Single Final State for NFAs and DFAs

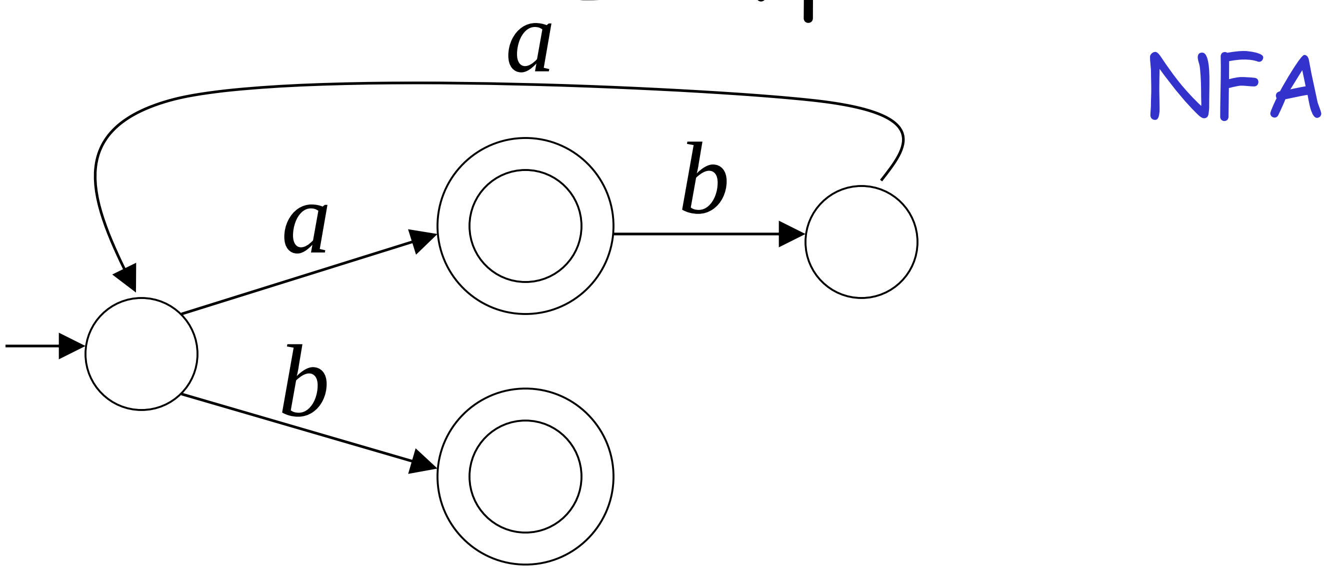
Observation

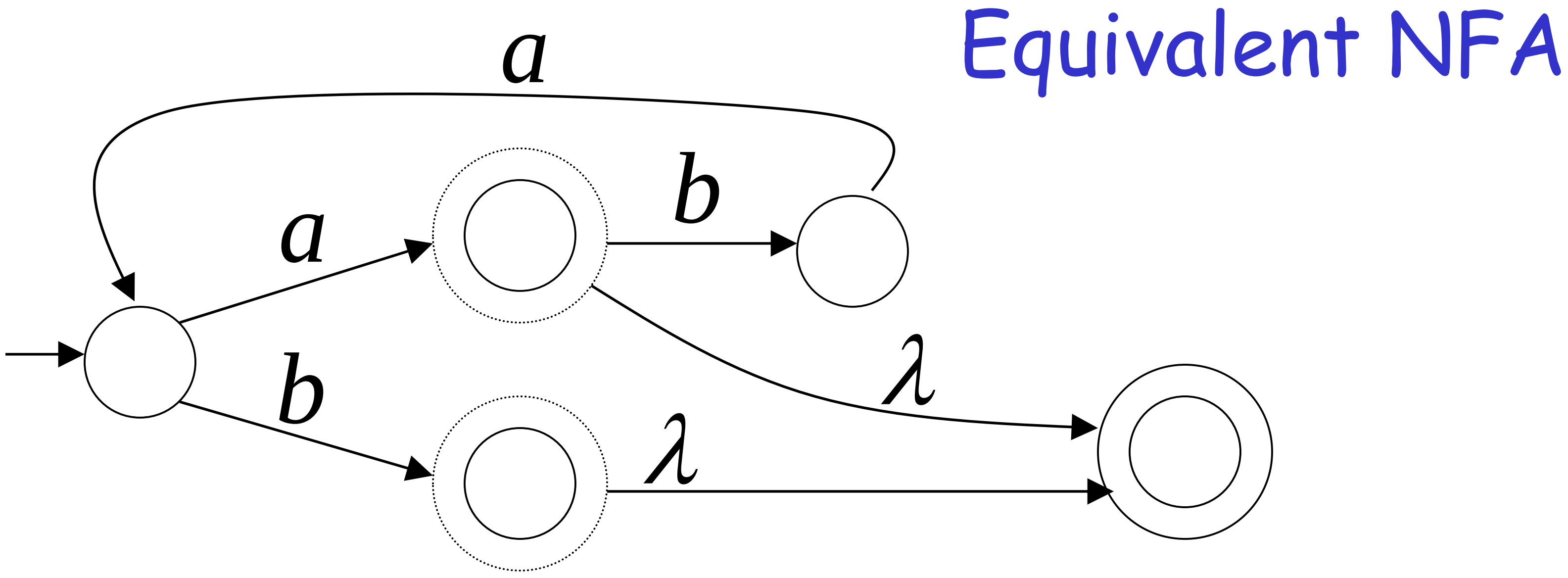
Any Finite Automaton (NFA or DFA)

can be converted to an equivalent NFA

with a single final state

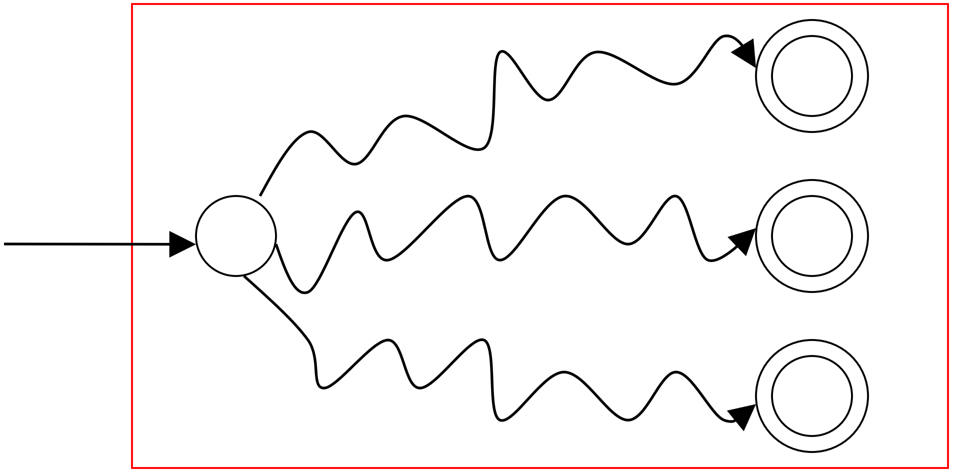
Example



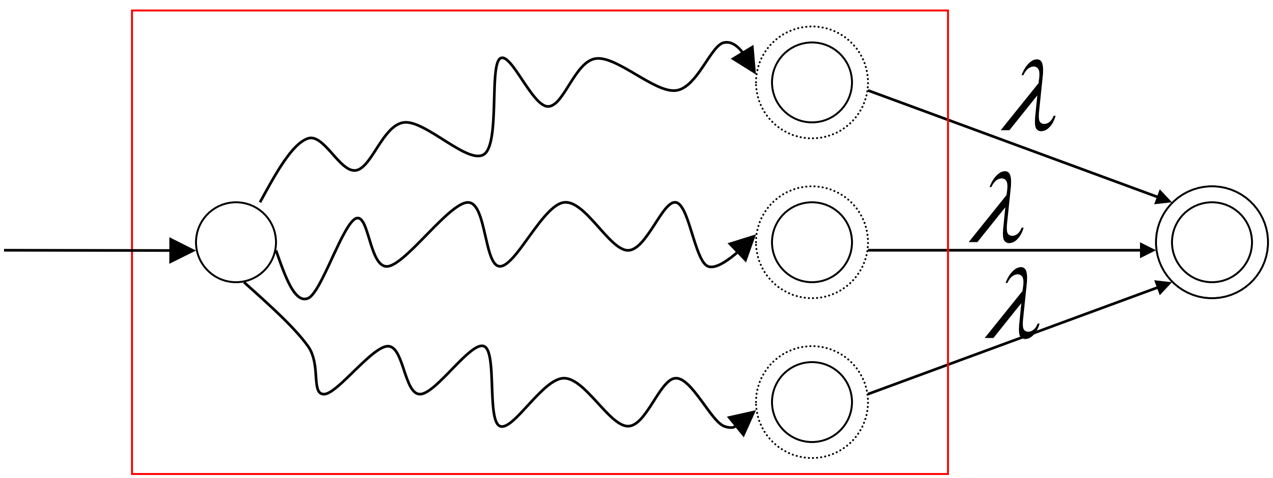


In General

NFA



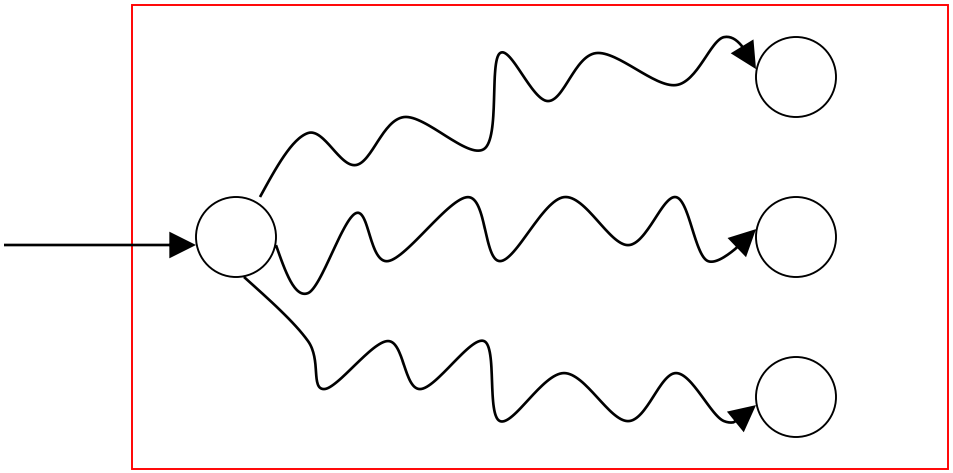
Equivalent NFA

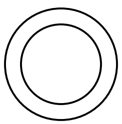


Single final state

Extreme Case

NFA without final state





Add a final state

Without transitions

Some Properties of Regular Languages

Properties

For regular languages L_1 and L_2
we will prove that:

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

Are regular Languages

We Say:

Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

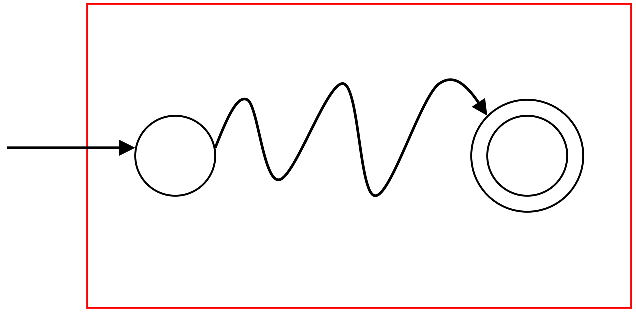
Regular language L_1

Regular language L_2

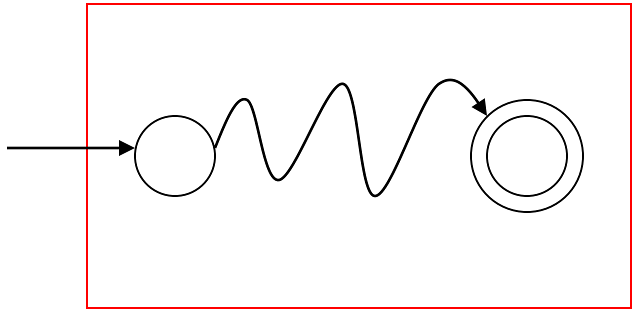
$$L(M_1) = L_1$$

$$L(M_2) = L_2$$

NFA M_1



NFA M_2



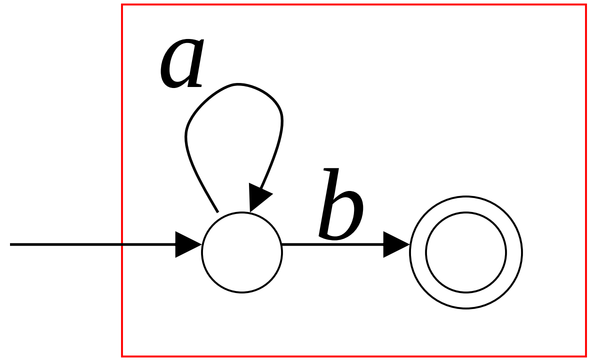
Single final state

Single final state

Example

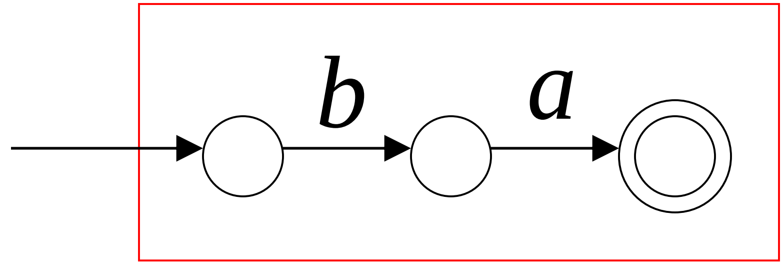
M_1

$L_1 = \{a^n b\}$



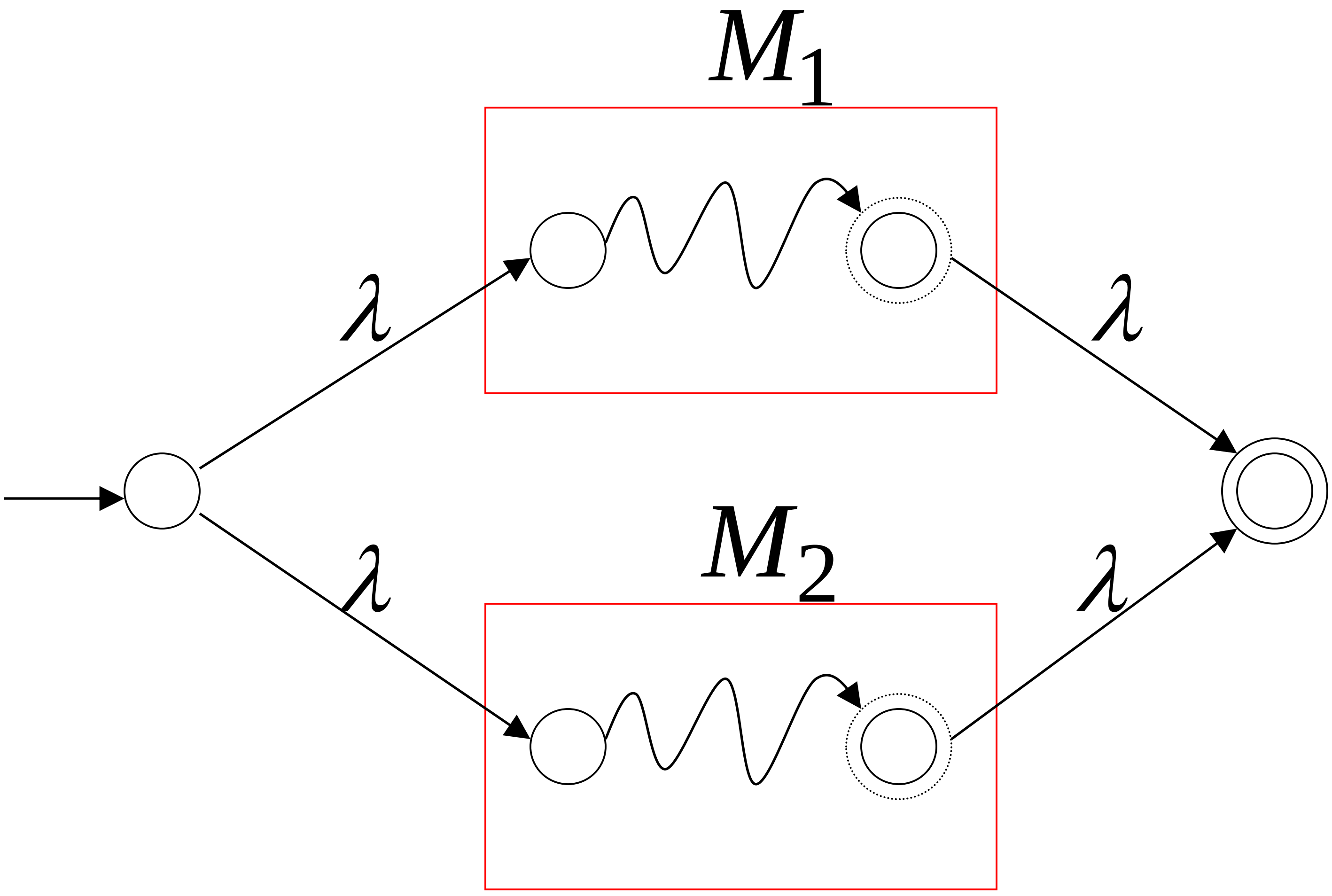
M_2

$L_2 = \{ba\}$



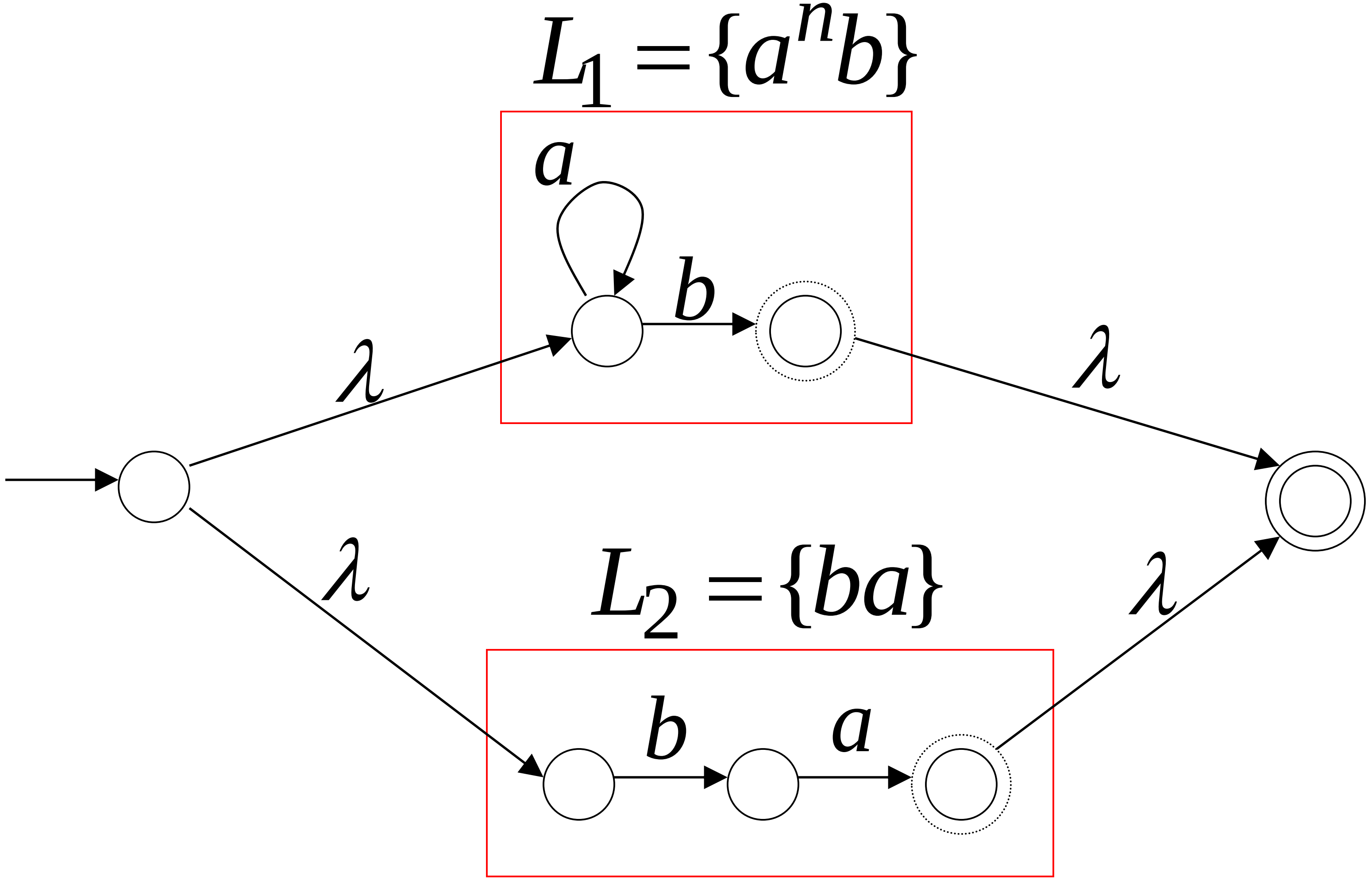
Union

NFA for $L_1 \cup L_2$



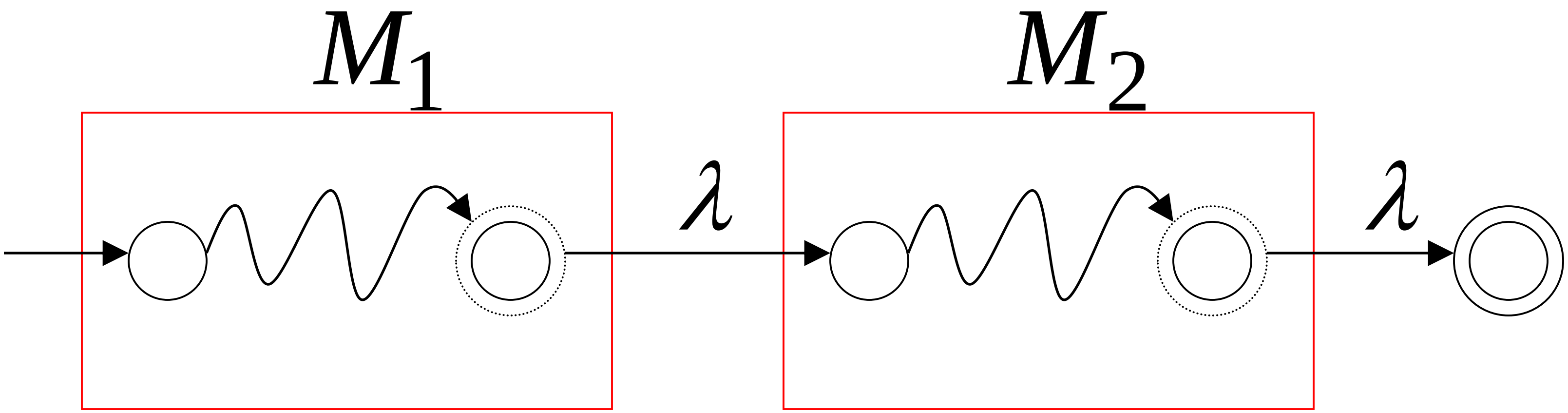
Example

NFA for $L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$



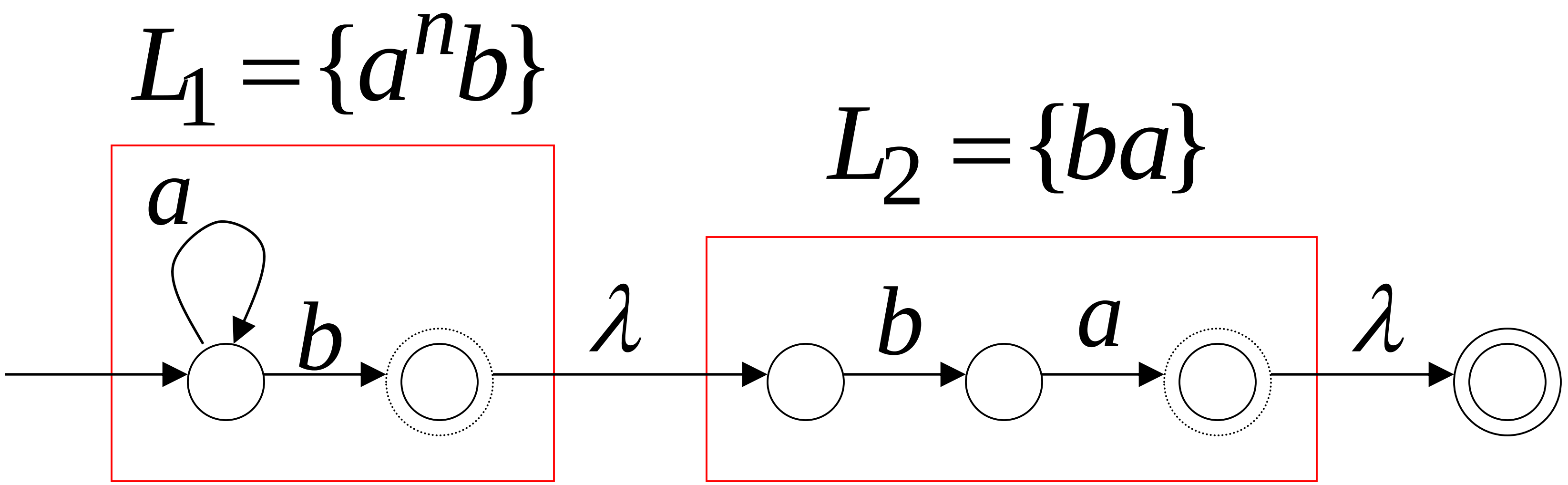
Concatenation

NFA for $L_1 L_2$



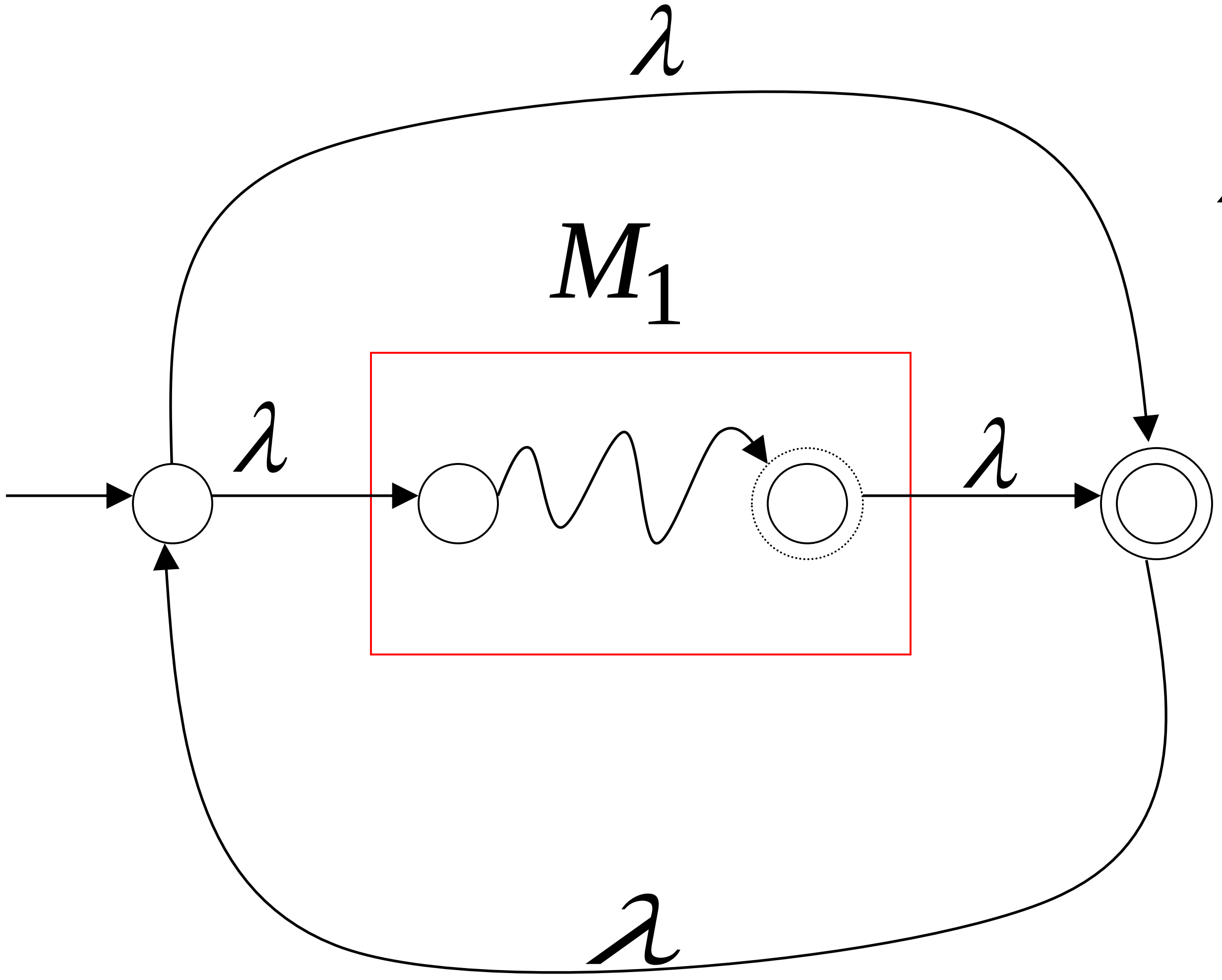
Example

NFA for $L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$



Star Operation

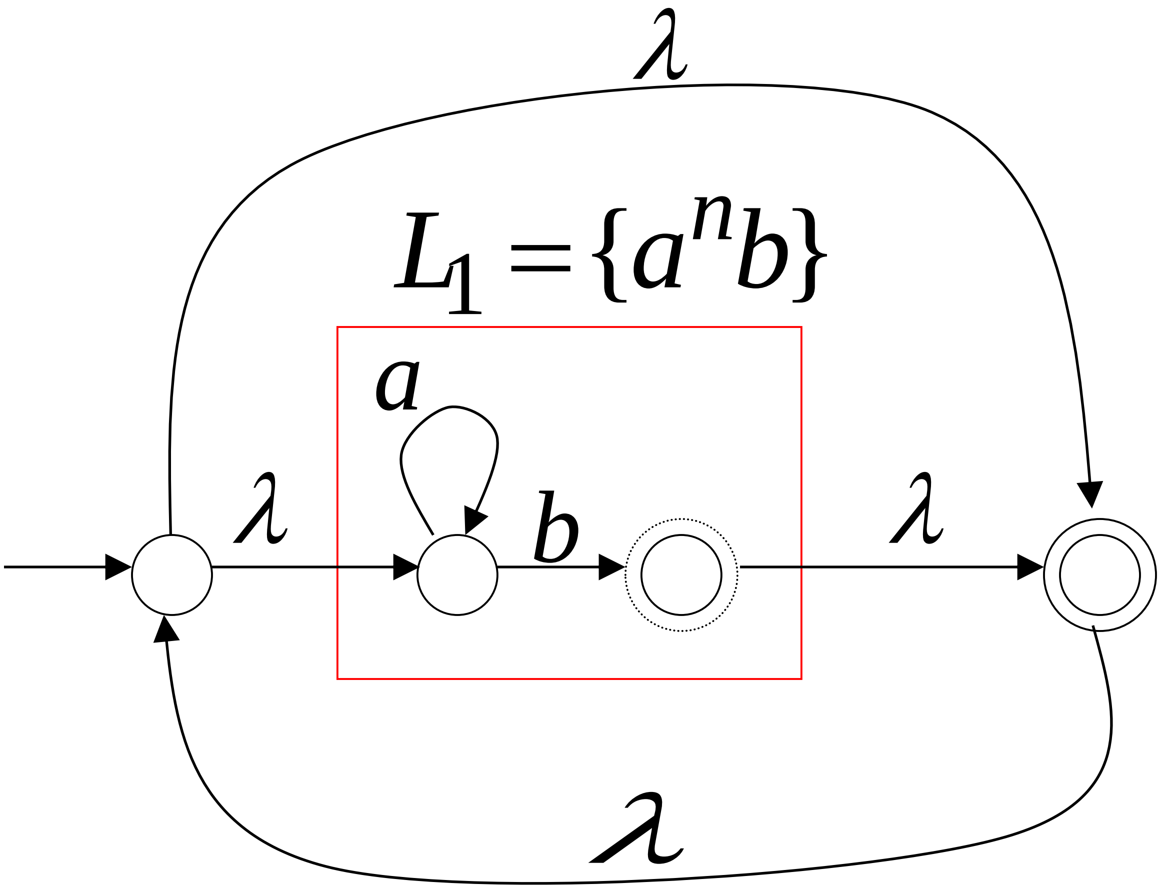
NFA for $L_1{}^*$



$\lambda \in L_1{}^*$

Example

NFA for $L_1{}^* = \{a^n b\}^*$



Regular Expressions

Regular Expressions

Regular expressions
describe regular languages

Example: $(a+b\cdot c)^*$

describes the language

$$\{a,bc\}^* = \{\lambda, a, bc, aa, abc, bca, \ldots\}$$

Recursive Definition

Primitive regular expressions: $\emptyset, \quad \lambda, \quad \alpha$

Given regular expressions r_1 and r_2

$$\left.\begin{array}{l} r_1 + r_2 \\ r_1 \cdot r_2 \\ r_1 * \\ (r_1) \end{array}\right\}$$ Are regular expressions

Examples

A regular expression: $(a+b\cdot c)^*\cdot(c+\varnothing)$

Not a regular expression: $(a+b+)$

Languages of Regular Expressions

$L(r)$: language of regular expression r

Example

$$L((a+b\cdot c)^*)=\{\lambda, a, bc, aa, abc, bca, \ldots\}$$

Definition

For primitive regular expressions:

$$L(\varnothing) = \varnothing$$

$$L(\lambda) = \{\lambda\}$$

$$L(a) = \{a\}$$

Definition (continued)

For regular expressions r_1 and r_2

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

Example

Regular expression: $(a+b)\cdot a^*$

$$
\begin{aligned}
L((a+b)\cdot a^*) &= L((a+b))\,L(a^*) \\
&= L(a+b)\,L(a^*) \\
&= (L(a)\cup L(b))\,(L(a))^* \\
&= (\{a\}\cup\{b\})\,(\{a\})^* \\
&= \{a,b\}\,\{\lambda,a,aa,aaa,\ldots\} \\
&= \{a,aa,aaa,\ldots,b,ba,baa,\ldots\}
\end{aligned}
$$

Example

Regular expression $r = (a+b)^*(a+bb)$

$$L(r) = \{a, bb, aa, abb, ba, bbb, \ldots\}$$

Example

Regular expression $\quad r = (aa)^*(bb)^*b$

$$L(r) = \{a^{2n}b^{2m}b : \quad n, m \geq 0\}$$

Example

Regular expression $r = (0+1)^*00(0+1)^*$

$L(r)$ = { all strings with at least two consecutive 0 }

Example

Regular expression $r = (1 + 01)^*(0 + \lambda)$

$L(r)$ = { all strings without two consecutive 0 }

Equivalent Regular Expressions

Definition:

Regular expressions r_1 and r_2

are **equivalent** if $L(r_1) = L(r_2)$

Example

L = { all strings without two consecutive 0 }

$$r_1 = (1+01)^*(0+\lambda)$$

$$r_2 = (1^*011^*)^*(0+\lambda)+1^*(0+\lambda)$$

$$L(r_1) = L(r_2) = L \quad \Longrightarrow$$

r_1 and r_2 are equivalent regular expr.

Regular Expressions and Regular Languages

Theorem

$$\left\{\begin{array}{l}\text{Languages}\\\text{Generated by}\\\text{Regular Expressions}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\\text{Languages}\end{array}\right\}$$

Theorem - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

1. For any regular expression r
   the language $L(r)$ is regular

Theorem - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Expressions} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

2. For any regular language L there is a regular expression r with $L(r) = L$

Proof - Part 1

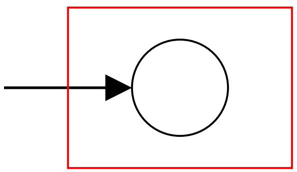
1. For any regular expression r
   the language $L(r)$ is regular

Proof by induction on the size of r

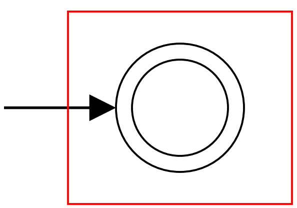
Induction Basis

Primitive Regular Expressions: $\emptyset, \quad \lambda, \quad \alpha$

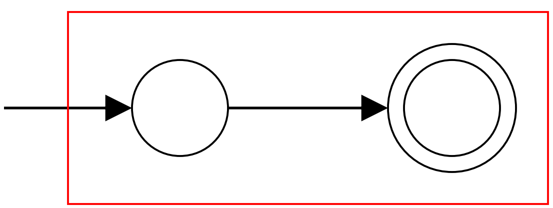
NFAs



$$L(M_1) = \emptyset = L(\emptyset)$$



$$L(M_2) = \{\lambda\} = L(\lambda)$$



$$L(M_3) = \{a\} = L(a)$$

regular languages

Inductive Hypothesis

Assume

for regular expressions r_1 and r_2

that

$L(r_1)$ and $L(r_2)$ are regular languages

Inductive Step

We will prove:

$$\left.\begin{array}{l} L(r_1 + r_2) \\ \\ L(r_1 \cdot r_2) \\ \\ L(r_1 *) \\ \\ L((r_1)) \end{array}\right\}$$ Are regular Languages

By definition of regular expressions:

$$L(r_1 + r_2) = L(r_1) \cup L(r_2)$$

$$L(r_1 \cdot r_2) = L(r_1)\, L(r_2)$$

$$L(r_1 *) = (L(r_1))*$$

$$L((r_1)) = L(r_1)$$

By inductive hypothesis we know:

$L(r_1)$ and $L(r_2)$ are regular languages

We also know:

Regular languages are closed under

union	$L(r_1) \cup L(r_2)$
concatenation	$L(r_1)\,L(r_2)$
star	$(L(r_1))^*$

Therefore:

$$\left.\begin{aligned} L(r_1 + r_2) &= L(r_1) \cup L(r_2) \\ L(r_1 \cdot r_2) &= L(r_1)\, L(r_2) \\ L(r_1 *) &= (L(r_1))* \end{aligned}\right\} \text{Are regular languages}$$

And trivially:

$L((r_1))$ is a regular language

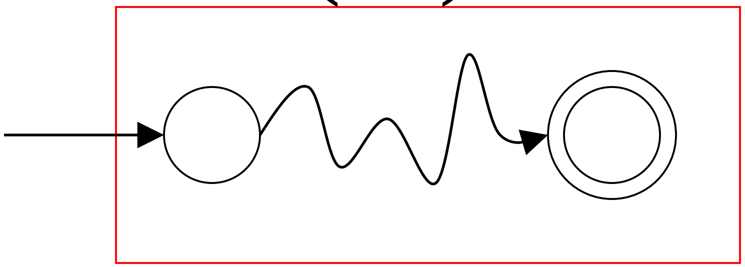
Proof - Part 2

2. For any regular language L there is a regular expression r with $L(r) = L$

Proof by construction of regular expression

Since L is regular take the NFA M that accepts it

$$L(M) = L$$



Single final state

From M construct the equivalent

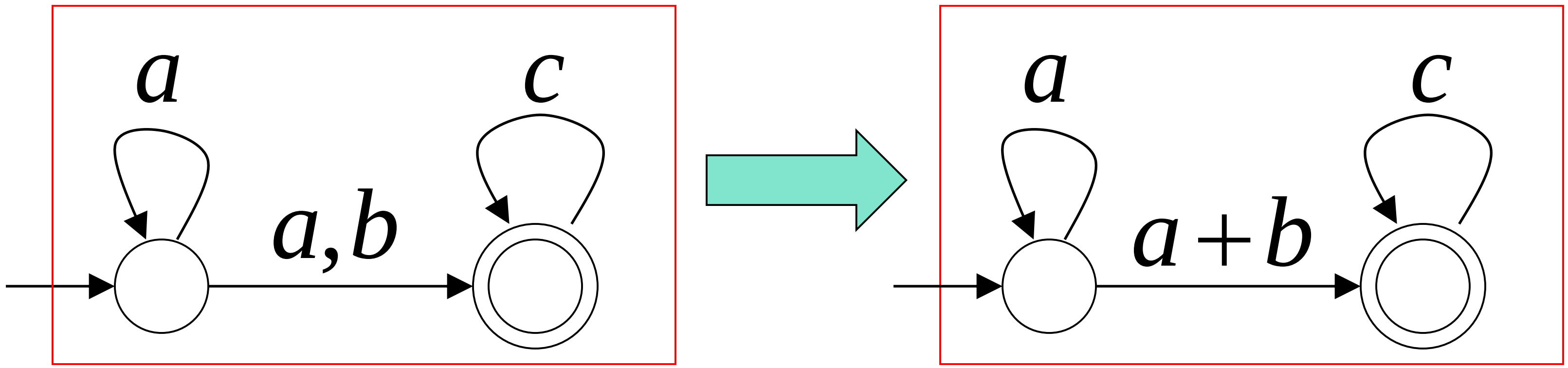
Generalized Transition Graph

transition labels

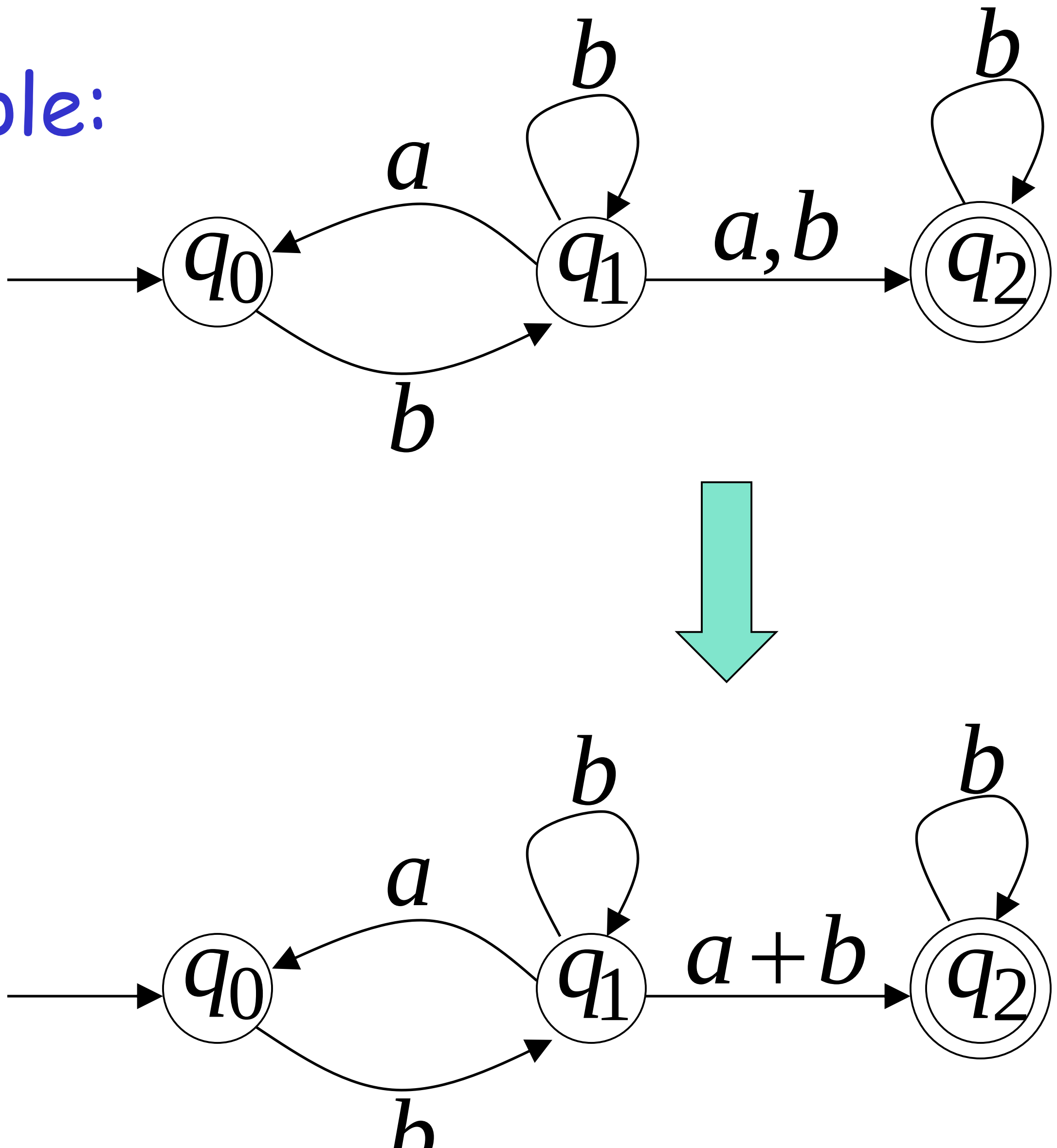
are regular expressions

Example:

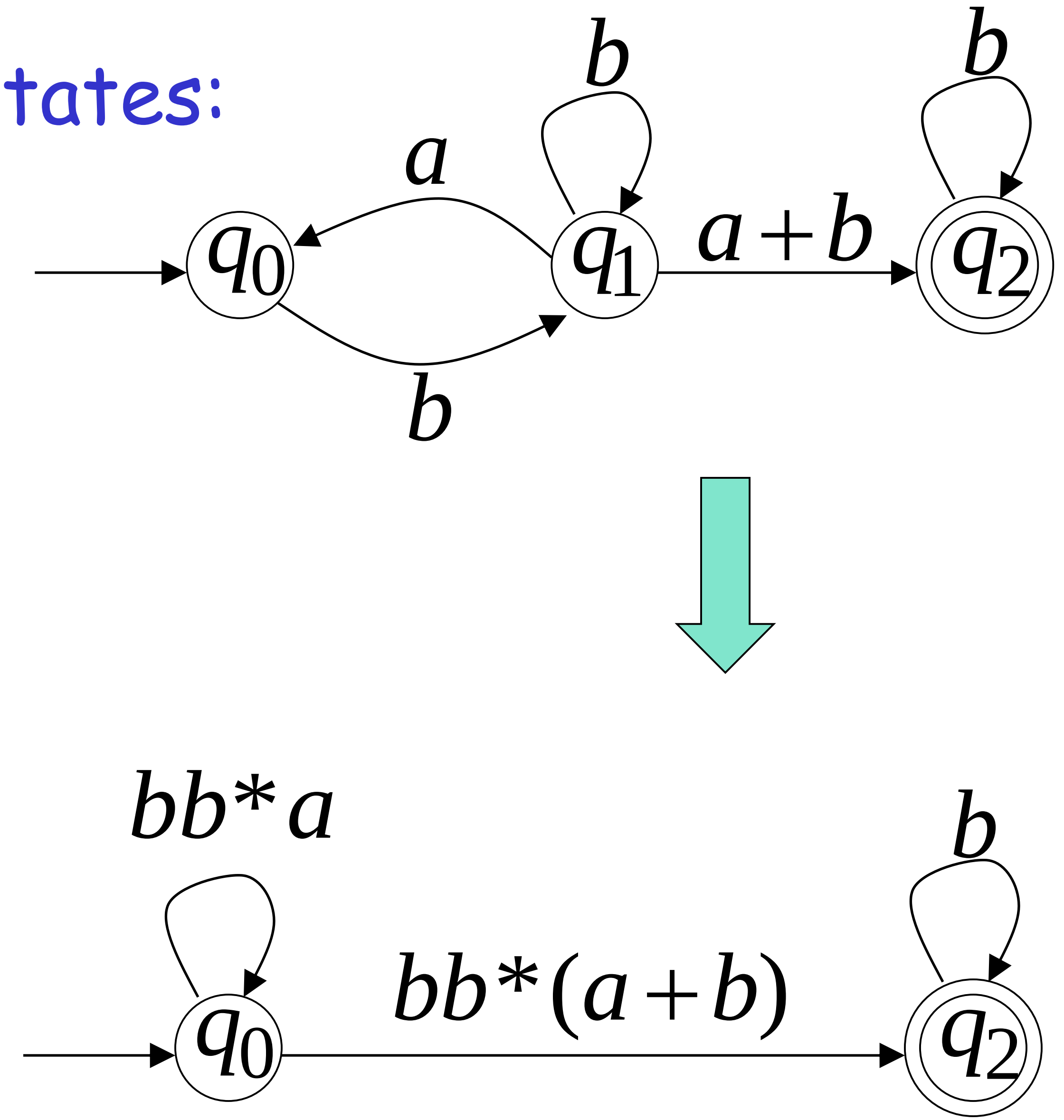
M



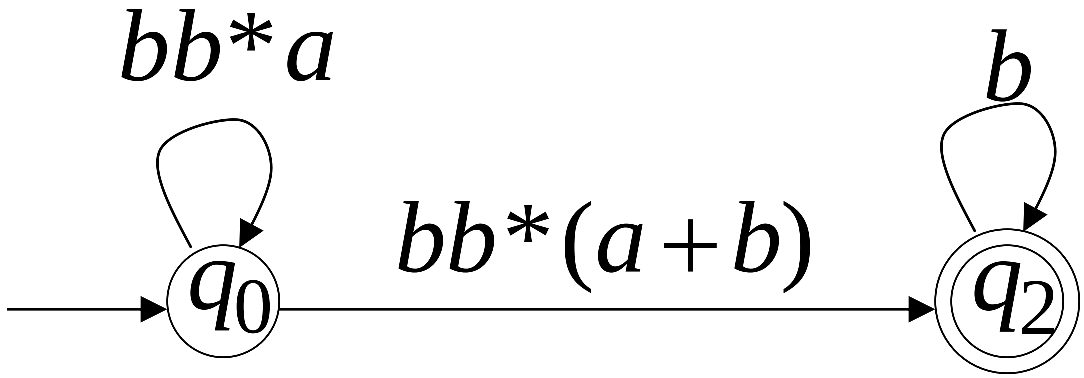
Another Example:



Reducing the states:



Resulting Regular Expression:

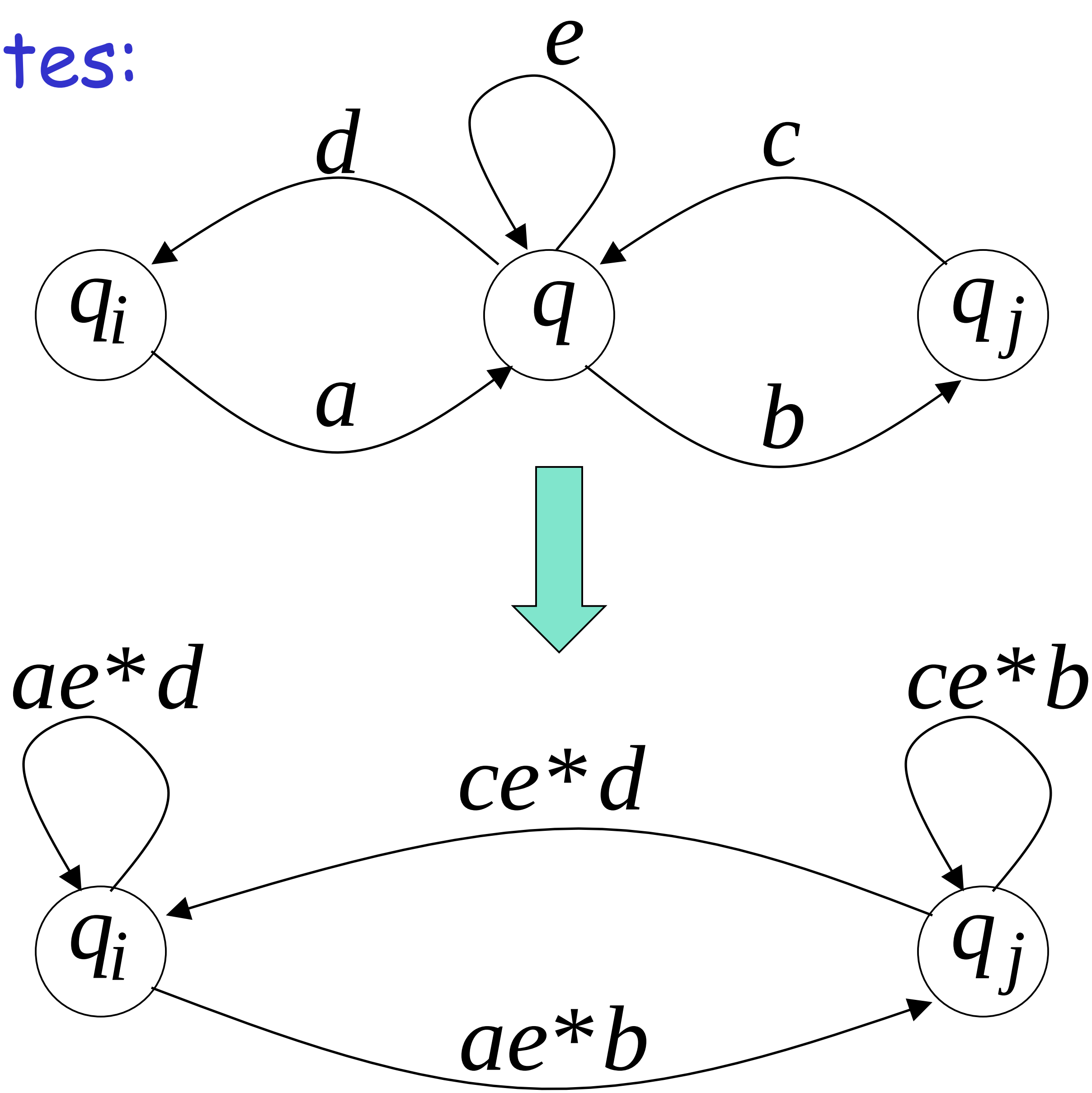


$$r = (bb^*a)^*bb^*(a+b)b^*$$

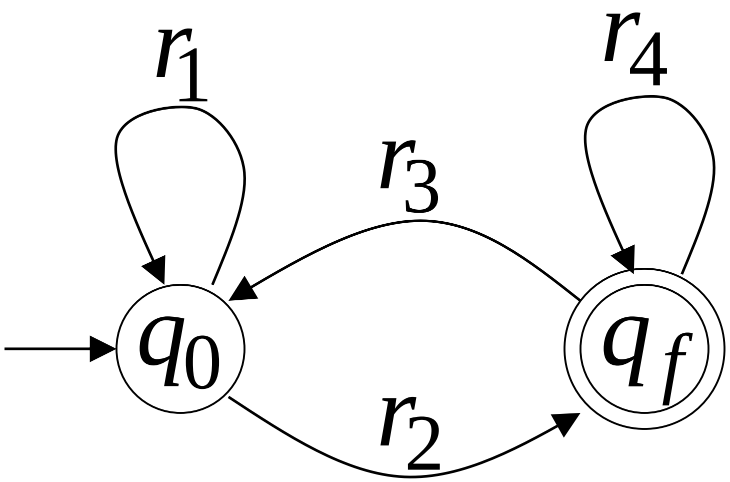
$$L(r) = L(M) = L$$

In General

Removing states:



The final transition graph:



The resulting regular expression:

$$r = r_1 * r_2 (r_4 + r_3 r_1 * r_2) *$$

$$L(r) = L(M) = L$$