$\mathbb{A}^3$, $T\mathbb{A}^3 = \mathbb{R}^3$ $\quad (\mathbb{R}^3, \langle \cdot, \cdot \rangle_{st})$

- rette affini : sottospazi affini di dim 1
- piani affini : sottospazi affini di dim 2

Fissiamo un sistema di riferimento affine ortogonale $(O; \{\underline{e}_1, \underline{e}_2, \underline{e}_3\})$

- rette affine: r retta, $P_0 \in r$

$\underline{v} \in \mathbb{R}^3$ $\quad \text{span}\{\underline{v}\} = Tr$

eq. parametrica $\quad P \in r \iff P - P_0 = \lambda \underline{v}$
$\lambda \in \mathbb{R}$

se $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad P_0 = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}$

$\underline{v} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ $\quad \lambda \in \mathbb{R}$

$$r: \begin{cases} x = a_0 + \lambda \alpha \\ y = b_0 + \lambda \beta \\ z = c_0 + \lambda \gamma \end{cases} \quad \lambda \in \mathbb{R}$$

eq. cartesiana: $P \in r \iff \langle P - P_0, \underline{u} \rangle = 0$

per ogni $\underline{u} \in (Tr)^\perp$

OSS: $\dim Tr = 1 \Rightarrow \dim (Tr)^\perp = 2$

perchè $\mathbb{R}^3 = Tr \oplus (Tr)^\perp$

da Grassmann $3 = 1 + 2$

se $(Tr)^\perp = \text{span}\{\underline{u}_1, \underline{u}_2\}$ allora

$P \in r \iff \langle P - P_0, \underline{u}_1 \rangle = \langle P - P_0, \underline{u}_2 \rangle = 0$

se $\underline{u}_1 = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ $\underline{u}_2 = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ allora

$$P \in r \iff \begin{cases} \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}, \begin{pmatrix} l \\ m \\ n \end{pmatrix} \right\rangle = 0 \\ \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}, \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\rangle = 0 \end{cases}$$

$$r : \begin{cases} (x-a_0)l + (y-b_0)m + (z-c_0)n = 0 \\ (x-a_0)p + (y-b_0)q + (z-c_0)r = 0 \end{cases}$$

$$\wr$$

$$\begin{cases} lx + my + nz + \delta = 0 \\ px + qy + rz + k = 0 \end{cases}$$

piani affini

$\pi \qquad \dim T\pi = 2$

$$P_0 = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} \in \pi \qquad \underline{v}_1, \underline{v}_2 \in \mathbb{R}^3$$

$$T\pi = \operatorname{span}\{\underline{v}_1, \underline{v}_2\}$$

$$\underline{v}_1 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix}$$

equazione parametrica:

$$P \in \pi \iff P - P_0 \in T\pi$$

$$\iff P - P_0 = \lambda \underline{v}_1 + \mu \underline{v}_2 \qquad \lambda, \mu \in \mathbb{R}$$

in coordinate

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \mu \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} \qquad \lambda, \mu \in \mathbb{R}$$

$$\pi : \begin{cases} x = a_0 + \lambda \alpha + \mu \tilde{\alpha} \\ y = b_0 + \lambda \beta + \mu \tilde{\beta} \\ z = c_0 + \lambda \gamma + \mu \tilde{\gamma} \end{cases} \qquad \lambda, \mu \in \mathbb{R}$$

$$\begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} = P_0 \in \pi, \qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \underline{v}_1, \qquad \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \end{pmatrix} = \underline{v}_2$$

<u>eq. cartesiana</u>:

<u>OSS</u>: $\dim (T\pi)^\perp = 1$

$$\underset{3}{\mathbb{R}^3} = \underset{2}{T\pi} \oplus \underset{1}{(T\pi)^\perp}$$

$P \in \pi \iff \langle P - P_0 , \underline{u} \rangle = 0$ per ogni $\underline{u} \in (T\pi)^\perp$

se $\quad \underline{u}_1 = \begin{pmatrix} m \\ n \\ \ell \end{pmatrix} \qquad \mathrm{span}\{\underline{u}_1\} = (T\pi)^{\perp}$

allora

$$P \in \pi \iff \langle P - P_0 , \underline{u}_1 \rangle = 0$$

in coordinate

$$\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix}, \begin{pmatrix} m \\ n \\ \ell \end{pmatrix} \right\rangle = 0$$

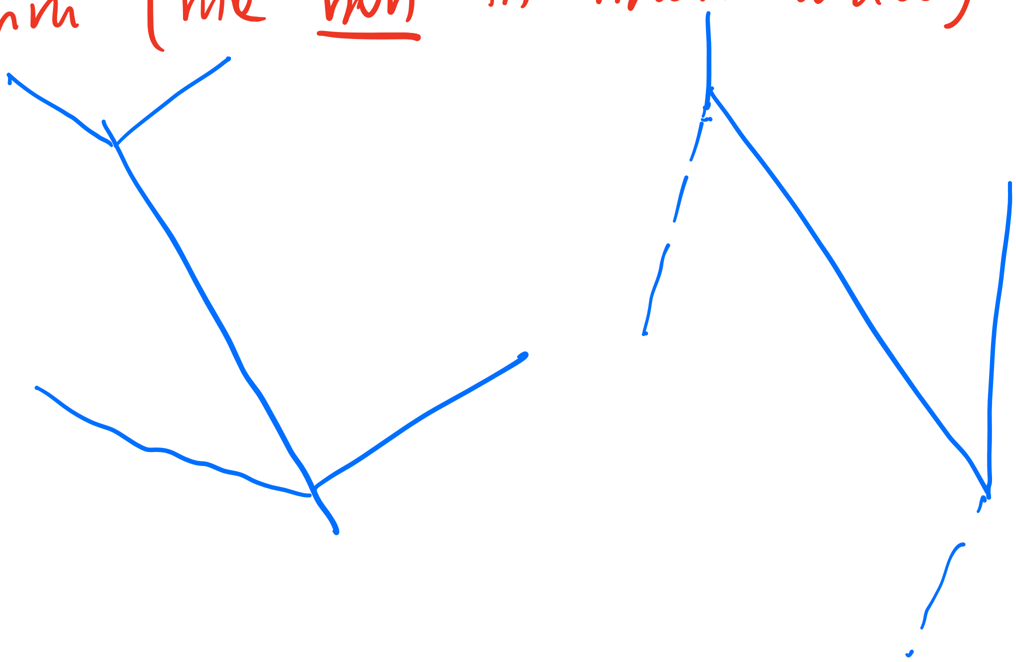
$$\pi : \ (x - a_0) \cdot m + (y - b_0) n + (z - c_0) \ell = 0$$

$$\Downarrow$$

$$\pi : \ \boxed{m} x + \boxed{n} y + \boxed{\ell} z + \delta = 0$$

$$\begin{pmatrix} m \\ n \\ \ell \end{pmatrix} = \underline{u}_1$$

OSS: comparando le eq. cartesiane di rette piani si vede che ogni retta affine è l'intersezione di due piani affini (ma <u>non</u> in modo unico)



PRODOTTO VETTORIALE IN $\mathbb{R}^3$

Fissiamo la base canonica $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$

siano $\underline{v}, \underline{w} \in \mathbb{R}^3$

$$\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\underline{v} \wedge \underline{w} := \det \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} :=$$

wedge
prodotto esterno
prodotto vettoriale

$$= b\gamma \, \underline{e}_1 + \alpha c \, \underline{e}_2 + a \beta \, \underline{e}_3$$

$$- \alpha b \, \underline{e}_3 - \beta c \, \underline{e}_1 - a \gamma \, \underline{e}_2 =$$

$$(b\gamma - \beta c) \underline{e}_1 + (\alpha c - a\gamma) \underline{e}_2 + (a\beta - \alpha b) \underline{e}_3$$

$$\underline{v} \wedge \underline{w} = \begin{pmatrix} b\gamma - \beta c \\ \alpha c - a\gamma \\ a\beta - \alpha b \end{pmatrix}$$

<u>OSS</u>: $\underline{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\langle \underline{v} \wedge \underline{w}, \underline{u} \rangle =$$

$$= \left\langle \begin{pmatrix} b\gamma - \beta c \\ \alpha c - a\gamma \\ a\beta - \alpha b \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle =$$

$$= \underbrace{b\gamma x} - \underbrace{\beta c x} + \underbrace{\alpha c y} - \underbrace{a\gamma y} + \underbrace{a\beta z} - \underbrace{\alpha b z}$$

$$\det \begin{pmatrix} x & y & z \\ a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} = \underbrace{xb\gamma} + \underbrace{yc\alpha} + \underbrace{z\beta a} - \underbrace{\alpha b z} - \underbrace{\beta c x} - \underbrace{\gamma y a}$$

$$\boxed{\langle \underline{v} \wedge \underline{w}, \underline{u} \rangle = \det \begin{pmatrix} x & y & z \\ a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{matrix} \to \underline{u} \\ \to \underline{v} \\ \to \underline{w} \end{matrix}}$$

da questa formula segue immediatamente:

Corollario:

$$\langle \underline{v} \wedge \underline{w}, \underline{v} \rangle = 0$$

$$\langle \underline{v} \wedge \underline{w}, \underline{w} \rangle = 0$$

quindi

$\underline{v} \wedge \underline{w}$ è ortogonale a $\operatorname{span}\{\underline{v}, \underline{w}\}$

OSS:

- $\underline{v} \wedge \underline{w} = - \underline{w} \wedge \underline{v}$
- se $\{\underline{v}, \underline{w}\}$ sono lin. dip.

  $\underline{v} \wedge \underline{w} = 0$
- $\underline{\tilde{v}} \in \mathbb{R}^3$

  $(\underline{v} + \underline{\tilde{v}}) \wedge \underline{w} = \underline{v} \wedge \underline{w} + \underline{\tilde{v}} \wedge \underline{w}$

- $\lambda \in \mathbb{R}$

$$(\lambda \underline{v}) \wedge \underline{w} = \lambda (\underline{v} \wedge \underline{w}) = \underline{v} \wedge (\lambda \underline{w})$$

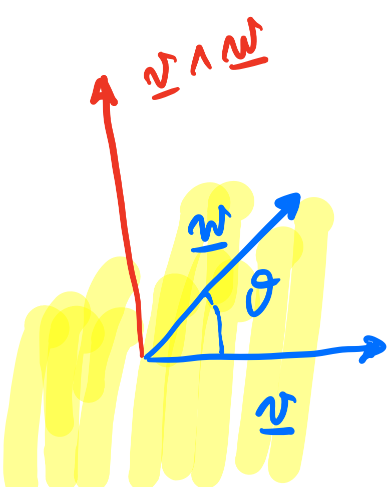
- $\underline{v} \wedge (\underline{w} + \underline{\tilde{w}}) = \underline{v} \wedge \underline{w} + \underline{v} \wedge \underline{\tilde{w}}$

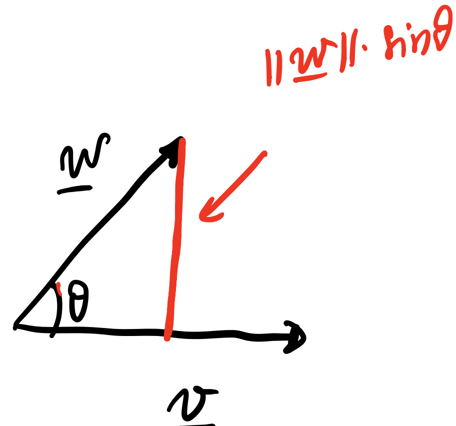
FATTO: $\underline{v}, \underline{w} \neq 0$

$$\|\underline{v} \wedge \underline{w}\| = \|\underline{v}\| \cdot \|\underline{w}\| \sin\theta$$

θ è l'angolo tra $\underline{v}$ e $\underline{w}$

(cioè $\cos\theta = \dfrac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{v}\| \cdot \|\underline{w}\|}$ se $\underline{v}, \underline{w} \neq \underline{0}$)

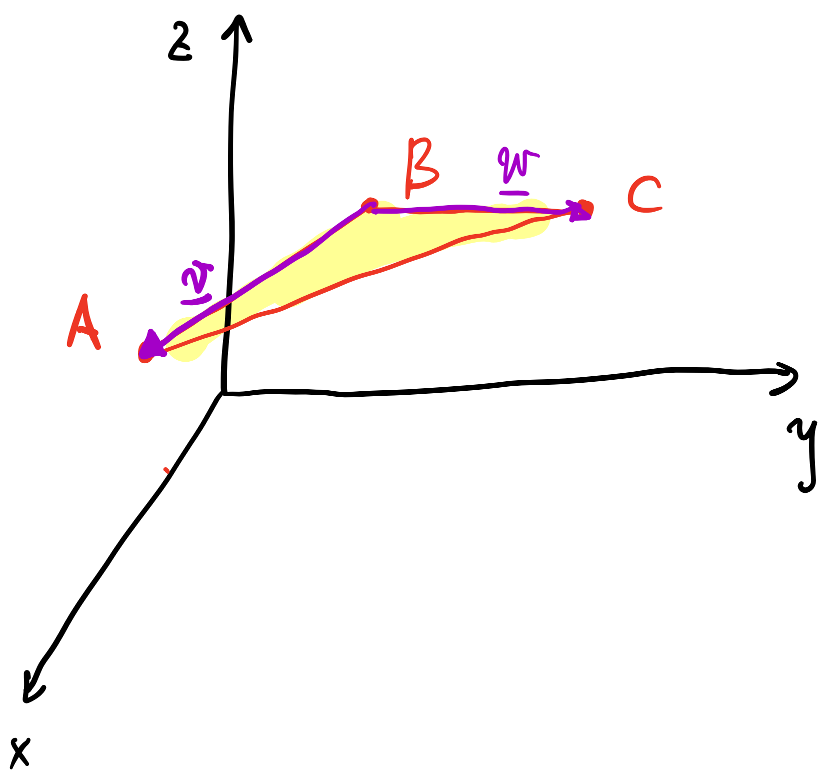




Esempio: Siano $A = (1,0,1)$

$B = (-1,1,1)$ $\quad C = (0,2,1)$

in $\mathbb{A}^3$ (in cui abbiamo fissato un sistema di rif. affine ortogonale) trovare l'area del triangolo con vertici A, B, C



Sia $\underline{v} = A - B \qquad \underline{w} = C - B$

Area del triangolo $= \frac{1}{2} \| \underline{v} \wedge \underline{w} \|$

$$\underline{v} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{w} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\underline{v} \wedge \underline{w} = \det \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 2 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix} =$$

$$= 2\underline{e}_3 + \underline{e}_3 = 3\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

$$\| \underline{v} \wedge \underline{w} \| = \sqrt{0^2 + 0^2 + 3^2} = 3$$

Area $= \frac{3}{2}$.

Esempio: Sia r la retta in $\mathbb{A}^3$

$$\begin{cases} x + y = 1 \\ x - z = 0 \end{cases}$$

determinare l'eq. parametrica di r

$$\begin{cases} 1 \cdot x + 1 \cdot y + 0 \cdot z = 1 \\ 1 \cdot x + 0 \cdot y - 1 \cdot z = 0 \end{cases}$$

i vettori $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ e $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ sono una

base di $(T_r)^{\perp}$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ è tangente a r

ovvero $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} = T_r$

$$\begin{pmatrix}1\\1\\0\end{pmatrix} \wedge \begin{pmatrix}1\\0\\-1\end{pmatrix} = \det \begin{pmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} =$$

$$= -\underline{e}_1 - \underline{e}_3 + \underline{e}_2 = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

prendiamo un qualunque punto di r:
ad esempio $(0,1,0)$ [ponendo nel sistema $x=0$]

eq. parametrica:

$$\begin{cases} x = 0 + \lambda \cdot (-1) \\ y = 1 + \lambda \cdot 1 \\ z = 0 + \lambda \cdot (-1) \end{cases} \qquad \lambda \in \mathbb{R}$$

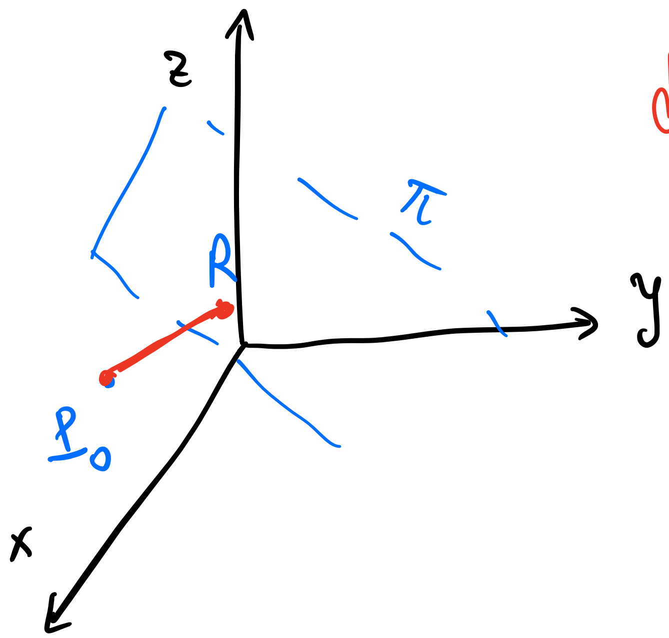
$$\begin{cases} x = -\lambda \\ y = 1 + \lambda \\ z = -\lambda \end{cases} \qquad \lambda \in \mathbb{R}$$

Esempio: Sia π il piano

$$x+y+z=-1$$

e sia $P_0=(1,0,1)$

Calcolare la distanza tra P_0 e π



$$\mathrm{dist}(P_0,\pi) = \min\{\mathrm{dist}(P_0,P) \mid P\in\pi\}$$

1°) troviamo la retta r ortogonale a π e passante per P_0.

Poi determiniamo $R=\pi\cap r$

$$\mathrm{dist}(P_0,\pi)=\|R-P_0\|$$

$\pi : x + y + z = -1$

$\underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \text{span}\{\underline{v}\} = (T\pi)^{\perp}$

$\underline{v}$ è un vettore tangente a r

$r : \begin{cases} x = 1 + \lambda \cdot 1 \\ y = 0 + \lambda \cdot 1 \\ z = 1 + \lambda \cdot 1 \end{cases}$ $\quad \lambda \in \mathbb{R}$

P_0

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in r \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 + \lambda \\ \lambda \\ 1 + \lambda \end{pmatrix}$ $\quad \lambda \in \mathbb{R}$

$R = \pi \cap r$

$\pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ t.c.

$x + y + z = -1$

sostituisco x, y, z con

$$(1+\lambda) + \lambda + (1+\lambda) = -1$$
$$(x + y + z = -1)$$

$$2 + 3\lambda = -1 \qquad \lambda = -1$$

$$R = \begin{pmatrix} 1 + (-1) \\ -1 \\ 1 + (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

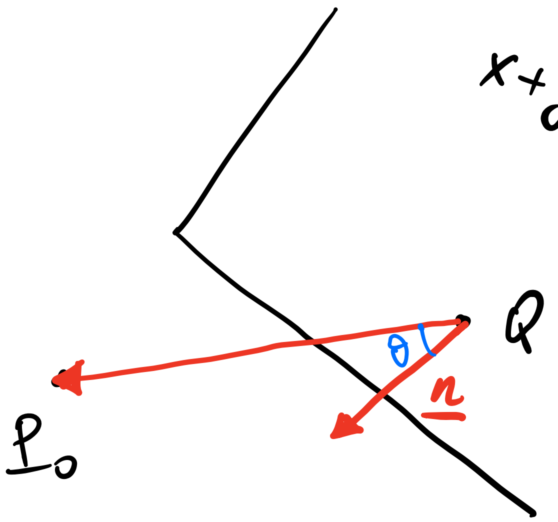
$$\|P_0 - R\| = \left\| \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\| =$$

$$\|R - P_0\|$$

$$= \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

2°)

$x+y+z=-1$

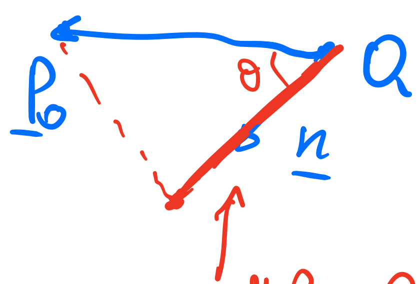


$$\operatorname{span}\{\underline{n}\} = (T\pi)^{\perp}$$

$$\|\underline{n}\| = 1$$

$$|\langle P_0 - Q, \underline{n} \rangle| = \|P_0 - Q\| \cdot \underbrace{\|\underline{n}\|}_{=1} |\cos\theta|$$

$$= \|P_0 - Q\| \, |\cos\theta|$$



$\|P_0 - Q\| \cdot |\cos\theta|$

Quindi $\operatorname{dist}(P_0, \pi) = |\langle P_0 - Q, \underline{n} \rangle|$

$$P_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\pi : \quad x + y + z = -1 \qquad \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \perp \pi$$

$$Q = (-1, 0, 0) \qquad \|\underline{v}\| = \sqrt{3}$$

$$\underline{n} = \frac{\underline{v}}{\|\underline{v}\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$P_0 - Q = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\left| \left\langle \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\rangle \right| = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$$