ON THE INVERSE GAUSSIAN DISTRIBUTION FUNCTION

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This note deals with a method of evaluating the distribution function of the Inverse Gaussian Distribution, from the Standard Normal Distribution.

TWEEDIE [1,2] and Wasan [3] have investigated certain properties of the Inverse Gaussian Distribution. The latter has found an approximation to its cumulative distribution function. Theorem 2 of this paper indicates a method to obtain the exact probabilities.

Definition: A random variable X follows the Inverse Gaussian Distribution with parameters $\mu>0$, $\lambda>0$, if it has density function:

$$\begin{aligned} f(x) &= \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left[-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right], && x>0, \\ &= 0, && x \leq 0. \end{aligned}$$

Let $F(c;\ \mu,\ \lambda)=\Pr(X<c)$ denote the cumulative distribution function of $f(x)$.

Theorem 1. Let X be a random variable with density function $f(x)$, defined above. Then $\lambda(X-\mu)^2/\mu^2 X$ is χ_1^2 distributed.

This follows immediately from the following transformations:

$$Y = \min(X,\ \mu^2/X) \quad \text{and} \quad Z = \lambda(Y-\mu)^2/\mu^2 Y.$$

One obtains that Z is χ_1^2 distributed; and $Z=\lambda(X-\mu)^2/\mu^2 X$.

Theorem 2.

(A) $F(c;\ \mu,\ \lambda)=\frac{1}{2}G(a)+\frac{1}{2}\exp(2\lambda/\mu)G(a+4\lambda/\mu)$, $0<c\leq\mu$;

(B) $F(c;\ \mu,\ \lambda)=1-\frac{1}{2}G(a)+\frac{1}{2}\exp(2\lambda/\mu)G(a+4\lambda/\mu)$, $\mu<c<\infty$,

where

$$a = \frac{\lambda(c-\mu)^2}{\mu^2 c} \quad \text{and} \quad G(z) = \int_z^\infty (2\pi t)^{-\frac{1}{2}} \exp(-t/2)dt,$$

the right tail probability of χ_1^2.

Proof: From the definition of $F(c;\ \mu,\ \lambda)$, given earlier, one has

$$F(c;\ \mu, \lambda) = F(c/\mu;\ 1, \lambda/\mu). \tag{1}$$

The results will be verified for the case $\mu=1$, and then will be generalized to any $\mu>0$, by means of (1).

(A) Let $c\leq 1$. Then

$$F(c;\ 1,\ \lambda) = \int_0^c \left(\frac{\lambda}{2\pi X^3}\right)^{\frac{1}{2}} \exp\left[-\frac{\lambda(X-1)^2}{2X}\right] dX.$$

Substitute $Y=X^{-\frac{1}{2}}$ to obtain

$$F(c;\ 1,\ \lambda) = \int_{c^{-\frac{1}{2}}}^{\infty} (2\lambda/\pi)^{\frac{1}{2}} \exp\left[-\frac{\lambda(Y^2-1)^2}{Y^2}\right] dY. \tag{2}$$

Next, let

$$Z = [\lambda(Y^2 - 1)^2/Y^2]. \tag{3}$$

Since Z is a monotone increasing function of Y in the region of integration, one has

$$\begin{aligned} F(c; 1, \lambda) &= \int_{\lambda(c-1)^2/c}^{\infty} (2\lambda/\pi)^{\frac{1}{2}} \frac{Y^3}{2\lambda(Y^4-1)} \exp(-Z/2)dZ, \\ &= \int_a^{\infty} (2\pi Z)^{-\frac{1}{2}} \frac{Y^2}{Y^2+1} \exp(-Z/2)dZ. \end{aligned} \tag{4}$$

Solving (3) for Y^2, using the admissable root $Y^2 > c^{-1}$, one has

$$Y^2/(Y^2+1) = \tfrac{1}{2} + \tfrac{1}{2}(Z/(4\lambda + Z))^{\frac{1}{2}}. \tag{5}$$

Substituting this result in (4), one obtains

$$\begin{aligned} F(c;\ 1,\ \lambda) &= \frac{1}{2}\int_a^{\infty} (2\pi Z)^{-\frac{1}{2}} \exp(-Z/2)dZ \\ &\quad + \frac{1}{2}\int_a^{\infty} (2\pi)^{-\frac{1}{2}}(4\lambda + Z)^{-\frac{1}{2}} \exp(-Z/2)dZ, \\ &= \tfrac{1}{2}G(a) + \tfrac{1}{2}\exp(2\lambda)G(a + 4\lambda). \end{aligned}$$

This is precisely the required expression, for $\mu = 1$, $c \leq 1$.

(B) Let $c > 1$. It is immediate that

$$a = \frac{\lambda(c-1)^2}{c} = \frac{\lambda\left(\frac{1}{c} - 1\right)^2}{1/c}\,. \tag{6}$$

By virtue of (A) and (6), one obtains

$$\begin{aligned} F(c; 1, \lambda) &= F(1/c; 1, \lambda) + (F(c; 1, \lambda) - F(1/c; 1, \lambda)) \\ &= \tfrac{1}{2}G(a) + \tfrac{1}{2}\exp(2\lambda)G(a + 4\lambda) + \Pr(1/c < X < c), \end{aligned} \tag{7}$$

where X is a random variable with cumulative distribution function $F(\cdot\,; 1, \lambda)$.

It is readily seen that $1/c < X < c$ iff $[\lambda(X-1)^2/X] < a$. Hence, by Theorem 1, one has

$$\Pr(1/c < X < c) = \Pr\left(\frac{\lambda(X-1)^2}{X} < a\right) = 1 - G(a).$$

Substituting this last result in (7), one obtains

$$F(c;\ 1,\ \lambda) = 1 - \tfrac{1}{2}G(a) + \tfrac{1}{2}\exp(2\lambda)G(a + 4\lambda).$$

The truth of Theorem 2, for $\mu = 1$, is thus complete. By virtue of (1), it is immediately evident that Theorem 2 holds for any $\mu > 0$. Q.E.D.

Corollary:

$$F(c;\mu,\lambda) = H\left[(\lambda/c)^{\frac{1}{2}}\left(1-\frac{c}{\mu}\right)\right] + \exp(2\lambda/\mu)H\left[(\lambda/c)^{\frac{1}{2}}\left(1+\frac{c}{\mu}\right)\right],$$

where

$$H(y) = \int_y^\infty (2\pi)^{-\frac{1}{2}}\exp(-t^2/2)dt - \infty < y < \infty,$$

for any $c>0$, $\mu>0$, $\lambda>0$.

This result follows directly from Theorem 2, and the elementary fact: if Z is χ_1^2 distributed and Y is Standard Normal, then

$$\Pr(Z > b) = 2\Pr(Y > b^{\frac{1}{2}}) = 2\Pr(Y < -b^{\frac{1}{2}}) \qquad \text{for } b > 0.$$

It is evident that from (i) Standard Normal tables, and (ii) Log tables, that one can obtain numerical values for $F(c;\mu,\lambda)$.

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