Challenge problems

Physics Discord
24-04-2018

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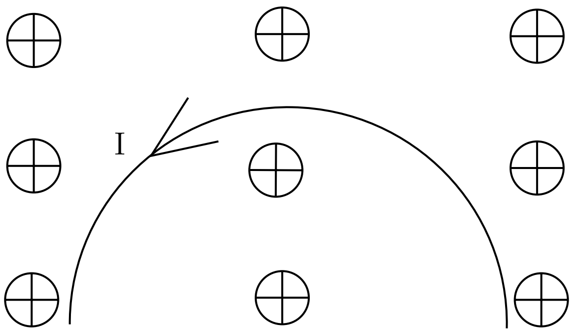
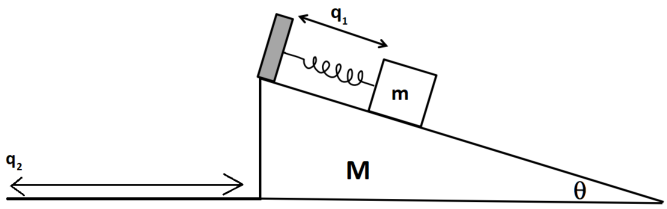
1) Prove that for a wire carrying current I in the shape of a semicircle with radius R in a magnetic field B experiences an overall force of $\mathbf{F} = 2BIR$



2) A block of mass m is attached to a spring (spring constant k) on larger inclined block of mass M. The inclined block rests without friction on a table and is thus free to move. Find the general solutions of the equations of motion for both blocks



3) If x^i, x'^i are coordinates of a general point in two coordinate systems of a manifold, establish the identity

$$x'^i_{jk} + x'^i_r x^r_{mn} x'^m_j x'^n_k = 0$$

where $x'^i_{jk} = \frac{\partial^2 x'^i}{\partial x^j \partial x^k}$, $x^r_{mn} = \frac{\partial^2 x^r}{\partial x'^m \partial x'^n}$, etc.

Solution

1)

$$d\vec{F_B} = Id\vec{L} \times \vec{B}$$
$$d\vec{F_B} = ||d\vec{F_B}||\cos(\theta)\mathbf{i} + ||d\vec{F_B}||\sin(\theta)\mathbf{j} = I||\vec{B}\vec{L}||$$

In polar coordinates,

$$||d\vec{F_B}|| = R\,d\theta,\ ||d\vec{B}|| = B$$
$$d\vec{F_B} = I[BR\cos(\theta)\mathbf{i} + BR\sin(\theta)\mathbf{j}]d\theta$$
$$\int d\vec{F_B} = IBR\int_0^{\pi}[\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}]d\theta$$
$$\vec{F_B} = IBR[\sin(\theta)\mathbf{i} - \cos(\theta)\mathbf{j}]_{\theta=0}^{\pi}$$
$$\vec{F_B} = 2IBR$$

2) Find the Lagrangian

$$T = \frac{1}{2}M\dot{q}_2^2 + \frac{1}{2}m((\dot{q}_1\cos\theta + \dot{q}_2)^2 + (\dot{q}^1\sin\theta)^2)$$
$$V = \frac{1}{2}kq_1^2 - mgq_1\sin\theta$$
$$\mathcal{L} = T - V$$

Finding the eqatuions of motion:

$$\frac{d\mathcal{L}}{dq_1} = -kq_1 + mg\sin\theta$$
$$\frac{d\mathcal{L}}{d\dot{q}_1} = m((\dot{q}_1\cos\theta + \dot{q}_2)\cos\theta + (\dot{q}_1\sin\theta)\sin\theta)$$
$$= m\dot{q}_1 + m\dot{q}_2\cos\theta$$
$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{q}_1}\right) = \frac{d\mathcal{L}}{dq_1}$$
$$m\ddot{q}_1 + m\ddot{q}_2\cos\theta = -kq_1 + mg\sin\theta$$
$$\to m\ddot{q}_1 + m\ddot{q}_2\cos\theta + kq_1 = mg\sin\theta$$

$$\frac{d\mathcal{L}}{dq_2} = 0, \frac{d\mathcal{L}}{d\dot{q}_2} = M\dot{q}_2 + m(\dot{q}_1\cos\theta + \dot{q}_2)$$
$$\frac{d}{dt}\frac{d\mathcal{L}}{d\dot{q}_2} \to M\ddot{q}_2 + m\ddot{q}_2 + m\ddot{q}_1\cos\theta = 0$$
$$(M+m)\ddot{q}_2 + m\ddot{q}_1\cos\theta = 0$$

Finding the general forms of the motions of the masses

$$m\ddot{q}_1 + m\ddot{q}_2\cos\theta + kq_1 = mg\sin\theta$$
$$(M+m)\dot{q}_2 + m\ddot{q}_1\cos\theta = 0$$
$$\ddot{q}_2 = \frac{-m\cos\theta}{M+m}\ddot{q}_1$$
$$m\ddot{q}_1 - \frac{m^2\cos^2\theta}{M+m}\ddot{q}_1 + kq_1 = mg\sin\theta$$
$$= \frac{m^2+mM}{m+M}\ddot{q}_1 - \frac{m^2\cos^2\theta}{M+m}\ddot{q}_1 + kq_1$$
$$= \frac{mM+m^2\sin^2\theta}{m+M}\ddot{q}_1 + kq_1 = mg\sin\theta$$

Let $M_0 = \frac{mM+m^2\sin^2\theta}{m+M}$

$$M_0\ddot{q}_1 + kq_2 = mg\sin\theta$$
$$\ddot{q}_1 + \frac{k}{M_0}q_1 = \frac{mg\sin\theta}{M_0}$$

Solution is $q_H + q_P$. $q_P = \frac{mg\sin\theta}{k}$ since RHS is constant. Letting $\frac{k}{M_0} = \omega^2$,

$$\ddot{q}_1 + \omega^2 q = 0$$
$$q_1 = C\cos(\omega t - \delta) + \frac{mg\sin\theta}{k}$$

Since $\ddot{q}_2 = \frac{-m\cos\theta}{M+m}q_1$,

$$q_2 = \frac{-m\cos\theta}{m+M}\left((\cos\omega t - \delta) + \frac{mg\sin\theta}{k}\right)$$

3) From $x^r_n x'^n_k = \delta^r_k$, take $\frac{\partial}{\partial x^j}$ on both sides:

$$\frac{\partial}{\partial x^j}(x^r_n x'^n_k) = 0$$
$$\frac{\partial(x^r_n)}{\partial x^j}x'^n_k + x^r_n x'^n_{jk} = 0$$
$$\frac{\partial(x^r_n)}{\partial x'^m}x'^m_k x'^n_k + x^r_n x'^n_{jk} = 0$$
$$x^r_{mn}x'^m_j x'^n_k + x^r_n x'^n_{jk} = 0$$
$$x'^i_r x^t_{mn} x'^m_j x'^n_k + \delta^i_n x'^n_{jk} = 0$$
$$x'^i_r x^r_{mn} x'^m_j x'^n_k + x'^i_{jk} = 0$$