$L : V \longrightarrow W$ lineare $\qquad V, W$ spazi vettoriali

$\mathcal{B}_V = \{\underline{v}_1, \dots, \underline{v}_n\}$ base di $V$

$\tilde{\mathcal{B}}_V = \{\tilde{\underline{v}}_1, \dots, \tilde{\underline{v}}_n\}$ base di $V$

$\mathcal{B}_W = \{\underline{w}_1, \dots, \underline{w}_m\}$ base di $W$

$\tilde{\mathcal{B}}_W = \{\tilde{\underline{w}}_1, \dots, \tilde{\underline{w}}_m\}$ base di $W$

$A \in Mat(m \times n)$ matrice associata a $L$

nelle basi $\mathcal{B}_V, \mathcal{B}_W$

$\tilde{A} \in Mat(m \times n)$ matrice associata a $L$

nelle basi $\tilde{\mathcal{B}}_V, \tilde{\mathcal{B}}_W$

$$\begin{tikzcd}
\underline{v} \arrow[r, mapsto] & \underline{v} \arrow[r] & L(\underline{v}) \arrow[r] & L(\underline{v}) \\
V \arrow[r, "id"] & V \arrow[r, "L"] & W \arrow[r, "id"] & W \\
\tilde{\mathcal{B}}_V & \mathcal{B}_V & \mathcal{B}_W & \tilde{\mathcal{B}}_W
\end{tikzcd}$$

$$\underline{x} \longrightarrow C\cdot \underline{x} \longmapsto A\cdot(C\underline{x}) \longrightarrow B(A(C\underline{x}))$$

$\underline{x} \in \mathbb{R}^n$

coordinate di $\underline{v}$ in $\widetilde{\mathcal{B}}_V$

$C$ è la matrice di cambiamento di base da $\widetilde{\mathcal{B}}_V$ a $\mathcal{B}_V$

$B$ matrice di cambiam. di base da $\mathcal{B}_W$ a $\widetilde{\mathcal{B}}_W$

OSS:

$C \in Mat(n\times n)$ invertibile

$B \in Mat(m\times m)$ invertibile

$$B(A(C\underline{x})) = \underbrace{(BAC)}\underline{x}$$

è la matrice associata a $L$ nelle basi $\widetilde{\mathcal{B}}_V$, $\widetilde{\mathcal{B}}_W$

ovvero

$$\boxed{\widetilde{A} = B\cdot A\cdot C}$$

# DETERMINANTE

**Def**: il determinante è una applicazione n-multilineare, alternante

$$\det : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n} \longrightarrow \mathbb{R}$$

tale che $\det(\underline{e}_1, \underline{e}_2, \dots, \underline{e}_n) = 1$.

n-multilineare significa che, fissati $n-1$ vettori è lineare nel vettore non fissato, ovvero fissati $\underline{v}_1, \dots, \underline{v}_{n-1} \in \mathbb{R}^n$ le funzioni:

$$\mathbb{R}^n \ni \underline{x} \longmapsto \det(\underline{x}, \underline{v}_1, \dots, \underline{v}_{n-1})$$

$$\mathbb{R}^n \ni \underline{x} \longmapsto \det(\underline{v}_1, \underline{x}, \underline{v}_2, \dots, \underline{v}_{n-1})$$

$$\vdots$$

$$\mathbb{R}^n \ni \underline{x} \longmapsto \det(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{x})$$

sono lineari da $\mathbb{R}^n$ a $\mathbb{R}$.

ovvero: $\lambda, \mu \in \mathbb{R}$ $\quad \underline{x}, \underline{y} \in \mathbb{R}^n$

$$\det(\lambda \underline{x} + \mu \underline{y}, \underline{v}_1, \dots, \underline{v}_{n-1}) =$$

$$= \lambda \det(\underline{x}, \underline{v}_1, \dots, \underline{v}_{n-1}) + \mu \det(\underline{y}, \underline{v}_1, \dots, \underline{v}_{n-1})$$

(e lo stesso per tutte le altre

$$\det(\underline{v}_1, \dots, \underline{v}_{n-1}, \lambda \underline{x} + \mu \underline{y}) =$$

$$= \lambda \det(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{x}) + \mu \det(\underline{v}_1, \dots, \underline{v}_{n-1}, \underline{y})$$

**ATTENZIONE**: det <u>non</u> lineare

$$\det(\lambda \underline{e}_1, \lambda \underline{e}_2) \neq \lambda \det(\underline{e}_1, \underline{e}_2)$$

infatti

$$\det(\lambda \underline{e}_1, \underbrace{\lambda \underline{e}_2}_{\text{fissiamo}}) \overset{\text{linearità nella prima entrata}}{=} \lambda \det(\underline{e}_1, \underbrace{\lambda \underline{e}_2}_{\text{fissiamo}})$$

$$\overset{\text{linearità nella seconda}}{=} \lambda \cdot (\lambda \det(\underline{e}_1, \underline{e}_2)) = \lambda^2 \det(\underline{e}_1, \underline{e}_2)$$

$$= \lambda^2 .$$

**alternante:**

$$\det(\underline{v}_1, \dots, \underline{v}_i, \underline{v}_{i+1}, \dots, \underline{v}_n) =$$

$$= -\det(\underline{v}_1, \dots, \underline{v}_{i+1}, \underline{v}_i, \dots, \underline{v}_n)$$

ovvero scambiando 2 vettori il determinante cambia segno

OSS:

$$\boxed{\det(\underline{v}_1, \dots, \underset{i}{\underline{v}}, \dots, \underset{j}{\underline{v}}, \dots, \underline{v}_n) = 0}$$

perchè per l'alternanza scambiando gli elementi di posto $i$ e $j$ si ottiene

$$\det(\underline{v}_1, \dots, \underline{v}, \dots, \underline{v}, \dots, \underline{v}_n) =$$

$$= -\det(\underline{v}_1, \dots, \underline{v}, \dots, \underline{v}, \dots, \underline{v}_n)$$

$\Longrightarrow$ è zero.

Es:

$$\det(\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_1) =$$

$$= -\det(\underline{e}_1, \underline{e}_2, \underline{e}_1, \underline{e}_3) =$$

$$= \det(\underline{e}_1, \underline{e}_1, \underline{e}_2, \underline{e}_3) =$$

$$= -\det(\underline{e}_1, \underline{e}_1, \underline{e}_2, \underline{e}_3) = 0$$

$$\Rightarrow \underbrace{\det(\underline{e}_1, \underline{e}_1, \underline{e}_2, \underline{e}_3)}_{\substack{\cap \\ \mathbb{R}}} = - \underbrace{\det(\underline{e}_1, \underline{e}_1, \underline{e}_2, \underline{e}_3)}_{\substack{\cap \\ \mathbb{R}}}$$

$$\Rightarrow \det(\underline{e}_1, \underline{e}_1, \underline{e}_2, \underline{e}_3) = 0$$

OSS: se $\underline{v} = \lambda_1 \underline{v}_1 + \dots + \lambda_{n-1} \underline{v}_{n-1}$

allora

$$\det(\underline{v}_1, \dots, \overset{\overset{i}{\downarrow}}{\underline{v}}, \dots, \underline{v}_{n-1}) = 0$$

perchè

$$\det(\underline{v}_1, \dots, \underbrace{\lambda_1 \underline{v}_1 + \dots + \lambda_{n-1} \underline{v}_{n-1}}_{i}, \dots, \underline{v}_{n-1})$$

$$\overset{\text{linearità nell'entrata } i}{=} \lambda_1 \underbrace{\det(\underline{v}_1, \dots, \underline{v}_1, \dots, \underline{v}_{n-1})}_{=0} + \dots$$

$$\dots + \lambda_{n-1} \underbrace{\det(\underline{v}_1, \dots, \underline{v}_{n-1}, \dots, \underline{v}_{n-1})}_{=0}$$

$$= 0$$

Ovvero :

se $\underline{v}_1, \dots, \underline{v}_n \in \mathbb{R}^n$ sono linearmente dipendenti allora $\det(\underline{v}_1, \dots, \underline{v}_n) = 0$

Esempio :

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{v}_3 = \underline{v}_1 + \underline{v}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$$

$$\det\left( \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right) =$$

$$= \det\left( \underbrace{\underline{e}_1 - \underline{e}_2}_{\text{linearità}}, \underline{e}_1, 2\underline{e}_1 - \underline{e}_2 \right) =$$

$$= \underbrace{\det\left( \underline{e}_1, \underline{e}_1, 2\underline{e}_1 - \underline{e}_2 \right)}_{=0} - \det\left( \underline{e}_2, \underline{e}_1, \underbrace{2\underline{e}_1 - \underline{e}_2}_{\text{linearità}} \right)$$

$$= -\left[2\underbrace{\det(\underline{e}_2, \underline{e}_1, \underline{e}_1)}_{=0} - \underbrace{\det(\underline{e}_2, \underline{e}_1, \underline{e}_2)}_{=0}\right] = 0$$

<u>Esempio:</u>

$$\det\left(\overset{\underline{e}_1 + 2\underline{e}_2}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \underset{-\underline{e}_1 + \underline{e}_2}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}\right) =$$

$$= \det(\underbrace{\underline{e}_1 + 2\underline{e}_2}_{\text{linearità}}, -\underline{e}_1 + \underline{e}_2) =$$

$$= \det(\underline{e}_1, \underbrace{-\underline{e}_1 + \underline{e}_2}_{\text{linearità}}) + 2\det(\underline{e}_2, \underbrace{-\underline{e}_1 + \underline{e}_2}_{\text{linearità}})$$

$$-\overbrace{\det(\underline{e}_1, \underline{e}_1)}^{=0} + \overbrace{\det(\underline{e}_1, \underline{e}_2)}^{=1} +$$

$$2\left[-\det(\underline{e}_2, \underline{e}_1) + \underbrace{\det(\underline{e}_2, \underline{e}_2)}_{=0}\right]$$

$$= 1 - 2 \underbrace{\det(\underline{e}_2, \underline{e}_1)}_{\substack{\text{alternanza} \\ = -\det(\underline{e}_1,\underline{e}_2) \\ = -1}} = 1 + 2 = 3$$

**DEF** determinante di matrici QUADRATE

Sia $A \in Mat(n \times n)$ n righe, n colonne

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$\underline{r}_1 := (a_{11} \ \cdots \ a_{1n})$ $1^a$ riga di A

$\vdots$

$\underline{r}_n := (a_{n1} \ \cdots \ a_{nn})$ $n^a$ riga di A

$$\boxed{\det A := \det(\underline{r}_1^t, \ldots, \underline{r}_n^t)}$$

ESEMPI:

$$\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \det\left(\underbrace{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}}_{=\,\alpha \underline{e}_1+\beta \underline{e}_2}, \underbrace{\begin{pmatrix}\gamma\\ \delta\end{pmatrix}}_{=\,\gamma \underline{e}_1+\delta \underline{e}_2}\right) =$$

$$= \det\left(\underbrace{\alpha \underline{e}_1 + \beta \underline{e}_2}_{\text{linearità}}, \gamma \underline{e}_1 + \delta \underline{e}_2\right) =$$

$$= \alpha \det\left(\underline{e}_1, \underbrace{\gamma \underline{e}_1 + \delta \underline{e}_2}_{\text{linearità}}\right) + \beta \det\left(\underline{e}_2, \underbrace{\gamma \underline{e}_1 + \delta \underline{e}_2}_{\text{linearità}}\right)$$

$$= \alpha\left[\gamma \underbrace{\det(\underline{e}_1, \underline{e}_1)}_{=\,0} + \delta \overbrace{\det(\underline{e}_1, \underline{e}_2)}^{=\,1}\right] +$$

$$\beta\left[\gamma \underbrace{\det(\underline{e}_2, \underline{e}_1)}_{=\,-\det(\underline{e}_1, \underline{e}_2) = -1} + \delta \underbrace{\det(\underline{e}_2, \underline{e}_2)}_{=\,0}\right] =$$

$$= \alpha\delta - \beta\gamma$$

ovvero

$$\boxed{\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha\delta - \beta\gamma}$$

OSS:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

$$\det\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^t\right) = \det\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \alpha\delta - \beta\gamma$$

$$= \det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Esempio:

$$\det \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} = 2 \cdot (-1) - 1 \cdot 0 = -2$$

$$\det \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} = 1 \cdot 1 - (2 \cdot (-1)) = 3$$

$$\det \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 1 \cdot 4 - 2 \cdot 2 = 0$$

osserviamo che i vettori riga (e i vettori colonna) sono lin. dip.

OSS: se $\det \neq 0 \Rightarrow$ i vettori riga (e quindi i vettori colonna) formano una base di $\mathbb{R}^n$.

infatti se $\underline{v}_1, \ldots, \underline{v}_n \in \mathbb{R}^n$ sono lin. dip. $\Rightarrow \det(\underline{v}_1, \ldots, \underline{v}_n) = 0$ (osservazione precedente) quindi se $\det(\underline{v}_1, \dots, \underline{v}_n) \neq 0$

$\Rightarrow \{\underline{v}_1, \dots, \underline{v}_n\}$ sono lin. indip.

essendo $n$ vettori in $\mathbb{R}^n$ lin. indip.

$\Rightarrow \{\underline{v}_1, \dots, \underline{v}_n\}$ base.

<u>Esempio</u>

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\{\underline{v}_1, \underline{v}_2\}$ sono una base di $\mathbb{R}^2$?

$$\det \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = 1 \cdot 2 - (-1 \cdot 3) = 5 \neq 0$$

$$\left( \overset{\parallel}{\det(\underline{v}_1, \underline{v}_2)} \right)$$

$\Rightarrow$ sono una base.

## $n = 3$

$$\det \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \\ u & v & w \end{pmatrix} =$$

$$= \det \left( \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) =$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \alpha \underline{e}_1 + \beta \underline{e}_2 + \gamma \underline{e}_3$$

$$= \det \Big( \underbrace{\alpha \underline{e}_1 + \beta \underline{e}_2 + \gamma \underline{e}_3}_{\text{linearità}}, a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3, u\underline{e}_1 + v\underline{e}_2 + w\underline{e}_3 \Big)$$

$$= \alpha \det \big( \underline{e}_1, \cancel{a\underline{e}_1} + b\underline{e}_2 + c\underline{e}_3, \cancel{u\underline{e}_1} + v\underline{e}_2 + w\underline{e}_3 \big)$$

$$+ \beta \det \big( \underline{e}_2, a\underline{e}_1 + \cancel{b\underline{e}_2} + c\underline{e}_3, u\underline{e}_1 + \cancel{v\underline{e}_2} + w\underline{e}_3 \big)$$

$$+ \gamma \det \big( \underline{e}_3, a\underline{e}_1 + b\underline{e}_2 + \cancel{c\underline{e}_3}, u\underline{e}_1 + v\underline{e}_2 + \cancel{w\underline{e}_3} \big)$$

$$= \alpha \left[ b \det(\underline{e}_1, \underline{e}_2, \cancel{v \underline{e}_2} + w \underline{e}_3) + c \det(\underline{e}_1, \underline{e}_3, v \underline{e}_2 + \cancel{w \underline{e}_3}) \right]$$

$$+ \beta \left[ a \det(\underline{e}_2, \underline{e}_1, \cancel{u \underline{e}_1} + w \underline{e}_3) + c \det(\underline{e}_2, \underline{e}_3, u \underline{e}_1 + \cancel{w \underline{e}_3}) \right] +$$

$$+ \gamma \left[ a \det(\underline{e}_3, \underline{e}_1, \cancel{u \underline{e}_1} + v \underline{e}_2) + b \det(\underline{e}_3, \underline{e}_2, u \underline{e}_1 + \cancel{v \underline{e}_2}) \right]$$

$$= \alpha b w \overbrace{\det(\underline{e}_1, \underline{e}_2, \underline{e}_3)}^{=1} + \alpha c v \overset{-1 \,=\, -\det(\underline{e}_1,\underline{e}_2,\underline{e}_3) \,=}{\det(\underline{e}_1, \underline{e}_3, \underline{e}_2)}$$

$$+ \beta a w \underbrace{\det(\underline{e}_2, \underline{e}_1, \underline{e}_3)}_{=-1} + \beta c u \underbrace{\det(\underline{e}_2, \underline{e}_3, \underline{e}_1)}_{=1}$$

$$+ \gamma a v \underbrace{\det(\underline{e}_3, \underline{e}_1, \underline{e}_2)}_{=1} + \gamma b u \underbrace{\det(\underline{e}_3, \underline{e}_2, \underline{e}_1)}_{=-1}$$

$$= \alpha b w - \alpha c v - \beta a w + \beta c u + \gamma a v - \gamma b u$$

REGOLA DI SARRUS [VALE SOLO PER N=3]

$$\det \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \\ u & v & w \end{pmatrix} \begin{matrix} \alpha & \beta \\ a & b \\ u & v \end{matrix}$$

$$\alpha b w + \beta c u + \gamma a v - \beta a w - \alpha c v - \gamma b u$$

<u>Esempi</u>:

$$\bullet \det \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{matrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{matrix}$$

$$= 1 \cdot 1 \cdot 0 + 1 \cdot 1 \cdot 2 + (-1) \cdot 0 \cdot 1 - 1 \cdot 0 \cdot 0 +$$

$$- 1 \cdot 1 \cdot 1 - (-1) \cdot 1 \cdot 2 = 0 + 2 + 0 - 0 - 1 + 2$$

$= 3.$

- Determinare se

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\} = \mathbb{R}^3$$

se

$$\det\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 1 & 1 & -1\end{pmatrix} \neq 0 \implies \text{i tre}$$

vettori formano una base di $\mathbb{R}^3$ e quindi il loro span è $\mathbb{R}^3$.

$$\det\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{matrix}1 & -1\\ 0 & 1\\ 1 & 1\end{matrix} =$$

$$= -1+(-1)+0-0-1-0 = -3 \neq 0.$$