A polynomial expansion to approximate the ultimate ruin probability in the Cramer-Lundberg ruin model

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Outlines

Executive summary

Main Goal

Work out a new numerical method to approximate ruin probabilities

Main Idea

Polynomial expansion of a function though orthogonal projection

- ↪ Change of measure via Natural Exponential Family with Quadratic Variance Function
- ↪ Construction of a polynomial orthogonal system w.r.t this probability measure

Achievement

Approximation of the ultimate ruin probability in the Cramer-Lundberg ruin model with phase type claim size

Notations

dF is an univariate Probability Measure (UPM). Denote by :

- F its Cumulated Distribution Function (CDF).
- $f = F'$ its Probability Density Function (PDF) w.r.t. to a positive measure.
- $\widehat{F}(\theta) = \int e^{\theta x} dF(x)$ its Moment Generating Function (MGF).
- $\kappa(\theta) = ln(\widehat{F}(\theta))$ its Cumulant Generating Function (CGF).

Denote by $L^2(dF)$ the function space such that :

- If $f \in L^2(dF)$ then $\int f^2(x)dF(x) < \infty$.

$L^2(F)$ is a normed vector space with :

$$||f||^2 =< f, f >= \int f^2(x)dF(x).$$

Natural Exponential Family

Let F be an univariate CDF possessing a MGF in a neighborhood of 0.

- $\{F_\theta; \theta \in \Theta\}$ is the NEF generated by F, with :

$$dF_\theta(x) = exp(\theta x - \kappa(\theta))dF(x),$$

  and Θ the definition domain of κ.

- The mean of F_θ is

$$\mu = \int x dF_\theta(x) = \kappa'(\theta),$$

where

- $\kappa' : \theta \to \kappa'(\theta)$ is a bijection.
  $\hookrightarrow$ $\phi : \mu \to \phi(\mu) = \kappa'^{-1}(\mu) = \theta$ its inverse function.
- $\{F_\mu; \mu \in \kappa'(\Theta) = M\}$ is an equivalent definition for NEF with :

$$dF_\mu(x) = exp(\phi(\mu)x - \kappa(\phi(\mu)))dF(x).$$

Quadratic Variance Function

The Variance Function (VF) of a NEF F_μ is :

$$V(\mu) = \int (x-\mu)^2 dF_\mu(x) = \kappa''(\mu).$$

The VF is said Quadratic if :

$$V(\mu) = a + b\mu + c\mu^2.$$

The NEF-QVF contains six types of distributions :

- Normal
- Gamma
- Hyperbolic
- Binomial
- Negative Binomial
- Poisson

Orthogonal polynomials for NEF-QVF Distributions

Define by $\{F_\mu; \mu \in M\}$ a NEF-QVF generated by F with mean μ_0.

- $f(x,\mu)$ a NEF-QVF PDF proportional to $exp(\phi(\mu)x - \kappa(\phi(\mu)))$ w.r.t. F :

$$Q_n(x,\mu) = V^n(\mu)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}/f(x,\mu),$$

  is a polynomial of degree n both in μ and x.
- Note that $f(x,\mu_0) = 1$. Then

$$Q_n(x) = Q_n(x,\mu_0) = V^n(\mu_0)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}_{\mu=\mu_0}.$$

$\{Q_n\}$ is an orthogonal polynomials system such that :

$$< Q_n(x), Q_m(x) >= \int Q_n(x)Q_m(x)dF(x) = ||Q_n||^2\delta_{nm}.$$

For a full description of the NEF-QVF see Morris [1982] [1].

Polynomial Expansion and Truncations

- The polynomials are dense in $L^2(F)$.
  $\hookrightarrow$ $\{Q_n\}$ is therefore an orthogonal basis of $L^2(F)$.
- dF_X a probability measure associated to some random variable X.
  $\hookrightarrow$ $\frac{dF_X}{dF}$ PDF w.r.t. dF
- If $\frac{dF_X}{dF} \in L^2(F)$ we have :

$$\frac{dF_X}{dF}(x) = \sum_{n\in\mathbb{N}} < \frac{dF_X}{dF}, \frac{Q_n}{||Q_n||} > \frac{Q_n(x)}{||Q_n||} = \sum_{n\in\mathbb{N}} E(Q_n(X)) \frac{Q_n(x)}{||Q_n||^2}.$$

- The CDF F_X is then :

$$F_X(x) = \sum_{n\in\mathbb{N}} E(Q_n(X)) \frac{\int_{-\infty}^x Q_n(y)dF(y)}{||Q_n||^2}.$$

Approximations are then obtained by truncation :

$$F_X^K(x) = \sum_{n=0}^{K} E(Q_n(X)) \frac{\int_{-\infty}^x Q_n(y)dF(y)}{||Q_n||^2}.$$

Definition and hypothesis

Denote by $\{R(t); t \geq 0\}$ the Risk Reserve Process :

$$R(t) = u + pt - \sum_{i=1}^{N(t)} U_i,$$

where

- u is the initial reserve.
- p is the constant premium rate per unit of time.
- $N(t)$ is a Poisson process with intensity β.
- $\{U_i\}_{i\in\mathbb{N}^*}$ are **i.i.d**. non-negative random variables with CDF F_U and mean μ.

Let $\{S(t); t \geq 0\}$ be the Surplus process :

$$S(t) = u - R(t).$$

$\eta > 0$ is the safety loading such that :

$$p = (1+\eta)\beta\mu.$$

Ultimate ruin probability

We denote by $M = Sup\{S(t); t > 0\}$ and we define the ultimate ruin probability by :

$$\psi(u) = P(M > u) = \overline{F_M}(u).$$

Theorem : Pollaczek-Khinchine formula

In the Cramer-Lundberg model, the ruin probability can be written as :

$$\begin{aligned}\psi(u) &= (1-\rho)\sum_{n=0}^{+\infty}\rho^n\overline{F_{U^I}^{*n}}(u),\\ M &\overset{D}{=} \sum_{i=1}^{N} U_i^I,\end{aligned}$$

where N is geometric with parameter $\rho = \frac{\beta\mu}{p} < 1$ and $F_{U^I}^{*n}$ denotes the nth convolution of F_{U^I}.

See *Ruin probabilities* by Asmussen and Albrecher [2001] [2].

Phase Type Distribution for Claim amounts

- $\{J(t)\}$ is a continuous time Markov process with state space $E = \{1, \dots, d\}$ and generator $\mathbf{T}$.
- $\{\tilde{J}(t)\}$ is a terminating Markov process with state space $E_{\Delta} = \{E \cup \Delta\}$. Δ is an absorbing state and the Markov process' generator is :

$$\left(\begin{array}{c|c} \mathbf{T} & \mathbf{t} \\ \hline 0 & 0 \end{array}\right).$$

F_U is of phase-type with representation $(\alpha, \mathbf{T}, E)$ if :

$$F_U(x) = P_{\alpha}(\zeta \leq x) = 1 - \alpha e^{\mathbf{T}x}\mathbf{e},$$

where

- $\zeta = inf\{t; \tilde{J}(t) = \Delta\}$.
- α is a vector of size d such as $\alpha_j = P(J(0) = j)$.
- $\mathbf{e}$ is a vector of size d with all componants equal to 1.

Ultimate ruin probability with phase type claim size

Theorem : An exponential matrix expression

Assume that F_U is of phase-type with representation $(\alpha, \mathbf{T})$ then :

- M is zero-modified phase-type with representation $(\alpha_+, \mathbf{T} + t\alpha_+)$, where $\alpha_+ = -\beta.\alpha.\mathbf{T}^{-1}$.
- $\psi(u) = \alpha_+.e^{(\mathbf{T}+\mathbf{t}.\alpha_+)u}.\mathbf{e}$.

In this case the ruin probability is asymptotically exponential :

$$\psi(u) \sim Ce^{-\lambda u},$$

with $-\lambda = max\{Sp(\mathbf{T} + \mathbf{t}.\alpha_+)\}$.

Polynomial Expansion for the ruin probability

Recall that $M = \sum_{i=1}^{N} U_i^I$. Then :

$$\begin{aligned} dF_M(x) &= (1-\rho)\delta_0(dx) + (1-\rho)\sum_{n=1}^{+\infty}\rho^n dF_{U^I}^{*n}(x) \\ &= (1-\rho)\delta_0(dx) + dG_M(x). \end{aligned}$$

If $\frac{dG_M}{dF} \in L^2(F)$ then :

$$\frac{dG_M}{dF}(x) = \sum_{n\in\mathbb{N}} < \frac{dG_M}{dF}, \frac{Q_n}{||Q_n||} > \frac{Q_n(x)}{||Q_n||}.$$

From this we deduce a polynomial expansion for the ruin probability :

$$\psi(u) = \sum_{n\in\mathbb{N}} < \frac{dG_M}{dF}, \frac{Q_n}{||Q_n||} > \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||}.$$

Approximation of the ruin probability though truncation of the polynomial expansion

Theorem : Approximation of the ultimate ruin probability

Let $\{F_\mu; \mu \in M\}$ be a NEF-QVF generated by F of mean μ_0 and $f(x,\mu)$ a NEF-QVF PDF proportional to $exp(\phi(\mu)x - \kappa(\phi(\mu)))$ w.r.t F.

- If $\frac{dG_M}{dF} \in L^2(dF)$ then :

$$
\begin{aligned}
\psi^K(u) &= \sum_{n=0}^{K} V_F(\mu_0)^n \left[\frac{\partial^n}{\partial\mu^n} e^{-\kappa(\phi(\mu))}\left(\widehat{G_M}(\phi(\mu))\right)\right]_{\mu=\mu_0} \\
&\times \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n(x)||^2}
\end{aligned}
$$

Choice of the NEF-QVF

dG_M is a defective probability measure supported on $[0,+\infty]$.
Among NEF-QVF the only one supported on $[0,+\infty]$ is generated by the exponential distribution. Then :

$$dF(x) = \xi e^{-\xi x} d\lambda(x)$$

The orthogonal polynomials linked to the exponential measure are the Laguerre ones see Szegö [1939] [3]

- Which value for ξ to complete the integrability condition ?
- $\frac{dG_M}{dF}(u) \sim De^{-u(\lambda-\xi)}$
  $\hookrightarrow \xi < 2\lambda$

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  $\hookrightarrow\ \xi < 2\lambda$

Simulation studies setting

For the ruin model we assume that :

- the premium rate p is equal to 1
- the safety loading η is equal to 33% and consequently $\rho = 0.75$
- The distributions of claim amount are of phase type given by $\mathbf{T}$ and α

A graphic visualization is proposed, we plot the quantity :

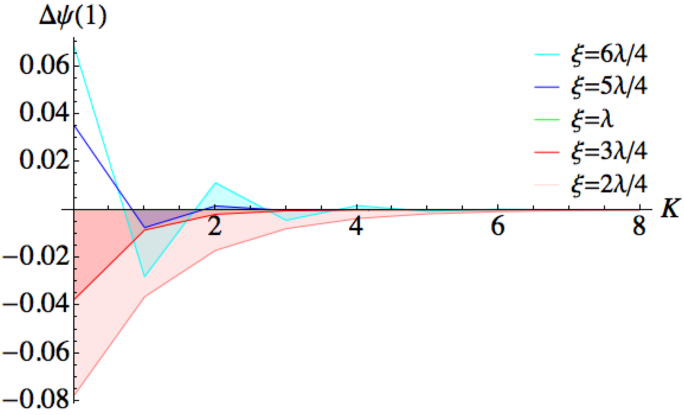
$$\Delta\psi(u) = \psi(u) - \psi^K(u),$$

for an initial reserve u and based on the truncation of order K.

$\hookrightarrow$ Different values for ξ are tested with one equal to λ.

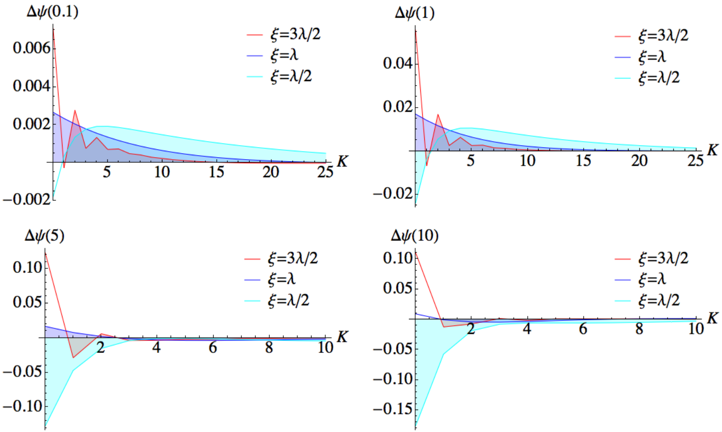
Exponentially distributed claim size

$$f_U(x) = e^{-x} 1_{[0,+\infty]}(x) \qquad \lambda = 0,25$$



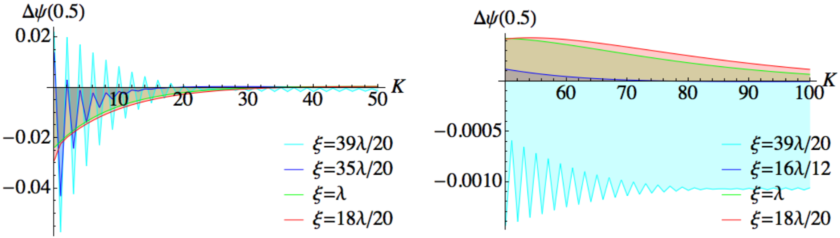
Mixture of Erlang distributed claim size

$$\mathbf{T} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & -\frac{2}{3} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \end{pmatrix}, \quad \lambda = 0.1205$$



HyperExponentially distributed claim size

$$\mathbf{T} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}, \ \alpha = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \ \lambda = 0.1495$$



Conclusion

$+$ A method quite efficient

$\hookrightarrow$ An approximation as precise as one might want

$+$ Easy to understand and to implement

$+$ No discretization of the claim size is needed

$-$ Limited to light tailed distribution

Outlooks :

- Theoritical sensitiveness study of the parameters
- Agregate claim amount distribution
- Finite time ruin probability

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Proof for the ruin probability approximation

Let's start from :

$$
\begin{aligned}
\psi(u) &= \sum_{n\in\mathbb{N}} < \frac{dG_M}{dF}, Q_n > \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2} \\
&= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} Q_n(x)\frac{dG_M}{dF}(x)dF(x)\frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2} \\
&= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} Q_n(x)dG_M(x)\frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2} \qquad (1)
\end{aligned}
$$

recall that :

$$
\begin{aligned}
Q_n(x) &= V_F(\mu_0)^n \left\{\frac{\partial^n}{\partial\mu^n} f(x,\mu)\right\}_{\mu=\mu_0} \\
&= V_F(\mu_0)^n \left\{\frac{\partial^n}{\partial\mu^n} exp(\phi(\mu)x-\kappa(\phi(\mu))\right\}_{\mu=\mu_0}
\end{aligned}
$$

Proof for the ruin probability approximation

Re-injecting in (1), one gets :

$$
\begin{aligned}
\psi(u) &= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}} V_F(\mu_0)^n \left\{ \frac{\partial^n}{\partial \mu^n} exp(\phi(\mu)x - \kappa(\phi(\mu))) \right\}_{\mu=\mu_0} dG_M(x) \\
&\times \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2} \\
&= \sum_{n\in\mathbb{N}} V_F(\mu_0)^n \left\{ \frac{\partial^n}{\partial \mu^n} e^{-\kappa(\phi(\mu))} \int_{\mathbb{R}} exp(\phi(\mu)x) dG_M(x) \right\}_{\mu=\mu_0} \\
&\times \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2} \\
&= \sum_{n\in\mathbb{N}} V_F(\mu_0)^n \left\{ \frac{\partial^n}{\partial \mu^n} e^{-\kappa(\phi(\mu))} \widehat{G_M}(\phi(\mu)) \right\}_{\mu=\mu_0} \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||^2}
\end{aligned}
$$