COMP2123-Assignment 5

May 20, 2024

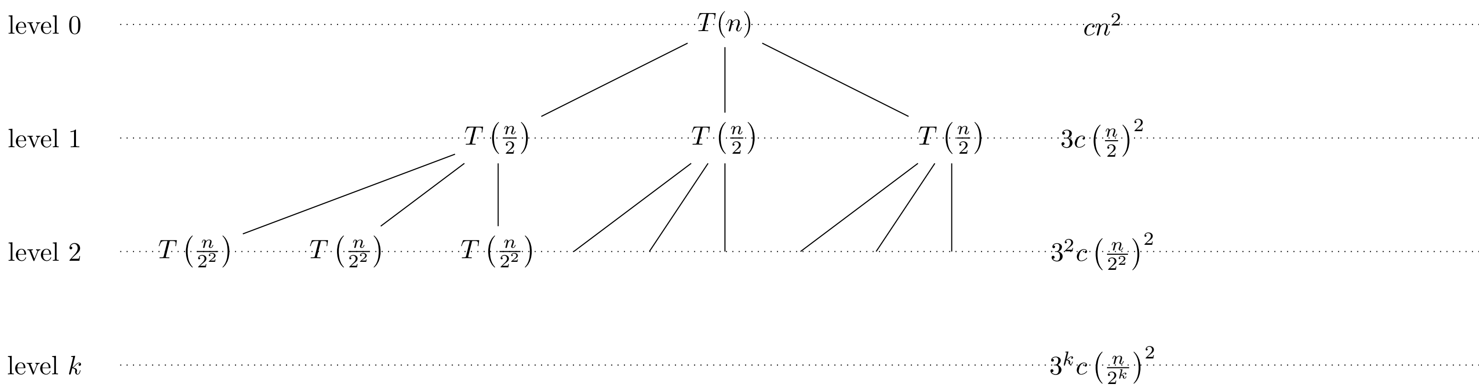
Notation Clarification

This section is to clarify the notations used throughout this assignment.

$[a : b]$	The sequence a, $a+1, \ldots, b-1$.
`int`	The data type representing integers.
`real`	The data type representing the real numbers.
`void`	Used to show that a function does not return anything.
`null`	The variable representing nothingness.
`bool`	The data type representing a boolean value which is either true or false.

Problem 1: Recurrence Relations Analysis

a) $T(n) = 3T(n/2) + n^2$



- Since we halve the size every time, there are totally $k = \log_2 n$ levels. Summing all this up, we get:

$$\begin{aligned}
T(n) &= cn^2 + 3c\left(\frac{n}{2}\right)^2 + 3^2c\left(\frac{n}{2^2}\right)^2 + 3^3c\left(\frac{n}{2^3}\right)^2 + \ldots + 3^kc\left(\frac{n}{2^k}\right)^2 && (1)\\
&= cn^2\left(3^0\left(\frac{1}{2^0}\right)^2 + 3^1\left(\frac{1}{2}\right)^2 + 3^2\left(\frac{1}{2^2}\right)^2 + 3^3\left(\frac{1}{2^3}\right)^2 + ... + 3^k\left(\frac{1}{2^k}\right)^2\right) && (2)\\
&= cn^2\left(3^0\left(\frac{1}{2^2}\right)^0 + 3\left(\frac{1}{2^2}\right) + 3^2\left(\frac{1}{2^2}\right)^2 + 3^3\left(\frac{1}{2^2}\right)^3 + ... + 3^k\left(\frac{1}{2^2}\right)^k\right) && (3)\\
&= cn^2\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots + \left(\frac{3}{4}\right)^k\right) && (4)\\
&= cn^2\left(\sum_{i=0}^{\log_2 n}\left(\frac{3}{4}\right)^i\right) && (5)
\end{aligned}$$

- When we apply the bound on the geometric series to (64) with $r = \frac{3}{4}$, we get:

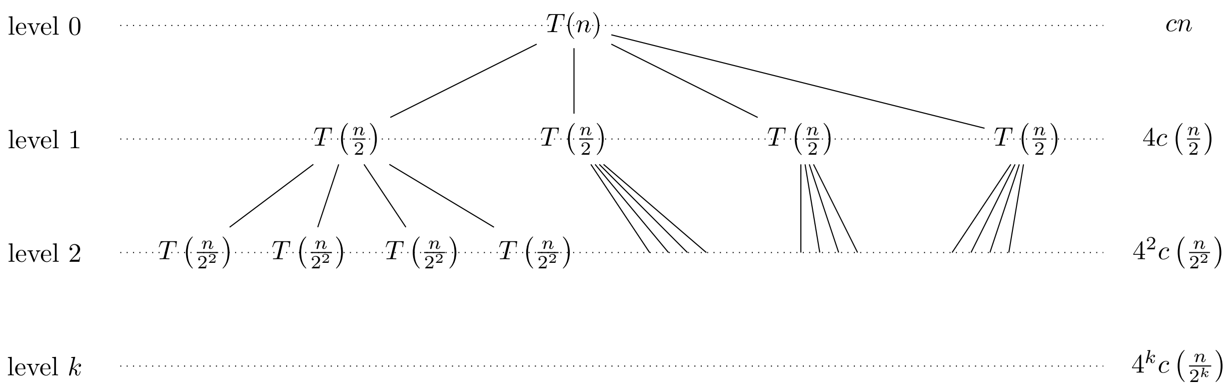
$$\begin{aligned}
T(n) &= cn^2\left(\frac{1 - \left(\frac{3}{4}\right)^{\log_2 n + 1}}{1 - \frac{3}{4}}\right) && (6)\\
&= cn^2\left(4\left(1 - \left(\frac{3}{4}\right)^{\log_2 n + 1}\right)\right) && (7)\\
&= cn^2\left(4 - 4\cdot\frac{3}{4}\cdot\left(\frac{3}{4}\right)^{\log_2 n}\right) && (8)\\
&= cn^2\left(4 - 3\cdot\left(\frac{3}{4}\right)^{\log_2 n}\right) && (9)
\end{aligned}$$

- Using the formula $a^{\log_2 b} = b^{\log_2 a}$, we have:

$$\begin{aligned}
T(n) &= cn^2 \left(4 - 3 \cdot (n)^{\log_2 \frac{3}{4}}\right) \\
&= cn^2 \left(4 - 3 \cdot (n)^{\log_2 3 - 2}\right) \\
&= cn^2 \left(4 - 3 \cdot \left(\frac{n^{\log_2 3}}{n^2}\right)\right) \\
&= 4cn^2 - 3cn^2 \cdot \frac{n^{\log_2 3}}{n^2} \\
&= 4cn^2 - 3cn^{\log_2 3}
\end{aligned}$$

Since $\log_2 3$ is a constant, the term $3cn^{\log_2 3}$ is much smaller than $4cn^2$ as n grows larger. Therefore, the dominant term is $4cn^2$, and we can conclude that $T(n) = \mathcal{O}(n^2)$.

b) $T(n) = 4T(n/2) + n$



- Since we halve the size every time, there are totally $k = \log_2 n$ levels. Summing all this up, we get:

$$T(n) = cn + 4c\left(\frac{n}{2}\right) + 4^2c\left(\frac{n}{2^2}\right) + 4^3c\left(\frac{n}{2^3}\right) + \ldots + 4^kc\left(\frac{n}{2^k}\right) \tag{10}$$

$$= cn\left(\left(\frac{4}{2}\right)^0 + \left(\frac{4}{2}\right)^1 + \left(\frac{4}{2}\right)^2 + \left(\frac{4}{2}\right)^3 + ... + \left(\frac{4}{2}\right)^k\right) \tag{11}$$

$$= cn(1 + 2 + 2^2 + 2^3 + ... + 2^k) \tag{12}$$

- When we apply the bound on the geometric series to (12) with $r = 2$, we get:

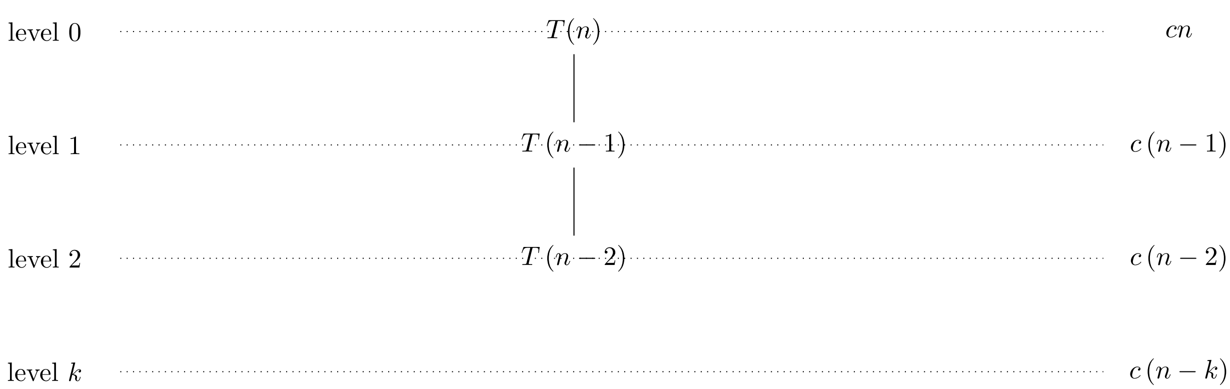
$$\begin{aligned}
T(n) &= cn(\frac{1-2^{k+1}}{1-2}) && (13)\\
&= cn(2^{k+1}-1) && (14)\\
&= cn\left(2 \cdot 2^{k}-1\right) && (15)\\
&= 2cn \cdot 2^{log_2 n} - cn && (16)\\
& && (17)
\end{aligned}$$

- Using the formula $a^{\log_2 b} = b^{\log_2 a}$, we have:

$$\begin{aligned}
T(n) &= 2cn \cdot n^{log_2 2} - cn && (18)\\
&= 2cn^2 - cn && (19)
\end{aligned}$$

the term cn is much smaller than $2cn^2$ as n grows larger. Therefore, the dominant term is $2cn^2$, and we can conclude that $T(n) = \mathcal{O}(n^2)$.

c) $T(n) = T(n-1) + n$



- Since we reduce the size by 1 every time, there are totally $k = n - 1$ levels. Summing all this up, we get:

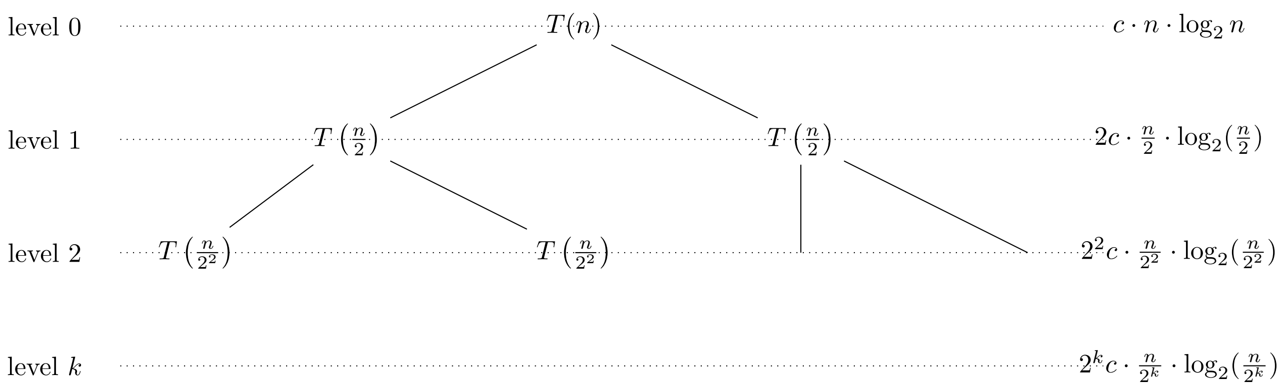
$$\begin{aligned}
T(n) &= cn + c(n-1) + c(n-2) + c(n-3) + ... + c(n-k) && (20)\\
&= c(n + (n-1) + ... + 3 + 2 + 1) && (21)
\end{aligned}$$

- We have formula $1+2+3+...+n = \frac{n(n+1)}{2} for$ (21), we have:

$$\begin{aligned}
T(n) &= c(n^2 - \frac{k(k+1)}{2}) \\
&= c(n^2 - \frac{(n-1)n}{2}) \\
&= c(n^2 - \frac{n^2-n}{2}) \\
&= cn^2 - \frac{1}{2}cn^2 - \frac{1}{2}cn \\
&= \frac{1}{2}cn^2 - \frac{1}{2}cn
\end{aligned}$$

The term $\frac{1}{2}cn$ is much smaller than $\frac{1}{2}cn^2$ as n grows larger. Therefore, the dominant term is $\frac{1}{2}cn^2$, and we can conclude that $T(n) = \mathcal{O}(n^2)$.

d) $T(n) = 2T(n/2) + n \log n$

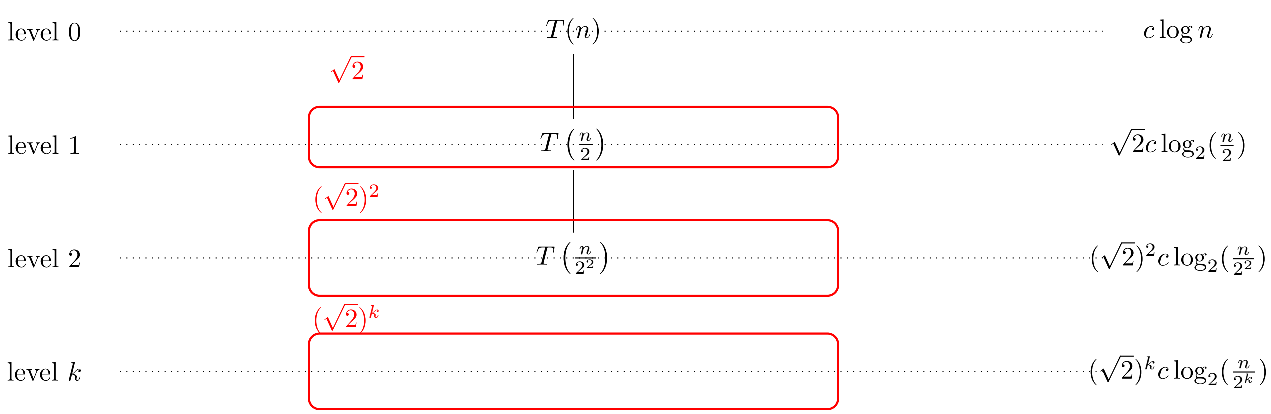


- Since we halve the size every time, there are totally $k = \log_2 n$ levels. Summing all this up, we get:

$$
\begin{aligned}
T(n) &= cn \cdot \log_2 n + 2c \cdot \frac{n}{2} \cdot \log_2(\frac{n}{2}) + 2^2 c \cdot \frac{n}{2^2} \cdot \log_2(\frac{n}{2^2}) + ... + 2^k c \cdot \frac{n}{2^k} \cdot \log_2(\frac{n}{2^k}) && (22)\\
&= cn \left(log_2 n + 2\frac{1}{2}\log_2(\frac{n}{2}) + 2^2 \frac{1}{2^2}\log_2(\frac{n}{2^2}) + ... + 2^k \frac{1}{2^k}\log_2(\frac{1}{2^k}) \right) && (23)\\
&= cn \left(log_2 n + log_2(\frac{n}{2}) + log_2(\frac{n}{2^2}) + ... + log_2(\frac{n}{2^k}) \right) && (24)\\
&= cn \left((log_2 n - log_2 2^0) + (log_2 n - log_2 2) + (log_2 n - log_2 2^2) ... + (log_2 n - log_2 2^k) \right) && (25)\\
&= cn \left((k-1) log_2 n + (log_2 2^0 + log_2 2^1 + log_2 2^2 + ... + log_2 2^k) \right) && (26)\\
&= cn \left(klog_2 n - log_2 n + (0log_2 2 + 1log_2 2 + 2log_2^2 + ... + klog_2 2) \right) \triangleright \textit{ Using Power rule of logarithms} && (27)\\
&= cn \left(klog_2 n - log_2 n + (0 + 1 + 2 + ...k) \right) && (28)\\
&= cn \left(klog_2 n - log_2 n + \frac{k(k+1)}{2} \right) && (29)\\
&= cn \left((log_2 n)^2 - log_2 n + \frac{log_2 n(log_2 n + 1)}{2} \right) && (30)\\
&= cn \left((log_2 n)^2 - log_2 n + \frac{(log_2 n)^2 + log_2 n}{2} \right) && (31)\\
&= cn(log_2 n)^2 - cnlog_2 n + \frac{1}{2}cn(log_2 n)^2 + \frac{1}{2}cn(log_2 n) && (32)\\
&= \frac{3}{2}cn(log2_n)^2 - \frac{1}{2}cn(log_2 n) && (33)\\
& && (34)
\end{aligned}
$$

The term $\frac{1}{2}cn(log_2 n)$ is much smaller than $\frac{3}{2}cn(log2_n)^2$ as n grows larger. Therefore, the dominant term is $\frac{1}{2}cn^2$, and we can conclude that $T(n) = \mathcal{O}(n(log_2 n)^2)$.

e) $T(n) = \sqrt{2}T(n/2) + \log n$



- Since we halve the size every time, there are totally $k = \log_2 n$ levels. Summing all this up, we get:

$$T(n) = c\log_2 n + \sqrt{2}c\left(\log_2\left(\frac{n}{2}\right)\right) + (\sqrt{(2)})^2 c\left(log_2\left(\frac{n}{2^2}\right)\right) + ... + (\sqrt{2^k})c\left(log_2\left(\frac{n}{2^k}\right)\right) \tag{35}$$
$$= c\left((\sqrt{2})^0\log_2\frac{n}{2^0} + (\sqrt{2})^1\log_2\frac{n}{2^1} + (\sqrt{2})^2\log_2\frac{n}{2^2} + ... + (\sqrt{2})^k\log_2\frac{n}{2^k}\right) \tag{36}$$
$$= c\left((\sqrt{2})^0\left(\log_2 n - log_2 2^0\right) + (\sqrt{2})^1\left(\log_2 n - log_2 2^1\right) + (\sqrt{2})^2\left(\log_2 n - log_2 2^2\right) + ... + (\sqrt{2})^k\left(\log_2 n - log_2 2^k\right)\right) \tag{37}$$
$$= c\left(log_2 n\left((\sqrt{2})^0 + (\sqrt{2})^1 + (\sqrt{2})^2 + ... + (\sqrt{2})^k\right) - \left((log_2 2)^0 + (log_2 2)^1 + (log_2 2)^2 + ... + (log_2 2)^k\right)\right) \tag{38}$$
$$= c\left(log_2 n\left((\sqrt{2})^0 + (\sqrt{2})^1 + (\sqrt{2})^2 + ... + (\sqrt{2})^k\right) - (0 + 1 + 2 + ... + k)\right) \tag{39}$$

- When we apply the bound on the geometric series to (39) with $r = \sqrt{2}$, we get:

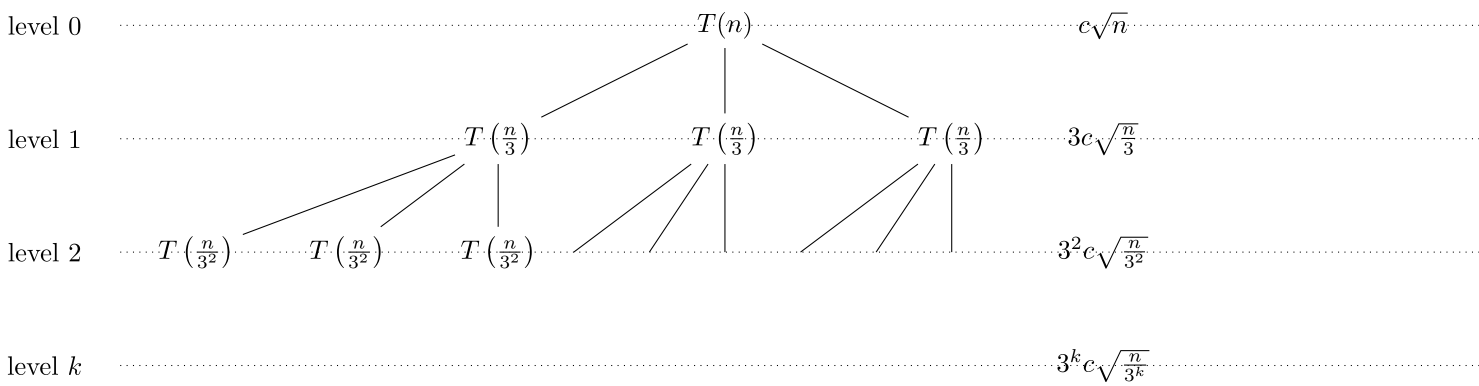
$$T(n) = c\left(\log_2 n\left(\frac{(\sqrt{2})^{k+1} - 1}{\sqrt{2} - 1}\right) - \frac{k(k+1)}{2}\right) \tag{40}$$
$$= c\left(\log_2 n\left(\frac{\sqrt{2}\cdot(\sqrt{2})^{log_2 n} - 1}{\sqrt{2} - 1}\right) - \frac{(\log_2 n)^2 + \log_2 n}{2}\right) \triangleright k = \log_2 n \tag{41}$$
$$= c\left(\log_2 n\left(\frac{\sqrt{2}\cdot(n)^{log_2\sqrt{2}} - 1}{\sqrt{2} - 1}\right) - \frac{(\log_2 n)^2 + \log_2 n}{2}\right) \tag{42}$$
$$= c\left(\log_2 n\left(\frac{\sqrt{2}n^{\frac{1}{2}} - 1}{\sqrt{2} - 1}\right) - \frac{(\log_2 n)^2 + \log_2 n}{2}\right) \tag{43}$$
$$= \frac{\sqrt{2}c}{\sqrt{2} - 1}\cdot n^{1/2}\cdot log_2 n - \frac{c}{\sqrt{2} - 1}\log_2 n - \frac{c}{2}(\log_2 n)^2 - \frac{c}{2}log_2 n \tag{44}$$
$$= \frac{\sqrt{2}c}{\sqrt{2} - 1}\cdot n^{1/2}\cdot log_2 n - \frac{(3 + 2\sqrt{2})c}{2}\log_2 n - \frac{c}{2}(\log_2 n)^2 \tag{45}$$

- **Complexity analysis**
  - First term: $\frac{\sqrt{2}c}{\sqrt{2}-1}\cdot n^{1/2}\cdot log_2 n$:
    * This term grows as $n^{1/2}\cdot log_2 n$
    * It represents the primary contributio to the complexity for large n because $n^{1/2}$ faster than $\log_2 n$ and $(\log_2 n)^2$
  - Second term $\frac{(3+2\sqrt{2})c}{2}\log_2 n$:
    * Thisd term grows linearly with $log_2 n$
    * Although significant, it grows slower compared to $\frac{\sqrt{2}c}{\sqrt{2}-1}\cdot n^{1/2}\cdot log_2 n$
  - Third term $\frac{c}{2}(\log_2 n)^2$:
    * It is smaller compared to $n^{1/2}\cdot\log_2 n$

- Thus, first tearm is the dominant term, the asymptotic complexity of $T(n)$ can be approximated by this term alone for large n

Thus, the complexity of $T(n) = \mathcal{O}(n^{1/2} log_2 n)$

f) $T(n) = 3T(n/3) + \sqrt{n}$



- 'Since we reduce the size by a factor of three each time, there are totally $k = \log_3 n$ levels. Summing all this up, we get:'

$$T(n) = c\sqrt{n} + 3c\sqrt{\frac{n}{3}} + 3^2c\sqrt{\frac{n}{3^2}} + ... + 3^kc\sqrt{\frac{n}{3^k}} \tag{46}$$

$$= c\sqrt{n}\left(3^0\sqrt{\frac{1}{3^0}} + 3^1\sqrt{\frac{1}{3^1}} + 3^2\sqrt{\frac{1}{3^2}} + ... + 3^k\sqrt{\frac{1}{3^k}}\right) \tag{47}$$

$$= c\sqrt{n}\left(\sqrt{\frac{(3^0)^2}{3^0}} + \sqrt{\frac{(3^1)^2}{3^1}} + \sqrt{\frac{(3^2)^2}{3^2}} + ... + \sqrt{\frac{(3^k)^2}{3^k}}\right) \tag{48}$$

$$= c\sqrt{n}\left(\sqrt{3^0} + \sqrt{3^1} + \sqrt{3^2} + ... + \sqrt{3^k}\right) \tag{49}$$

$$= c\sqrt{n}\left((3^{(1/2)})^0 + (3^{(1/2)})^1 + (3^{(1/2)})^2 + ... + (3^{(1/2)})^k\right) \tag{50}$$

- When we apply the bound on the geometric series to (68) with $r = 3^{1/2}$, we get:

$$
\begin{aligned}
T(n) &= c\sqrt{n}\frac{(3^{1/2})^{k+1}-1}{3^{1/2}-1} && (51)\\
&= c\sqrt{n}\frac{(3^{1/2})^{\log_3 n+1}-1}{3^{1/2}-1} \triangleright\ k = \log_3(n) && (52)\\
&= c\sqrt{n}\frac{\sqrt{3}(3^{1/2})^{\log_3 n}-1}{3^{1/2}-1} && (53)\\
&= c\sqrt{n}\frac{\sqrt{3}(n)^{\log_3 3^{1/2}}-1}{3^{1/2}-1} && (54)\\
&= c\sqrt{n}\frac{\sqrt{3}(n)^{1/2}-1}{\sqrt{3}-1} && (55)\\
&= c\sqrt{n}\frac{\sqrt{3n}-1}{\sqrt{3}-1} && (56)\\
&= \frac{\sqrt{3}c}{\sqrt{3}-1}\cdot\sqrt{n}\cdot\sqrt{n} - \frac{c\sqrt{n}}{\sqrt{3}-1} && (57)\\
&= \frac{\sqrt{3}c}{\sqrt{3}-1}\cdot n - \frac{c\sqrt{n}}{\sqrt{3}-1} && (58)\\
& && (59)
\end{aligned}
$$

The term $\frac{c\sqrt{n}}{\sqrt{3}-1}$ is much smaller than $\frac{\sqrt{3}cn}{\sqrt{3}-1}$ as n grows larger. Therefore, the dominant term is $\frac{\sqrt{3}cn}{\sqrt{3}-1}$, and we can conclude that $T(n) = \mathcal{O}(n)$.

g) $T(n) = 7T(n/3) + n^2$

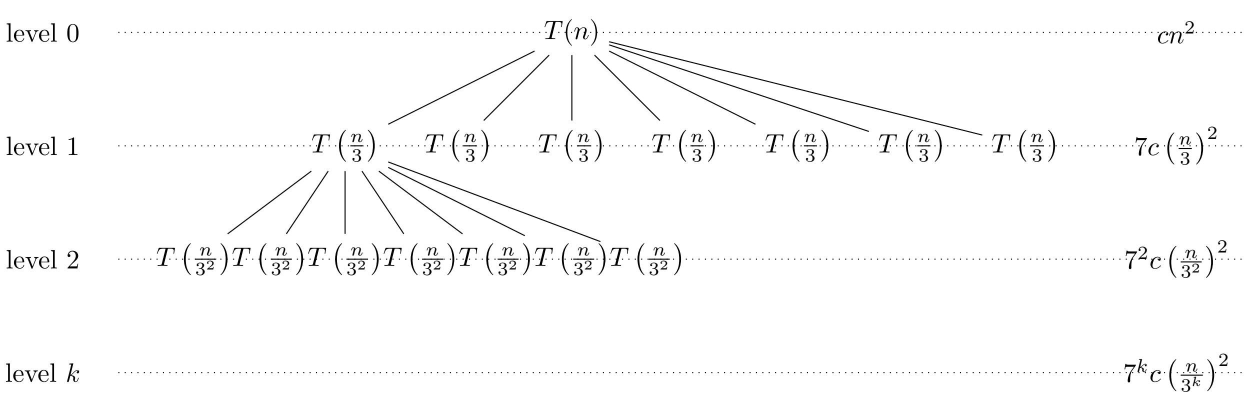


Figure 1: Recurrence tree for $T(n) = 7T(n/3) + n^2$

- Since we reduce the size by a factor of three each time, there are totally $k = \log_3 n$ levels. Summing all this up, we get:

$$
\begin{aligned}
T(n) &= cn^2 + 7c\left(\frac{n}{3}\right)^2 + 7^2c\left(\frac{n}{3^2}\right)^2 + 7^3c\left(\frac{n}{3^3}\right)^2 + \ldots + 7^kc\left(\frac{n}{3^k}\right)^2 && (60)\\
&= cn^2\left(7^0\left(\frac{1}{3^0}\right)^2 + 7^1\left(\frac{1}{3}\right)^2 + 7^2\left(\frac{1}{3^2}\right)^2 + 7^3\left(\frac{1}{3^3}\right)^2 + ... + 7^k\left(\frac{1}{3^k}\right)^2\right) && (61)\\
&= cn^2\left(7^0\left(\frac{1}{3^2}\right)^0 + 7\left(\frac{1}{3^2}\right) + 7^2\left(\frac{1}{3^2}\right)^2 + 7^3\left(\frac{1}{3^2}\right)^3 + ... + 7^k\left(\frac{1}{3^2}\right)^k\right) && (62)\\
&= cn^2\left(1 + \left(\frac{7}{9}\right) + \left(\frac{7}{9}\right)^2 + \left(\frac{7}{9}\right)^3 + \ldots + \left(\frac{7}{9}\right)^k\right) && (63)\\
&= cn^2\left(\sum_{i=0}^{\log_3 n}\left(\frac{7}{9}\right)^i\right) && (64)
\end{aligned}
$$

- When we apply the bound on the geometric series to (64) with $r = \frac{7}{9}$, we get:

$$
\begin{aligned}
T(n) &= cn^2\left(\frac{1-\left(\frac{7}{9}\right)^{\log_3 n+1}}{1-\frac{7}{9}}\right) && (65)\\
&= cn^2\left(\frac{9}{2}\left(1-\left(\frac{7}{9}\right)^{\log_3 n+1}\right)\right) && (66)\\
&= cn^2\left(\frac{9}{2} - \frac{9}{2}\cdot\frac{7}{9}\cdot\left(\frac{7}{9}\right)^{\log_3 n}\right) && (67)\\
&= cn^2\left(\frac{9}{2} - \frac{7}{2}\cdot\left(\frac{7}{9}\right)^{\log_3 n}\right) && (68)
\end{aligned}
$$

- Using the formula $a^{\log_3 b} = b^{\log_3 a}$, we have:

$$
\begin{aligned}
T(n) &= cn^2\left(\frac{9}{2} - \frac{7}{2}\cdot(n)^{\log_3 \frac{7}{9}}\right)\\
&= cn^2\left(\frac{9}{2} - \frac{7}{2}\cdot(n)^{\log_3 7-2}\right)\\
&= cn^2\left(\frac{9}{2} - \frac{7}{2}\cdot\left(\frac{n^{\log_3 7}}{n^2}\right)\right)\\
&= \frac{9}{2}cn^2 - \frac{7}{2}cn^2\cdot\frac{n^{\log_3 7}}{n^2}\\
&= \frac{9}{2}cn^2 - \frac{7}{2}cn^{\log_3 7}
\end{aligned}
$$

Since $\log_3 7$ is a constant, the term $\frac{7}{2}cn^{\log_3 7}$ is much smaller than $\frac{9}{2}cn^2$, and we can conclude that $T(n) = \mathcal{O}(n^2)$.

h) $T(n) = T(\sqrt{n}) + \log n$

level 0	$T(n)$	$c\log_2 n$
level 1	$T(\sqrt{n})$	$c(\log_2 \sqrt{n})$
level 2	$T(\sqrt[4]{n})$	$c(\log_2 \sqrt[4]{n})$
level k		$c\left(log_2 \sqrt[(2^k)]{n}\right)$

- From $T(n) = T(\sqrt{n}) + \log_2 n$ we can also write as:

$$T(n) = T(n^{1/2}) + \log_2 n$$

- This does not show a constant where we can stop. Therefore we need to subtitute n (Let $n = 2^m$), we have:

$$\begin{aligned} T(2^m) &= T(2^{m/2}) + \log_2(2^m) && (69)\\ &= T(2^{m/2}) + m \triangleright \textit{ Using Power rule} && (70)\\ & && (71) \end{aligned}$$

- Let $S(m) = T(2^m)$ so $S(m/2) = T(2^{m/2})$. Thus, we have:

$$S(m) = S(m/2) + m \tag{72}$$

- Now we will find the complexity of (72)
  - Since we halve the size everytime, there are totally $k = \log_2 m$ levels. Summing all this up,we get:

$$\begin{aligned} S(m) &= c(\frac{m}{2^0}) + c(\frac{m}{2}) + c(\frac{m}{2^2}) + ... + c(\frac{m}{2^k}) && (73)\\ &= cm\left((\frac{1}{2})^0 + (\frac{1}{2})^1 + (\frac{1}{2})^2 + ... + (\frac{1}{2})^k\right) && (74) \end{aligned}$$

– When we apply the bound on the geometric series to (74) with $r = \frac{1}{2}$, we get:

$$
\begin{aligned}
S(m) &= cm\left(\frac{(\frac{1}{2})^{k+1} - 1}{\frac{1}{2} - 1}\right) && (75)\\
&= -2cm\left(\left(\frac{1}{2}\right)^{\log_2(m)+1} - 1\right) && (76)\\
&= -2cm\left(\frac{1}{2}\left(\frac{1}{2}\right)^{\log_2 m} - 1\right) && (77)\\
&= -2cm\left(\frac{1}{2}m^{\log_2(\frac{1}{2})} - 1\right) && (78)\\
&= -2cm\left(\frac{1}{2m} - 1\right) && (79)\\
&= -c + 2cm && (80)\\
& && (81)
\end{aligned}
$$

– $2cm$ is a dominant term. Thus, the complexity of $S(m) = \mathcal{O}(m)$

- We started with the recurrence $T(n) = T(\sqrt{n}) + \log_2 n$ and transformed n to 2^m, which led us to a new recurrence $S(m) = S(m/2) + m$, where $S(m) = T(2^m)$. At the same time, complexity of $S(m) = \mathcal{O}(m)$ and $m = \log_2 n$, substituting back, we get:

$$S(\log_2 n) = \mathcal{O}(\log_2 n)$$

- By our transformation definition $S(m) = T(2^m)$. Setting $m = \log_2 n$ implies $2^m = n$. Thus, we have:

$$S(\log_2 n) = T(n)$$

- Therefore, substituting our expression for $\log_2 n$:

$$T(n) = \mathcal{O}(\log_2 n)$$

Problem 2