MATH1061 Assignment 2

SID: 530328265 - Tutorial_1: 9.00 WED - Tutorial_2: 10.00 FRI

Due Date: 12/5/2024

1.

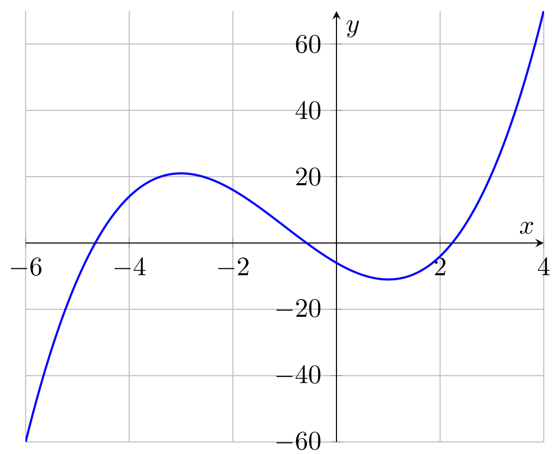

Figure 1: Graph of the function $y = x^3 + 3x^2 - 9x - 6$

(a) **A critical point** of $f(x)$ is a number c in the domain of f, where either $f'(c) = 0$ or where f is not differentiable at c. To find the critical points for $f(x) = x^3 + 3x^2 - 9x - 6$, we take the first derivative and set it to zero.

$$
\begin{aligned}
f'(x) &= 3x^2 + 6x - 9 = 0 && (1)\\
&= 3(x^2 + 2x - 3) = 0 && (2)\\
&= 3(x^2 + 3x - x - 3) = 0 && (3)\\
&= 3((x^2 + 3x) - (x + 3)) = 0 && (4)\\
&= 3(x(x + 3) - 1(x + 3)) = 0 && (5)\\
&= 3(x - 1)(x + 3) = 0 && (6)\\
&= 3(x + 3)(x - 1) = 0 && (7)\\
& && (8)
\end{aligned}
$$

Thus, $x = -3$ and $x = 1$ are the roots of $f^{'}(x)$. At the same time, both of them on $[-4, 5]$, so $x = -3$ and $x = 1$ are the critical points for $f(x)$ on $[-4, 5]$

Classifying: Taking the second derivative test

We have the first derivatition of $f(x)$ is $3x^2 + 6x - 9$. Thus, second derivatition of $f(x)$ is:

$$f^{''}(x) = 6x + 6$$

We have $f^{''}(-3) = -12 < 0$ and $f^{''}(1) = 12 > 0$, so we have an conclustion that

There is a local minimum of f at x = 1 and there is a local maximum of f at x = -3

(b) From a), we have $x = -3$ and $x = 1$ are the critical points for $f(x)$ on $[-4, 5]$. To find the global maximum and global minimum values, first we find the values of $f(x)$ critical points on $[-4, 5]$, and we calculate the values of f at the endpoints of the interval(-4 and 5). After that, we compare all the numbers obtained in the first two steps. The largest of all the values is the **global maximum** value and the smallest of these values is the **global minimum** value.

- The value of f when $x = -3$ is:

$$f(-3) = (-3)^3 + 3 \times (-3)^2 - 9 \times (-3) - 6 = 21$$

- The value of f when $x = 1$ is:

$$f(1) = 1^3 + 3 \times (1)^2 - 9 \times (1) - 6 = -11$$

- The value of f when $x = -4$ is:

$$f(-4) = (-4)^3 + 3 \times (-4)^2 - 9 \times (-4) - 6 = 14$$

- The value of f when $x = 5$ is:

$$f(5) = 5^3 + 3 \times 5^2 - 9 \times 5 - 6 = 149$$

As we can see, with $x = -3$, $x = 1$, $x = -4$ and $x = 5$, the largest value of $f(x)$ is 149 when $x = 5$ and the smallest value of $f(x)$ is -11 when $x = 1$. Thus, the global maximum of $f(x)$ on $[-4, 5]$ is $x = 5$, and the global minimum of $f(x)$ on $[-4, 5]$ is $x = 1$

(c) A points of inflection is where the function changes its concavity. For example, from concave up to concave down or vice versa. To find the inflection points we do the following steps

- Step 1: Find the second derivative of f
- Step 2: Set the second derivative to zero to find the potenial points of inflection
- Step 3: Check the concavity change around that potential points

Step 1: We have the first derivatition of $f(x)$ is $3x^2 + 6x - 9$. Thus, second derivatition of $f(x)$ is:

$$f^{''}(x) = 6x + 6$$

Step 2: The potenial point of inflection is:

$$f^{''}(x) = 6x + 6 = 0 \leftrightarrow x = -1$$

Step 3: To check the concavity change around that potential points, we have the following table to check the concavity change around $x = -1$:

x	$-\infty$	-1	$+\infty$
y''	$-$	0	$+$

As we can see, $f^{''}(x)$ change from negative to positive as x increase through -1, the concavity changes from concave down to concave up. Therefore, $x = -1$ is a point of inflection. At the same time, -1 in $[-4, 5]$, so $f(x)$ have a point of iflection on $[-4, 5]$

2.

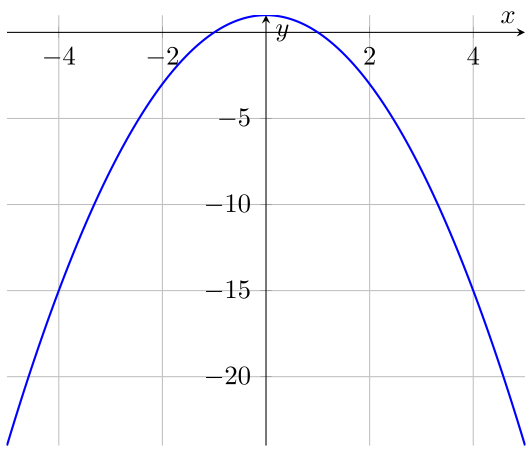

Figure 2: Graph of the function $y = 1 - x^2$

(a) With integer n ≤ 1, each subinterval has length

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

The partion points are given by:

$$x_i = a + \Delta x \times i = 0 + \frac{1}{n} \times i \text{ for } 0 \leq i \leq n$$

The ith subinterval will then be the interval $[x_{i-1}, x_i]$

- Find the lower Riemann sum L_n
  - Since $f(x) = 1 - x^2$ is decreasing in $[0, 1]$, the minimum values of $f(x)$ on any subinterval $[x_{i-1}, x_i]$ occurs at the right endpoint x_i. Therefore the lower sum is

$$L_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(1 - \left(\frac{i}{n}\right)^2\right)\frac{1}{n} \tag{9}$$

$$= \frac{1}{n} \times \sum_{i=1}^{n} \left(1 - \left(\frac{i}{n}\right)^2\right) \tag{10}$$

$$= \frac{1}{n} \times \left(\sum_{i=1}^{n}(1) - \sum_{i=1}^{n}(\frac{i}{n})^2\right) \tag{11}$$

$$= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \sum_{i=1}^{n} i^2\right) \tag{12}$$

– We have the following equation:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{13}$$

– From (12) and (13), we have:

$$\begin{aligned}
L_n &= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \frac{n(n+1)(2n+1)}{6} \right) \\
&= 1 - \frac{1}{n^2} \times \frac{(n+1)(2n+1)}{6} \\
&= \frac{6n^2 - (n+1)(2n+1)}{6n^2} \\
&= \frac{6n^2 - (2n^2 + 3n + 1)}{6n^2} \\
&= \frac{4n^2 - 3n - 1}{6n^2}
\end{aligned}$$

- Find the upper Riemann sum L_n
  – Since $f(x) = 1 - x^2$ is decreasing in $[0, 1]$, the maximum values of $f(x)$ on any subinterval $[x_{i-1}, x_i]$ occurs at the left endpoint x_{i-1}. Therefore the lower sum is

$$U_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \left(1 - \left(\frac{i-1}{n} \right)^2 \right) \frac{1}{n} \tag{14}$$

$$= \frac{1}{n} \times \sum_{i=1}^{n} \left(1 - \left(\frac{i-1}{n} \right)^2 \right) \tag{15}$$

$$= \frac{1}{n} \times \left(\sum_{i=1}^{n} (1) - \sum_{i=1}^{n} (\frac{i-1}{n})^2 \right) \tag{16}$$

$$= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \sum_{i=1}^{n} (i-1)^2 \right) \tag{17}$$

$$= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \sum_{i-1=0}^{n-1} (i-1)^2 \right) \tag{18}$$

$$= \frac{1}{n} \times \left(\left(n - \frac{1}{n^2} \times 0 \right) + \left(-\frac{1}{n^2} \times \sum_{i-1=1}^{n-1} (i-1)^2 \right) \right) \tag{19}$$

$$= \frac{1}{n} \times \left(n + \left(-\frac{1}{n^2} \times \sum_{i-1=1}^{n-1} (i-1)^2 \right) \right) \tag{20}$$

$$= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \sum_{i-1=1}^{n-1} (i-1)^2 \right) \tag{21}$$

$$\tag{22}$$

– From (21) and (13), we have:

$$\begin{aligned}
U_n &= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \frac{(n-1)n(2(n-1)+1)}{6} \right) \\
&= \frac{1}{n} \times \left(n - \frac{1}{n^2} \times \frac{(n-1)n(2n-1)}{6} \right) \\
&= 1 - \frac{(n-2)(2n-1)}{6n^2} \\
&= \frac{6n^2 - (n-1)(2n-1)}{6n^2} \\
&= \frac{6n^2 - (2n^2 - 3n + 1)}{6n^2} \\
&= \frac{6n^2 - (2n^2 - 3n + 1)}{6n^2} \\
&= \frac{4n^2 + 3n - 1}{6n^2}
\end{aligned}$$

(b) We compute the limit as $n \to \infty$ of L_n and U_n:

- Calculate the limit as $n \to \infty$ of L_n

$$\lim_{n\to\infty} (L(n)) = \lim_{n\to\infty} \frac{4n^2 - 3n - 1}{6n^2} = \lim_{n\to\infty} \frac{4 - \frac{3}{n} - \frac{1}{n^2}}{6} \tag{23}$$

$$= \frac{4-0-0}{6} = \frac{4}{6} = \frac{2}{3} \tag{24}$$

- Calculate the limit as $n \to \infty$ of U_n

$$\lim_{n\to\infty} (U(n)) = \lim_{n\to\infty} \frac{4n^2 + 3n - 1}{6n^2} = \lim_{n\to\infty} \frac{4 + \frac{3}{n} - \frac{1}{n^2}}{6} \tag{25}$$

$$= \frac{4+0-0}{6} = \frac{4}{6} = \frac{2}{3} \tag{26}$$

- As we can see, $\lim_{n\to\infty} L(n) = \lim_{n\to\infty} U(n) = \frac{2}{3}$.
- Not only that, $f(x)$ is decreasing on the interval $[0, 1]$ so we have:

$$L(n) \leq \int_0^1 (1 - x^2)\,dx \leq U(n)$$

- Applying squeeze law, then we have

$$\int_0^1 (1 - x^2)\,dx = \frac{2}{3}$$

3.

(a) First we find the first and second derivative of $f(x)$:

$$f^{'}(x) = (2x+5)e^{-x} - e^{-x}(9+5x+x^2)$$
$$f^{''}(x) = 2e^{-x} - (5+2x)e^{-x} - (2x+5)e^{-x} + e^{-x}(9+5x+x^2)$$

- Second, we find the value of $f^{'}(0)$ and $f^{''}(0)$ and $f(0)$

$$f(0) = 9 + 5\times 0 + 0 \times 2 \times e^{-0} = 9 \quad (27)$$
$$f^{'}(0) = (2\times 0+5)e^{-0} + e^{-0}(9+5\times 0+0^2) = -4 \quad (28)$$
$$f^{''}(0) = 2e^{-0} - (5+2\times 0)e^{-0} - (2\times 0+5)e^{-0} + e^{-0}(9+5\times 0+0^2) = 1 \quad (29)$$

From that, we can find $P_2(x)$, which is:

$$P_2(x) = \frac{f(0)}{0!} + \frac{f^{'}(0)\times x}{1!} + \frac{f^{''}(0)\times x^2}{2!} \quad (30)$$
$$= 9 - 4x + \frac{x^2}{2!} \quad (31)$$

(b) The lagrange form of the remainder $R_n(x)$ for Taylor series of order n centered at a is given by:

$$R_n(X) = \frac{f^{n+1}(c)}{(n+1)!} \times (x-a)^{n+1}$$

We have $R_2(x) = \frac{f^{(3)}(c)\times x^3}{3!}$ for $0 \le c \le 1$

$$f^{''}(x) = 2e^{-x} - (5+2x)e^{-x} - (2x+5)e^{-x} + e^{-x}(9+5x+x^2) \quad (32)$$
$$= e^{-x}\left(2 - (5+2x) - (2x+5) + (9+5x+x^2)\right) \quad (33)$$
$$= e^{-x}\left(2 - 5 - 2x - 2x - 5 + (9+5x+x^2)\right) \quad (34)$$
$$= e^{-x}\left(-4x - 8 + (9+5x+x^2)\right) \quad (35)$$
$$= e^{-x}(x^2+x+1) \quad (36)$$

From (36), we can find the $f^{(3)}(x)$ and we call $g(x) = f^{(3)}(x)$

$$f^{(3)}(x) = (e^{-x}(x^2+x+1))^{'} \quad (37)$$
$$= (2x+1)e^{-x} - e^{-x}(x^2+x+1) \quad (38)$$
$$= e^{-x}(2x+1-x^2-x-1) \quad (39)$$
$$= e^{-x}(-x^2+x) \quad (40)$$
$$(41)$$

Now, we will find the global maximum values of g(x) on $[0,1]$

Find critical points for g(x) on $[0,1]$

- The first derivatition of $g(x)$ is:

$$\begin{aligned} g^{'}(x) &= e^{-x}(-2x+1) - e^{-x}(-x^2+x) \\ &= -e^{-x} \times 2x + e^{-x} + e^{-x} \times x^2 - e^{-x} \times x \\ &= e^{-x} \times x^2 - 3x \times e^{x} + e^{-x} \\ &= e^{-x} \times (x^2 - 3x + 1) \end{aligned}$$

- Therefore the critical points of $g(x)$ are the roots of $g^{'}(x)$:

$$e^{-x} \times (x^2 - 3x + 1) = 0$$

- Because e^{-x} always larger than 0 for all $x \in \mathbb{R}$. Thus the roots of $x^2 - 3x + 1$ are also the root of $g^{'}(x)$

$$\Delta = \sqrt{(-3)^2 - 4 \times 1 \times 1} = \sqrt{5} \tag{42}$$

$$\rightarrow x_{1,2} = \frac{-(-3) \pm \sqrt{5}}{2} \tag{43}$$

- Thus, $x = \frac{3-\sqrt{5}}{2}$is the critical points for $g(x)$ on $[0, 1]$.
- The global minimum of $g(x)$ on $[0, 1]$ is:
  - The value of g when x = 0 is $g(0) = 0$
  - The value of g when x = $\frac{3-\sqrt{5}}{2}$ is $g(\frac{3-\sqrt{5}}{2}) \approx 0.16112$
  - The value of g when x = 1 is $g(1) = 0$
- Thus, the global maximum of g is ≈ 0.360 when x = $\frac{3-\sqrt{5}}{2}$

We have:

$$f^{(3)}(c) = (-c^2 + c)e^{-c} \leq (-1^2 + 1)e^{-1}$$

Thus,

$$f^{(3)}(c) \leq 0$$

At the same time,

$$R_2(x) = \frac{f^{(3)}(c) \times x^{(3)}}{3!} = \frac{g(c) \times x^{(3)}}{3!} \leq \frac{0.16112 \times x^3}{3!}$$

With $x = 1$, we have

$$R_2(1) \leq \frac{0.16112 \times 1^3}{3!} \approx 0.02685$$

Thus, the upperbound is 0.02685

4.

Let ℓ_1 and ℓ_2 be two lines in space defined by the parametric equations:

$$\ell_1 : \begin{cases} x = 3 - s \\ y = 4 + 5s \\ z = 3 + s \end{cases} \qquad (s \in \mathbb{R})$$

$$\ell_2 : \begin{cases} x = 2 + t \\ y = -3 + t \\ z = -2 + 2t \end{cases} \qquad (t \in \mathbb{R})$$

(a)

To find the point of intersection, equate the parametric equations:

$$\begin{aligned} 3 - s &= 2 + t \\ 4 + 5s &= -3 + t \\ 3 + s &= -2 + 2t \end{aligned}$$

Solving the system:

$$\begin{aligned} s + t &= 1 \quad \text{(from x-coordinates)} \\ 5s - t &= -7 \quad \text{(from y-coordinates)} \\ s - 2t &= -5 \quad \text{(from z-coordinates)} \end{aligned}$$

From xcoordinates and ycoordinates, we find $s = -1$ and $t = 2$. Thus, we plugging back to zcoordinates to check that s and t is the valid solution or not.

$$s - 2t = -1 - 2 \times 2 = -5 \text{ (VALID SOLUTION)}$$

We find $s = -1$ and $t = 2$. Plugging these back into the line equations gives:

$$x = 3 - s = 3 + 1 = 4 \tag{44}$$
$$y = 4 + 5 \times (-1) = -1 \tag{45}$$
$$z = 3 + s = 3 - 1 = 2 \tag{46}$$
$$\tag{47}$$

$$\text{Intersection Point: } (4, -1, 2)$$

(b)

To find a general equation for the plane P that contains both ℓ_1 and ℓ_2, use the direction vectors of the lines $\vec{d_1} = (-1, 5, 1)$ and $\vec{d_2} = (1, 1, 2)$ and a point on the plane. The cross product of $\vec{d_1}$ and $\vec{d_2}$ gives a normal vector $\vec{n}$

to the plane:

$$\vec{n} = \vec{d_1} \times \vec{d_2} = \begin{vmatrix} -1 & 5 & 1 \\ \\ 1 & 1 & 2 \end{vmatrix} \tag{48}$$

$$= ((5 \times 2 - 1 \times 1), (1 \times 1 - (-1 \times 2)), (1 \times -1 - 5 \times 1)) \tag{49}$$

$$= (9, 3, -6) \tag{50}$$

$$= (3, 1, -2) \tag{51}$$

The normal vector simplifies to $\vec{n} = (3, 1, -2)$. Using the point of intersection (4, -1, 2), the plane equation is:

$$3(x - 4) + 1(y + 1) - 2(z - 2) = 0$$

This simplifies to:

$$3x - 12 + y + 1 - 2z + 4 = 0$$

Thus, the general equation for plane P is:

$$3x + y - 2z = 7$$

5.

We start with the augmented matrix of the system:

$$\begin{bmatrix} 1 & 0 & -5 & | & -4 \\ 1 & -\lambda & -2 & | & 2 \\ 1 & 2 & \lambda & | & 2 \end{bmatrix}.$$

The goal is to use elementary row operations to transform this matrix into row echelon form. We will perform the following steps:

Step 1: Eliminate x from the second and third rows.

- $R_2 \leftarrow R_2 - R_1$: Subtract the first row from the second row.
- $R_3 \leftarrow R_3 - R_1$: Subtract the first row from the third row.

This results in:

$$\begin{bmatrix} 1 & 0 & -5 & | & -4 \\ 0 & -\lambda & 3 & | & 6 \\ 0 & 2 & \lambda+5 & | & 6 \end{bmatrix}.$$

Then, we take $R_3 \leftarrow R_3/2$, after that we swap R_3 with R_2

$$\begin{bmatrix} 1 & 0 & -5 & | & -4 \\ 0 & 1 & \frac{\lambda+5}{2} & | & 3 \\ 0 & -\lambda & 3 & | & 6 \end{bmatrix}.$$

We take $R_3 \leftarrow \lambda R_2 + R_3$. Then we will have the row echelon form.

$$\begin{bmatrix} 1 & 0 & -5 & | & -4 \\ 0 & 1 & \frac{\lambda+5}{2} & | & 3 \\ 0 & 0 & \frac{\lambda^2+5\lambda+6}{2} & | & 3\lambda+6 \end{bmatrix}.$$

b) For $\frac{\lambda^2+5\lambda+6}{2}z = 6 + 3\lambda$

- To make the system has no solution, we have the following equations

- We have

$$\frac{\lambda^2 + 5\lambda + 6}{2} z = 6 + 3\lambda \tag{52}$$

$$(\lambda + 2)(\lambda + 3)z = 12 + 6\lambda \tag{53}$$

$$(\lambda + 2)(\lambda + 3)z = 6(\lambda + 2) \tag{54}$$

- To make the system has no solution, we need to make z not exists, which means $\lambda = -3$ and $\lambda \neq -2$(because at this situation $(0z = -6)$)
- Thus the system has no solution if and only if $\lambda = -3$

• To make the system has infinite solutions, we need to make the equation $(\lambda + 2)(\lambda + 3)z = 6(\lambda + 2)$ always True. Thus, λ equals to -2. The reason for that is when $\lambda = -2$, we have:

-
$$0z = 0 \text{ (This one always True)} \tag{55}$$

• To make the system has an unique solution, $\lambda \neq -3$ and $\lambda \neq -2$. At that time, we have the value of z is $\frac{6\lambda+12}{\lambda^2+5\lambda+6}$

6

(a) We hav the following transition probalities

- From state 1 (linear algebra)
  (a) Stay in State 1: 80% ($P_{1,1} = 0.8$)
  (b) Move to State 2 (Calculus): 10% ($P_{2,1} = 0.1$)
  (c) Move to State 3 (Netflix): 10% ($P_{3,1} = 0.1$)
- From State 2(Calculus)
  - Move to State 1: 20% ($P_{1,2} = 0.2$)
  - Stay in State 2: 60% ($P_{2,2} = 0.6$)
  - Move to State 3: 20% ($P_{3,2} = 0.2$)
- From State 3(Netflix)
  - Move to State 1: 40% ($P_{1,3} = 0.4$)
  - Stay to State 2: 40% ($P_{2,3} = 0.4$)
  - Stay in State 3: 20% ($P_{3,3} = 0.2$)
- Hence, we have the following matrix

$$P = \begin{bmatrix} 0.8 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.2 \end{bmatrix}$$

(b) we have:

- $$\vec{x_0} = \begin{bmatrix} 1000 \\ 1000 \\ 800 \end{bmatrix}$$
- Let x_2 is the number of each state on day 2. Thus, x_2 equals to:

$$\vec{x_2} = P \times \vec{x_1} = P \times P \times \vec{x_0} = P^2 \times \vec{x_0}$$

- Calculate P^2

$$P^2 = \begin{bmatrix} 0.8 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} \cdot \begin{bmatrix} 0.8 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} \tag{56}$$

$$= \begin{bmatrix} (0.8 \cdot 0.8 + 0.2 \cdot 0.1 + 0.4 \cdot 0.1) & (0.8 \cdot 0.2 + 0.2 \cdot 0.6 + 0.4 \cdot 0.2) & (0.8 \cdot 0.4 + 0.2 \cdot 0.4 + 0.4 \cdot 0.2) \\ (0.1 \cdot 0.8 + 0.6 \cdot 0.1 + 0.4 \cdot 0.1) & (0.1 \cdot 0.2 + 0.6 \cdot 0.6 + 0.4 \cdot 0.2) & (0.1 \cdot 0.4 + 0.6 \cdot 0.4 + 0.4 \cdot 0.2) \\ (0.1 \cdot 0.8 + 0.2 \cdot 0.1 + 0.2 \cdot 0.1) & (0.1 \cdot 0.2 + 0.2 \cdot 0.6 + 0.2 \cdot 0.2) & (0.1 \cdot 0.4 + 0.2 \cdot 0.4 + 0.2 \cdot 0.2) \end{bmatrix} \tag{57}$$

$$= \begin{bmatrix} 0.7 & 0.36 & 0.48 \\ 0.18 & 0.46 & 0.36 \\ 0.12 & 0.18 & 0.16 \end{bmatrix} \tag{58}$$

- Therefore, $\vec{x_2}$ is calculated as follows:

$$\begin{aligned}
\vec{x_2} &= P^2 \cdot \vec{x_0} \\
&= \begin{bmatrix} 0.7 & 0.36 & 0.48 \\ 0.18 & 0.46 & 0.36 \\ 0.12 & 0.18 & 0.16 \end{bmatrix} \cdot \begin{bmatrix} 1000 \\ 1000 \\ 800 \end{bmatrix} \\
&= \begin{bmatrix} (0.7 \cdot 1000 + 0.36 \cdot 1000 + 0.48 \cdot 800) \\ (0.18 \cdot 1000 + 0.46 \cdot 1000 + 0.36 \cdot 800) \\ (0.12 \cdot 1000 + 0.18 \cdot 1000 + 0.16 \cdot 800) \end{bmatrix} \\
&= \begin{bmatrix} 700 + 360 + 384 \\ 180 + 460 + 288 \\ 120 + 180 + 128 \end{bmatrix} \\
&= \begin{bmatrix} 1444 \\ 928 \\ 428 \end{bmatrix}
\end{aligned}$$

- Thus, there are
  - 1444 students in state 1
  - 928 students in state 2
  - 428 students in state 3

(c) Expanding $(P - I_3)\vec{x} = 0$, we have:

$$\begin{aligned}
(P - I_3)\vec{x} &= 0 \\
Px - I_3\vec{x} &= 0
\end{aligned}$$

Since $I_3\vec{x} = \vec{x}$(identity matrix leaves vectors unchanged), the equation simplifies to $Px = x$

Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We have:

$$P\vec{x} = \vec{x}$$
$$\begin{bmatrix} 0.8 & 0.2 & 0.4 \\ 0.1 & 0.6 & 0.4 \\ 0.1 & 0.2 & 0.2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then, we have the equations

$$\begin{aligned} 0.8x + 0.2y + 0.4z &= x \\ 0.1x + 0.6y + 0.4z &= y \\ 0.1x + 0.2y + 0.2z &= z \end{aligned}$$

Which equals to:

$$\begin{aligned} -0.2x + 0.2y + 0.4z &= 0 \\ 0.1x - 0.4y + 0.4z &= 0 \\ 0.1x + 0.2y - 0.8z &= 0 \end{aligned}$$

From that equations, we have the an augmented matrix:

$$\begin{bmatrix} -0.2 & 0.2 & 0.4 & | & 0 \\ 0.1 & -0.4 & 0.4 & | & 0 \\ 0.1 & 0.2 & -0.8 & | & 0 \end{bmatrix}.$$

First, we take $R_1 \leftarrow R_1 / -2$ and $R_2 \leftarrow R_2 - R_3$, then we have

$$\begin{bmatrix} 0.1 & -0.1 & -0.2 & | & 0 \\ 0 & -0.6 & 1.2 & | & 0 \\ 0.1 & 0.2 & -0.8 & | & 0 \end{bmatrix}.$$

Then, we take $R_2 \leftarrow \frac{R_2}{2}$ and $R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 0.1 & -0.1 & -0.2 & | & 0 \\ 0 & -0.3 & 0.6 & | & 0 \\ 0 & 0.3 & -0.6 & | & 0 \end{bmatrix}.$$

Finally, we take $R_3 \leftarrow R_3 + R_2$

$$\begin{bmatrix} 0.1 & -0.1 & -0.2 & | & 0 \\ 0 & -0.3 & 0.6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

As we can see R_3 has infinite solutions, since 0 always $= 0$(z can be anything)

Hence we have infinite solution for $\vec{x}$

We have equations that:

$$x - y - 2z = 0$$

$$y - 2z = 0$$

Let $z = t$, then $x = 3t$ and $y = 2t$

Thus, we got vector x:

$$\vec{x} = \begin{bmatrix} 4t \\ 2t \\ t \end{bmatrix}$$

(d) We have:

$$x_0 = \begin{bmatrix} 1600 \\ 800 \\ 400 \end{bmatrix} = \begin{bmatrix} 4 \times 400 \\ 2 \times 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 4t \\ 2t \\ t \end{bmatrix} \text{(With t = 400)}$$

Because we have x has the form in question c, we have the properties that have been proved in 6b, which is:

$$Px = x$$

We have $x_1 = Px_0 = x_0$, which implies to:

$$x_{100} = Px_{99} = P(Px_98) = P(P(Px_{97})) = P(P(P...(Px_0)))\text{(We will have } 100P_s)$$

Thus, x_{100} equals to:

$$x_{100} = \begin{bmatrix} 1600 \\ 800 \\ 400 \end{bmatrix}$$