# Two-Sided Matching Model

by

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Dissertation submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy in the Department of Department of Political Science
in the Graduate School of Duke University
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# Abstract

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# List of Abbreviations and Symbols

## Abbreviations

| | |
|---|---|
| EM | Expectation Maximization. |
| FDI | Foreign Direct Investment. |
| MCMC | Markov Chain Monte Carlo. |
| MH | Metropolis-Hastings. |
| MLE | Maximum Likelihood Estimation. |
| MNC | Multinational Corporation. |
| MVN | Multivariate Normal. |

# 1

# Introduction

Much of our social, economic, and political life is governed by two-sided matching markets. In these matching markets, actors from two disjoint sets evaluate the characteristics of someone on the other side and voluntarily form a match if both deem each other satisfactory.[1] Marriage is a prominent example of such matching process. Others include the matching between firms and workers, federal judges and law clerks, the *formateur* of a coalition government and other minority parties, or countries and multinational corporations (MNCs) that are looking for a location to invest.

Two-sided matching market is substantively consequential because it often involves scarce, indivisible goods, such as life commitment to a marital partner or political allegiance in a coalition government. It is also intellectually interesting because the market outcome depends on the actions of both sides, demanding a different analytical approach from what's used for one-sided markets.

This chapter will proceed as follows. First, I discuss the game theory literature

[1] Throughout the dissertation, I use "two-sided matching market" and "matching market" interchangeably. On the other hand, note that a two-sided market is not necessarily a matching market (Rysman, 2009).

of two-sided matching models, where much of the terminology and insight originate. I will highlight key results that are relevant to our goal of estimating actors' preference in matching markets. Second, I examine existing empirical studies of matching markets. Where existing studies have not taken into account the market's two-sided nature, I discuss how doing so can improve our understanding of the subject area. Where existing studies do model the two-sided dynamics, I discuss how they may or may not be used to study subjects that political scientists are interested in.

## 1.1 Game theory models of matching markets

Gale and Shapley (1962) was the first to study the matching market, using marriage as an example. In this market, there are two finite and disjoint sets of actors: men and women. Each man has preferences over the women, and vice versa. Each man's preference can be represented as an ordered list, ranking each woman based on how much he likes her.

The outcome of this market is a set of marriages, with none of some of people prefer to remain single. We call such a set of marriages a *matching* $\mu$, which is a one-to-one function that matches a man with a woman. We refer to $\mu(x)$ as the *mate* of $x$. For convenience, we say that if an individual decides to remain single, they are matched with themselves.

We define a matching $\mu$ as *stable* if it cannot be improved by any individual or any pair of agents. A matching can be improved in two ways. First, an individual may prefer to remain single than to be matched with his or her mate $\mu(x)$ under the current matching $\mu$. Second, a man and a woman may prefer to be with one another rather than whom they are currently matched with. Therefore, if a matching is stable, no one has a better option than their current situation.

The first key result from the game theory literature is that for any set of preference, there always exists a stable matching (Gale and Shapley, 1962). The proof is constructive, describing the "deferred acceptance" procedure that is guaranteed to produce a stable matching.[2] This result provides some justifications for us to assume that the matching we observe in real matching market is stable, and that the agents' utility cannot be further improved. Our empirical model of two-sided matching markets thus needs to describe a process that produces a stable matching.

While a central coordinator employing the *deferred acceptance* algorithm is guaranteed to come up with a stable matching, it is unclear whether decentralized markets, such as the labor market or the FDI market, would be able to reach this outcome by themselves.[3] The second key result from the game theory literature is that stable matching in decentralized matching market is indeed possible, even likely. For example, Roth and Vate (1990) show that, starting from an arbitrary matching, the market can converge to a stable matching with probability 1 if we allow random blocking pairs, i.e. two individuals that are not matched but prefer each other to their current match, to break off and form their own match. In addition, Adachi (2003) shows that a random search process, in which pairs of man and woman randomly meet and decide whether each other is better than their current mates, will

[2] The "deferred acceptance" procedure works as follows. In the first stage, every man proposes to his preferred mate. Every woman rejects all of her suitors except the one that she most prefers. However, she does not yet accept her (so far) favorite suitor, but keeps him along. In the second stage, every man that was rejected in the previous round proposes to his second choice. Every woman then picks her favorite from the set of new proposers and the man she keeps along from the previous round. The procedure continues until there is no longer any woman that is unmatched, at which point women finally accept their current favorite choices. (This procedure is called *deferred acceptance* to capture the fact that women defer accepting her favorite choice until the last round in case better options become available.) The resulting match is stable because, throughout the procedure, every woman has received all the offers that would have been made to her, and she has chosen her favorite among all of those offers. If there were any other man that she would prefer to her current match, that man would not have been available to her. Therefore, the final match cannot be further improved by any man or woman.

[3] The deferred acceptance procedure was used in the market for US medical residency with enthusiastic participation from medical students and hospitals. The high participation rate indicates that the matching produced is stable enough to entice students and hospitals away from arranging their own matches outside of the centralized market.

converge towards a stable matching if the search cost is negligible.[4] These results further suggest that the matching we observe in decentralized markets is likely stable. Therefore, our empirical model of matching markets to describe a process that produces a stable matching.

The third key result is that all conclusions regarding the one-to-one matching market (e.g. marriage) generalize to the many-to-one matching market (e.g. college admission, labor market), albeit requiring additional assumptions Roth and Sotomayor (1992). One important assumption is that firms treat workers as substitutes, not complements. In other words, firms never regret hiring a worker even if another worker is no longer available. Therefore, when we conduct empirical analysis of many-to-one markets, we should focus on markets where agents have such "substitutable preference." Otherwise, a stable matching is not guaranteed, agents' utility functions are interdependent, and it becomes unclear what kind of matching process our empirical model should approximate.

## 1.2 Empirical models of matching markets

The game theory literature takes the agents' preference as given and proves the existence of a stable matching. In contrast, empirical models of matching markets takes the observed matching as given and attempt to estimate the agents' preference.

Unfortunately, most extant empirical models fail to adequately account for the structure of a two-sided matching market. Often, researchers simply analyze the market from one side, e.g. estimating a firm's preference by looking at the type of workers it hires. This approach does not take into account the fact that a match depends not only on the agent's preference but also his opportunity. For example, a farm may prefer to hire highly-educated workers but cannot do so because highly-

[4] In this model, searching has a time cost. Thus, negligible search cost is modeled as agents having a time discount close to 1.

educated workers do not want to work on farms. Modeling this interaction between preference and opportunity is the key contribution of this dissertation.

Alternatively, some researchers measure agents' preferences by surveying them directly (Posner, 2001; Sprecher et al., 1994). While this approach circumvents the need to disentangle preference and opportunity, it can only measure agents' *stated* preference. In addition, such surveys require a high data collection effort while data on final matching (e.g. married couples, workers' current job, country location of MNCs) are widely available. This dissertation aims to make use of such available data to estimate agents' *revealed* preference.

Below I discuss existing empirical models of matching markets. First, I discuss two markets of interest to political scientists: the US federal clerkship market and the "market" for forming a coalition government. Researchers in both subject areas have not approached the problem with an empirical model that adequately captures its two-sided dynamics.

Second, I examine models from other disciplines that do take into account the two-sided dynamics of matching markets. I start with machine learning models applied to online marketplaces and dating sites. Then, I discuss the statistical models of the labor market (Logan, 1996) and the marriage market (Logan et al., 2008), which are most relevant to our goal of estimating agents' preference based on observed match data. These statistical models serve as the foundation of my empirical approach.

### *1.2.1 US federal clerkship market*

In the US, graduates at top law schools vie for the best federal clerkships every year. These temporary, one-to-two-year positions are the launching pad for Supreme Court clerkships, prestigious teaching jobs, or employment at top law firms. On the other side, federal judges also compete for the best law graduates, who help reduce the judges' workload from copy-editing to drafting opinions (Gulati and Posner, 2016; Posner, 2001). Because the first clerkship tends to have an outsized ideological influence on law graduates, this matching market has important implications for the polarization of the judicial branch (Ditslear and Baum, 2001; Liptak, 2007).

The market for US federal clerkship has been noted as a classic case of a two-sided market. Clerks look for positions that provide not only prestige and connection but also comfortable quality of life (Posner, 2001). Judges select law graduates based on not only academic credentials but also, some argue, ideology, gender, and race (Slotnick, 1984). This market also suffers from strategic behavior emblematic of a matching market, such as offers being made aggressively early and with a short time to accept (Posner, 2001; Posner et al., 2007).

One approach to estimating the preference of agents in this market is to survey clerks and judges directly (Peppers et al., 2008). However, as discussed, this approach only measures stated preference, which is likely to suffer from social desirability bias when it comes to dimensions that we care about most such as matching based on ideology, gender, or race.

Other approaches estimate revealed preference by using observed hiring outcome. However, no existing study has properly taken into account the two-sided nature of the market, thus confusing the effects of preference and opportunity. For example, Bonica et al. (2017) use political contribution data (DIME dataset) to measure political ideology, then correlate the ideology of the hiring judge and the ideology of his clerks. This approach does not take into account the pool of applicants, leading to conclusions such as conservative judges hire more liberal clerks than conservative clerks (Bonica et al., 2017, 31). This curious finding has a potentially simple explanation: the pool of top law graduates tend to be overwhelmingly liberal, leaving conservative judges with no choice. Despite this issue, the authors proceed to measure judges' ideology by taking the average of their clerks' ideology. Without taking the pool of applicants into account, they may wrongly conclude that conservative judges are more liberal than they actually are.

In another approach, Rozema and Peng (2016) model the process as a discrete choice problem, in which clerks are differentiated products that Supreme Court justices select to maximize their utilities. Their model does not consider what clerks think about the offer because of their focus on Supreme Court clerkships, whose unparalleled prestige ensures that any offer made will be accepted. However, if we want to extend the model to the broader market of federal clerkship, such assumption is untenable.

### *1.2.2 The market for forming a coalition government*

Besides election, government formation is the most consequential political process in determining which government people are subject to. Most extant studies of government formation are either game theoretic models or thick, "inside-the-Beltway" narratives. Potential advances can be made if we consider government formation as a many-to-one matching market, with the *formateur* party on one side and other minority parties on the other.[5]

A two-sided matching model of government formation would complement the game theory literature that models politicians as policy-seeking (as opposed to office-seeking) (Laver, 1998). When politicians are policy-seeking, parties have policy positions that can be modeled as their characteristics. Then, parties choose one another to form a coalition based on their policy positions, akin to men and women choosing one another to form a marriage based on their height or income.[6] As the game theory literature suggests, ideologically compact coalitions are more valuable because they

[5] The *formateur* party could be the one with the procedural power to set up the coalition, e.g. the incumbent party, or the largest party in established coalitions.

[6] In contrast, when politicians are office-seeking, the only coin of the realm is the number of legislative seats that a party controls. It determines both the inclusion of the party in the government and its portfolio allocation. In this framework, concepts like power indices and dominant parties are all about how parties can bring its controlled seats to a coalition to turn it into a winning coalition.

entail a smaller cost in terms of policy compromises (De Swaan, 1973). With the empirical matching model, we can test if parties do indeed prefer others that are ideologically close to themselves.

In addition, an advantage of the two-sided matching approach is its ability to consider multidimensional policy spaces. By considering a party's positions on various policies as their covariates, we would be able to estimate parties' relative preference for ideological proximity across policy dimensions.

#### *1.2.3 The FDI market*

To be introduced here, or kept until its own empirical chapter?

#### *1.2.4 Recommender system for online two-sided markets*

In recent years, the Internet underwent a proliferation of two-sided matching markets such as online marketplaces (e.g. AirBnB), dating sites (e.g. eHarmony), or job board (e.g. Elance). To help their users discover a match quicker, these sites often build a recommender system that suggests potential matches.[7] To maximize user engagement and profitability, these sites are incentivized to make recommendations that resemble a stable matching so that their users get the best match possible. And fo find the stable matching, they have to first estimate the preferences of their users.

While most of these algorithms are proprietary, some academic publications have addressed this problem. An interesting approach is the paper by Tu et al. (2014), which uses the Latent Dirichlet allocation (LDA) model to uncover the latent types of users based on their activities on an online dating platform.[8] In the original

[7] To clarify, the term "recommender system" typically refers to systems that recommend items to users based on the reviews of users like them. That is not what we are discussing here. Instead, we focus on matching markets where the recommender system recommends users to one another.

[8] Besides Tu et al. (2014), Hitsch et al. (2010); Goswami et al. (2014) are two other attempts to estimate users' preference in online matching markets. However, these papers take a simple one-sided approach, ignoring the interplay between preference and opportunity. Therefore, I don't discuss them further here.

application of LDA model in topic modeling, each document is a mixture of latent topics, and each topic is a distribution over words. In this application, each user is a mixture of latent "types," and each type is a distribution signifying relative preference over various mates' features. For example, the "outdoor type" may have higher preference for athleticism or dog ownership over other traits.

While the LDA model works well for the online dating market, it is not applicable to most social science problems for two reasons. First, this model requires data of users reaching out to multiple partners rather than just the final match. Second, while the LDA model uncovers users' latent types, most social scientists want to estimate the preference of specific, known types (e.g. how different regime types may prefer different characteristics of an MNC).

#### *1.2.5 Two-sided models for the labor and marriage markets*

To be introduced here, or in their own chapters?

## 1.3 Conclusion

Roadmap for the rest of the dissertation

# 2

# Two-Sided Matching Model

Here I present a behavioral model of the two-sided matching market, focusing on the case of many-to-one matching, proposed by Logan (1996). For easier exposition, throughout the chapter I will use the example of the labor market, where many workers can be matched to one firm.

We assume that the matching process in the labor market happens in two stages. In the first stage, each firm evaluates each worker in the sample, then decides whether to hire that worker or not. At the end of this stage, each worker will have received a set of offers from firms, which we call his *opportunity set*. In the second stage, each worker evaluates the firms in his opportunity set and chooses the firm that he likes best. This constitutes the final, observed match between a worker and a firm. This is a many-to-one matching problem because a firm can make offers to multiple workers, none, some, or all of which can be accepted by workers.

Our model only needs data on 1) the covariates of firms and workers, and 2) the job that workers accept. Such data is widely available in many social science surveys of the job market. Importantly, we do not need data on the opportunity set. Therefore, our model obviates the need to follow the matching process and record who makes offer to whom, which is rarely possible for researchers.

If we assume that firms and workers are utility-maximizing agents, at the end of the matching process, no firm or worker would voluntarily change their final matches. As discussed in Section 1.1, this property is called *stability* in the game theoretic two-sided matching literature. We want our model to have this property because matching markets tend to produce stable matching. Indeed, Roth and Sotomayor (1992) show that for any given set of preferences, a stable match always exist. Furthermore, Roth and Vate (1990) and Adachi (2003) show that a decentralized market with agents making independent, utility-maximizing decisions can also reach a stable match by itself.

This stability property does not imply that the matches will never change. Indeed, if actors' preference shifts, their characteristics change, or new actors enter the market, the matches will also change as a result of actors' recalculating their utility and adjusting their decisions. Therefore, since we are estimating actors' preference using only a snapshot of matching market, we are making the assumption that on a systemic level, the average characteristics of the actors and their preference remain sufficiently static for our estimates to be meaningful.

This chapter will proceed as follows. First, I discuss the utility model for how firms make offers to workers. Second, I discuss the utility model for how workers choose the best offer among those extended by firms. Third, I show why estimating the model with MLE is difficult, and how we can use Bayesian MCMC approach for estimation. Fourth, I analyze US labor data and demonstrate how to interpret the model's result.

## 2.1 Modeling firms' decision making

A firm $j$'s decision on whether to hire worker $i$ rests on two utility functions. First, firm $j$'s utility for hiring worker $i$ is:

$$U_j(i) = \boldsymbol{\beta}_j' X_i + \epsilon_{1ij} \tag{2.1}$$

where $\beta_j$ is a vector of firm $j$'s preference for worker characteristics, $x_i$ is a vector of worker $i$'s measured values on those characteristics, and $\epsilon_{1ij}$ is the unobserved component that influences firm $j$'s utility.

On the other hand, the utility of not hiring worker $i$ is:

$$U_j(\neg i) = b_j + \epsilon_{0ij} \tag{2.2}$$

where $b_j$ is the baseline utility of firm $j$, and $\epsilon_{0ij}$ is the unobserved component that influences firm $j$'s utility.

Firm $j$ will make an offer to hire worker $i$ if $U_j(i) > U_j(\neg i)$. Relevant worker characteristics (i.e. $X_i$) that a firm may consider are age, education, or experience. The corresponding $\beta$'s represent the firm's preference for these characteristics.

This model makes two important assumptions about firms' hiring process. First, whether a firm decides to hire worker A depends on the characteristics of worker A alone, and it will continue to hire worker A even if worker B is no longer available. In other words, firms regard workers as substitutes rather than complements.[1] This assumption is not universally true. A Hollywood producer may want to hire two specific lead actors for their chemistry, and if one is unavailable, the other also has to be replaced. However, for large firms where workers are closer to swappable cogs than unique superstars, this assumption is reasonable.

Second, the model assumes that the utility of hiring a worker does not depend on how many other workers accept the offer. In other words, the firm is large enough

[1] In the terminology of Roth and Sotomayor (1992), firms are assumed to have "substitutable preference," or firms' preference is assumed to have the property of substitutability. As discussed in Section 1.1, this assumption is necessary to prove the existence of stable matching in the case of many-to-one matching.

to employ all the workers to whom it extends offer without feeling the effect of diminishing marginal productivity of labor. This assumption is less restrictive than it may seem. Indeed, we can model the fact that the workers under consideration are less productive than the previous batch of workers by allowing firm $j$ to have a high baseline utility $b_j$. Therefore, this assumption does not require that there is never any diminishing marginal productivity of labor, only that there is negligible diminishing effect between the first and the last of the workers under consideration. This assumption is a reasonable approximation if the firm's labor force is large compared to the number of workers being considered.[2]

In addition to two above assumptions about the process of firm's decision making, we make three parametric assumptions that are standard in the discrete choice literature. First, we assume a linear utility function. Second, we assume that the error terms $\epsilon_{1ij}, \epsilon_{0ij}$ are uncorrelated with one another and across firms. Third, we assume that the as error terms $\epsilon_{1ij}, \epsilon_{0ij}$ follow the Gumbel distribution.[3] The choice of the Gumbel distribution is largely motivated by convenience since it allows us to derive the probability of firm $j$ making an offer to worker $i$ as the familiar binomial logit form:

$$
\begin{aligned}
Pr(o_{ij} = 1) &= Pr(U_j(i) > U_j(\neg i)) && (2.3)\\
&= Pr(\epsilon_{0ij} - \epsilon_{1ij} < \boldsymbol{\beta}'_j X_i - b_j) && (2.4)\\
&= \frac{\exp(\boldsymbol{\beta}'_j X_i)}{1 + \exp(\boldsymbol{\beta}'_j X_i)} && (2.5)
\end{aligned}
$$

[2] While not concerned with diminishing marginal productivity, Roth and Sotomayor (1992) also assume that firms' quota, i.e. the number of workers they can accept, is sufficiently large to hire everyone in the set of workers under consideration. This assumption simplifies the proof that a stable match always exists in the case of many-to-one matching.

[3] The Gumbel distribution is very similar to the normal, only with a slightly fatter tail that allows for slightly more extreme variation in the unobserved utility. Its density function is $\exp^{-(x+\exp^{-x})}$, with mode 0, mean 0.5772, and fixed variance $\frac{\pi^2}{6}$. In practice, the difference between using Gumbel and independent normal error terms is small (Train, 2009).

The term $b_j$ is absorbed into $\boldsymbol{\beta}$ when we add an intercept term to the covariate matrix $X$.

Once firms have made their offers, each worker $i$ will have a set of offers from which to pick her favorite. We call this set of offers the *opportunity set* of worker $i$, denoted $O_i$. Since unemployment is always an available option, every opportunity set includes unemployment as an "offer."[4]

The probability of worker $i$ obtaining the opportunity set $O_i$ is:

$$p(O_i|\boldsymbol{\beta}) = \prod_{j \in O_i} p(o_{ij} = 1|\boldsymbol{\beta}) \prod_{j \notin O_i} p(o_{ij} = 0|\boldsymbol{\beta}) \tag{2.6}$$

$$= \prod_{j \in O_i} \frac{\exp(\boldsymbol{\beta_j'} X_i)}{1 + \exp(\boldsymbol{\beta_j'} X_i)} \prod_{j \notin O_i} \frac{1}{1 + \exp(\boldsymbol{\beta_j'} X_i)} \tag{2.7}$$

## 2.2 Modeling workers' decision making

Worker $i$'s utility for the accepting an offer from firm $j$ is:

$$V_i(j) = \alpha' W_j + v_{ij} \tag{2.8}$$

where $\alpha$ is a vector of workers' preference for relevant characteristics of firms, $W_j$ is a vector of firm $j$'s measured values on those characteristics, and $v_{ij}$ is the unobserved component that influences worker $i$'s utility.

Worker $i$ evaluates all the firms in her opportunity set and selects the offer that brings the highest utility. This decision of worker $i$ concludes the matching process, resulting in the observed final match between a worker and her chosen firm in our data.

[4] In our model setup, firms and workers decide sequentially, with firms making offers first in order for workers to have opportunity sets to choose from. While firms and workers in real life certainly do not act in this sequential manner, the idea of the opportunity set is still applicable. Workers in the real labor market may not know their exact set of offers, but they can certainly guess which firms are within their reach based on their characteristics and on guesses about firms' preference.

We make two assumptions in modeling the worker's decision making. First, for simplicity, we assume that all workers share the same set of preferences–hence $\alpha$ does not have a subscript $i$. The model can be extended so that there is heterogeneous preference among workers, either by estimating a separate model for each worker type (i.e. no pooling) or by building a hierarchical model for worker preference (i.e. partial pooling).

Second, we assume that the error term $v_{ij}$ are uncorrelated across $j$. In other words, the unobserved factors in the utility of one job offer is uncorrelated to the unobserved factors in the utility of another job offer.[5] This assumption is most likely not true: if worker $i$ values some unobserved factors of an offer, she is likely to consider those same factors in another offer as well. The hope is that we have modeled the observed portion sufficiently well that the remaining unobserved factors are close to white noise. In any case, this issue afflicts any application of discrete choice models and is not unique to our setup.[6]

Similar to our model of firm's utility, our model of worker's utility has three additional parametric assumptions that are standard in the literature. First, we assume that utility is linear. Second, the error term $v_{ij}$ are uncorrelated across $i$. Third, we model $v_{ij}$ having a Gumbel distribution so that the probability that worker $i$ will accept the offer of firm $j$ out of all the offers in its opportunity set $O_i$ takes the conditional logit form (Cameron and Trivedi, 2005):

[5] This assumption also gives rise to the Independence of the Irrelevant Alternatives (IIA) property. IIA implies that the relative odds of choosing between two alternatives depend only on the two alternatives under consideration. It does not depend on whether other alternatives are available or what their characteristics may be. Hence, other alternatives are considered "irrelevant."

[6] The discrete choice literature has developed solutions for such correlated error structure, such as nested logit, probit, and mixed logit, that can be applied here if we suspect that the unobserved portion is strongly correlated.

$$p(A_i = a_i | O_i, \alpha_i) = \frac{\exp(\alpha' W_{a_i})}{\sum\limits_{j: j \in O_i} \exp(\alpha' W_j)} \tag{2.9}$$

where $a_i$ is the index of the firm that $i$ accepts to to work for. Unemployment is indexed as 0.

## 2.3 Model estimation

Our goal is to estimate the preference of firms and workers, i.e. $\beta_j$ and $\alpha$. The key insight is that, conditional on the opportunity set being observed, the model of firms' and workers' decision making is a straightforward application of the binary logit and conditional logit model. Both models can be estimated with familiar tools like Maximum Likelihood Estimation (MLE).

However, in most social science research problems, the researcher only observes the final match $A$ and not the opportunity set $O$. For example, labor market data typically does not include the set of offers a worker received (or would have received if she had applied), while data on her current job is widely available. Similarly in the marriage market or the FDI market, researchers often do not have the data on the offers being made, and only observe the final matching between men and women (i.e. who is married to whom) and between MNCs and countries (i.e. which factory is located where).

Logan (1998)'s solution to this problem is to use the Expectation-Maximization (EM) algorithm, an iterative method capable of finding the maximum likelihood estimates when the model depends on unobserved latent variables (i.e. the unobserved opportunity set in this case) (Dempster et al., 1977). Our innovation is to estimate the model using a Bayesian MCMC approach, which offers several advantages. First, our MCMC approach produces the full posterior distribution, making inference easy.

In contrast, EM only produces point estimates out of the box.[7] Second, our MCMC approach can be faster than EM when the latent variable, i.e. the opportunity set, is high dimensional (Rydén, 2008).[8] Third, within the Bayesian framework, we can naturally put a hierarchical structure on firms' preference. This allows us to borrow information across firms, producing more precise estimates even when there is not a lot of data for a specific firm.

The rest of this section describes how we conduct model estimation.

### *2.3.1 Estimating the model using Metropolis-Hastings*

We are interested in the posterior distribution of workers' and firms' preference given the observed final match, i.e. $p(\alpha, \beta|A)$. Unconditioned on the opportunity set, this posterior is difficult to derive or sample from. Therefore, we instead sample from the augmented posterior $p(\alpha, \beta, O|A)$, whose density is much simpler to derive.[9] Specifically,

$$p(\alpha, \beta, O|A) = \frac{p(A|\alpha, \beta, O)p(\alpha, \beta, O)}{p(A)} \tag{2.10}$$

$$\propto p(A|O, \alpha)p(O|\beta)p(\alpha)p(\beta) \tag{2.11}$$

where $p(A|O, \alpha)$ is derived in (2.9), $p(O|\beta)$ is derived in (2.7), $p(\alpha)$ and $p(\beta)$ are prior distributions for $\alpha$ and $\beta$. A key insight of this equation is that the acceptance of offers, i.e. $p(A|O, \alpha)$, depends only on the opportunity set and on the

[7] Jamshidian and Jennrich (2000) propose a method for estimating the standard error of EM estimates. However, for hypothesis testing, we need further assumptions about the distribution of the EM estimates.

[8] Indeed, our opportunity set $O$ is a $(I \times J)$ matrix of 0s and 1s, where $I$ is the number of workers and $J$ is the number of firms. Thus, there are $2^{IJ}$ potential values for the opportunity set, which quickly becomes untenable even for a small number of $I$ and $J$. The high dimension of $O$ forces Logan (1998) to reduce the data dimension by aggregating 17 employers in the data into 5 employer types, e.g. professional or blue collar jobs.

[9] See Tanner and Wong (1987) for a discussion of such data augmentation techniques.

workers' preference. Similarly, the opportunity sets, i.e. $p(O|\beta)$, depend only on firms' preference.

Because the opportunity set $O$ is a discrete matrix of 0's and 1's, there is not any convenient conjugate model for (2.11), making Gibbs sampling impossible. Therefore, we use Metropolis-Hastings instead, a technique to sample from an arbitrary distribution $p(\theta)$ using the following steps:

1. Start from an arbitrary value of $\theta$
2. Generate a proposal value $\theta^*$ from the proposal distribution $q(\theta^*|\theta)$
3. Calculate the acceptance ratio $MH_\theta = \frac{p(\theta^*)q(\theta|\theta*)}{p(\theta)q(\theta^*|\theta)}$
4. Accept the proposed value $\theta^*$ with probability $\max(1, MH_\theta)$
5. Repeat step 2-4 until convergence

In our case, we will use symmetric proposal distributions, i.e. $p(\theta^*|\theta) = p(\theta|\theta^*) \forall \theta, \theta^*$, so that the MH acceptance ratio simplifies to $MH_\theta = \frac{p(\theta^*)}{p(\theta)}$. In addition, because preference parameters tend to be correlated, we use an adaptive proposal distribution so that our MCMC samples have a faster convergence rate (Haario et al., 1999, 2001).[10]

Below we describe how to sample from the posterior of each parameter in the model using the Metropolis-Hastings (MH) algorithm. More detailed derivation of the Metropolis acceptance ratio is included in Appendix A. We ensure that our derivation and implementation of the acceptance ratio is correct using the unit-testing approach suggested by Grosse and Duvenaud (2014).[11]

[10] Description of the Adaptive Metropolis procedure?

[11] Describe the unit-testing framework to ensure the correctness of MCMC code?

### 2.3.2 *Posterior of the opportunity set $p(O|A, \alpha, \beta)$*

For each worker $i$, we propose a new value $O_i^*$ by flipping random cells in the current value $O_i$ from 0 to 1 and 1 to 0. Substantively, this is equivalent to perturbing the opportunity set by randomly making new offers or withdrawing existing offers. Note that this proposal distribution is indeed symmetric because proposing $O_i^*$ from $O_i$ and proposing $O_i$ from $O_i^*$ both involve flipping the same cells in the opportunity set. Hence, $p(O_i^*|O_i) = p(O_i|O_i^*) =$ the probability of selecting these particular cells out of the opportunity set.

The Metropolis acceptance ratio for the proposed opportunity set $O_i^*$ is

$$MH_O = \frac{p(O_i^*|A_i, \alpha, \boldsymbol{\beta})}{p(O_i|A_i, \alpha, \boldsymbol{\beta})} \tag{2.12}$$

$$= \frac{\sum\limits_{j:j \in O_i} \exp(\alpha' W_j)}{\sum\limits_{j:j \in O_i} \exp(\alpha' W_j) \pm \exp(\alpha' W_{j*})} \times \exp(\pm \boldsymbol{\beta}'_{j*} X_i) \tag{2.13}$$

where $\pm$ evaluates to $+$ if $j^*$ is a new offer being added to the current opportunity set, and evaluates to $-$ if $j^*$ is an existing offer being withdrawn from the current opportunity set.

To understand the intuition behind this formula for $MH_O$, consider the scenario in which we propose a new opportunity set for worker $i$ by adding an offer from firm $j$. Since worker $i$ now has one more choice to choose from, it becomes less likely that worker $i$'s accepted job is the best choice. This makes the proposed opportunity set less consistent with the observed data than the current opportunity set, and $MH_O$ should decrease accordingly. This is reflected in the formula for $MH_O$ by the $\exp(\alpha' W_{j*})$ term in the denominator.

On the other hand, whether we should add the offer to the opportunity set also depends on firm $j$'s preference for worker $i$. If hiring worker $i$ bring firm $j$ net positive utility (i.e. $\boldsymbol{\beta}'_{j*}X_i > 0$), we should add the offer. This is reflected in the formula for $MH_O$ by the multiplier $\exp(\boldsymbol{\beta}'_{j*}X_i)$, which is larger than 1 when $\boldsymbol{\beta}'_{j*}X_i > 0$.

### 2.3.3 *Posterior of firms' preference $p(\alpha|A, O, \beta)$*

At the beginning of the MCMC chain, we propose a new $\alpha^*$ using a Normal proposal distribution centered on the current value $\alpha$ with a hand-tuned diagonal covariance matrix. Later in the MCMC chain, the covariance matrix of the proposal distribution is adapted based on past samples to take into account the correlations across preference parameters (Haario et al., 2001).

The Metropolis acceptance ratio for the proposed $\alpha^*$ is[12]

$$\begin{aligned}\log MH_\alpha = \sum_i \Big[ &(\alpha^* - \alpha)' W_{a_i} + \\ &\log\left(\sum_{j:j\in O_i} \exp(\alpha' W_j)\right) - \log\left(\sum_{j:j\in O_i} \exp(\alpha^{*\prime} W_j)\right) + \\ &\log p(\alpha^*) - \log p(\alpha)\Big] \end{aligned} \tag{2.14}$$

### 2.3.4 *Posterior of workers' preference $p(\beta|A, O, \alpha)$*

We propose a new $\boldsymbol{\beta}^*$ using a Normal, adaptive proposal distribution similar to $\alpha$. Because $\boldsymbol{\beta}$ is high dimensional, with one set of $\beta$ for each employer, in each MCMC iteration we randomly choose and update only a part of $\boldsymbol{\beta}$.

The Metropolis acceptance ratio for the proposed $\beta$ is

$$\begin{aligned}\log MH_\beta = \sum_i \left[ \sum_{j\in O_i} \left(\beta_j^{*\prime} X_i - \beta_j' X_i\right) + \sum_j \left(\log(1+\exp(\beta_j' X_i)) - \log(1+\exp(\beta_j^{*\prime} X_i))\right) \right] \\ + \log p(\boldsymbol{\beta}^*) - \log p(\boldsymbol{\beta}) \end{aligned} \tag{2.15}$$

[12] We log-transform the Metropolis acceptance ratio for better numerics.

### 2.3.5 *Posterior of $\beta$'s hyperparameters $\mu_\beta, \tau_\beta$*

As discussed above, the Bayesian approach to estimating our two-sided model allows us to put a hierarchical structure on the preference parameter. Here, we model firms' preference $\boldsymbol{\beta}$ as being drawn from the multivariate normal distribution $MVN(\mu_\beta, \tau_\beta)$, where $\mu_\beta$ is the mean and $\tau_\beta$ is the precision.

When the prior $p(\beta)$ is also normal, we have a conjugate multivariate normal model, where $\mu_\beta$ and $\tau_\beta$ are the parameters while $\beta$ is considered the "data".

Since the model is conjugate, we can sample from the posterior of $\mu_\beta$ and $\tau_\beta$ with Gibbs sampling. Their full conditional distribution of $\mu_\beta$ is:

$$p(\mu_\beta) \sim MVN(\mu_0, \Sigma_0) \tag{2.16}$$

$$p(\mu_\beta|\beta, \tau_\beta) \sim MVN(m, V) \text{ where} \tag{2.17}$$

$$V = (\Sigma_0^{-1} + n\tau_\beta)^{-1} \tag{2.18}$$

$$m = (\Sigma_0^{-1} + n\tau_\beta)^{-1}(\Sigma_0^{-1}\mu_0 + n\tau_\beta\bar{\beta}) \tag{2.19}$$

The full conditional distribution of $\tau_\beta$ is:

$$p(\tau_\beta) \sim \text{Wishart}(\nu_0, S_0^{-1}) \tag{2.20}$$

$$p(\tau_\beta|\beta, \mu_\beta) \sim \text{Wishart}(\nu, S^{-1}) \text{ where} \tag{2.21}$$

$$\nu = \nu_0 + n \tag{2.22}$$

$$S^{-1} = \left(S_0 + \sum(\beta - \mu_\beta)(\beta - \mu_\beta)'\right)^{-1} \tag{2.23}$$

## 2.4 Results for US labor data

To be written ...

# Appendix A

## Derivation of the Metropolis-Hastings Acceptance Ratio

#### *A.0.1 Opportunity sets O*

Target distribution for a firm $i$

$$p(O_i|A_i,\alpha,\boldsymbol{\beta}) = \frac{p(O_i,A_i,\alpha,\boldsymbol{\beta})}{p(A_i,\alpha,\boldsymbol{\beta})} \tag{A.1}$$

$$MH_O = \frac{p(O_i^*|A_i,\alpha,\boldsymbol{\beta})}{p(O_i|A_i,\alpha,\boldsymbol{\beta})} = \frac{p(O_i^*,A_i,\alpha,\boldsymbol{\beta})}{p(A_i,\alpha,\boldsymbol{\beta})} \times \frac{p(A_i,\alpha,\boldsymbol{\beta})}{p(O_i,A_i,\alpha,\boldsymbol{\beta})} \tag{A.2}$$

$$= \frac{p(O_i^*,A_i,\alpha,\boldsymbol{\beta})}{p(O_i,A_i,\alpha,\boldsymbol{\beta})} \tag{A.3}$$

$$= \frac{p(A_i|O_i^*,\alpha)p(O_i^*|\boldsymbol{\beta})}{p(A_i|O_i,\alpha)p(O_i|\boldsymbol{\beta})} \tag{A.4}$$

$$\tag{A.5}$$

where the factorization of the likelihood in (A.4) is due to the fact that the acceptance of firm $i$ only depends on what is offered to it and what is its preference, $p(A_i|O_i^*,\alpha)$; what is offered to $i$ depends on the preferences of all countries, $p(O_i^*|\boldsymbol{\beta})$.

If we plug in (2.9) and (2.7)

$$\frac{p(O_i^*|A_i,\alpha,\boldsymbol{\beta})}{p(O_i|A_i,\alpha,\boldsymbol{\beta})} = \frac{\sum\limits_{j:j\in O_i}\exp(\alpha' W_j)}{\sum\limits_{j:j\in O_i}\exp(\alpha' W_j) + \exp(\alpha' W_{j*})} \times \exp(\boldsymbol{\beta}'_{j*}X_i) \tag{A.6}$$

where $j^*$ is the index of the newly sampled job. This is the case when the newly proposed job is not already offered, so it's added to the opportunity set.

When the newly proposed job is already offered, so it's removed from the opportunity set, we have

$$\frac{p(O_i^*|A_i,\alpha,\boldsymbol{\beta})}{p(O_i|A_i,\alpha,\boldsymbol{\beta})} = \frac{\sum\limits_{j:j\in O_i}\exp(\alpha' W_j)}{\sum\limits_{j:j\in O_i}\exp(\alpha' W_j) - \exp(\alpha' W_{j*})} \times \exp(-\boldsymbol{\beta}'_{j*}X_i) \tag{A.7}$$

*A.0.2 Workers' parameters, $\alpha$*

Target distribution:

$$p(\alpha|A,O,\boldsymbol{\beta}) = \frac{p(O,A,\alpha,\boldsymbol{\beta})}{p(A,O,\boldsymbol{\beta})} \tag{A.8}$$

Metropolis-Hasting acceptance ratio:

$$MH_\alpha = \frac{p(\alpha^*|A,O,\boldsymbol{\beta})}{p(\alpha|A,O,\boldsymbol{\beta})} = \frac{p(A_i|O_i,\alpha^*)p(O_i|\boldsymbol{\beta})}{p(A_i|O_i,\alpha)p(O_i|\boldsymbol{\beta})} \tag{A.9}$$

$$= \frac{p(A_i|O_i,\alpha^*)}{p(A_i|O_i,\alpha)} \tag{A.10}$$

where (A.10) is due to the flat prior (so $\frac{p(\alpha^*)}{p(\alpha)} = 1$) and the symmetric proposal distribution (so $\frac{p(\alpha^*|\alpha)}{p(\alpha|\alpha^*)} = 1$)

If we plug in (2.9),

$$MH_\alpha = \prod_i \left[ \frac{\exp(\alpha^{*\prime} W_{a_i})}{\exp(\alpha' W_{a_i})} \times \frac{\sum\limits_{j:j\in O_i} \exp(\alpha' W_j)}{\sum\limits_{j:j\in O_i} \exp(\alpha^{*\prime} W_j)} \right] \tag{A.11}$$

$$= \prod_i \left[ \exp(\epsilon_\alpha' W_{a_i}) \times \frac{\sum\limits_{j:j\in O_i} \exp(\alpha' W_j)}{\sum\limits_{j:j\in O_i} \exp(\alpha^{*\prime} W_j)} \right] \tag{A.12}$$

Finally, we log transform the MH acceptance ratio for numerical stability.

$$\log MH_\alpha = \sum_i \left[ \epsilon_\alpha' W_{a_i} + \log\left( \sum_{j:j\in O_i} \exp(\alpha' W_j) \right) - \log\left( \sum_{j:j\in O_i} \exp(\alpha^{*\prime} W_j) \right) \right] \tag{A.13}$$

*A.0.3 Firms' parameters, $\beta$*

Target distribution:

$$p(\boldsymbol{\beta}|A, O, \alpha) = \frac{p(O, A, \alpha, \boldsymbol{\beta})}{p(A, O, \alpha)} \tag{A.14}$$

Metropolis-Hasting acceptance ratio:

$$MH_\beta = \frac{p(\beta^*|A, O, \alpha)}{p(\beta|A, O, \alpha)} = \frac{p(A_i|O_i, \alpha)p(O_i|\boldsymbol{\beta}^*)p(\boldsymbol{\beta}^*|\mu_\beta, \tau_\beta)}{p(A_i|O_i, \alpha)p(O_i|\boldsymbol{\beta})p(\boldsymbol{\beta}|\mu_\beta, \tau_\beta)} \tag{A.15}$$

$$= \frac{p(O_i|\boldsymbol{\beta}^*)p(\boldsymbol{\beta}^*|\mu_\beta, \tau_\beta)}{p(O_i|\boldsymbol{\beta})p(\boldsymbol{\beta}|\mu_\beta, \tau_\beta)} \tag{A.16}$$

where (A.15) is due to the flat prior on $\beta$ and the symmetric proposal distribution. We plug in (2.7),

$$MH_\beta = \prod_i \left[ \prod_{j \in O_i} \frac{\exp(\beta_j^{*\prime} X_i)}{\exp(\beta_j' X_i)} \times \prod_j \frac{1 + \exp(\beta_j^{*\prime} X_i)}{1 + \exp(\beta_j' X_i)} \right] \times \frac{MVN(\boldsymbol{\beta}^* | \mu_\beta, \tau_\beta)}{MVN(\boldsymbol{\beta} | \mu_\beta, \tau_\beta)} \quad \text{(A.17)}$$

$$\log MH_\beta = \sum_i \left[ \sum_{j \in O_i} \beta_j^{*\prime} X_i - \beta_j' X_i + \sum_j \log(1 + \exp(\beta_j^{*\prime} X_i)) - \log(1 + \exp(\beta_j' X_i)) \right] \quad \text{(A.18)}$$

$$+ \log MVN(\boldsymbol{\beta}^* | \mu_\beta, \tau_\beta) - \log MVN(\boldsymbol{\beta} | \mu_\beta, \tau_\beta)$$

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