Governing Equations of Classical Gas Dynamics

Characteristics form and Simple Waves

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Outline

CONSERVATION FORM

- Euler Equations expresed in classical conservation form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial (\rho u^2 + p)}{\partial x} = 0, \tag{2}$$

$$\frac{\partial \rho e_T}{\partial t} + \frac{\partial (\rho u e_T + p u)}{\partial x} = 0, \tag{3}$$

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho u s}{\partial x} \geq 0 \tag{4}$$

VECTOR NOTATION

Vectors of conserved quantities

- Define the Vectors of conserved quantities:

$$\vec{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho e_T \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{5}$$

$$\vec{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e_T + p) u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{6}$$

VECTOR NOTATION

of the conservation equation

- By using the vectors of conserved quantities, we can express in a very compact form the Euler Equations:

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0 \tag{7}$$

- But we know:

$$\vec{f}(\vec{u}) \tag{8}$$

VECTOR NOTATION

- The by the chain rule

$$\frac{\partial \vec{u}}{\partial x} = \frac{\partial \vec{f}}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial x} \tag{9}$$

where

$$\frac{\partial \vec{u}}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{bmatrix} \tag{10}$$

MATRIX NOTATION

- To simplify, we call the Jacobian Matrix: A

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0 \tag{11}$$

Computing A we obtain:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ \gamma u e_T + (\gamma-1)u^3 & \gamma e_T - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{bmatrix} \tag{12}$$

THE MATERIAL DERIVATE

- The Primite variable from is not commonly used in gasdynamics.
- The Primite variables are those flow variable that we can dyrectly measure.
- This is a lagrangean description of the Euler Equations.

The Material Derivate:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \tag{13}$$

PRIMITIVE VARIABLE FORM

- The material derivate is rate of change a long the pathlines.
- Using the material derivate we rewrite the Euler Equations as:

$$\frac{D\rho}{Dt} + \rho\frac{\partial u}{\partial x} = 0 \tag{14}$$

$$\frac{Du}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{15}$$

$$\frac{Dp}{Dt} + \rho a^2\frac{\partial u}{\partial x} = 0 \tag{16}$$

$$\frac{Ds}{Dt} \geq 0 \tag{17}$$

VECTOR-MATRIX FORM

- Define the vector of primitive variables:

The Material Derivate:

$$\vec{w} = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \tag{18}$$

VECTOR-MATRIX FORM

- Then primitive form of the Euler equations can be written as:

$$\frac{\partial \vec{w}}{\partial t} + C \frac{\partial \vec{w}}{\partial x} = 0 \tag{19}$$

Where:

$$C = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix} \tag{20}$$

IMPORTANT RELATIONS

- Relations between A and C: First notice that:

$$d\vec{u} = Qd\vec{w} \tag{21}$$

where

$$Q = \frac{d\vec{u}}{d\vec{w}} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{1}{2}u^2 & \rho u & \frac{1}{\gamma-1} \end{bmatrix} \tag{22}$$

IMPORTANT RELATIONS

- Relations between A and C: Or:

$$d\vec{w} = Qd^{-1}\vec{u} \tag{23}$$

where

$$Q^{-1} = \frac{d\vec{w}}{d\vec{u}} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\rho}u & \frac{1}{\rho} & 0 \\ 1/2(\rho-1)u^2 & -(\rho-1)u & \gamma-1 \end{bmatrix} \tag{24}$$

IMPORTANT RELATIONS

- Relations between A and C:

$$Q\frac{\partial \vec{w}}{\partial t} + AQ\frac{\partial \vec{w}}{\partial x} = 0 \tag{25}$$

$$\frac{\partial \vec{w}}{\partial t} + Q^{-1}AQ\frac{\partial \vec{w}}{\partial x} = 0 \tag{26}$$

$$\frac{\partial \vec{w}}{\partial t} + C\frac{\partial \vec{w}}{\partial x} = 0 \tag{27}$$

- In other words, A and C are similar matrices!

VECTOR-MATRIX FORM

- Let's look again to the Primitive Variable Form:

$$\frac{\partial \vec{w}}{\partial t} + C\frac{\partial \vec{w}}{\partial x} = 0 \tag{28}$$

Where:

$$C = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho a^2 & u \end{bmatrix} \tag{29}$$

CHARACTERISTIC FORM

- Matrix C is a diagonizable and therefore:
- $Q_c^{-1} C Q_c = \Lambda$

Where:

$$Q_c^{-1} = \begin{bmatrix} 1 & \rho/2a & -\rho/2a \\ 0 & 1/2 & 1/2 \\ 0 & \rho a/2 & -\rho a/2 \end{bmatrix} \tag{30}$$

$$Q_c = \begin{bmatrix} 1 & 0 & -1/2a \\ 0 & 1 & 1/2 \\ 0 & 1 & -1/\rho a \end{bmatrix} \tag{31}$$

and

$$\Lambda = \begin{bmatrix} u & 0 & 0 \\ 0 & u+a & 0 \\ 0 & 0 & u-a \end{bmatrix} \tag{32}$$

CHARACTERISTIC FORM

- using the relation $d\vec{v} = Q_c^{-1} d\vec{w}$ we can formulate a new form of the Euler equation:

$$Q_c^{-1} \frac{\partial \vec{w}}{\partial t} + Q_c^{-1} C \frac{\partial \vec{w}}{\partial x} = 0 \tag{33}$$

- a form that involves characteristics rather than primitives variables:

$$\frac{\partial \vec{v}}{\partial t} + \Lambda \frac{\partial \vec{v}}{\partial x} = 0 \tag{34}$$

CHARACTERISTIC FORM

- This is a decouple sistem of differential equations:

$$\frac{\partial v_0}{\partial t} + u\frac{\partial v_0}{\partial x} = 0 \tag{35}$$

$$\frac{\partial v_+}{\partial t} + (u+a)\frac{\partial v_+}{\partial x} = 0 \tag{36}$$

$$\frac{\partial v_-}{\partial t} + (u-a)\frac{\partial v_-}{\partial x} = 0 \tag{37}$$

Where

$$dv_0 = d\rho - \frac{dp}{a^2} \tag{38}$$

$$dv_+ = du + \frac{dp}{\rho a} \tag{39}$$

$$dv_- = du - \frac{dp}{\rho a} \tag{40}$$

CHARACTERISTIC FORM

- Euler Equations can be written as:

$$\begin{aligned} dv_0 &= d\rho - \frac{dp}{a^2} = 0 \quad for \quad dx = udt \\ dv_+ &= du + \frac{dp}{\rho a} = 0 \quad for \quad dx = (u+a)dt \\ dv_- &= du - \frac{dp}{\rho a} = 0 \quad for \quad dx = (u-a)dt \end{aligned} \tag{41}$$

- integrating the compatibility relations, these equations become:

$$\begin{aligned} s &= const. \quad for \quad dx = udt \\ v_+ &= u + \int \frac{dp}{\rho a} = const. \quad for \quad dx = (u+a)dt \\ v_- &= u - \int \frac{dp}{\rho a} = const. \quad for \quad dx = (u-a)dt \end{aligned} \tag{42}$$

CHARACTERISTIC FORM

- by asumming homentropic conditions (isoentropic flow) we can express $dp/\rho a$ in terms only of a:

$$\int \frac{dp}{\rho a} = \frac{2a}{\gamma - 1} + const. \tag{43}$$

- Then characteristic form can be written as:

$$\begin{aligned} s &= const & \text{for} \quad dx &= udt \\ v_+ = u + \tfrac{2a}{\gamma-1} &= const & \text{for} \quad dx &= (u+a)dt \\ v_- = u - \tfrac{2a}{\gamma-1} &= const & \text{for} \quad dx &= (u-a)dt \end{aligned} \tag{44}$$

- The characteristic Variables $v_\pm = u \pm 2a/(\gamma - 1)$ are also called Reimann invariants.

CHARACTERISTIC FORM

In short:
"All flow variables are constant along the caracteristics"
Thus the characteristics are strait lines.
To summarize:

- Assuming $s = const.$ and $v_- = u - 2a/(\gamma - 1) = conts.$ then all flow properties are constant along lines $x = (u + a)t + const.$ (formulation 1)
- Assuming $s = const.$ and $v_+ = u + 2a/(\gamma - 1) = conts.$ then all flow properties are constant along lines $x = (u - a)t + const.$ (formulation 2)
- Assuming $v_- = const.$ and $v_+ = conts.$ then all flow properties are constant along lines $x = ut + const.$ (formulation 3)

SIMPLE WAVES

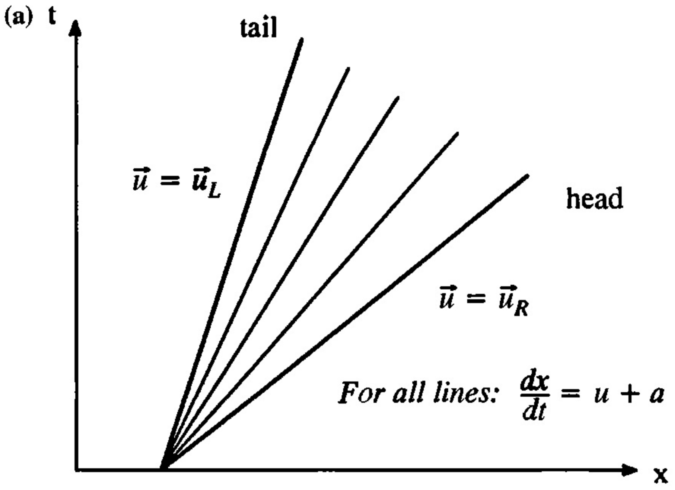
- Any region Governed by Formulation 1 to 3 are called a simple waves.
- equations 1 and 2 describe acoustic waves.
- equations 3 describe entropy waves.
- two regions of steady uniform flow are always separated by simple waves, steadily moving shocks, or steady moving contacts.

EXPANSION WAVES

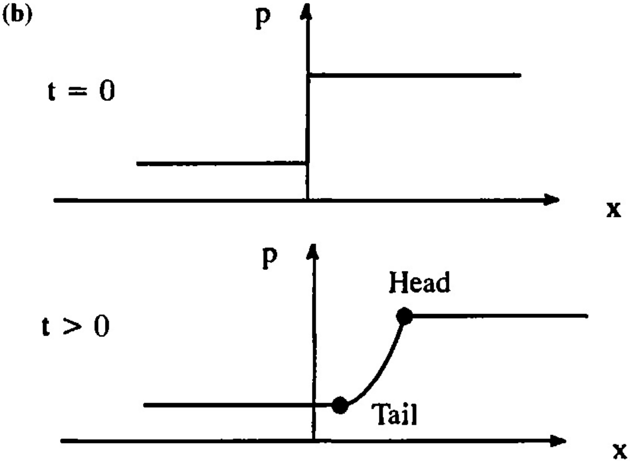
- It decreases presure and density.
- In any region in which the wave speed $\lambda_2 = u + a$ or $\lambda_3 = u - a$ increases monotonically from left to right.
- Mathematically:

$$
\begin{gathered}
u(x,t) + a(x,t) \leq u(y,t) + a(y,t), \quad b_1(t) \leq x \leq y \leq b_2(t) \\
or \\
u(x,t) - a(x,t) \leq u(y,t) - a(y,t), \quad b_1(t) \leq x \leq y \leq b_2(t)
\end{gathered} \tag{45}
$$

EXPANSION WAVES



EXPANSION WAVES



COMPRESION WAVES

- It increases presure and density.
- In any region in which the wave speed $\lambda_2 = u + a$ or $\lambda_3 = u - a$ decreases monotonically from left to right.
- Mathematically:

$$
\begin{gathered}
u(x,t) + a(x,t) \geq u(y,t) + a(y,t), \quad b_1(t) \leq x \leq y \leq b_2(t) \\
\mathit{or} \\
u(x,t) - a(x,t) \geq u(y,t) - a(y,t), \quad b_1(t) \leq x \leq y \leq b_2(t)
\end{gathered}
\tag{46}
$$

SHOCK WAVES

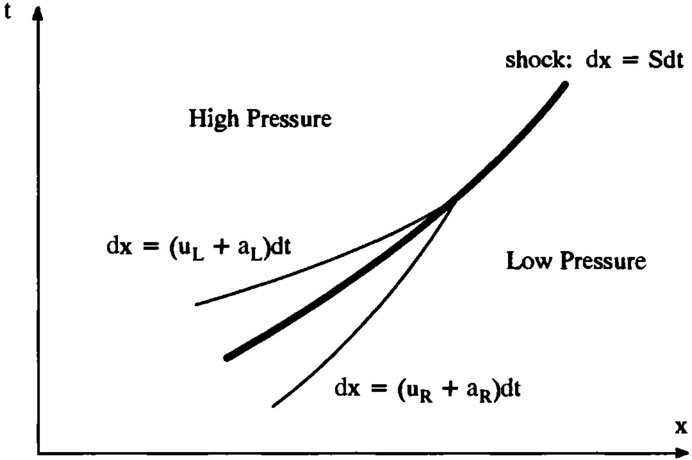


Figure 3.8 Wave diagram for a shock wave in the Euler equations.

SHOCK WAVES

As we have seen:

- The characteristics in a expansion diverge.
- The characteristics in a compresion diverge.
- An intersection between two or more characteristics from the same family creates a shock wave.
- A shock wave is jump discontinuity governed by Rankine-Hugoniot relations:

$$\vec{f}_R - \vec{f}_L = S(\vec{u}_R - \vec{u}_L) \tag{47}$$

where:

- $\vec{f}_{L,R}$ are flux vectors on the left- and right- side of the shock.
- $\vec{u}_{L,R}$ are the conserved quantities on the left- and right- side of the shock.

CONTACT DISCONTINUITIES

- Nearby Characteristics must diverge, converge or be precisely parallel each other.
- Contact discontinuities are parallel entropy waves that neither create and compression or expansion. They separate regions of different Entropy.
- They ocurr when $lambda_1 = u$ and pressure are continuous while other flow properties jump.
- In other words:

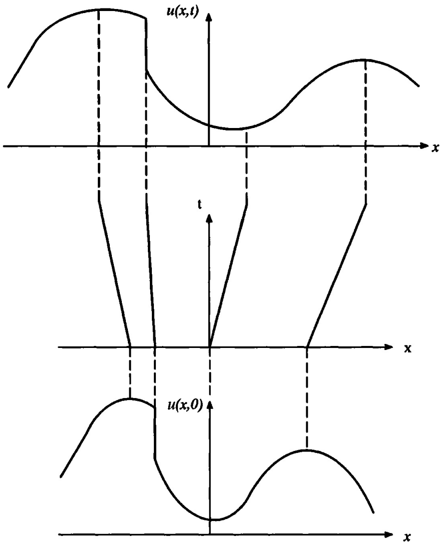
$$u_L = u_R, \tag{48}$$

$$p_L = p_R \tag{49}$$

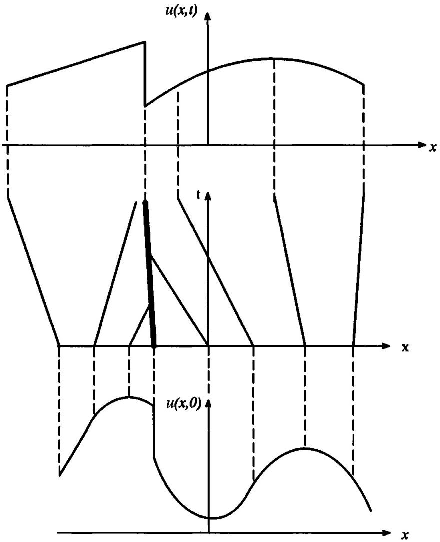
CONTACT DISCONTINUITIES

- No fluid passes through a contact; thus the second law does not apply across a contact.
- - Thus, entropy, density, energy, and all other other flow properties may either increase or decrease acroos the contact.
- Like shocks, contacts discontinuities obey the Rankine-Hugoniot relations.
- Unlike shock, contacts cannot form spontaneously: They must originate either in the initial condition or in the intersection of two shocks.

WAVEFORM EXAMPLE 1



WAVEFORM EXAMPLE 2



WAVEFORM EXAMPLE 3