Def: $(V, \langle \cdot, \cdot \rangle)$ spazio metrico

$U \subseteq V$ sottospazio

$U^{\perp} := \{ \underline{v} \in V$ tali che $\langle \underline{v}, \underline{u} \rangle = 0$ per ogni $\underline{u} \in U \}$

Prop: $U^{\perp}$ è un sottospazio di V,

$U \cap U^{\perp} = \{ \underline{0} \}$

$V = U \oplus U^{\perp}$

Dim: per verificare che $U^{\perp}$ è un sottospazio, dobbiamo verificare che è chiuso rispetto alla somma e al prodotto per uno scalare:

$\underline{v}_1, \underline{v}_2 \in U^{\perp} \overset{?}{\Longrightarrow} \underline{v}_1 + \underline{v}_2 \in U^{\perp}$

$\underline{v}_1 + \underline{v}_2 \in U^{\perp}$ se per ogni $\underline{u} \in U$ si verifica $\langle \underline{v}_1 + \underline{v}_2, \underline{u} \rangle = 0$

$$\langle \underline{v}_1 + \underline{v}_2, \underline{u} \rangle = \underbrace{\langle \underline{v}_1, \underline{u} \rangle}_{\substack{= \\ 0 \\ \underline{v}_1 \in U^\perp}} + \underbrace{\langle \underline{v}_2, \underline{u} \rangle}_{\substack{= \\ 0 \\ \underline{v}_2 \in U^\perp}} = 0.$$

- $U \cap U^\perp = \{\underline{0}\}$

sia $\underline{v} \in U \cap U^\perp \overset{?}{\Longrightarrow} \underline{v} = \underline{0}$

$$\|\underline{v}\|^2 = \langle \underset{\substack{\cap \\ U^\perp}}{\underline{v}}, \underset{\substack{\cap \\ U}}{\underline{v}} \rangle = 0 \Longrightarrow \underline{v} = \underline{0}.$$

- $V = U \oplus U^\perp$

Sia $\{\underline{v}_1, \dots, \underline{v}_m\}$ base di U

completiamola ad una base

$$\{\underline{v}_1, \dots, \underline{v}_m, \underline{v}_{m+1}, \dots, \underline{v}_n\} \text{ di } V$$

Utilizziamo Gram-Schmidt a partire da

$$\{\underline{v}_1, \dots, \underline{v}_n\}$$

otteniamo una base <u>ortonormale</u> di V

$\{\underline{u}_1, \dots, \underline{u}_n\}$ tale che

$\mathrm{span}\{\underline{u}_1, \dots, \underline{u}_m\} = \mathrm{span}\{\underline{v}_1, \dots, \underline{v}_m\} = U$

quindi $\{\underline{u}_1, \dots, \underline{u}_m\}$ sono una base ortonormale di U

Oss: $\underline{v} \in U^\perp$ se e solo se

$$<\underline{v}, \underline{u}_1> = \dots = <\underline{v}, \underline{u}_m> = 0$$

infatti $\underline{v} \in U^\perp$ per definizione significa

$<\underline{v}, \underline{u}> = 0$ per ogni $\underline{u} \in U$

dato $\underline{u} \in U$, $\underline{u} = \alpha_1 \underline{u}_1 + \dots + \alpha_m \underline{u}_m$

e dunque

$$0 = <\underline{v}, \underline{u}> = <\underline{v}, \alpha_1 \underline{u}_1 + \dots + \alpha_m \underline{u}_m> =$$

$$\alpha_1 <\underline{v}, \underline{u}_1> + \dots + \alpha_m <\underline{v}, \underline{u}_m> = 0$$

Da questa osservazione, poiché

$$\langle \underline{u}_j, \underline{u}_1\rangle = \dots = \langle \underline{u}_j, \underline{u}_m\rangle = 0$$

per $j = m+1, \dots, n$

$$\Rightarrow \underline{u}_{m+1}, \dots, \underline{u}_n \in U^\perp$$

poiché $\underline{u}_{m+1}, \dots, \underline{u}_n$ sono lin. indip.

$$\Rightarrow \underbrace{\text{span}\{\underline{u}_{m+1}, \dots, \underline{u}_n\}}_{\text{ha dim } n-m} \subseteq U^\perp$$

$$\Rightarrow \dim U^\perp \geq n-m$$

da Grassmann: $\left(\dim U \cap U^\perp = 0 \text{ perché } U \cap U^\perp = \{\underline{0}\}\right)$

$$\underbrace{\dim(U + U^\perp)}_{\leq n} = \underbrace{\underbrace{\dim U}_{= m} + \underbrace{\dim U^\perp}_{\geq n-m}}_{\geq n}$$

$\Rightarrow$ "$\leq$" "$\geq$" sono "$=$"

$\Rightarrow \dim U^\perp = n - m$ (e in particolare

$U^\perp = \text{span}\{\underline{u}_{m+1}, \dots, \underline{u}_n\}$

e $V = U + U^\perp$ (poiché $U \cap U^\perp = \{\underline{0}\}$

$\Rightarrow V = U \oplus U^\perp$) $\blacksquare$

DEF: $(V, \langle \cdot, \cdot \rangle)$ spazio metrico

$U \subseteq V$ sottospazio

la proiezione ortogonale di V su U

$P_U : V \longrightarrow U$

è $P_U = \pi_{U, U^\perp}$, cioè è la proiezione di V su U lungo $U^\perp$.

$V = U \oplus U^\perp$, dato $\underline{v} \in V$ allora esistono unici $\underline{u} \in U$, $\underline{w} \in U^\perp$ tali che $\underline{v} = \underline{u} + \underline{w}$,

$$P_U(\underline{v}) := \underline{u}$$

- Se $\{\underline{u}_1, \dots, \underline{u}_n\}$ è una base ortonormale di V tale che

$$\boxed{U = \operatorname{span}\{\underline{u}_1, \dots, \underline{u}_m\}}$$

allora abbiamo visto

$$\boxed{U^\perp = \operatorname{span}\{\underline{u}_{m+1}, \dots, \underline{u}_n\}}$$

Pertanto, se $\underline{v} \in V$

$$\underline{v} = \underbrace{(\alpha_1 \underline{u}_1 + \dots + \alpha_m \underline{u}_m)} + \underbrace{(\alpha_{m+1} \underline{u}_{m+1} + \dots + \alpha_n \underline{u}_n)}$$

$$\overset{\Uparrow}{U} \qquad\qquad \overset{\Uparrow}{U^\perp}$$

dunque

$$\boxed{P_U(\underline{v}) = \alpha_1 \underline{u}_1 + \dots + \alpha_m \underline{u}_m}$$

Inoltre

$$\boxed{\begin{aligned} \alpha_1 &= \langle \underline{v}, \underline{u}_1 \rangle \\ &\vdots \\ \alpha_m &= \langle \underline{v}, \underline{u}_m \rangle \\ \alpha_{m+1} &= \langle \underline{v}, \underline{u}_{m+1} \rangle \\ &\vdots \\ \alpha_n &= \langle \underline{v}, \underline{u}_n \rangle \end{aligned}}$$

perché

$$\langle \underline{v}, \underline{u}_1 \rangle = \langle \alpha_1 \underline{u}_1 + \dots + \alpha_n \underline{u}_n, \underline{u}_1 \rangle =$$

$$= \alpha_1 \underbrace{\langle \underline{u}_1, \underline{u}_1 \rangle}_{=} + \alpha_2 \underbrace{\langle \underline{u}_2, \underline{u}_1 \rangle}_{=} + \dots + \alpha_n \underbrace{\langle \underline{u}_n, \underline{u}_1 \rangle}_{=}$$

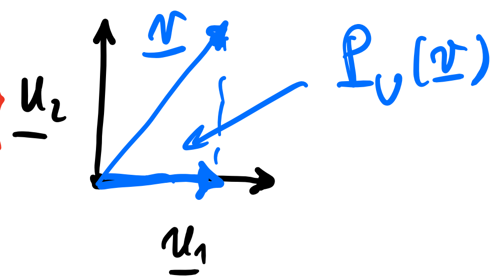
1 0 0

$= \alpha_1 .$

Pertanto:

$$\boxed{P_U(\underline{v}) = \langle \underline{v}, \underline{u}_1 \rangle \underline{u}_1 + \cdots + \langle \underline{v}, \underline{u}_m \rangle \underline{u}_m}$$

$U^\perp = \mathrm{span}\{\underline{u}_2\}$



$U = \mathrm{span}\{\underline{u}_1\}$

OSS: $P_U : V \longrightarrow U$ lineare

$$\boxed{\mathrm{Ker}\, P_U = U^\perp}$$

$$\boxed{\mathrm{Im}\, P_U = U}$$

DIM: $\underline{u} \in U$ $\qquad \underline{u} = \underset{\underset{U}{\cap}}{\underline{u}} + \underset{\underset{U^\perp}{\cap}}{\underline{0}}$

$P_U(\underline{u}) = \underline{u}$ dunque $\operatorname{Im} P_U \supseteq U$

ma poiché $\operatorname{Im} P_U \subseteq U \Rightarrow \operatorname{Im} P_U = U$.

dal teo delle dimensione

$$\underbrace{\dim V}_{\underset{n}{\parallel}} = \dim \ker P_U + \underbrace{\dim \operatorname{Im} P_U}_{\underset{\underbrace{\dim U}_{\underset{m}{\parallel}}}{\parallel}}$$

$\Rightarrow \dim \ker P_U = n - m.$

(sappiamo che $\dim U^\perp = n-m$)

d'altra parte se $\underline{v} \in U^\perp$ allora

$\underline{v} = \underset{\cap}{\underline{0}} + \underset{\cap}{\underline{v}}.$

$$U \qquad U^\perp$$

$$P_U(\underline{v}) = \underline{0} \implies \underline{v} \in \ker P_U$$

poiché $\underline{v}$ è un generico vettore di $U^\perp$

$$\implies U^\perp \subseteq \ker P_U$$

ma $\dim U^\perp = \dim \ker P_U \implies U^\perp = \ker P_U$ $\square$

$$V = U \oplus U^\perp = \underbrace{\ker P_U \oplus \operatorname{Im} P_U}$$

sia $\underline{v} \in V$

$$\underline{v} = \underbrace{P_U(\underline{v})}_{\substack{\in \\ \operatorname{Im} P_U}} + \underline{w} \qquad \text{con } \underline{w} \in \ker P_U$$

$$\underline{w} = \underline{v} - P_U(\underline{v})$$

$$\underline{v} = \underbrace{P_U(\underline{v})}_{\in} + \underbrace{\left(\underline{v} - P_U(\underline{v})\right)}_{\in}$$

$$\operatorname{Im} P_U = U \qquad \operatorname{Ker} P_U = U^\perp$$

Prop: $\underline{v} \in V$ allora

$$\|\underline{v}\|^2 = \|P_U(\underline{v})\|^2 + \|\underline{v} - P_U(\underline{v})\|^2$$

Dim:

$$\|\underline{v}\|^2 = \langle \underline{v}, \underline{v} \rangle = \langle P_U(\underline{v}) + (\underline{v} - P_U(\underline{v})), P_U(\underline{v}) + (\underline{v} - P_U(\underline{v})) \rangle =$$

$$= \langle P_U(\underline{v}), P_U(\underline{v}) + (\underline{v} - P_U(\underline{v})) \rangle + \langle (\underline{v} - P_U(\underline{v})), P_U(\underline{v}) + (\underline{v} - P_U(\underline{v})) \rangle$$

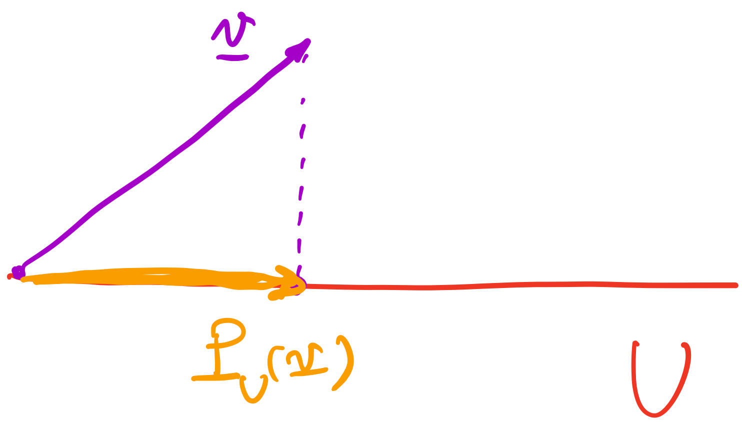
$$= \langle P_U(\underline{v}), P_U(\underline{v}) \rangle + \underbrace{\langle P_U(\underline{v}), (\underline{v} - P_U(\underline{v})) \rangle}_{=0} + \underbrace{\langle (\underline{v} - P_U(\underline{v})), P_U(\underline{v}) \rangle}_{=0} + \langle \underline{v} - P_U(\underline{v}), \underline{v} - P_U(\underline{v}) \rangle$$

$\Big(= 0$ perchè $P_U(\underline{v}) \in U$, $\underline{v} - P_U(\underline{v}) \in U^\perp \Big)$ $\square$

Corollario: $\underline{v} \in V$ allora

$$\| P_U(\underline{v}) - \underline{v} \| \leq \| \underline{u} - \underline{v} \| \quad \text{per ogni } \underline{u} \in U$$

è " $=$ " se e solo se $\underline{u} = P_U(\underline{v})$



DIM: $\underline{v} = \underset{\substack{\cap \\ U}}{\underline{u}_0} + \underset{\substack{\cap \\ U^\perp}}{\underline{u}_0^\perp}$, sia $\underline{u} \in U$

$$\| \underline{v} - \underline{u} \|^2 = \| \underline{u}_0 + \underline{u}_0^\perp - \underline{u} \|^2 =$$

$$= \| \underbrace{(\underline{u}_0 - \underline{u})}_{\in} + \underbrace{\underline{u}_0^\perp}_{\in} \|^2 \underset{\text{Prop. precedente}}{=}$$

$\mathrm{Im}\, P_U = U \qquad U^\perp = \mathrm{Ker}\, P_U$

$$\| \underline{u}_0 - \underline{u} \|^2 + \| \underline{u}_0^\perp \|^2$$

$$\underset{\substack{\Uparrow \\ \mathrm{Ker}\, P_U}}{\underline{u}_0^\perp} = \underline{v} - P_U(\underline{v})$$

pertanto

$$\| \underline{v} - \underline{u} \|^2 = \underbrace{\| \underline{u}_0 - \underline{u} \|^2}_{\geq 0} + \| P_U(\underline{v}) - \underline{v} \|^2$$

$$\geq \| P_U(\underline{v}) - \underline{v} \|^2$$

"=" se $\underline{u}_0 - \underline{u} = \underline{0}$ cioè $\underline{u}_0 = \underline{u}$

ma allora, poiché $\underline{u}_0 = P_U(\underline{v})$, si ha

$\underline{u} = P_U(\underline{v})$ ∎

Esempio:

Sia U il sottospazio di $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_{st})$ definito da

$$U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ tali che } x-y=0 \right\}$$

determinare la proiezione ortogonale di $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ su U.

Troviamo una base di U:

$x-y=0 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U$ se e solo se $x=y$

ovvero i vettori di U sono della forma

$$\begin{pmatrix} x \\ x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

da Rouché-Capelli, $\dim U = 2$, poichè $\begin{pmatrix}1\\1\\0\end{pmatrix}$ e $\begin{pmatrix}0\\0\\1\end{pmatrix}$ generano U

$\Rightarrow \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ sono una base di U

utilizziamo Gram-Schmidt sulla base

$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

$\underline{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\qquad \|\underline{v}_1\| = \sqrt{\left\langle \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\rangle} = \sqrt{2}$

$\underline{u}_1 := \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}$

$\underline{v}_2 := \begin{pmatrix}0\\0\\1\end{pmatrix} - \underbrace{\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} \right\rangle}_{= 0} \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\|\underline{v}_2\| = 1$

$\underline{u}_2 = \underline{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

Pertanto $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ sono una base ortonormale di U

Pertanto

$$P_U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} +$$

$$+ \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \left(\frac{1}{\sqrt{2}} x + \frac{1}{\sqrt{2}} y \right) \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1}{2}x + \frac{1}{2}y \\ \frac{1}{2}x + \frac{1}{2}y \\ z \end{pmatrix}$$

$$P_U \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} .$$

se vogliamo trovare $U^\perp$ (cioè una base)

$U^\perp = \ker P_U$ quindi troviamo

una base di $\ker P_U$:

$$\ker P_U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \begin{cases} \frac{1}{2}x + \frac{1}{2}y = 0 \\ z = 0 \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{cases} x = -y \\ z = 0 \end{cases} \right\} =$$

$$= \text{span}\left\{\begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$$

OPPURE:

$$U^\perp = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix}\right\rangle = 0 \quad \text{per ogni} \begin{pmatrix}a\\b\\c\end{pmatrix} \in U\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \underline{u}_1\right\rangle = \left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \underline{u}_2\right\rangle = 0\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle = 0\right\}$$

$$= \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{cases}\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y = 0\\ z = 0\end{cases}\right\}$$

$$= \begin{cases} x = -y \\ z = 0\end{cases}$$

in $\mathbb{R}^3$

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \quad x - y + z = 0 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle = 0 \right\} =$$

$$= \left(\operatorname{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} \right)^{\perp}$$

per trovare una base di W

1°) risolvere $x - y + z = 0$ $\quad y = x + z$

$$W = \left\{ \begin{pmatrix} x \\ x+z \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} =$$

$$= \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

2°] $W = \left(\text{span}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}\right)^{\perp}$

troviamo una base ortonormale $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ di $\mathbb{R}^3$

tale che $\text{span}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\right\} = \text{span}\{\underline{v}_1\}$

$\Rightarrow W = \text{span}\{\underline{v}_2, \underline{v}_3\}$

$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix}$ t.c. $\left\langle \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rangle = 0$

lin. indip. $\left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\rangle = 0$