MATH1061 Assignment 1

SID: 530328265 - Tutorial_1: 9.00 WED - Tutorial_2: 10.00 FRI

Due Date: 18/3/2024

1.

(a) The condition require for $\lim_{x\to -2} f(x)$ to exist is:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) \tag{1}$$

We also have $\lim_{x\to -2^- f(x)}$, which equals to:

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (2+2^x) = \lim_{x\to -2^-} (2) + \lim_{x\to -2^-} (2^x) = 2 + 2^{-2} = \frac{9}{4} \tag{2}$$

From (1) and (2), we have $\lim_{x\to -2^+} f(x)$:

$$\begin{aligned}
&\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} (ax+1) = \lim_{x\to -2^-} f(x) = \frac{9}{4} \\
&\Leftrightarrow \lim_{x\to -2^+} (ax+1) = \frac{9}{4} \\
&\Leftrightarrow \lim_{x\to -2^+} (ax) + \lim_{x\to -2^+} (1) = \frac{9}{4} \\
&\Leftrightarrow a(-2) + 1 = \frac{9}{4} \\
&\Leftrightarrow a = \frac{-5}{8} \quad \blacksquare
\end{aligned}$$

Thus, The value of a is require for $\lim_{x\to -2} f(x)$ to exist is $\frac{-5}{8}$

(b) Let c $\in \mathbb{R}$ (The domain of the function). $f(x)$ is a continious function if and only if $f(c)$ is define, $\lim_{x\to c}(f(x))$ exists and is finite, and $\lim_{x\to c}(f(x)) = f(c)$.

Because $-2 \in \mathbb{R}$ and $1 \in \mathbb{R}$ so we have:

$$\lim_{x\to -2}(f(x)) \text{ exists and equal to } f(-2) \tag{3}$$

$$\lim_{x\to 1}(f(x)) \text{ exists and equal to } f(1) \tag{4}$$

From (3), we have:

$$\begin{aligned}
&\lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = \lim_{x\to -2} f(x) = f(-2)\\
&\Leftrightarrow \lim_{x\to -2^+}(ax+1) = \lim_{x\to -2^-}(2+2^x) = 2+2^{-2} = \frac{9}{4}\\
&\Leftrightarrow \lim_{x\to -2^+}(ax+1) = \frac{9}{4}\\
&\Leftrightarrow a\times(-2)+1 = \frac{9}{4}\\
&\Leftrightarrow a = \frac{-5}{8} \quad \blacksquare
\end{aligned}$$

From (4), we have:

$$\begin{aligned}
&\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)\\
&\Leftrightarrow \lim_{x\to 1^+}(\frac{c-2x}{x}) = \lim_{x\to 1^-}(\frac{-5}{8}(x)+1) = \frac{-5}{8}\times 1+1 = \frac{3}{8}\\
&\Leftrightarrow \lim_{x\to 1^+}(\frac{c-2x}{x}) = \frac{3}{8}\\
&\Leftrightarrow \frac{c-2}{1} = \frac{3}{8}\\
&\Leftrightarrow c = \frac{19}{8} \quad \blacksquare
\end{aligned}$$

Because $1 \in \mathbb{R}$, we have:

$$\lim_{x\to 1} f(x) = f(1)$$

At the same time, $\lim_{x\to 1} f(x) = \frac{3}{8}$ and $f(x) = b$ if and only if $x = 1$ so we have:

$$f(1) = b = \frac{3}{8} \quad \blacksquare$$

Thus, If $f(x)$ is a continious function $a = \frac{-5}{8}$, $b = \frac{3}{8}$, $c = \frac{19}{8}$

(c)

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}(2+2^x)\\
&= \lim_{x\to-\infty}(2) + \lim_{x\to-\infty}(2^x)\\
&= 2+0\\
&= 2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty}(\frac{c-2x}{x})\\
&= \lim_{x\to\infty}(\frac{\frac{c}{x}-\frac{2x}{x}}{\frac{x}{x}})\\
&= \lim_{x\to\infty}(\frac{\frac{c}{x}-2}{1})\\
&= \lim_{x\to\infty}(\frac{c}{x}-2)\\
&= \lim_{x\to\infty}(\frac{c}{x}) - \lim_{x\to\infty}(2) = 0-2\\
&= -2
\end{aligned}$$

2.

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(a)

$$g : D \to \mathbb{R},\ g(x) = \sqrt{\frac{1}{2} + \sin x}, \text{ which is define if and only if:}$$

$$\frac{1}{2} + \sin x \geq 0$$
$$\Leftrightarrow \sin x \geq \frac{-1}{2}$$

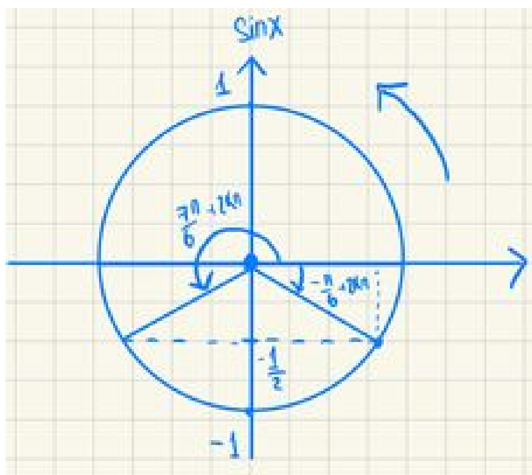

Figure 1: Ex: 2a

As we can see in Figure 1, we can see that:

$$\sin x = \frac{-1}{2} \leftrightarrow x = \frac{-\pi}{6} + 2k\pi \text{ or } x = \frac{7\pi}{6} + 2k\pi \tag{5}$$

From Figure 1 and (5), we can find D of $g(x)$
This condition holds for some integer $k \in \mathbb{Z}$

$$\frac{-\pi}{6} + 2k\pi \leq x \leq \frac{7\pi}{6} + 2k\pi \tag{6}$$

Thus, the natural domain of $g(x)$ is:

$$D = [\frac{-\pi}{6} + 2k\pi, \frac{7\pi}{6} + 2k\pi], \text{ with k} \in \mathbb{Z}$$

(b) With the natural domain of $g(x)$, we can see that:

$$\frac{-1}{2} \leq \sin x \leq 1$$
$$0 \leq \frac{1}{2} + \sin x \leq \frac{3}{2}$$
$$\sqrt{0} \leq \sqrt{\frac{1}{2} + \sin x} \leq \sqrt{\frac{3}{2}}$$
$$0 \leq g(x) \leq \sqrt{\frac{3}{2}}$$

Thus, the range of g is $[0, \sqrt{\frac{3}{2}}]$

(c) Let $a = \sin x$, we can see that the function $h(x)$ will become:

$$h(x) = \sqrt{\frac{1}{2} + a}$$

For each unique value of a, the output of $h(x)$ will vary accordingly.
Thus, to make $h(x)$ bijective, $\sin x$ have to be bijective.

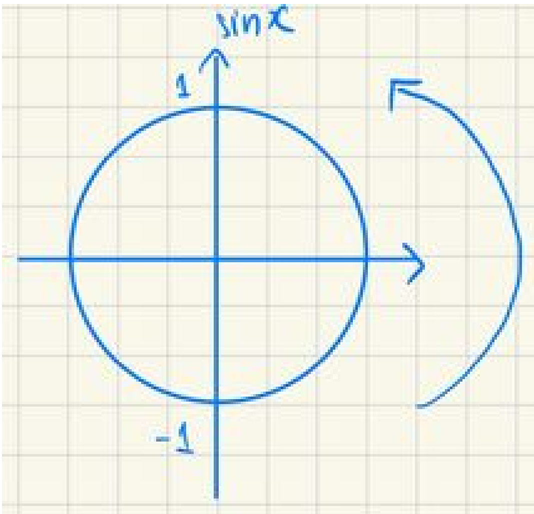

Figure 2: Ex: 2c

In the Figure 2, we can see that $\sin x$ is biject tive $\leftrightarrow x \in [\frac{-\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi]$, with $k \in \mathbb{Z}$
We can choose only one k to make $\sin x$ bijective. At the same time, we have:

$$h(0) = \sqrt{\frac{1}{2}}$$

Which means:

$$0 \in [\frac{-\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi]$$

Thus, we have to choose $k = 0$, so we have:

$$x \in [\frac{-\pi}{2}, \frac{\pi}{2}] \tag{7}$$

However, to make $\sqrt{\frac{1}{2} + \sin x}$ exists, so:

$$x \in [\frac{-\pi}{6}, \frac{7\pi}{6}] \tag{8}$$

From (7) and (8), the natural domain of $h(x)$ is:

$$x \in [\frac{-\pi}{6}, \frac{\pi}{2}] \tag{9}$$

Thus, α, β are:

$$[\alpha, \beta] = [\frac{-\pi}{6}, \frac{\pi}{2}] \tag{10}$$

From (10) and $h(x)$ is bijective, γ and δ are:

$$[\gamma, \delta] = [h(\frac{-\pi}{6}), h(\frac{\pi}{2})] = [0, \sqrt{\frac{3}{2}}] \tag{11}$$

From, (10) and (11), $\alpha = \frac{-\pi}{6}$, $\beta = \frac{\pi}{2}$, $\gamma = 0$ and $\delta = \sqrt{\frac{3}{2}}$

(d) We have $h : [\frac{-\pi}{6}, \frac{\pi}{2}] \to [0, \sqrt{\frac{3}{2}}]$, $h(x) = \sqrt{\frac{1}{2} + \sin x}$

$$\begin{aligned} y = h(x) &= \sqrt{\frac{1}{2} + \sin x} \\ \Leftrightarrow \sin x &= y^2 - \frac{1}{2} \\ \Leftrightarrow x &= \sin^{-1}(y^2 - \frac{1}{2}) \\ \Leftrightarrow h^{-1}(x) &= \sin^{-1}(x^2 - \frac{1}{2}) \end{aligned}$$

As we know, $h(x)$ is bijective. Thus, domain of $h(x)$ will be the Co-domain of $h^{-1}(x)$ and co-domain of $h(x)$ will be the domain of $h^{-1}(x)$. Since the domain and the Co-domain of $h^{-1}(x)$ is:

$$\text{Domain of } h^{-1}(x) \text{ is: } [0, \sqrt{\frac{3}{2}}]$$

$$\text{Co-Domain of } h^{-1}(x) \text{ is: } [\frac{-\pi}{6}, \frac{\pi}{2}]$$

3.

(a) We have sine function to the complex numbers:

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i} \text{ for } z \in \mathbb{C}$$

with $\sin z = -\alpha i$, we have:

$$\begin{aligned} \sin z = \frac{e^{iz} - e^{-iz}}{2i} &= -\alpha i \\ \Leftrightarrow e^{iz} - e^{-iz} &= -2\alpha i^2 \\ \Leftrightarrow e^{iz} - e^{-iz} &= 2\alpha \\ \Leftrightarrow e^{iz} &= 2\alpha + e^{-iz} \end{aligned}$$

Using Euler's formula ($e^{i\alpha} = \cos\alpha + i\sin\alpha$), we have:

$$\begin{aligned} e^{iz} = 2\alpha + e^{-iz} &= 2\alpha + \cos(-z) + i\sin(-z) \\ &= 2\alpha + \cos(z) - i\sin(z) \\ &= 2\alpha + \cos(z) - \alpha \text{ (Because } \sin z = -\alpha i) \\ &= \alpha + \cos(z) \end{aligned}$$

We have:

$$\begin{gathered} \sin^2 x + \cos^2 x = 1 \\ \Leftrightarrow \cos^2 x = 1 - \sin^2 x \\ \Leftrightarrow \cos^2 x = 1 + \alpha^2 \text{ (Because } \sin z = -\alpha i) \\ \Leftrightarrow \cos x = \pm\sqrt{1+\alpha^2} \end{gathered} \tag{12}$$

From (12), e^{iz} equals to:

$$e^{iz} = \alpha \pm \sqrt{1+\alpha^2} \quad \blacksquare$$

(b) Let $z = a + bi$ From a), we have:

$$e^{iz} = \alpha \pm \sqrt{1+\alpha^2} \text{ with } \alpha > 1 \text{ and } \sin z = -\alpha i$$

Then:

$$\begin{aligned} iz &= \ln(a \pm \sqrt{1+a^2}) \\ \Leftrightarrow i(a+bi) &= \ln(a \pm \sqrt{1+a^2}) \\ \Leftrightarrow ai - b &= \ln(a \pm \sqrt{1+a^2}) \end{aligned}$$

Thus,

$$\begin{cases} a & = 0 \\ -b & = \ln(a+\sqrt{1+a^2}) \text{ or } \ln(a-\sqrt{1+a^2}) \end{cases} \Leftrightarrow \begin{cases} a & = 0 \\ b & = -\ln(a+\sqrt{1+a^2}) \text{ or } -\ln(a-\sqrt{1+a^2}) \end{cases} \tag{13}$$

With $a > 1$ so $a+\sqrt{1+a^2} > 1$, Thus the first solution $z \in \mathbb{C}$ is:

$$z = a + bi = 0 + (-\ln(a+\sqrt{1+a^2}))i = 0 - \ln(a+\sqrt{1+a^2})i \tag{14}$$

We have:

$$\begin{aligned} a^2 &< a^2+1 \text{ Because } a > 1 \\ \Leftrightarrow \sqrt{a^2} &< \sqrt{a^2+1} \\ \Leftrightarrow a &< \sqrt{a^2+1} \\ \Leftrightarrow a - \sqrt{a^2+1} &< 0 \end{aligned}$$

However $ln(x)$ exists if and only if $x > 0$
Thus, it does not exist $b = -\ln(a-\sqrt{1+a^2})$
As a result, we can find only one solution $z \in \mathbb{C}$ such that $sin(z) = -\alpha i$, which is:

$$z = 0 - \ln(a+\sqrt{1+a^2})i \qquad \blacksquare$$

4.

(a) We have $A = (1,2,3)$, $B = (-1,4,1)$, $C = (3,2,-2)$, So $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are:

$$\overrightarrow{AB} = \begin{bmatrix} x_B - x_A \\ \\ y_B - y_A \\ \\ z_B - z_A \end{bmatrix} = \begin{bmatrix} -1-1 \\ \\ 4-2 \\ \\ 1-3 \end{bmatrix} = \begin{bmatrix} -2 \\ \\ 2 \\ \\ -2 \end{bmatrix}$$

$$\overrightarrow{AC} = \begin{bmatrix} x_C - x_A \\ \\ y_C - y_A \\ \\ z_C - z_A \end{bmatrix} = \begin{bmatrix} 3-1 \\ \\ 2-2 \\ \\ -2-3 \end{bmatrix} = \begin{bmatrix} 2 \\ \\ 0 \\ \\ -5 \end{bmatrix}$$

Let c_1 and c_2 are the coefficients of the linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
Thus, we have:

$$\mathbf{v} = c_1\overrightarrow{AB} + c_2\overrightarrow{AC}$$
$$\Leftrightarrow [k, k-2, 2k-5] = c_1[-2,2,-2] + c_2[2,0,-5]$$

Comparing components gives:

$$k = -2c_1 + 2c_2 \tag{15}$$
$$k - 2 = 2c_1 + 0c_2 \tag{16}$$
$$2k - 5 = -2c_1 - 5c_2 \tag{17}$$

From (16), we have:

$$k = 2c_1 + 2 \tag{18}$$

From (15) and (18), we have:

$$k = -2c_1 + 2c_2 = 2c_1 + 2$$
$$\Leftrightarrow 2c_2 = 4c_1 + 2$$
$$\Leftrightarrow c_2 = \frac{4c_1 + 2}{2}$$
$$\Leftrightarrow c_2 = 2c_1 + 1 \tag{19}$$

From (19) and (17), we have:

$$2k - 5 = -2c_1 - 5(2c_1 + 1)$$
$$\Leftrightarrow 2k - 5 = -12c_1 - 5$$

$$\Leftrightarrow 2k = -12c_1$$
$$\Leftrightarrow k = -6c_1 \tag{20}$$

From (18) and (20), we have

$$k = 2c_1 + 2 = -6c_1$$
$$\Leftrightarrow 8c_1 = -2$$
$$\Leftrightarrow c_1 = \frac{-2}{8} = \frac{-1}{4} \tag{21}$$

From (21) and (20), we have:

$$k = -6 \times \frac{-1}{4} = \frac{3}{2} \tag{22}$$

From (21) and (19), we have:

$$c_2 = 2 \times \frac{-1}{4} + 1 = \frac{1}{2} \tag{23}$$

From (21), (22) and (23),we have:

$$c_1 = \frac{-1}{4}$$
$$c_2 = \frac{1}{2}$$
$$k = \frac{3}{2}$$

Thus, $\mathbf{v}$ is:

$$\mathbf{v} = [k, k-2, 2k-5] = [\frac{3}{2}, \frac{-1}{2}, -2]$$

And write $\mathbf{v}$ as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is:

$$[k, k-2, 2k-5] = c_1[-2, 2, -2] + c_2[2, 0, -5]$$
$$\Leftrightarrow [\frac{3}{2}, \frac{-1}{2}, -2] = \frac{-1}{4}[-2, 2, -2] + \frac{1}{2}[2, 0, -5]$$

(b) In a), we have:

$$\overrightarrow{AB} = [-2, 2, -2]$$
$$\overrightarrow{AC} = [2, 0, -5]$$

$$\overrightarrow{BC} = \begin{bmatrix} x_C - x_B \\ y_C - y_B \\ z_C - z_B \end{bmatrix} = \begin{bmatrix} -3 - (-1) \\ 2 - 4 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

Then, length of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{CB}$ are:

$$\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + 2^2 + (-2)^2} = 2\sqrt{3}$$
$$\|\overrightarrow{AC}\| = \sqrt{(-2)^2 + 0^2 + (-5)^2} = \sqrt{29}$$
$$\|\overrightarrow{BC}\| = \sqrt{(4)^2 + (-2)^2 + (-3)^2} = \sqrt{29}$$

We can see that $\|\overrightarrow{AC}\| = \|\overrightarrow{BC}\|$

Thus, $\triangle$ABC isoceles at vertex C

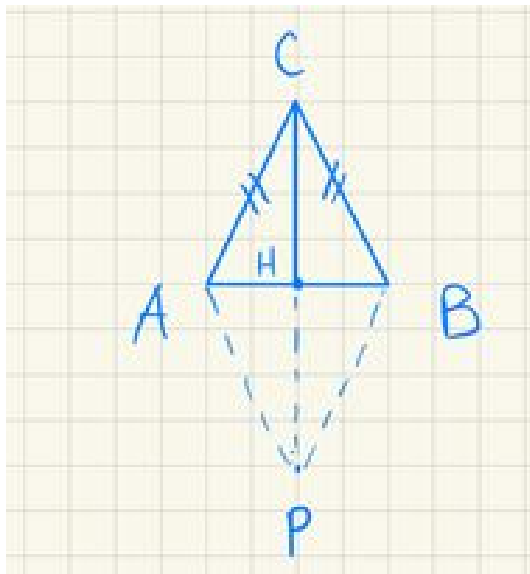

Figure 3: Ex: 4b

A rhombus is a quadrilateral with all sides of equal length. By definition, the diagonals of a rhombus intersect at their midpoint. Thus, I call H is a middle point of AB, then H also have to be the middle point of CP.

The coordinate of H is:

$$H = (\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}) = (\frac{1 + (-1)}{2}, \frac{2+4}{2}, \frac{3+1}{2}) = (0, 3, 2)$$

Let coordinate of $P = (x_P, y_P, z_P)$. Because H also the middle point of CP, so we have:

$$H = (0,3,2) = (\frac{x_C + x_P}{2}, \frac{y_C + y_P}{2}, \frac{z_C + z_P}{2}) = (\frac{x_C + x_P}{2}, \frac{y_C + y_P}{2}, \frac{z_C + z_P}{2}) = (\frac{3 + x_P}{2}, \frac{2 + y_P}{2}, \frac{-2 + z_P}{2})$$

Thus we have the equation.

$$0 = \frac{3 + x_P}{2} \tag{24}$$
$$\Leftrightarrow x_P = -3$$

$$3 = \frac{2 + y_P}{2} \tag{25}$$
$$\Leftrightarrow y_P = 4$$
$$2 = \frac{-2 + z_P}{2} \tag{26}$$
$$\Leftrightarrow z_P = 6$$

Thus, coordinate of P is $P = (-3, 4, 6)$