More Properties of Regular Languages

We have proven

Regular languages are closed under:

Union

Concatenation

Star operation

Reverse

Namely, for regular languages L_1 and L_2 :

Union $L_1 \cup L_2$

Concatenation $L_1 L_2$

Star operation $L_1{}^*$

Reverse $L_1{}^R$

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ Regular Languages

We will prove

Regular languages are closed under:

Complement

Intersection

Namely, for regular languages L_1 and L_2:

Complement $\overline{L_1}$

Intersection $L_1 \cap L_2$

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ Regular Languages

Complement

Theorem: For regular language L
the complement $\overline{L}$ is regular

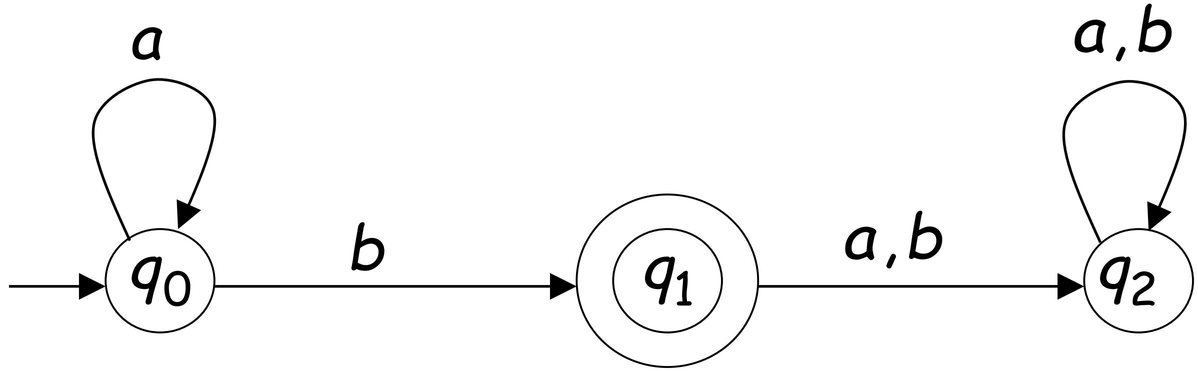
Proof: Take DFA that accepts L and make

- nonfinal states final
- final states nonfinal

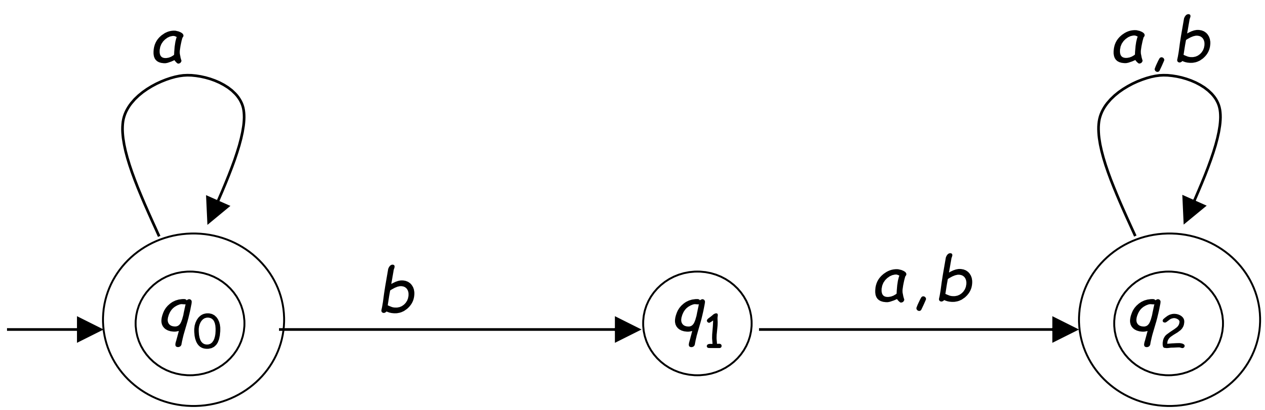
Resulting DFA accepts $\overline{L}$

Example:

$$L = L(a^*b)$$



$$\overline{L} = L(a^* + a^*b(a+b)(a+b)^*)$$



Intersection

Theorem: For regular languages L_1 and L_2 the intersection $L_1 \cap L_2$ is regular

Proof: Apply DeMorgan's Law:

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

$L_1 , \ L_2$ regular

$\Rightarrow \overline{L_1} , \ \overline{L_2}$ regular

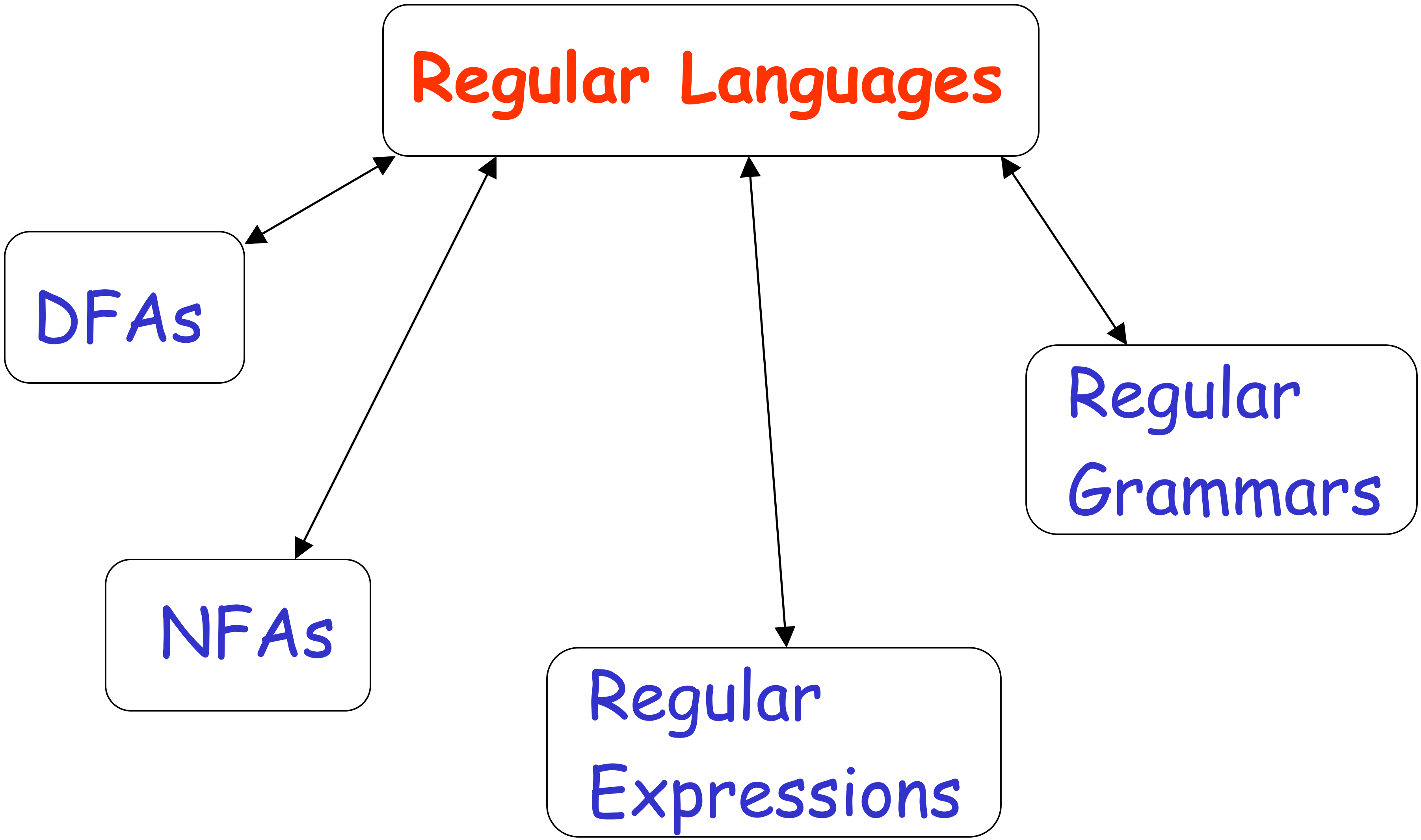
$\Rightarrow \overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow \overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow L_1 \cap L_2$ regular

Standard Representations of Regular Languages

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

We may assume a regular language can be represented as a DFA, an NFA, a regular expression, or a regular grammar, whatever we find convenient.

Elementary Questions

about

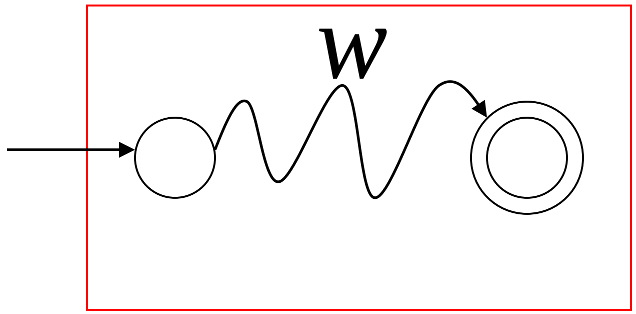
Regular Languages

Membership Question

Question: Given regular language L
and string w
how can we check if $w \in L$?

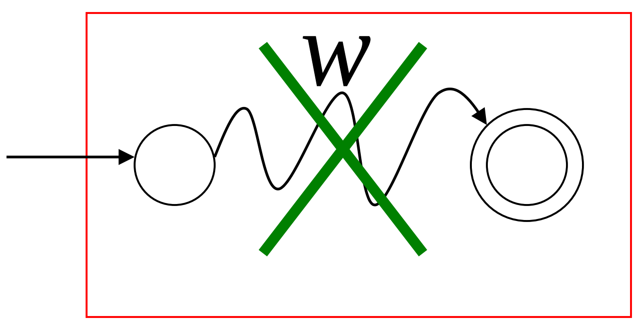
Answer: Take the DFA that accepts L
and check if w is accepted

DFA



$$w \in L$$

DFA



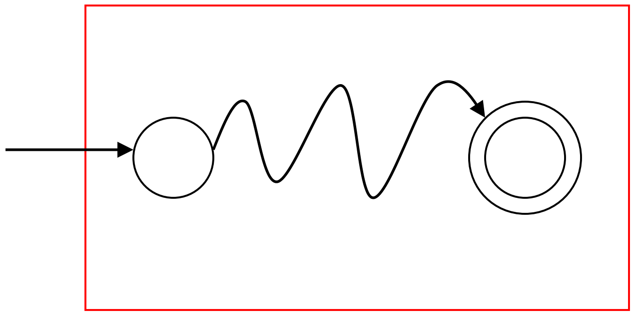
$$w \notin L$$

Question: Given regular language L
how can we check
if L is empty: $(L = \emptyset)$?

Answer: Take the DFA that accepts L

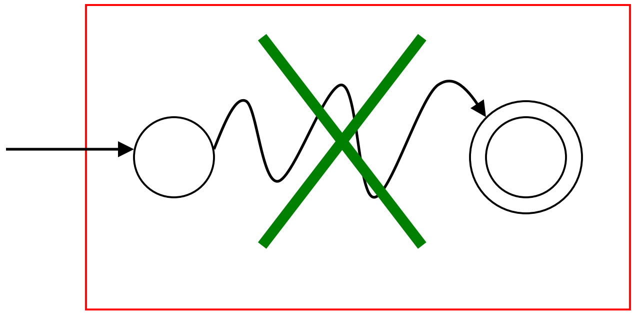
Check if there is a path from
the initial state to a final state

DFA



$$L \neq \emptyset$$

DFA



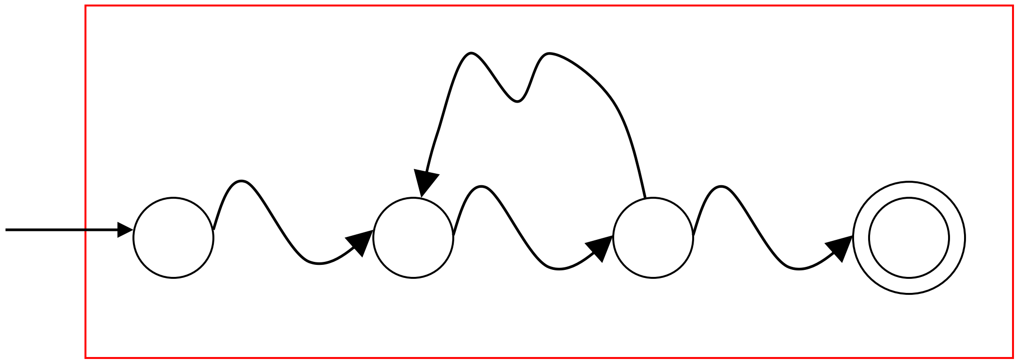
$$L = \emptyset$$

Question: Given regular language L
how can we check
if L is finite?

Answer: Take the DFA that accepts L

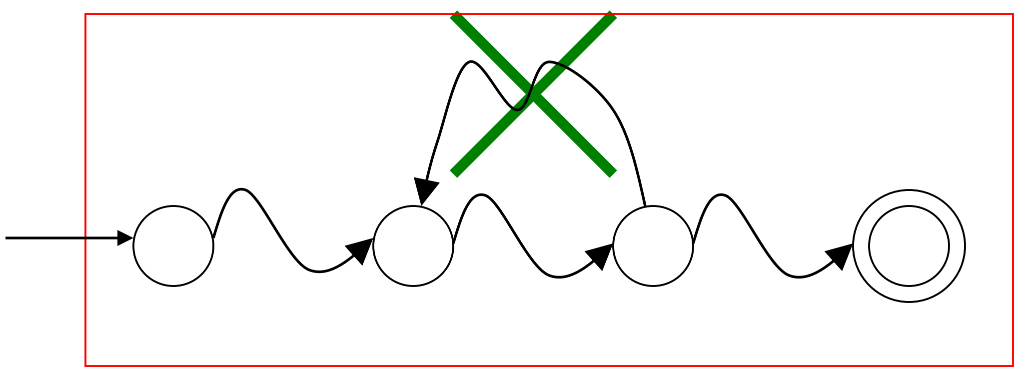
Check if there is a walk with cycle
from the initial state to a final state

DFA



L is infinite

DFA



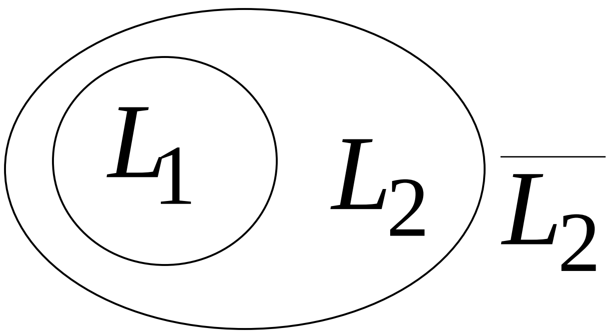
L is finite

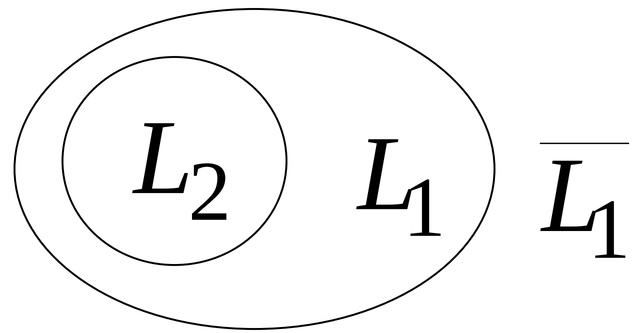
Question: Given regular languages L_1 and L_2 how can we check if $L_1 = L_2$?

Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$$

$$L_1 \cap \overline{L_2} = \emptyset \quad \text{and} \quad \overline{L_1} \cap L_2 = \emptyset$$



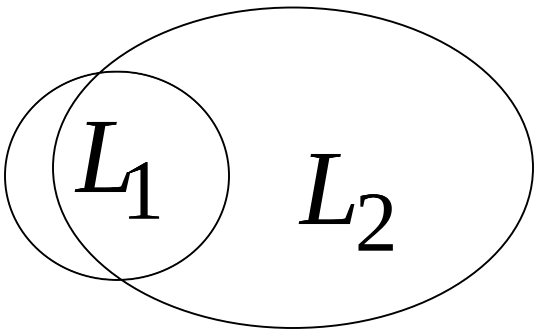


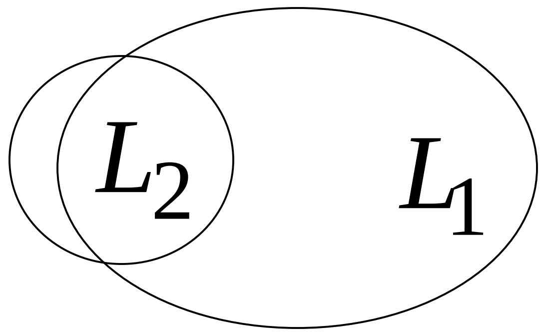
$$L_1 \subseteq L_2 \qquad L_2 \subseteq L_1$$

$$L_1 = L_2$$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \varnothing$$

$L_1 \cap \overline{L_2} \neq \varnothing$ or $\overline{L_1} \cap L_2 \neq \varnothing$





$L_1 \not\subset L_2$ $\qquad$ $L_2 \not\subset L_1$

$$L_1 \neq L_2$$

Non-regular languages

Non-regular languages

$$\{a^n b^n: \quad n \geq 0\}$$

$$\{ww^R: \quad w \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c + a$$

$$b + c(a+b)^*$$

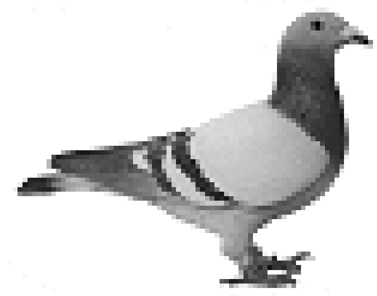
$$etc...$$

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

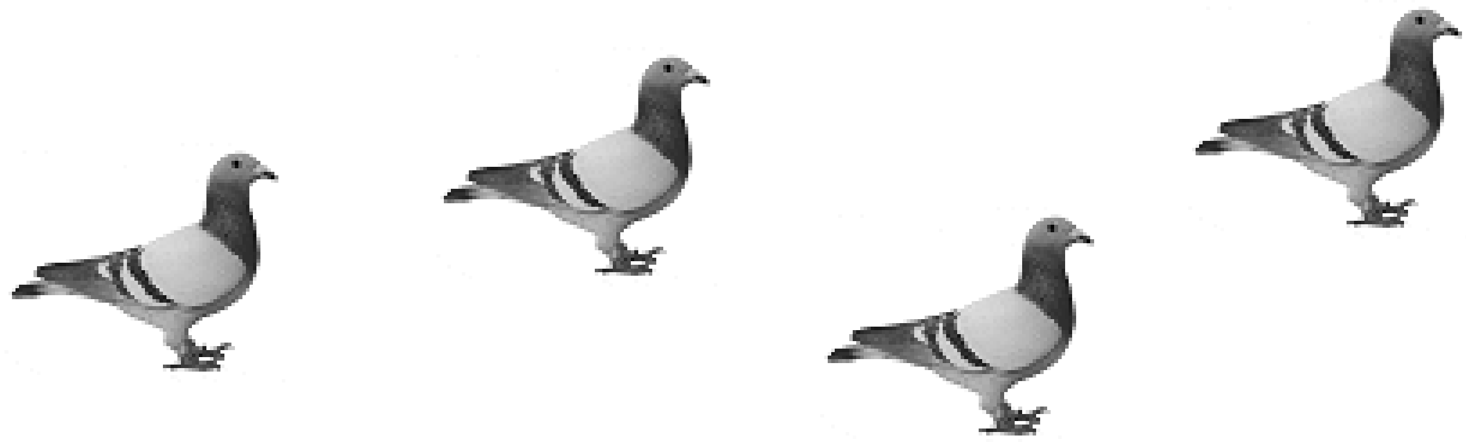
Problem: this is not easy to prove

Solution: the Pumping Lemma !!!

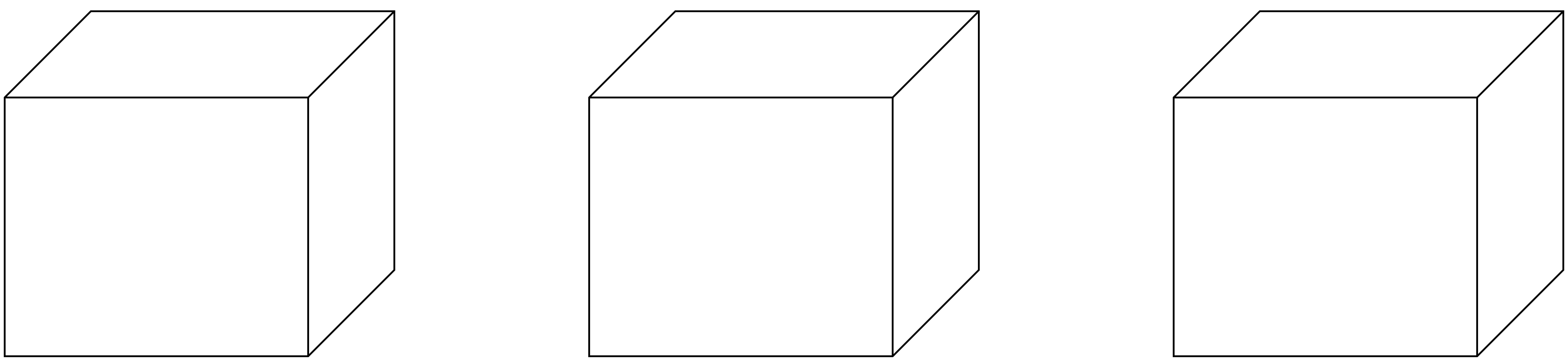


The Pigeonhole Principle

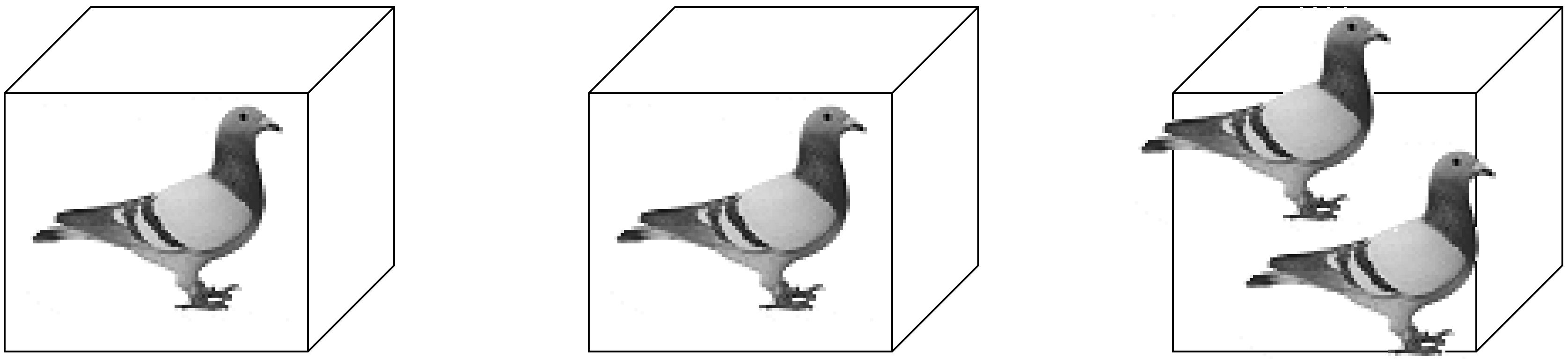
4 pigeons



3 pigeonholes

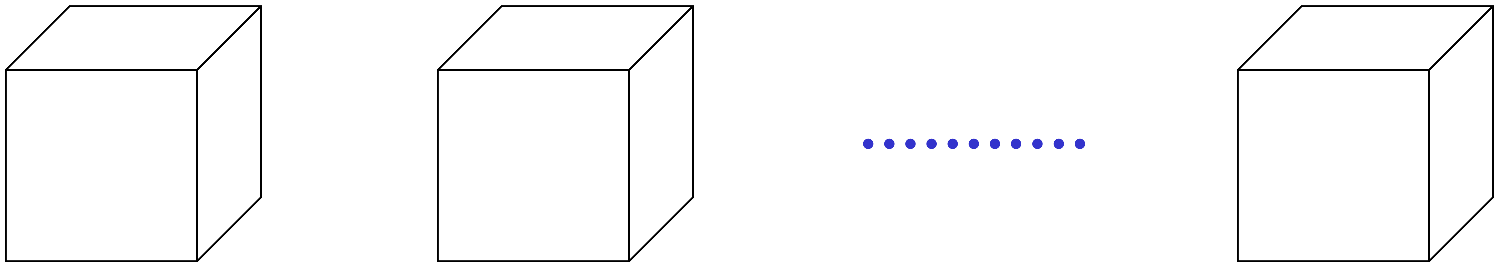


A pigeonhole must
contain at least two pigeons



n pigeons

m pigeonholes $\quad n > m$



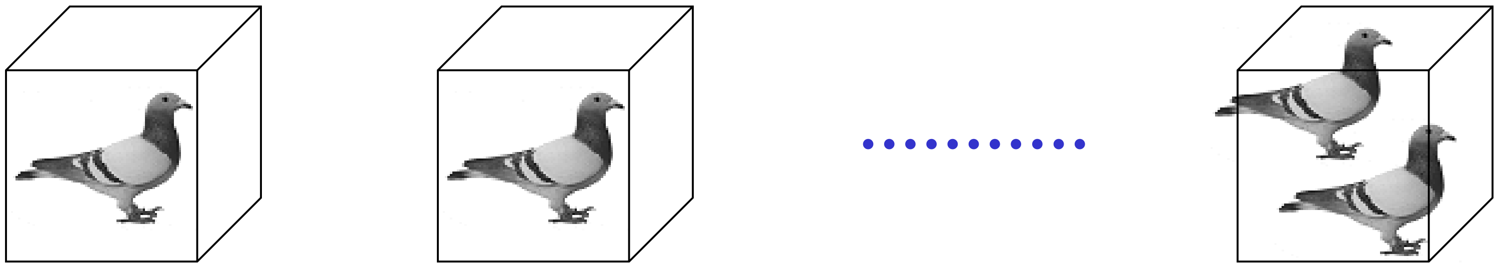
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

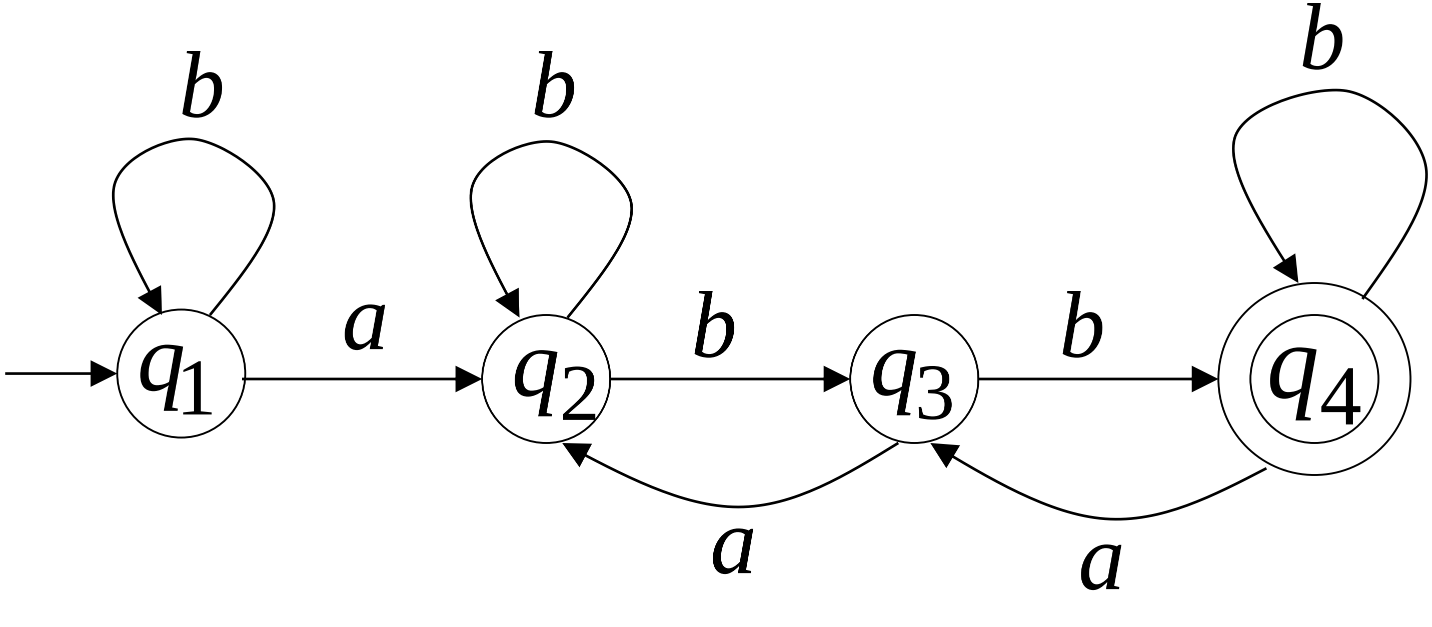


The Pigeonhole Principle

and

DFAs

DFA with 4 states

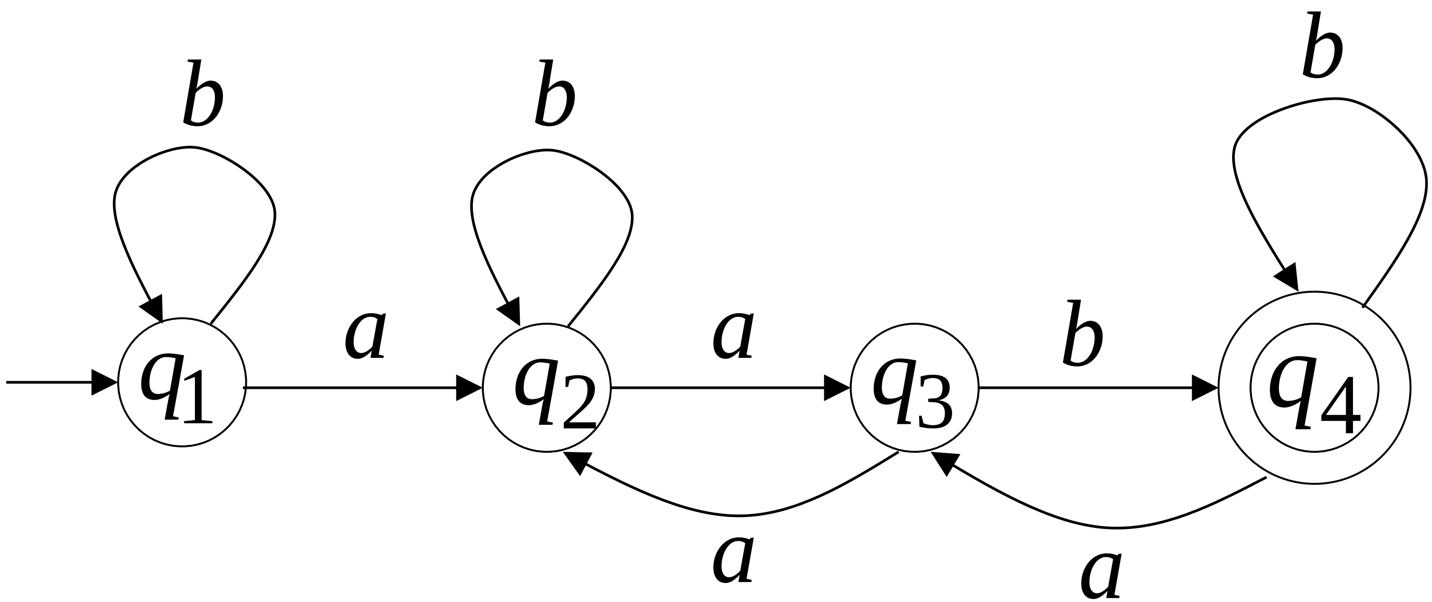


In walks of strings: a

aa

aab

no state is repeated



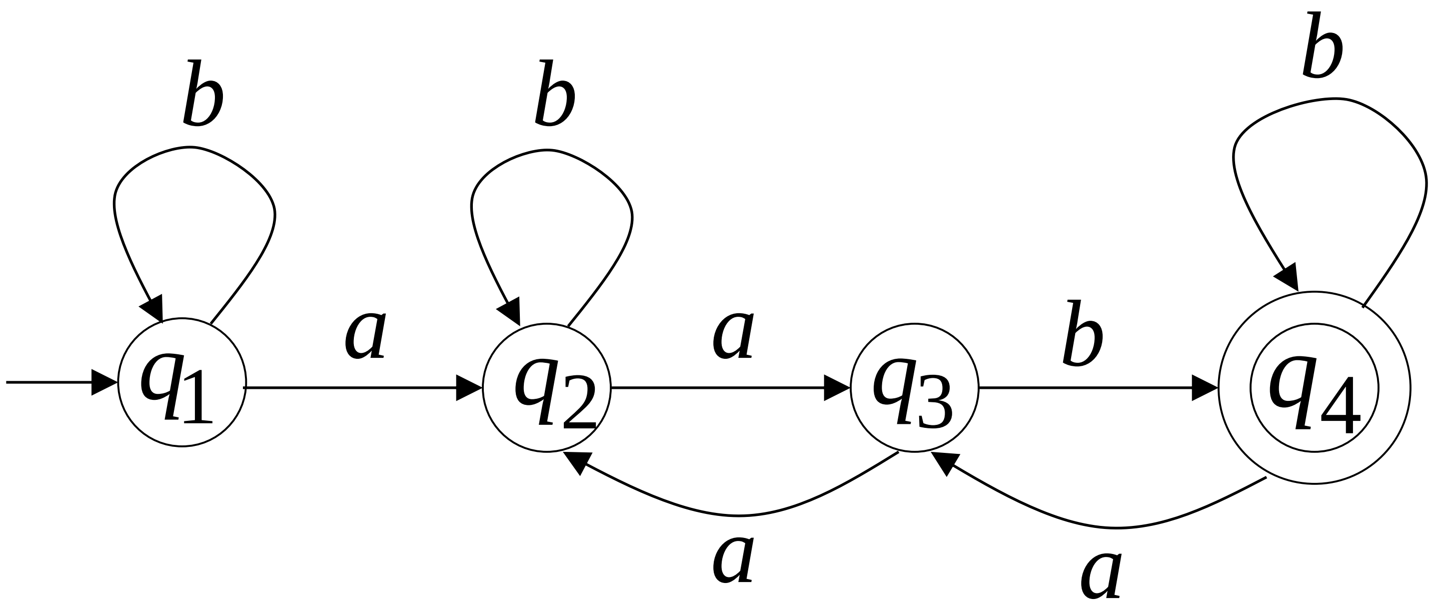
In walks of strings: $aabb$

$bbaa$

$abbabb$

$abbbabbabb...$

a state is repeated



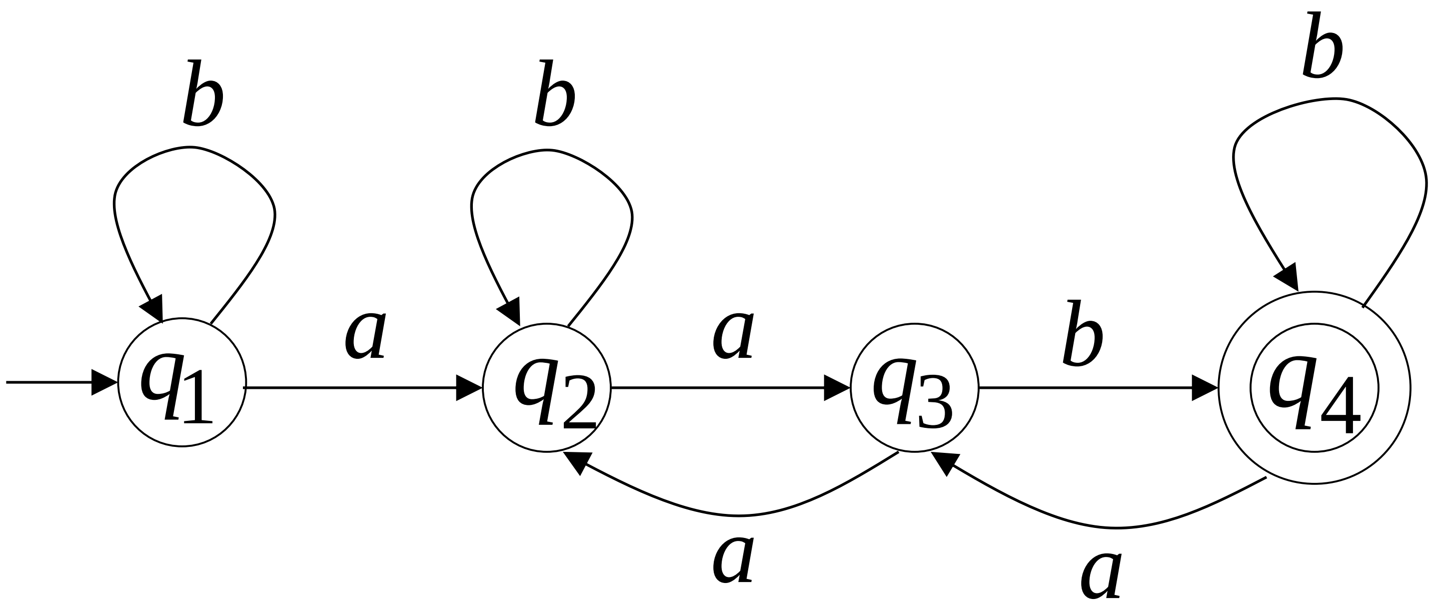
In walks of strings: $aabb$

$bbaa$

$abbabb$

$abbbabbabb...$

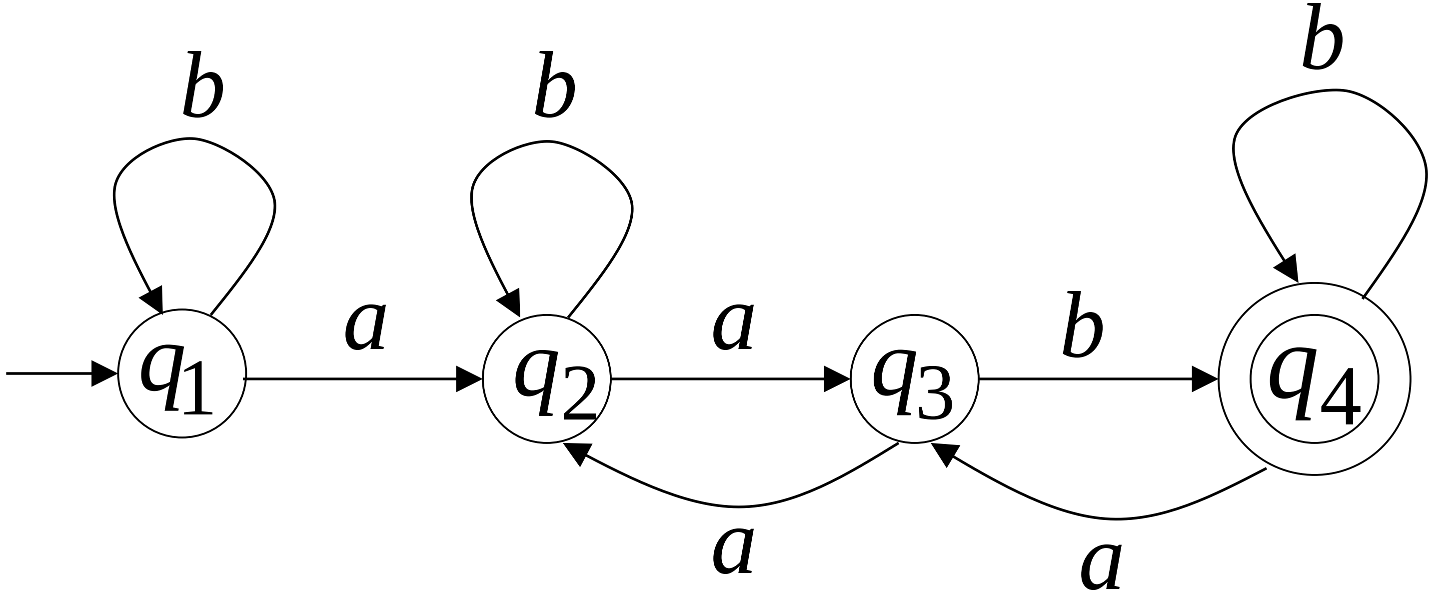
a state is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

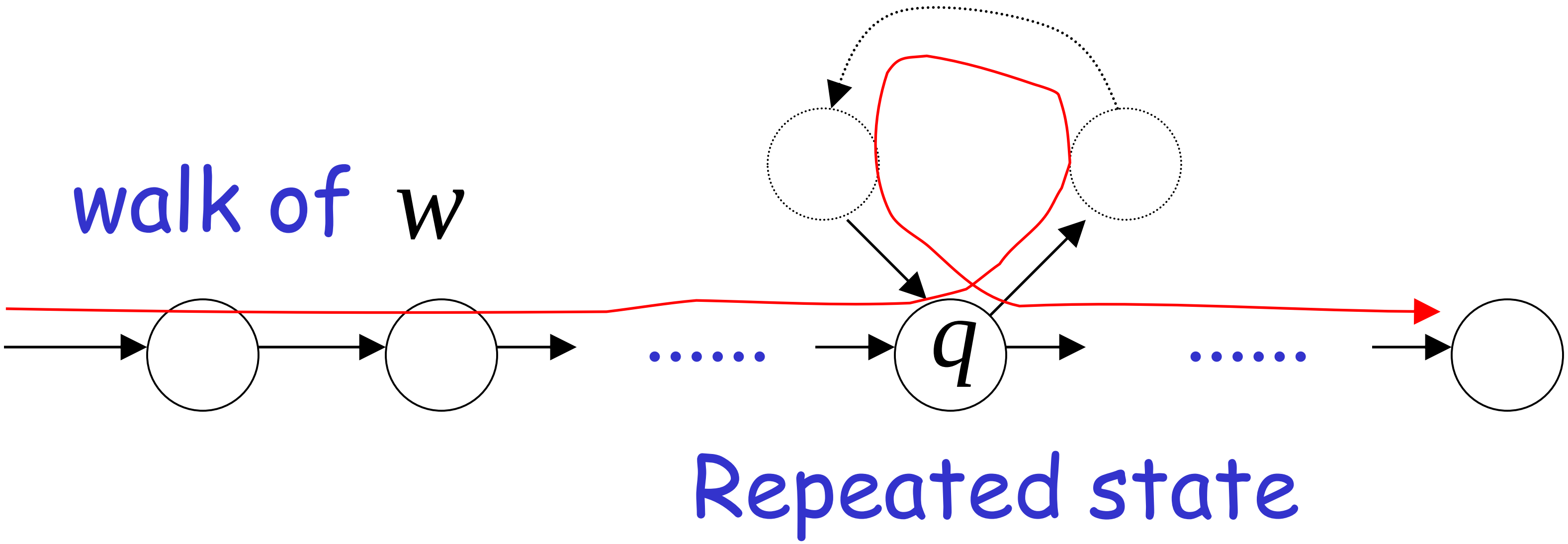
Thus, a state must be repeated



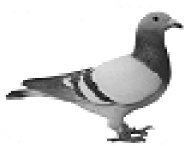
In general, for any DFA:

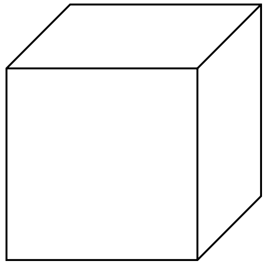
String w has length $\geq$ number of states

A state q must be repeated in the walk of w

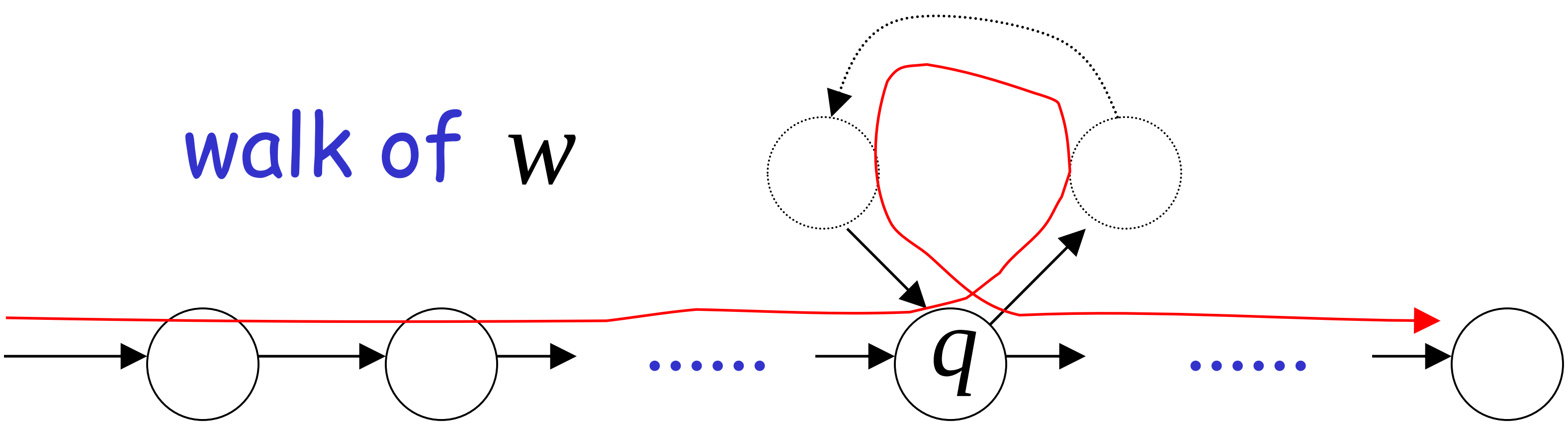


In other words for a string w:

$\xrightarrow{a}$ transitions are pigeons 

q states are pigeonholes 

walk of w

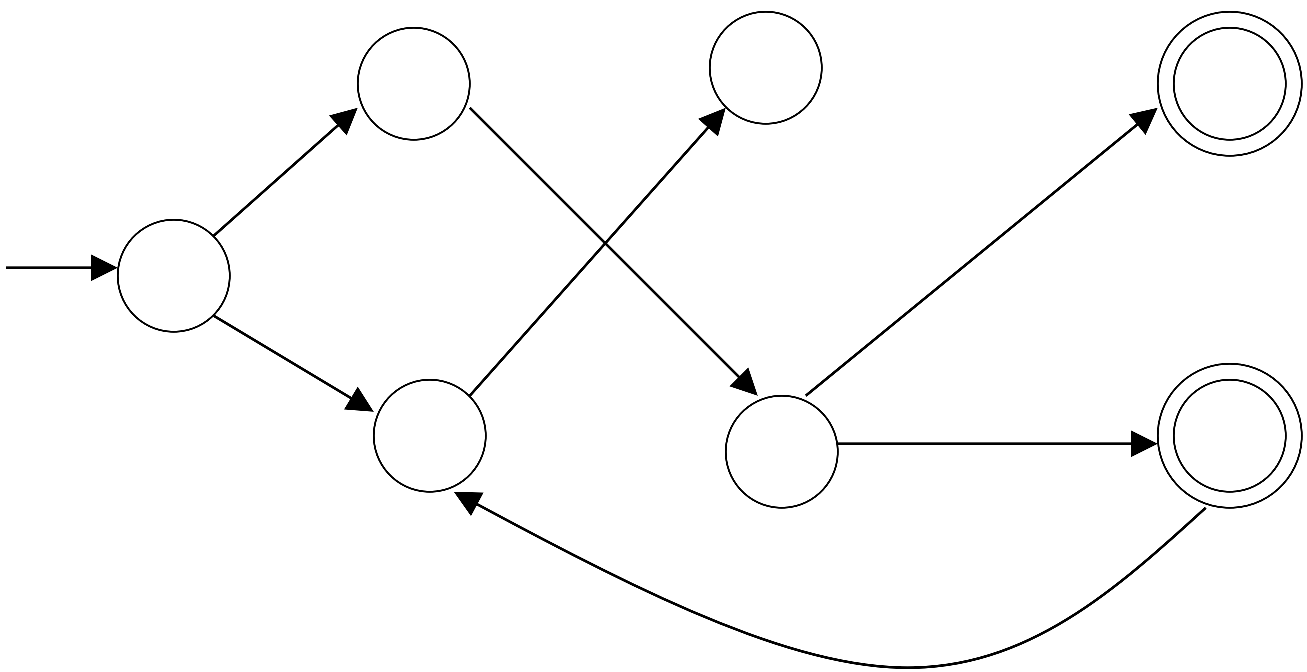


Repeated state

The Pumping Lemma

Take an ~~infinite~~ regular language L

DFA that accepts L



m
states

Take string w with $w \in L$

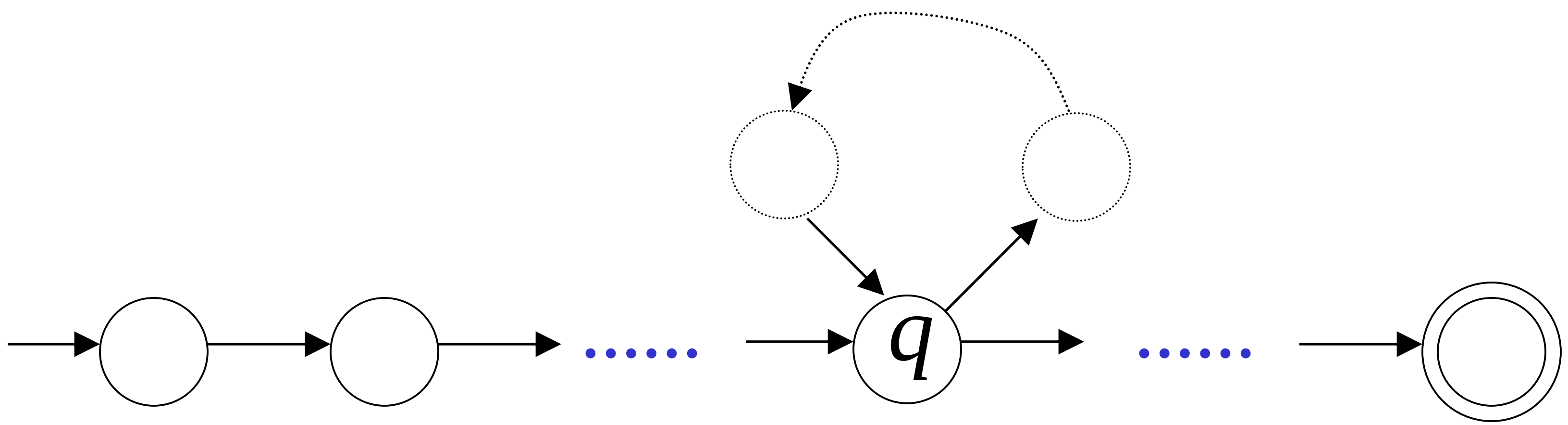
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ number of states of DFA

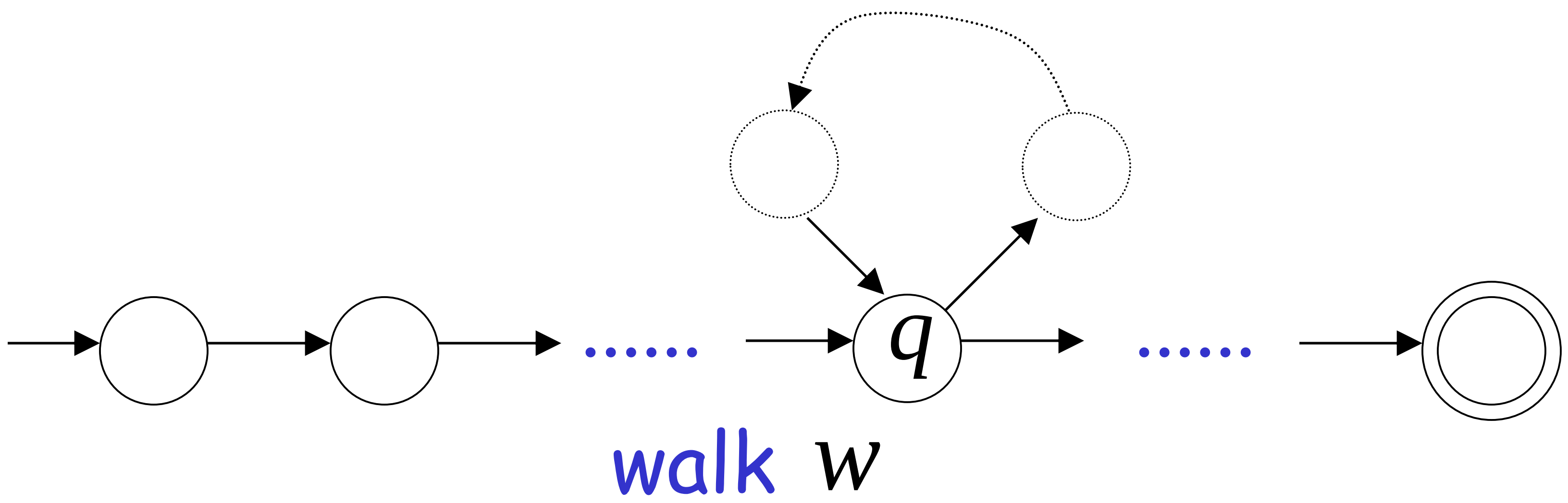
then, from the pigeonhole principle:

a state q is repeated in the walk w

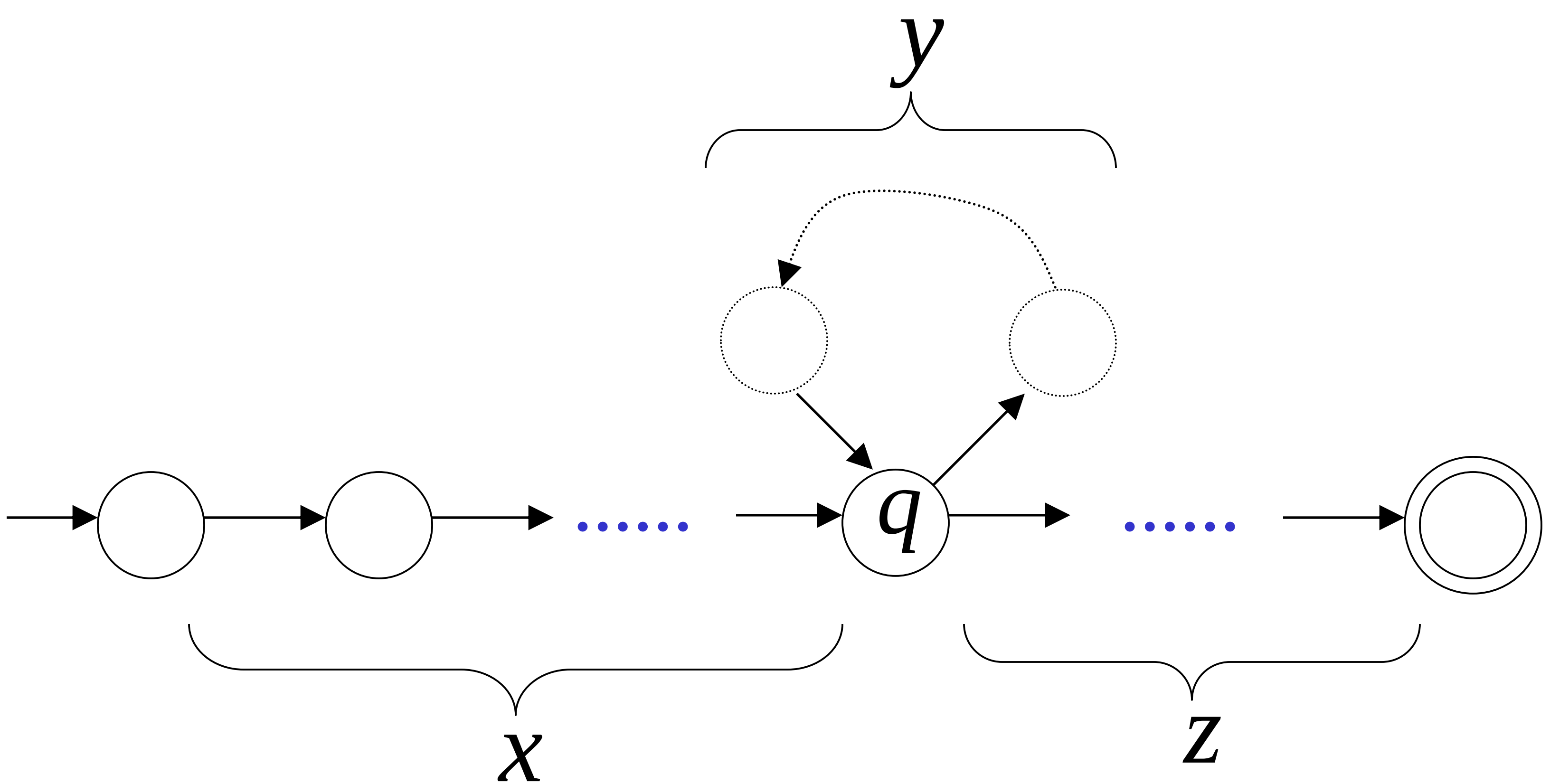


walk w

Let q be the first state repeated

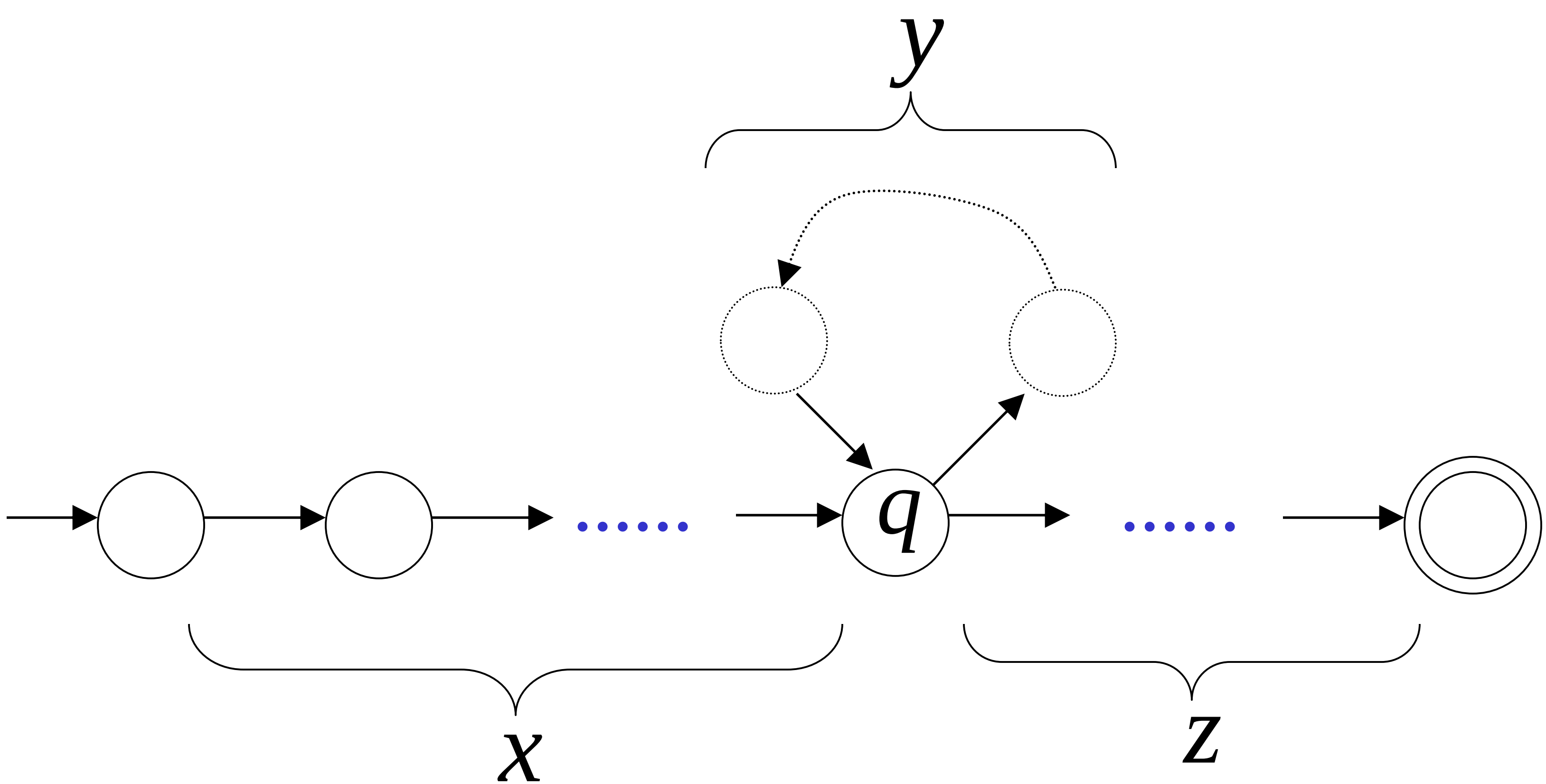


Write $w = x\ y\ z$

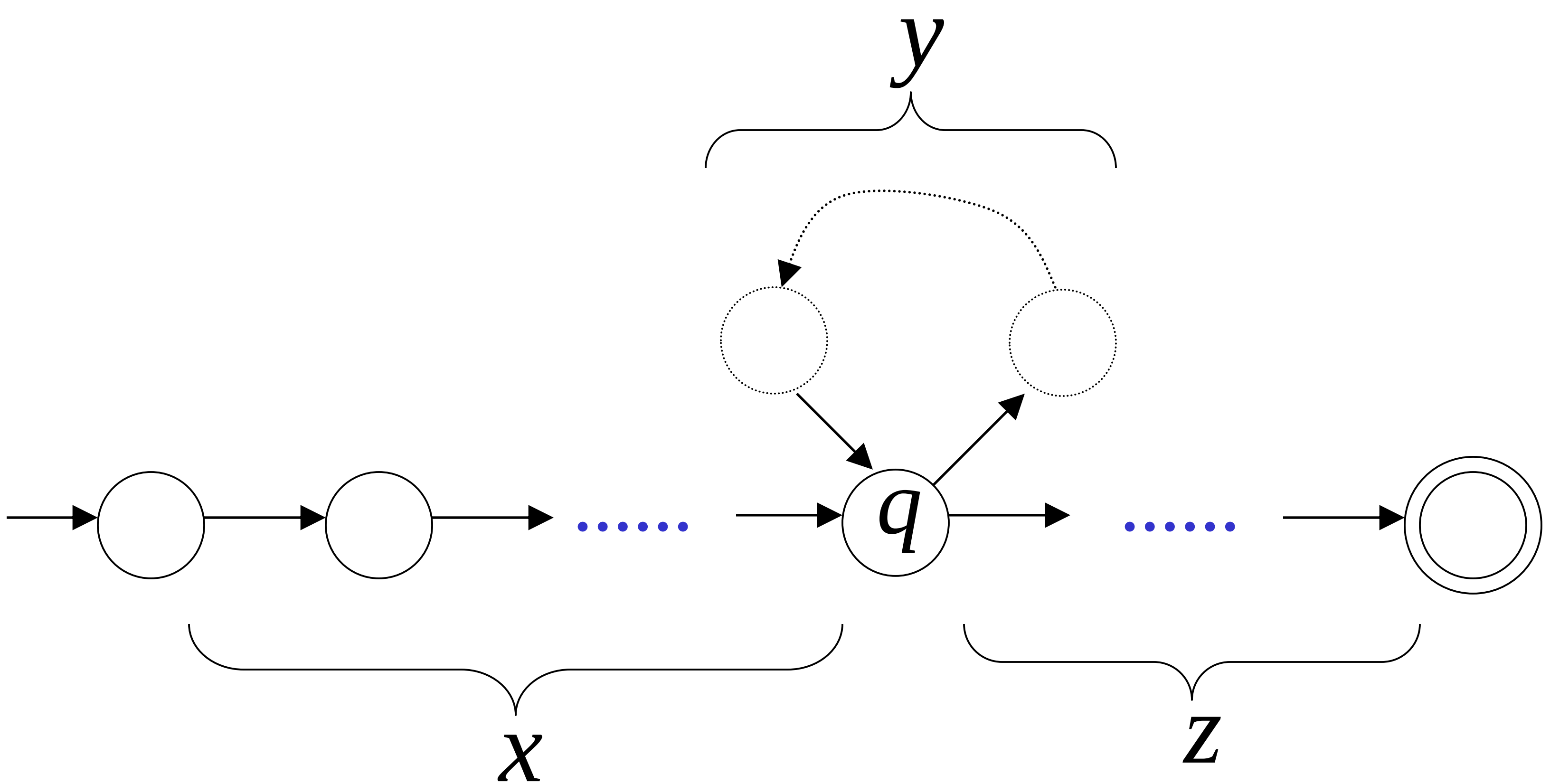


Observations: length $|x\,y| \leq m$ number of states of DFA

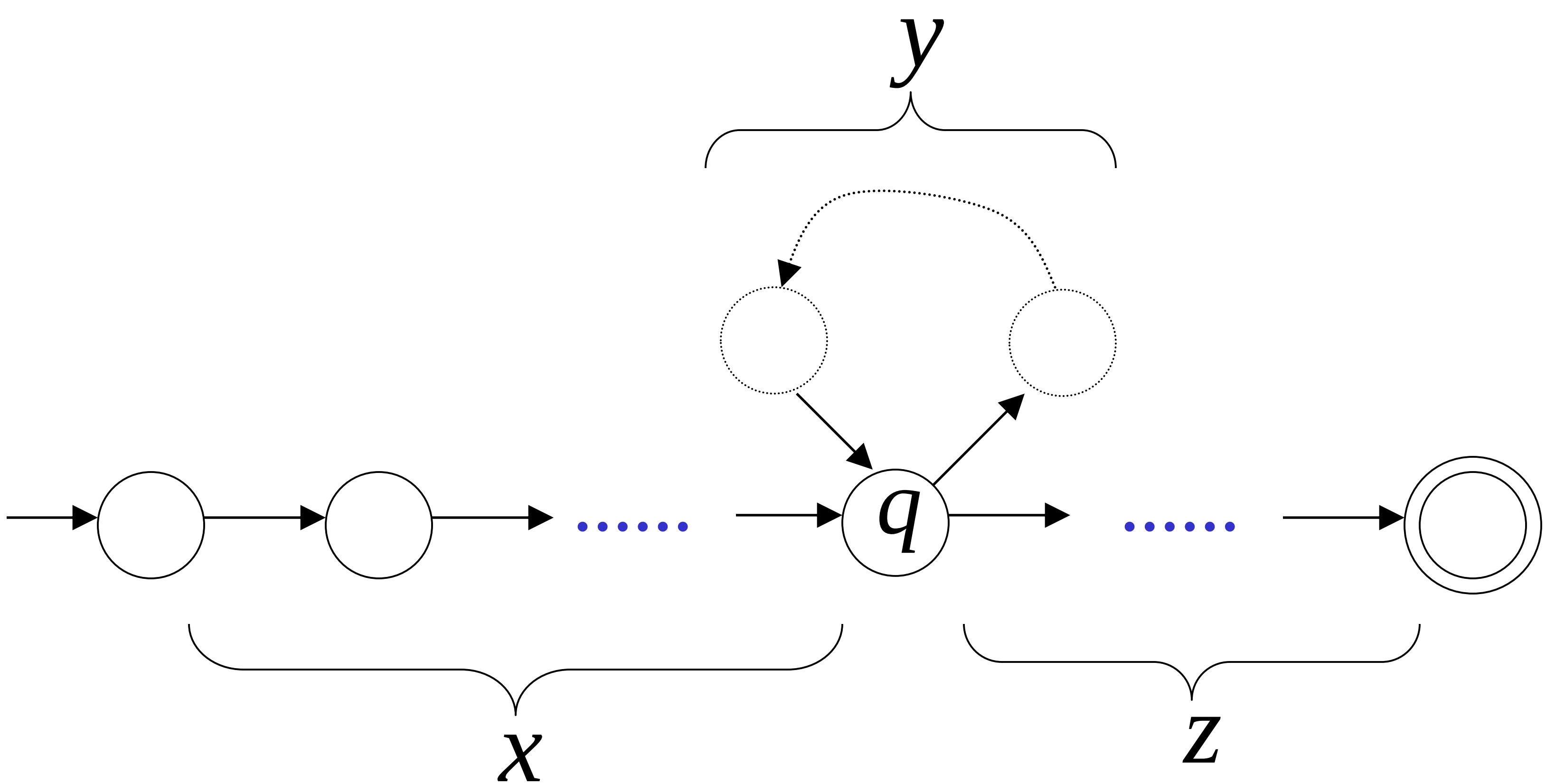
length $|y| \geq 1$



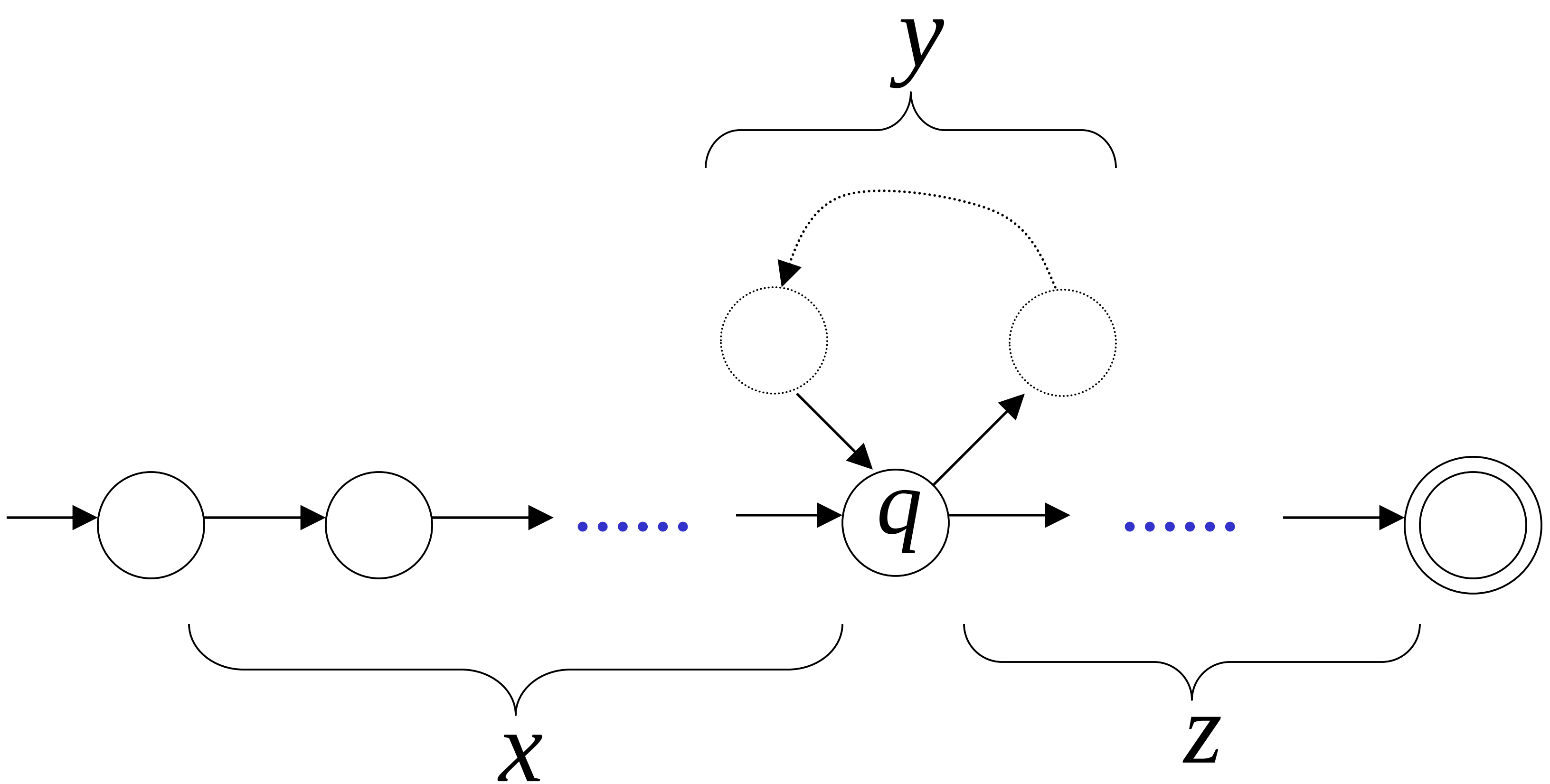
Observation: The string $x\,z$ is accepted



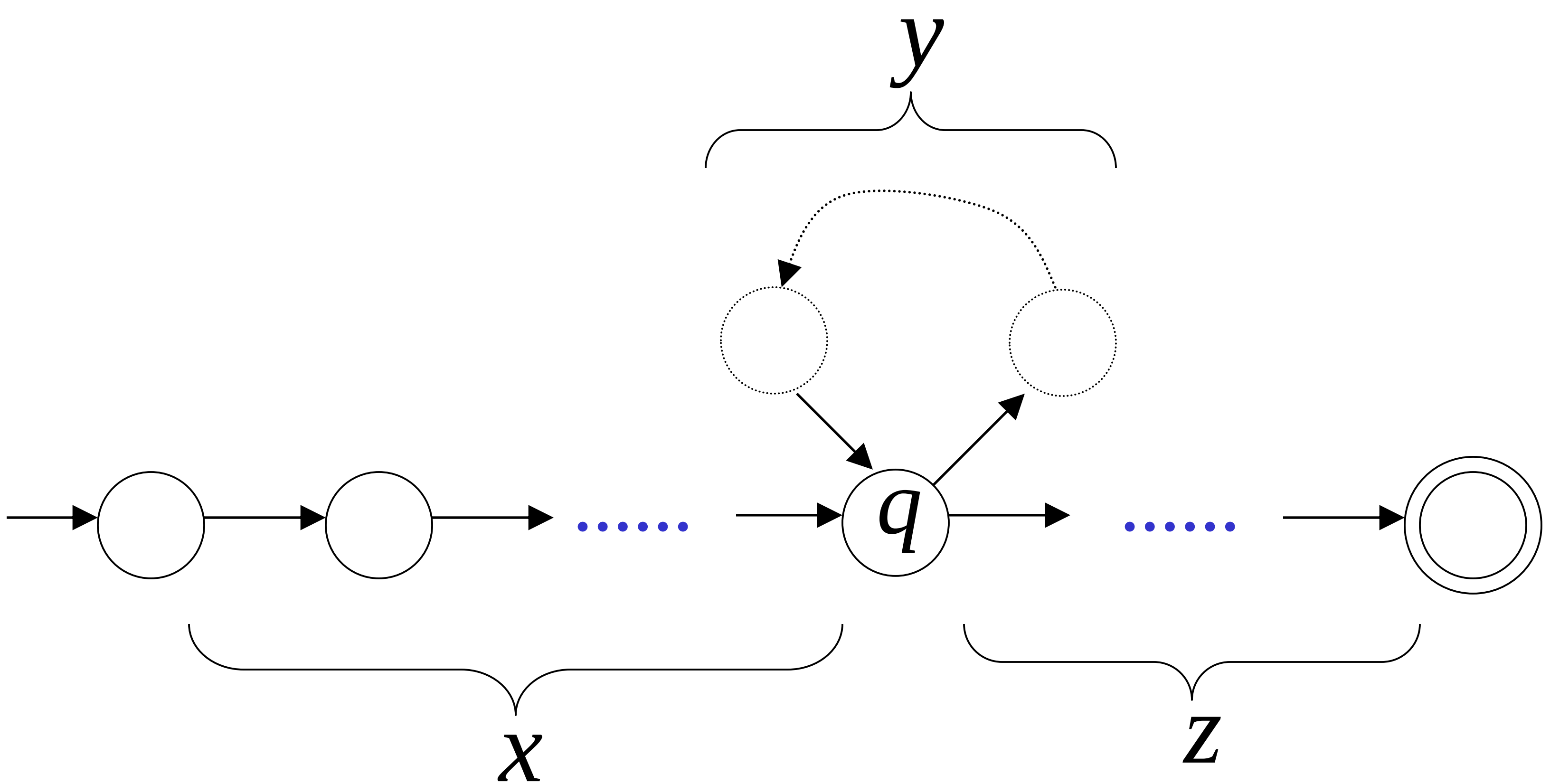
Observation: The string $x\ y\ y\ z$ is accepted



Observation: The string $x\,y\,y\,y\,z$ is accepted

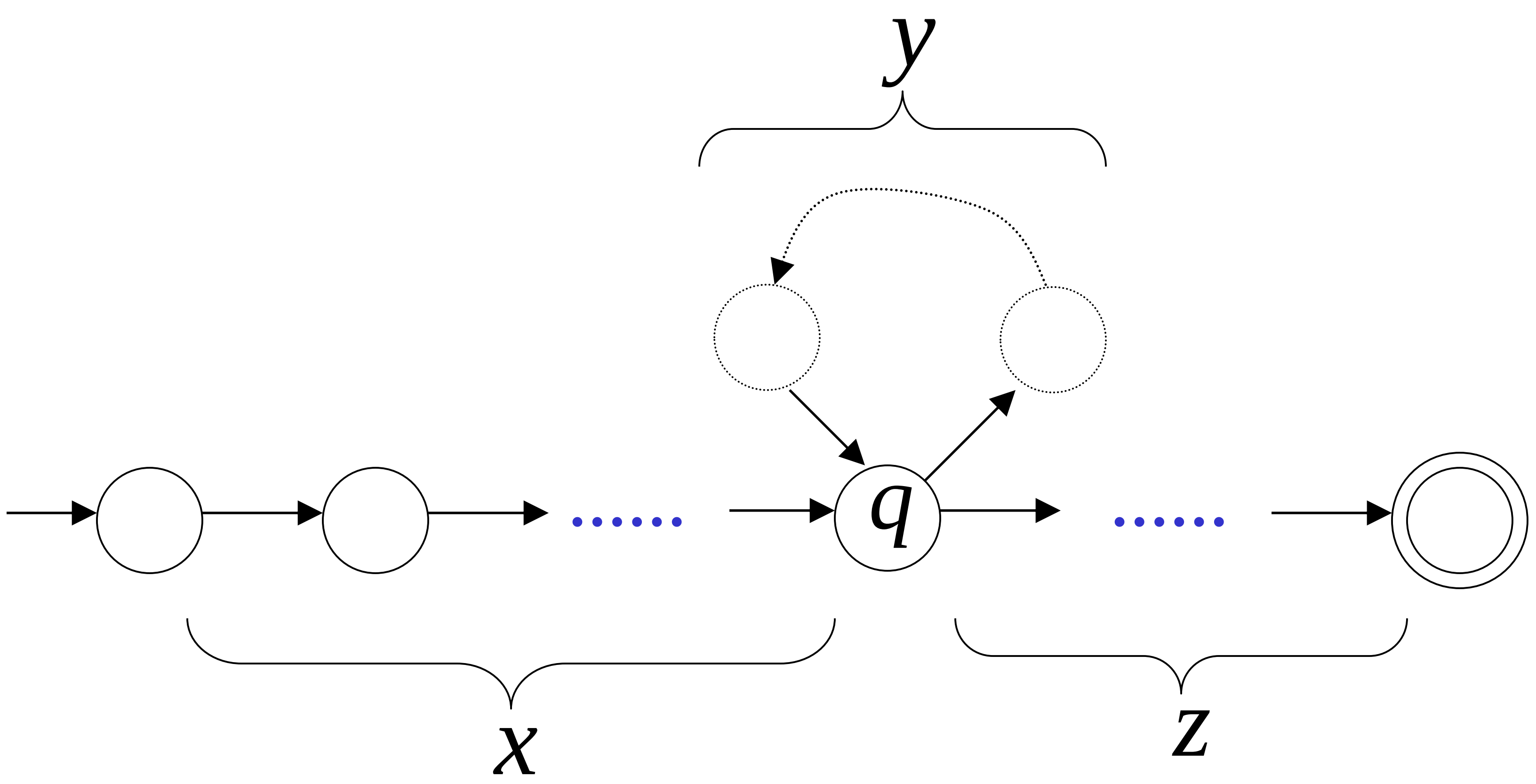


In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In General: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

The original language



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a ~~infinite~~ regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq m$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

~~Since L is infinite~~
we can apply the Pumping Lemma

$$L=\{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^m b^m \;=\; \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m}$$

$$\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \;\in\; L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \;\in\; L$

$$x\ y\ z = a^m b^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}a...a}^{m+k}\overbrace{b...b}^{m} \in L$$

(with z spanning $a...ab...b$)

Thus: $a^{m+k}b^m \in L$

$$a^{m+k}b^m \in L \qquad k \geq 1$$

BUT: $L = \{a^n b^n : n \geq 0\}$

$$\Downarrow$$

$$a^{m+k}b^m \notin L$$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular language $\{a^n b^n : \ n \geq 0\}$