	DAYANANDA SAGAR COLLEGE OF ENGINEERING *(An Autonomous Institute Affiliated to VTU, Belagavi)* Shavige Malleshwara Hills, Kumaraswamy Layout, Bengaluru-560078 **DEPARTMENT OF MATHEMATICS**

Course Material

COURSE	**MATHEMATICS FOR COMPUTER ENGINEERS**
COURSE CODE	**21MAT31A**
MODULE	**2**
MODULE NAME	**Eigen Values & Eigen Vectors**
Module Coordinator	**Dr. SANJAY OLI**
Course Coordinator	**Ms. Padmaja C**

Objectives:

- Student will be able to verify & find the Eigenvalues and Eigen vectors of square matrix of order n.
- Student will be able diagonalizable to square matrix $\boldsymbol{A}$ & find the powers of $\boldsymbol{A}$, that is, $\boldsymbol{A^n}$.
- Student will be able to find the quadratic forms of matrix $\boldsymbol{A}$ and orthogonal transform which transforms to quadratic form to canonical form.

INTRODUCTION

- **MATRIX REPRESENTATION OF LINEAR TRANSFORMATION**

Let us consider matrix equation $\boldsymbol{A\vec{X}} = \boldsymbol{\vec{b}}$ **.** Here, $\boldsymbol{A}$ is matrix of $m \times n$ order, $\boldsymbol{\vec{X}}$ is column vector $n \times 1$ order, then we are getting column vector $\boldsymbol{\vec{b}}$ of $m \times 1$ order. Thus, here the matrix $\boldsymbol{A}$ of $m \times n$ order is a linear transformation which maps the elements in $\boldsymbol{R^n}$ into the elements in $\boldsymbol{R^m}$. Let us take the examples

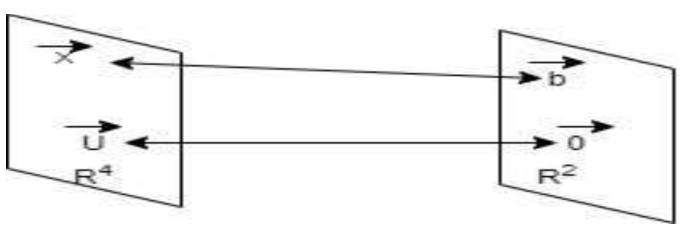
(i)

$$\underset{\boldsymbol{A}\uparrow}{\begin{bmatrix} \mathbf{4} & \mathbf{-3} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{0} & \mathbf{5} & \mathbf{1} \end{bmatrix}} \underset{\boldsymbol{\vec{X}}\uparrow}{\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}} = \underset{\boldsymbol{\vec{b}}\uparrow}{\begin{bmatrix} \mathbf{5} \\ \mathbf{8} \end{bmatrix}}$$

(ii)

$$\underset{\boldsymbol{A}\uparrow}{\begin{bmatrix} \mathbf{4} & \mathbf{-3} & \mathbf{1} & \mathbf{3} \\ \mathbf{2} & \mathbf{0} & \mathbf{5} & \mathbf{1} \end{bmatrix}} \underset{\boldsymbol{\vec{U}}\uparrow}{\begin{bmatrix} \mathbf{1} \\ \mathbf{4} \\ \mathbf{-1} \\ \mathbf{3} \end{bmatrix}} = \underset{\boldsymbol{\vec{0}}\uparrow}{\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}}$$

Thus, the multiplication by $\boldsymbol{A}$ transforms $\boldsymbol{\vec{X}}$ into $\boldsymbol{\vec{b}}$ and transforms $\boldsymbol{\vec{U}}$ into $\boldsymbol{\vec{0}}$ vector. Thus, it is a linear transformation which maps the elements in $\boldsymbol{R^4}$ into the elements in $\boldsymbol{R^2}$ and represented as



- **TRANSFORMS VECTORS VIA MATRIX MULTIPLICATION**

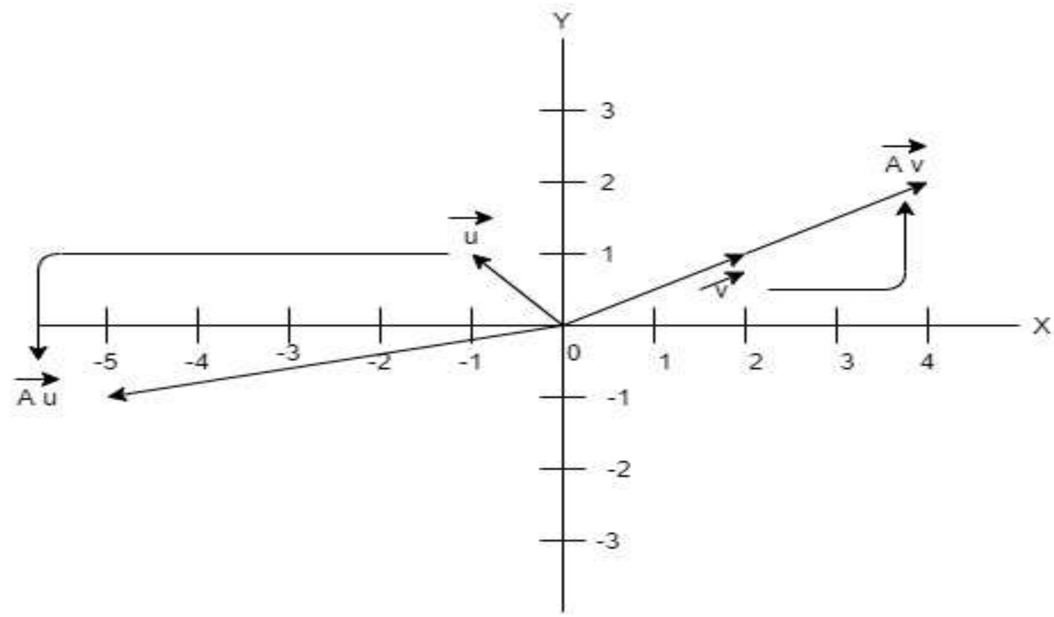
Let us consider the linear transformation $\boldsymbol{X} \to \boldsymbol{A\,X}$, which move vectors in a variety of directions. Let us take the examples

(i) If $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \boldsymbol{\vec{u}} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix}$, then

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix} \neq \vec{u}$$

(ii) If $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\vec{v}$$



Thus, we are interested here the vectors that are transformed by matrix A into a scalar multiple of themselves.

Characteristic Equation: Let A be a square matrix of order n. The matrix A may be singular or non-singular. Consider the homogeneous systems of equations

$$AX = \lambda X \text{ or } (A - \lambda I)X = 0$$

where λ is a scalar and I is an identity matrix of order n. It represent the homogeneous system of equations & always have trivial solution. If the homogeneous system of equations have non-trivial solution, then the coefficient matrix $(A - \lambda I)$ must be singular or its determinant value is zero. That is

$$det(\boldsymbol{A}-\boldsymbol{\lambda I})=\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}=\mathbf{0}.$$

Expanding the determinant given above, we obtain a polynomial of degree $\boldsymbol{n}$ in $\boldsymbol{\lambda}$, which is of the form

$$\boldsymbol{P_n(\lambda)}=det(\boldsymbol{A}-\boldsymbol{\lambda I})=\boldsymbol{\lambda^n-c_1\lambda^{n-1}+c_2\lambda^{n-2}-\cdots\cdots+(-1)^n c_n=0}$$

This polynomial equation $\boldsymbol{P_n(\lambda)=0}$ is called the characteristic equation of matrix $\boldsymbol{A}$.

Eigenvalue & Eigenvectors: The value of $\boldsymbol{\lambda}$, for which non-trivial solutions of the homogeneous systems exist, are called the eigenvalues or characteristic value or latent root of matrix $\boldsymbol{A}$. The corresponding non-trivial solution vector $\boldsymbol{X}$ are called the eigenvectors or the characteristic or latent vector of matrix $\boldsymbol{A}$.

Solving the polynomial equation $\boldsymbol{P_n(\lambda)=0}$, we obtain n roots which can be real or complex conjugate in pairs, simple or repeated. The n roots $\boldsymbol{\lambda_1,\lambda_2,\lambda_3,\cdots\cdots,\lambda_n}$ of the polynomial equation $\boldsymbol{P_n(\lambda)=0}$ are called the eigenvalues of matrix $\boldsymbol{A}$. The corresponding eigenvectors are given by the solving the homogeneous system $(\boldsymbol{A}-\boldsymbol{\lambda I})\boldsymbol{X}=\mathbf{0}$ for each value of eigenvalues $\boldsymbol{\lambda_1,\lambda_2,\lambda_3,\cdots\cdots,\lambda_n}$.

Also we have

$$trace(\boldsymbol{A})=\boldsymbol{Sum\ of\ eigenvalues}=\boldsymbol{\lambda_1+\lambda_2+\lambda_3+\cdots\cdots+\lambda_n}$$

$$det(\boldsymbol{A})=|\boldsymbol{A}|=\boldsymbol{Product\ of\ eigenvalues}=\boldsymbol{\lambda_1.\lambda_2.\lambda_3.\cdots\cdots.\lambda_n}$$

Problems

Q-1 Find the characteristic polynomial and the eigenvalue of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{3} & \mathbf{8} \end{bmatrix}$.

ANSWER: The characteristic equation of A is given by

$$|A - \lambda I| = \begin{bmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} = 0 \,,$$

$$= (3-\lambda)(8-\lambda) - 6 = \lambda^2 - 11\lambda + 18 = 0 \,.$$

Solving for λ, eigenvalues are 9 & 2.

Q-2 Find the eigenvalues and the corresponding eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{6} & \mathbf{-3} & \mathbf{1} \\ \mathbf{3} & \mathbf{0} & \mathbf{5} \\ \mathbf{2} & \mathbf{2} & \mathbf{6} \end{bmatrix}$.

Answer: The characteristic polynomial is

$$|A - \lambda I| = \begin{bmatrix} \mathbf{6}-\lambda & \mathbf{-3} & \mathbf{1} \\ \mathbf{3} & \mathbf{0}-\lambda & \mathbf{5} \\ \mathbf{2} & \mathbf{2} & \mathbf{6}-\lambda \end{bmatrix} = 0 \,,$$

$$= (6-\lambda)[-\lambda(6-\lambda) - 10] + 3[3(6-\lambda) - 10] + 1[6 + 2\lambda] = 0 \,,$$

$$= \lambda^3 - 12\lambda^2 + 29\lambda + 6 = 0.$$

Solving the cubic equation for λ, the eigenvalues are 8.5089 , -0.1914 & 3.6825.

Q-3 Find the eigenvectors of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$.

ANSWER: The characteristic equation of A is given by

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix} = 0 \,,\ \lambda^2 - 2\lambda + 2 = 0 \,.$$

Solving for λ, eigenvalues are 1+i & 1-i. The eigenvectors are:

$\boldsymbol{For\ \lambda = 1 + i}$, the characteristic equation is $[A-(1+i)I]X = 0$. It implies that $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It gives the two equation $-ix_1 + x_2 = 0$ & $-x_1 - ix_2 = 0$ but it reduce to only one equation. Taking one of the equation, we obtain $x_2 = ix_1$. For $x_1 = k, x_2 = ik$. For $k = 1, X = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is the eigenvector.

$\boldsymbol{For\ \lambda = 1 - i}$, the characteristic equation is $[A-(1-i)I]X = 0$. It implies that $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

It gives the two equations $ix_1 + x_2 = 0$ & $-x_1 + ix_2 = 0$ but reduce to a one equation. Taking one of the equation, we obtain $x_1 = ix_2$. For $x_2 = k, x_1 = ik, X = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is the eigenvector.

Q-4 Find the eigenvalues and the corresponding eigenvectors of the matrix $\boldsymbol{A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$**.**

Answer: The characteristic polynomial is $|A - \lambda I| = 0$, that is $(1-\lambda)^3 = 0$. Solving the cubic equation for λ, the eigenvalues are 1 , 1 & 1.

Solving the characteristic equation $(A - \lambda I)X = 0$, for each value of λ , we obtain the eigenvectors as follows:

$For\ \lambda = 1,, (A - I)X = 0, where\ \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.It implies that $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Taking x_1 is free variable and $x_2 = x_3 = 0$, we obtain $X = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$. For $k = 1, X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the eigenvector $for\ \lambda = 1$.

Q-5 Is $\boldsymbol{\lambda = 4}$ **an eigenvalue of matrix** $\boldsymbol{= \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}}$**? If so, find one corresponding eigenvector.**

Answer: The number 4 is an eigenvalue of A if and only if the equation $AX = 4X$ have a nontrivial solution. It implies that $(A - 4I)X = 0$, for vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Thus

$$(A - 4I)X = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \ .$$

The columns of A are obviously linearly dependent, since $|A - 4I| = 0$ and it has a nontrivial solution, and so that 4 is an eigenvalue of A.

The eigenvector corresponding to eigenvalue $\lambda = 4$ is given by the solution for row reduce the augmented matrix $(A - 4I)X = 0$, that is

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & 1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $(A - 4I)X = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$, it implies that $-x_1 - x_3 = 0$ & $-x_2 - x_3 = 0$.

Taking x_3 as free variable or $x_3 = k$. For $k = 1$, the eigenvector is $X = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Q-6 Find a basis for the eigenspace corresponding to eigenvalue $\boldsymbol{\lambda = 1, 5}$ for the matrix

$\boldsymbol{A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}}$.

Answer: For $\boldsymbol{\lambda} = 1$, $(A - I)X = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$. It gives two equations $4x_1 + 0x_2 = 0$ & $2x_1 + 0x_2 = 0$ .Taking one of the equation, it implies that $x_1 = 0$ & taking x_2 as free variable or $x_2 = k$.

For $k = 1$, the eigenvector is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

For $\boldsymbol{\lambda} = 5$, $(A - 5I)X = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$.It gives two equations $0x_1 + 0x_2 = 0$ & $2x_1 - 4x_2 = 0$ .From second equation, it implies that $x_1 = 2x_2$ & taking x_2 as free variable or $x_2 = k$.

For $k = 1$, the eigenvector is $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Thus the basis for eigenspace is two dimensional eigenspace of R^2, that is $\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

Q-7 Find a basis for the eigenspace corresponding to eigenvalue $\boldsymbol{\lambda = 2}$ for the matrix

$\boldsymbol{A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}}$**.**

Answer: The $\lambda = 2$ is an eigenvalue of A if and only if the equation $AX = 2X$ have a nontrivial solution. It implies that $(A - 2I)X = 0$ & for vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

$$(A - 2I)X = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 .$$

The row reduce augmented matrix $(A - 2I)$, that is

$$\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $(A - 2I)X = \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$,it reduce to the one equation, that is

$2x_1 - x_2 + 6x_3 = 0$. Taking $x_2 = k_2$ & $x_3 = k_3$ as free variable, then

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/2)k_2 - 3k_3 \\ k_2 \\ k_3 \end{bmatrix} = k_2 \begin{bmatrix} (\frac{1}{2}) \\ 1 \\ 0 \end{bmatrix} + k_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

For $k_2 = 1$ & $k_3 = 0, X = \begin{bmatrix} \left(\frac{1}{2}\right) \\ 1 \\ 0 \end{bmatrix}$ $. Also, for\ k_2 = 0$ & $k_3 = 1, X = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Thus, the two dimensional eigenspace of R^3 is the basis $\left\{ \begin{bmatrix} \left(\frac{1}{2}\right) \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Q-8 Show that λ^{-1} is an eigenvalue of A^{-1}, If λ be an eigenvalue of an invertible matrix A .

ANSWER: Here, matrix A is invertible, so that A^{-1} exists. Since, λ be an eigenvalue of an invertible matrix , then by definition $AX = \lambda X$.

Pre multiplying by A^{-1} both sides, we obtain

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$(A^{-1}A)X = \lambda(A^{-1}X)$$

$$IX = \lambda(A^{-1}X)\ implies\ X = \lambda(A^{-1}X)$$

$$\lambda^{-1}X = A^{-1}X, divide\ by\ \lambda.$$

Thus, by definition of eigenvalues it implies that λ^{-1} is an eigenvalue of A^{-1}.

Q-9 Find the value of h in the matrix $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ such that the eigenspace for $\lambda = 5$ is two dimensional.

ANSWER: The row reduce augmented matrix for the equation $(A - 5I)X = 0$ is

$$(A-5I)=\begin{bmatrix}0 & -2 & 6 & -1\\ 0 & -2 & h & 0\\ 0 & 0 & 0 & 4\\ 0 & 0 & 0 & -4\end{bmatrix}\begin{matrix}R_2\to R_2-R_1\\ R_4\to R_4+R_3\end{matrix}\sim\begin{bmatrix}0 & -2 & 6 & -1\\ 0 & 0 & h-6 & 1\\ 0 & 0 & 0 & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\to R_3/4\sim\begin{bmatrix}0 & -2 & 6 & -1\\ 0 & 0 & h-6 & 1\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{matrix}R_2\to R_2-R_3\\ R_1\to R_1+R_3\end{matrix}\sim\begin{bmatrix}0 & -2 & 6 & 0\\ 0 & 0 & h-6 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

For a two-dimensional eigenspace, the system above needs two free variables or two non-zero rows. This can be possible, if and only if $h=6$.

EXCERSISE

1. Is $\lambda = 3$ an eigenvalue of matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$? If so, find one corresponding eigenvector. Answer: $X = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

2. Find the characteristic polynomial and the eigenvalue of the matrix $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. Also, find that corresponding eigenvectors of matrix A. Answer: $\lambda = -2{,}5$ & Eigenvector: $\lambda = -2\ is\ X = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ & for $\lambda = 5\ is\ X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Find the characteristic polynomial and the eigenvalue of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. Also, find that corresponding eigenvectors of matrix A. Answer: $= 1{,}2\ 3$ & $Eigenvectors: \lambda = 3,\ X = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \lambda = 2,\ X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \&\ \lambda = 1,\ X = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

4. Find the characteristic polynomial and the eigenvalue of the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. Also, verify that eigenvalues of A^2 are squares of those of eigenvalues of matrix A .

Answer: Eigenvalues of A are $3, i\ \&\ -i$.

Similar Matrices: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrix of same order. The matrix $\boldsymbol{A}$ is similar to the matrix $\boldsymbol{B}$ if there exist an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} \text{ or } \boldsymbol{P}\boldsymbol{A} = \boldsymbol{B}\boldsymbol{P}.$$

The matrix $\boldsymbol{P}$ is called the similarity matrix. Similar matrix have the same characteristic equation and hence the same eigenvalue.

Diagonalizable Matrices: A square matrix $\boldsymbol{A}$ is diagonalizable, if it is similar to diagonal matrix, that is there exist an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$$

where $\boldsymbol{D}$ is diagonal matrix. Since, similar matrix have the same eigenvalue, the diagonal elements of $\boldsymbol{D}$ are the eigenvalue of matrix $\boldsymbol{A}$.

Thus, square matrix $\boldsymbol{A}$ of order n is diagonalizable if and only if it has n linearly independent eigenvectors. If the n eigenvalues are distinct, then eigenvectors are linearly independent and square matrix $\boldsymbol{A}$ of order n is diagonalizable. If the n eigenvalues are not distinct, then we have to check the eigenvectors of square matrix $\boldsymbol{A}$ of each value of $\boldsymbol{\lambda}$.

If matrix $\boldsymbol{A}$ is diagonalizable, then

$$\boldsymbol{A}^{\boldsymbol{n}} = \boldsymbol{P}\boldsymbol{D}^{\boldsymbol{n}}\boldsymbol{P}^{-1} \; for\ any\ positive\ integer\ n.$$

PROBLEMS

Q1 Diagonalizable the matrix $= \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{6} & \mathbf{-1} \end{bmatrix}$ **, if possible.**

ANSWER: The characteristic polynomial is $|A - \lambda I| = 0$, that is

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{bmatrix} = 0 \text{ , } \lambda^2 - 1 = 0 \text{ .}$$

Solving for λ, eigenvalues are 1 & -1. The eigenvalues are distinct, so that matrix A is diagonalizable. The eigenvectors are:

$\boldsymbol{For\ \lambda = 1}$, the characteristic equation is $(A - I)X = 0,$. It $implies\ that$ $\begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It gives the one equation $6x_1 - 2x_2 = 0$ or $x_2 = 3x_1$. Taking $x_1 = k, x_2 = 3k.$

We obtain $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 3k \end{bmatrix}$ & for $k = 1, X = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is the eigenvector.

$\boldsymbol{For\ \lambda = -1}$, the characteristic equation is $(A + I)X = 0,$. It $implies\ that$ $\begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It gives the two equation $2x_1 - 0x_2 = 0$ & $6x_1 - 0x_2 = 0$. It implies that

$x_1 = 0$ & $x_2\ is\ free\ variable.$

For $x_2 = k, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix}$ & $k = 1, X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus the matrix P which diagonalizes A is $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ &

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus the diagonal matrix D contains eigen values 1 & -1 as diagonal elements.

Q2 Diagonalizable the matrix $= \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{1} & \mathbf{5} \end{bmatrix}$ **, if possible.**

ANSWER: The characteristic polynomial is $|A-\lambda I|=0$, that is

$$|A-\lambda I|=\begin{bmatrix}3-\lambda & -1\\ 1 & 5-\lambda\end{bmatrix}=0.$$

$$(3-\lambda)(5-\lambda)+1=0, \lambda^2-8\lambda+16=0 \text{ or } (\lambda-4)^2=0\,.$$

Solving for λ, eigenvalues are 4 & 4. Eigenvalues are not distinct, but we check the eigenvector for the eigenvalue $\lambda=4$. The eigenvectors is:

$\boldsymbol{For\ \lambda=4}, (A-4I)X=0, implies \begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$.

It gives $-x_1-x_2=0$ & $x_1+x_2=0$ .It reduce to single equation $x_1+x_2=0$ or $x_2=-x_1$. Taking $x_1=k, x_2=-k$. Thus eigenvector is $X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}k\\ -k\end{bmatrix}$ & for $k=1, X=\begin{bmatrix}1\\ -1\end{bmatrix}$.

Thus matrix A is not diagonalizable since only one eigenvector $X=\begin{bmatrix}1\\ -1\end{bmatrix}$ exists.

Q3 Show that the matrix $A=\begin{bmatrix}3 & 1 & -1\\ -2 & 1 & 2\\ 0 & 1 & 2\end{bmatrix}$ is diagonalizable. Hence, find D such that $P^{-1}AP$ is a diagonal matrix.

ANSWER: The characteristic polynomial is $|A-\lambda I|=0$, that is $\lambda^3-6\lambda^2+11\lambda-6=0$. Since, the eigenvalues are 1, 2, & 3 and all are distinct. Hence matrix A is diagonalizable.

$\boldsymbol{For\ \lambda=1}$, the characteristic equation is$(A-I)X=0$.It corresponds to eigenvector$X=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$.

$\boldsymbol{For\ \lambda=2}$, the characteristic equation is$(A-2I)X=$. It corresponds to eigenvector $X=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$.

$\boldsymbol{For\ \lambda=3}$, the characteristic equation is$(A-3I)X=0$.It corresponds to eigenvector$X=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$.

The diagonal matrix D is given by $D = P^{-1}AP$

$$\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{bmatrix} = \begin{bmatrix}-1 & -1 & 1\\2 & 1 & -1\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}3 & 1 & -1\\-2 & 1 & 2\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\-1 & 0 & 1\\1 & 1 & 1\end{bmatrix}.$$

Q4 Show that the matrix $A = \begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}$ is diagonalizable. Hence, find D such that $P^{-1}AP$ is a diagonal matrix.

ANSWER: The characteristic polynomial is $|A - \lambda I| = 0$, that is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$. Since, eigenvalues are 5, -3, -3 are not distinct, we have to check the eigenvector for $= -3$.

$\boldsymbol{For\ \lambda = 5}$, the characteristic equation is $(A - 5I)X = 0$. It corresponds to eigenvector $X = \begin{bmatrix}-1\\-2\\1\end{bmatrix}$.**For $= -3$** , the eigen vectors are $X = \begin{bmatrix}3\\0\\1\end{bmatrix}$ & $X = \begin{bmatrix}-2\\1\\0\end{bmatrix}$. Since eigenvalue $\lambda = -3$ represents two distinct eigenvectors. Hence matrix A is diagonalizable.

The diagonal matrix D is given by $D = P^{-1}AP$

$$\begin{bmatrix}-3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 5\end{bmatrix} = \begin{bmatrix}1/8 & 1/4 & 5/8\\-1/4 & 1/2 & 3/4\\-1/8 & -1/4 & 3/8\end{bmatrix}\begin{bmatrix}-2 & 2 & -3\\2 & 1 & -6\\-1 & -2 & 0\end{bmatrix}\begin{bmatrix}3 & -2 & -1\\0 & 1 & -2\\1 & 0 & 1\end{bmatrix}.$$

Q5 Find the matrix A, whose eigenvalues are $1, 1, 1$ and corresponding eigenvectors are $[-1, 1, 1]^T$, $[1, -1, 1]^T$ and $[1, 1, -1]^T$ respectively.

ANSWER: The diagonal matrix D has the elements of the eigenvalues of A at diagonal and modal matrix P is the corresponding to these eigenvectors. Thus

$$D = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \ \& \ P = \begin{bmatrix}-1 & 1 & 1\\1 & -1 & 1\\1 & 1 & -1\end{bmatrix}.$$

Then P^{-1} is given by $P^{-1} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$.

The matrix A is given by

$$A = PDP^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Q6 Find a formula for A^n, given that $A = PDP^{-1}$, where $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

ANSWER: Here $P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ & $D^n = \begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix}$.

Thus, $A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2.5^n - 3^n & 5^n - 3^n \\ 2.3^n - 2.5^n & 2.3^n - 5^n \end{bmatrix}$.

Q7 Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

ANSWER: Here the characteristic polynomial is

$$|A - \lambda I| = \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = 0 \ , \ \lambda^2 - 3\lambda + 2 = 0 \ .$$

Solving for λ, eigenvalues are 2 & 1. Eigenvalues are distinct, so that matrix A is diagonalizable.

Eigenvectors : $\boldsymbol{For\ \lambda = 2}, (A - 2I)X = 0, implies \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Taking any one equation it gives $2x_1 - 3x_2 = 0$ or $x_2 = 2x_1/3$. Taking $x_1 = 3k, x_2 = 2k$. Thus $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \end{bmatrix}$, for $k = 1, X = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Eigenvectors: $\boldsymbol{For\ \lambda = 1}, (A - I)X = 0, implies \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Taking any one equation it gives $3x_1 - 3x_2 = 0$ or $x_2 = x_1$. Taking $x_1 = k, x_2 = k$. Thus $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$, for $k = 1, X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Here $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\ and\ P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$.

$A^8 = PD^8P^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$.

EXCERSISE

1. Show that the matrix $A = \begin{bmatrix} 1 & 1 & i \\ 1 & 0 & i \\ -i & -i & 1 \end{bmatrix}$ is diagonalizable.

ANSWER: For $\lambda = 0$ and $X = \begin{bmatrix} i \\ 0 \\ -1 \end{bmatrix}$; $\lambda = 1 + \sqrt{3}$ and $X = \begin{bmatrix} 1 \\ \sqrt{3} - 1 \\ -i \end{bmatrix}$; $\lambda = 1 - \sqrt{3}$ and

$X = \begin{bmatrix} 1 \\ -(\sqrt{3} + 1) \\ -i \end{bmatrix}$. Matrix A is diagonalizable.

2. Show that the matrix $A = \begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix}$ is diagonalizable.

ANSWER: For $= -i, -i$, $X = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; $\lambda = 2i$ and $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Matrix A is diagonalizable.

3. Find the matrix A, if the eigenvectors of a 3×3 matrix A corresponding to eigenvalues 1,1,3 are $[1,0,-1]^T$, $[0,1,-1]^T$ and $[1,1,0]^T$ respectively.

ANSWER: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

4. Compute A^n, given that $A = PD\, P^{-1}$, where $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.

ANSWER: $A^n = \begin{bmatrix} 4 - 3.2^n & 12.2^n - 12 \\ 1 - 2^n & 4.2^n - 3 \end{bmatrix}$

5. Compute A^4, where $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$.

ANSWER: $A^4 = \begin{bmatrix} -159 & 480 \\ -80 & 241 \end{bmatrix}$.

Inner Product(dot product) of vectors: Let $x = (x_1, x_2, \dots \dots \dots .., x_n)^T$ and $y = (y_1, y_2, \dots \dots \dots .., y_n)^T$ be two vectors in n-dimensional space. Then
$x.y = x^T y = \sum_{i=1}^{i=n} x_i y_i$.
Is called inner product of the vectors x and y and is a scalar quantity.

Orthogonal and Orthonormal Vectors: The vectors x and y for which $x.y = 0$ are said to be orthogonal vectors. The orthogonal vectors said to be orthonormal If length of vector is unit.

Orthogonal Matrix : A real matrix A is orthogonal matrix if $A^{-1} = A^T$.

Quadratic Forms: Let $X = (x_1, x_2, \dots \dots \dots .., x_n)^T$ be a vector in n-dimensional space. A real quadratic form is an homogeneous expression of the form

$$Q = X^T A\, X = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} a_{ij} x_i x_j$$

Orthogonal Transformation: If A is the orthogonal matrix , then orthogonal transforms is given by $X = A\, Y$ and it geometrically represents the rotation.

Transformation of Quadratic Form to Canonical Form: Let Q be the quadratic form given above, then coefficient matrix A is real symmetric. If P is an orthogonal matrix and the transformation $X = PY$ is an orthogonal transformation. Then

$$Q = X^T A\, X = Y^T D\, Y$$

$$= [y_1, y_2, \dots \dots \dots .., y_n] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \cdot\cdot \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$Q = \lambda_1 {y_1}^2 + \lambda_1 {y_2}^2 + \lambda_1 {y_3}^2 + \cdots \dots \dots \dots \dots \dots + \lambda_1 {y_n}^2$$

Here, D is diagonal matrix whose elements are eigenvalues of matrix A, that is $\lambda_1, \lambda_2, \dots \dots \dots .., \lambda_n$ of nth order square matrix.

This quadratic form is known as the canonical form or sum of the squares form or principal axes form.

PROBLEMS

Q1 Show that S = { v_1, v_2, v_3} forms a orthogonal and orthonormal set of R^3, where

$$\boldsymbol{v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix},\ v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix},\ v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}.}$$

Answer: Here , $S = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$ is square matrix of order 3. S is orthogonal matrix if $S^{-1} = S^T\ OR\ SS^T = I$.

$$SS^T = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix} \begin{bmatrix} 3/\sqrt{11} & 1/\sqrt{11} & 1/\sqrt{11} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ -1/\sqrt{66} & -4/\sqrt{66} & 7/\sqrt{66} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus, S = **{ v_1, v_2, v_3}** forms a orthogonal set of R^3. Also

$v_1.v_1 = {}^9/_{11} + {}^1/_{11} + {}^1/_{11} = 1$, $v_2.v_2 = {}^1/_6 + {}^4/_6 + {}^1/_6 = 1$, $v_3.v_3 = {}^1/_{66} + {}^{16}/_{66} + {}^{49}/_{66} = 1$. It implies that $v_1,\ v_2,\ v_3$ are unit vectors.

Thus, $S = \{v_1, v_2, v_3\}$is orthogonal set and each vector is unit length, so that S is an orthonormal set for R^3.

Q2 Show that $|A| = \pm 1$, if A is an orthogonal matrix.

ANSWER: Here A is orthogonal matrix so that $A^{-1} = A^T$. Taking the determinant both sides we obtain

$\det(A^{-1}) = \det(A^T)$

$(1/\det A) = \det(A) \therefore \det(A^{-1}) = (1/\det A)\ \&\ \det(A^T) = \det(A)$

$[\det(A)]^2 = 1\ OR\ |A| = \pm 1$.

Q3 Find the symmetric matrix B for the quadratic form $Q = 2x_1{}^2 + x_2{}^2 + 3x_1x_2$.

ANSWER: The quadratic form is given by $Q = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} a_{ij}x_ix_j$. Comparing we obtain

$a_{11} = 2, a_{12} + a_{21} = 3$ & $a_{22} = 1$. Therefore

$b_{11} = a_{11} = 2, b_{12} = b_{21} = \frac{a_{12}+a_{21}}{2} = 3/2$ & $b_{22} = a_{22} = 1$. Thus

$B = \begin{bmatrix} 2 & 3/2 \\ 3/2 & 1 \end{bmatrix}$.

Q4 Find the symmetric matrix B for the quadratic form $Q = x_1{}^2 - 5x_2{}^2 + 4x_3{}^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$.

ANSWER: The quadratic form is given by $Q = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} a_{ij}x_ix_j$. Comparing we obtain

$a_{11} = 1, a_{12} + a_{21} = 2, a_{13} + a_{31} = -4, a_{23} + a_{32} = 6, a_{22} = -5$ & $a_{33} = 4$.

Therefore, $b_{11} = a_{11} = 1, b_{12} = b_{21} = \frac{a_{12}+a_{21}}{2} = 1, b_{13} = b_{31} = \frac{a_{13}+a_{31}}{2} = -2$.

$b_{23} = b_{32} = \dfrac{a_{23} + a_{32}}{2} = 3, b_{22} = a_{22} = -5$ & $b_{33} = a_{33} = 4$.

Thus, $B = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 3 \\ -2 & 3 & 4 \end{bmatrix}$.

Q5 Find the canonical form which transforms the quadratic form $Q = x_1{}^2 + 3x_2{}^2 + 3x_3{}^2 - 2x_2x_3$.

ANSWER: The symmetric matrix of the quadratic form Q is given by $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

The characteristic polynomial is $|B - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = 0$,

$|B - \lambda I| = \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$. Solving the cubic equation for λ, the eigenvalues are 1,2 & 4.

The canonical form is given by $Q = X^T B\, X = Y^T D\, Y$, where $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore,

$$Q = [y_1, y_2, y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1{}^2 + 2y_2{}^2 + 4y_3{}^2 \ .$$

Q6 Find the orthogonal transform which transforms the quadratic form $\boldsymbol{Q = x_1{}^2 + 3x_2{}^2 + 3x_3{}^2 - 2x_2x_3}$ to canonical form.

ANSWER: The symmetric matrix of the quadratic form Q is given by $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$.

The characteristic polynomial is $|B - \lambda I| = \begin{bmatrix} \mathbf{1-\lambda} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3-\lambda} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-1} & \mathbf{3-\lambda} \end{bmatrix} = 0$,

$|B - \lambda I| = \lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$. Solving the cubic equation for λ, the eigenvalues are 1,2 & 4.

The eigenvector corresponding to eigenvalues 1,2 & 4 are given by $X_1 = [1,0,0]^T, X_2 = [0,1,1]^T$ & $X_3 = [0,1,-1]^T$.

Thus the modal matrix P is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and normalized modal matrix $\hat{P}$ is

$\hat{P} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Thus the orthogonal transformation is given by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \hat{P}Y = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ (y_2 + y_3)/\sqrt{2} \\ (y_2 - y_3)/\sqrt{2} \end{bmatrix}$$

Thus, $x_1 = y_1, x_2 = \frac{(y_2+y_3)}{\sqrt{2}}$ & $x_3 = \frac{(y_2-y_3)}{\sqrt{2}}$ is the orthogonal transform which reduces quadratic form to canonical form.

EXCERSISE

1. Prove that the eigenvectors of symmetric matrix corresponding to distinct eigenvalues are orthogonal.

2. Show that the matrices A and A^T have the same eigenvalues and for distinct eigenvalues the eigenvectors corresponding to A and A^T are mutually orthogonal.

3. Find the symmetric matrix B for the quadratic form $Q = {x_1}^2 - {x_2}^2 + {x_3}^2 - 2x_1x_2 + 4x_2x_3$.

ANSWER: $\begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

4. Find the canonical form which transforms the quadratic form $= 2{x_1}^2 + {x_2}^2 - 3{x_3}^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$.

ANSWER: $Q = {y_1}^2 - {y_2}^2 - {y_3}^2$

VIEDO URL for more information about Eigenvalue & Eigenvectors:

(i) https://youtu.be/oz0bUB44LDg

(ii) https://youtu.be/P2pL5VThrzQ

(iii) https://youtu.be/OELTJdaU8aA

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