

# Tutorial 3: Comparisons and Inference

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September 11, 2015

**Question:** sometimes R gives you output like this:

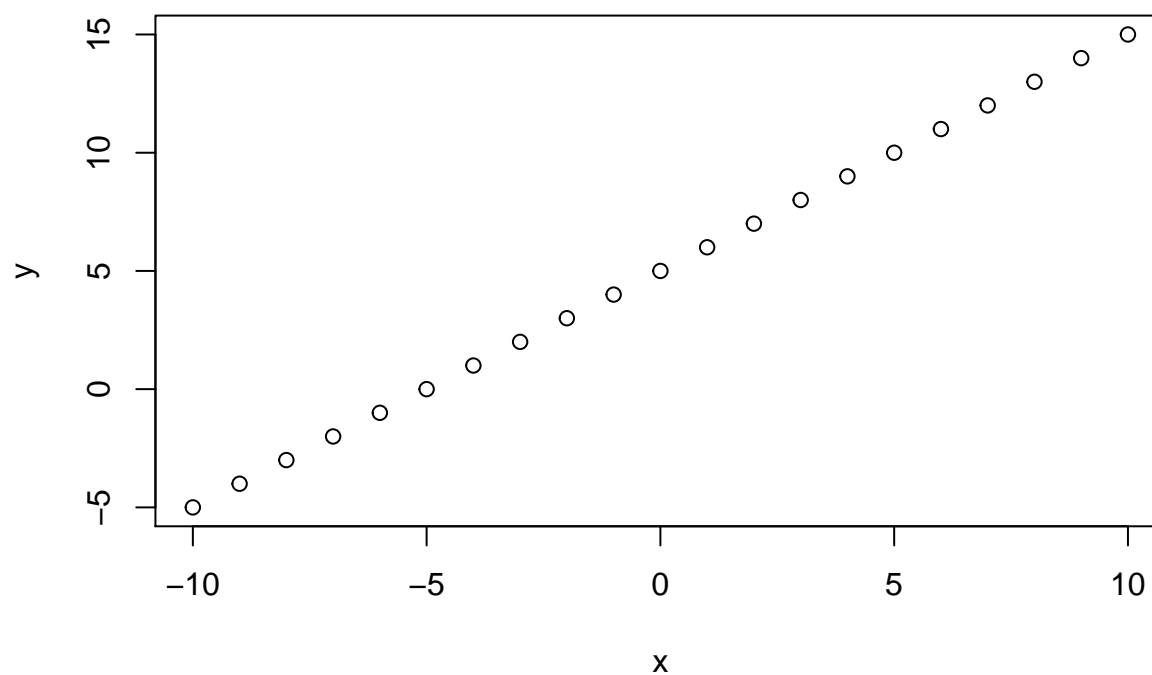
2.43e-05

**What does this mean?**

## Topic 1: Covariance

Let us create two variables that are clearly linearly dependent on each other.

```
x=seq(-10,10)
y=(x+5)
plot(x,y)
```



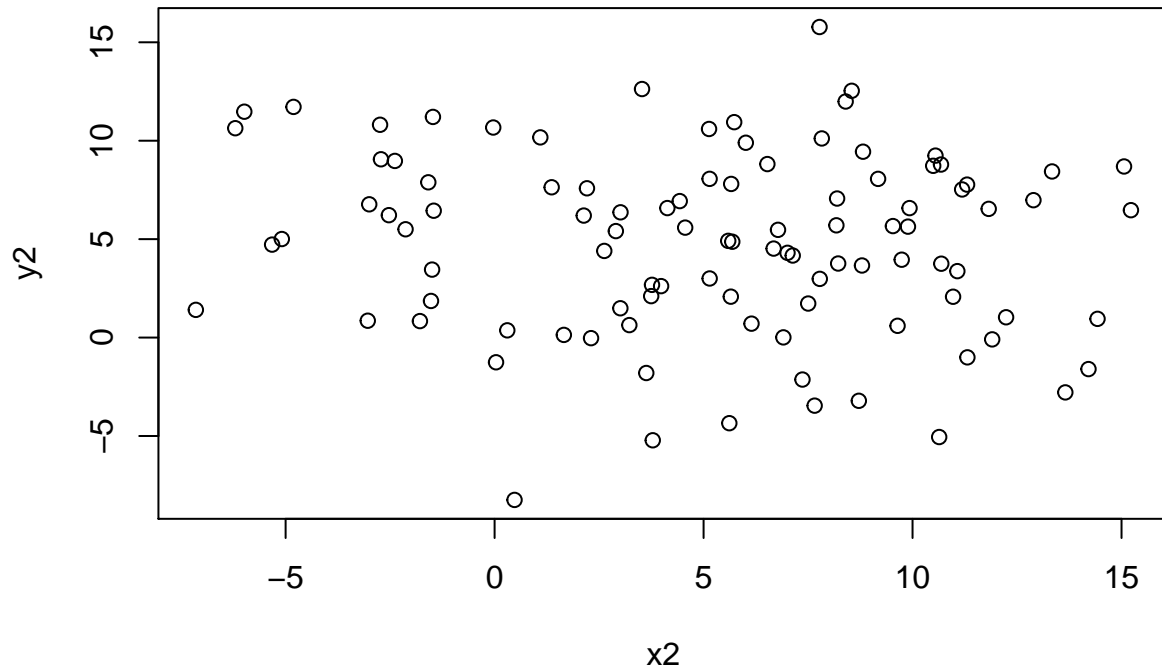
The covariance of these two variables is positive - as x increases so does y.

```
cov(x,y)
```

```
## [1] 38.5
```

Let us create two variables that have no relationship to each other:

```
x2=rnorm(100,mean=5,sd=5)
y2=rnorm(100,mean=5,sd=5) # Both variables are just random draws from the normal distribution
plot(x2,y2)
```



The covariance should be close to zero - due to the randomness of the data it is most likely not exactly zero though.

```
cov(x2,y2)
```

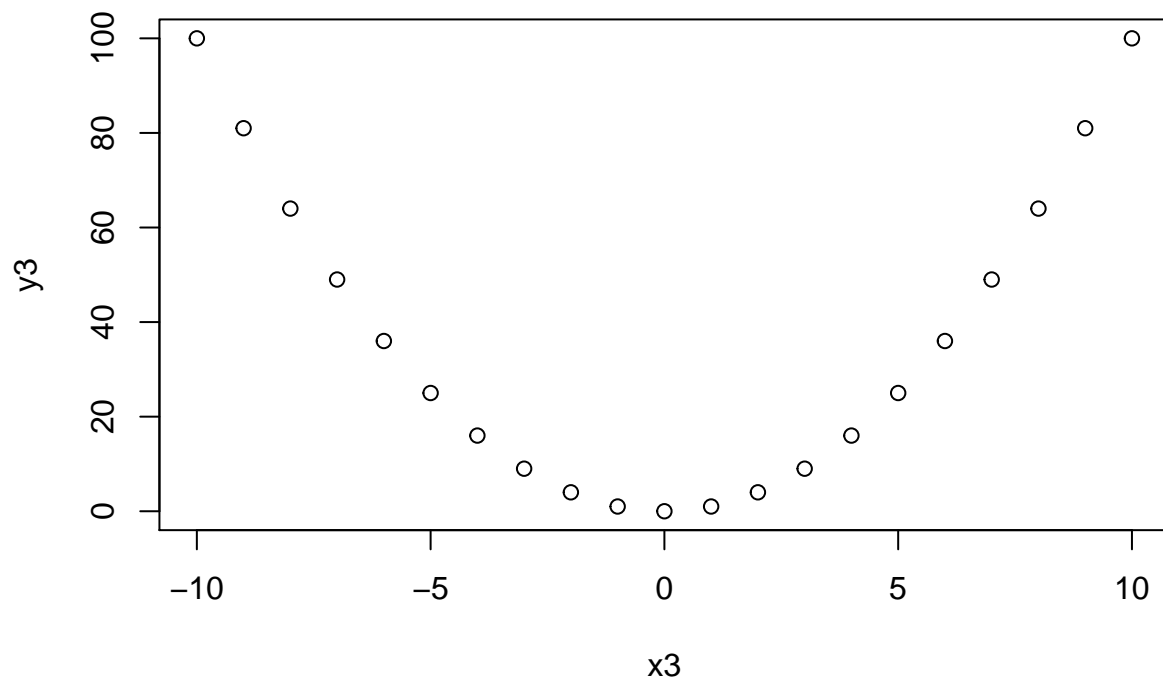
```
## [1] -2.91383
```

Note that this means even variables that are completely random and not related to each other may produce a non-zero covariance. However, the  $E(\text{Cov}(x_2, y_2)) = 0$ , so the distribution of the covariance is centered on the value 0.

Independence implies that the expected value of the covariance is zero.

Does a covariance of zero imply independence?

```
x3=seq(-10,10)
y3=x3^2
plot(x3,y3)
```



As we can clearly see from the plot, there is a curvilinear relationship of the two variables - they are not independent.

```
cov(x3,y3)
```

```
## [1] 0
```

The formula tells us that the covariance is zero. Why?

Covariance captures linear relationships.

When  $x_3$  is below its mean, the values of  $y_3$  vary in the exact same way as when  $x_3$  is above its mean.

The formula can't capture the curvilinear relationship because it looks at the variation of  $y_3$  relative to  $x_3$ 's deviation from its mean.  $y_3$  varies in the exact same way when  $x_3$  moves above and below its mean. Thus, there is no linear relationship that could be captured.

## Topic 2: Correlation

As you've learned in the lecture, the problem with covariance is that it is not to scale. It doesn't really tell us that much about how much variables vary with each other because it doesn't account for their individual variation magnitudes. However, correlation standardizes covariance by the standard deviation of the two variables. The result is that the measure of correlation is bound between -1 and 1.

```
cor(x,y) ### Why does this produce "1". What is the meaning of this value?
```

```
## [1] 1
```

How about x2 and y2 that are completely random?

```
cor(x2,y2) # The correlation is extremely close to zero, indicating that there is no systematic linear
```

```
## [1] -0.1160971
```

Does correlation capture non-linear relationships equally well?

Correlation is a mathematical concept. We cannot find the correlation between a numeric and a character vector.

```
y4=rep(c("a","b","c"),7)
y4
```

```
## [1] "a" "b" "c" "a" "b" "c" "a" "b" "c" "a" "b" "c" "a" "b" "c" "a" "b"
## [18] "c" "a" "b" "c"
```

```
is.numeric(y4) # Checks whether y4 is numeric and returns the argument FALSE.
```

```
## [1] FALSE
```

```
cor(x,y4) # Gives us the error message "y must be numeric".
```

```
## Error in cor(x, y4): 'y' must be numeric
```

Interestingly, however, R allows us to find the correlation between a numeric and a logical vector, although a logical vector is not numeric.

```
y5=rep(c(T,F,F),7)
y5
```

```
## [1] TRUE FALSE FALSE TRUE FALSE FALSE TRUE FALSE FALSE TRUE FALSE
## [12] FALSE TRUE FALSE FALSE TRUE FALSE FALSE TRUE FALSE FALSE
```

```
is.numeric(y5) # Checks whether y5 is numeric and returns the argument FALSE.
```

```
## [1] FALSE
```

```
class(y5) # Returns the class of the vector.
```

```
## [1] "logical"
```

```
cor(x,y5) # Returns a value.
```

```
## [1] -0.1167748
```

How do we have to think about this?

Assume that T=1 and F=0.

```
y6=rep(c(1,0,0),7)
cor(x,y6) # Returns the same value as above, meaning that R views T=1 and F=0
```

```
## [1] -0.1167748
```

### Topic 3: Cross-tabs

R has several built-in datasets, let's have a look at them.

```
library(datasets)
data(occupationalStatus)
occupationalStatus
```

```
##      destination
## origin  1  2  3  4  5  6  7  8
##      1  50 19 26  8  7 11  6  2
##      2  16 40 34 18 11 20  8  3
##      3  12 35 65 66 35 88 23 21
##      4  11 20 58 110 40 183 64 32
##      5   2  8 12 23 25 46 28 12
##      6  12 28 102 162 90 554 230 177
##      7   0  6 19 40 21 158 143 71
##      8   0  3 14 32 15 126 91 106
```

According to the documentation this is “Cross-classification of a sample of British males according to each subject’s occupational status and his father’s occupational status.”

The source is a journal article from 1979: “Goodman, L. A. (1979) Simple Models for the Analysis of Association in Cross-Classifications having Ordered Categories.”

Let us assume that 1 is a low occupational status and 8 is a high occupational status (it might be the opposite). Is there a relationship between the status of the father and the son?

Before using the command below, use `install.packages(“gmodels”)`.

```
library(gmodels)
CrossTable(occupationalStatus)
```

```
##
##
##      Cell Contents
## |-----|
## |                      N |
## | Chi-square contribution |
## |          N / Row Total |
## |          N / Col Total |
## |          N / Table Total |
## |-----|
##
##
## Total Observations in Table:  3498
##
##
```

##	destination							
##	origin	1	2	3	4	5	6	7
##								
##	1	50	19	26	8	7	11	6
##		561.961	29.430	15.717	4.708	0.444	24.504	11.515
##		0.388	0.147	0.202	0.062	0.054	0.085	0.047
##		0.485	0.119	0.079	0.017	0.029	0.009	0.010
##		0.014	0.005	0.007	0.002	0.002	0.003	0.002
##								
##	2	16	40	34	18	11	20	8
##		30.377	161.485	27.842	0.144	0.028	18.723	11.946
##		0.107	0.267	0.227	0.120	0.073	0.133	0.053
##		0.155	0.252	0.103	0.039	0.045	0.017	0.013
##		0.005	0.011	0.010	0.005	0.003	0.006	0.002
##								
##	3	12	35	65	66	35	88	23
##		0.334	23.798	32.359	9.492	4.969	7.176	21.531
##		0.035	0.101	0.188	0.191	0.101	0.255	0.067
##		0.117	0.220	0.197	0.144	0.143	0.074	0.039
##		0.003	0.010	0.019	0.019	0.010	0.025	0.007
##								
##	4	11	20	58	110	40	183	64
##		1.186	0.534	1.707	25.988	0.414	0.309	6.458
##		0.021	0.039	0.112	0.212	0.077	0.353	0.124
##		0.107	0.126	0.176	0.240	0.164	0.154	0.108
##		0.003	0.006	0.017	0.031	0.011	0.052	0.018
##								
##	5	2	8	12	23	25	46	28
##		1.464	0.117	0.502	0.313	18.318	0.898	0.091
##		0.013	0.051	0.077	0.147	0.160	0.295	0.179
##		0.019	0.050	0.036	0.050	0.102	0.039	0.047
##		0.001	0.002	0.003	0.007	0.007	0.013	0.008
##								
##	6	12	28	102	162	90	554	230
##		19.508	18.320	5.219	1.404	0.216	19.474	0.000
##		0.009	0.021	0.075	0.120	0.066	0.409	0.170
##		0.117	0.176	0.309	0.353	0.369	0.467	0.388
##		0.003	0.008	0.029	0.046	0.026	0.158	0.066
##								
##	7	0	6	19	40	21	158	143
##		13.486	10.547	13.563	6.721	3.751	0.047	55.016
##		0.000	0.013	0.041	0.087	0.046	0.345	0.312
##		0.000	0.038	0.058	0.087	0.086	0.133	0.241
##		0.000	0.002	0.005	0.011	0.006	0.045	0.041
##								
##	8	0	3	14	32	15	126	91
##		11.395	12.103	13.878	6.946	5.330	0.207	9.829
##		0.000	0.008	0.036	0.083	0.039	0.326	0.235
##		0.000	0.019	0.042	0.070	0.061	0.106	0.153
##		0.000	0.001	0.004	0.009	0.004	0.036	0.026
##								
##	Column Total	103	159	330	459	244	1186	593
##		0.029	0.045	0.094	0.131	0.070	0.339	0.170
##								

```
##  
##
```

How can we interpret this table?

#### Topic 4: Central Limit Theorem

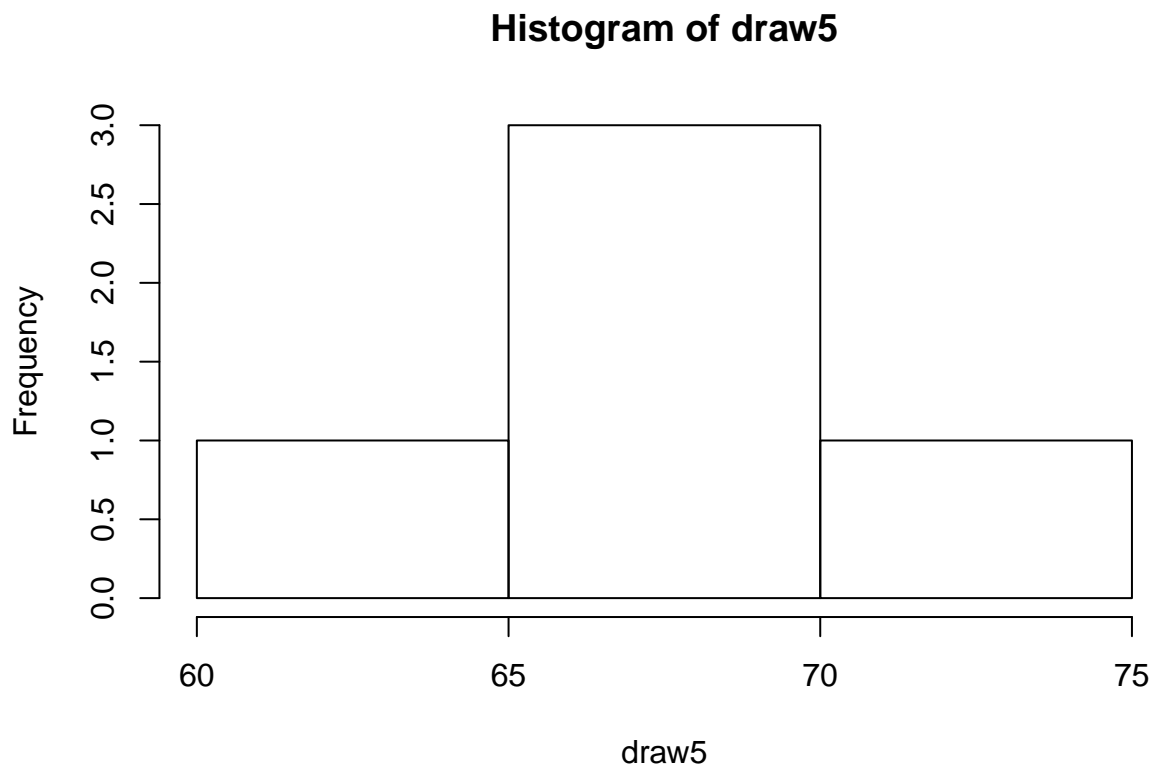
The Central Limit Theorem says that if we have infinitely many draws of the same size from a specific distribution, the mean of this distribution will be approximately normally distributed.

Let us illustrate this with a simple example of the binomial distribution.

```
draw1=rbinom(1,size=100,p=0.67)  
draw1
```

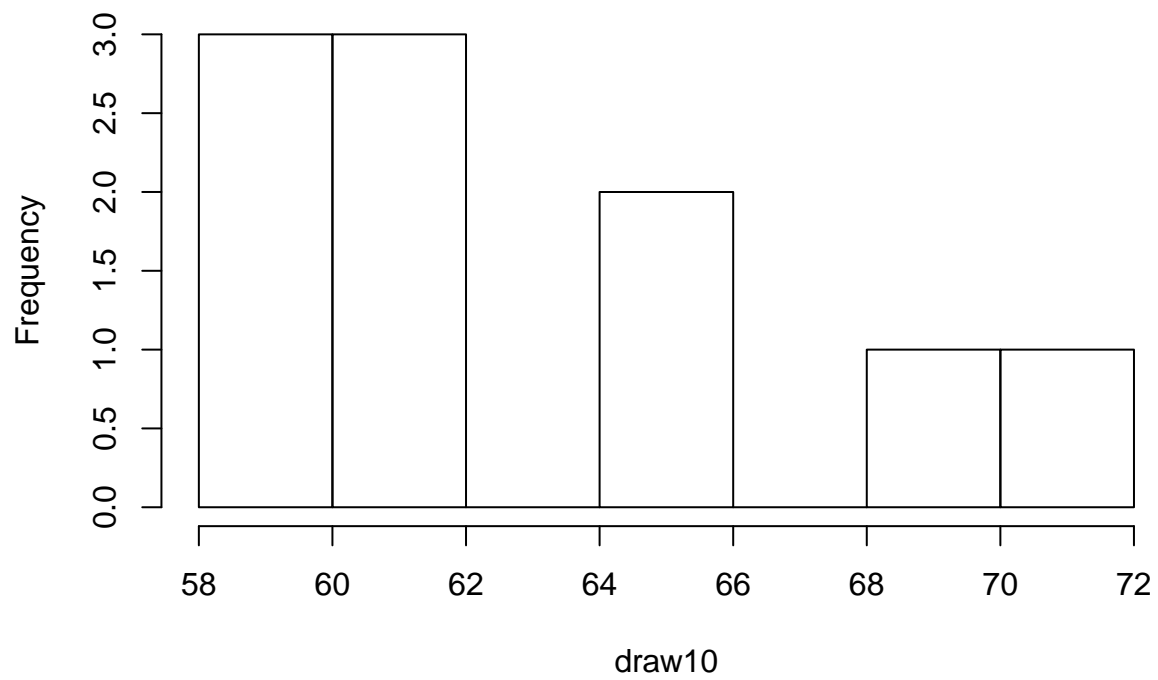
```
## [1] 55
```

```
draw5=rbinom(5,size=100,p=0.67)  
hist(draw5)
```



```
draw10=rbinom(10,size=100,p=0.67)  
hist(draw10)
```

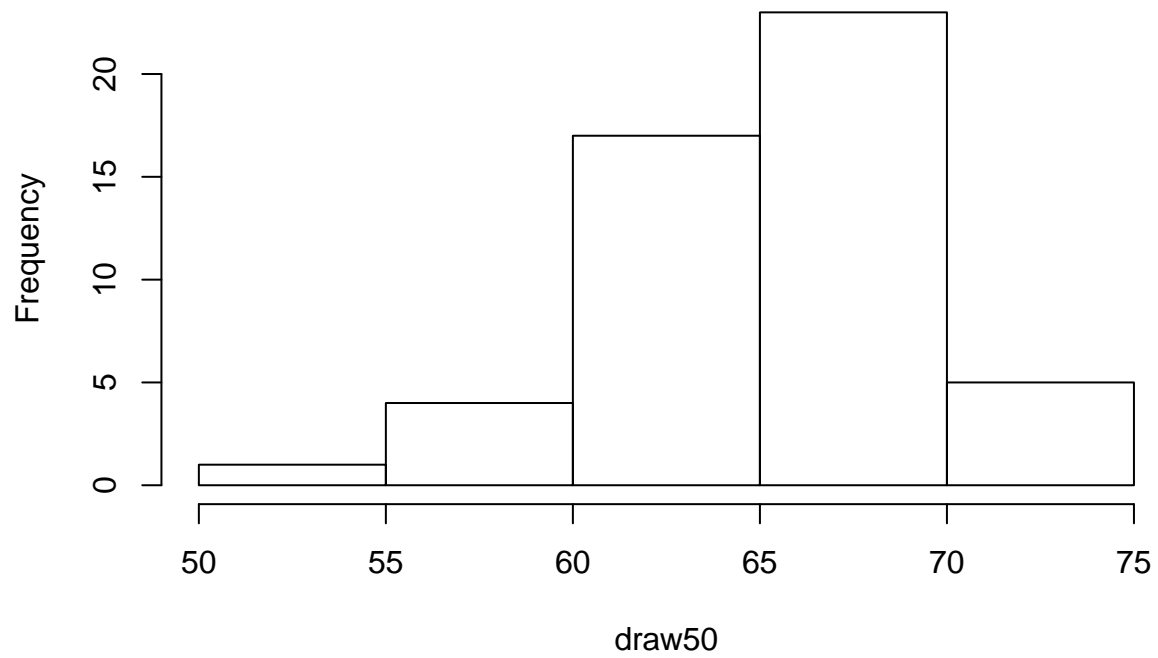
**Histogram of draw10**



```
draw50=rbinom(50,size=100,p=0.67)
hist(draw50)
```

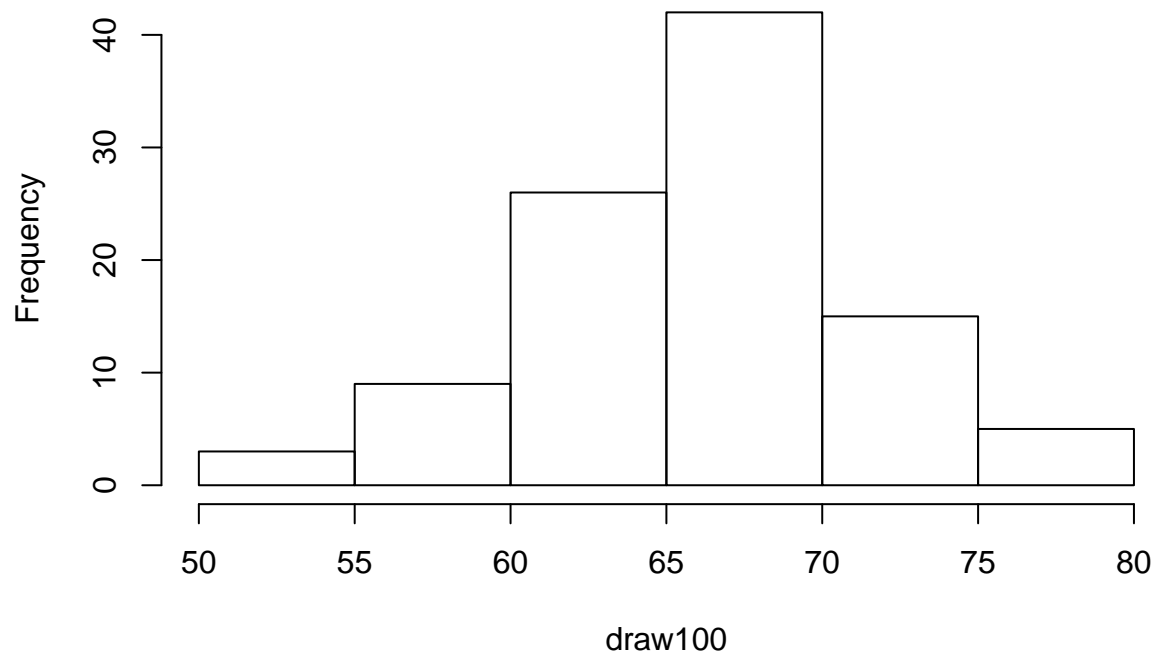


**Histogram of draw50**



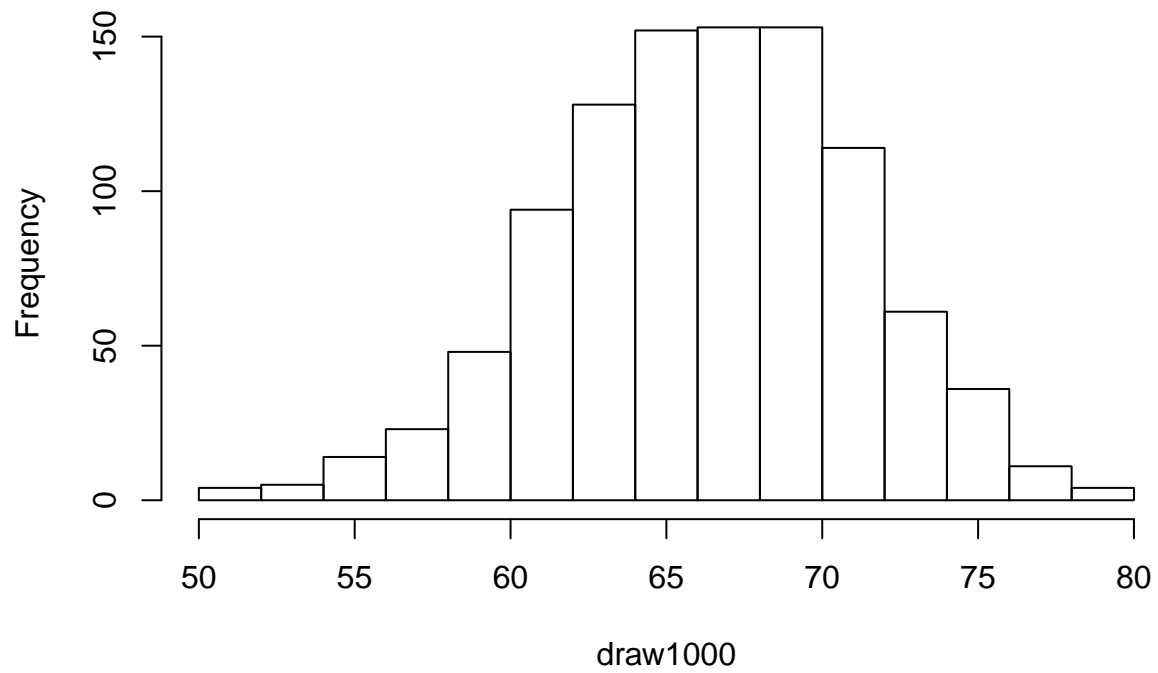
```
draw100=rbinom(100,size=100,p=0.67)
hist(draw100)
```

**Histogram of draw100**



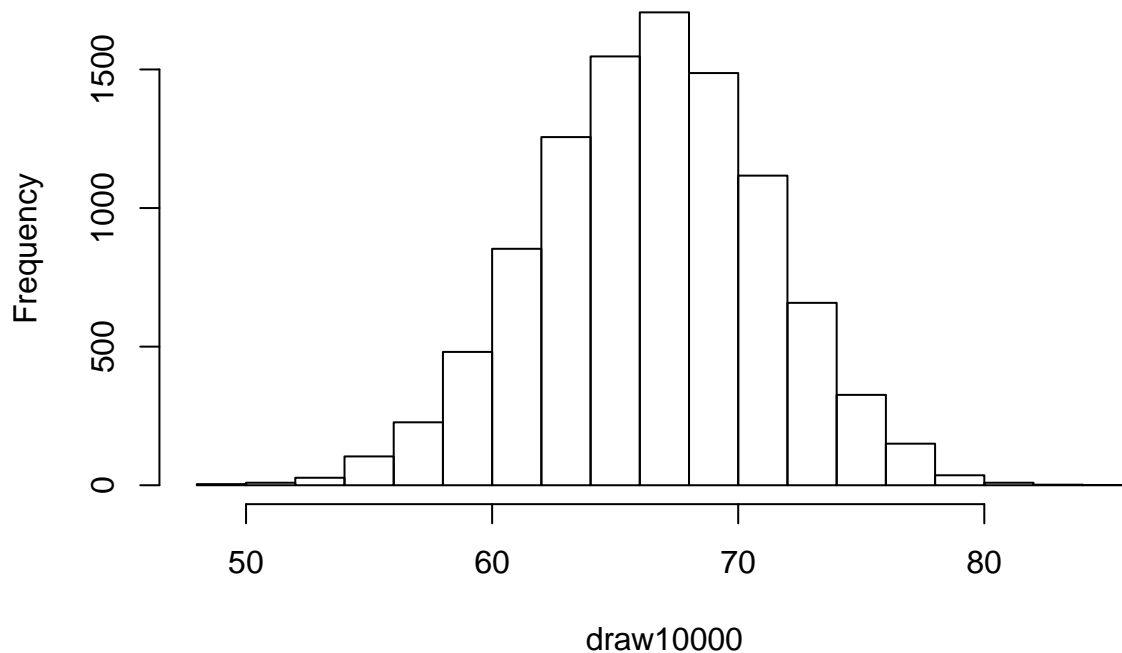
```
draw1000=rbinom(1000,size=100,p=0.67)
hist(draw1000)
```

**Histogram of draw1000**



```
draw10000=rbinom(10000,size=100,p=0.67)
hist(draw10000) # This really looks like a normal distribution
```

## Histogram of draw10000

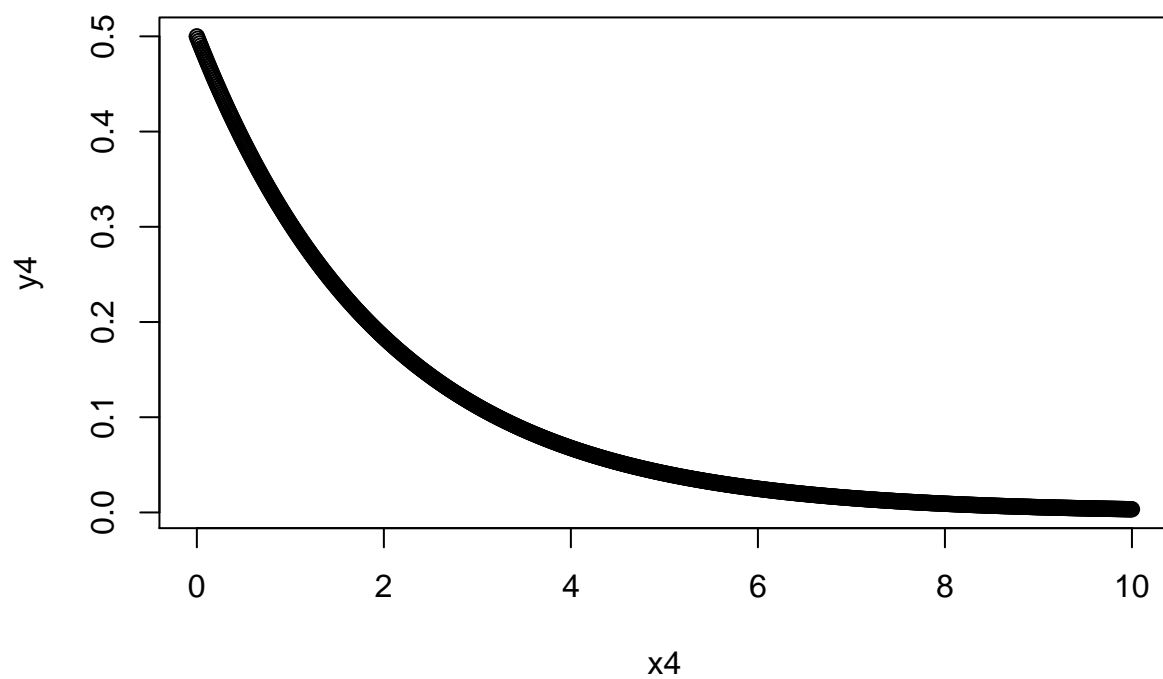


Note that this is true for any distribution, even those that are NOT normally distributed themselves (the distribution of a binomial looks similar to a normal distribution for large N).

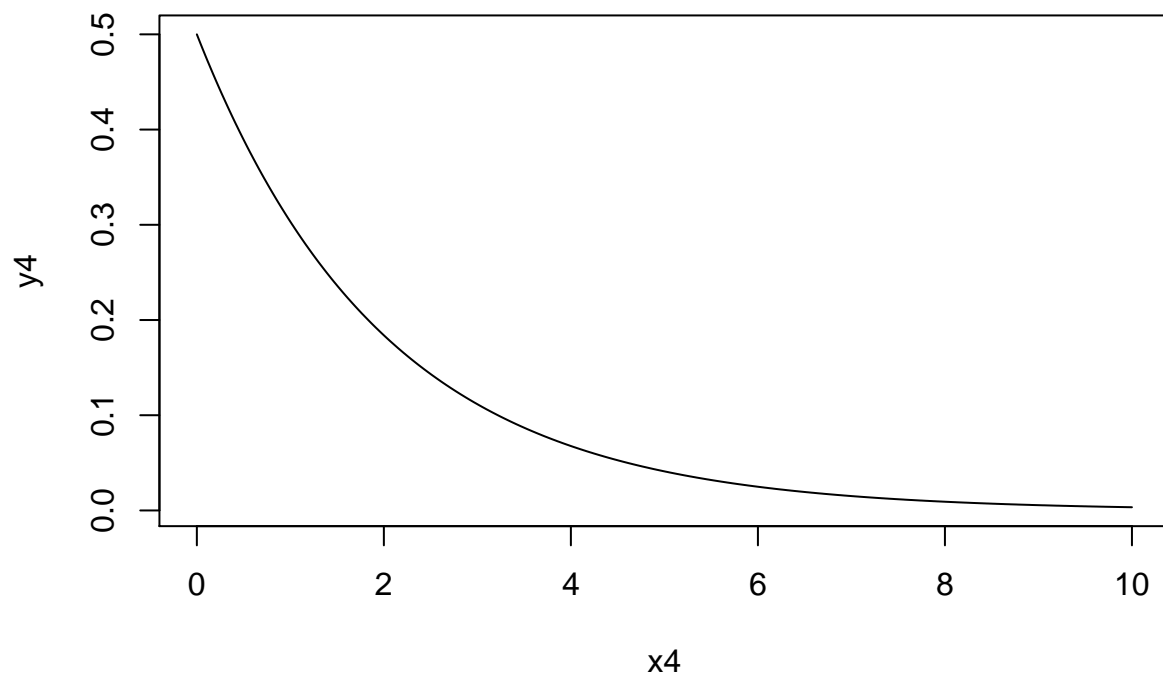
Let's try a similar example with an exponential distribution. The exponential distribution doesn't look like a normal distribution.

How does an exponential distribution look like?

```
x4=seq(0,10,by=0.01)
y4=dexp(x4, rate=0.5) # Returns the density
plot(x4,y4) # This doesn't look nice
```



```
plot(x4,y4, type="l") # Use type="l" for a line plot
```



We can also use the ggplot2 package to make it look even nicer.

Use the command `install.packages("ggplot2")` before you run this code.

```
library(ggplot2)
plot1=qplot(x4,y4) # Now that looks even nicer
```

Recall: The Central Limit Theorem states that if we have multiple samples of the same size, their mean will be approximately normally distributed.

So, what happens if we draw 1000 times 10 samples from this distribution, how will their mean be distributed?

```
meanstore=rep(0,1000)
for (i in 1:1000){
  expdraw=rexp(10, rate=0.5)
  meanstore[i]=mean(expdraw)
}
hist(meanstore, breaks=20) # It is approximately normally distributed, as predicted by the CLT.
```



### Topic 5: t-tests

t-tests allow us to either compare the mean of two populations or to compare the mean of one population against a theoretical example

Let us create two sets of numbers that come from normal distributions with different means.

```
vec1=rnorm(30,mean=2,sd=1)
vec2=rnorm(30,mean=3,sd=1)
```

The t-test allows us to find out the likelihood that these two come from the same distribution:

```
t.test(vec1,vec2)
```

```
##
##  Welch Two Sample t-test
##
## data:  vec1 and vec2
## t = -3.411, df = 56.608, p-value = 0.0012
## alternative hypothesis: true difference in means is not equal to 0
## 95 percent confidence interval:
##  -1.3800777 -0.3589924
## sample estimates:
## mean of x mean of y
##  1.927965  2.797500
```

What does this t-value mean? What does this p-value mean?

We can also compare a single sample against a mean that we define to be our  $H_0$ .

```
t.test(vec1,mu=2)

##
## One Sample t-test
##
## data:  vec1
## t = -0.37156, df = 29, p-value = 0.7129
## alternative hypothesis: true mean is not equal to 2
## 95 percent confidence interval:
##  1.531453 2.324478
## sample estimates:
## mean of x
##  1.927965
```

What does this t-value mean? What does this p-value mean?

### Topic 6: p-values

Typically, a p-value is related to a type-1 error rate ( $\alpha$ ) that we define in advance. Type-1 errors refer to the incorrect rejection of a true null hypothesis. Often we want  $\alpha$  to be smaller than 0.05.

Typically, our null hypothesis ( $H_0$ ) is that there is no relationship between two variables.

The p-value is the probability that we get data with evidence that is such strong AGAINST  $H_0$  if  $H_0$  was true. Think about what this means. If we define a threshold of  $p$  to be  $p < 0.05$ , then we have a type-1 error rate of  $\alpha = 0.05$ .

Let's load another R dataset that can illustrate this. The "airquality" dataset. According to the documentation, this is "Daily air quality measurements in New York, May to September 1973."

More details can be found here:

```
data(airquality)
airquality
```

```
##      Ozone Solar.R Wind Temp Month Day
## 1      41      190  7.4   67     5   1
## 2      36      118  8.0   72     5   2
## 3      12      149 12.6   74     5   3
## 4      18      313 11.5   62     5   4
## 5      NA       NA 14.3   56     5   5
## 6      28       NA 14.9   66     5   6
## 7      23      299  8.6   65     5   7
## 8      19       99 13.8   59     5   8
## 9       8       19 20.1   61     5   9
## 10     NA      194  8.6   69     5  10
## 11      7       NA  6.9   74     5  11
## 12     16      256  9.7   69     5  12
## 13     11      290  9.2   66     5  13
## 14     14      274 10.9   68     5  14
## 15     18       65 13.2   58     5  15
## 16     14      334 11.5   64     5  16
## 17     34      307 12.0   66     5  17
## 18      6       78 18.4   57     5  18
```



## 19	30	322	11.5	68	5	19
## 20	11	44	9.7	62	5	20
## 21	1	8	9.7	59	5	21
## 22	11	320	16.6	73	5	22
## 23	4	25	9.7	61	5	23
## 24	32	92	12.0	61	5	24
## 25	NA	66	16.6	57	5	25
## 26	NA	266	14.9	58	5	26
## 27	NA	NA	8.0	57	5	27
## 28	23	13	12.0	67	5	28
## 29	45	252	14.9	81	5	29
## 30	115	223	5.7	79	5	30
## 31	37	279	7.4	76	5	31
## 32	NA	286	8.6	78	6	1
## 33	NA	287	9.7	74	6	2
## 34	NA	242	16.1	67	6	3
## 35	NA	186	9.2	84	6	4
## 36	NA	220	8.6	85	6	5
## 37	NA	264	14.3	79	6	6
## 38	29	127	9.7	82	6	7
## 39	NA	273	6.9	87	6	8
## 40	71	291	13.8	90	6	9
## 41	39	323	11.5	87	6	10
## 42	NA	259	10.9	93	6	11
## 43	NA	250	9.2	92	6	12
## 44	23	148	8.0	82	6	13
## 45	NA	332	13.8	80	6	14
## 46	NA	322	11.5	79	6	15
## 47	21	191	14.9	77	6	16
## 48	37	284	20.7	72	6	17
## 49	20	37	9.2	65	6	18
## 50	12	120	11.5	73	6	19
## 51	13	137	10.3	76	6	20
## 52	NA	150	6.3	77	6	21
## 53	NA	59	1.7	76	6	22
## 54	NA	91	4.6	76	6	23
## 55	NA	250	6.3	76	6	24
## 56	NA	135	8.0	75	6	25
## 57	NA	127	8.0	78	6	26
## 58	NA	47	10.3	73	6	27
## 59	NA	98	11.5	80	6	28
## 60	NA	31	14.9	77	6	29
## 61	NA	138	8.0	83	6	30
## 62	135	269	4.1	84	7	1
## 63	49	248	9.2	85	7	2
## 64	32	236	9.2	81	7	3
## 65	NA	101	10.9	84	7	4
## 66	64	175	4.6	83	7	5
## 67	40	314	10.9	83	7	6
## 68	77	276	5.1	88	7	7
## 69	97	267	6.3	92	7	8
## 70	97	272	5.7	92	7	9
## 71	85	175	7.4	89	7	10
## 72	NA	139	8.6	82	7	11

## 73	10	264	14.3	73	7	12
## 74	27	175	14.9	81	7	13
## 75	NA	291	14.9	91	7	14
## 76	7	48	14.3	80	7	15
## 77	48	260	6.9	81	7	16
## 78	35	274	10.3	82	7	17
## 79	61	285	6.3	84	7	18
## 80	79	187	5.1	87	7	19
## 81	63	220	11.5	85	7	20
## 82	16	7	6.9	74	7	21
## 83	NA	258	9.7	81	7	22
## 84	NA	295	11.5	82	7	23
## 85	80	294	8.6	86	7	24
## 86	108	223	8.0	85	7	25
## 87	20	81	8.6	82	7	26
## 88	52	82	12.0	86	7	27
## 89	82	213	7.4	88	7	28
## 90	50	275	7.4	86	7	29
## 91	64	253	7.4	83	7	30
## 92	59	254	9.2	81	7	31
## 93	39	83	6.9	81	8	1
## 94	9	24	13.8	81	8	2
## 95	16	77	7.4	82	8	3
## 96	78	NA	6.9	86	8	4
## 97	35	NA	7.4	85	8	5
## 98	66	NA	4.6	87	8	6
## 99	122	255	4.0	89	8	7
## 100	89	229	10.3	90	8	8
## 101	110	207	8.0	90	8	9
## 102	NA	222	8.6	92	8	10
## 103	NA	137	11.5	86	8	11
## 104	44	192	11.5	86	8	12
## 105	28	273	11.5	82	8	13
## 106	65	157	9.7	80	8	14
## 107	NA	64	11.5	79	8	15
## 108	22	71	10.3	77	8	16
## 109	59	51	6.3	79	8	17
## 110	23	115	7.4	76	8	18
## 111	31	244	10.9	78	8	19
## 112	44	190	10.3	78	8	20
## 113	21	259	15.5	77	8	21
## 114	9	36	14.3	72	8	22
## 115	NA	255	12.6	75	8	23
## 116	45	212	9.7	79	8	24
## 117	168	238	3.4	81	8	25
## 118	73	215	8.0	86	8	26
## 119	NA	153	5.7	88	8	27
## 120	76	203	9.7	97	8	28
## 121	118	225	2.3	94	8	29
## 122	84	237	6.3	96	8	30
## 123	85	188	6.3	94	8	31
## 124	96	167	6.9	91	9	1
## 125	78	197	5.1	92	9	2
## 126	73	183	2.8	93	9	3

```
## 127    91    189  4.6   93    9    4
## 128    47     95  7.4   87    9    5
## 129    32     92 15.5   84    9    6
## 130    20    252 10.9   80    9    7
## 131    23    220 10.3   78    9    8
## 132    21    230 10.9   75    9    9
## 133    24    259  9.7   73    9   10
## 134    44    236 14.9   81    9   11
## 135    21    259 15.5   76    9   12
## 136    28    238  6.3   77    9   13
## 137     9     24 10.9   71    9   14
## 138    13    112 11.5   71    9   15
## 139    46    237  6.9   78    9   16
## 140    18    224 13.8   67    9   17
## 141    13     27 10.3   76    9   18
## 142    24    238 10.3   68    9   19
## 143    16    201  8.0   82    9   20
## 144    13    238 12.6   64    9   21
## 145    23     14  9.2   71    9   22
## 146    36    139 10.3   81    9   23
## 147     7     49 10.3   69    9   24
## 148    14     20 16.6   63    9   25
## 149    30    193  6.9   70    9   26
## 150    NA    145 13.2   77    9   27
## 151    14    191 14.3   75    9   28
## 152    18    131  8.0   76    9   29
## 153    20    223 11.5   68    9   30
```

```
summary(airquality) # Use this command if you don't want to see the whole dataset but just a summary of
```

```
##      Ozone      Solar.R      Wind      Temp
## Min.   : 1.00   Min.   : 7.0   Min.   : 1.700   Min.   :56.00
## 1st Qu.: 18.00   1st Qu.:115.8   1st Qu.: 7.400   1st Qu.:72.00
## Median : 31.50   Median :205.0   Median : 9.700   Median :79.00
## Mean   : 42.13   Mean   :185.9   Mean   : 9.958   Mean   :77.88
## 3rd Qu.: 63.25   3rd Qu.:258.8   3rd Qu.:11.500   3rd Qu.:85.00
## Max.   :168.00   Max.   :334.0   Max.   :20.700   Max.   :97.00
## NA's   :37      NA's   :7
##      Month      Day
## Min.   :5.000   Min.   : 1.0
## 1st Qu.:6.000   1st Qu.: 8.0
## Median :7.000   Median :16.0
## Mean   :6.993   Mean   :15.8
## 3rd Qu.:8.000   3rd Qu.:23.0
## Max.   :9.000   Max.   :31.0
##
```

Our question is: is there a linear relationship between the Ozone measures and the Solar.R measures?

Let us use linear regression to answer this question:

```
lm1=lm(Ozone ~ Solar.R, data=airquality)
```

The summary of this linear regression will return a t-value and a p-value for the intercept and all coefficients.

```
summary(lm1)
```

```
##
## Call:
## lm(formula = Ozone ~ Solar.R, data = airquality)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -48.292 -21.361  -8.864   16.373  119.136
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  18.59873     6.74790   2.756 0.006856 **
## Solar.R       0.12717     0.03278   3.880 0.000179 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 31.33 on 109 degrees of freedom
## (42 observations deleted due to missingness)
## Multiple R-squared:  0.1213, Adjusted R-squared:  0.1133
## F-statistic: 15.05 on 1 and 109 DF,  p-value: 0.0001793
```

The last column  $\text{Pr}(>|t|)$  is the p-value of the t-test. How do we interpret this finding?

The probability to find this data if  $H_0$  is true (i.e. there is no relationship between Ozone and Solar.R) is 0.1%. This is strong evidence against  $H_0$ . Therefore we reject  $H_0$  under  $p < 0.001$ . Please pay close attention to the difference between percent (0-100) and proportions (0-1).

How would we interpret the finding with respect to the linear relationship between the two variables? The interpretation would look like this:

There is a positive linear relationship between Ozone and Solar.R. For a 1-point increase in Solar.R, we would expect a 0.13 increase in Ozone (in a multivariate model we would have to add: “holding all other variables constant”). The associated t-value is 3.880. This t-value implies a p-value of 0.0002. This  $p < 0.001$  corresponds to a type-1 error rate of  $\alpha < 0.001$ , meaning that the relationship is significant at all common levels of statistical significance.

Note that there are four important levels of statistical significance:

$p \leq 0.001$ , corresponds to a type-1 error rate ( $\alpha$ ) of 0.001  $p \leq 0.01$ , corresponds to a type-1 error rate ( $\alpha$ ) of 0.01  $p \leq 0.05$ , corresponds to a type-1 error rate ( $\alpha$ ) of 0.05  $p \leq 0.1$ , corresponds to a type-1 error rate ( $\alpha$ ) of 0.1

If a coefficient has a p-value of  $p < 0.001$ , the linear relationship is significant at all common levels of statistical significance.

Note: If you have a linear regression with multiple independent variables, then the code you need to use looks like the following:

```
lm(y ~ x1 + x2 + x3)
```