NeRF Uncertainty

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1. Uncertainty

1.1. Gaussian Approximation

In [1], the weight α_i of i'th sample is given as:

$$\begin{aligned}
\alpha_i &= \exp\left(-\sum_{j=1}^{i-1}\delta_j\rho_j\right)\left(1-\exp\left(-\delta_i\rho_i\right)\right) \\
&= \exp\left(-\sum_{j=1}^{i-1}\delta_j\rho_j\right) - \exp\left(-\sum_{j=1}^{i}\delta_j\rho_j\right) \\
&= T_i - T_{i+1},
\end{aligned} \tag{1}$$

where ρ is the volume density and $\delta_k = t_{k+1} - t_k$ is the distance between adjacent samples. T denotes the accumulated transmittance along the ray $\boldsymbol{r}$. Using α_i, expected color $C(\boldsymbol{r})$ and expected depth $D(\boldsymbol{r})$ of the ray $\boldsymbol{r}$ can be derived as [1, 2, 3, 4, 5, 6]:

$$C(\boldsymbol{r}) = \sum_{i=1}^{N_s}\boldsymbol{c}_i\alpha_i, \tag{2}$$

$$D(\boldsymbol{r}) = \sum_{i=1}^{N_s} d_i\alpha_i = \sum_{i=1}^{N_s}\frac{t_{i+1}+t_i}{2}\alpha_i. \tag{3}$$

In Eqn. (1), assume that $\sum\delta_j\rho_j \ll 1$. That is, samples are close to each other, and their volume densities are restricted to less than 1, by using activation functions such as sigmoid. Then we can rewrite Eqn. (1) as the following:

$$\alpha_i \approx \left(1-\sum_{j=1}^{i-1}\delta_j\rho_j\right) - \left(1-\sum_{j=1}^{i}\delta_j\rho_j\right) = \delta_i\rho_i. \tag{4}$$

Suppose the k'th volume density ρ_k follows the Gaussian distribution $\rho_k \sim \mathcal{N}\left(\mu_k, \sigma_k^2\right)$ and the densities are i.i.d.. Using Eqn. (4), we approximately have $C(\boldsymbol{r}) \approx \sum_{i=1}^{N_s}\boldsymbol{c}_i\delta_i\rho_i$. We then have $C(\boldsymbol{r}) \sim \mathcal{N}\left(\sum_{i=1}^{N_s}\boldsymbol{c}_i\delta_i\mu_i, \sum_{i=1}^{N_s}\boldsymbol{c}_i^2\delta_i^2\sigma_i^2\right)$, and we can obtain the approximated solution of the volume density by the following MLE:

$$\{(\hat{\mu}_i,\hat{\sigma}_i)\} = \underset{\{(\hat{\mu}_i,\hat{\sigma}_i)\}}{\operatorname{argmin}}\ \ln\sum_{i=1}^{N_s}\boldsymbol{c}_i^2\delta_i^2\sigma_i^2 + \frac{\left(C(\boldsymbol{r}) - \sum_{i=1}^{N_s}\boldsymbol{c}_i\delta_i\mu_i\right)^2}{\sum_{i=1}^{N_s}\boldsymbol{c}_i^2\delta_i^2\sigma_i^2}. \tag{5}$$

Similarly, we have the following MLE for $D(\boldsymbol{r})$ as:

$$\{(\hat{\mu}_i,\hat{\sigma}_i)\} = \underset{\{(\hat{\mu}_i,\hat{\sigma}_i)\}}{\operatorname{argmin}}\ \ln\sum_{i=1}^{N_s}d_i^2\delta_i^2\sigma_i^2 + \frac{\left(D(\boldsymbol{r}) - \sum_{i=1}^{N_s}d_i\delta_i\mu_i\right)^2}{\sum_{i=1}^{N_s}d_i^2\delta_i^2\sigma_i^2}. \tag{6}$$

If $\boldsymbol{c}_k$ also follows the Gaussian distribution $\boldsymbol{c}_k \sim \mathcal{N}\left(\mu_{\boldsymbol{c}_k}, \sigma^2_{\boldsymbol{c}_k}\right)$, then Eqn. (2) is the summation of the product of two Gaussian random variables ρ and $\boldsymbol{c}$. The distribution of this product $\rho\boldsymbol{c}$ is infeasible, however when $\mu \gg \sigma$ holds for those two variables we can have the approximated distribution as the following [7]:

$$\rho_k \boldsymbol{c}_k \sim \mathcal{N}\left(\mu_k \mu_{\boldsymbol{c}_k}, \sigma_k^2 \mu^2_{\boldsymbol{c}_k} + \sigma^2_{\boldsymbol{c}_k} \mu_k^2 + \sigma_k^2 \sigma^2_{\boldsymbol{c}_k}\right).$$

We then have the approximated distribution as $C(\boldsymbol{r}) \sim \mathcal{N}\left(\sum_{i=1}^{N_s} \delta_i \mu_i \mu_{\boldsymbol{c}_i}, \sum_{i=1}^{N_s} \delta_i^2 \left(\sigma_i^2 \mu^2_{\boldsymbol{c}_i} + \sigma^2_{\boldsymbol{c}_i} \mu_i^2 + \sigma_i^2 \sigma^2_{\boldsymbol{c}_i}\right)\right)$. Using this approximated distribution, similar to Eqn. (5), we can have the approximated solution of the volume density and color by the following MLE:

$$\left\{\left(\hat{\mu}_i, \hat{\sigma}_i\right), \left(\hat{\mu}_{\boldsymbol{c}_i}, \hat{\sigma}_{\boldsymbol{c}_i}\right)\right\} = \underset{\left\{\left(\hat{\mu}_i, \hat{\sigma}_i\right), \left(\hat{\mu}_{\boldsymbol{c}_i}, \hat{\sigma}_{\boldsymbol{c}_i}\right)\right\}}{\operatorname{argmin}} \ln \sum_{i=1}^{N_s} \delta_i^2 \left(\sigma_i^2 \mu^2_{\boldsymbol{c}_i} + \sigma^2_{\boldsymbol{c}_i} \mu_i^2 + \sigma_i^2 \sigma^2_{\boldsymbol{c}_i}\right) + \frac{\left(C(\boldsymbol{r}) - \sum_{i=1}^{N_s} \delta_i \mu_i \mu_{\boldsymbol{c}_i}\right)^2}{\sum_{i=1}^{N_s} \delta_i^2 \left(\sigma_i^2 \mu^2_{\boldsymbol{c}_i} + \sigma^2_{\boldsymbol{c}_i} \mu_i^2 + \sigma_i^2 \sigma^2_{\boldsymbol{c}_i}\right)}. \tag{7}$$

1.2. Lognormal Approximation

The approximation in Eqn. (4) hardly holds when the ray $\boldsymbol{r}$ travels the long distance, or the volume density ρ takes large value, thereby the approximated solutions obtained from Eqn. (5) or Eqn. (7) become inaccurate. Since $T_i = \exp\left(-\sum_{j=1}^{i-1} \delta_j \rho_j\right)$, we have $T_i \sim Lognormal\left(\mu_{T_i}, \sigma^2_{T_i}\right)$ where $\mu_{T_i} = -\sum_{j=1}^{i-1} \delta_j \mu_j$ and $\sigma^2_{T_i} = \sum_{j=1}^{i-1} \delta_j^2 \sigma_j^2$. From (1), α is given as the difference of two lognormal random variables T_i and T_{i+1}. It is challenging to have the closed-form representation of the difference of two $lognormal$ random variables, even the two random variables are independent thus uncorrelated. However, [8] showed that the distribution of shifted variable $s_i + T_j - T_k$, where T_j and T_k are two $lognormal$ random variables and s_i is a constant to be estimated, can be approximated by a $lognormal$ distribution when $\sigma_{T_j} > \sigma_{T_k}$. Since $\sigma_{T_{i+1}} > \sigma_{T_i}$, we can approximately have the distribution of $s_i + T_{i+1} - T_i = s_i - \alpha_i$ as [8, 9]:

$$s_i - \alpha_i \sim Lognormal\left(\mu_{\alpha_i}, \sigma^2_{\alpha_i}\right) \tag{8}$$

where

$$s_i = \frac{\sigma^2_{T_{i+1}} + \sigma^2_{T_i} - 2\sigma_{T_{i+1}}\sigma_{T_i} Corr\left(T_{i+1}, T_i\right)}{\sigma^2_{T_{i+1}} - \sigma^2_{T_i}} \left[\exp\left(\mu_{T_{i+1}} + \frac{\sigma^2_{T_{i+1}}}{2}\right) + \exp\left(\mu_{T_i} + \frac{\sigma^2_{T_i}}{2}\right)\right], \tag{9}$$

$$\sigma_{\alpha_i} = \frac{\sigma^2_{T_{i+1}} - \sigma^2_{T_i}}{2\sqrt{\sigma^2_{T_{i+1}} + \sigma^2_{T_i} - 2\sigma_{T_{i+1}}\sigma_{T_i} Corr\left(T_{i+1}, T_i\right)}}, \tag{10}$$

$$\mu_{\alpha_i} = \ln\left[\exp\left(\mu_{T_{i+1}} + \frac{\sigma^2_{T_{i+1}}}{2}\right) - \exp\left(\mu_{T_i} + \frac{\sigma^2_{T_i}}{2}\right) + s_i\right] - \frac{\sigma^2_{\alpha_i}}{2}. \tag{11}$$

Using the definition of μ_{T_i} and $\sigma^2_{T_i}$ and plugging Eqn. (9) into Eqn. (10) and Eqn. (11), we have:

$$s_i = \frac{\sigma^2_{T_{i+1}} + \sigma^2_{T_i} - 2\sigma_{T_{i+1}}\sigma_{T_i} Corr\left(T_{i+1}, T_i\right)}{\delta_i^2 \sigma_i^2} \left[\exp\left(\cdot\right) + \exp\left(\cdot\right)\right], \tag{12}$$

$$\sigma_{\alpha_i} = \frac{\delta_i^2 \sigma_i^2}{2\sqrt{\sigma^2_{T_{i+1}} + \sigma^2_{T_i} - 2\sigma_{T_{i+1}}\sigma_{T_i} Corr\left(T_{i+1}, T_i\right)}}, \tag{13}$$

$$\mu_{\alpha_i} = \mu_{T_i} + \frac{\sigma^2_{T_i}}{2} - \ln\frac{\delta_i^2 \sigma_i^2}{2} + \ln\left[\exp\left(-\delta_i \mu_i + \frac{\delta_i^2 \sigma_i^2}{2}\right) \cdot \sigma^2_{T_{i+1}} + \sigma^2_{T_i}\right] - \frac{\sigma^2_{\alpha_i}}{2}. \tag{14}$$

Assume that $\delta_j = const$, the above equations can be further simplified to:

$$s_i \simeq \frac{\sum_{j=1}^{i} \sigma_j^2 + \sum_{j=1}^{i-1} \sigma_j^2}{\sigma_i^2} \left[\exp\left(\cdot\right) + \exp\left(\cdot\right)\right] \tag{15}$$

$$\sigma_{\alpha_i} \simeq \frac{\delta_i \sigma_i^2}{2\sqrt{\sum_{j=1}^{i} \sigma_j^2 + \sum_{j=1}^{i-1} \sigma_j^2}} \tag{16}$$

$$\mu_{\alpha_i} \simeq \mu_{T_i} + \frac{\sigma_{T_i}^2}{2} - \ln \frac{\sigma_i^2}{2} + \ln \left[\exp\left(-\delta_i \mu_i + \frac{\delta_i^2 \sigma_i^2}{2}\right) \cdot \sum_{j=1}^{i} \sigma_j^2 + \sum_{j=1}^{i-1} \sigma_j^2\right] - \frac{\sigma_{\alpha_i}^2}{2} \tag{17}$$

Using this $lognormal$ random variable $s - \alpha$, we can redefine $C\left(\boldsymbol{r}\right)$ in Eqn. (2). Let $\lambda = \sum_{i=1}^{N_s} \boldsymbol{c}_i s_i$ and $A\left(\boldsymbol{r}\right) = \sum_{i=1}^{N_s} \boldsymbol{c}_i \left(s_i - \alpha_i\right)$. Then we can rewrite Eqn. (2) as:

$$\lambda - C\left(\boldsymbol{r}\right) = A\left(\boldsymbol{r}\right). \tag{18}$$

Here, $A\left(\boldsymbol{r}\right)$ is the linear combination of $lognormal$ random variables. The distribution of this linear combination has no closed-form expression, however can be reasonably approximated by another $lognormal$ distribution at the right tail [10]. Thus we have $A\left(\boldsymbol{r}\right) \sim Lognormal\left(\mu_A, \sigma_A^2\right)$, where

$$\begin{aligned}
\sigma_A^2 &= \ln \left[\frac{\sum_{i=1}^{N_s} \boldsymbol{c}_i^2 \exp\left(2\mu_{\alpha_i} + \sigma_{\alpha_i}^2\right)\left(\exp\left(\sigma_{\alpha_i}^2\right) - 1\right)}{\left(\sum_{i=1}^{N_s} \boldsymbol{c}_i \exp\left(\mu_{\alpha_i} + \frac{\sigma_{\alpha_i}^2}{2}\right)\right)^2} + 1\right], \\
\mu_A &= \ln \left[\sum_{i=1}^{N_s} \boldsymbol{c}_i \exp\left(\mu_{\alpha_i} + \frac{\sigma_{\alpha_i}^2}{2}\right)\right] - \frac{\sigma_A^2}{2}.
\end{aligned} \tag{19}$$

Therefore, we can define the probability density function $f\left(A\left(\boldsymbol{r}\right)\right)$ as the following:

$$f\left(A\left(\boldsymbol{r}\right)\right) = \frac{1}{A\left(\boldsymbol{r}\right)\sigma_A\sqrt{2\pi}} \exp\left(-\frac{\left(\ln A\left(\boldsymbol{r}\right) - \mu_A\right)^2}{2\sigma_A^2}\right). \tag{20}$$

The negative loglikelihood $\mathcal{L}$ of $A\left(\boldsymbol{r}\right)$ then can be given as:

$$\mathcal{L}\left(A\left(\boldsymbol{r}\right); \mu_A, \sigma_A\right) = const + \ln A\left(\boldsymbol{r}\right) + \ln \sigma_A + \frac{\left(\ln A\left(\boldsymbol{r}\right) - \mu_A\right)^2}{2\sigma_A^2} \tag{21}$$

From Eqn. (18) and Eqn. (21), we can derive the loglikelihood of $C\left(\boldsymbol{r}\right)$ and find the optimal values of μ_A, σ_A as the following:

$$\begin{aligned}
\hat{\mu}_A, \hat{\sigma}_A &= \underset{\mu_A, \sigma_A}{\operatorname{argmin}} \ \mathcal{L}\left(C\left(\boldsymbol{r}\right); \mu_A, \sigma_A\right) \\
&= \underset{\mu_A, \sigma_A}{\operatorname{argmin}} \ \ln\left(\lambda - C\left(\boldsymbol{r}\right)\right) + \ln \sigma_A + \frac{\left(\ln\left(\lambda - C\left(\boldsymbol{r}\right)\right) - \mu_A\right)^2}{2\sigma_A^2}.
\end{aligned} \tag{22}$$

Since μ_A, σ_A are determined by $\{(\mu_k, \sigma_k)\}$, by Eqn. (22) we can perform MLE for estimating the volume density $\rho_k \sim \mathcal{N}\left(\mu_k, \sigma_k\right)$.

Note that for $D\left(\boldsymbol{r}\right)$, by replacing $\boldsymbol{c}_i$ to d_i of λ and $A\left(\boldsymbol{r}\right)$, we can have the similar formulations for the depth estimation.

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