Corollario: $\underline{v}, \underline{w} \in V$ $\quad (V, \langle \cdot , \cdot \rangle)$ sp. metrico

$$\big| \|\underline{v}\| - \|\underline{w}\| \big| \leq \|\underline{v} - \underline{w}\|$$

Dim: sia $\underline{\tilde{w}} = -\underline{w}$

dalla disuguaglianza triangolare

$$\big| \|\underline{v}\| - \|\underline{\tilde{w}}\| \big| \leq \|\underline{v} + \underline{\tilde{w}}\| = \|\underline{v} - \underline{w}\|$$

$\|$

$$\big| \|\underline{v}\| - \|-\underline{w}\| \big|$$

$\|$ $\quad \|-\underline{w}\| = \langle -\underline{w}, -\underline{w} \rangle = \langle \underline{w}, \underline{w} \rangle$

$$\big| \|\underline{v}\| - \|\underline{w}\| \big|$$

$\blacksquare$

dalla disuguaglianza di Cauchy-Schwarz

$$|\langle \underline{v}, \underline{w} \rangle| \leq \|\underline{v}\| \cdot \|\underline{w}\|$$

in particolare, se $\underline{v}, \underline{w} \neq \underline{0}$ (e dunque $\|\underline{v}\| > 0$, $\|\underline{w}\| > 0$):

$$\frac{|\langle \underline{v}, \underline{w} \rangle|}{\|\underline{v}\| \cdot \|\underline{w}\|} \leq 1$$

$$\frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{v}\| \cdot \|\underline{w}\|} \in [-1, 1]$$

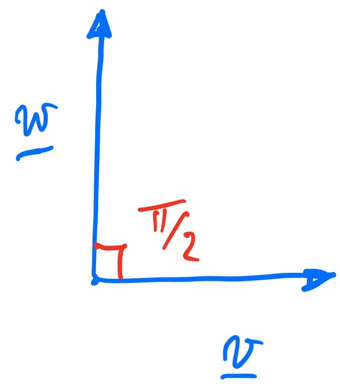
Def: $\theta \in [0, \pi]$ è l'angolo tra $\underline{v}, \underline{w}$

se

$$\cos\theta = \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{v}\| \cdot \|\underline{w}\|}$$

OSS: • $\langle \underline{v}, \underline{w} \rangle = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$

$\underline{v}, \underline{w} \neq \underline{0}$



• se $\underline{v}, \underline{w}$ sono lin. dip. $(\underline{v}, \underline{w} \neq \underline{0})$

$\alpha \underline{v} + \beta \underline{w} = \underline{0}$ (se $\alpha = 0$ allora poiché $\underline{w} \neq \underline{0} \Rightarrow \beta = 0$!)

α o $\beta \neq 0$ (se $\beta = 0$ allora $\alpha \underline{v} = \underline{0}$ ma $\underline{v} \neq \underline{0} \Rightarrow \alpha = 0$!)

$\Rightarrow \alpha$ e $\beta \neq 0$

$$\Rightarrow \quad \underline{v} = \left(-\frac{\beta}{\alpha}\right)\underline{w}$$

ovvero $\underline{v}$ è multiplo di $\underline{w}$,

$$\underline{v} = \lambda \underline{w} \qquad (\lambda \neq 0)$$

$$\langle \underline{v}, \underline{w} \rangle = \langle \lambda \underline{w}, \underline{w} \rangle = \lambda \|\underline{w}\|^2$$

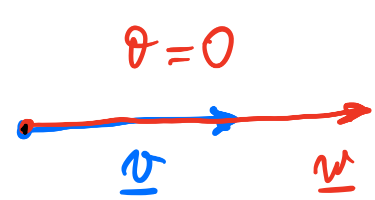
$$\|\underline{v}\| = \|\lambda \underline{w}\| = \sqrt{\langle \lambda \underline{w}, \lambda \underline{w} \rangle} =$$

$$= \sqrt{\lambda \langle \underline{w}, \lambda \underline{w} \rangle} = \sqrt{\lambda^2 \langle \underline{w}, \underline{w} \rangle} =$$

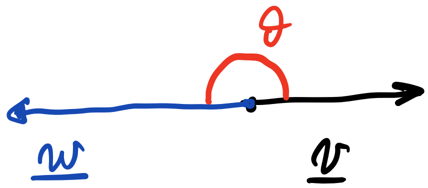
$$|\lambda| \, \|\underline{w}\|$$

$$\cos\theta = \frac{\langle \underline{v}, \underline{w} \rangle}{\|\underline{v}\| \, \|\underline{w}\|} = \frac{\lambda \cancel{\|\underline{w}\|^2}}{|\lambda| \cancel{\|\underline{w}\|} \cdot \cancel{\|\underline{w}\|}} = \pm 1$$

$$\theta = 0 \ \text{ o } \ \pi$$



$\theta = \pi$



OSS: se $\langle \underline{v}, \underline{w} \rangle = 0$ (e $\underline{v}, \underline{w} \neq \underline{0}$)

allora $\underline{v}, \underline{w}$ sono lin. indip.

Dim: $\alpha \underline{v} + \beta \underline{w} = \underline{0} \overset{?}{\Longrightarrow} \alpha, \beta = 0$

$$0 = \langle \underbrace{\alpha \underline{v} + \beta \underline{w}}_{= \underline{0}}, \underline{v} \rangle = \alpha \langle \underline{v}, \underline{v} \rangle + \beta \underbrace{\langle \underline{w}, \underline{v} \rangle}_{= \langle \underline{v}, \underline{w} \rangle = 0}$$

$$= \alpha \cdot \underbrace{\|\underline{v}\|^2}_{\neq 0} \Longrightarrow \alpha = 0$$

dunque $\underline{0} = \alpha \underline{v} + \beta \underline{w} = \underline{0} + \overset{= \underline{0}}{0 \cdot \underline{v}} + \beta \underline{w}$

$$= \beta \underline{w} \underset{\underline{w} \neq \underline{0}}{\Longrightarrow} \beta = 0 \qquad \square$$

Def: $(V, \langle \cdot, \cdot \rangle)$ sp. metrico

$\underline{v}, \underline{w} \in V$ si dicono ortogonali (o perpendicolari) e si scrive $\underline{v} \perp \underline{w}$ se $\langle \underline{v}, \underline{w} \rangle = 0$

Prop: Se $\{\underline{v}_1, \dots, \underline{v}_m\} \subset V$ tali che $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ $i \neq j$, $i, j \in \{1, \dots, m\}$ e $\underline{v}_i \neq \underline{0}$ $i = 1, \dots, m$ allora $\{\underline{v}_1, \dots, \underline{v}_m\}$ sono lin. indip.

es: $m = 3$, $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ se:

$\langle \underline{v}_1, \underline{v}_2 \rangle = 0$

$\langle \underline{v}_1, \underline{v}_3 \rangle = 0$

$\langle \underline{v}_2, \underline{v}_3 \rangle = 0$

e $\underline{v}_1 \neq \underline{0}$, $\underline{v}_2 \neq \underline{0}$, $\underline{v}_3 \neq \underline{0}$

allora $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ sono lin. indip.

DEF: $(V, \langle \cdot, \cdot \rangle)$ sp. metrico

Una base $\{\underline{v}_1, \ldots, \underline{v}_n\}$ di V si dice una base ortogonale se

- $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ per ogni $i \neq j$

si dice una base ortonormale se

- $\langle \underline{v}_i, \underline{v}_j \rangle = 0$ per ogni $i \neq j$
- $\| \underline{v}_i \| = 1$ per $i = 1, \ldots, n$

ESEMPIO: $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{st})$ prodotto scalare standard

la base canonica $\{\underline{e}_1, \ldots, \underline{e}_n\}$ è una base ortonormale.

$$\langle \underline{e}_i, \underline{e}_j \rangle \underset{i \neq j}{=} \langle i \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \vdots \end{pmatrix} j \rangle =$$

$$0\cdot 0+\cdots+0\cdot 1+\cdots+1\cdot 0+\cdots+0\cdot 0=0$$

$$\left(\left\langle \begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle = 1\cdot 0+0\cdot 1+0\cdot 0=0\right)$$

$$\|\underline{e}_j\| = 1$$

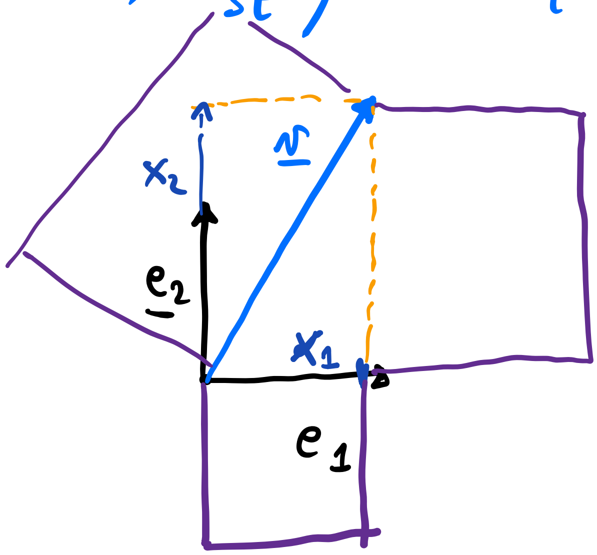
$$\left(\left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\| = \sqrt{\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\rangle} = \sqrt{0\cdot 0+1\cdot 1+0\cdot 0} = \sqrt{1}=1\right)$$

Teorema di Pitagora $(V, \langle \cdot,\cdot\rangle)$ sp. metrico

Sia $\{\underline{v}_1, \ldots, \underline{v}_n\}$ una base ortonormale di V. Sia $\underline{v} \in V$ con coordinate $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$ in $\{\underline{v}_1, \ldots, \underline{v}_n\}$. Allora

$$\|\underline{v}\|^2 = x_1^2 + \cdots + x_n^2$$

$(\mathbb{R}^2, \langle \cdot,\cdot \rangle_{st})$ $\{\underline{e}_1, \underline{e}_2\}$ base canonica



$\underline{v} = x_1 \underline{e}_1 + x_2 \underline{e}_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ coord. di $\underline{v}$

nella base $\{\underline{e}_1, \underline{e}_2\}$

$$\|\underline{v}\|^2 = x_1^2 + x_2^2$$

<u>Dim</u>: $\underline{v} = x_1 \underline{v}_1 + \dots + x_n \underline{v}_n$

$$\|\underline{v}\|^2 = \langle \underline{v}, \underline{v} \rangle = \langle x_1 \underline{v}_1 + \dots + x_n \underline{v}_n, \; x_1 \underline{v}_1 + \dots + x_n \underline{v}_n \rangle =$$

$$x_1 \langle \underline{v}_1, x_1 \underline{v}_1 + \dots + x_n \underline{v}_n \rangle + \dots + x_n \langle \underline{v}_n, \; x_1 \underline{v}_1 + \dots + x_n \underline{v}_n \rangle =$$

$$x_1^2 \underbrace{\langle \underline{v}_1, \underline{v}_1 \rangle}_{=\|\underline{v}_1\|^2 = 1} + \dots + x_1 x_n \underbrace{\langle \underline{v}_1, \underline{v}_n \rangle}_{=0} + \dots$$

$$\dots + x_n x_1 \underbrace{\langle \underline{v}_n, \underline{v}_1 \rangle}_{=0} + \dots + x_n^2 \underbrace{\langle \underline{v}_n, \underline{v}_n \rangle}_{=\|\underline{v}_n\|^2 = 1}$$

$$= x_1^2 + \dots + x_n^2 \qquad \blacksquare$$

TEOREMA: Ogni spazio metrico ammette basi ortonormali

PROCEDIMENTO DI ORTONORMALIZZAZIONE DI GRAM-SCHMIDT

$(V, \langle \cdot, \cdot \rangle)$

Sia $\{\underline{w}_1, \dots, \underline{w}_n\}$ base di V

1°] $\underline{u}_1 := \dfrac{\underline{w}_1}{\|\underline{w}_1\|} = \dfrac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1 \quad \left(\underline{w}_1 \neq 0 \Rightarrow \|\underline{w}_1\| \neq 0 \right)$

- $\operatorname{span}\{\underline{u}_1\} = \operatorname{span}\{\underline{w}_1\}$

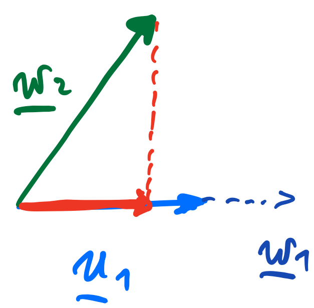
- $\|\underline{u}_1\|^2 = \langle \underline{u}_1, \underline{u}_1 \rangle = \langle \frac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1, \frac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1 \rangle$

$= \frac{1}{\|\underline{w}_1\|} \langle \underline{w}_1, \frac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1 \rangle =$

$\frac{1}{\|\underline{w}_1\|^2} \cdot \langle \underline{w}_1, \underline{w}_1 \rangle = \frac{1}{\|\underline{w}_1\|^2} \cdot \|\underline{w}_1\|^2 = 1$

$\Rightarrow \|\underline{u}_1\| = 1.$

2°]



$\underline{v}_2 := \underline{w}_2 - \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1$

- $\langle \underline{v}_2, \underline{u}_1 \rangle = \langle \underline{w}_2 - \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1, \underline{u}_1 \rangle$

$$= \langle \underline{w}_2, \underline{u}_1 \rangle - \langle \underline{w}_2, \underline{u}_1 \rangle \underbrace{\langle \underline{u}_1, \underline{u}_1 \rangle}_{\|\underline{u}_1\|^2 = 1}$$

$$= \langle \underline{w}_2, \underline{u}_1 \rangle - \langle \underline{w}_2, \underline{u}_1 \rangle = 0$$

$\underline{v}_2 \perp \underline{u}_1$

- $\underline{v}_2 \neq \underline{0}$ perchè altrimenti

$$\underline{0} = \underline{w}_2 - \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1$$

$$\Rightarrow \underline{w}_2 = \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1$$

$$\Rightarrow \text{span}\{\underline{w}_1, \underline{w}_2\} = \text{span}\{\underline{u}_1, \underline{w}_2\}$$

$$= \text{span}\{\underline{u}_1\}$$ ↯ perchè

$\underline{w}_1, \underline{w}_2$ sono lin. indip.

- $\text{span}\{\underline{u}_1, \underline{v}_2\} = \text{span}\{\underline{w}_1, \underline{w}_2\}$

$$\underline{u}_2 := \frac{1}{\|\underline{v}_2\|} \cdot \underline{v}_2$$

$\underline{u}_1 \perp \underline{u}_2$, span $\{\underline{w}_1, \underline{w}_2\} =$ span $\{\underline{u}_1, \underline{u}_2\}$

3°] $\underline{v}_3 := \underline{w}_3 - \langle \underline{w}_3, \underline{u}_1 \rangle \underline{u}_1 - \langle \underline{w}_3, \underline{u}_2 \rangle \underline{u}_2$

come prima $\underline{v}_3 \neq \underline{0}$, $\langle \underline{v}_3, \underline{u}_1 \rangle = 0$

$\langle \underline{v}_3, \underline{u}_2 \rangle = 0$

verifichiamo $\langle \underline{v}_3, \underline{u}_1 \rangle = 0$

$$\langle \underline{v}_3, \underline{u}_1 \rangle = \langle \underline{w}_3 - \langle \underline{w}_3, \underline{u}_1 \rangle \underline{u}_1 - \langle \underline{w}_3, \underline{u}_2 \rangle \underline{u}_2, \underline{u}_1 \rangle$$

$$= \langle \underline{w}_3, \underline{u}_1 \rangle - \langle \underline{w}_3, \underline{u}_1 \rangle \underbrace{\langle \underline{u}_1, \underline{u}_1 \rangle}_{= \|\underline{u}_1\|^2 = 1} +$$

$$- \langle \underline{w}_3, \underline{u}_2 \rangle \underbrace{\langle \underline{u}_2, \underline{u}_1 \rangle}_{= 0 \quad \underline{u}_1 \perp \underline{u}_2} = 0$$

$$\mathrm{span}\{\underline{w}_1, \underline{w}_2, \underline{w}_3\} = \mathrm{span}\{\underline{u}_1, \underline{u}_2, \underline{v}_3\}$$

$$\underline{u}_3 := \frac{1}{\|\underline{v}_3\|} \cdot \underline{v}_3$$

$$\mathrm{span}\{\underline{w}_1, \underline{w}_2, \underline{w}_3\} = \mathrm{span}\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$$

$$\underline{u}_1 \perp \underline{u}_2, \quad \underline{u}_1 \perp \underline{u}_3, \quad \underline{u}_2 \perp \underline{u}_3$$

$$\|\underline{u}_1\|^2 = \|\underline{u}_2\|^2 = \|\underline{u}_3\|^2 = 1$$

⋮ si itera il processo

$i°]$ $$\underline{v}_i := \underline{w}_i - \langle \underline{w}_i, \underline{u}_1 \rangle \underline{u}_1 + \dots$$

$$\dots - \langle \underline{w}_i, \underline{u}_{i-1} \rangle \underline{u}_{i-1}$$

$$\underline{u}_i := \frac{1}{\|\underline{v}_i\|} \cdot \underline{v}_i$$

OTTENIAMO:

$\{\underline{u}_1, \ldots, \underline{u}_n\}$ base ortonormale t.c.

$$\text{span}\{\underline{u}_1, \ldots, \underline{u}_j\} = \text{span}\{\underline{w}_1, \ldots, \underline{w}_j\}$$

$j = 1, \ldots, n$

Esempio: In $\mathbb{R}^3$, $\langle \cdot, \cdot \rangle_{st}$

$$\underline{w}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \underline{w}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \quad \underline{w}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

utilizzare GS (Gram-Schmidt) per determinare una base ortonormale.

$$\underline{u}_1 := \frac{1}{\|\underline{w}_1\|} \cdot \underline{w}_1$$

$$\|\underline{w}_1\|^2 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 1\cdot 1 + 0\cdot 0 + 1\cdot 1 = 2$$

$$\|\underline{w}_1\| = \sqrt{2}$$

$$\underline{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\underline{v}_2 = \underline{w}_2 - \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1$$

$$\langle \underline{w}_2, \underline{u}_1 \rangle = \left\langle \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\rangle = 0\cdot \frac{1}{\sqrt{2}} + (-1)\cdot 0 + 1\cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\underline{v}_2 = \underline{w}_2 - \langle \underline{w}_2, \underline{u}_1 \rangle \underline{u}_1 =$$

$$= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix}$$

$$\|\underline{v}_2\|^2 = \left\langle \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix} \right\rangle = \frac{1}{4} + 1 + \frac{1}{4} = \frac{3}{2}$$

$$\|\underline{v}_2\| = \sqrt{\frac{3}{2}}$$

$$\underline{u}_2 := \frac{1}{\|\underline{v}_2\|} \cdot \underline{v}_2 = \sqrt{\frac{2}{3}} \begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{pmatrix}$$

$$\underline{v}_3 := \underline{w}_3 - \langle \underline{w}_3, \underline{u}_1 \rangle \underline{u}_1 - \langle \underline{w}_3, \underline{u}_2 \rangle \underline{u}_2$$

$$\langle \underline{w}_3, \underline{u}_1 \rangle = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\rangle = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$\langle \underline{w_3}, \underline{u_2} \rangle = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{pmatrix} \right\rangle =$$

$$-\frac{1}{\sqrt{6}} - \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{\sqrt{6}} = -\frac{\sqrt{2}}{\sqrt{3}}$$

$$\underline{v_3} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} + \frac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} -1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/3 \\ -2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\underline{u_3} = \frac{1}{\|\underline{v_3}\|} \cdot \underline{v_3}$$

$$\|\underline{v_3}\|^2 = \left\langle \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix} \right\rangle = \frac{1}{3}$$

$$\|\underline{v_3}\| = \frac{1}{\sqrt{3}}$$

$$\underline{u_3} = \sqrt{3}\begin{pmatrix} -1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

la base ortonormale è:

$$\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{6} \\ -\sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\}$$