Duality in ruin problems for ordered risk models

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November 17, 2016

Abstract

The dual risk model is considered when the gain arrivals are governed by an order statistic point process (OSPP). The **p.d.f.** of the ruin time is obtained in terms of a remarkable family of polynomials. By duality, the **p.d.f.** of the ruin time is deduced for a Sparre-Andersen insurance risk model where the claim sizes are distributed as the inter-arrival times in an OSPP. On the other hand, duality is used again to derive the finite-time ruin probability in a dual model where the gains correspond to the inter-arrival times of an OSPP.

MSC 2010: 60G55, 60G40, 12E10.
Keywords: Order statistic property; Appell and Abel-Gontcharoff polynomials; Dual risk model; Time to ruin; Risk theory.

1 Introduction

Dual risk model. In this model, a company (not necessarilly operating on the insurance market) is able to follow its financial reserves' at any time. The company holds an initial capital $v > 0$ and faces operational expenses as time goes on. The financial reserves are therefore decreasing linearly in time with a slope $a > 0$. The company experiences capital gains that form a sequence $\{Y_i \,;\, i \geq 0\}$ of **i.i.d.** non-negative random variables. These capital gains occur at random times $\{S_n \,;\, n \geq 0\}$ that correspond to the jump times of an independent counting process $\{M(s) \,;\, s \geq 0\}$. The dual risk reserve process $\{U(s) \,;\, s \geq 0\}$ is given by

$$U(s) = v - as + \sum_{i=1}^{M(s)} Y_i,\ s \geq 0. \tag{1}$$

This model suits a company that is investing money in various business sectors such as oil prospection, pharmaceutical research or new technology development. The expected return on such investments is often uncertain.

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The dual risk model has been less treated in the literature than the insurance risk model. It is discussed, however, in several books, e.g. by Cramér [10, Section 5.13], Seal [33, pages 116-119] and Takàcs [36, pages 152-154]. An application in life insurance is mentioned in the monograph of Grandell [18, page 8]. Here, a life insurance company pays annuities on a regular basis and earns from time to time part of the reserve when a policyholder dies. The practical use of the dual model is also pointed out by Bayraktar and Egami [6] to model the dynamic of the financial reserves of venture capital funds that invest in start-ups. Another use of the dual rmodel is to describe the exposure to a given food contaminant (see Bertail et al. [7]).

The compound Poisson dual model, i.e. when the gain arrival process $\{M(s)\ ;\ s \geq 0\}$ in (1) is a homogeneous Poisson process, has received an increasing attention. In particular, much research effort has been made to solving dividend payment problems (e.g. Avanzi et al. [5], Gerber and Smith [16], Albrecher et al. [2], Ng [28], Dai et al. [11], Wen [37] and Afonso et al. [1]). The ruin time in the dual risk model is the first instant σ_v at which the reserve $U(.)$ reaches the level 0, i.e.

$$\sigma_v = \inf\{s \geq 0\ ;\ U(s) = 0\,(\leq 0)|U(0) = v\}. \tag{2}$$

For the compound Poisson case, Kendall's identity (see e.g. Borovkov and Burq [8]) provides us directly with the **p.d.f.** of σ_v. For the Sparre-Andersen dual risk model, where $\{M(s)\ ;\ s \geq 0\}$ is a renewal process, the Laplace transform of the ruin time was obtained by Sendova and Yang [39]. In Zhu and Yang [40], Lundberg type bounds were derived for the ultimate ruin probability in a Markov-modulated dual risk model.

In the present paper, we first focus on a dual risk model where the gain arrival process is an order statistic point process (OSPP). This means that conditionally on the number of gain arrivals up to time $t \geq 0$, the jump times are distributed as the order statistics for a random sample drawn from some probability distribution with support $(0, t)$. The OSPP family includes classical point processes such as the homogeneous Poisson process, the linear birth process with immigration and the linear death process. This family has been extensively used to model claim frequencies in insurance (see e.g. Willmot [38], De Vylder and Goovaerts [12, 13], Lefèvre and Picard [23, 25], Sendova and Zitikis [34] and Dimitrova et al. [14]).

Our purpose is to derive an explicit formula for the **p.d.f.** of the ruin time σ_v for such a dual risk model. For that, we take advantage of the polynomial representation of the joint distribution of the order statistics associated to a random sample drawn from a uniform distribution. The involved polynomials are of the Appell or Abel-Gontcharov type. A review of these polynomials with applications in risk and epidemic modelling is given by Lefèvre and Picard [26]. In the compound Poisson case, Kendall's identity is retrieved, of course.

Insurance risk model. In insurance, a classical risk process is the Sparre-Andersen model (Andersen [3]). The risk reserve process $\{R(t)\ ;\ t \geq 0\}$ is here given by

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i,\ \ t \geq 0, \tag{3}$$

where $u \geq 0$ is the initial capital of the company, the premiums are collected linearly in time at a rate $c > 0$, the claim amounts form a sequence $\{X_i \ ; \ i \geq 0\}$ of **i.i.d.** non-negative random variables, and the number of claims up to time $t \geq 0$ is a renewal counting process $\{N(t) \ ; \ t \geq 0\}$. The time to ruin is defined as the first time τ_u at which the reserve $R(.)$ becomes negative, i.e.

$$\tau_u = \inf\{t \geq 0 \ ; \ R(t) < 0 | R(0) = u\}. \tag{4}$$

For an overview of results for this model, we refer to the book of Asmussen and Albrecher [4].

It is well established that to the insurance risk model (3) is associated a dual risk model (1) whose characteristics are inverted in a certain sense. Specifically, the capital gains in the dual model correspond to the inter-arrival times in the insurance model while the inter-arrival times in the dual model correspond to the claim sizes in the insurance model, and the cost rate in the dual model is the inverse of the premium rate in the insurance model. A close connection between the ruin time distributions in both models was established by e.g. Mazza and Rullière [27, Theorem 2] and Dimitrova et al. [15, Lemma 2.1] within a more general framework.

This link between the ruin times can lead to new results by simply making a passage between the dual and insurance models. This idea was well embodied in Mazza and Rullière [27]. Starting with the dual model, they worked out a recursive formula for the ruin probability in a compound Poisson dual risk model with lattice claim sizes [27, Theorem 5] (see e.g. Picard and Lefèvre [30], Loisel and Rullière [32] and Lefèvre and Loisel [22] for a similar result in insurance). Passing to the insurance model, they then deduced a formula for the ruin probability when the claim arrival process is generated by a sequence of lattice inter-arrival times and exponentially distributed claim sizes [27, Theorem 6]. On the other hand, Dimitrova et al. [15] derived the finite-time ruin probability for a generalized dual model by using results obtained in Ignatov and Kaishev [19, 20] for the insurance model; see also Lefèvre and Picard [24].

We consider here a round trip between the two models. We start by deriving the ruin time **p.d.f.** for an OSPP dual risk model. From this formula, we deduce the **p.d.f.** of the ruin time in the Sparre-Andersen insurance model when the claim arrivals are governed by a renewal process and the successive claim sizes may be dependent and are distributed as the inter-arrival times in an OSPP. As a special case, we recover a result derived by Borovkov and Dickson [9] for the case of the Sparre-Andersen model with **i.i.d.** exponential claim amounts.

Our second journey is from the insurance model to the dual model. We begin from an insurance model where the claim arrivals are described by an OSPP and the claim sizes are **i.i.d.** non-negative random variables. The finite-time ruin probability in that case was obtained by Lefèvre and Picard [23, Proposition 4.1]. By duality, we get the finite-time ruin probability in the corresponding dual risk model where gains are distributed as the inter-arrival times of an OSPP and their arrivals are governed by a renewal process.

Summary. The paper is organized as follows. In Section 2, we give an overview of the order statistic point processes. In Section 3, we obtain the ruin time **p.d.f.** in the dual risk model when the gain arrivals are governed by an OSPP. In Section 4, we deduce the ruin time **p.d.f.** in a Sparre-Andersen insurance risk model where the claim sizes are distributed as the inter-arrival times of an OSPP. In Section 5, we obtain the finite-time ruin probability in the dual model when the gains correspond to the inter-arrival times of an OSPP.

2 Order statistic property

The Poisson process is a common model for counting events that arise randomly in the course of time. Of simple construction, it has also many desirable properties. In particular, it belongs to the class of order statistic point processes.

Definition 2.1. *A point process $\{N(t),\, t \geq 0\}$ with $N(0) = 0$ is an OSPP if for every $n \geq 1$, provided $\mathbb{P}[N(t) = n] > 0$, then conditioned upon $[N(t) = n]$, the successive jump times $(T_1, T_2, \ldots, T_n)$ are distributed as the order statistics $[U_{1:n}(t), \ldots, U_{n:n}(t)]$ of a sample of n **i.i.d.** random variables, distributed as $U(t)$ say, with distribution function $\mathbb{P}[U(t) \leq s] = F_t(s)$ for $0 \leq s \leq t$.*

De Vylder and Goovaerts [12, 13] introduced a risk model, named homogeneous, that generalizes the classical Cramér-Lundberg risk model. When considered on an infinite-time horizon, the homogeneous model assumes that the claim arrival process satisfies the order statistic property with $U(t)$ uniformly distributed on $(0, t)$ (as for the Poisson process). Their research was made independently of the existing literature on OSPP. More recently, Lefèvre and Picard [23] developed a risk model in which claim arrivals are modelled by a general OSPP. This paper was continued in Lefèvre and Picard [25, 26] for evaluating ruin probabilities over a finite horizon; see also Sendova and Zitikis [34] and Ignatov and Kaishev [21]. Recently, Goffard and Lefèvre [17] studied the first-crossing problem of a (simple) OSSP through general boundaries.

A complete representation of the class of OSPP was derived by Puri [31], following on earlier works.

Proposition 2.2. *(Puri [31])*
Let $\{N(t),\, t \geq 0\}$ be an OSPP where $\mu(t) = \mathbb{E}[N(t)]$ is finite for all t.

(i) If $\lim_{t\to\infty} \mu(t) = \infty$, then $\{N(t),\, t \geq 0\}$ is a mixed Poisson process up to a time-scale transformation. So, it can be represented as

$$N(t) = \mathcal{P}\left[W\nu(t)\right], \quad t \geq 0, \quad a.s., \tag{5}$$

where $\{\mathcal{P}(t),\, t \geq 0\}$ is a Poisson process with rate 1, W is an independent non-negative random variable and $\nu(t)$ is a deterministic time function.

*(ii) If $\lim_{t\to\infty} \mu(t) = \mu < \infty$, then $\{N(t),\, t \geq 0\}$ is a death counting process in which the individual lifetimes are **i.i.d.** random variables of distribution function $\mu(t)/\mu$, $t \geq 0$, and there is initially an independent random number Z of individuals in the population. So, it can be represented as is a mixed binomial type process:*

$$N(t) = \mathcal{B}\left[Z, \mu(t)/\mu\right], \quad t \geq 0, \quad a.s.. \tag{6}$$

For both cases, the order statistic property holds with

$$F_t(s) = \mu(s)/\mu(t), \quad 0 \leq s \leq t. \tag{7}$$

Here are some simple special cases used in various applications.

Particular OSPP.

(1) A Poisson process of parameter λ. Here, $N(t)$ has a Poisson distribution of mean $\mu(t) = \lambda t$. So, (5) holds with $W = 1$ **p.s.** and $\nu(t) = \lambda t$, and (7) gives $F_t(s) = s/t$.

(2) An inhomogeneous Poisson process of continuous intensity function $\lambda(t)$. Here, $N(t)$ has a Poisson distribution of mean $\mu(t) = \int_0^t \lambda(z)dz$. So, (5) holds with $\nu(t) = \mu(t)$ and $W = 1$, and $F_t(s)$ is given by (7).

(3) A mixed Poisson process of mixing variable Λ. Here, $N(t)$ has a mixed Poisson distribution of random parameter Λ, with mean $\mu(t) = E(\Lambda)t$. So, (5) holds with $W = \Lambda$ and $\nu(t) = t$, and (7) gives $F_t(s) = s/t$ (independently of Λ).

For instance, if Λ has a gamma distribution $\Gamma(\gamma, \beta)$, then the mixed Poisson process is a negative binomial process of parameters γ and β. So, $N(t)$ has a negative binomial distribution:

$$\mathbb{P}[N(t) = n] = \binom{\gamma + n - 1}{n} \left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{t+\beta}\right)^\gamma, \quad n \geq 0,$$

with mean $\mu(t) = (\gamma/\beta)t$.

(4) A linear birth process of rate α and with immigration of rate λ. Here, $N(t)$ has a negative binomial distribution:

$$\mathbb{P}[N(t) = n] = \binom{\lambda/\alpha + n - 1}{n} (1 - e^{-\alpha t})^n \, e^{-\lambda t}, \quad n \geq 0,$$

with mean $\mu(t) = (\lambda/\alpha)(e^{\lambda t} - 1)$.

This process can also be considered as a inhomogeneous mixed Poisson process (5) for which W has a gamma distribution $\Gamma(\lambda/\alpha, 1)$ and $\nu(t) = e^{\lambda t} - 1$.

(5) A linear death counting process of rate α and initial size z. Here, $N(t)$ has a binomial distribution:

$$\mathbb{P}[N(t) = n] = \binom{z}{n} (1 - e^{-\alpha t})^n e^{-\alpha t(z-n)}, \quad 0 \leq n \leq z,$$

with mean $\mu(t) = z(1 - e^{-\alpha t})$, of finite limit $\mu = z$ as $t \to \infty$. So, (6) holds with $Z = z$ a.s. and the lifetimes are **i.i.d.** exponentials of parameter α.

In the sequel, we are also interested in the distributions of the inter-arrival times $\Delta T_i = T_i - T_{i-1}$, $i \geq 1$ in an OSPP ($T_0 = 0$). They are named here level spacing distributions (a level spacing in physics is the difference between two consecutive elements). A general expression for such disributions is not available, unfortunately. So, let us reexamine the previous special cases.

Particular level spacings.

(1) For a Poisson process, the ΔT_i's are **i.i.d.** exponentials of parameter λ.

(2) For an inhomogeneous Poisson process, the ΔT_i's are dependent and each vector $(\Delta T_1, \ldots, \Delta T_n)$ has density

$$f_{\Delta T_1,\ldots,\Delta T_n}(x_1, \ldots, x_n) = e^{-\mu(x_1+\ldots+x_n)} \prod_{i=1}^n \lambda(x_1 + \ldots + x_i), \quad x_i \geq 0.$$

(3) For a mixed Poisson process, the ΔT_i's are mixed exponentials such that each vector $(\Delta T_1, \ldots, \Delta T_n)$ has density (for $x_i \geq 0$)

$$f_{\Delta T_1,\ldots,\Delta T_n}(x_1,\ldots,x_n) = \mathbb{E}\left\{e^{-\Lambda(x_1+\ldots+x_n)}\left[\Lambda(x_1+\ldots+x_n)\right]^n/(n!)\right\}.$$

(4) For a linear birth process with immigration, the ΔT_i's, $i \geq 1$, are independent exponentials of parameter $\lambda + \alpha(i-1)$.

(5) For a linear death counting process, the ΔT_i's, $1 \leq i \leq z$, are independent exponentials of parameter $\alpha(z-i+1)$.

3 Dual model with ordered gain arrivals

Consider the dual risk model described in the Introduction. Its reserves are given by

$$U(s) = v - as + \sum_{i=1}^{M(s)} Y_i \;\equiv\; v - as + V(s), \quad s \geq 0, \tag{8}$$

where $U(0) = v > 0$, $a > 0$, $\{M(s)\ ;\ s \geq 0\}$ is an OSPP and $\{Y_i\ ;\ i \geq 1\}$ are **i.i.d.** non-negative random variables. Put $V_n = Y_1 + \ldots + Y_n$, $n \geq 1$, with $V_0 = 0$. The ruin time is

$$\sigma_v = \inf\{s \geq 0\ ;\ U(s) = 0 | U(0) = v\}. \tag{9}$$

Note that from (8), we have $\sigma_v \geq v/a$, i.e. ruin cannot arise before time v/a.

Evidently, σ_v is the first-meeting time of the process $\{V(s)\ ;\ s \geq 0\}$ with the lower linear boundary $y = -v + as$, $s \geq 0$. Figure 1 illustrates this first-meeting problem. Let S_i (resp. $\Delta S_i = S_i - S_{i-1}$) be the i-th arrival time (resp. inter-arrival period) in $\{M(s)\ ;\ s \geq 0\}$, for $i \geq 1$, with $S_0 = 0$.

Suppose that the Y_i's have a **p.d.f.** f_Y. Its n-th convolution is denoted by f_Y^{*n}, $n \geq 1$, and we put $f_Y^{*0}(y) = \mathbf{1}_{(y=0)}$. The following result expresses the ruin time **p.d.f.** in terms of a family of polynomials, named Abel-Gontcharoff (A-G). A short presentation of these polynomials is given in Appendix A. Given a set of reals $U = \{u_i,\ i \geq 1\}$, the A-G polynomial of degree n in x is denoted by $G_n(x|U)$.

Theorem 3.1. *The ruin time σ_v has a **p.d.f.** at point s $(\geq v/a)$ of the form*

$$f_{\sigma_v}(s) = a\,\mathbb{E}\left[(-1)^{M(s)}\, f_Y^{*M(s)}(as - v)\, h_{M(s)}(s,v)\right], \tag{10}$$

where if $[M(s) = n]$ $(n \geq 0)$, the function $h_n(s,v)$ is the conditional expectation

$$h_n(s,v) = \mathbb{E}\left\{G_n\left[0\middle|F_s\left(\frac{V_0+v}{a}\right),\ldots,F_s\left(\frac{V_{n-1}+v}{a}\right)\right]\middle|V_n = as - v\right\}. \tag{11}$$

Proof. Consider the event $[\sigma_v \in (s, s+\mathrm{d}s)]$ where $\mathrm{d}s$ is small enough. We can express it as

$$[\sigma_v \in (s, s+\mathrm{d}s)] = \bigcup_{n=0}^{+\infty}\{[M(s)=n] \cap [\sigma_v \in (s,s+\mathrm{d}s)]\} \tag{12}$$

$$= \bigcup_{n=0}^{+\infty}\left\{[M(s)=n]\bigcap_{k=1}^{n}\left[S_k \leq \frac{V_{k-1}+v}{a}\right] \cap \left[\frac{V_n+v}{a} \in (s,s+\mathrm{d}s)\right]\right\}. \tag{13}$$

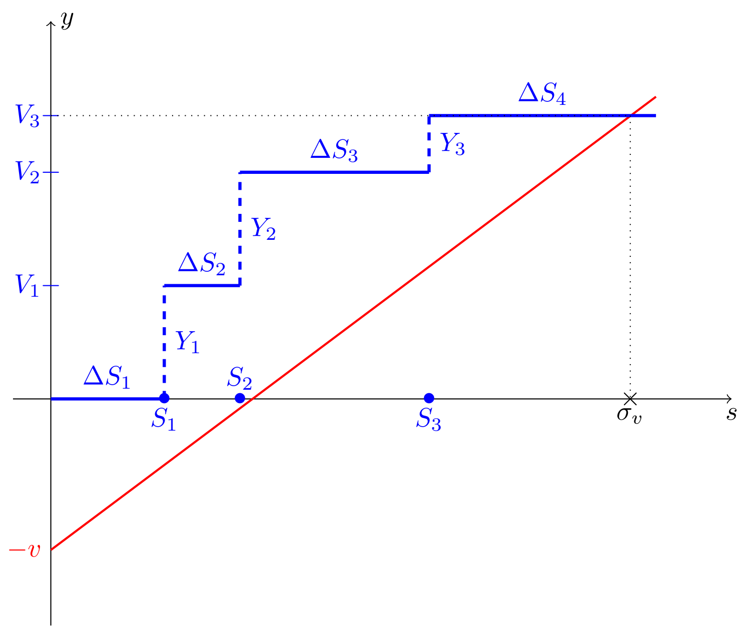

Figure 1: Ruin time in the dual risk model. The solid red line represents the cost function $y = -v + as$, and the dashed blue line corresponds to a trajectory of the aggregated capital gains.

Conditioning on $[M(s) = n]$ $(n \geq 0)$, (12) gives

$$\mathbb{P}[\sigma_v \in (s, s + \mathrm{d}s)] = \sum_{n=0}^{+\infty} \mathbb{P}[\sigma_v \in (s, s + \mathrm{d}s) \big| M(s) = n] \mathbb{P}[M(s) = n], \tag{14}$$

and from (13), we get

$$\begin{aligned} &\mathbb{P}[\sigma_v \in (s, s + \mathrm{d}s) \big| M(s) = n] \\ &\quad = \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[S_k \leq \frac{V_{k-1} + v}{a} \right] \cap \left[\frac{V_n + v}{a} \in (s, s + \mathrm{d}s) \right] \Big| M(s) = n \right\}. \end{aligned} \tag{15}$$

By the order statistic property, given $[M(s) = n]$ $(n \geq 1)$, the vector $(S_1, \dots, S_n)$ is distributed as the order statistics $[U_{1;n}(s), \dots, U_{n;n}(s)]$ of a sample of n **i.i.d.** random variables with distribution function F_s on $(0, s)$. This implies that

$$[F_s(U_{1:n}(s)), \dots, F_s(U_{n:n}(s))] \overset{\mathcal{D}}{=} (U_{1;n}, \dots, U_{n;n}), \tag{16}$$

where $(U_{1:n}, \dots, U_{n:n})$ are the order statistics of n independent uniform variables on $(0,1)$. Thanks to (16), we may rewrite (15) as

$$
\begin{aligned}
&\mathbb{P}[\sigma_v \in (s,s+\mathrm{d}s) \big| M(s)=n] \\
&= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n}(s) \leq \frac{V_{k-1}+v}{a} \right] \cap \left[\frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right] \right\} \\
&= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n} \leq F_s\left(\frac{V_{k-1}+v}{a} \right) \right] \cap \left[\frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right] \right\} \\
&= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n} \leq F_s\left(\frac{V_{k-1}+v}{a} \right) \right] \Big| \frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{P}\left[\frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right]. \qquad (17)
\end{aligned}
$$

At this point, the key step is the probabilistic interpretation of the A-G polynomials given in (A.1). Using this property, we obtain that

$$
\begin{aligned}
&\mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n} \leq F_s\left(\frac{V_{k-1}+v}{a} \right) \right] \Big| \frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right\} \\
&= \mathbb{E}\left\{ \mathbb{P}\left[\bigcap_{k=1}^{n} \left[U_{k:n} \leq F_s\left(\frac{V_{k-1}+v}{a} \right) \right] \Big| V_1,\ldots,V_n \right] \Big| V_n = as-v \right\} \\
&= (-1)^n \mathbb{E}\left\{ G_n\left[0 \Big| F_s\left(\frac{V_0+v}{a} \right),\ldots,F_s\left(\frac{V_{n-1}+v}{a} \right) \right] \Big| V_n = as-v \right\} \\
&= (-1)^n h_n(s,v), \quad n \geq 1, \qquad (18)
\end{aligned}
$$

in the notation (11). On the other hand, we have

$$
\mathbb{P}\left[\frac{V_n+v}{a} \in (s,s+\mathrm{d}s) \right] = f_Y^{*n}(as-v)\mathrm{d}s, \quad n \geq 0. \qquad (19)
$$

Inserting (18), (19) in (17), (14) then provides (10). □

The formulas (10), (11) show clearly the algebraic structure underlying the density of σ_v. Of course, their numerical implementation can be rather complex but remain quite practicable. We now show that the result becomes simple and explicit in the important case where the OSPP is a mixed Poisson process (see case (3) of OSPP in Section 2).

Corollary 3.2. *If $\{M(s)\ ;\ s \geq 0\}$ is a mixed Poisson process, then*

$$
f_{\sigma_v}(s) = \frac{v}{s}\,\mathbb{E}\left[f_Y^{*M(s)}(as-v) \right], \quad s \geq v/a. \qquad (20)
$$

Proof. We noted earlier that if the OSPP is a mixed Poisson process, then $F_s(x) = x/s$ for $x \in (0,s)$. Thus, $h_n(s,v)$ in (11) becomes

$$
\begin{aligned}
h_n(s,v) &= \mathbb{E}\left[G_n\left(0 \Big| \frac{V_0+v}{as},\ldots,\frac{V_{n-1}+v}{as} \right) \Big| V_n = as-v \right] \\
&= \frac{1}{(as)^n}\mathbb{E}\left[G_n(-v|V_0,\ldots,V_{n-1}) \Big| V_n = as-v \right], \qquad (21)
\end{aligned}
$$

using the identity (52) for G_n. Now, the Y_i's being **i.i.d.** variables, Property A.2 for the conditional expectation of G_n is applicable to (21), which yields

$$h_n(s, v) = \frac{1}{(as)^n}(-v)(-v - as + v)^{n-1} = (-1)^n \frac{v}{as}. \tag{22}$$

Combining (22) with (10) then gives the result (20). $\square$

As explained below, formula (20) is closely related to Kendall's identity. An analogous result was obtained by Stadje and Zacks [35].

Remark. Let us focus on the special case where $\{M(s)\ ;\ s \geq 0\}$ is a standard Poisson process. Then, the process $\{X^*(s) = as - V_{M(s)}\ ;\ s \geq 0\}$ is a Lévy process which is skip free in the positive direction. In other words, it has no positive jumps and its increments are stationnary and independent. The stopping time is $\sigma_v = \inf\{t \geq 0\ ;\ X^*(s) = v\}$. From Kendall's identity for such processes (see Borovkov and Burq [8]), we know that

$$\frac{1}{v} f_{\sigma_v}(s) = \frac{1}{s} f_{X^*(s)}(v).$$

This matches exactly formula (20).

4 Insurance model with spacing claim amounts

Consider the Sparre-Andersen risk model described in the Introduction. Its reserves are given by

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i \quad \equiv \quad u + ct - W(t), \qquad t \geq 0, \tag{23}$$

where $R(0) = u \geq 0$, $c \geq 0$, $\{N(t)\ ;\ t \geq 0\}$ is a renewal process and $\{X_i\ ;\ i \geq 1\}$ are non-negative variables, possibly dependent, with a level spacing distribution. The X_i's generate an OSPP $\{M(s)\ ;\ s \geq 0\}$, say. Put $W_n = X_1 + \ldots + X_n$, $n \geq 1$, with $W_0 = 0$. The ruin time is

$$\tau_u = \inf\{t \geq 0\ ;\ R(t) < 0 | R(0) = u\}. \tag{24}$$

This time, τ_u corresponds to the first-crossing time of the stochastic process $\{W(t)\ ;\ t \geq 0\}$ through the upper linear boundary $x = u + ct$, $t \geq 0$. Figure 2 illustrates this first-crossing problem. Observe that the crossing is not a first-meeting as in the previous model since the trajectory is jumping over the boundary. Let T_i (resp. $\Delta T_i = T_i - T_{i-1}$) be the i-th arrival time (resp. inter-arrival period) in $\{N(t)\ ;\ t \geq 0\}$, for $i \geq 1$, with $T_0 = 0$.

Suppose that the ΔT_i's have a **p.d.f.** $f_{\Delta T}$. Using a duality argument, we will deduce the **p.d.f.** of τ_u from Theorem 3.1 obtained above for the dual model. Let us introduce a random variable Δ_0 distributed as ΔT_i and independent of the renewal process $\{N(t)\ ;\ t \geq 0\}$.

Theorem 4.1. *The ruin time τ_u has a **p.d.f.** at point t (≥ 0) of the form*

$$f_{\tau_u}(t) = \mathbb{E}\left[(-1)^{M(u+ct)}\, f_{\Delta T}^{*M(u+ct)}(t - \Delta_0)\, h_{M(u+ct)}(u + ct, u/c + \Delta_0)\, \mathbf{1}_{\{t \geq \Delta_0\}}\right], \tag{25}$$

where if $[M(u+ct)=n]$ $(n \geq 0)$ *and* $[\Delta_0 = d_0]$ $(d_0 \geq 0)$, *the function* $h_n(u+ct, u/c+d_0)$ *is the conditional expectation (similar to (11))*

$$h_n\left(u+ct, u/c+d_0\right) =$$
$$\mathbb{E}\left\{G_n\left[0\middle|F_{u+ct}\left[u+c(T_0+d_0)\right],\ldots,F_{u+ct}\left[u+c(T_{n-1}+d_0)\right]\right]\middle|T_n = t-d_0\right\}. \tag{26}$$

Proof. As the first-crossing happens at a jump of the stochastic process, it is difficult to derive the **p.d.f.** of τ_u by a direct argument. A simple trick consists in passing to an associated dual model. Such an approach is rather standard and was used e.g. by Borovkov and Dickson [9].

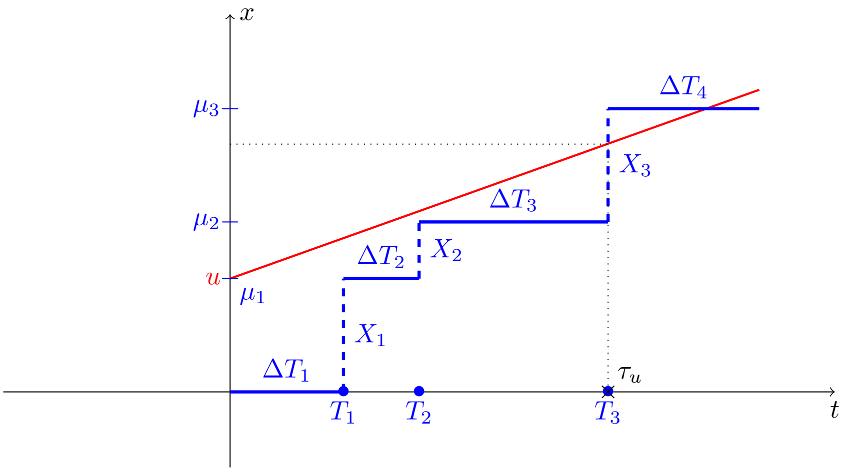

Figure 2: Ruin time in the insurance risk model. The solid red line represents the premium income $x = u + ct$, and the dashed blue line corresponds to a trajectory of the aggregated claim amounts.

More precisely, let us construct a new system of coordinates, denoted by (s, y), where the roles of space and time are swapped. This means that now the abscissa is the space axis and the ordinate is the time axis. First, we put the new origin of these coordinates at the point $(0, \Delta T_1)$ of the original coordinates. Then, we make an anticlockwise rotation of 90° of the whole graph in Figure 2. This yields the corresponding Figure 3. We notice that the straight line is now of equation $y = -V + s/c$, $s \geq 0$, where $V = u/c + \Delta T_1$ is a random variable. By that operation, we have built a dual risk model as in Section 3 whose characteristics are the inverse of those in the insurance model. Thus, this dual risk model is defined as

$$U(s) = V - \frac{s}{c} + \sum_{i=1}^{M(s)} \Delta T_{i+1}, \quad s \geq 0, \tag{27}$$

where the cost function is linear with slope $1/c$ and intercept $V = u/c + \Delta T_1$, the inter-arrival times ΔT_i become the capital gains and the claim amounts X_i become the inter-arrival times in an OSPP $\{M(s)\ ;\ s \geq 0\}$.

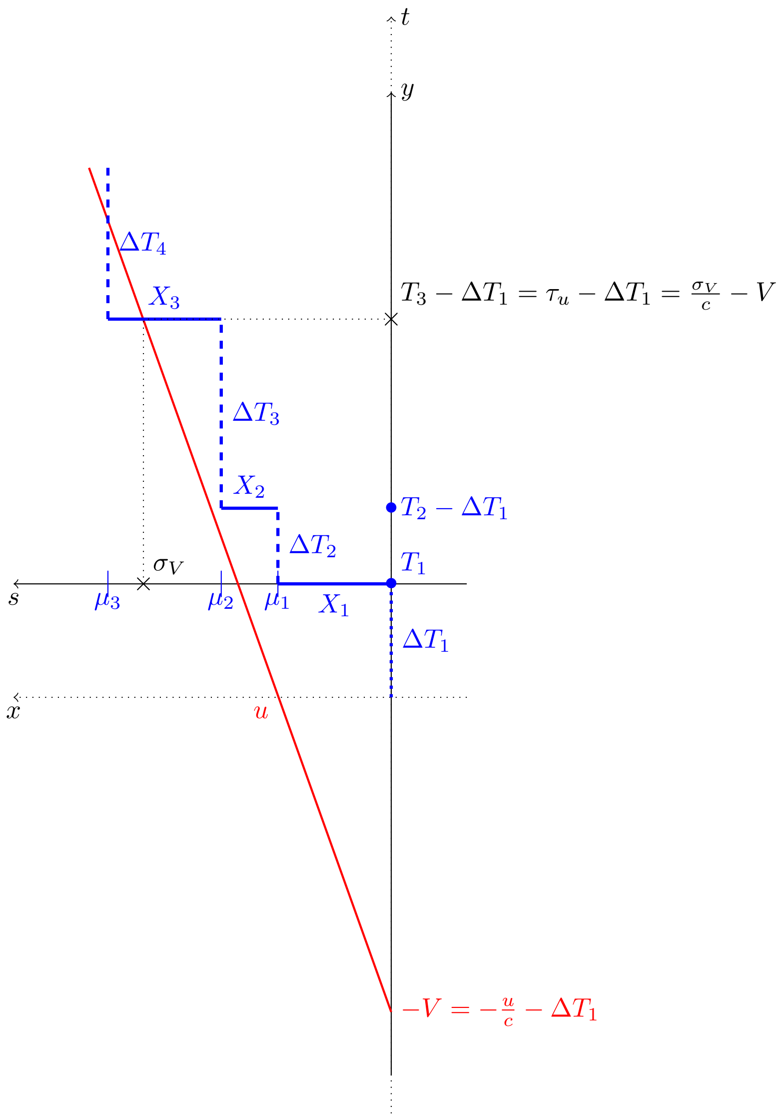

Figure 3: Ruin time in the dual risk model with inverted characteristics. The solid red line represents the cost function $y = -V + s/c$ with $V = u/c + \Delta T_1$, and the dashed blue line corresponds to a trajectory of the aggregated capital gains.

From Figure 3, we see that the first-crossing problem in the insurance model is equivalent to a first-meeting problem in the dual model. In fact, we have the simple identity

$$\tau_u - \Delta T_1 = \sigma_V / c - V,$$

where σ_V is the first-meeting time in the dual model. From the definition of V, this becomes

$$\tau_u = \left(\sigma_{u/c+\Delta T_1} - u\right) / c, \tag{28}$$

so that the **p.d.f.** of τ_u can be expressed as

$$f_{\tau_u}(t) = c\, f_{\sigma_{u/c+\Delta T_1}}(u+ct), \quad t \geq 0. \tag{29}$$

Note that the random variables ΔT_1 and V are independent of the ΔT_{i+1}'s.

The dual risk model (27) satisfies all the assumptions made for the model of Section 3. Note also that $\sigma_V \geq cV$, otherwise ruin is not possible. Thus, Theorem 3.1 provides us with the **p.d.f.** of σ_V, namely

$$f_{\sigma_V}(s) = \mathbb{E}\left[(-1)^{M(s)} f_{\Delta T}^{*M(s)}(s/c - V)\, h_{M(s)}(s,V)\, \mathbf{1}_{\{s \geq cV\}}\right], \tag{30}$$

where if $[M(s) = n]$ $(n \geq 0)$ and $[V = v = u/c + t_1]$ $(t_1 \geq 0)$, the function $h_n(s,v)$ is given by

$$h_n(s,v) = \mathbb{E}\left\{G_n\left[0 \middle| F_s[c(\tilde{T}_0 + v)], \ldots, F_s[c(\tilde{T}_{n-1} + v)]\right] \middle| \tilde{T}_n = s/c - v\right\}. \tag{31}$$

In accordance with (11), we introduced in (31) a sequence of new variables $\tilde{T}_i$ defined by $\tilde{T}_0 = 0$ and $\tilde{T}_i = \sum_{j=1}^{i} \Delta T_{j+1}$ for $i \geq 1$. Combining (30), (31) with (29), we then deduce that the **p.d.f.** of τ_u at point t is

$$\begin{aligned} f_{\tau_u}(t) = \mathbb{E}\Big[&(-1)^{M(u+ct)} f_{\Delta T}^{*M(u+ct)}(t - \Delta T_1)\, h_{M(u+ct)}(u+ct, u/c + \Delta T_1) \\ &\mathbf{1}_{\{t \geq \Delta T_1\}}\Big], \end{aligned} \tag{32}$$

where if $[M(u+ct) = n]$ $(n \geq 0)$ and $[\Delta T_1 = t_1]$ $(t_1 \geq 0)$,

$$\begin{aligned} h_n(u+ct, u/c + t_1) = \mathbb{E}\Big\{G_n\Big[0 \Big| &F_{u+ct}[u + c(\tilde{T}_0 + t_1)], \ldots, \\ &F_{u+ct}[u + c(\tilde{T}_{n-1} + t_1)]\Big] \Big| \tilde{T}_n = t - t_1\Big\}. \end{aligned} \tag{33}$$

To close, note that $\tilde{T}_i = T_{i+1} - t_1$, $i \geq 1$, and remember the definition of Δ_0 given before (25). Substituting Δ_0 for ΔT_1 above, we see that formulas (32), (33) may be rewritten as (25), (26). □

Here again, formulas (25), (26) point out the algebraic structure underlying the density of τ_u. Let us show that they become quite explicit in the case where the claim amounts are mixed exponentials, i.e when $\{M(s)\ ;\ s \geq 0\}$ is a mixed Poisson process (see case (3) of level spacings in Section 2).

Corollary 4.2. *If the X_i's have a mixed exponential distribution, then*

$$f_{\tau_u}(t) = \frac{1}{u+ct}\, \mathbb{E}\left[(u + c\Delta_0)\ f_{\Delta T}^{*M(u+ct)}(t - \Delta_0)\, \mathbf{1}_{\{t \geq \Delta_0\}}\right], \quad t \geq 0. \tag{34}$$

Proof. We proceed as for Corollary 3.2. By assumption, $F_s(x) = x/s$, $x \in (0,s)$. Thus, inside (26), we have

$$\begin{aligned} &h_n\left(u+t, \frac{u}{c} + d_0\right) \\ &\quad = \mathbb{E}\left\{G_n\left[0 \middle| \frac{u + c(T_0 + d_0)}{u+ct}, \ldots, \frac{u + c(T_{n-1} + d_0)}{u+ct}\right] \middle| T_n = t - d_0\right\} \\ &\quad = \left(\frac{c}{u+ct}\right)^n \mathbb{E}\left[G_n\left(-\frac{u}{c} - d_0 \middle| T_0, \ldots T_{n-1}\right) \middle| T_n = t - d_0\right], \end{aligned} \tag{35}$$

using the relation (52) for G_n. Since the ΔT_i's are **i.i.d.** variables, Property A.2 can be applied to the conditional expectation of G_n in (35), which yields

$$\begin{aligned} h_n(u+ct, u/c+d_0) &= [c/(u+ct)]^n \, (-u/c-d_0) \, (-u/c-d_0-t+d_0)^{n-1} \\ &= (-1)^n (u+cd_0)/(u+ct). \end{aligned} \tag{36}$$

Inserting (36) in (25) then gives (34) as announced. □

By formula (34), we retrieve the result obtained by Borovkov and Dickson [9, Theorem 1, Formula 3] in the case where the claim sizes are **i.i.d.** and exponentially distributed (i.e. correspond in our framework to the inter-arrival times in a Poisson process).

5 Dual model with spacing capital gains

Let us go back to the dual risk model described in Section 3. By (8), the reserves are modeled through

$$U(s) = v - as + \sum_{i=1}^{M(s)} Y_i, \quad s \geq 0. \tag{37}$$

using the same notations as before. This time, however, we consider a different set of assumptions. On one side, $\{M(s)\ ;\ s \geq 0\}$ is a renewal process generated by its **i.i.d.** inter-arrival times $\{\Delta S_i\ ;\ i \geq 1\}$. On the other side, the capital gains $\{Y_i\ ;\ i \geq 1\}$ are non-negative random variables, possibly dependent, with a level spacing distribution. These Y_i's generate an OSPP $\{N(t)\ ;\ t \geq 0\}$, say.

Instead of focusing on the **p.d.f.** of the ruin time, we are now interested in the non-ruin probability over any finite time horizon. So, let

$$\varphi_v(s) = \mathbb{P}(\sigma_v > s), \quad s \geq 0, \tag{38}$$

be the probability of non-ruin until time s. Obviously, $\varphi_v(s) = 1$ if $s < v/a$, so from now on we will suppose that $s \geq v/a$. Proposition 5.1 below expresses $\varphi_v(s)$ in terms of a family of Appell polynomials. We refer to the Appendix A for a short presentation of these polynomials. Given a set of reals $U = \{u_i\ ;\ i \geq 1\}$, the Appell polynomial of degree n in x is denoted by $A_n(x|U)$.

For the proof, we will use again a duality argument to take advantage of a result known for the insurance risk model. Indeed, Lefèvre and Picard [23] derived a formula for the finite-time non-ruin probability when the claim amounts are **i.i.d.** and the claim arrival process is an OSPP. Similarly in what was done in Section 4, we introduce a random variable Δ_0 distributed as ΔS_i and independent of the renewal process $\{M(s)\ ;\ s \geq 0\}$.

Theorem 5.1. *The non-ruin probability $\varphi_v(s)$ $(s \geq v/a)$ can be expressed as*

$$\begin{aligned} &\varphi_v(s) = \\ &\mathbb{E}\left[\mathbf{1}_{\{\Delta_0 > v/a\}} + g_{N(as-v)}(as-v, v/a-\Delta_0)\,\mathbf{1}_{\{\Delta_0 \leq v/a,\, S_{N(as-v)} \leq s-\Delta_0\}}\right], \end{aligned} \tag{39}$$

where if $[N(as-v) = n]$ $(n \geq 0)$ and $[\Delta_0 = d_0]$ $(0 \leq d_0 \leq v/a)$, the function $g_n(as-v, v/a-d_0)$ is the conditional expectation

$$\begin{aligned} &g_n(as-v, v/a-d_0) \\ &= \mathbb{E}\left\{A_n\left[1\middle|F_{as-v}\left[a(S_1+d_0-v/a)_+\right], \ldots, F_{as-v}\left[a(S_n+d_0-v/a)_+\right]\right]\right\}. \end{aligned} \tag{40}$$

Proof. As announced, we apply duality to convert the first-meeting problem under study for the dual model (39) into an equivalent first-crossing problem in the insurance model with inverted characteristics. Such a reasoning was followed recently by Dimitrova et al. [15] in the same purpose. The results used here for the insurance model, however, were obtained by Lefèvre and Picard [23] and their form is especially simple and compact.

Let us start by looking again at Figure 1. We now construct a new system of coordinates, denoted by (t, x), in which space and time are swapped. So, the abscissa is the space axis and the ordinate is the time axis. The new origin of these coordinates is put at the point $(0, \Delta S_1)$ of the original coordinates. Then, an anticlockwise rotation of 90° is operated on the whole graph of Figure 1. This yields the corresponding Figure 4. The new straight line is of equation $x = U + t/a$, $t \geq 0$, where $U = v/a - \Delta S_1$ is a random variable. In that way, we have built an insurance model whose characteristics are the inverse of those in the dual model. This insurance risk model is defined as

$$R(t) = U + \frac{t}{a} - \sum_{i=1}^{N(t)} \Delta S_{i+1}, \tag{41}$$

where the premium rate is equal to $1/a$ and the initial reserves are of amount $U = v/a - \Delta S_1$, the inter-arrival times ΔS_i become the claim sizes and the capital gains Y_i become the inter-arrival times in an OSPP $\{N(t)\ ;\ t \geq 0\}$.

It is clear from Figure 4 that the first-meeting problem in the dual model is equivalent to a first-crossing time in the insurance risk model. Specifically, we see that

$$\sigma_v - \Delta S_1 = \tau_U/a + U,$$

where τ_U is the first-crossing time in the insurance model. Using the definition of U, this yields

$$\sigma_v = (\tau_{v/a-\Delta S_1} + v)/a. \tag{42}$$

From (38) and (42), we can the rewrite the probability $\varphi_v(s)$ as

$$\begin{aligned} \varphi_v(s) &= \mathbb{P}\left(\tau_{v/a-\Delta S_1} > as - v\right) \\ &= \mathbb{E}\left[\phi_{v/a-\Delta S_1}(as - v)\right] \quad \text{say}, \end{aligned} \tag{43}$$

where the expectation is with respect to the variable ΔS_1, and if $\Delta S_1 = S_1 = s_1$ (≥ 0) is fixed, then $\phi_{v/a-s_1}(as - v) = \mathbb{P}(\tau_{v/a-s_1} > t)$ denotes the probability of non-ruin until time $as - v$ in the corresponding insurance model with initial reserves $v/a - s_1$. Note that if $s_1 > v/a$, then the initial reserves are negative so that ruin arises at time 0.

Since the counting process $\{N(t)\ ;\ t \geq 0\}$ is an OSPP, the result of Lefèvre and Picard [23, Proposition 4.1] is applicable. We have, however to account for the possibility that the initial reserves $U = v/a - \Delta S_1$ are here random and can be negative. Then, the non-ruin probability until time $t = as - v$ can be expressed as

$$\mathbb{E}[\phi_U(t)] = \mathbb{E}\left[\mathbf{1}_{\{U<0\}} + g_{N(t)}(t, U)\, \mathbf{1}_{\{U \geq 0, \tilde{S}_{N(t)} \leq U + t/a\}}\right], \tag{44}$$

where if $[N(t) = n]$ $(n \geq 0)$ and $[U = u]$ $(u \geq 0)$, $g_n(t, u)$ is given by

$$g_n(t, u) = \mathbb{E}\left\{A_n\left[1 \middle| F_t[a(\tilde{S}_1 - u)_+], \ldots, F_t[a(\tilde{S}_n - u)_+]\right]\right\}. \tag{45}$$

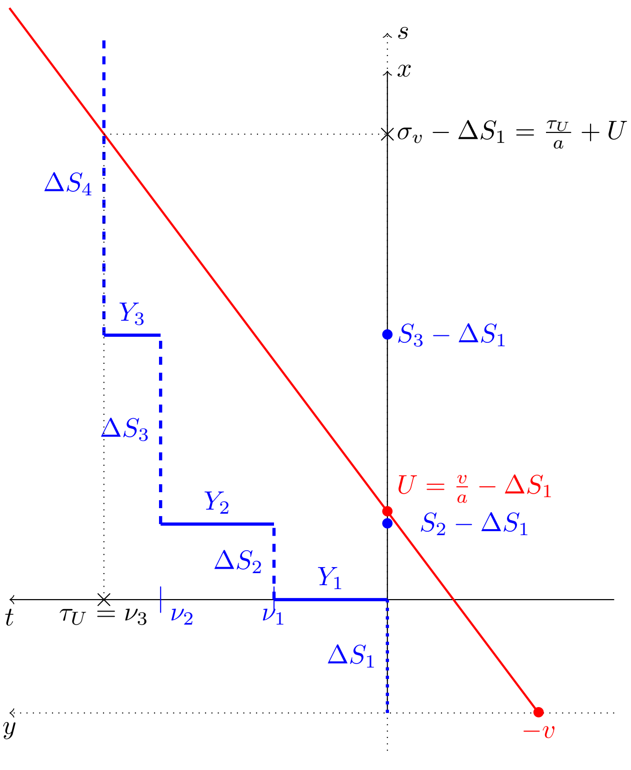

Figure 4: Ruin time in the insurance risk model with inverted characteristics. The solid red line represents the premium function $x = U + t/a$ with $U = v/a - \Delta S_1$, and the dashed blue line corresponds to a trajectory of the aggregated claim amounts.

In (44), (45), we introduced a sequence of new variables $\tilde{S}_i$ defined by $\tilde{S}_0$ and $\tilde{S}_i = \sum_{j=1}^{i} \Delta S_{j+1}$ for $i \geq 1$. Thus, from (43), (44), (45), we obtain

$$\begin{aligned}\varphi_v(s) = \mathbb{E}\Big[&\mathbf{1}_{\{\Delta S_1 > v/a\}} + \\ &g_{N(as-v)}(as-v, v/a - \Delta S_1)\, \mathbf{1}_{\{\Delta S_1 \leq v/a,\, \tilde{S}_{N(as-v)} \leq s - \Delta S_1\}}\Big],\end{aligned} \tag{46}$$

where if $[N(as-v) = n]$ $(n \geq 0)$ and $[\Delta S_1 = s_1]$ $(0 \leq s_1 \leq v/a)$,

$$\begin{aligned}&g_n(as-v, v/a - s_1) = \\ &\mathbb{E}\left\{A_n\left[1 \Big| F_{as-v}[a(\tilde{S}_1 - v/a + s_1)_+], \ldots, F_{as-v}[a(\tilde{S}_n - v/a + s_1)_+]\right]\right\}.\end{aligned} \tag{47}$$

Finally, as $\tilde{S}_i = S_{i+1} - s_1$, $i \geq 1$, and using the definition of Δ_0, we see that (46), (47) match formulas (39), (40). □

The result of Theorem 5.1 looks a little like the one obtained by Dimitrova et al. [15, Proposition 2.2]. The risk process examined there is another dual model where the capital gain arrival process is arbitrary and their amounts are distributed as independent linear combinations of exponential variables.

6 Concluding remarks

Our study deals with three ruin problems for different risk models with dependence. The first risk process is the dual model where the capital gains are **i.i.d.** but their arrival process satisfies the order statistic property. Using a direct analysis, we derived the **p.d.f.** of the ruin time for the model. The second risk process is the Sparre-Andersen insurance model where claims arrive according to a renewal process and their amounts have a level spacing distribution. We obtained here the **p.d.f.** of the ruin time by applying a duality argument to the previous problem. The third risk process is again the dual model but where the capital gains have a level spacing distribution and their arrival is modeled by a renewal process. By exploiting duality, we derived the finite-time non-ruin probability for the model. In all cases, the formulas have a clear algebraic structure and can be used for numerical computation. Some illustrations for simpler variants of these models can be found in e.g. Goffard and Lefèvre [17] and Dimitrova et al. [15]. Of course, other ruin topics could be studied by a similar duality approach. To close, we mention that an alternative approach to such problems would consist in working with Laplace transforms. This is the method followed e.g. in Perry et al. [29] and the references therein.

Acknowledgements

This research received partial support from the *ARC project IAPAS* of the Fédération Wallonie-Bruxelles.

A Appell and A-G polynomials

Appell and Abel-Gontcharov (A-G) polynomials are well-known in mathematics. Recently, they were used to solve various problems in statistics and applied probabilty. A short presentation is provided below. We refer to Lefèvre and Picard [26] for a review with many references.

Let $U = \{u_i,\, i \geq 1\}$ be a sequence of reals, non-decreasing in our context. To U is attached a (unique) family of Appell polynomials of degree n in x, $\{A_n(x|U),\, n \geq 0\}$, defined as follows. Starting with $A_0(x|U) = 1$, the $A_n(x|U)$'s satisfy the differential equations

$$A_n^{(1)}(x|U) = nA_{n-1}(x|U),$$

with the border conditions

$$A_n(u_n|U) = 0, \quad n \geq 1.$$

So, each A_n, $n \geq 1$, has the integral representation

$$A_n(x|U) = n! \int_{u_n}^{x} \left[\int_{u_{n-1}}^{y_n} \mathrm{d}y_{n-1} \ldots \int_{u_1}^{y_1} \mathrm{d}y_1 \right] \mathrm{d}y_n. \tag{48}$$

In parallel, to U is attached a (unique) family of Abel-Gontcharov (A-G) polynomials of degree n in x, $\{G_n(x|U),\, n \geq 0\}$. Starting with $G_0(x|U) = 1$, the $G_n(x|U)$'s satisfy the differential equations

$$G_n^{(1)}(x|U) = nG_{n-1}(x|\mathcal{E}U),$$

where $\mathcal{E}U$ is the shifted family $\{u_{i+1},\, i \geq 1\}$, and with the border conditions

$$G_n(u_1|U) = 0, \quad n \geq 1.$$

So, each G_n, $n \geq 1$, has the integral representation

$$G_n(x|U) = n! \int_{u_1}^{x} \left[\int_{u_2}^{y_1} \mathrm{d}y_2 \ldots \int_{u_n}^{y_{n-1}} \mathrm{d}y_n \right] \mathrm{d}y_1. \tag{49}$$

Note that the Appell and A-G polynomials are sometimes defined without the factor $n!$ in (48) and (49). Of course, these polynomials are related through the identity

$$G_n(x|u_1, \ldots, u_n) = A_n(x|u_n, \ldots, u_1), \quad n \geq 1. \tag{50}$$

However, the two families, i.e. considered for all $n \geq 0$, are distinct and enjoy different properties (see Lefèvre and Picard [26]).

From (48) and (49), we see that the polynomials A_n and G_n can be interpreted in terms of the joint distribution of the vector $(U_{1:n}, \ldots, U_{n:n})$ of order statistics for n independent uniform variables on $(0, 1)$.

Proposition A.1. *For $0 \leq u_1 \leq \ldots \leq u_n \leq x \leq 1$, and $n \geq 1$,*

$$P[U_{1:n} \geq u_1, \ldots, U_{n:n} \geq u_n \text{ and } U_{n:n} \leq x] = A_n(x|u_1, \ldots, u_n),$$

while for $0 \leq x \leq u_1 \leq \ldots \leq u_n \leq 1$, and $n \geq 1$,

$$P[U_{1:n} \leq u_1, \ldots, U_{n:n} \leq u_n \text{ and } U_{1:n} \geq x] = (-1)^n\, G_n(x|u_1, \ldots, u_n). \tag{51}$$

These representations play a key role in the first-passage problems discussed in the paper. We will also use the simple relation

$$A_n(x|a+bU) = b^n A_n\left[(x-a)/b\,|U\right], \quad n \geq 1, \tag{52}$$

and similarly for G_n. An important particular case in our study is when the parameters in U are random and correspond to partial sums of exchangeable random variables.

Proposition A.2. *Let $\{X_n \,;\, n \geq 1\}$ be a sequence of exchangeable random variables, with partial sums $S_n = \sum_{k=1}^n X_k$ $(S_0 = 0)$. Then, for $n \geq 1$,*

$$\mathbb{E}\left[A_n(x|S_1, \ldots, S_n)|S_n\right] = x^{n-1}(x - S_n), \tag{53}$$

$$\mathbb{E}\left[G_n(x|S_0, \ldots, S_{n-1})|S_n\right] = x(x - S_n)^{n-1}. \tag{54}$$

Proof. The identity (53) was derived in Lefèvre and Picard [23, Property A.1]. For (54), we write

$$\begin{aligned}\mathbb{E}\left[G_n(x|S_0, \ldots, S_{n-1})|S_n\right] &= \mathbb{E}\left[G_n(x - S_n|S_0 - S_n, \ldots, S_{n-1} - S_n)|S_n\right] \\ &= (-1)^n \mathbb{E}\left[G_n(S_n - x|S_n, \ldots, S_n - S_{n-1})|S_n\right] \\ &= (-1)^n \mathbb{E}\left[A_n(S_n - x|S_n - S_{n-1}, \ldots, S_n)|S_n\right], \end{aligned} \tag{55}$$

using the relations (52) and (50). As the X_n's are exchangeable, we deduce from (55) and (53) the desired formula (54). □

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