OSS: se $d(\underline{v},\underline{w}) := \|\underline{v}-\underline{w}\|$
allora $P_U(\underline{v})$ è il punto di minima distanza di $\underline{v}$ da U.

Prodotti scalari e metrici

$(V, \langle \cdot, \cdot \rangle)$ spazio metrico

$\{\underline{v}_1, \dots, \underline{v}_n\}$ base di V

$a_{ij} := \langle \underline{v}_i, \underline{v}_j \rangle \qquad i,j = 1, \dots, n$

OSS: poiché il prodotto scalare è simmetrico $\langle \underline{v}_i, \underline{v}_j \rangle = \langle \underline{v}_j, \underline{v}_i \rangle$

e quindi $\boxed{a_{ij} = a_{ji}}$. $(*)$

Sia

$$A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$\underline{\text{OSS}}$: $(*) \Rightarrow A$ è simmetrica, ovvero

$$\boxed{A = A^t}$$

$$\underline{v} = \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n$$

$$\underline{w} = \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n$$

$$\langle \underline{v}, \underline{w} \rangle = \langle \alpha_1 \underline{v}_1 + \dots + \alpha_n \underline{v}_n, \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n \rangle$$

$$= \alpha_1 \langle \underline{v}_1, \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n \rangle + \dots + \alpha_n \langle \underline{v}_n, \beta_1 \underline{v}_1 + \dots + \beta_n \underline{v}_n \rangle$$

$$= \alpha_1 \beta_1 \langle \underline{v}_1, \underline{v}_1 \rangle + \dots + \alpha_1 \beta_n \langle \underline{v}_1, \underline{v}_n \rangle + \dots$$

$$+ \alpha_n \beta_1 \langle \underline{v}_n, \underline{v}_1 \rangle + \dots + \alpha_n \beta_n \langle \underline{v}_n, \underline{v}_n \rangle$$

$$= \alpha_1 \beta_1 a_{11} + \dots + \alpha_1 \beta_n a_{1n} + \dots + \alpha_n \beta_1 \underset{}{\overset{a_{1n}}{\overset{\shortparallel}{a_{n1}}}} + \dots$$

$$\dots + \alpha_n \beta_n a_{nn} =$$

$$= (\alpha_1, \dots, \alpha_n) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$$

cioè:

$\underline{v} \rightsquigarrow \underline{x} := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ coordinate in $\{\underline{v}_1, \dots, \underline{v}_n\}$

$\underline{w} \rightsquigarrow \underline{y} := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$ coordinate in $\{\underline{v}_1, \dots, \underline{v}_n\}$

allora

$$\boxed{\langle \underline{v}, \underline{w} \rangle = \underline{x}^t \cdot A \cdot \underline{y}}$$

OSS:

$$\langle \underline{v}, \underline{w} \rangle = \underline{x}^t A \underline{y}$$
$$\parallel$$
$$\langle \underline{w}, \underline{v} \rangle = \underline{y}^t \cdot A \cdot \underline{x}$$

$$\left[\begin{array}{l} (\underline{x}^t A \underline{y})^t = \underline{x}^t A \underline{y} \\ \parallel \\ \underline{y}^t A^t \underline{x} \underset{A = A^t}{=} \underline{y}^t \cdot A \cdot \underline{x} \end{array} \right.$$

Lemma: A, B matrici $n \times n$. Se

$$\underline{x}^t A \underline{y} = \underline{x}^t B \cdot \underline{y} \qquad \text{per ogni } \underline{x}, \underline{y} \in \mathbb{R}^n$$

allora $A = B$

Dim:

$$\underline{e}_i^t A \underline{e}_j =$$

$$\underline{e}_i^t \cdot A = (0, \dots, \underset{\substack{\uparrow \\ i}}{1}, 0 \dots, 0) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$= (a_{i1} \cdots a_{in}) \qquad \text{i-ma riga della matrice}$$

$$\left(A \cdot \underline{e}_j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \quad \text{j-ma colonna} \right)$$

$$\underline{e}_i^t A \cdot \underline{e}_j = (a_{i1} \cdots a_{in}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j = a_{ij}$$

se $\underline{x}^t A \underline{y} = \underline{x}^t B \underline{y}$ per ogni $\underline{x}, \underline{y}$

in particolare

$$\underline{e}_i^t A \underline{e}_j = \underline{e}_i^t B \underline{e}_j \quad \text{per ogni } i, j$$

$$\Rightarrow \quad a_{ij} = b_{ij}$$

∎

Assumiamo che $\{\underline{v}_1, \dots, \underline{v}_n\}$ sia base ortonormale di V

$$\Rightarrow A = I$$

<u>OSS</u>: se $A = I \Rightarrow \{\underline{v}_1, \dots, \underline{v}_n\}$ è base ortonormale perché:

$$\underline{v}_1 \rightsquigarrow \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix} \quad \text{Coordinate nella base } \{\underline{v}_1, \dots, \underline{v}_n\}$$

$$\vdots$$

$$\underline{v}_n \rightsquigarrow \begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix} \quad \text{coord. nella base } \{\underline{v}_1, \dots, \underline{v}_n\}$$

$$\langle \underline{v}_i, \underline{v}_j \rangle = \begin{array}{r} \\ \\ i \rightarrow \\ \\ \\ \end{array} \begin{pmatrix}0\\ \vdots\\1\\0\\ \vdots\\0\end{pmatrix}^t \cdot I \begin{pmatrix}0\\ \vdots\\1\\0\\ \vdots\\0\end{pmatrix} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \end{array} =$$

$$= \underline{e}_i^t \cdot I \cdot \underline{e}_j = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$

Sia $\{\underline{w}_1, \dots, \underline{w}_n\}$ un'altra base <u>ortonormale</u> di V

$\mathcal{V} := \{\underline{v}_1, \dots, \underline{v}_n\}$ ortonormale

$\mathcal{W} := \{\underline{w}_1, \dots, \underline{w}_n\}$ ortonormale

Sia C la matrice di cambiamento di base da $\mathcal{V}$ a $\mathcal{W}$

ovvero se $\underline{v} \in V$ $\quad \underline{v} \longrightarrow \underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_{\mathcal{V}}$

$$\underline{v} \longrightarrow \underline{\tilde{x}} = \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix}_{\mathcal{W}}$$

$$\boxed{\underline{\tilde{x}} = C\underline{x}}$$

$\underline{w} \in V \quad \underline{w} \to \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}_{\mathcal{V}}, \quad \underline{\tilde{y}} = \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_n \end{pmatrix}_{\mathcal{W}}$

$$\boxed{\underline{\tilde{y}} = C\underline{y}}$$

$\mathcal{V}$ ortonormale

$$\langle \underline{v}, \underline{w} \rangle \overset{\downarrow}{=} \underline{x}^t \cdot I \cdot \underline{y} = \underline{x}^t \cdot \underline{y}$$

$\| \leftarrow \mathcal{W}$ ortonormale

$$\underline{\tilde{x}}^t \cdot I \cdot \underline{\tilde{y}} = \underline{\tilde{x}}^t \cdot \underline{\tilde{y}}$$

$$\Rightarrow \quad \underline{\tilde{x}}^t \cdot \underline{\tilde{y}} = \underline{x}^t \cdot \underline{y}$$

$$\| $$

$$(C\underline{x})^t \cdot (C \cdot \underline{y}) = (\underline{x}^t \cdot C^t)(C \cdot \underline{y}) =$$

$$\underbrace{(C\underline{x})^t}_{= \underline{x}^t \cdot C^t} \qquad = \underline{x}^t (C^t \cdot C) \cdot \underline{y}$$

pertanto per ogni $\underline{x}, \underline{y} \in \mathbb{R}^n$

$$\underline{x}^t \cdot I \cdot y = \underline{x}^t \underline{y} = \underline{x}^t (C^t \cdot C)\, \underline{y}$$

$$\Rightarrow \quad \boxed{C^t C = I}$$

In altri termini $\boxed{C^{-1} = C^t}$

Def: Una matrice A $n \times n$ si dice <u>ortogonale</u> se $A^{-1} = A^t$ (cioè se $AA^t = I$)

l'argomento precedente implica:

Prop: $(V, \langle\cdot\rangle)$ spazio metrico. Siano $\mathcal{B} = \{\underline{v}_1, \dots, \underline{v}_n\}$, $\mathcal{W} = \{\underline{w}_1, \dots, \underline{w}_n\}$ due basi ortonormali di V.
Allora la matrice di cambiamento di coordinate da $\mathcal{B}$ a $\mathcal{W}$ (e quella da $\mathcal{W}$ a $\mathcal{B}$) è una matrice ortogonale

OSS: A è una matrice ortogonale se e solo se $\underset{\underset{A^{-1}}{\shortparallel}}{A^t}$ è ortogonale.

A è ortogonale se e solo se i vettori colonna di A formano una

base ortonormale di $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{st})$

$$A^t A = I$$

$$A = \begin{pmatrix} \underline{e}_1 & \dots & \underline{e}_n \end{pmatrix} \qquad A^t = \begin{pmatrix} \underline{e}_1^t \\ \vdots \\ \underline{e}_n^t \end{pmatrix}$$

$\uparrow$ vettori colonne

$$A^t \cdot A = \begin{pmatrix} \underline{e}_1^t \\ \vdots \\ \underline{e}_n^t \end{pmatrix} \cdot \begin{pmatrix} \underline{e}_1 & \dots & \underline{e}_n \end{pmatrix} = \begin{pmatrix} \langle \underline{e}_1, \underline{e}_1 \rangle & \dots & \langle \underline{e}_1, \underline{e}_n \rangle \\ \vdots & & \\ \langle \underline{e}_n, \underline{e}_1 \rangle & \dots & \langle \underline{e}_n, \underline{e}_n \rangle \end{pmatrix}$$

$$A^t \cdot A = I \iff \langle \underline{e}_i, \underline{e}_j \rangle = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}$$