Stochastic Models for Blockchain Analysis

Pierre-O. Goffard

Université de Strasbourg
goffard@unistra.fr

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Agenda

1. Introduction
2. Consensus protocol
3. Stochastic models

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1 Introduction

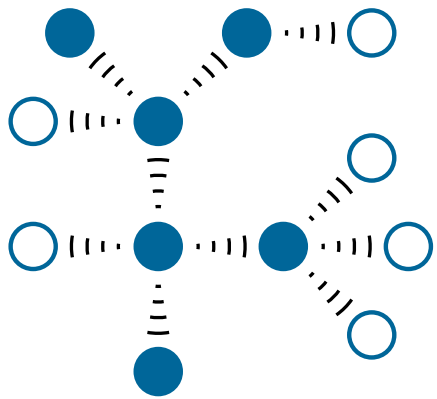
2 Consensus protocol

3 Stochastic models

Blockchain

A data ledger made of a sequence of blocks maintained by a achieving consensus in a Peer-To-Peer network.

- Decentralized
- Public/private
- Permissionned/permissionless
- Immutable
- Incentive compatible



Consensus protocols

The mechanism to make all the nodes agree on a common data history.

The four dimensions of blockchain systems analysis

1. Efficiency (Queueing theory)
   - Throughputs
   - Transaction confirmation time
2. Decentralization (Entropy)
   - Fair distribution of the accounting right
3. Security (Fluctuation Theory)
   - Resistance to attacks
4. Incentive mechanism (Risk theory)
   - Incentive for the nodes to contribute

Applications of blockchain : Cryptocurrency

S. Nakamoto, “Bitcoin : A peer-to-peer electronic cash system.” Available at https://bitcoin.org/bitcoin.pdf, 2008.

- Transaction anonymity
- No need for a thrusted third party

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Consensus protocol

Definition

Algorithm to allows the full nodes to agree on a common data history

It must rely on the scarce resources of the network

- bandwidth
- computational power
- storage (disk space)

Types of consensus protocols

1. Voting based

L. Lamport, R. Shostak, and M. Pease, "The byzantine generals problem," *ACM Transactions on Programming Languages and Systems*, pp. 382–401, July 1982.

⚠ Communication overhead

⚠ Denial of service

2. Leader based
   - Proof-of-Work (computational power)
   - Proof-of-Capacity and Proof-of-Spacetime (storage)
   - Proof-of-Interaction (bandwidth, locally grown)
   - Proof-of-Stake (tokens)

Proof-of-Work

Consensus protocol

Objective

Elect a leader based on computational effort to append the next block.

What's inside a block ?

Consensus protocol

A block consists of

- a header
- a list of "transactions" that represents the information recorded through the blockchain.

The header usually includes

- the date and time of creation of the block,
- the block height which is the index inside the blockchain,
- the hash of the block
- the hash of the previous block.

Question

What is the hash of a block ?

`https://blockchain.info/rawblock/0`

Cryptographic Hash function

Consensus protocol

A function that maps data of arbitratry size (message) to a bit array of fixed size (hash value)

$$h : \{0,1\}^* \mapsto \{0,1\}^d.$$

A good hash function is

- deterministic
- quick to compute
- One way
  - ↪ For a given hash value $\overline{h}$ it is hard to find a message m such that

$$h(m) = \overline{h}$$

- Colision resistant
  - ↪ Impossible to find m_1 and m_2 such that

$$h(m_1) = h(m_2)$$

- Chaotic

$$m_1 \approx m_2 \Rightarrow h(m_1) \neq h(m_2)$$

SHA-256

Consensus protocol

The SHA-256 function which converts any message into a hash value of 256 bits.

Example

The hexadecimal digest of the message

```
Is DeFi the future?
```

is

```
60a147c28568dc925c347bce20c910ef90f3774e2501ac63344f3411b6a6bf79
```

Hidden prediction

Consensus protocol

Matt Levine 
@matt_levine

Here is a SHA-256 hash of a prediction I am making:
64b70b0494580b278d7f1f551d482a3fb952a4b018b43090ffeb87b662d34847.

M. Levine, "The crypto story." Bloomberg business week, Oct. 2022.

Mining a block

Consensus protocol

```
Block Hash: 1fc23a429aa5aaf04d17e9057e03371f59ac8823b1441798940837fa2e318aaa
Block Height: 0
Time:2022-02-25 12:42:04.560217
Nonce:0
Block data: [{'sender': 'Coinbase', 'recipient': 'Satoshi', 'amount': 100, 'fee': 0}, {'sen
der': 'Satoshi', 'recipient': 'Pierre-O', 'amount': 5, 'fee': 2}]
Previous block hash: 0
Mined: False
--------------
```

Figure – A block that has not been mined yet.

Mining a block

Consensus protocol

The maximum value for a 256 bits number is

$$T_{\max} = 2^{256} - 1 \approx 1.16e^{77}.$$

Mining consists in drawing at random a nonce

$$\text{Nonce} \sim \text{Unif}(\{0, \ldots, 2^{32} - 1\}),$$

until

$$h(\text{Nonce}|\text{Block info}) < T,$$

where T is referred to as the target.

Difficulty of the cryptopuzzle

$$D = \frac{T_{\max}}{T}.$$

`https://pierre-olivier.goffard.me/bitcoin-hashing/`

Mining a block

Consensus protocol

If we set the difficulty to $D = 2^4$ then the hexadecimal digest must start with at least 1 leading 0

```
Block Hash: 0869032ad6b3e5b86a53f9dded5f7b09ab93b24cd5a79c1d8c81b0b3e748d226
Block Height: 0
Time:2022-02-25 13:41:48.039980
Nonce:2931734429
Block data: [{'sender': 'Coinbase', 'recipient': 'Satoshi', 'amount': 100, 'fee': 0}, {'sen
der': 'Satoshi', 'recipient': 'Pierre-O', 'amount': 5, 'fee': 2}]
Previous block hash: 0
Mined: True
--------------
```

Figure – A mined block with a hash value having on leading zero.

The number of trial is geometrically distributed

- Exponential inter-block times
- Lenght of the blockchain = Poisson process

Conflict resolution in blockchain

Consensus protocol

Fork

A fork arises when there is a disagreement between the nodes resulting in several branches in the blockchain.

LCR

The *Longest Chain Rule* states that if there exist several branches of the blockchain then the longest should be trusted.

In practice

- A branch can be considered legitimate if it is $k \in \mathbb{N}$ blocks ahead of its pursuers.
- Fork can be avoided when

$$\text{block appending time} > \text{propagation delay}$$

Bitcoin protocol

Consensus protocol

- One block every 10 minutes on average
- Depends on the hashrate of the network
- Difficulty adjustment every 2,016 blocks (≈ two weeks)
- Reward halving every 210,000 blocks

. `https://www.bitcoinblockhalf.com/`

Mining equipments

Consensus protocol

How it started

- CPU, GPU

How it is going

- Application Specific Integrated Chip (ASIC)
  - `Network electricity consumption`
  - `E-Waste`
  - `Centralization issue`

Proof of Stake

Consensus protocol

PoW is slow and ressource consuming. Let $\{1,\ldots,N\}$ be a set of miners and $\{\pi_1,\ldots,\pi_N\}$ be their share of cryptocoins.

PoS

1. Node $i \in \{1,\ldots,N\}$ is selected with probability π_i to append the next block

Nodes are chosen according to what they own.

- Nothing at stake problem
- Rich gets richer ?
- `https://www.peercoin.net/`

F. Saleh, "Blockchain without waste : Proof-of-stake," *The Review of Financial Studies*, vol. 34, pp. 1156–1190, jul 2020.

Ethereum Blockchain

Consensus protocol

The world-computer

- Smart contracts, decentralized applications
- Proof-of-stake (since 2022)
- a block is created every 12 seconds [1]
- Decentralized Finance (DeFi)

1. `https://txcity.io/v/eth-btc`

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Efficiency

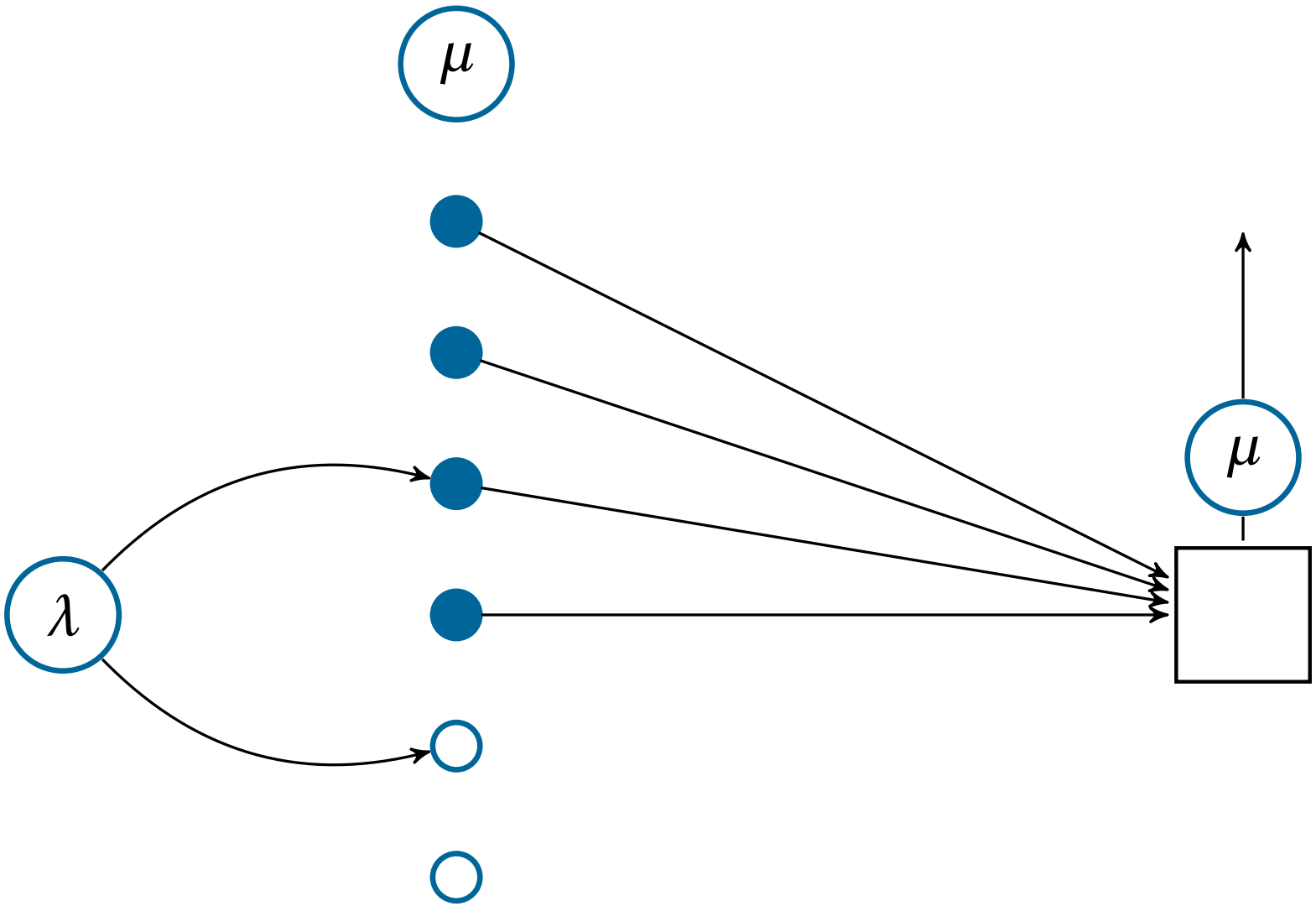
Stochastic models

Efficiency is characterized by

- Throughputs : Number of transaction being processed per time unit
- Latency : Average transaction confirmation time

Efficiency

Stochastic models



Queueing setting

Stochastic models

- Poisson arrival with rate $\lambda > 0$ for the transactions
- Poisson arrival with rate $\mu > 0$ for the blocks
- Block size $b \in \mathbb{N}^*$ $\Rightarrow$Batch service

⚠ The server is always busy

This is somekind of $M/M^b/1$ queue.

Y. Kawase and S. Kasahara, "Transaction-confirmation time for bitcoin : A queueing analytical approach to blockchain mechanism," in *Queueing Theory and Network Applications*, pp. 75–88, Springer International Publishing, 2017.

N. T. J. Bailey, "On queueing processes with bulk service," *Journal of the Royal Statistical Society : Series B (Methodological)*, vol. 16, pp. 80–87, jan 1954.

D. R. Cox, "The analysis of non-markovian stochastic processes by the inclusion of supplementary variables," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 51, pp. 433–441, jul 1955.

Queue length distribution

The queueuing process eventually reaches stationarity if

$$\mu \cdot b > \lambda. \tag{1}$$

We denote by N^q the length of the queue upon stationarity.

The blockchain efficiency theorem

Assume that (1) holds then N^q is geometrically distributed

$$\mathbb{P}(N^q = n) = (1-p) \cdot p^n,$$

where $p = 1/z^*$ and z^* is the only root of

$$-\frac{\lambda}{\mu} z^{b+1} + z^b \left(\frac{\lambda}{\mu} + 1 \right) - 1,$$

such that $|z^*| > 1$.

Latency and throughputs

Stochastic models

Little's law

Consider a stationary queueing system and denote by

- $1/\lambda$ the mean of the unit inter-arrival times
- L be the mean number of units in the system
- W be the mean time spent by units in the system

We have

$$L = \lambda \cdot W$$

J. D. C. Little, "A proof for the queuing formula :L= λW," *Operations Research*, vol. 9, pp. 383–387, jun 1961.

- Latency is the confirmation time of a transaction

$$\text{Latency} = W + \frac{1}{\mu} = \frac{\mathbb{E}(N^q)}{\lambda} + \frac{1}{\mu} = \frac{p}{(1-p)\lambda} + \frac{1}{\mu}.$$

- Throughput is the number of transaction confirmed per time unit

$$\text{Throughput} = \mu\mathbb{E}(N^q \mathbb{I}_{N^q \leq b} + b\mathbb{I}_{N^q > b}) = \mu \sum_{n=0}^{b} n(1-p)p^n + bp^{b+1}.$$

Perspectives

Stochastic models

1. Include some priority consideration to account for the transaction fees

Y. Kawase, , and S. Kasahara, "Priority queueing analysis of transaction-confirmation time for bitcoin," *Journal of Industrial & Management Optimization*, vol. 16, no. 3, pp. 1077–1098, 2020.

2. Go beyond the Poisson process framework

Q.-L. Li, J.-Y. Ma, and Y.-X. Chang, "Blockchain queue theory," in *Computational Data and Social Networks*, pp. 25–40, Springer International Publishing, 2018.

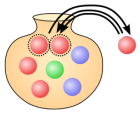
Q.-L. Li, J.-Y. Ma, Y.-X. Chang, F.-Q. Ma, and H.-B. Yu, "Markov processes in blockchain systems," *Computational Social Networks*, vol. 6, jul 2019.

Risk of centralization ?

Stochastic models

PoS algorithm

- Draw a coin at random
- The owner of the coin append a block and collect the reward
- The block appender is more likely to get selected during the next round

Similar to Polya's urn 

- Consider an urn of N balls of color in $E = \{1, \ldots, p\}$
- Draw a ball of color $x \in E$
- Replace the ball together with r balls of color x

p is the number of peers and r is the size of the block reward.

Theorem

The proportion of coins owned by each peer is stable on average over the long run

I. Roșu and F. Saleh, "Evolution of shares in a proof-of-stake cryptocurrency," *Management Science*, vol. 67, pp. 661–672, feb 2021.

Extensions and perspectives

Stochastic models

- How to include more peers along the way ?
- What if the peers are not simply buy and hold investors ?
- Find ways to monitor decentralization and take action if necessary

Double spending attack

1. Mary transfers 10 BTCs to John
2. The transaction is recorded in the public branch of the blockchain and John ships the good.
3. Mary transfers to herself the exact same BTCs
4. The malicious transaction is recorded into a private branch of the blockchain
   - Mary has friends among the miners to help her out
   - The two chains are copycat up to the one transaction

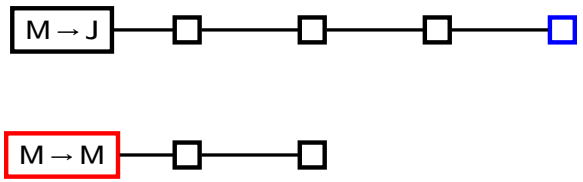
Fact (Bitcoin has only one rule)

The longest chain is to be trusted

Double Spending attack

Stochastic models

Vendor are advised to wait for $\alpha \in \mathbb{N}$ of confirmations so that the honest chain is ahead of the dishonest one.



In the example, vendor awaits $\alpha = 4$ confirmations, the honest chain is ahead of the dishonest one by $z = 2$ blocks.

Fact (PoW is resistant to double spending)

- Attacker does not own the majority of computing power
- Suitable α

Double spending is unlikely to succeed.

S. Nakamoto, "Bitcoin : A peer-to-peer electronic cash system." Available at https://bitcoin.org/bitcoin.pdf, 2008.

Double Spending Attack

Assume that

- $R_0 = z \geq 1$ (the honest chain is z blocks ahead)
- at each time unit a block is created
  - ↪ in the honest chain with probability p
  - ↪ in the dishonest chain with probability $q = 1 - p$

The process $(R_n)_{n \geq 0}$ is a random walk on $\mathbb{Z}$ with

$$R_n = z + Y_1 + \ldots + Y_n,$$

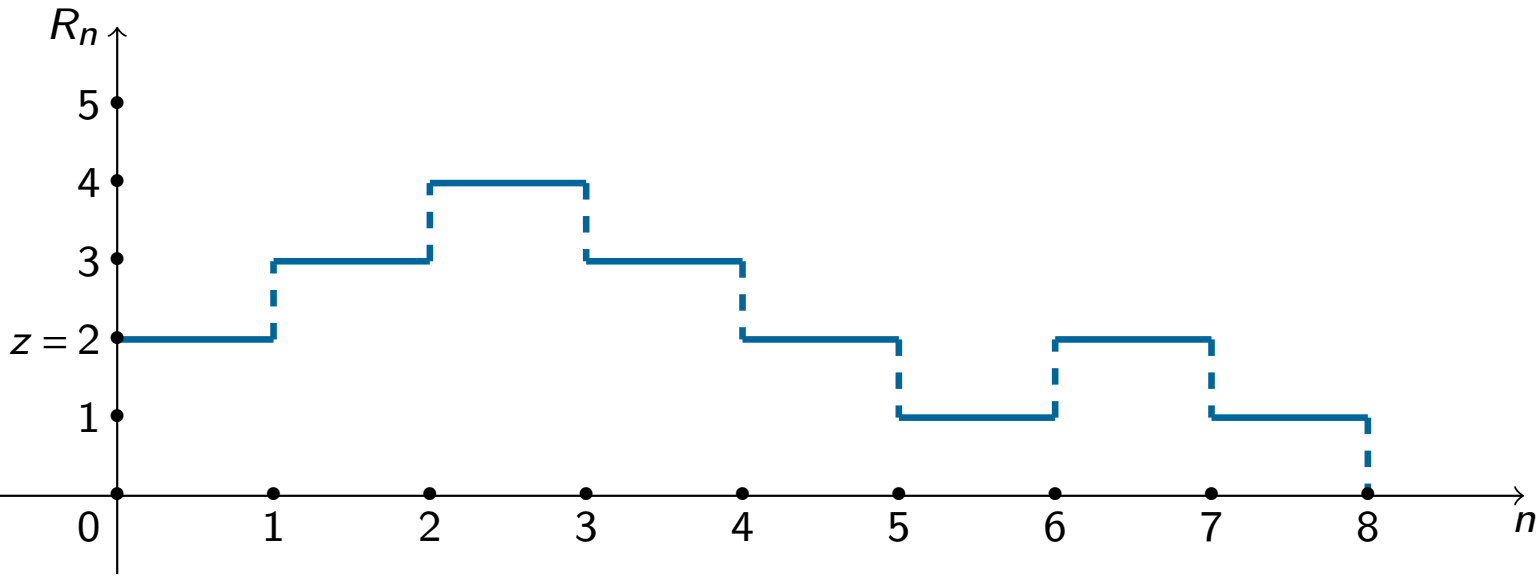
where $Y_1, \ldots, Y_n$ are the **i.i.d.** steps of the random walk.

Double Spending Attack

Stochastic models

Double spending occurs at time

$$\tau_z = \inf\{n \in \mathbb{N};\ R_n = 0\}.$$



Double spending theorem

If $p > q$ then the double-spending probability is given by

$$\psi(z) = \mathbb{P}(\tau_z < \infty) = \left(\frac{q}{p}\right)^z.$$

Double spending with Poisson processes

Stochastic models

Let the length of honest and dishonest chain be driven by counting processes

- Honest chain $\Rightarrow z + N_t$, $t \geq 0$, where $z \geq 1$.
- Malicious chain $\Rightarrow M_t$, $t \geq 0$
- Study the distribution of the first-*rendez-vous* time

$$\tau_z = \inf\{t \geq 0 \ , \ M_t = z + N_t\}.$$

P.-O. Goffard, "Fraud risk assessment within blockchain transactions," *Advances in Applied Probability*, vol. 51, pp. 443–467, jun 2019.
https://hal.archives-ouvertes.fr/hal-01716687v2.

R. Bowden, H. P. Keeler, A. E. Krzesinski, and P. G. Taylor, "Modeling and analysis of block arrival times in the bitcoin blockchain," *Stochastic Models*, vol. 36, pp. 602–637, jul 2020.

Blockchain as a research topic

Stochastic models

- Computer science
  - Peer-to-peer networks and consensus algorithm
  - Cryptography and security
- Economics
  - Game theory to study the incentive mechanism at play
  - Nature of the cryptoassets
- Operations research
  - Optimization of complex system
- Financial math
  - Valuation models for cryptoassets
- Machine learning and statistics
  - Open data
  - Interaction between blockchain users
  - (Social) network analysis
  - Clustering of public keys and addresses in the bitcoin blockchain.

TradFi Pain Points I

1 Access barrier

TradFi Criteria ⇒ bank account
DeFi No Barrier

2 Centralization

TradFi Bank are record keepers ⇒ Cyber risk
DeFi Decentralized Ledger

3 High costs and intermediation

TradFi Transaction fees, account maintenance fees, wire transfer fees,...
DeFi Intermediaries ⇒ Smart contracts (Gas fees)

4 Slow transaction settlement

TradFi Cross border transactions ⇒ Takes day to be settled
DeFi Near instant settlement (30 min on the bitcoin blockchain)

5 Transparency and auditability

TradFi Difficulty to verify the accuracy of transactions and asset holding
DeFi Publicly available ledger and open source code

TradFi Pain Points II

6 Censorship and restrictions

TradFi Asset in custody, censorhip by governments
DeFi No interference of central authority

7 Global accessibility

TradFi Respect the operating hours
DeFi Operates 24/7

8 Fractional ownership

TradFi Real estate and art work are not divisible
DeFi Tokenized real worl assets

9 Innovation and interoperability

TradFi Use outdated IT solutions
DeFi Interoperability between platforms and project

A. Lipton and A. Treccani, *Blockchain and Distributed Ledgers*. WORLD SCIENTIFIC, apr 2021.

Cryptocurrencies

1. No central authority (Decentralized network)
2. Ledger to record all the transactions and coin ownership (blockchain)
3. A coin generation process (block finding reward)
   ↪ Incentive to the full nodes
4. Ownership can be proved cryptographically (wallet associated to a public/private key)
5. Transactions can be issued by an entity proving ownership of the cryptographic unit (through the private key)
6. The system cannot process more than one transaction associated to the same cryptographic unit (double spending)

J. Lansky, "Possible state approaches to cryptocurrencies," *Journal of Systems Integration*, vol. 9, pp. 19–31, jan 2018.

Cryptocurrency implementation

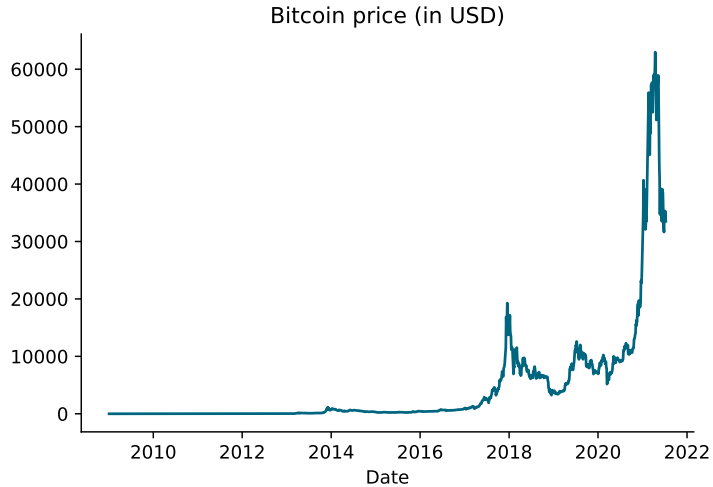
Blockchain parameters

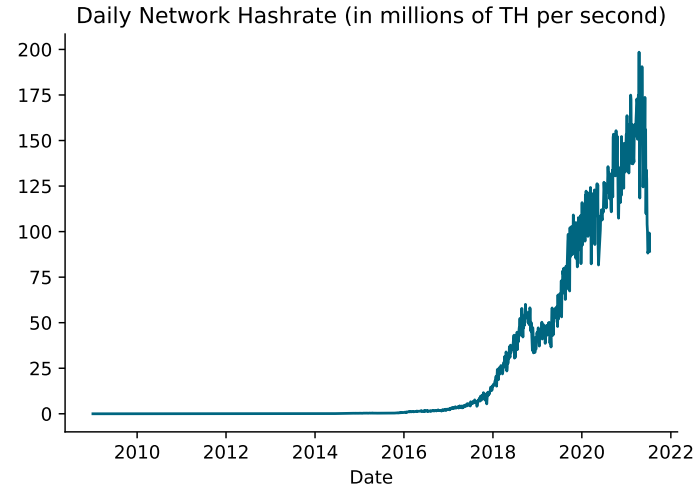
- Consensus protocol (PoW or PoS)
  - ↪ Hash function (SHA-256 for Bitcoin and scrypt for LiteCoin)
  - ↪ Hybrid PoW/PoS (PeerCoin)
- Block generation time (`https://txcity.io/v/eth-btc`)
  - ↪ every 10 minutes for Bitcoin
  - ↪ every 12 sec for Ethereum
- Block finding reward
  - ↪ Halved every 210,000 blocks in Bitcoin. It started at 50 BTC, is now 6.25 BTC `https://www.bitcoinblockhalf.com/`
- Total coin supply
  - ↪ 21,000,000 in total for Bitcoin
- Transaction fees
  - ↪ GAS in Ethereum

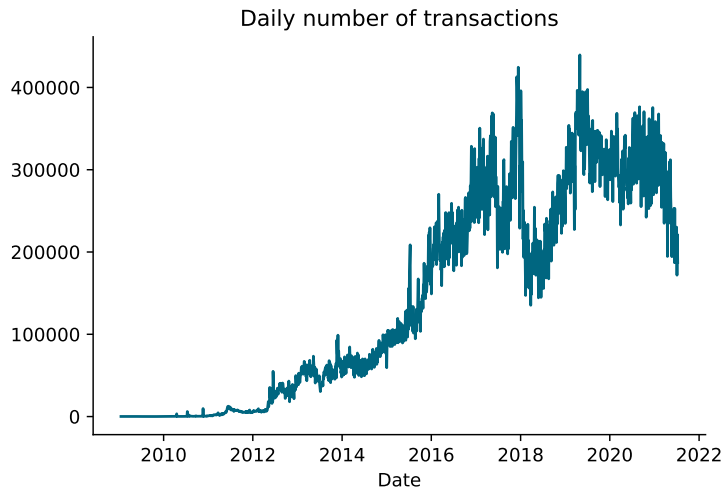
These choices lead to the creation of multiple cryptocurrencies

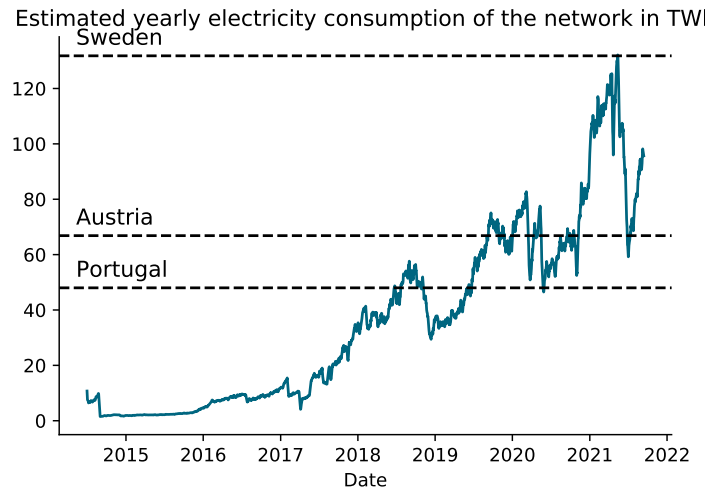
Examples

Bitcoin and AltCoins (Ethereum, LiteCoin, DogeCoin, Ripple...), see `https://en.wikipedia.org/wiki/List_of_cryptocurrencies`









. **Sources** : `https://www.blockchain.com/` & `https://cbeci.org/`

Decentralized finance

DeFi extends the Bitcoin promises to more complex financial operations

- Collateralized lending
- Decentralized Exchange Platform
- Tokenized assets

Thanks to Smart Contract on the Ethereum blockchain.

S. M. Werner, D. Perez, L. Gudgeon, A. Klages-Mundt, D. Harz, and W. J. Knottenbelt, "Sok : Decentralized finance (defi)," 2021.

. `https://uniswap.org/`

Decentralized Exchange platforms

Centralized Exchange

Binance, Coinbase, Kraken,...

$\Rightarrow$ Order book

Decentralized Exchange

Exchange where you trade one token for another through a smart contract on the Ethereum blockchain (ETH).

$\Rightarrow$ Automated Market Makers (AMM) and Liquidity Providers (LPs)

Order book

Table – Buy Orders

Price	Quantity	Total
9,950	10	99,500
9,900	5	49,500

Table – Sell Orders

Price	Quantity	Total
10,100	2	20,200
10,200	15	153,000

and a market maker in the middle to offer liquidity...

⚠ Lots of transaction must be issued

↪ Slow (10-15 txs/s)

↪ Expensive (gas fees)

Order book

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Automated Market Makers (AMM)

A blockchain requires an algorithm ⇒ AMM

- Exchange one token against another, usually Crypto/Stable Coins
- Constant Function Market Makers
- Liquidity Providers

Stable Coin

A bridge from fiat to crypto currency

- Fiat-Collateralized Stablecoins (e.g. USDC backed by one USD)
- Crypto-Collateralized Stablecoins (e.g. DAI backed by crypto locked in a smart contract)

Constant Product Market Maker

Let k be a constant such that

$$x \cdot y = k,$$

- x is the amount of token X
- y is the amount of token Y

Example

ETH/DAI even Constant Product Market Maker

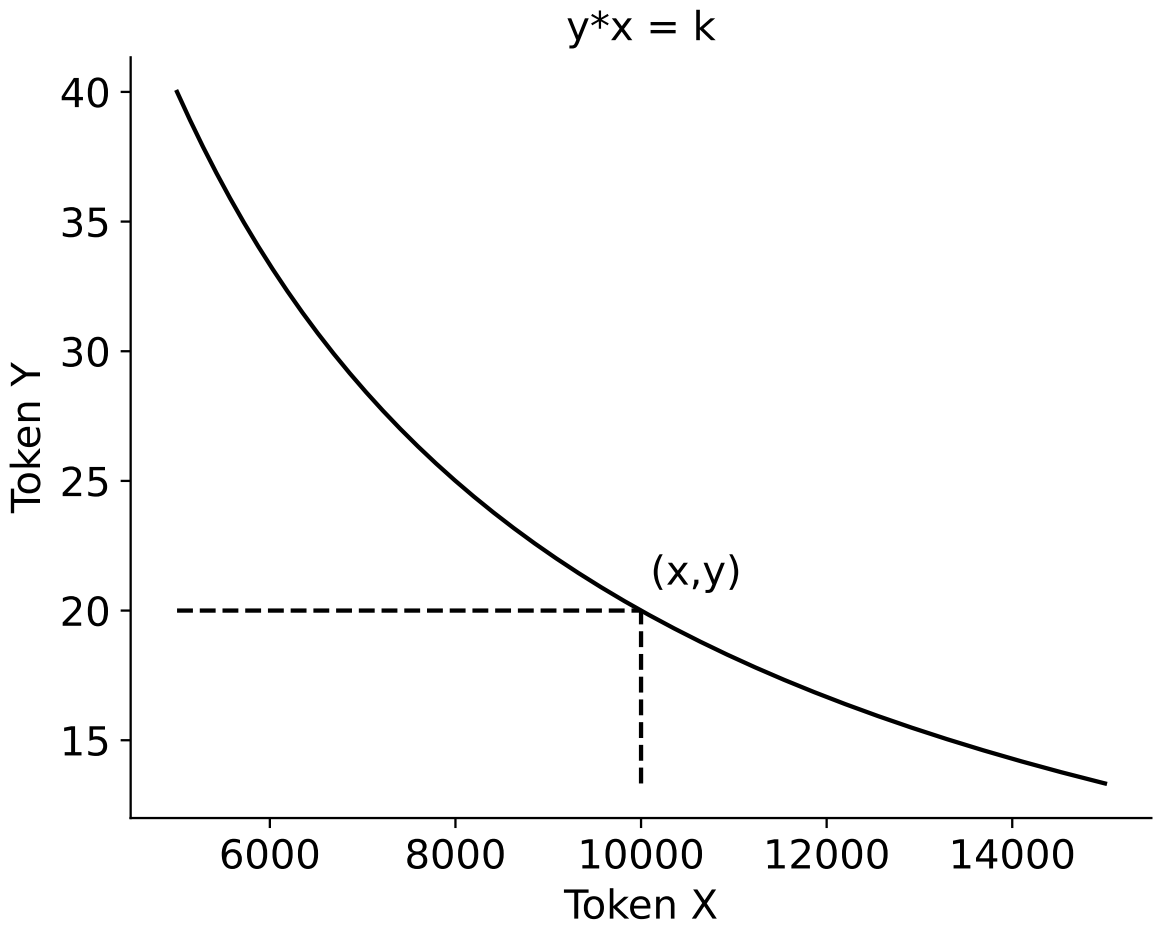
- Say that the price for one ETH is $P = \$500$
- A first LP provides 20 ETH plus 10,000 DAI which sets

$$k = 20{,}000$$

- The liquidity provided is measured by

$$L = \sqrt{x \cdot y} \approx 447 \text{ (geometric mean)}$$

Trade on a curve



. The pool never runs out of X nor Y

Swap X for Y

Acquire dy of token Y, then deposit dx that solves

$$(x+dx)(y-dy)=k \Leftrightarrow dx=\frac{x\cdot dy}{y-dy}$$

and pay a fee $\alpha\cdot dx$ to the LPs.

$\Rightarrow$ Price of Y rise in the pool

Example

Arbitrageur takes 2 ETH then deposits

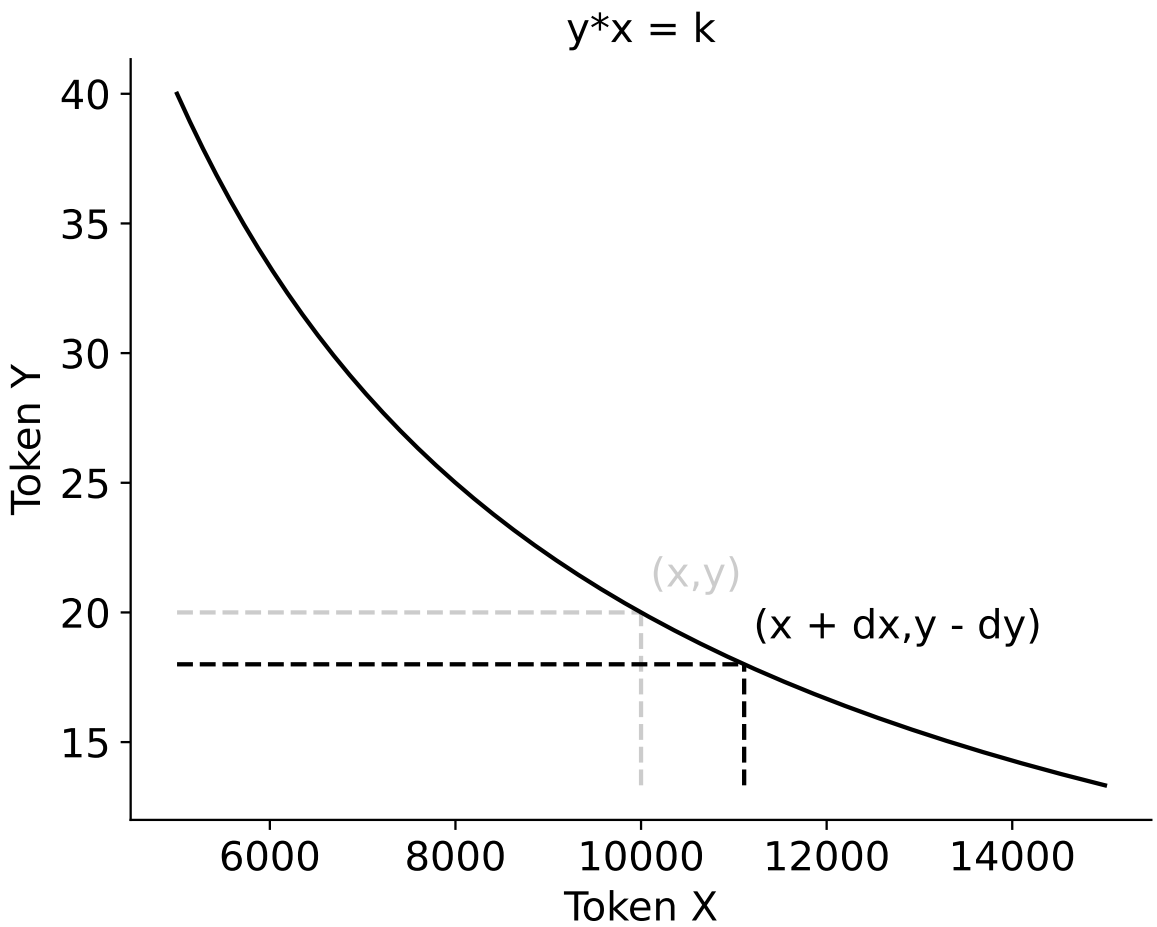
$$dx=1,111$$

and give out $0.3\cdot 1,111=333$ worth of fees

- The price of ETH in the pool rises to

$$\frac{x+dx}{y-dy}=\$617.$$

Trade on a curve



Add Liquidity

Another LP provides dx of token X and thus $dy = \frac{y}{x} dx$ (the price must not change)

- New level $k' = (x + dx)(y + dy)$
- Liquidity rises $L' = \sqrt{x \cdot y} + \sqrt{dx \cdot dy}$
- LPs are weighted according to the liquidity they provide
- Fees are distributed according to these weights

Example

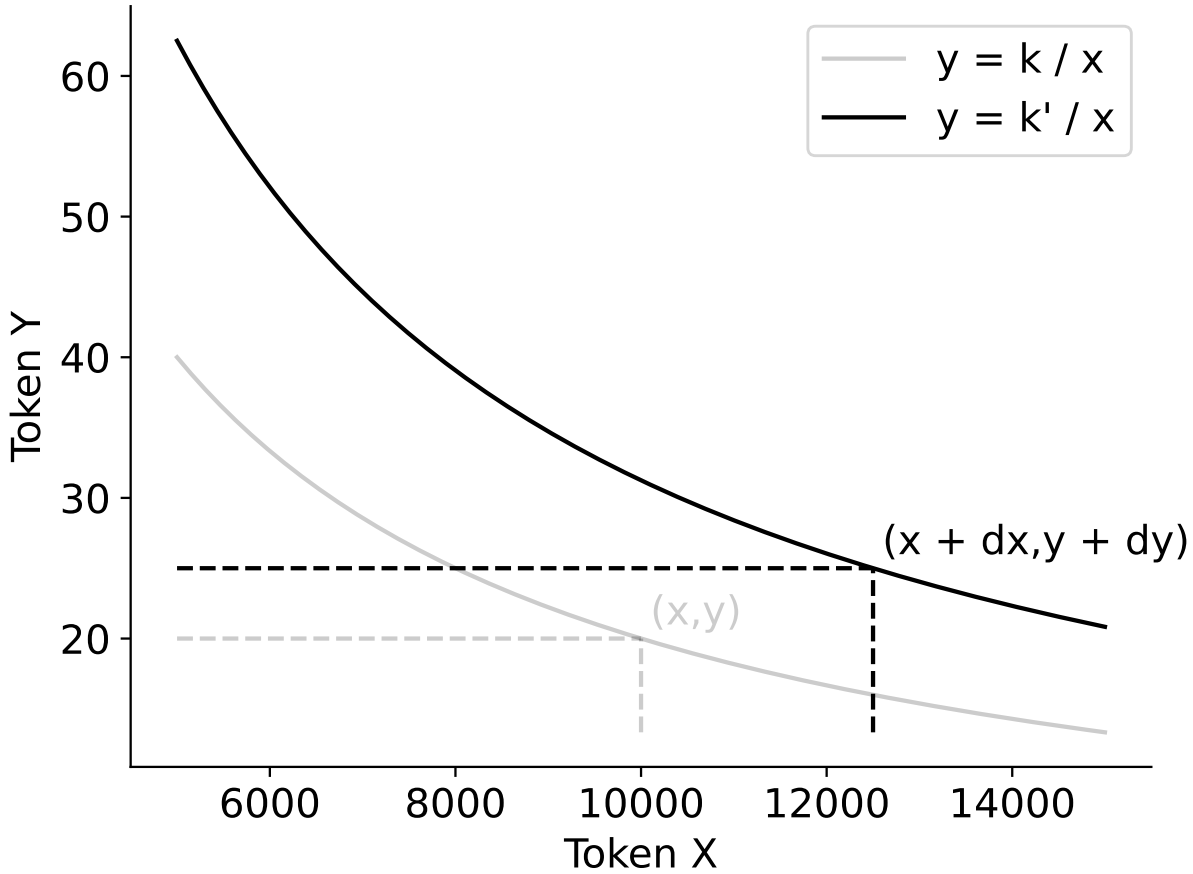
New LP deposits \$5,000 worth of tokens to the pool then

- $dx = 2{,}500$
- $dy = 5$
- $k' = 312{,}500$
- $L' \approx 559$

The weights are

$$\frac{L}{L'} = 0.8 \text{ and } \frac{L' - L}{L'} = 0.2.$$

Trade on a new curve



Impermanent Loss

Holding tokens VS Locking Tokens in a Smart Contract

Impermanent Loss

The price of Y in the pool is given by

$$P = \frac{x}{y},$$

in terms of token X.

- Price of Y becomes $P' > P$ on another trading venue then arbitrageurs wish to find

$$\underset{dy>0}{\operatorname{argmax}}\ dy \cdot P' - dx \cdot (1+\alpha) = \underset{dy}{\operatorname{argmax}}\ dy \cdot P' - \frac{x \cdot dy}{y - dy}(1+\alpha)$$

- We have

$$dy^* = y - \sqrt{\frac{k(1+\alpha)}{P'}}$$

- The arbitrageurs profit is

$$dy^* \cdot P' - dx^*(1+\alpha)$$

- It coincides with the impermanent loss of the LPs

$$y \cdot P' + x - [(y - dy^*) \cdot P' + x + dx^*(1+\alpha)]$$

Impermanent Loss

The loss becomes permanent if

- The LPs withdraw their funds
- The price of the asset is not mean reverting

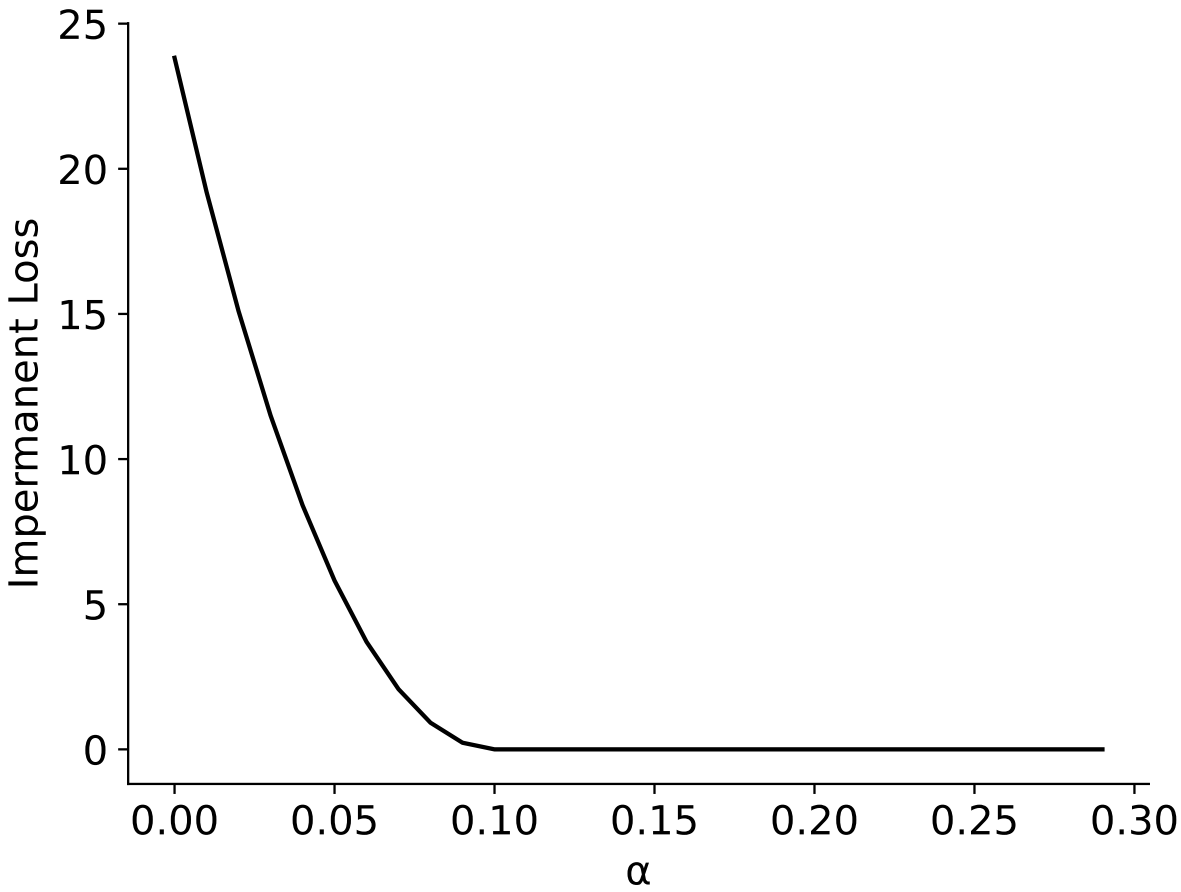
Example

Suppose that $P' = \$550$ and $\alpha = 0$ then

- $dy^* = 0.93$
- The impermanent loss is then $dy^* \cdot P' - \frac{x dy^*}{y - dy^*} = \$23$

Impermanent loss

The trading fee allows a pool to mitigate the impermanent loss



Decentralized insurance

Parametric insurance

Compensation if a measurable quantity reaches a threshold

- Example : Flight delay insurance
  - `https://etherscan.io/address/0xdc3d8fc2c41781b0259175bdc19516f7da11cba7`
- Use smart contract and off-chain data through oracles
- Transparent and automatic