Somma e intersezione di sottospazi

V sp. vettoriale, $U, W \subset V$ sottospazi

<u>Def</u>:

intersezione

- $U \cap W := \{ \underline{v} \in V$ tali che $\underline{v} \in U$ e $\underline{v} \in W \}$

somma:

- $U + W := \{ \underline{v} \in V$ tali che esistono $\underline{u} \in U$, $\underline{w} \in W$ tali che $\underline{v} = \underline{u} + \underline{w} \}$

$= \{ \underline{u} + \underline{w}$ tali che $\underline{u} \in U, \underline{w} \in W \}$

<u>Oss</u>:

$U \subseteq U + W, \quad W \subseteq U + W$

perchè $\underline{0} \in W$ allora $\underline{u} \in U$

$$\underline{u} = \underline{u} + \underline{0} \in U + W$$

allo stesso modo $\underline{0} \in U$ allora $\underline{w} \in W$

$$\underline{w} = \underline{0} + \underline{w} \in U + W$$

Prop: $U \cap W$, $U + W$ sono sottospazi di V

Dim: $U \cap W$, $U + W$ sono chiusi rispetto alla somma e al prodotto per uno scalare.

- $U \cap W$ $\quad \underline{v}_1, \underline{v}_2 \in U \cap W \overset{?}{\Longrightarrow} \underline{v}_1 + \underline{v}_2 \in U \cap W$

$\underline{v}_1 \in U \cap W \Longrightarrow \underline{v}_1 \in U$ e $\underline{v}_1 \in W$

$\underline{v}_2 \in U \cap W \Longrightarrow \underline{v}_2 \in U$ e $\underline{v}_2 \in W$

d'altra parte U e W sono sottospazi e quindi chiusi rispetto alla somma pertanto

$$\underset{\underset{U}{\Uparrow}}{\underline{v}_1} + \underset{\underset{U}{\Uparrow}}{\underline{v}_2} \in U \qquad \underset{\underset{W}{\Uparrow}}{\underline{v}_1} + \underset{\underset{W}{\Uparrow}}{\underline{v}_2} \in W$$

$\Longrightarrow \underline{v}_1 + \underline{v}_2 \in U \cap W$.

- $\lambda \in \mathbb{R}$, $\underline{v} \in U \cap W \overset{?}{\Longrightarrow} \lambda \cdot \underline{v} \in U \cap W$

$\underline{v} \in U, \quad \underline{v} \in W$

U, W sono chiusi rispetto al prodotto per uno scalare:

$$\lambda \cdot \underline{v} \in U \quad , \quad \lambda \underline{v} \in W \implies \lambda \cdot \underline{v} \in U \cap W.$$

- $\underline{v_1}, \underline{v_2} \in U + W \overset{?}{\implies} \underline{v_1} + \underline{v_2} \in U + W$

$\underline{v_1} \in U + W \implies$ esistono $\underline{u} \in U$, $\underline{w} \in W$ tali che $\underline{v_1} = \underline{u} + \underline{w}$

$\underline{v_2} \in U + W \implies$ esistono $\underline{\tilde{u}} \in U$, $\underline{\tilde{w}} \in W$ tali che $\underline{v_2} = \underline{\tilde{u}} + \underline{\tilde{w}}$

$$\underline{v_1} + \underline{v_2} = (\underline{u} + \underline{w}) + (\underline{\tilde{u}} + \underline{\tilde{w}}) \overset{\text{somma associativa commutativa}}{=}$$

$$\underbrace{(\underline{u} + \underline{\tilde{u}})}_{\in U} + \underbrace{(\underline{w} + \underline{\tilde{w}})}_{\in W} \implies \underline{v_1} + \underline{v_2} \in U + W$$

- **Esercizio**: dimostrare che se $\underline{v} \in U+W$, $\lambda \in \mathbb{R} \Rightarrow \lambda \cdot \underline{v} \in U+W$.

OSS: $U \cup W$ **non** è (in genere) un sottospazio.

Es: $V = \mathbb{R}^2$ $\quad U = \text{span}\{\underline{e}_1\} =$

$$= \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix}, \; x \in \mathbb{R} \right\}$$

$$W = \text{span}\{\underline{e}_2\} = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix}, \; y \in \mathbb{R} \right\}$$

$$U \cup W = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \text{ o } \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad x, y \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in U \cup W \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in U \cup W$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U \cup W$$

Teorema di Grassmann Se $U, W \leq V$ sottosp.

$$\boxed{\dim U + \dim W = \dim(U+W) + \dim(U \cap W)}$$

Esempio: $V = \mathbb{R}^3$

$$U = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad W = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\dim U = 2$

$\dim W = 2$

$$U + W = \operatorname{span}\Big\{ \underbrace{\overbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{\text{(base) generatori di } U}, \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}^{\text{(base) generatori di } W}}_{\text{generatori di } U+W} \Big\}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ sono lin. indip.}$$

$$e \quad \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} =$$

$$= \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\} = \mathbb{R}^3$$

$$\Rightarrow \dim(U+W) = 3$$

$$\dim(U \cap W)$$

$$U \cap W = \left\{ \begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \text{ tali che } \begin{pmatrix}x\\y\\z\end{pmatrix} \in U \quad e \quad \begin{pmatrix}x\\y\\z\end{pmatrix} \in W \right\}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \in \underset{\text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}}{\underset{\parallel}{U}} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \in \underset{\text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}}{\underset{\parallel}{W}} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \mu_1\begin{pmatrix}0\\1\\0\end{pmatrix} + \mu_2\begin{pmatrix}0\\0\\1\end{pmatrix}$$

allora

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mu_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\Longrightarrow$

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mu_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\Longrightarrow \lambda_1 = 0, \quad \lambda_2 = \mu_1, \quad \mu_2 = 0$$

ovvero $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underset{\overset{\shortparallel}{0}}{\lambda_1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

ovvero $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U \cap W \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda \\ 0 \end{pmatrix}$ $\lambda \in \mathbb{R}$

$$= \lambda \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

cioè $\quad U \cap W = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

$\dim(U \cap W) = 1$

Formula di Grassmann:

$$\underbrace{\dim U}_{=2} + \underbrace{\dim W}_{=2} = \underbrace{\dim(U+W)}_{=3} + \underbrace{\dim(U \cap W)}_{=1}$$

<u>Dim</u> (teo di Grassmann):

Sia $\{\underline{v}_1, \dots, \underline{v}_m\}$ base di $U \cap W$ $(\Leftrightarrow \dim U \cap W = m)$

$U \cap W \subseteq U \xRightarrow[\text{teo del completam. della base}]{}$ esistono $\underline{u}_1, \dots, \underline{u}_k \in U$

tali che $\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k\}$ sono

una base di U $\quad (\dim U = m+k)$

$U \cap W \subseteq W \xRightarrow[\text{come sopra}]{}$ esistono $\underline{w}_1, \ldots, \underline{w}_n \in W$

tali che $\{\underline{v}_1, \ldots, \underline{v}_m, \underline{w}_1, \ldots, \underline{w}_n\}$ sono

una base di W $\Rightarrow$ $\dim W = m+n$

$\boxed{\text{\underline{Affermazione}: } \{\underline{v}_1, \ldots, \underline{v}_m, \underline{u}_1, \ldots, \underline{u}_k, \underline{w}_1, \ldots, \underline{w}_n\} \text{ è una base di } U+W}$

da questa affermazione discende il teorema:

$$\dim(U+W) \underset{\text{affermazione}}{=} m+k+n$$

$$\dim(U \cap W) = m$$

$$\dim U = m+k$$

$$\dim W = m+n$$

$$\underbrace{\dim U}_{m+k} + \underbrace{\dim W}_{m+n} = \underbrace{\dim(V+W)}_{m+k+n} + \underbrace{\dim(U\cap V)}_{m}$$

Dim (affermazione):

- $U+W = \operatorname{span}\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k, \underline{w}_1, \dots, \underline{w}_n\}$

ogni $\underline{u}$ è combinazione lineare di

$\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k\}$

ogni $\underline{w}$ è combinazione lineare di

$\{\underline{v}_1, \dots, \underline{v}_m, \underline{w}_1, \dots, \underline{w}_n\}$

$\Rightarrow$ $\underline{u}+\underline{w}$ è combin. lineare di

$\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k, \cancel{\underline{v}_1, \dots, \underline{v}_m}, \underline{w}_1, \dots, \underline{w}_n\}$

- $\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k, \underline{w}_1, \dots, \underline{w}_n\}$ sono linearmente indipendenti:

$$\lambda_1 \underline{v_1} + \dots + \lambda_m \underline{v_m} + \mu_1 \underline{u_1} + \dots + \mu_k \underline{u_k} + \theta_1 \underline{w_1} + \dots + \theta_n \underline{w_n} = \underline{0}$$

$\Longrightarrow$

$$(A) \quad \underbrace{\theta_1 \underline{w_1} + \dots + \theta_n \underline{w_n}}_{\in W} = -\underbrace{\left(\lambda_1 \underline{v_1} + \dots + \lambda_m \underline{v_m} + \mu_1 \underline{u_1} + \dots + \mu_k \underline{u_k}\right)}_{\in U}$$

$\Longrightarrow \quad \theta_1 \underline{w_1} + \dots + \theta_n \underline{w_n} \in U \cap W$

$\{\underline{v_1}, \dots, \underline{v_m}\}$ base di $U \cap W$

$\Longrightarrow$

$$\theta_1 \underline{w_1} + \dots + \theta_n \underline{w_n} = \nu_1 \underline{v_1} + \dots + \nu_m \underline{v_m}$$

$\Longrightarrow$

$$\nu_1 \underline{v_1} + \dots + \nu_m \underline{v_m} - \theta_1 \underline{w_1} + \dots - \theta_n \underline{w_n} = \underline{0}$$

però $\{\underline{v_1}, \dots, \underline{v_m}, \underline{w_1}, \dots, \underline{w_n}\}$ sono una base di W $\Longrightarrow$ lin. indip.

$\Longrightarrow \nu_1 = \dots = \nu_m = \theta_1 = \dots = \theta_n = 0.$

sostituendo in (A)

$$\underline{0} = -\left(\lambda_1 \underline{v}_1 + \dots + \lambda_m \underline{v}_m + \mu_1 \underline{u}_1 + \dots + \mu_k \underline{u}_k\right)$$

$\{\underline{v}_1, \dots, \underline{v}_m, \underline{u}_1, \dots, \underline{u}_k\}$ base di U

$\Longrightarrow$ lin. indip. $\Rightarrow \lambda_1 = \dots = \lambda_m = 0$

$\mu_1 = \dots = \mu_k = 0.$ $\square$

DEF:

Siano $U, W \subseteq V$ sottospazi.

Diciamo che V è somma diretta di U e W, e scriviamo

$$\boxed{V = U \oplus W}$$

se:

1) $V = U + W$

2) $U \cap W = \{\underline{0}\}$ (ovvero $\dim(U \cap W) = 0$)

OSS: dalla formula di Grassmann:

$$\dim U + \dim W = \underbrace{\dim(U+W)}_{= \dim V} + \underbrace{\dim(U \cap W)}_{= 0}$$

$$\Rightarrow \dim V = \dim U + \dim W$$

DEF: $U_1, \dots, U_m \subseteq V$

$$V = U_1 \oplus \cdots \oplus U_m$$

se:

1) $U_1 + \dots + U_m = V$

2) $U_j \cap U_k = \{\underline{0}\}$ $\quad j, k \in \{1, \dots, m\}$, $j \neq k$

$$U_1 + \dots + U_m = \{ \underline{u}_1 + \dots + \underline{u}_m, \quad \underline{u}_j \in U_j \}$$

$$= U_1 + \underbrace{(U_2 + \dots + U_m)}$$

$$= U_2 + (\underbrace{U_3 + \dots + U_m})$$

$$= U_{m-1} + U_m$$

<u>esempio:</u>

$$U_1 + U_2 + U_3 = U_1 + (U_2 + U_3)$$

<u>Prop</u>: $V = U \oplus W$ allora ogni elemento $\underline{v}$ di V si scrive in modo unico come

$$\underline{v} = \underline{u} + \underline{w}$$

<u>Dim</u>: $V = U \oplus W \Longrightarrow V = U + W$ quindi dato $\underline{v} \in V$ esistono $\underline{u} \in U$, $\underline{w} \in W$

tali che $\underline{v} = \underline{u} + \underline{w}$

supponiamo che $\underline{v} = \tilde{\underline{u}} + \tilde{\underline{w}}$ allora

$$\underline{u} + \underline{w} = \underline{v} = \tilde{\underline{u}} + \tilde{\underline{w}}$$

$$\underline{u} + \underline{w} = \tilde{\underline{u}} + \tilde{\underline{w}}$$

$$\underbrace{\underline{u} - \tilde{\underline{u}}}_{U} = \underbrace{\tilde{\underline{w}} - \underline{w}}_{W}$$

perché $V = U \oplus W$

$$\Rightarrow \quad \underline{u} - \tilde{\underline{u}} \in U \cap W = \{\underline{0}\}$$
$$\tilde{\underline{w}} - \underline{w} \in U \cap W = \{\underline{0}\}$$

$$\Rightarrow \quad \underline{u} - \tilde{\underline{u}} = \underline{0}, \quad \tilde{\underline{w}} - \underline{w} = \underline{0}$$

$$\Rightarrow \quad \underline{u} = \tilde{\underline{u}}, \quad \tilde{\underline{w}} = \underline{w} \qquad \blacksquare$$

OSS: se $V = U_1 \oplus \dots \oplus U_m$ allora ogni vettore di V si scrive in modo unico come somma di vettori di $U_1, \dots, U_m$.

Esempio:

$$\mathbb{R}^3 = \text{span}\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$$

$$U_1 = \text{span}\{\underline{e}_1\}$$

$$U_2 = \text{span}\{\underline{e}_2\}$$

$$U_3 = \text{span}\{\underline{e}_3\}$$

Allora $\mathbb{R}^3 = U_1 \oplus U_2 \oplus U_3$

$U = \text{span}\{\underline{e}_1, \underline{e}_2\}$ $\quad W = \text{span}\{\underline{e}_3\}$

$$\mathbb{R}^3 = U \oplus W$$

Esempio: $\mathbb{R}^4$

$$U = \text{span}\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\end{pmatrix} \right\} \Rightarrow \dim U \leq 3$$

$$W = \text{span}\left\{ \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\} \Rightarrow \dim W \leq 2$$

determinare una base di $U+W$ e di $U \cap W$ e dire se $\mathbb{R}^4 = U \oplus W$

Un sistema di generatori di $U+W$ è dato

$$\left\{ \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix} \right\}$$

estraiamo una base:

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$R_4 \longrightarrow R_4 - R_1$

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & -2 & -2 & 2 & -1 \\ 0 & -1 & -1 & 1 & -1 \\ 0 & -1 & -1 & 2 & 0 \end{pmatrix}$$

$R_3 \longrightarrow R_3 - \frac{1}{2} R_2$

$R_4 \longrightarrow R_4 - \frac{1}{2} R_2$

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & -2 & -2 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{pmatrix}$$

$R_3 \longleftrightarrow R_4$

$$\begin{pmatrix} 1 & 1 & 2 & -1 & 1 \\ 0 & -2 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & -1/2 \end{pmatrix}$$

quindi una base di $V+W$ è

$$\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$$

$$\Rightarrow \dim(V+W) = 4 = \dim \mathbb{R}^4$$

$$U = \text{span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\\1\end{pmatrix}\right\}$$

per trovare una base

$$\begin{pmatrix}1 & 1 & 2\\1 & -1 & 0\\1 & 0 & 1\\1 & 0 & 1\end{pmatrix} \quad \text{EG} \downarrow$$

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\\0\end{pmatrix} \quad \text{base di } U$$

Formula di Grassmann:

$$\underbrace{\underbrace{\dim U}_{=2} + \underbrace{\dim W}_{\leq 2}}_{\leq 4} = \underbrace{\underbrace{\dim(U+W)}_{=4} + \underbrace{\dim(U\cap W)}_{\geq 0}}_{\geq 4}$$

$$\Rightarrow \quad = \qquad\qquad =$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\dim W = 2 \qquad\qquad \dim(U\cap W) = 0$$

$$\Rightarrow \quad \mathbb{R}^4 = U \oplus W$$