (Hint: Tutorial) and color it red (Some Googling involved)