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ON THE INVERSE GAUSSIAN DISTRIBUTION FUNCTION

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This note deals with a method of evaluating the distribution function of the Inverse Gaussian Distribution, from the Standard Normal Distribution.

TWEEDIE [1,2] and Wasan [3] have investigated certain properties of the Inverse Gaussian Distribution. The latter has found an approximation to its cumulative distribution function. Theorem 2 of this paper indicates a method to obtain the exact probabilities.

Definition: A random variable X follows the Inverse Gaussian Distribution with parameters $\mu > 0$, $\lambda > 0$, if it has density function:

$$f(x) = \left(\frac{\lambda}{2\pi x^3} \right)^{\frac{1}{2}} \exp \left[- \frac{\lambda(x - \mu)^2}{2\mu^2 x} \right], \quad x > 0, \\ = 0, \quad x \leq 0.$$

Let $F(c; \mu, \lambda) = \Pr(X < c)$ denote the cumulative distribution function of $f(x)$.

Theorem 1. Let X be a random variable with density function $f(x)$, defined above. Then $\lambda(X - \mu)^2 / \mu^2 X$ is χ_1^2 distributed.

This follows immediately from the following transformations:

$$Y = \min(X, \mu^2/X) \quad \text{and} \quad Z = \lambda(Y - \mu)^2 / \mu^2 Y.$$

One obtains that Z is χ_1^2 distributed; and $Z = \lambda(X - \mu)^2 / \mu^2 X$.

Theorem 2.

(A) $F(c; \mu, \lambda) = \frac{1}{2}G(a) + \frac{1}{2} \exp(2\lambda/\mu)G(a + 4\lambda/\mu)$, $0 < c \leq \mu$;

(B) $F(c; \mu, \lambda) = 1 - \frac{1}{2}G(a) + \frac{1}{2} \exp(2\lambda/\mu)G(a + 4\lambda/\mu)$, $\mu < c < \infty$,

where

$$a = \frac{\lambda(c - \mu)^2}{\mu^2 c} \quad \text{and} \quad G(z) = \int_z^\infty (2\pi t)^{-\frac{1}{2}} \exp(-t/2) dt,$$

the right tail probability of χ_1^2 .

Proof: From the definition of $F(c; \mu, \lambda)$, given earlier, one has

$$F(c; \mu, \lambda) = F(c/\mu; 1, \lambda/\mu). \quad (1)$$

The results will be verified for the case $\mu = 1$, and then will be generalized to any $\mu > 0$, by means of (1).

(A) Let $c \leq 1$. Then

$$F(c; 1, \lambda) = \int_0^c \left(\frac{\lambda}{2\pi X^3} \right)^{\frac{1}{2}} \exp \left[- \frac{\lambda(X - 1)^2}{2X} \right] dX.$$

Substitute $Y = X^{-1}$ to obtain

$$F(c; 1, \lambda) = \int_{c^{-1/2}}^{\infty} (2\lambda/\pi)^{1/2} \exp\left[-\frac{\lambda(Y^2 - 1)^2}{Y^2}\right] dY. \quad (2)$$

Next, let

$$Z = [\lambda(Y^2 - 1)^2/Y^2]. \quad (3)$$

Since Z is a monotone increasing function of Y in the region of integration, one has

$$\begin{aligned} F(c; 1, \lambda) &= \int_{\lambda(c-1)^2/c}^{\infty} (2\lambda/\pi)^{1/2} \frac{Y^3}{2\lambda(Y^4 - 1)} \exp(-Z/2) dZ, \\ &= \int_a^{\infty} (2\pi Z)^{-1/2} \frac{Y^2}{Y^2 + 1} \exp(-Z/2) dZ. \end{aligned} \quad (4)$$

Solving (3) for Y^2 , using the admissible root $Y^2 > c^{-1}$, one has

$$Y^2/(Y^2 + 1) = \frac{1}{2} + \frac{1}{2}(Z/(4\lambda + Z))^{1/2}. \quad (5)$$

Substituting this result in (4), one obtains

$$\begin{aligned} F(c; 1, \lambda) &= \frac{1}{2} \int_a^{\infty} (2\pi Z)^{-1/2} \exp(-Z/2) dZ \\ &\quad + \frac{1}{2} \int_a^{\infty} (2\pi)^{-1/2} (4\lambda + Z)^{-1/2} \exp(-Z/2) dZ, \\ &= \frac{1}{2} G(a) + \frac{1}{2} \exp(2\lambda) G(a + 4\lambda). \end{aligned}$$

This is precisely the required expression, for $\mu = 1$, $c \leq 1$.

(B) Let $c > 1$. It is immediate that

$$a = \frac{\lambda(c-1)^2}{c} = \frac{\lambda\left(\frac{1}{c} - 1\right)^2}{1/c}. \quad (6)$$

By virtue of (A) and (6), one obtains

$$\begin{aligned} F(c; 1, \lambda) &= F(1/c; 1, \lambda) + (F(c; 1, \lambda) - F(1/c; 1, \lambda)) \\ &= \frac{1}{2} G(a) + \frac{1}{2} \exp(2\lambda) G(a + 4\lambda) + \Pr(1/c < X < c), \end{aligned} \quad (7)$$

where X is a random variable with cumulative distribution function $F(\cdot; 1, \lambda)$.

It is readily seen that $1/c < X < c$ iff $[\lambda(X-1)^2/X] < a$. Hence, by Theorem 1, one has

$$\Pr(1/c < X < c) = \Pr\left(\frac{\lambda(X-1)^2}{X} < a\right) = 1 - G(a).$$

Substituting this last result in (7), one obtains

$$F(c; 1, \lambda) = 1 - \frac{1}{2} G(a) + \frac{1}{2} \exp(2\lambda) G(a + 4\lambda).$$

The truth of Theorem 2, for $\mu = 1$, is thus complete. By virtue of (1), it is immediately evident that Theorem 2 holds for any $\mu > 0$. Q.E.D.

Corollary:

$$F(c; \mu, \lambda) = H\left[(\lambda/c)^{\frac{1}{2}}\left(1 - \frac{c}{\mu}\right)\right] + \exp(2\lambda/\mu)H\left[(\lambda/c)^{\frac{1}{2}}\left(1 + \frac{c}{\mu}\right)\right],$$

where

$$H(y) = \int_y^{\infty} (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) dt \quad -\infty < y < \infty,$$

for any $c > 0$, $\mu > 0$, $\lambda > 0$.

This result follows directly from Theorem 2, and the elementary fact: if Z is χ_1^2 distributed and Y is Standard Normal, then

$$\Pr(Z > b) = 2 \Pr(Y > b^{\frac{1}{2}}) = 2 \Pr(Y < -b^{\frac{1}{2}}) \quad \text{for } b > 0.$$

It is evident that from (i) Standard Normal tables, and (ii) Log tables, that one can obtain numerical values for $F(c; \mu, \lambda)$.

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REFERENCES

- [1] Tweedie, M. C. K., "Statistical Properties of the Inverse Gaussian Distribution, I," *Annals of Mathematical Statistics*, 28 (1957), 362-77.
- [2] ———, "Statistical Properties of the Inverse Gaussian Distribution, II," *Annals of Mathematical Statistics*, 28 (1957), 696-705.
- [3] Wasan, M. T., "Monograph on Inverse Gaussian Distribution" (1966).