A polynomial expansion to approximate the ultimate ruin probability in the compound Poisson ruin model

Pierre-Olivier GOFFARD*, Stéphane LOISEL** and Denys POMMERET***

* AXA France et Université de Aix-Marseille, pierreolivier.goffard@axa.fr.

** Université de Lyon, Université de Lyon 1, Institut de Science Financière et d'Assurance, stephane.loisel@univ-lyon1.fr.

*** Université de Aix-Marseille, denys.pommeret@univ-amu.fr.

Abstract

A numerical method to approximate ruin probabilities is proposed within the frame of a compound Poisson ruin model. The defective density function associated to the ruin probability is projected in an orthogonal polynomial system. These polynomials are orthogonal with respect to a probability measure that belongs to a Natural Exponential Family with Quadratic Variance Function (NEF-QVF). The method is convenient in at least four ways. Firstly, it leads to a simple analytical expression of the ultimate ruin probability. Secondly, the implementation does not require strong computer skills. Thirdly, our approximation method does not necessitate any preliminary discretisation step of the claim sizes distribution. Finally, the coefficients of our formula do not depend on initial reserves.
Keywords: compound Poisson model, ultimate ruin probability, natural exponential families with quadratic variance functions, orthogonal polynomials, gamma series expansion, Laplace transform inversion.

1 Introduction

A non-life insurance company is assumed to be able to follow the financial reserves' evolution associated with one of its portfolios in continuous time. The number of claims until time t is assumed to be an homogeneous Poisson process $\{N_t\}_{t\geq 0}$, with intensity β. The successive claim amounts $(U_i)_{i\in\mathbb{N}^*}$, form a sequence of positive i.i.d. continuous random variables and independent of $\{N_t\}_{t\geq 0}$, with density function f_U and mean μ. The initial reserves are of amounts $u \geq 0$, and the premium rate is constant and equal to $p \geq 0$. The risk reserve process is therefore defined as

$$R_t = u + pt - \sum_{i=1}^{N_t} U_i,$$

the associated claims surplus process is defined as $S_t = u - R_t$. In this work, we focus on the evaluation of ultimate ruin probabilities (or infinite-time ruin probabilities) defined as

$$\psi(u) = P\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right) = P\left(\sup_{t\geq 0} S_t > u | S_0 = 0\right). \quad (1.1)$$

This model is called a compound Poisson model (also known as Cramer-Lundberg ruin model) and has been widely studied in the risk theory literature. For general a background about ruin theory, we refer to [21] and [5].
A useful technique in applied mathematics consists of determining a probability density function from the knowledge of its Laplace transform. We give here a brief review of the literature involving numerical inversion of Laplace transform and ruin probability approximations. In a few particular cases, the inversion of the Laplace transform associated with ruin probabilities is manageable analytically and leads to closed formula. But in most cases numerical methods are needed. The Laguerre method is an old established method based on the 1935 Tricomi-Widder Theorem. The recovered function takes the form of a sum of Laguerre functions derived through orthogonal projections. The numerical inversion of Laplace transform using Laguerre series has been originally described in [24] and improved in [1]. In the wake of Laguerre series method, we found attempts in the actuarial science literature to write probability density functions as sum of gamma densities. For instance the early work of Bowers [9] that gave rise to the so-called Beekman-Bowers approximation for the ultimate ruin probability, derived in [8]. The idea is to approximate the ultimate ruin probability by the survival function of a gamma distribution using moments fitting. Gamma series expansion has been employed in [23] and later in [4]. The authors highlight that it is useful to carry out both analytical calculations and numerical approximations. They show that the direct injection of the gamma series expression into integro-differential equations leads to reccurence relations between the expansion's coefficients and therefore characterize them. They focus on the finite-time ruin probability but the results are valid in the infinite-time case by letting the time t tend to infinity. We explain later that our method, within the frame of ruin probabilities approximation, is closely related to the Laguerre method and represents in fact an improvement. The numerical inversion via Fourier-series techniques (Fast Fourier Transform) received a great deal of interest. These techniques have been presented for instance in [2] within a queueing theory setting. For an application within an actuarial framework, we refer to [13] and [21] Chapter 5 Section 5.5. There is also a great body of literature dealing with Laplace transform inversion linked the Hausdorff moment problem. Probability density function are recovered from different kind of moments. The use of exponential moments and scaled Laplace transform is presented in [17] and has been performed for ruin probabilities computations in [18]. In the work of Gzyl et al [15], the maximum entropy applied to fractional exponential moments is employed to determine the probability of ultimate ruin. Recently, Albrecher et al. [3] and Avram et al. [6] discuss different methods for computing the inverse Laplace transform. The first one consider a numerical inversion procedure based on a quadrature rule that uses as stepping stone a rational approximation of the exponential function in the complex plane. The second one implements and reviews much of the work done using Padé approximants to invert Laplace transform.
There are several usual techniques for calculation of ultimate ruin probabilities. We want to mention a classical iterative method that we will use for comparison purposes. The so called Panjer's algorithm introduced in [20], has been widely used in the actuarial field. One can find an application to the computation of the probability of ultimate ruin in [11]. The method that we propose here consists in an orthogonal projection of the defective probability density function, associated with the probability of ultimate ruin, with respect to a reference probability measure that belongs to the Natural Exponential Families with Quadratic Variance Function. The desired defective PDF takes the form of an infinite serie of orthonormal polynomials (orthogonal with respect to the aforementioned reference probability measure). The coefficients of the expansion are defined by a scalar product

and are computed from the Laplace transform or equivalently from the moments of the distribution. Ruin probability approximations are obtained through truncation of the infinite serie followed by integration. This method permits the recovery of functions from the knowledge of their Laplace transform. Once the set of coefficients of the expansion has been evaluated, ultimate ruin probabilities can be approximated for any initial reserve. It is easy to implement, does no necessitate large computation time and is competitive in terms of accuracy. The approximation of the ruin probability allows manipulations such as integration or reinjection in formulas to derive approximations of distributions that governed other quantities of interest in ruin theory. For instance, the probability density function of the surplus prior to ruin involves the ultimate ruin probability, we refer to [10, 14] for more details. This work is also a theoretical background in view of a future statistical application. Many papers deals with statistical estimation of ruin probabilities when observations of the claim sizes are available. The use of a Laplace inversion formula as basis for a nonparametric estimator has already been employed for ruin probabilities estimation in [16, 25], scaled laplace transform and the maximum of entropy may offer the same possibility and would be based on empirical estimation of the moments. The definition of the coefficients in our method are based on quantities that are well adapted to empirical estimations. By pluging in the estimators of the coefficients, we will obtain a nonparametric estimator taking the form of an orthogonal serie. We aim to investigate ruin probability statistical estimation in future work, for now we only consider ruin probability approximations. In Section 2, we introduce a density expansion formula based on orthogonal projection within the frame of NEF-QVF. Our main results are developed in Section 3: the expansion for ultimate ruin probabilities is derived, a sufficient condition of applicability is given and the goodness of the approximation is discussed. Section 4 is devoted to numerical illustrations. Just like what is done in [15], we compare our method to other existing method, namely Panjer's algorithm, Fast Fourrier Transform and scaled Laplace transform inversion.

2 Polynomial expansions of a probability density function

Let $F = \{P_\theta, \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}$ be a Natural Exponential Family (NEF), see [7], generated by a probability measure ν on $\mathbb{R}$ such that

$$\begin{aligned} P_\theta(X \in A) &= \int_A exp\{x\theta - \kappa(\theta)\} d\nu(x) \\ &= \int_A f(x,\theta) d\nu(x), \end{aligned}$$

where

- $A \subset \mathbb{R}$,
- $\kappa(\theta) = log\left(\int_{\mathbb{R}} e^{\theta x} d\nu(x)\right)$ is the Cumulant Generating Function (CGF),
- $f(x,\theta)$ is the density of P_θ with respect to ν.

Let X be a random variable P_θ distributed. We have

$$\begin{array}{ccccccc} \mu &=& \mathrm{E}_\theta(X) &=& \int x dF_\theta(x) &=& \kappa'(\theta), \\ \mathrm{V}(\mu) &=& \mathrm{Var}_\theta(X) &=& \int (x-\mu)^2 dF_\theta(x) &=& \kappa''(\theta). \end{array}$$

The application $\theta \to \kappa'(\theta)$ is one to one. Its inverse function $\mu \to h(\mu)$ is defined on $\mathcal{M}$ $= \kappa'(\Theta)$. With a slight change of notation, we can rewrite $F = \{P_\mu, \mu \in \mathcal{M}\}$, where P_μ

has mean μ and density $f(x,\mu) = exp\{h(\mu)x - \kappa(h(\mu))\}$ with respect to ν. A NEF has a Quadratic Variance Function (QVF) if there exists reals v_0, v_1, v_2 such that

$$V(\mu) = v_0 + v_1\mu + v_2\mu^2. \tag{2.1}$$

The Natural Exponential Families with Quadratic Variance Function (NEF-QVF) include the normal, gamma, hyperbolic, Poisson, binomial and negative binomial distributions. Define

$$P_n(x,\mu) = V^n(\mu)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}/f(x,\mu), \tag{2.2}$$

for $n \in \mathbb{N}$. Each $P_n(x,\mu)$ is a polynomial of degree n in both μ and x. Moreover, if F is a NEF-QVF, $\{P_n\}_{n\in\mathbb{N}}$ is a family of orthogonal polynomials with respect to P_μ in the sense that

$$< P_n, P_m >= \int P_n(x,\mu)P_m(x,\mu)dP_\mu(x) = \delta_{nm}||P_n||^2, \qquad m,n \in \mathbb{N},$$

where δ_{mn} is the Kronecker symbol equal to 1 if $n = m$ and 0 otherwise. For the sake of simplicity, we choose $\nu = P_{\mu_0}$. Then, $f(x,\mu_0) = 1$ and we write

$$P_n(x) = P_n(x,\mu_0) = V^n(\mu_0)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}_{\mu=\mu_0}. \tag{2.3}$$

We also consider in the rest of the paper a normalized version of the polynomials defined in (2.3) with $Q_n(x) = P_n(x)/||P_n||$. For an exhaustive review regarding NEF-QVF and their properties, we refer to [19].
We will denote by $L^2(\nu)$ the space of functions square integrable with respect to ν.

Proposition 1. *Let ν be a probability measure that generates a NEF-QVF, with associated orthonormal polynomials $\{Q_n\}_{n\in\mathbb{N}}$. Let X be a random variable with density function $\frac{dP_X}{d\nu}$ with respect to ν. If $\frac{dP_X}{d\nu} \in L^2(\nu)$ then we have the following expansion*

$$\frac{dP_X}{d\nu}(x) = \sum_{n=0}^{+\infty} E(Q_n(X))Q_n(x). \tag{2.4}$$

Proof. By construction $\{Q_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis of $L^2(\nu)$, and by orthogonal projection we get

$$\frac{dP_X}{d\nu}(x) = \sum_{n=0}^{+\infty} < Q_n, \frac{dP_X}{d\nu} > Q_n(x).$$

It follows that

$$\begin{aligned} < Q_n, \frac{dP_X}{d\nu} > Q_n(x) &= \int Q_n(y)\frac{dP_X}{d\nu}(y)d\nu(y) \times Q_n(x) \\ &= \int Q_n(y)dP_X(y) \times Q_n(x) \\ &= E(Q_n(X))Q_n(x). \end{aligned}$$

□

Denote by f_ν and f_X the probability density functions of ν and X respectively. Proposition 1 gives an expansion of f_X that takes the following simple form

$$f_X(x) = \sum_{n=0}^{+\infty} a_n Q_n(x) f_\nu(x), \tag{2.5}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of real number called coefficients of the expansion in the rest of the paper, $\{Q_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence of polynomials with respect to ν, and f_ν is the PDF of ν. The polynomials $\{Q_n\}_{n\in\mathbb{N}}$ are of degree n in x and can therefore be written as $Q_n(x) = \sum_{i=0}^{n} q_{i,n} x^i$. Using this last remark, we rewrite the coefficients of the expansion as

$$\begin{aligned} a_n &= E(Q_n(X)) \\ &= \sum_{i=1}^{n} q_{i,n} E\left(X^i\right) && (2.6) \\ &= \sum_{i=1}^{n} q_{i,n} (-1)^i \left[\frac{d^i \widehat{f_X}(s)}{ds^i}\right]_{s=0}, && (2.7) \end{aligned}$$

where $\widehat{f_X}(s) = \int e^{-sx} dP_X(x)$ is the Laplace transform of the random variable X. The approximation of the PDF of X is simply obtained by truncation of the infinite serie in (2.5). We need to choose a NEF-QVF then a member of the choosen family characterized by its parameters. These choices are to be made wisely so as to ensure the validity of the expansion and to reach an acceptable level of accuracy that goes along with an order of truncation as small as possible. In the light of the expression of the coefficients (2.6), it seems natural to consider a statistical extension that would lead to a nonparametric estimator of the probability density function. However, it is of interest to start with the probabilistic problem as it is the theoretical basis for a statistical application. The next section shows how to use the orthogonal polynomials and NEF-QVF framework to approximate ruin probabilities.

3 Application to the ruin problem

3.1 General formula

The ultimate ruin probability in the Cramer-Lundberg ruin model is the survival function of a geometric compound distributed random variable

$$M = \sum_{i=1}^{N} U_i^I, \tag{3.1}$$

where N is an integer valued random variable having a geometric distribution with parameter $\rho = \frac{\beta\mu}{p}$, and $(U_i^I)_{i\in\mathbb{N}^*}$ is a sequence of independent and identically distributed nonnegative random variables having CDF $F_{U^I}(x) = \frac{1}{\mu}\int_0^x \overline{F_U}(y)dy$. The distribution of M has an atom at 0 with probability mass $P(N=0) = 1-\rho$. The probability measure that governs M is

$$dP_M(x) = (1-\rho)\,\delta_0(x) + dG_M(x), \tag{3.2}$$

where dG_M is the continuous part of the probability measure associated to M which admits a defective probability density function with respect to the Lebesgue measure. We denote by g_M the defective probability density function. The ultimate ruin probability is then obtained by integrating the continuous part as the discrete part vanishes

$$\psi(u) = P(M > u) = \int_u^{+\infty} dG_M(x).$$

Theorem 1. *Let ν be an univariate distribution having a probability density function with respect to the Lebesgue measure, and that generates a NEF-QVF. Let $\{Q_n\}_{n\in\mathbb{N}}$ be the sequence of orthonormal polynomials with respect to ν. If $\frac{dG_M}{d\nu} \in L^2(\nu)$ then*

$$\psi(u) = \sum_{n=0}^{+\infty} a_n \int_u^{+\infty} Q_n(x) d\nu(x), \tag{3.3}$$

where $\{a_n\}_{n\in\mathbb{N}}$ is defined as in (2.7). Recall that $a_n = \sum_{i=1}^{n} q_{i,n}(-1)^i \left[\frac{d^i \widehat{g_M}(s)}{ds^i}\right]_{s=0}$ and $Q_n(x) = \sum_{i=0}^{n} q_{i,n} x^i$.

Proof. We simply apply Proposition 1 to get the result.

3.2 Approximation with Laguerre polynomials

We derive an approximation for the ultimate ruin probability, using Theorem 1, combined with truncations of the infinite series (3.3). For $K \in \mathbb{N}$, we will denote by

$$\psi_K(u) = \sum_{n=0}^{K} a_n \int_u^{+\infty} Q_n(x) d\nu(x), \tag{3.4}$$

the approximated ruin probability with truncation order K. In practice, as the distribution of M is supported on $\mathbb{R}^+$, we will choose the gamma distribution with mean parameter m and scale parameter r, that is:

$$d\nu(x) = f_\nu(x)\mathbf{1}_{\mathbb{R}^+}(x) d\lambda(x) = \frac{x^{r-1}e^{-x/m}}{\Gamma(r)m^r}\mathbf{1}_{\mathbb{R}^+}(x) d\lambda(x).$$

The associated orthogonal polynomials are the generalized Laguerre polynomials. By definition, they satisfy the following orthogonality condition

$$\int_0^{+\infty} L_n^{r-1}(x) L_m^{r-1}(x) x^{r-1} e^{-x} dx = \binom{n+r-1}{n} \delta_{nm}.$$

The polynomials involved in the ruin probability approximation in (3.4) are the generalized Laguerre polynomials with a slight change in comparison to the definition given in [22], namely

$$Q_n(x) = (-1)^n \binom{n+r-1}{n}^{-1/2} L_n^{r-1}(x/\mu). \tag{3.5}$$

Remark 1. *The Laguerre functions are defined in [1] as*

$$l_n(x) = e^{-x/2} L_n(x), \ x \geq 0. \tag{3.6}$$

The application of the Laguerre method consists in representing g_M as a Laguerre serie

$$g_M(x) = \sum_{n=0}^{+\infty} a_n l_n(t). \tag{3.7}$$

One can note that the representation (3.7) *is close to the expansion proposed in this paper, indeed it is exactly the same if we choose for our expansion* $r = 1$ *and* $m = 2$. *The difference lies in the possibility to change the parameter in our expansion. We will see later that the parametrization is of prime importance.*

The defective probability density function associated to G_M has the following expression

$$g_M(x) = \sum_{n=1}^{+\infty} (1-\rho)\rho^n f_{U^I}^{*n}(x). \tag{3.8}$$

By Taking the Laplace transform of (3.8), we get

$$\widehat{g_M}(s) = \frac{(1-\rho)\rho \widehat{f_{U^I}}(s)}{1 - \rho \widehat{f_{U^I}}(s)}, \tag{3.9}$$

with $\widehat{f_{U^I}}(s) = \int e^{-sx} f_{U^I}(x)dx$ is the Laplace transform of f_{U^I}. The Laplace transform of the claim size distribution appears in the formula. This fact limits the application to claim sizes distributions that admit a well defined Laplace transform, namely light-tailed distributions. We want to mention that this problem might be reconsidered once the study of the statistical extension will be done. In [17], approximations of the ruin probability in case of heavy tail claim amounts are derived from the associated nonparametric estimator computed with simulated data.

3.3 Integrability condition

We illustrate here how the applicability of the method is subject to the parametrization i.e. the choice of m and r. The parametrization permits in this problem to ensure the integrability condition. First we define the adjustment coefficient as the unique positive solution of the so-called Cramer-Lundberg equation

$$\widehat{m_{U^I}}(s) = \frac{1}{\rho}, \tag{3.10}$$

where $\widehat{m_{U^I}}(s) = \int_0^{+\infty} e^{sx} f_{U^i}(x)dx$ is the moment generating function of U^I. The integrability condition $\frac{dG_M}{d\nu} \in L^2(d\nu)$ is equivalent to

$$\int_0^{+\infty} g_M(x)^2 e^{-x/m} x^{1-r} dx < \infty. \tag{3.11}$$

In order to ensure this condition, we need the following results.

Theorem 2. *Assume that* U^I *admits a bounded density function and that the equation* (3.10) *admits a positive solution, then for all* $x \geq 0$

$$g_M(x) \leq C(s_0)e^{-s_0 x}, \tag{3.12}$$

with $s_0 \in [0, \gamma)$ *and* $C(s_0) \geq 0$, *where* γ *is the adjustment coefficient.*

Proof. In order to prove the theorem we need the following lemma regarding the survival function $\overline{F_U}$ of the claim sizes distribution.

Lemma 1. *Let* U *be a non-negative random variable with bounded density function* f_U. *Assume there exists* $s_0 > 0$ *such that* $\widehat{m_U}(s_0) < +\infty$. *Then there exists* $A(s_0) > 0$ *such that for all* $x \geq 0$

$$\overline{F_U}(x) \leq A(s_0)e^{-s_0 x}. \tag{3.13}$$

Proof. As $\widehat{m_U}(s_0) < +\infty$, we have

$$\begin{aligned}
\widehat{m_U}(s_0) - 1 &= \int_0^{+\infty} (e^{s_0 x} - 1) f_U(x) dx \\
&= s_0 \int_0^{+\infty} \int_0^x e^{s_0 y} f_U(x) dy dx \\
&= s_0 \int_0^{+\infty} e^{s_0 y} \overline{F_U}(y) dy \\
&\geq s_0 \int_0^x e^{s_0 y} \overline{F_U}(y) dy \\
&\geq \overline{F_U}(x)(e^{s_0 x} - 1).
\end{aligned}$$

thus, we deduce that $\forall x \geq 0$

$$\overline{F_U}(x) \leq (\widehat{m_U}(s_0) - 1 + \overline{F_U}(x)) e^{-s_0 x}. \tag{3.14}$$

□

The equation (3.10) is equivalent to

$$\rho \widehat{m_U}(s) = 1 + s\mu. \tag{3.15}$$

The fact that γ is a solution of the equation (3.10) implies that $\widehat{m_U}(s) < +\infty$, $\forall s_0 \in [0, \gamma)$ and by application of Lemma 1, we get the following inequality upon the density function of U^I

$$f_{U^I}(x) = \frac{\overline{F_U}(x)}{\mu} \leq B(s_0) e^{-s_0 x}. \tag{3.16}$$

In view of (3.8), it is easily checked that g_M satisfies the following defective renewal equation,

$$g_M(x) = \rho(1-\rho) f_{U^I}(x) + \rho \int_0^x f_{U^I}(x-y) g_M(y) dy. \tag{3.17}$$

We can therefore bound g_M as in (3.12),

$$\begin{aligned}
g_M(x) &\leq \rho(1-\rho) f_{U^I}(x) + \int_0^{+\infty} f_{U^I}(x-y) g_M(y) dy \\
&\leq \rho(1-\rho) B(s_0) e^{-s_0 x} + B(s_0) e^{-s_0 x} \int_0^{+\infty} e^{s_0 y} g_M(y) dy \\
&= (\rho(1-\rho) + \widehat{g_M}(-s_0)) B(s_0) e^{-s_0 x} \\
&= C(s_0) e^{-s_0 x}.
\end{aligned}$$

□

The application of Theorem 2 yields a sufficient condition in order to use the polynomial expansion.

Corollary 1. *For $\frac{1}{m} < 2\gamma$ and $r = 1$, the integrability condition* (3.11) *is satisfied.*

We note the importance of the choice of the parameter m. The Laguerre method, briefly described in Remark 1, does not offer the possibility of changing the parameter. In the next subsection, we shed light on another key aspect of parametrization.

3.4 On the goodness of the approximation

The approximation is obtained through the truncation of an infinite serie. Obviously, the higher the order of truncation gets, the better the approximation is. Our goal is to work out an efficient numerical method that combines high accuracy and small computation time. We want to minimize the number of coefficients to compute so the coefficients' sequence must decrease as fast as possible. First, we assume that $\frac{dG_M}{d\nu} \in L^2(\nu)$ in order to apply the method. Consequently

$$< \frac{dG_M}{d\nu}, \frac{dG_M}{d\nu} >= \sum_{n=0}^{+\infty} a_n^2 < +\infty, \tag{3.18}$$

and the sequence of coefficients is approaching 0. The question is how fast is the decay? To answer this question we use generating function theory, just like what is done in [1]. We take the Laplace transform of g_M defined as a polynomial expansion.

$$\begin{aligned}
\widehat{g_M}(s) &= \int_0^{+\infty} e^{-sx} \sum_{n=0}^{+\infty} a_n Q_n(x) f_\mu(x) dx \\
&= \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-sx} Q_n(x) f_\mu(x) dx.
\end{aligned}$$

The orthonormal polynomials $\{Q_n\}_{n\in\mathbb{N}}$ are Laguerre's one

$$Q_n(x) = \frac{(-1)^n}{\sqrt{\binom{n+r-1}{n}}} L_n^{r-1}\left(\frac{x}{\mu}\right), \tag{3.19}$$

where $L_n^{r-1}(x) = \sum_{i=0}^n \binom{n+r-1}{n-i} \frac{(-x)^i}{i!}$, we refer to [22] for this last definition. We then have

$$\begin{aligned}
\widehat{g_M}(s) &= \sum_{n=0}^{+\infty} a_n \int_0^{+\infty} e^{-sx} \frac{(-1)^n}{\sqrt{\binom{n+r-1}{n}}} L_n^{r-1}\left(\frac{x}{m}\right) \frac{x^{r-1}e^{-x/m}}{\Gamma(r)m^r} dx \\
&= \sum_{n=0}^{+\infty} a_n \frac{(-1)^n}{\sqrt{\binom{n+r-1}{n}}} \int_0^{+\infty} e^{-sx} \sum_{i=0}^{n} \binom{n+r-1}{n-i} \frac{(-x)^i}{i!m^r} \frac{x^{r-1}e^{-x/m}}{\Gamma(r)m^r} dx \\
&= \sum_{n=0}^{+\infty} a_n \frac{(-1)^n}{\sqrt{\binom{n+r-1}{n}}} \sum_{i=0}^{n} \frac{(n+r-1)!}{(n-i)!(r+i-1)!(r-1)!i!} (-1)^i \int_0^{+\infty} e^{-sx} \frac{x^{r+i-1}e^{-x/m}}{m^{r+i}} dx \\
&= \sum_{n=0}^{+\infty} a_n (-1)^n \sqrt{\binom{n+r-1}{n}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \int_0^{+\infty} e^{-sx} \frac{x^{r+i-1}e^{-x/m}}{m^{r+i}\Gamma(r+i)} dx \\
&= \sum_{n=0}^{+\infty} a_n (-1)^n \sqrt{\binom{n+r-1}{n}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left(\frac{1}{sm+1}\right)^{r+i} \\
&= \sum_{n=0}^{+\infty} a_n (-1)^n \sqrt{\binom{n+r-1}{n}} \left(\frac{1}{sm+1}\right)^{r} \left(\frac{sm}{sm+1}\right)^{n} \\
&= \sum_{n=0}^{+\infty} a_n I_\nu(n) G_\nu(s) H_\nu(s)^n && (3.20) \\
&= G_\nu(s) B(H_\nu(s)), && (3.21)
\end{aligned}$$

where $B(z) = \sum_{n=0}^{+\infty} b_n z^n$ is the generating function of the sequence $\{b_n\}_{n\in\mathbb{N}}$ with $b_n = I_\nu(n)a_n, \forall n \in \mathbb{N}$. The generating function B is expressed with the Laplace transform of g_M via a change of variable, we get

$$\begin{aligned} B(z) &= \frac{\widehat{g_M}(H_\nu^{-1}(z))}{G_\nu(H_\nu^{-1}(z))} \\ &= (1-z)^{-r}\widehat{g_M}\left(\frac{z}{m(1-z)}\right). \end{aligned} \tag{3.22}$$

In generating function theory, it is possible to study the decay of a sequence by considering its radius of convergence. If the radius of convergence of $B(z)$ is greater than one then the sequence of coefficients admits a decay that is geometrically fast. Non-geometric convergence occurs when B admits a singularity on the unit circle $\{z : |z| = 1\}$. The parameters of the expansion permits to alter the form of the generating function in order to make it simpler. Sometimes it gets so simple that an exact formula is obtained. The two following examples may help convince the reader.

Example 1. *In [1], several attempts are made to expand PDF when the convergence is not geometrically fast. One of those cases is the expansion of a simple Gamma PDF, recall that the PDF is given by*

$$f_X(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, \tag{3.23}$$

with associated Laplace transform

$$\widehat{f_X}(s) = \left(\frac{1}{1+\beta s}\right)^\alpha. \tag{3.24}$$

The generating function of the coefficient defined in (3.22) *is*

$$B(z) = \frac{m^\alpha (1-z)^{\alpha-r}}{(m - z(\beta - m))^\alpha}. \tag{3.25}$$

It is easily check that taking $r = \alpha$ and $m = \beta$ lead to $B(z) = 1$. Then $a_0 = 1$ and $a_n = 0, \forall n \geq 1$.

Example 2. *The exact ultimate ruin probability for the classical ruin model is available in a closed form when the claim sizes are governed by an exponential distribution. In this particular case, the integrated tail distribution is also an exponential distribution with the same parameter as the claim amounts. Assume that the claim sizes are exponentially distributed with parameter δ, the Laplace transform of the defective PDF g_M is*

$$\widehat{g_M}(s) = \frac{\rho}{1 + \frac{\delta}{(1-\rho)}s}. \tag{3.26}$$

The generating function of the coefficients is then

$$B(z) = \frac{\rho m (1-z)^{1-r}}{m - z\left(\frac{\delta}{1-\rho} - m\right)}. \tag{3.27}$$

It is easily check that taking $r = 1$ and $m = \frac{\delta}{1-\rho}$ lead to $B(z) = \rho$. Then $a_0 = \rho$ and $a_n = 0, \forall n \geq 1$. Thus $g_M(x) = \rho\frac{1-\rho}{\delta}e^{-\frac{1-\rho}{\delta}x}$ and $\psi(u) = \rho e^{-\frac{1-\rho}{\delta}u}$ which is indeed the exact ultimate probability in the studied case.

This two examples shows how to play on the parameters to make the hole thing looks better. We see that this method is a mathematical tool to perform analytical Laplace transform inversion. The choice of the parameter is not automatic as one has to look at the generating function and choose wisely the parameters to have good results. It happens that the generating function B takes a tedious form which leave us with a difficult calll regarding the parameters.

3.5 Computation of the coefficients of the expansion

The coefficients are obtained from the derivative of the generating function defined in (3.22)

$$a_n = \frac{I_\nu(n)}{n!}\left[\frac{d^n}{dz^n}B(z)\right]_{z=0}. \tag{3.28}$$

Direct evaluation is doable using a computational software program. However if the expression of B is tedious then one might use an approximation procedure. The derivative can be expressed via Cauchy contour integral

$$a_n = \frac{I_\nu(n)}{2\pi i}\int_{C_r}\frac{B(z)}{z^{n+1}}dz, \tag{3.29}$$

where C_r is a circle about the origin of radius $0 < r < 1$. We make the change of variable $z = re^{iw}$ to get

$$a_n = \frac{I_\nu(n)}{2\pi r^n}\int_0^{2\pi} B(re^{iw})e^{-iw}dw. \tag{3.30}$$

The integrals in (3.30) are approximated through a trapezoidal rule

$$\begin{aligned} a_n &\approx \bar{a}_n \\ &= \frac{I_\nu(n)}{2\pi r^n}\sum_{j=1}^{2n}(-1)^j\Re(B(re^{\pi ji/n})) \\ &= \frac{I_\nu(n)}{2\pi r^n}\left\{B(r) + (-1)^n Q(-r) + 2\sum_{j=1}^{n}(-1)^j\Re(B(re^{\pi ji/n}))\right\}, \end{aligned}$$

where $\Re(z)$ denotes the real part of some complex number z. The goodness of this approximation procedure is widely studied in [1].

4 Numerical illustrations

We analyse the convergence of the sum in our method toward known exact values of ruin probabilities with gamma distributed claim sizes. For those claim sizes distribution we have explicit formulas that allow us to assess the accuracy of our approximated ruin probabilities. The goodness of the approximation depends on the order of truncation K. Our method also enables us to approximate ruin probabilities in cases that are relevant for applications but where no formulas are currently available. A comparison is done with the results obtained with the Fast Fourrier Transform, the scaled Laplace transform inversion and Panjer's algorithm. We plot the difference between the exact ruin probability value and its approximation

$$\Delta\psi(u) = \psi(u) - \psi_{Approx}(u). \tag{4.1}$$

We considerered $\Gamma(\alpha,\beta)$ and $U[0,1]$ distributed claim sizes. Regarding the ruin model settings, we fix a safety loading at 20% which is a standard value. The results are displayed on Figure 1 and 2 where the claim amount are $\Gamma(1/2,1/2)$-distributed and $\Gamma(3,1)$-distributed. The generating function of the coefficients does not take a simple form and it is really hard to choose an optimal parametrization. We then choose $r=1$ and $m=1/\gamma$ in order to ensure the convergence. We also set $K=40$, which sounds like a good compromise between accuracy, computation time and numerical difficulties. The Panjer algorithm is applied in its basic form as we are trying to recover probabilities of a geometric compound distribution. The distribution of the random variable M, defined in (3.1), is approximated as follows

$$
\begin{aligned}
P(M=nh) &\approx g_n \\
&= \frac{\rho}{1-\rho f_0}\sum_{j=1}^{n} f_j g_{n-j} \ \ \forall n\geq 1,
\end{aligned} \tag{4.2}
$$

where h is the bandwidth and $f_j = F_{U^I}\left(jh+\frac{h}{2}\right) - F_{U^I}\left(jh-\frac{h}{2}\right), \forall j\in\mathbb{N}$. This arithmetization design has been recommended in a recent paper [12]. The algorithm is initialized with $g_0 = G_N(0)$, where G_N is the probability generating function of N also defined in (3.1). The probability of ultimate ruin is then approximated by

$$
\tilde{\psi}(u) = 1 - \sum_{i=0}^{\lfloor u/h\rfloor} g_i.
$$

The bandwidth is set to $h=0.01$. The scaled Laplace transform technique has been presented in [17]. The ultimate ruin probability is approximated by

$$
\psi_{\alpha,b}(u) = \sum_{k=0}^{\lfloor \alpha e^{-xlnb}\rfloor}\sum_{j=k}^{\alpha}\binom{\alpha}{j}\binom{j}{k}(-1)^{j-k}\widehat{\psi}(jlnb), \tag{4.3}
$$

where $\widehat{\psi}(s) = \frac{1-\rho}{1-\rho\widehat{f_{U^I}}(s)}$ is the Laplace transform of the ruin probability. We set $\alpha=27$ and $b=1.25$ as it is recommended in [18]. Regarding the Fourier series method, we apply exactly the procedure described in [21] Chapter 5 Section 5.5. The fast Fourrier transform offers the best accuracy for the two examples, our method is competitive in comparison to the two other methods.

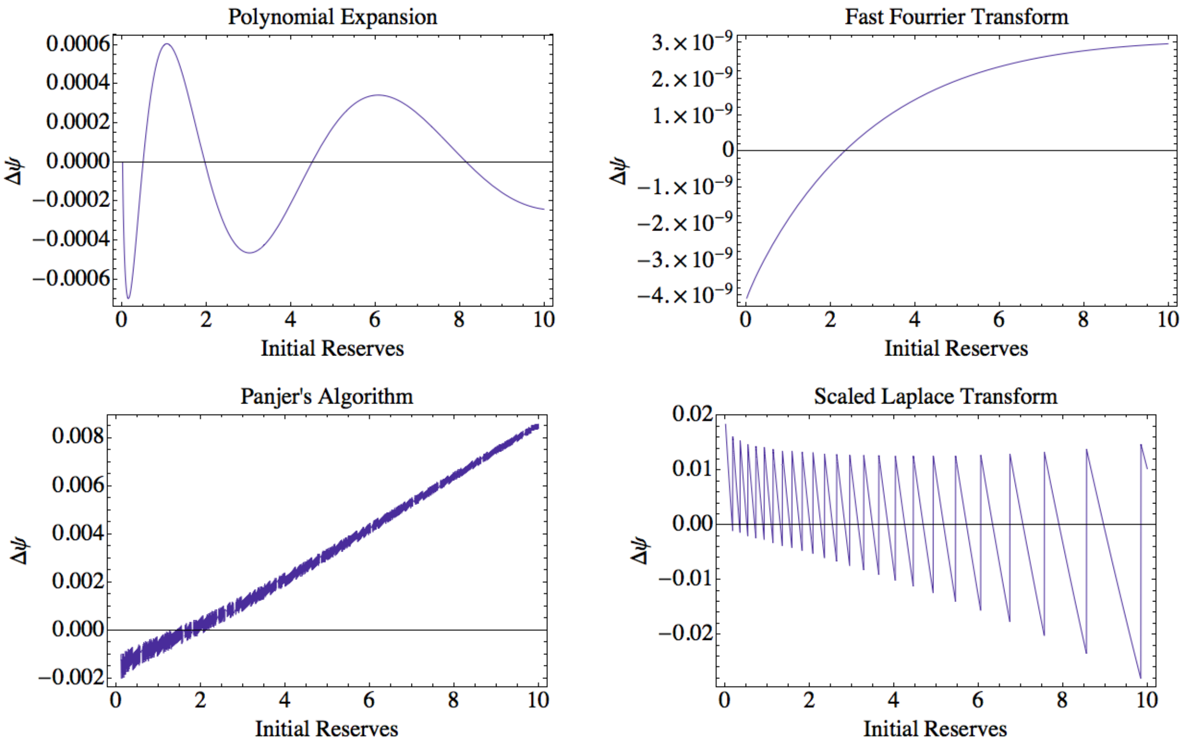

Figure 1: Difference between exact and approximated ruin probabilities for $\Gamma(1/2, 1/2)$-distributed claim sizes

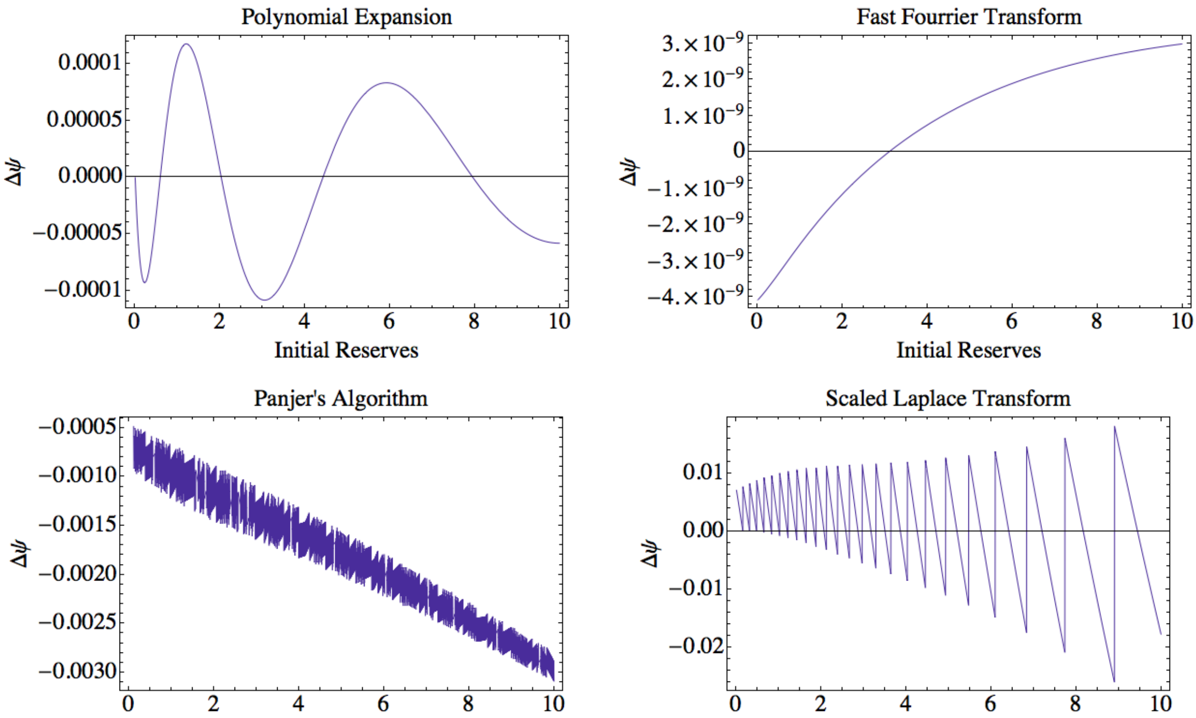

Figure 2: Difference between exact and approximated ruin probabilities for $\Gamma(3, 1)$-distributed claim sizes

In the case of uniformly distributed claim amounts, we do not have an exact formula for the ultimate ruin probability. As the fast Fourier transform did a tremendous job on the two other cases we decide to take the ruin probabilty approximated via the FFT as a benchmark value to assess the accuracy of the three other methods. We also plot the ultimate ruin probability obtained with each method, see Figure 3. One can note that the polynomial expansion give again satisfying results.

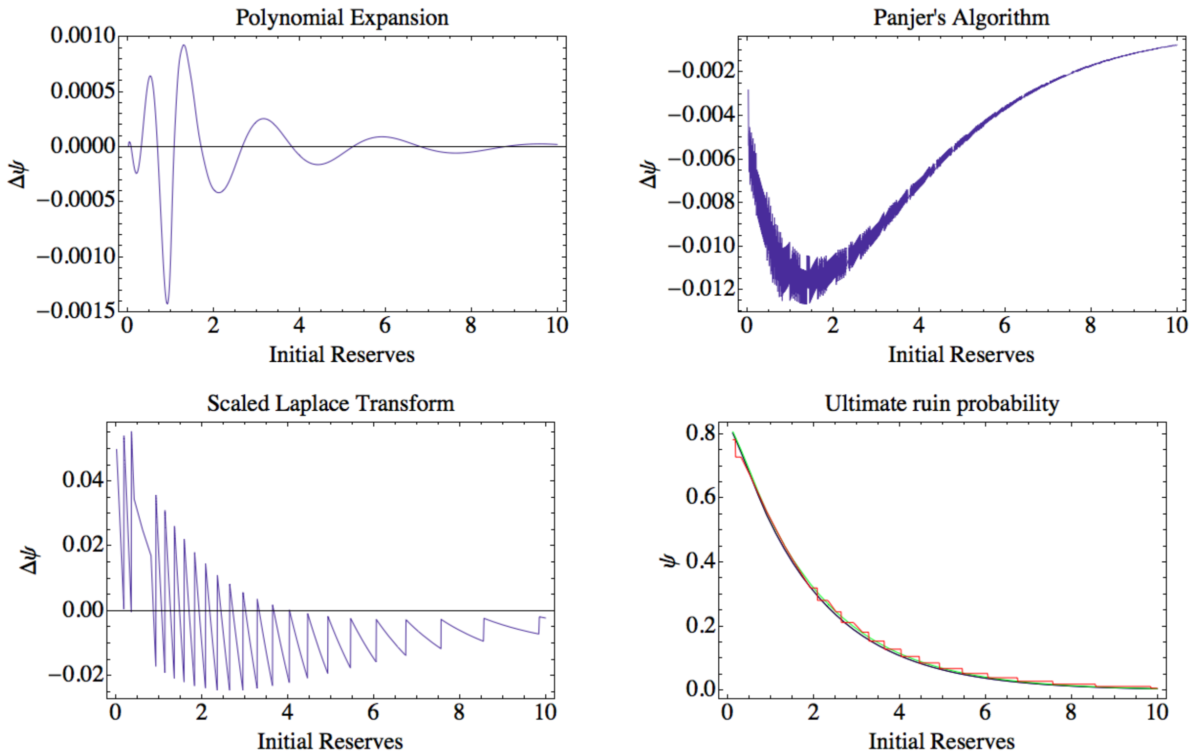

Figure 3: Difference between FFT approximation and approximated ruin probabilities with other method for $U[0, 1]$-distributed claim sizes

5 Conclusion

Our method provides a very good approximation of the ruin probability when the claim sizes distribution is light-tailed. We obtained a theoretical result that ensures the validity of our expansions. As expected, the numerical results show the superiority of the Fourier series based method in term of accuracy. Nevertheless, our method provides an approximation of a simple form for the hole ruin function and allows reinjection to derive approximations of other quantities of interest in ruin theory. Another advantage is the possibility of a statistical extension that will lead to a nonparametric estimation of ruin probabilities just like the scaled Laplace transform inversion and maximum entropy methods. The great results in terms of accuracy are promising and it will be interesting to consider statistical application. It is also worth noting that this method can be easily adapted to a multivariate problem. The inversion of a bivariate Laplace transform will be at the center of a forthcoming paper.

Acknowledgement

The authors also would like to thank H. Albrecher and X. Guerrault for helpful discussion and an anonymous referee for his comments which improve greatly the quality of this paper. This work is partially funded by the Research Chair *Actuariat Durable* sponsored by Miliman, and the insurance company AXA France.

References

[1] J. Abate, G.L. Choudhury, and W. Whitt. On the laguerre method for numerically inverting laplace transforms. *INFORMS Journal on Computing*, 8(4):413–427, 1995.

[2] J. Abate and W. Whitt. The fourier-series method for inverting transforms of probability distributions. *Queueing Systems*, 10:5–88, 1992.

[3] H. Albrecher, F. Avram, and D. Kortschak. On the efficient evaluation of ruin probabilities for completely monotone claim distributions. *Journal of computational and applied mathematics*, 233(10):2724–2736, 2010.

[4] H. Albrecher, J. Teugels, and R. Tichy. On a gamma series expansion for the time dependent probability of collective ruin. *Insurance: Mathematics and Economics*, (29):345–355, 2001.

[5] S. Asmussen and H. Albrecher. *Ruin Probabilities*, volume 14 of *Advanced Series on Statistical Science Applied Probability*. World Scientific, 2010.

[6] F. Avram, D.F. Chedom, and A. Horvarth. On moments based padé approximations of ruin probabilities. *Journal of computational and applied mathematics*, 235(10):321–3228, 2011.

[7] O. Barndorff-Nielsen. *Information and exponential Families in Statistical Theory*. Wiley, 1978.

[8] J.A. Beekman. Ruin function approximation. *Transaction of society of actuaries*, 21(59 AB):41–48, 1969.

[9] N.L. Bowers. Expansion of probability density functions as a sum of gamma densities with applications in risk theory. *Transaction of society of actuaries*, 18(52):125–137, 1966.

[10] D. Dickson. On the distribution of the surplus prior to ruin. *Insurance: Mathematics and Economics*, 11(3):191–207, 1992.

[11] D. C. M. Dickson. A review of panjer's recursion formula and it's applications. *British Actuarial Journal*, 1(1):107–124, April 1995.

[12] P. Embrechts and M. Frei. Panjer recursion versus fft for compound distribution. *Mathematical Methods of Operations Research*, 69:497–508, July 2009.

[13] P. Embrechts, P. Grübel, and S. M. Pitts. Some applications of the fast fourrier transform algorithm in insurance mathematics. *Statistica Neerlandica*, 41:59–75, March. 1993.

[14] H. U. Gerber and E.S.W Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–72, 1998.

[15] H. Gzyl, P.L. Novi-Inverardi, and A. Tagliani. Determination of the probability of ultimate ruin probability by maximum entropy applied to fractional moments. *Insurance: Mathematics and Economics*, 53(2):457–463, 2013.

[16] R. Mnatsakanov, L.L. Ruymgaart, and F.H. Ruymgaart. Nonparametric estimation of ruin probabilities given a random sample of claims. *Mathematical Methods of Statistics*, 17(1):35–43, 2008.

[17] R. M. Mnatsakanov and K. Sarkisian. A note on recovering the distribution from exponential moments. *Applied Mathematics and Computation*, 219:8730–8737, 2013.

[18] R. M. Mnatsakanov, K. Sarkisian, and A. Hakobyan. Approximation of the ruin probability using the scaled laplace transform inversion. *Working Paper*, 2014.

[19] Carl N. Morris. Natural exponential families with quadratic variance functions. *The Annals of Mathematical Statistics*, 10(1):65–80, 1982.

[20] H. H. Panjer. Recursive evaluation of a family of compound distributions. *Astin Bulletin*, 12(1):22–26, 1981.

[21] T. Rolski, H. Schmidli, V. Schmidt, and J.Teugels. *Stochastic Processes for Insurance and Finance*. Wiley series in probability and statistics, 1999.

[22] G. Szegö. *Orthogonal Polynomials*, volume XXIII. American mathematical society Colloquium publications, 1939.

[23] G. Taylor. Representation and explicit calculation of finite-time ruin probabilities. *Scandinavian Actuarial Journal*, pages 1–18, 1978.

[24] W. T. Weeks. Numerical inversion of laplace transforms using laguerre functions. *Journal of the ACM*, 13(3):419–429, 1966.

[25] Z. Zhang, H. Yang, and H. Yang. On a nonparametric estimator for ruin probability in the classical risk model. *Scandinavian Actuarial Journal*, 2014(4):309–338, 2014.