Introduction
Simple Waves

Important Relations
Rankine-Hugoniot Relations
Shockwave Relations
Remarks

Summary

The Physics of Simple Waves

Shock Waves

Manuel Diaz[1]

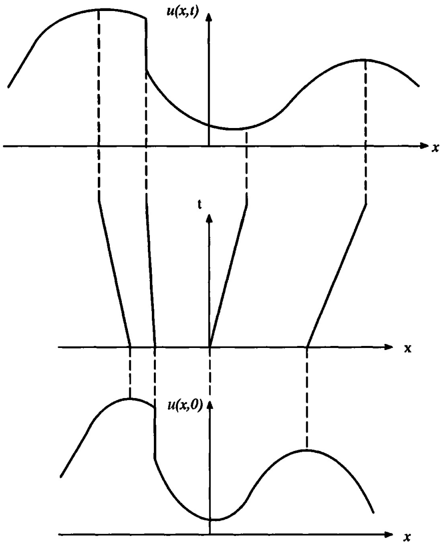
[1]National Taiwan University
Institute of Applied Mechanics

September 29th, 2011 / Weekly Meeting

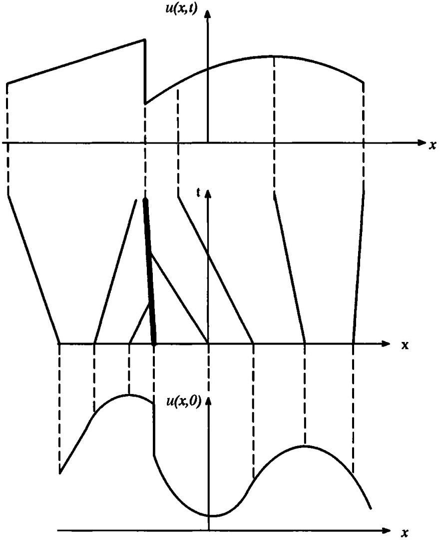
Outline

1. Introduction
   - Simple Waves
2. Important Relations
   - Rankine-Hugoniot Relations
   - Shockwave Relations
   - Remarks

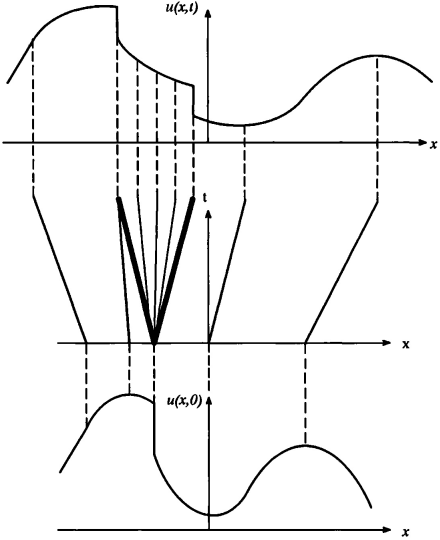
WAVEFORM EXAMPLE 1



WAVEFORM EXAMPLE 2

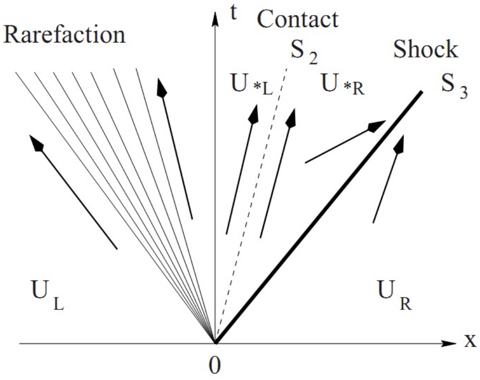


WAVEFORM EXAMPLE 3



INTRODUCTION

- Introduction to Simple Waves



RANKINE HUGONIOT RELATIONS

- Rankine Hugoniot Relation help us to describe the behavior of shockwaves traveling normal to the prevaling flow.

$$\vec{f}_R - \vec{f}_L = S(\vec{u}_R - \vec{u}_L) \tag{1}$$

RANKINE HUGONIOT RELATIONS

Vectors of conserved quantities

- Vectors of conserved quantities:

$$\vec{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho e_T \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{2}$$

$$\vec{f} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e_T + p)u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{3}$$

RANKINE HUGONIOT RELATIONS

Describing Shock Waves

- Graphically:

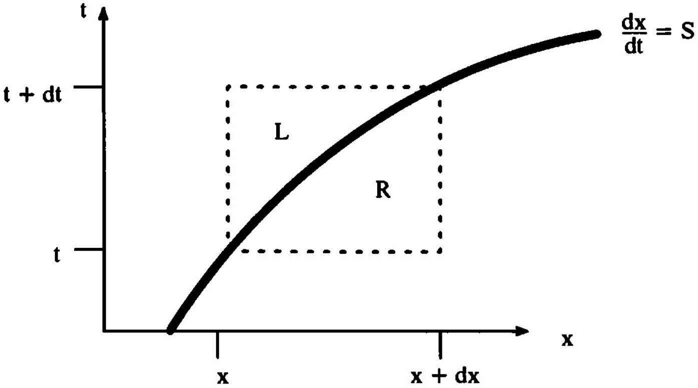


Figure 2.2 The derivation of the Rankine–Hugoniot relations.

RANKINE HUGONIOT RELATIONS

Describing Shock Waves

- Suppose that a shock that separates two regions of uniform flow u_l and u_R is traveling at a constant speed.

$$\rho_L(u_L - S) = \rho_R(u_R - S) \tag{4}$$

$$\rho_L(u_L - S)^2 + p_L = \rho_R(u_R - S)^2 + p_R \tag{5}$$

$$\rho_L(u_L - S)(E_L) = \rho_R(u_R - S)(E_R)$$

Introduction
Simple Waves
Important Relations
Rankine-Hugoniot Relations
Shockwave Relations
Remarks
Summary

RANKINE HUGONIOT RELATIONS

Describing Shock Waves

- we notice that the energy relation can be simplified by using relation (4) and:

$$E = \frac{1}{2}u^2 + h$$

- Which would lead us to a prefered form of the energy description:

$$E_L = E_R$$

$$\frac{1}{2}(u_L - S)^2 + h_L = \frac{1}{2}(u_R - S)^2 + h_R$$

RANKINE HUGONIOT RELATIONS

Describing Shock Waves

- Suppose that a shock that separates two regions of uniform flow u_l and u_R is traveling at a constant speed S.

$$\begin{aligned} \rho_L(u_L - S) &= \rho_R(u_R - S) \\ \rho_L(u_L - S)^2 + p_L &= \rho_R(u_R - S)^2 + p_R \\ \frac{1}{2}(u_L - S)^2 + h_L &= \frac{1}{2}(u_R - S)^2 + h_R \end{aligned} \tag{6}$$

SHOCKWAVE RELATIONS

- We are basically targeting three main relations to describe the physics of the shock wave:
  - Energy Relation: ρ_R and ρ_L
  - Flux Relation: u_R and u_L
  - Pressure Relation: P_R and P_L

- To do so, we start with Equations that we derive from the Hugoniot Relation, to formulate a way to describe this relations.

SHOCKWAVE RELATIONS

- Combining the eqns (4) & (5)

$$\rho_L \left(\frac{\rho_R}{\rho_L} (u_R - S)^2 \right) = \rho_R (u_R - S)^2 + P_R - P_L$$

- we can develop the following two relations:

$$(u_R - S)^2 = \frac{\rho_L}{\rho_R} \left(\frac{P_R - P_L}{\rho_R - \rho_L} \right) \tag{7}$$

$$(u_L - S)^2 = \frac{\rho_R}{\rho_L} \left(\frac{P_R - P_L}{\rho_R - \rho_L} \right) \tag{8}$$

SHOCKWAVE RELATIONS

- Substituting (7) and (8) in (6):

$$\frac{1}{2}(u_L - S)^2 + h_L = \frac{1}{2}(u_R - S)^2 + h_R$$

- we can get now get a relation of enthalpies:

$$\frac{1}{2}(P_L - P_R)\left(\frac{1}{\rho_L} + \frac{1}{\rho_R}\right) = h_R - h_L \tag{9}$$

SHOCKWAVE RELATIONS

- Now using the general formulation of enthalpies:

$$h = e + \frac{P}{\rho}$$

- where e is internal energy,
- By performing some algebraic manipulations we can obtain:

$$\frac{1}{2}(P_L + P_R)\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right) = e_R - e_L \tag{10}$$

- we have obtain the first of the relations we need.

Introduction
Simple Waves
Important Relations
Rankine-Hugoniot Relations
Shockwave Relations
Remarks
Summary

SHOCKWAVE RELATIONS

- but we know that:

$$e = c_v T, \quad c_v = \frac{R}{\gamma - 1}, \quad P = \rho R T$$

- Therefore we can describe internal energy as:

$$e = \frac{P}{(\gamma - 1)\rho}$$

SHOCKWAVE RELATIONS

- Substituting the new formulation of internal energy in (10)

$$\frac{1}{2}(P_L + P_R)\left(\frac{1}{\rho_L} - \frac{1}{\rho_R}\right) = e_R - e_L$$

- and performing some algebraic manipulations we can obtain:

$$\frac{\rho_L}{\rho_R} = \frac{\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{P_L}{P_R}\right) + 1} \tag{11}$$

- Wich stablish a very useful relation between the density ratio ρ_L/ρ_R

SHOCKWAVE RELATIONS

- We wish to derivate a relation P_L & P_R; but first, it is convenient to develope a relation a_L/a_R using the ρ_L/ρ_R ratio that we already now.
- using the general fomulation of sound speed "a" for a perfect gas:

$$a = \gamma RT = \frac{\gamma P}{\rho}$$

$$\frac{a_R^2}{a_L^2} = \frac{P_R}{P_L}\left(\frac{\rho_L}{\rho_R}\right)$$

SHOCKWAVE RELATIONS

- and performing some algebraic manipulations we can obtain:

$$\frac{a_R^2}{a_L^2} = \frac{P_R}{P_L}\left(\frac{\left(\frac{P_L}{P_R}\right)+\left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{P_L}{P_R}\right)+1}\right) \tag{12}$$

- We now introduce Mach Numbers

$$M_R = u_R/a_R \quad M_{shock} = S/a_R$$

- Which would leads to:

$$M_R - M_{shock} = \frac{u_R - S}{a_R} \tag{13}$$

SHOCKWAVE RELATIONS

- By manipulating eqns: (7), (11) and (13):

$$(u_R - S)^2 = \frac{\rho_L}{\rho_R}\left(\frac{P_R - P_L}{\rho_R - \rho_L}\right)$$

$$\frac{\rho_L}{\rho_R} = \frac{\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{P_L}{P_R}\right) + 1}$$

$$M_R - M_{shock} = \frac{u_R - S}{a_R}$$

SHOCKWAVE RELATIONS

- theory leads us to the density and pressure ratios across the shock as functions of the relative Mach Number $M_R - M_{Shock}$, namely

$$\frac{\rho_L}{\rho_R} = \frac{(\gamma+1)(M_R - M_{shock})^2}{(\gamma-1)(M_R - M_{shock})^2 + 2} \tag{14}$$

$$\frac{P_L}{P_R} = \frac{2\gamma(M_R - M_{shock})^2 - (\gamma-1)}{\gamma+1} \tag{15}$$

SHOCKWAVE RELATIONS

- from (15) we can notice the following relation:

$$M_R - M_{shock} = -\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \quad (16)$$

- which leads to an expresion for the shock speed as a function of the pressure ratio across the shock, namely

$$S = u_R + a_R\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)} \quad (17)$$

- Is important to notice that at P_L/P_R aproaches unity it aproaches the characteristic speed $\lambda_+ = u_R + a_R$ as expected.

SHOCKWAVE RELATIONS

- we can relate the u_L and u_R by using (4)

$$\rho_L(u_L - S) = \rho_R(u_R - S) \tag{18}$$

- which leads to an expresion for u_L, namely

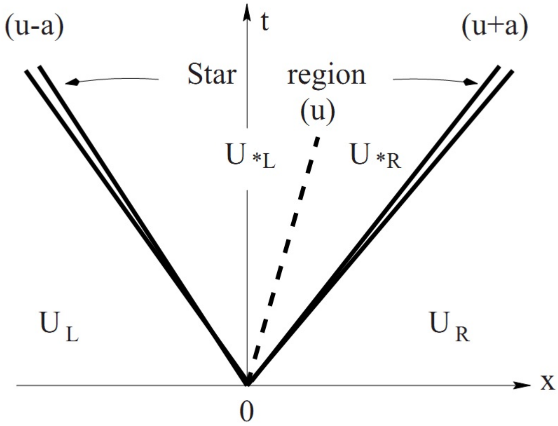
$$u_L = \left(1 - \frac{\rho_R}{\rho_L}\right) S + u_R \left(\frac{\rho_R}{\rho_L}\right) \tag{19}$$

- which leads to today's last relation:

$$u_L = u_R + \frac{a_R}{\gamma} \frac{\frac{P_L}{P_R} - 1}{\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}} \tag{20}$$

Remarks

- Shockwaves to the left or to the right?



Remarks

- Some literature like to use the following notation for the Rigth running Shockwave:

$$\frac{\rho_*}{\rho_R} = \frac{\left(\frac{P_*}{P_R}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{P_*}{P_R}\right) + 1}$$

$$\frac{\rho_*}{\rho_R} = \frac{(\gamma+1)(M_R - M_{shock})^2}{(\gamma-1)(M_R - M_{shock})^2 + 2}$$

$$\frac{P_*}{P_R} = \frac{2\gamma(M_R - M_{shock})^2 - (\gamma-1)}{\gamma+1}$$

$$u_* = u_R + \frac{a_R}{\gamma}\frac{\frac{P_*}{P_R} - 1}{\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_*}{P_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}}$$

Remarks

- For the left running Shockwave, the equations are totally analogous:

$$\frac{\rho_*}{\rho_L} = \frac{\left(\frac{P_*}{P_L}\right) + \left(\frac{\gamma-1}{\gamma+1}\right)}{\left(\frac{\gamma-1}{\gamma+1}\right)\left(\frac{P_*}{P_L}\right) + 1}$$

$$\frac{\rho_*}{\rho_L} = \frac{(\gamma+1)(M_R - M_{shock})^2}{(\gamma-1)(M_R - M_{shock})^2 + 2}$$

$$\frac{P_*}{P_L} = \frac{2\gamma(M_L - M_{shock})^2 - (\gamma-1)}{\gamma+1}$$

$$u_* = u_L + \frac{a_L}{\gamma}\frac{\frac{P_*}{P_L} - 1}{\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_*}{P_L}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}}$$

Summary

- Equations (14), (15) and (20) define a shock for given initial conditions $(\rho_R, u_R, P_R)^T$ ahead of the shock and a chosen mach number M_s or equivalently a shock Speed S.

$$\frac{\rho_L}{\rho_R} = \frac{(\gamma+1)(M_R - M_{shock})^2}{(\gamma-1)(M_R - M_{shock})^2 + 2}$$

$$\frac{P_L}{P_R} = \frac{2\gamma(M_R - M_{shock})^2 - (\gamma-1)}{\gamma+1}$$

$$u_L = u_R + \frac{a_R}{\gamma} \frac{\frac{P_L}{P_R} - 1}{\sqrt{\left(\frac{\gamma+1}{2\gamma}\right)\left(\frac{P_L}{P_R}\right) + \left(\frac{\gamma-1}{2\gamma}\right)}}$$