Governing Equations of Classical Gas Dynamics

From Euler form to the Characteristics form

Manuel Diaz[1]

[1]National Taiwan University
Institute of Applied Mechanics

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Outline

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INTEGRAL FORM

Conservation of Mass: “All mass in the universe is constant”

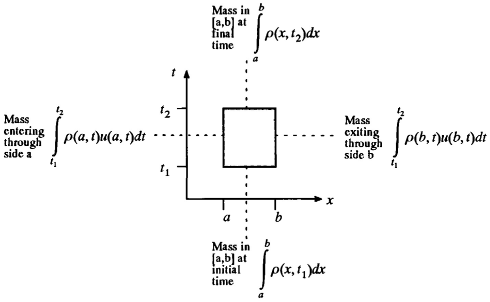

Figure 2.1 An illustration of conservation of mass.

In a space-time `(x,t)` plane a control volume is depicted.

INTEGRAL FORM

for mass

Change in total mass in `[a,b]` in time interval `[`t_1`,`t_2`]` = net mass passing through boundaries of `[a,b]` in time interval `[`t_1`,`t_2`]`.

$$\int_a^b [\rho(x,t_2) - \rho(x,t_1)dx = \\ -\int_{t_1}^{t_2} [\rho(b,t)u(b,t) - \rho(a,t)u(a,t)]dt \qquad (1)$$

INTEGRAL FORM

for momentum

Change in total momentum in `[a,b]` in time interval $[t_1, t_2]$ = net momentum passing through boundaries of `[a,b]` in time interval $[t_1, t_2]$ + net momentum change due to pressure on boundaries of `[a,b]`.

$$
\begin{aligned}
&\int_a^b [\rho(x,t_2)u(x,t_2) - \rho(x,t_1)u(x,t_1)]dx = \\
&- \int_{t_1}^{t_2} [\rho(b,t)u^2(b,t) - \rho(a,t)u^2(a,t)]dt \\
&- \int_{t_1}^{t_2} [p(b,t) - p(a,t)]dt
\end{aligned} \tag{2}
$$

INTEGRAL FORM

for Energy

Change in total Energy in `[a,b]` in time interval $[t_1, t_2]$ = net Energy passing through boundaries of `[a,b]` in time interval $[t_1, t_2]$ + net Energy change due to pressure on boundaries of `[a,b]`.

$$\int_a^b [\rho(x,t_2)e_T(x,t_2) - \rho(x,t_1)e_T(x,t_1)]dx =$$

$$-\int_{t_1}^{t_2} [\rho(b,t)u(b,t)e_T(b,t) - \rho(a,t)u(a,t)e_T(b,t)]dt$$

$$-\int_{t_1}^{t_2} [p(b,t)u(b,t) - p(b,t)u(b,t)]dt \qquad (3)$$

By 2nd Law of Thermodynamics we know: The total entropy of the universe never decreases.

- how is Entropy defined for an Ideal Gas?

$$\Delta S = \int_{T_0}^{T} \frac{Cv}{T} dT + \int_{V_0}^{V} \left(\frac{\partial p}{\partial T} \right)_V dV$$

$$\Delta S = C_v Nkln \left(\frac{T}{T_0} \right) + Nkln \left(\frac{V}{V_0} \right)$$

$$\Delta S = C_v Nkln(T) + Nkln(V)$$

INTEGRAL FORM

Ideal Gas relations

- Ideal gas equation of state:

$$p = \rho R T$$
$$e = c_v T$$
$$h = c_p T$$
$$\gamma = \frac{c_p}{c_v}$$
$$c_p = R + c_v$$

- the equation of state giving specific entropy s as a fuction of specific internal energy and density:

$$
\begin{aligned}
s &= c_v lne - Rln\rho + const. \\
s &= c_v lnp - c_p ln\rho + const.
\end{aligned}
\tag{4}
$$

- for homentropic conditions, we conclude:

$$p = (const.)\rho^{\gamma}$$
$$T = (const.)\rho^{\gamma-1}$$
$$a = (const.)\rho^{(\gamma-1)/2}$$

INTEGRAL FORM

Entropy and 2nd Law

The change in total entropy in `[a,b]` in time interval $[t_1, t_2] \geq$ net entropy passing through boundaries of `[a,b]` in time interval $[t_1, t_2]$.

$$\int_a^b [\rho(x,t_2)s(x,t_2) - \rho(x,t_1)s(x,t_1)]dx \geq$$

$$-\int_{t_1}^{t_2} [\rho(b,t)u(b,t)s(b,t) - \rho(a,t)u(a,t)s(b,t)]dt \quad (5)$$

CONSERVATION FORM

Vector Notation

- Define the Vectors of conserved quantities:

$$\vec{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho e_T \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{6}$$

$$\vec{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e_T + p)u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{7}$$

CONSERVATION FORM

- We can rewrite more compactly the conservation equations 1 to 3:

$$\int_a^b [\vec{u}(x, t_2) - \vec{u}(x, t_1) dx = - \int_{t_1}^{t_2} [\vec{f}(b, t) - \vec{f}(a, t) dx \tag{8}$$

CONSERVATION FORM

- Take the mass integral conservation (Ec.1)
- Asume that $\rho(x,t)$ is differentiable in time.
- Using the fundamental theorem of calculus we can rewrite:

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial \rho}{\partial t} dt \tag{9}$$

CONSERVATION FORM

- Similarly, if $\rho(x,t)u(x,t)$ is diffentiable in space the we can rewrite:

$$\rho(b,t)u(b,t) - \rho(a,t)u(a,t) = \int_a^b \frac{\partial \rho u}{\partial t} dt \qquad (10)$$

CONSERVATION FORM

- Assuming integration in space is reversible with integration in time, Ec. 1 becomes:

$$\int_a^b \int_{t_1}^{t_2} \left[\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} \right] dtdx = 0 \tag{11}$$

CONSERVATION FORM

- The conservation form of the Euler Equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \tag{12}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0, \tag{13}$$

$$\frac{\partial \rho e_T}{\partial t} + \frac{\partial (\rho u e_T + p u)}{\partial x} = 0, \tag{14}$$

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho u s}{\partial x} = 0. \tag{15}$$

Outline

VECTOR NOTATION

- Using againg the vector notation the conservation equation can be written as:

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0 \tag{16}$$

VECTOR NOTATION

- The by the chain rule

$$\frac{\partial \vec{u}}{\partial x} = \frac{d\vec{f}}{d\vec{u}}\frac{\partial \vec{u}}{\partial x} \tag{17}$$

where

$$\frac{\partial \vec{u}}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix} \tag{18}$$

MATRIX NOTATION

- To simplify, we call the Jacobian Matrix: A

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0 \tag{19}$$

Computing A we obtain:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ \gamma u e_T + (\gamma - 1)u^3 & \gamma e_T - \frac{3}{2}(\gamma - 1)u^2 & \gamma u \end{bmatrix} \tag{20}$$

Outline

THE MATERIAL

- The Primite variable from is not commonly used in gasdynamics.
- The Primite variables are those flow variable that we can dyrectly measure.
- This is a lagrangean description of the variables.

The Material Derivate:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{21}$$

LAGRANGE EQUATIONS

- The material derivate is rate of change a long the pathlines.
- Using the material derivate we rewrite the Euler Equations as:

The Material Derivate:

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \qquad (22)$$

$$\frac{D\rho}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (23)$$

$$\frac{D\rho}{Dt} + \rho a^2\frac{\partial u}{\partial x} = 0 \qquad (24)$$

$$\frac{Ds}{Dt} \geq 0 \qquad (25)$$

VECTOR-MATRIX FORM

- Define the vector of primitive variables:

The Material Derivate:

$$\vec{w} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \qquad (26)$$

VECTOR-MATRIX FORM

- Then primitive form of the Euler equations can be written as:

$$\frac{\partial \vec{w}}{\partial t} + C\frac{\partial \vec{w}}{\partial x} = 0 \tag{27}$$

Where:

$$C = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix} \tag{28}$$

IMPORTANT RELATIONS

- Relations between A and C: First notice that:

$$d\vec{u} = Qd\vec{w} \tag{29}$$

where

$$Q = \frac{d\vec{u}}{d\vec{w}} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{1}{2}u^2 & \rho u & \frac{1}{\gamma-1} \end{bmatrix} \tag{30}$$

IMPORTANT RELATIONS

- Relations between A and C: Or:

$$d\vec{w} = Qd^{-1}\vec{u} \tag{31}$$

where

$$Q^{-1} = \frac{d\vec{w}}{d\vec{u}} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\rho}u & \frac{1}{\rho} & 0 \\ 1/2(\rho-1)u^2 & -(\rho-1)u & \gamma-1 \end{bmatrix} \tag{32}$$

IMPORTANT RELATIONS

- Relations between A and C:

$$Q\frac{\partial \vec{w}}{\partial t} + AQ\frac{\partial \vec{w}}{\partial x} = 0 \tag{33}$$

$$\frac{\partial \vec{w}}{\partial t} + Q^{-1}AQ\frac{\partial \vec{w}}{\partial x} = 0 \tag{34}$$

$$\frac{\partial \vec{w}}{\partial t} + C\frac{\partial \vec{w}}{\partial x} = 0 \tag{35}$$

- In other words, A and C are similar matrices!

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Outline

Introduction: Euler Equations
- Integral and Conservation form
- Vector-Matrix Notation
- Primitive Variable Form

Wave Models
- Scalar Wave Model
- Vector Wave Model
- The Characteristic Form

Simple Waves
- Isotropic Flow
- Expansion Waves
- Compresion Waves/Shock Waves
- Contact Discontinuities

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Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.