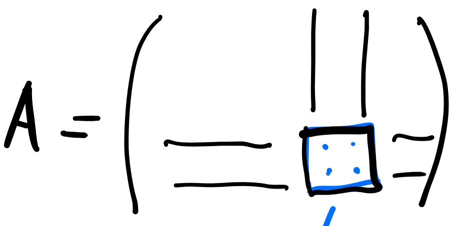
TEO DEGLI ORLATI



$m \times n$

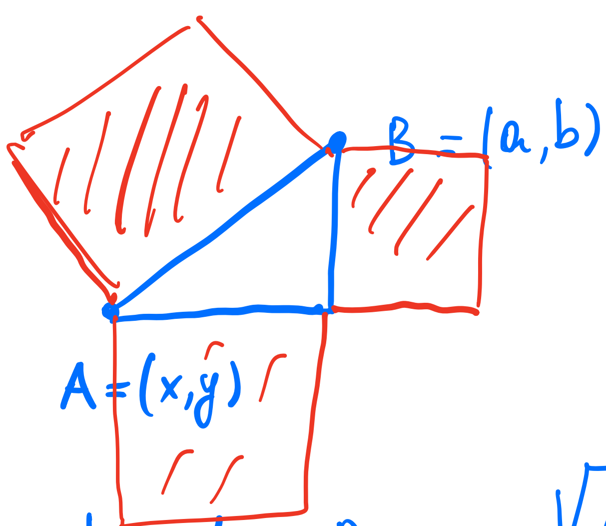
A' $k \times k$

$\det \neq 0$

se i determinanti di tutte le matrici $(k+1)\times(k+1)$ che contengono A' sono 0

allora $\operatorname{rg} A = k$

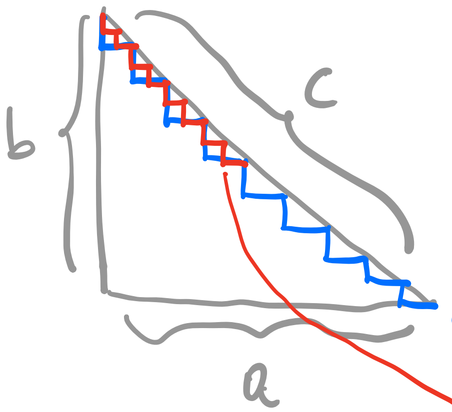
TEOREMA DI PITAGORA:



distanza tra A e B $= \sqrt{(x-a)^2+(y-b)^2}$

$[c^2 = a^2 + b^2]$

Il TEOREMA di PITAGORA!



ha lunghezza $a+b$

ha lunghezza $a+b$

⋮

passando al limite

$$\cancel{c = a + b}$$

<u>FALSO</u>

PRODOTTO SCALARE

Def: V sp. vettoriale

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

funzione che dati $\underline{v}, \underline{w} \in V$ restituisce un numero reale che indichiamo con $\langle \underline{v}, \underline{w} \rangle$

si dice un prodotto scalare se:

1) $\langle \cdot , \cdot \rangle$ è 2-multilineare (bilineare)

ovvero • se $\underline{u}, \underline{v}, \underline{w} \in V$ allora

$$\langle \underline{u} + \underline{v}, \underline{w} \rangle = \langle \underline{u}, \underline{w} \rangle + \langle \underline{v}, \underline{w} \rangle$$

$$\langle \underline{w}, \underline{u} + \underline{v} \rangle = \langle \underline{w}, \underline{u} \rangle + \langle \underline{w}, \underline{v} \rangle$$

• se $\lambda \in \mathbb{R}$, $\underline{u}, \underline{v} \in V$ allora

$$\langle \lambda \underline{u}, \underline{v} \rangle = \lambda \langle \underline{u}, \underline{v} \rangle$$

$$\langle \underline{u}, \lambda \underline{v} \rangle = \lambda \langle \underline{u}, \underline{v} \rangle$$

$$\left[\langle \lambda \underline{u}, \mu \underline{w} \rangle = \lambda \langle \underline{u}, \mu \underline{w} \rangle = \lambda \cdot \mu \langle \underline{u}, \underline{w} \rangle \right]$$

2) $\langle \cdot, \cdot \rangle$ è simmetrico, cioè

se $\underline{v}, \underline{w} \in V$ allora

$$\langle \underline{v}, \underline{w} \rangle = \langle \underline{w}, \underline{v} \rangle$$

3°) $\langle \cdot, \cdot \rangle$ è definito positivo

ovvero per ogni $\underline{v} \in V$ vale

$$\begin{cases} \langle \underline{v}, \underline{v} \rangle \geq 0 \\ \text{e } \langle \underline{v}, \underline{v} \rangle = 0 \text{ se e solo se } \underline{v} = \underline{0} \end{cases}$$

$\left(\Downarrow \text{ se } \underline{v} \neq \underline{0} \quad \langle \underline{v}, \underline{v} \rangle > 0 \right)$

ESEMPIO

in $\mathbb{R}^n$ definiamo il prodotto scalare standard

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \underline{y} \in \mathbb{R}^n$$

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle := x_1 \cdot y_1 + x_2 \cdot y_2 + \dots + x_n \cdot y_n =$$

prodotto righe per colonne

$$= (x_1, \dots, x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}^t \cdot I_n \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

- $\langle \cdot, \cdot \rangle$ è bilineare (perché è il prodotto righe per colonne di "matrici" $(1\times n)$ per "matrice" $(n\times 1)$

$$(\underline{x}+\underline{\tilde{x}})^t\cdot\underline{y} = (\underline{x}^t+\underline{\tilde{x}}^t)\cdot\underline{y} = \underline{x}^t\cdot\underline{y}+\underline{\tilde{x}}^t\cdot\underline{y}$$

$$\langle \underline{x}+\underline{\tilde{x}}, \underline{y}\rangle \qquad\qquad \langle \underline{x},\underline{y}\rangle+\langle \underline{\tilde{x}},\underline{y}\rangle$$

- $\langle\cdot,\cdot\rangle$ è simmetrico:

$$\left\langle \begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}, \begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix}\right\rangle = x_1\cdot y_1+\dots+x_n\cdot y_n =$$

$$= y_1\cdot x_1+\dots+y_n\cdot x_n = \left\langle \begin{pmatrix} y_1\\ \vdots\\ y_n\end{pmatrix}, \begin{pmatrix} x_1\\ \vdots\\ x_n\end{pmatrix}\right\rangle$$

Altra dim:

$$\langle \underline{x},\underline{y}\rangle = \overbrace{\underline{x}^t\cdot I_n\cdot\underline{y}}^{\in\mathbb{R}} = (\underline{x}^t\cdot I_n\cdot\underline{y})^t =$$

$$= \underline{y}^t\cdot I_n^t\cdot(\underline{x}^t)^t = \underline{y}^t\cdot I_n\cdot\underline{x} = \langle \underline{y},\underline{x}\rangle$$

• $\langle \cdot , \cdot \rangle$ è definito positivo:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\langle \underline{x}, \underline{x} \rangle = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\rangle = x_1 \cdot x_1 + \dots + x_n \cdot x_n$$

$$= x_1^2 + \dots + x_n^2 \geq 0$$

(somma di quadrati di numeri reali)

è $= 0 \iff x_1 = \dots = x_n = 0$.

ESERCIZIO:

$V = Pol_{\leq d}[x]$ $\qquad d \geq 1$

$p(x), q(x) \in Pol_{\leq d}[x]$

$$\langle p(x), q(x) \rangle := \int_0^1 p(x) \cdot q(x) \, dx$$

provare che è un prodotto scalare.

Def: Uno spazio vettoriale V munito di un prodotto scalare $\langle \cdot , \cdot \rangle$ si dice uno spazio metrico $(V, \langle \cdot , \cdot \rangle)$

OSS: $\langle \underline{0}, \underline{v} \rangle = 0$ per ogni $\underline{v} \in V$

perchè $\underline{0} = 0 \cdot \underline{0}$ e quindi

$$\langle \underline{0}, \underline{v} \rangle = \langle 0 \cdot \underline{0}, \underline{v} \rangle = 0 \cdot \langle \underline{0}, \underline{v} \rangle = 0$$

Def: $(V, \langle \cdot , \cdot \rangle)$ spazio metrico

La norma (o lunghezza) di $\underline{v} \in V$ è (o modulo)

$$\| \underline{v} \| := \sqrt{\langle \underline{v}, \underline{v} \rangle} \geq 0$$

OSS: • $\| \underline{v} \| = 0 \iff \underline{v} = \underline{0}$

• $\| -\underline{v} \| = \| \underline{v} \|$

$$\| -\underline{v} \| = \sqrt{\langle -\underline{v}, -\underline{v} \rangle} = \sqrt{\langle \underline{v}, \underline{v} \rangle} = \| \underline{v} \|.$$

Prop disuguaglianza di Cauchy_Schwarz

$(V, \langle \cdot, \cdot \rangle)$ sp. metrico. Allora

per ogni $\underline{v}, \underline{w} \in V$ vale

$$\boxed{|\langle \underline{v}, \underline{w} \rangle| \leq \|\underline{v}\| \cdot \|\underline{w}\|}$$

inoltre è $=$ se e solo se $\underline{v}$ e $\underline{w}$ sono linearmente dipendenti.

DIM: se $\underline{v} = \underline{0}$ (o $\underline{w} = \underline{0}$) è ok.

Supponiamo $\underline{v} \neq \underline{0} \neq \underline{w}$

Siano $a, b \in \mathbb{R}$

$$0 \leq \|a\underline{v} + b\underline{w}\|^2 = \langle a\underline{v} + b\underline{w}, a\underline{v} + b\underline{w} \rangle$$

$$= a \langle a\underline{v} + b\underline{w}, \underline{v} \rangle + b \langle a\underline{v} + b\underline{w}, \underline{w} \rangle$$

$$= a^2 \langle \underline{v}, \underline{v} \rangle + ab \underbrace{\langle \underline{w}, \underline{v} \rangle}_{= \langle \underline{v}, \underline{w} \rangle} + ba \langle \underline{v}, \underline{w} \rangle$$

$$+ b^2 \langle \underline{w}, \underline{w} \rangle$$

$$= a^2 \|\underline{v}\|^2 + 2ab \langle \underline{v}, \underline{w} \rangle + b^2 \|\underline{w}\|^2$$

dunque per ogni $a, b \in \mathbb{R}$

$$a^2 \|\underline{v}\|^2 + 2ab \langle \underline{v}, \underline{w} \rangle + b^2 \|\underline{w}\|^2 \geq 0$$

prendiamo

$$a = \|\underline{w}\|^2$$

$$b = - \langle \underline{v}, \underline{w} \rangle$$

$$0 \leq$$

$$\|\underline{w}\|^4 \|\underline{v}\|^2 + 2 \|\underline{w}\|^2 \left(- \langle \underline{v}, \underline{w} \rangle \right) \langle \underline{v}, \underline{w} \rangle +$$

$$+ \left| \langle \underline{v}, \underline{w} \rangle \right|^2 \|\underline{w}\|^2 =$$

$$= \|\underline{w}\|^2 \left(\|\underline{v}\|^2 \|\underline{w}\|^2 - \cancel{2} \left| \langle \underline{v}, \underline{w} \rangle \right|^2 + \cancel{\left| \langle \underline{v}, \underline{w} \rangle \right|^2} \right)$$

$$= \|\underline{w}\|^2 \left(\|\underline{v}\|^2 \cdot \|\underline{w}\|^2 - \left| \langle \underline{v}, \underline{w} \rangle \right|^2 \right)$$

poichè $\|\underline{w}\| > 0$ (perchè $\underline{w} \neq \underline{0}$)

$$\|\underline{v}\|^2 \cdot \|\underline{w}\|^2 - |\langle \underline{v}, \underline{w} \rangle|^2 \geq 0$$ ovvero

$$|\langle \underline{v}, \underline{w} \rangle| \leq \|\underline{v}\| \cdot \|\underline{w}\|$$

inoltre "$=$" $\Longleftrightarrow$ $\underline{v}, \underline{w}$ lin. dip. $\boxtimes$

Prop DISUGUAGLIANZA TRIANGOLARE

$\underline{v}, \underline{w} \in V$ allora

$$\boxed{\big|\|\underline{v}\| - \|\underline{w}\|\big| \leq \|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|}$$

"$=$" $\Longleftrightarrow$ $\underline{v}, \underline{w}$ sono lin. dip.

Dim:

$$(\|\underline{v}\| - \|\underline{w}\|)^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2 - 2\|\underline{v}\| \cdot \|\underline{w}\|$$

Cauchy-Schwarz: $\|\underline{v}\| \cdot \|\underline{w}\| \geq |\langle \underline{v}, \underline{w} \rangle|$

$$\Longrightarrow \quad -\|\underline{v}\| \cdot \|\underline{w}\| \leq -|\langle \underline{v}, \underline{w} \rangle|$$

$$\leq \|\underline{v}\|^2 + \|\underline{w}\|^2 + 2\langle \underline{v}, \underline{w}\rangle = \|\underline{v}+\underline{w}\|^2$$

$$\|\underline{v}+\underline{w}\|^2 = \langle \underline{v}+\underline{w}, \underline{v}+\underline{w}\rangle =$$

$$= \|\underline{v}\|^2 + \langle \underline{v}, \underline{w}\rangle + \langle \underline{w}, \underline{v}\rangle + \|\underline{w}\|^2$$

$$= \|\underline{v}\|^2 + 2\langle \underline{v}, \underline{w}\rangle + \|\underline{w}\|^2$$

$$\leq \|\underline{v}\|^2 + 2\|\underline{v}\| \cdot \|\underline{w}\| + \|\underline{w}\|^2$$

Cauchy-Schwarz

$$= \left(\|\underline{v}\| + \|\underline{w}\|\right)^2$$

$$\leq \left(\|\underline{v}\| + \|\underline{w}\|\right)^2$$

$\blacksquare$