Unit - II

Elementary Combinatories

Basics of counting

If S is a set, let us use length of S ($|S|$) to denote the number of elements in S. There are two elementary principles act as building blocks for counting problems. The two principles are:-

(i) Sum rule

(ii) Product rule

Sum rule (Principle of disjenctive counting):-

If a set S is the union of disjoint non-empty subsets $S_1, S_2, \ldots, S_n$, then

$$|S| = |S_1| + |S_2| + |S_3| + \ldots + |S_n|$$

is called sum rule

In another words, According to the events if $E_1, E_2, E_3, \ldots E_n$ are mutually exclusive events and E_1 can happen e_1 ways, E_2 can happen e_2 ways ... E_n can happen e_n ways. Then E_1 or E_2 or ... or E_n can happen $e_1 + e_2 + e_3 + \ldots + e_n$ is called sum rule

Product rule (Principle of sequential counting):-

If $S_1, S_2, \ldots, S_n$ are non-empty sets then the number of elements in the cartesian product i.e

$$|S| = |S_1| \times |S_2| \times |S_3| \times \ldots \times |S_n|$$

is called Product rule

Ex-1
It can be defined according to the events if $E_1, E_2, \ldots, E_n$ are mutually exclusive events and E_1 can happen in e_1 ways, E_2 can happen in e_2 ways ... E_n can happen in e_n ways. Then E_1 and E_2 and ... E_n can happen in $e_1 \times e_2 \times \ldots \times e_n$

In how ways can be drawn from an ordinary deck of playing cards

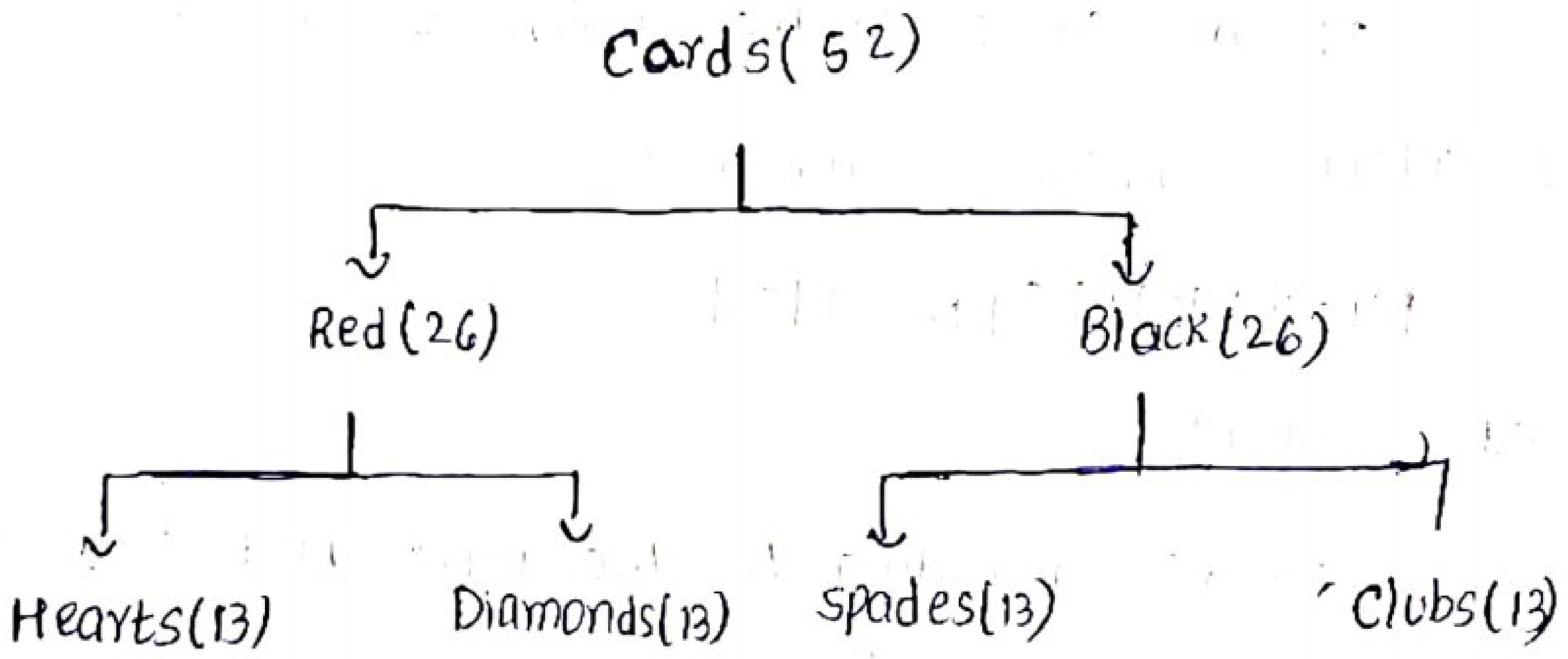
a) A heart or spade

b) A heart or an ace

c) An ace or a King

d) A card number 2 through 10

e) A number card or a King.



Line-up:- King, Queen, Jack, Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10

a) Since there are 13 hearts and 13 spades. Now we draw heart or spade is $13+13 = 26$

b) Now we may draw a heart (or) ace is $13+3 = 16$

(b) Since there are only 3 aces which are not hearts

c) Now we may draw an ace (or) King is $4+4=8$

d) 36

e) $36+4 = 40$

Ex-2

How manys ways can we get a (a) sum of 4 or eight when two distingushable dice are rolled

(b) an even sum when two distinguishable dice are rolled

Sol-

(a) To get the sum of 4, the ordered pairs are (1,3),(3,1) (2,2)

To get the sum of 8, the ordered pairs are (2,6),(6,2), (3,5), (5,3), (4,4)

∴ Total no. of ways = 8

(b) To get an even sum, the ordered pairs are:-

(2,2), (3,3), (1,1), (4,4), (5,5), (6,6)

(2,4), (3,5), (1,3), (4,2), (5,1), (6,2)

(2,6), (3,1), (1,5), (4,6), (5,3), (6,4)

∴ Total no. of ways = 18

(c) To get an odd sum, the ordered pairs are:

(1,2), (2,1), (1,4), (4,1), (1,6), (6,1)

(2,3), (2,5), (5,2), (3,2)

(3,4), (4,3), (3,6), (6,3)

(4,5), (5,4), (5,6), (6,5)

∴ Total no. of ways = 18

Permutations:-

Permutation relates to the act of arranging all the members of a set into some sequence or order. In other words, if the set is already ordered then the rearranging of its elements is called the process of permuting.

Permutation Formula :-

* A permutation is the choice of r things from a set of n things without replacement & where the order matters.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

where, n = total no. of objects

r = no. of selected objects

Ex: ① Find the no. of words, with or without meaning that can be formed with the letters of the word "chair"

Sol: CHAIR contains 5 letters. Therefore, the no. of words that can be formed with these 5 letters is $5! = 120$ (or)

Here $n = 5$, $r = 5$

No. of words $= {}^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{5!}{0!} = 120$

② Find the no. of words, with or without meaning that can be formed with the letters of the word "INDIA"

Sol: INDIA contains 5 letters. Therefore, the no. of words that can be formed is $5!$. But letter 'I' repeated twice. So no. of words $= \frac{5!}{2!} = 60$

(3) Find the no. of words with or without meaning that can be formed with the letters of the word "SWIMMING"

Sol: No. of words $= \frac{8!}{2!\,2!} = \frac{\overset{2}{\cancel{8}} \times 7!}{\cancel{4}} = 2 \times 5040 = 10080$

(4) How many different words can be formed with the letters of the word "Super" such that the vowels always come together.

Sol:- Since vowels come together all the vowels in the word super (U, E) are considered as single unit. Total no. of letters are 4 which can be arranged in 4! ways and the two ~~lett~~ vowels can internally be arranged in 2! ways

$\therefore$ Total no. of words $= 4!\,2! = 48$

(5) Find the no. of different words that can be formed with the letters of the word "BUTTER" so that the vowels are always come together.

Sol:- Vowels (u, e) are considered as single unit and can internally arranged in 2! ways. Total no. of letters are 5 with Two T's repeated

$\therefore$ Total no. of words $= \frac{5!\,2!}{2!} = 5! = 120$

(6) Find the no. of permutations of the word "REMAINS" such that the vowels always in odd places.

Sol:

E A I $\rightarrow {}^4P_3$

1 2 3 4 5 6 7

R M N S $\rightarrow {}^4P_4$

No. of Permutations $= {}^4P_3 \cdot {}^4P_4 = 24 \times 24$

$= 576$

7. How many ways the word PLAYGROUND arranged so that

(i) The word start with letter Y

(ii) The word start with Y & ends with G

(iii) The word always end with vowel

(iv) The word start with vowel & ends with consonant

(v) Vowels occupy odd positions

Sol: i) Y _ _ _ _ _ _ _ _ _

$9! \cdot 1! = 362880$

(ii) Y _ _ _ _ _ _ _ _ G

$1! \cdot 1! \cdot 8! = 40320$

(iii) _ _ _ _ _ _ _ _ _ _

$9! \cdot {}^3P_1 = 1088640$

(iv) _ _ _ _ _ _ _ _ _ _

${}^3P_1 \cdot {}^7P_1 \cdot 8! = 846720$

(v) _ _ _ _ _ _ _ _ _ _

${}^5P_3 \cdot 7! = 302400$

Combination:-

The combination is a way of selecting items from a collection such that the order of selection does not matter. In smaller cases it is possible to count the no. of combinations. Combination refers to the combination of n things taken k at a time without repetition.

Combination Formula:-

A combination is the choice of r things from a set of n things without replacement and where order does not matter.

$$^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{^nP_r}{r!}$$

where, r: no. of selected objects;

n: total no. of objects

Example:- In a comittee total 8 students, out of 8 students, 5 students are AI & ML and 3 students are CSE

(i) A commitee of 4 where 3 are AI & ML & 1 is CSE

(ii) A commitee of 5 where 3 are AI&ML & 2

(iii) A commitee of two there are no AI&ML students.

(iv) A commitee of 3 in which there are no CSE students

(v) A comitte of 2 in which either both are AI&ML and both are CSE

(vi) A commitee of 3 in which atleast one AI&ML is present

(i) $5c_3 \cdot 3c_1 = 10\times3 = 30$ ways , (ii) $5c_3 \cdot 3c_2 = 10\times3 = 30$ ways

(iii) $3c_2 = 3$ ways (iv) $5c_3 = 10$ ways (v) $5C_2 + 3C_2 = 10+3 =$ 13 ways

(vi) $5c_1 \cdot 3c_2 + 5c_2 \cdot 3c_1 + 5c_3 = 15 + 30 + 10 = 55$ ways

~~(vi) $3c_1+3c_2+3c_3 = 3+3+1 = 7$ ways~~

→ A commitee 5 members is to be formed out of 3 Assistant Proffesors, 4 Associate professors and 6 professors. In how many ways

(i) The commitee should have 4 associate professors and one Professor or 3 assistant professors & 2 associate proffers

(ii) The commitee should have 2 assistant profers & 3 professors

(iii) The commitee should have 2 assistant professors.

(iv) A commitee should not contain any associate professors.

Sol:- (i) $4C_4 \cdot 6C_1 = 1 \times 6 = 6$

(or)

$3C_3 \cdot 4C_2 = 1 \times 6 = 6$

(ii) $3C_2 \cdot 6C_3 = 3 \times 20 = 60$

(iii) $3C_2 \cdot 10C_3$
$= 3 \times 120$
$= 360$

(iv) $9C_5 = 126$ ways

→ Among a set of 5 black balls, 3 red balls. How many selections of 5 balls can be made such that atleast 3 of them are black balls.

Sol:- $5C_3 \cdot 3C_2 =$

→ In a group we have 10 men & 8 women. Out of 10 mens, 5 men are RVR, 3 men are KHIT and 2 are CHIPS. Out of 8 womens 3 are RVR, 2 are KHIT, 2 are KITS and 1 is VVIT.

i) A group of 5 in which 3 are men and 2 are women

ii) A group of 4 in which atleast 2 womens are there.

iii) A group of 2 members

iv) A group of 3 in which there is no RVR & KHIT.

-> A Q.P has 2 parts. Part A & Part B each containing 10 ~~Questions~~ Questions. If the students has to choose 8 questions from Part A and 5 questions from part B. In how many ways we can choose the questions.

-> How many 4 digit numbers that are divisible by 10 can be formed from the numbers 3, 5, 7, 8, 9, 0 such that no number repeats

Sol:- Number should end with zero (∵ divisible by 10)

_ _ _ 0

First place:- Should not start with 0, So we have 5 numbers to choose from

Second place:- Can be any digit => 6 ways

Third place:- Can be any digit => 6 ways

Fourth place:- should be zero => 1 way

Total $= 5 \times 6 \times 6 \times 1 = 180$ ways

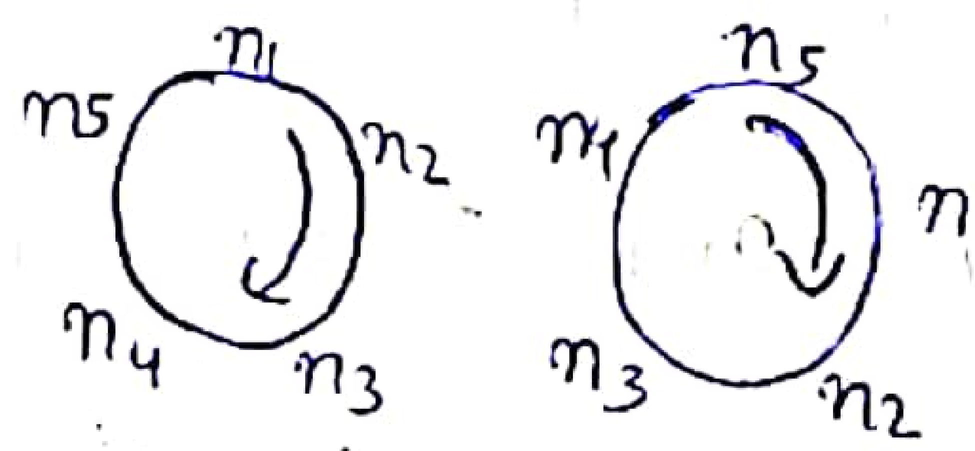
Enumeration of combinations and Permutations

1) Enumerating r permutations without repetition is $P(n,r) = {}^{n}P_{r} = \frac{n!}{(n-r)!}$

when $n=r$, $P(n,n) = n!$

There are $n!$ Permutations of n distinct objects arranged in linear

2) There are $(n-1)!$ Permutations of n distinct objects arranged in a circle or wheel.



3) Enumerating r combinations without repetition is $C(n,r) = \frac{n!}{(n-r)!\,r!} = {}^{n}C_{r}$

Problems

1. In how manys can 10 People arrange themselves (a) in (a) row of 10 chairs. :- $10!$ or ${}^{10}P_{10}$

b) In a row of 7 chairs :- ${}^{10}P_{7} = \frac{10!}{7!} = 10\times9\times8 = 720$

(c) In a circle of 10 chairs :- $(10-1)! = 9!$

2. (i) How many ways are there to seat 10 boys and 10 girls around a circular table. :- $(20-1)! = 19!$

(ii) If boys and girls alternate How many ways are there :- $10!\;9!$ ~~9!~~

3. A collection of 100 bulbs contains 8 defective ones

i) in how many ways can a sample of 10 bulbs to be selected

ii) in how many ways can a sample of 10 bulbs to be selected which contain 6 good bulbs & 4 defective bulbs:

iii) In how many can sample of 10 bulbs to be selected so that either the sample contains 6 good & 4 defective or 5 good & 5 defective

i) $n = 100$

$\Rightarrow {}^{100}C_{10} = \dfrac{100!}{10! \times 90!}$

(ii) ${}^{92}C_6 \times {}^8C_4$

(iii) ${}^{92}C_6 \times {}^8C_4 + {}^{92}C_5 \times {}^8C_5$

4. There are 30 females, 35 males in the junior class while there are 25 females, 20 males in the senior class. In how many ways can a commite of 10 to be choosen such that there are exactly 5 females and 3 juniors on the commitee.

Sol:- ${}^{45}C_5$

Case 1:

3 juniors including 3 females & 7 seniors (2f & 5m)

No. of ways $= ({}^{30}C_3)({}^{25}C_2 \cdot {}^{20}C_5)$

2: 3 Juniors includes 2 females, 1 male & 7 seniors (3f, 4m)

No. of ways $= ({}^{30}C_2 \cdot {}^{35}C_1)({}^{25}C_3 \cdot {}^{20}C_4)$

3: 3 Juniors (1f, 2m) & 7 Seniors (4f, 3m)

No. of ways $= ({}^{30}C_1 \cdot {}^{35}C_2)({}^{25}C_4 \cdot {}^{20}C_3)$

4:- 3 Juniors (3m) & 7 Seniors (5f, 2m)

No. of ways $= ({}^{35}C_3)({}^{25}C_5 \cdot {}^{20}C_2)$

Total $= {}^{30}C_3 \cdot {}^{25}C_2 \cdot {}^{20}C_5 + {}^{30}C_2 \cdot {}^{35}C_1 \cdot {}^{25}C_3 \cdot {}^{20}C_4 + {}^{30}C_1 \cdot {}^{35}C_2 \cdot {}^{25}C_4 \cdot {}^{20}C_3$
$+ {}^{35}C_3 \cdot {}^{25}C_3 \cdot {}^{20}C_2$

5. (i) In how many a comitlee of 5 to be chosen from 9 people,

(ii) How many comitlee of 5 or more can be choosen from 9 people

(iii) In how many ways can a comitlee of 5 teachers & 4 students can be choosen from 9 teachers & 15 students

(iv) How many ways can a comitle in (iii)

(i) $^9C_5$ $= 126$

(ii) $^9C_5 + {^9C_6} + {^9C_7} + {^9C_8} + {^9C_9}$ $= 126 + 84 + 36 + 9 + 1 = 256$

(iii) $^9C_5 \cdot {^{15}C_4}$ $= 126 \cdot 1365 = 171990$

(iv) How many ways can a committe in (iii) to be fomed if A teacher refuses to serve students $\cancel{^8C_5 \cdot {^{15}C_4}} + {^9C_5} \cdot {^{15}C_4} - {^8C_4} \cdot {^{14}C_3}$

6. There are 21 consononts, 5 vowels in the english alphabet. Consider only 8 letter words with 3 different vowels and 5 different consonants.

(i) How mony such words can be fomed :- $^{21}C_5 \cdot {^5C_3}$

(ii) How mony such words that contain letter 'A' :- $^4C_2 \cdot {^{21}C_5}$

(iii) How mony such words contains the letters 'A', 'B', 'C' :-

(iv) How mony begin with 'A' and end with 'B'

(v) How many begin with 'B' and end with 'C'

7. There are 50 distinguishable books including 18 english books, 17 french books and 15 spanish books

(i) How many ways can two books to be selected

(ii) How many ways can three books to be selected so that there is one book from each of 3 languages

(iii) How many ways to select 3 books where exactly one language is missing.

Enumerating combinations and permutations with repetitions

Let $U(n,r)$ denote the no. of r permutations of n objects with unlimited repetitions and let $V(n,r)$ denote the number of r combinations of n objects with unlimited repetitions.

i.e, if $a_1, a_2, a_3, \ldots, a_n$ are the n objects, we are counting r combinations and r permutations of $\{\infty.a_1, \infty.a_2, \ldots, \infty.a_n\}$

Enumerating r-permutations with unlimited repetitions is $U(n,r) = n^r$

Enumerating r-combinations with unlimited repetitions is ~~V(n,r)~~ =

$V(n,r)$ = The no. of r combinations of n distinct objects with unlimited repetitions

= The no. of non-negative integral solutions to $x_1 + x_2 + \ldots + x_n = r$ where $x_i \geq 0$

= the no. of ways of distributing r similar balls into n numbered boxes

= The no. of binary numbers with $(n-1)$ one's and r zero's

$= {}^{(n-1+r)}C_r = {}^{(n-1+r)}C_{n-1} = \dfrac{(n-1+r)!}{r!\,(n-1)!}$

Problems

→ They are 25 true or false questions on an examination. How many different ways can a student do the examination if he/she can also choose to leave the answer black.

Sol:- Each of the 3 positions (T, F, -) can be filled. We apply n permutations with unlimited repetition of 'r' objects is

$$U(n,r) = n^r$$

Here, $n = 3$ (T, F, -) and $r = 25$ $\therefore$ Total no. of ways $= 3^{25}$

→ The results of 50 football games (win, lose, tie) are to be predicted.
How many different forecasts can contain exactly 28 correct results.

Sol:- $n = 50,\ r = 28$

Total $= {}^{50}C_{28} \cdot 2^{22}$

→ The no. of 4 combinations of $\{\infty a_1, \infty a_2, \infty a_3, \infty a_4, \infty a_5\}$

Sol: $n = 5,\ r = 4$

$V(n,r) = {}^{(n+r-1)}C_r = {}^{5-1+4}C_4 = {}^{8}C_4 = 70$

→ The no. of 3 combinations of 5 objects with unlimited repetitions is

Sol: $n = 5,\ r = 3$

$V(n,r) = {}^{(n-1+r)}C_r = {}^{(3-1+5)}C_3 = {}^{7}C_3 = {}^{7}C_3 = 35$ $\frac{7 \times 6 \times 5!}{5! \times 2} = 21$

→ The no. of non-ve integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 50$

Sol: $r = 50,\ n = 5$

$V(n,r) = {}^{(n-1+r)}C_r = {}^{(5-1+50)}C_r = {}^{54}C_{50} = {}^{54}C_4 = 316251$

→ The no. of ways of placing 10 similar balls in 6 numbered boxes is

Sol: $n = 6,\ r = 10$

$V(n,r) = {}^{(n-1+r)}C_r = {}^{15}C_{10} = {}^{15}C_5 = 3003$

→ The no. of binary nos with 10 one's & 5 zero's

Sol: $n = 11,\ r = 5$

$V(n,r) = {}^{15}C_5 = 3003$

→ How many integral solutions $x_1+x_2+x_3+x_4+x_5=20$ where each $x_i \geq 2$

Sol:- $r=10,\ n=5$

$v(n,r) = {}^{(n-1+r)}C_r = \ldots = {}^{14}C_{10}$

$= {}^{14}C_4 = 1001$

→ How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where each $x_1 \geq 3,\ x_2 \geq 2,\ x_3 \geq 4,\ x_4 \geq 6,\ x_5 \geq 0$

Sol:- $3+2+4+6+0 = 15$

Total $= 20$

$= 20-15 \Rightarrow 5$

$n=5,\ r=5$

$v(n,r) = {}^{(n-1+r)}C_r = {}^{(5-1+5)}C_5 = {}^{9}C_5 = 126$

→ How many integral sol'ns are there to $x_1+x_2+x_3+x_4+x_5=20$ where $x_1 \geq -3,\ x_2 \geq 0,\ x_3 \geq 4,\ x_4 \geq 2$ and $x_5 \geq 2$

Sol:- Here, we interpret placing -3 balls in box 1 as actually increasing total no. of balls from 20 to 23. Then placing 4 balls in box 3, 2 balls in box 4, 2 balls in box 5

$23-(4+4) = 23-8 = 15$

$r=15,\ n=5$; $v(n,r) = {}^{(5-1+15)}C_{15} = {}^{19}C_{15} = 3876$

→ How many integral sol'ns are there to $x_1+x_2+x_3+x_4+x_5=30$, where each i

i) $x_i \geq 0$

ii) $x_i \geq 1$

(iii) $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$

(iv) $x_i \geq i$

Sol: i) $V(n,r) = {}^{(5-1+30)}C_{30} = {}^{34}C_{30} = {}^{34}C_4$

(ii) $V(n,r) = {}^{(5-1+25)}C_{25} = {}^{29}C_{25} = {}^{29}C_4$

(iii) $V(n,r) = {}^{(5-1+19)}C_{19} = {}^{23}C_{19} = {}^{23}C_4$

(iv) $V(n,r) = {}^{(5-1+15)}C_{15} = {}^{19}C_{15} = {}^{19}C_4$

→ Enumerate no. of non-ve integral solutions to

$x_1+x_2+x_3+x_4+x_5 \leq 19$

Sol: To count the no. of non -ve integral solutions to the $x_1+x_2+x_3+x_4+x_5 = K$, where K can be any integer from 0 to 19.

∴ We count the no. of integral solutions in this approach where $r = 0$ to 19 and $n=5$

$$\sum_{r=0}^{19} {}^{n-1+r}C_{n-1}$$

$$r = 0 \Rightarrow = {}^{4}C_4 + {}^{5}C_4 + {}^{6}C_4 + {}^{7}C_4 + {}^{8}C_4 + {}^{9}C_4 + {}^{10}C_4 + {}^{11}C_4 + {}^{12}C_4$$

$$+ {}^{13}C_4 + {}^{14}C_4 + {}^{15}C_4 + {}^{16}C_4 + {}^{17}C_4 + {}^{18}C_4 + {}^{19}C_4 + {}^{20}C_4 + {}^{21}C_4 + {}^{22}C_4 + {}^{23}C_4$$

$$= 1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + 1001$$

$$+ 1365 + 1820 + 2380 + 3060 + 3876 + 4845 + 5985 +$$

$$7315 + 8855$$

$$= 42504$$

→ If K is some integer b/w 0 to 19 then for every distribution of K balls into 5 boxes. One could distribute the remaining (19-K) balls into a 6th box. Hence the no. of non negative integral solutions of $x_1+x_2+x_3+x_4+x_5 \le 19$ is same as the no. of non negative integral solutions of

$y_1+y_2+y_3+y_4+y_5+y_6 = 19$

Sol: $n=6,\ r=19$

$\Rightarrow \quad {}^{(n-1+r)}C_r = {}^{24}C_{19} = 42504$

→ Find the no. of distinct triples (x_1, x_2, x_3) of non -ve integer satisfying the inequality $x_1+x_2+x_3 < 6$

Sol:- $r =$ 0 to 5 , $n = 3$

$\sum_{r=0}^{5} {}^{(n-1+r)}C_{n-1} = {}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + {}^{5}C_2 + {}^{6}C_2 + {}^{7}C_2$

$= 1 + 3 + 6 + 10 + 15 + 21$

$= 56$

→ Find the no. of soln's to $x_1+x_2+x_3+x_4 = 50$ where $x_1 \ge -4$, $x_2 \ge 7$, $x_3 \ge -14$, $x_4 \ge 10$

Enumerating n-permutations with constrained repetitions:-

$$P(n; q_1, q_2, q_3, \ldots q_t)$$

i.e To count the n permutations of $\{q_1 \cdot a_1, q_2 \cdot a_2, \ldots, q_t \cdot a_t\}$

$$\therefore P(n; q_1, q_2, \ldots, q_t) = \frac{n!}{q_1!\, q_2! \ldots q_t!}$$

→ Find the number of arrangements of letters in the word "TALLAHASSEE"

Sol:- $n = 11$

T - 1 time

A - 3 times

L - 2 times

H - 1 time

S - 2 times

E - 2 times

$$\therefore P(11; 1,3,2,1,2,2) = \frac{11!}{1!\, 3!\, 2!\, 1!\, 2!\, 2!} = \frac{11!}{3!(2!)^3}$$

→ Find the number of arrangements of letters in the word "TALLAHASSEE" that begin with T and End with E.

Sol:- $n = 11$

$\underset{\text{T}}{\overset{1}{\uparrow}}$ $\underset{\text{E}}{\overset{2}{\uparrow}}$

A - 3 times

L - 2 times

S - 2 times

$$\therefore P(9; 3,2,2) = \frac{9!}{3!\,2!\,2!} \times a = \frac{9!}{3!(2!)^2}$$

→ Find the Permutation of $\{3.a, 4.b, 2.c, 1.d\}$

Sol: Total no. of arrangements $= \dfrac{(3+4+2+1)!}{3!\,4!\,2!\,1!} = \dfrac{10!}{3!\,4!\,2!}$