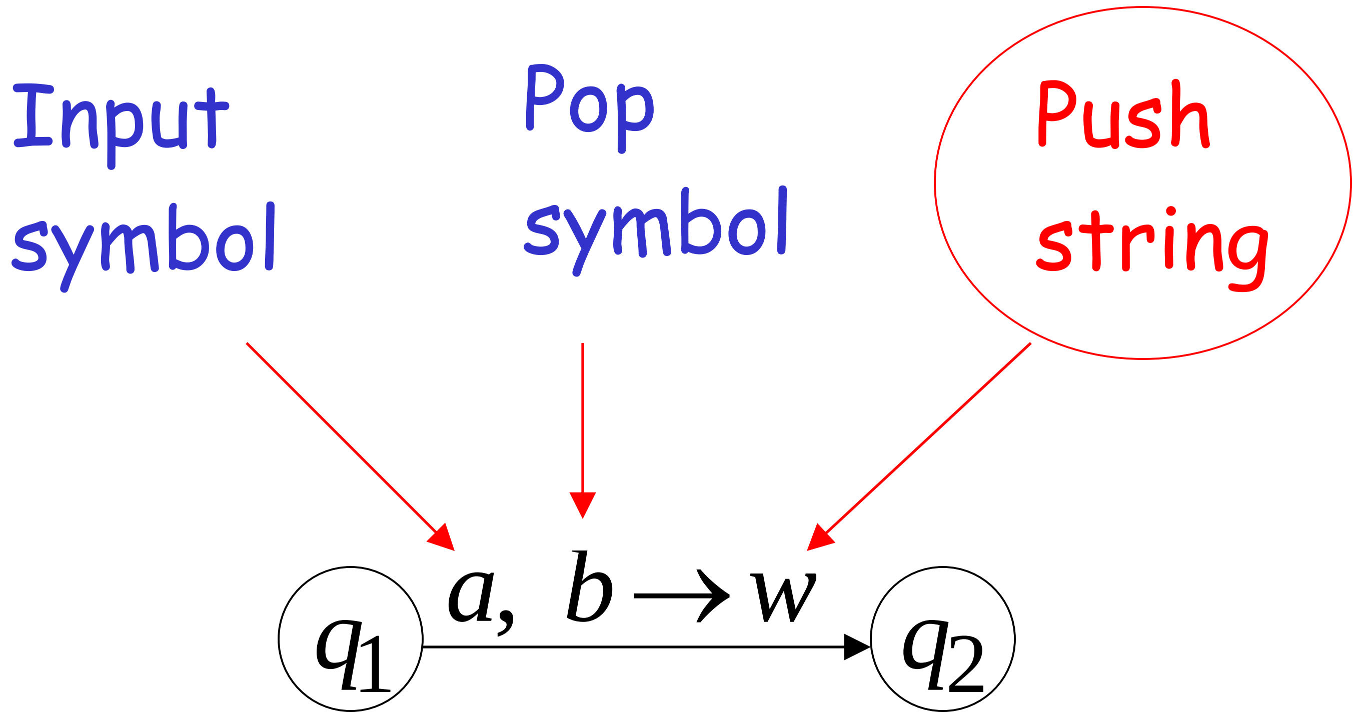
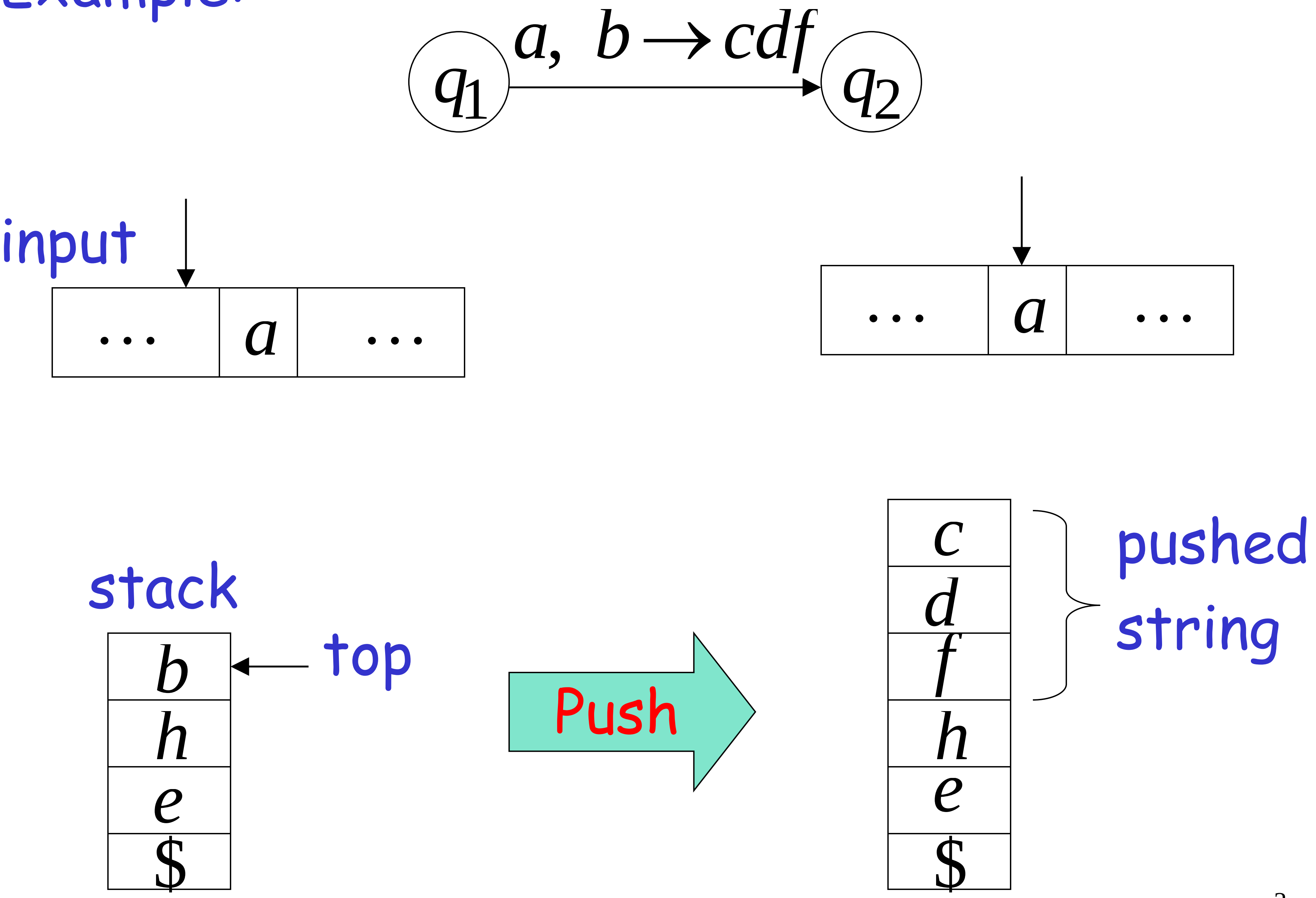
… NPDAs continued

Pushing Strings



Example:



$a, \ b \to cdf$
q_1
q_2
input
$\ldots$ a $\ldots$
$\ldots$ a $\ldots$
stack
top
b
h
e
$\$$
Push
c
d
f
h
e
$\$$
pushed string

Another NPDA example

Linz 6th, Section 7.1,
Example 7.4, page 187.

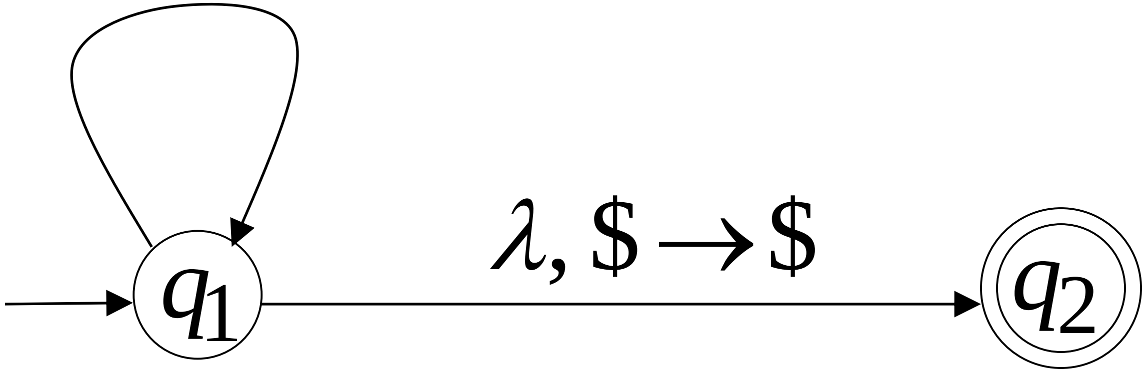
NPDA M

$$L(M) = \{w: \ n_a = n_b\}$$

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Execution Example: Time 0

Input

a	b	b	a	a	b

$\$$

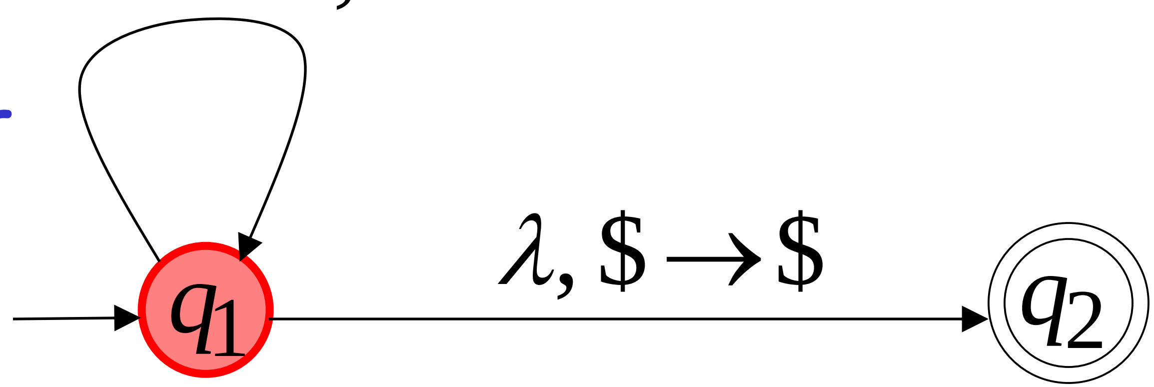
Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$

current state



Time 1

Input

a	b	b	a	a	b

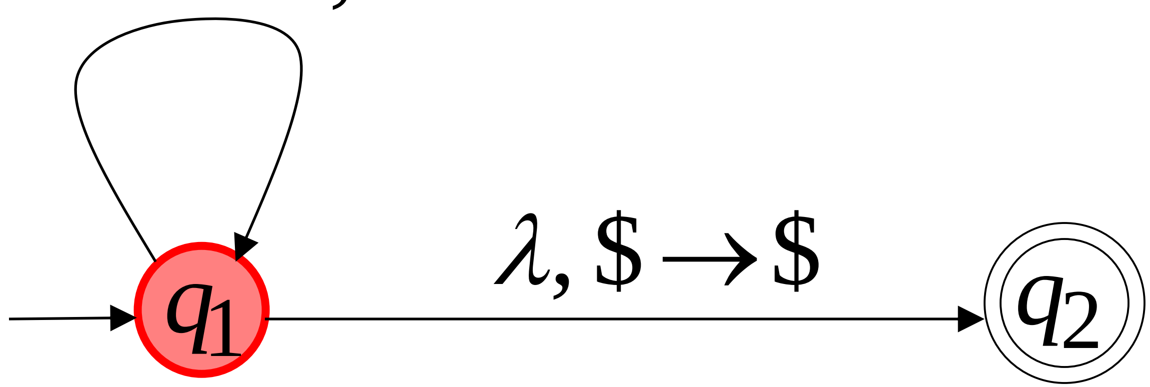
Stack
0
$\$$

Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Time 3

Input

a	b	b	b	a	a

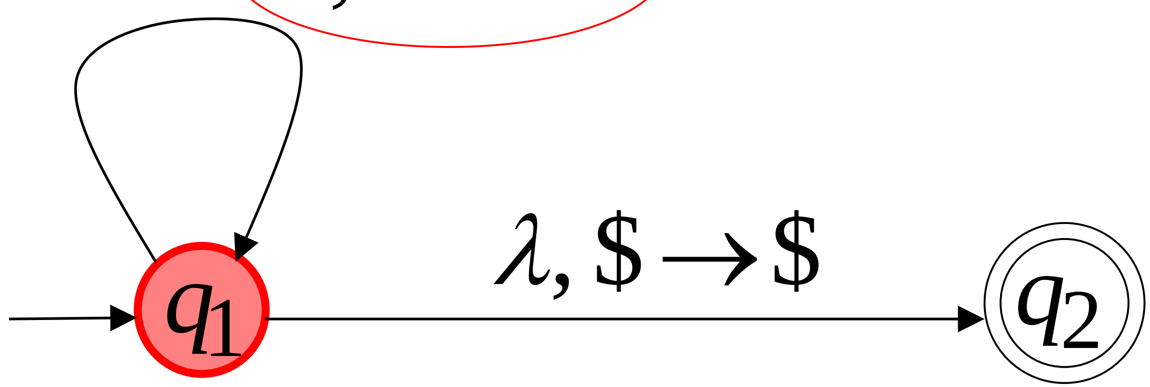
Stack

$\cancel{0}$
$\$$

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Time 4

Input

a	b	b	b	a	a

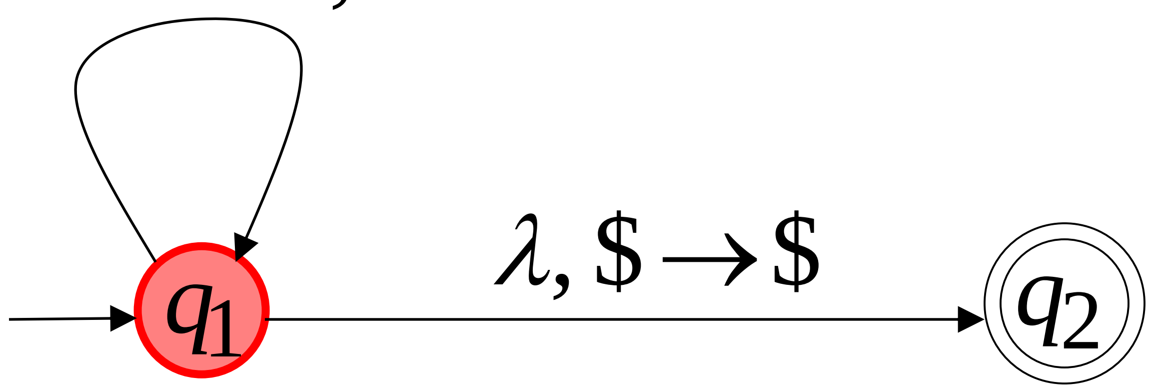
1
$

Stack

$$a,\$ \rightarrow 0\$ \qquad b,\$ \rightarrow 1\$$$

$$a,0 \rightarrow 00 \qquad b,1 \rightarrow 11$$

$$a,1 \rightarrow \lambda \qquad b,0 \rightarrow \lambda$$



Time 5

Input

a	b	b	b	a	a

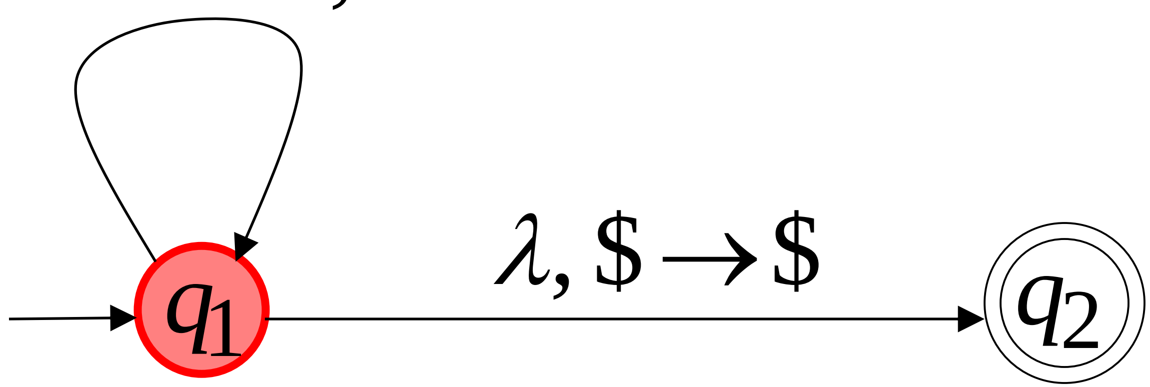
1
1
\$

Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Time 6

Input

a	b	b	b	a	a

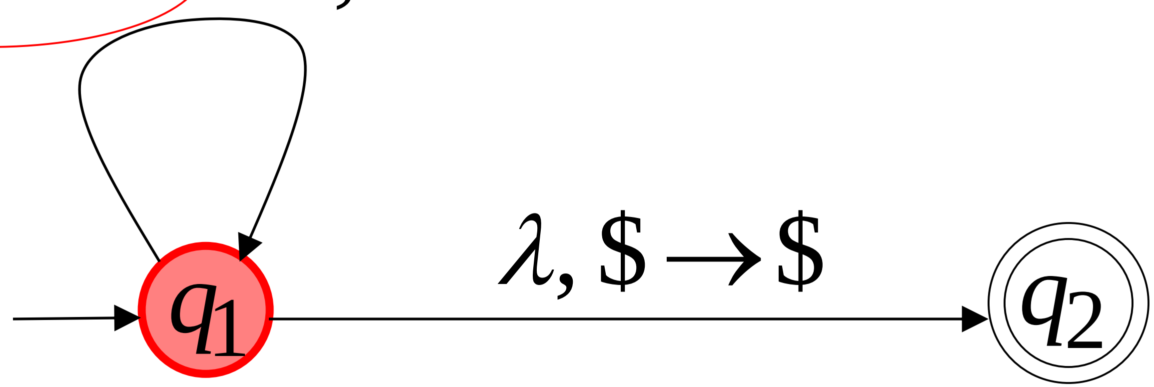
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

$\cancel{1}$
1
$\$$

Stack



Time 7

Input

a	b	b	b	a	a

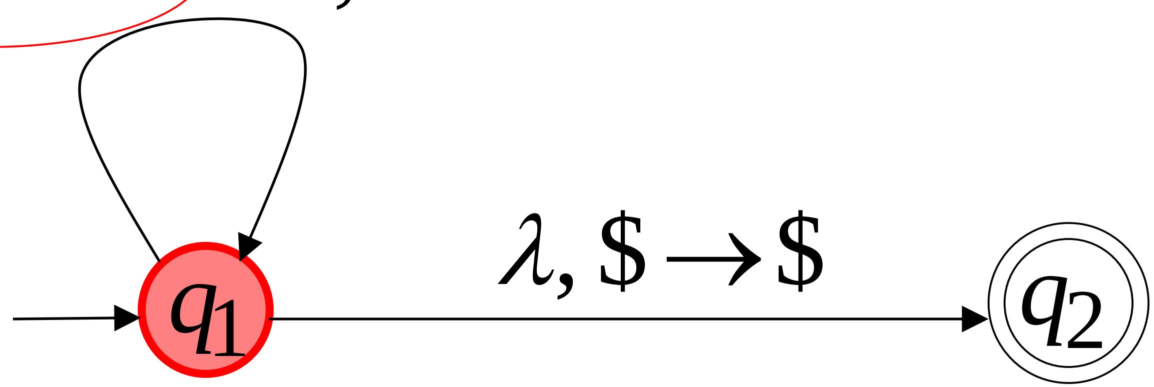
$\cancel{1}$
$\$$

Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Time 8

Input

a	b	b	b	a	a

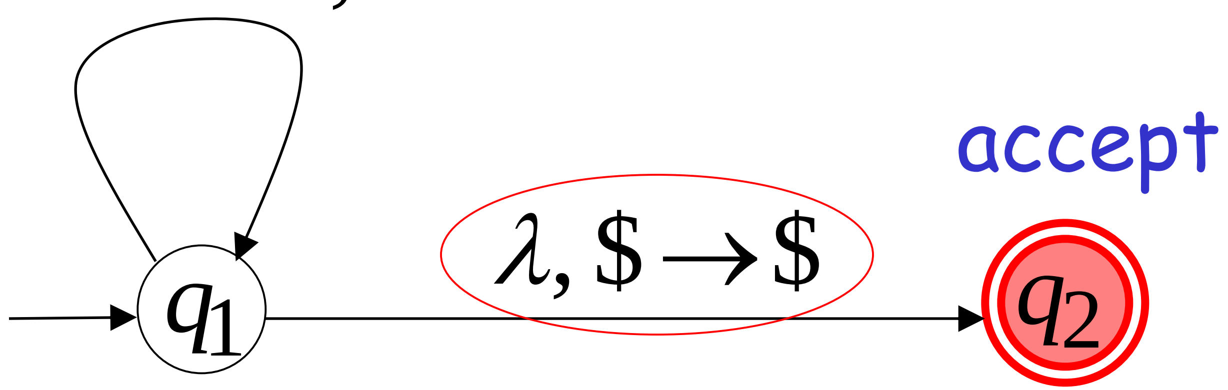
$\$$

Stack

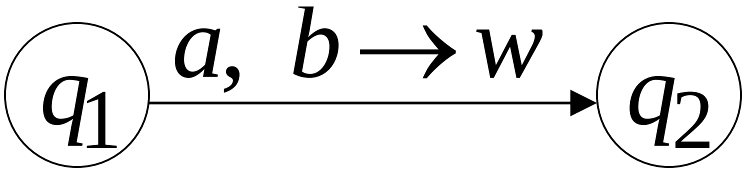
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

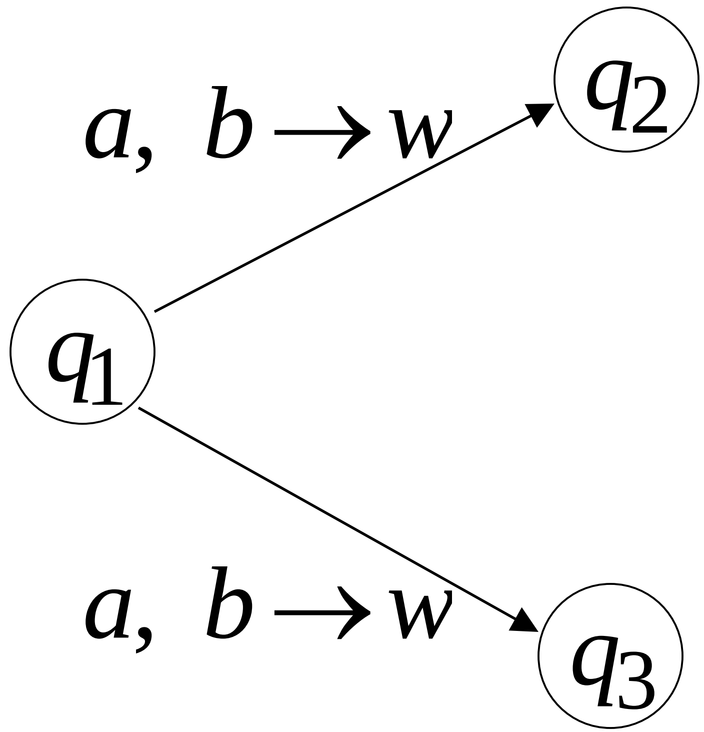


Formalities for NPDAs



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w)\}$$



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w), \ (q_3, w)\}$$

Formal Definition

Non-Deterministic Pushdown Automaton

NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

Transition function

Initial state

Stack start symbol

Final states

Instantaneous Description

$$(q, \ u, \ s)$$

Current state

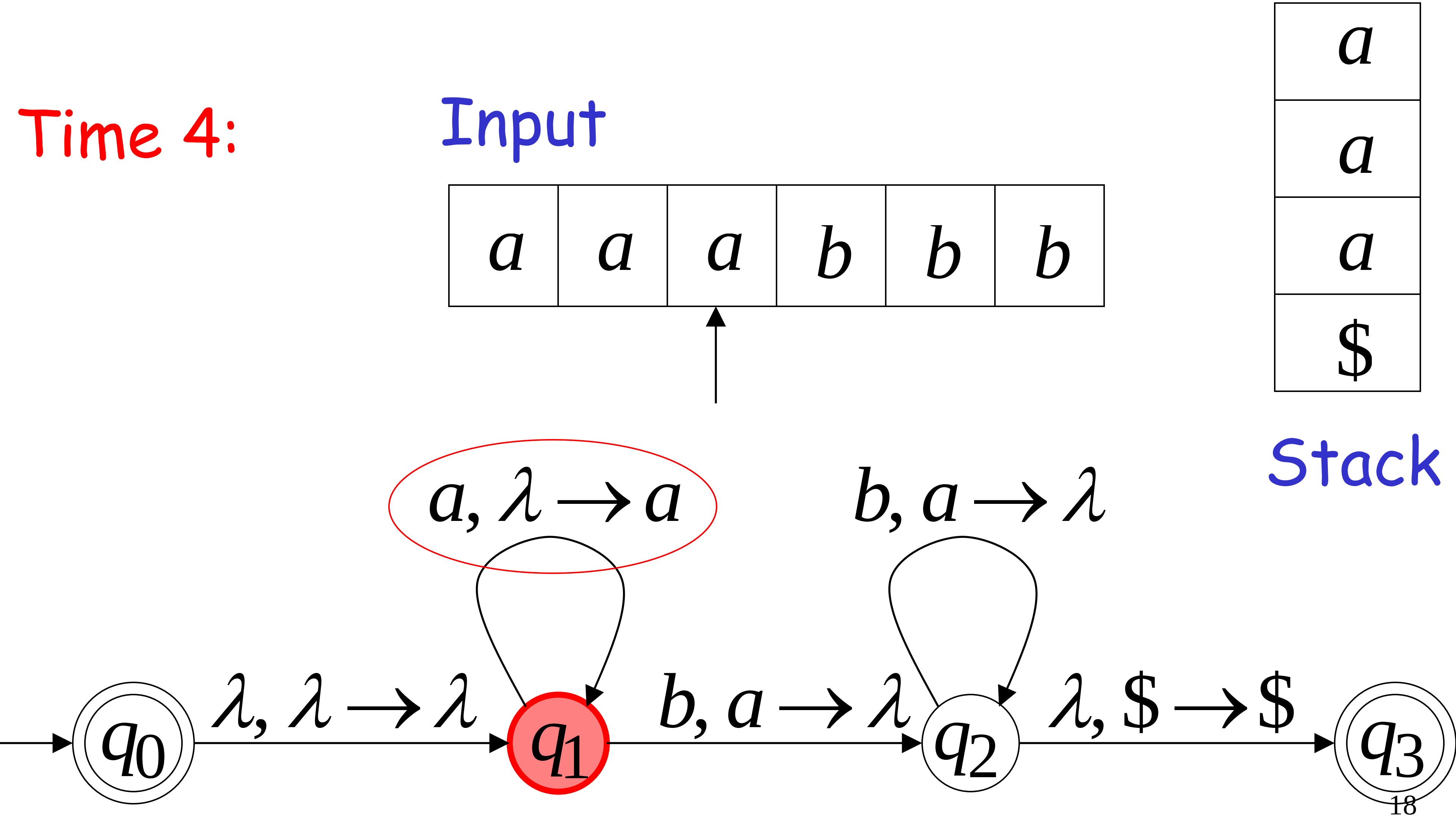
Remaining input

Current stack contents

Example: **Instantaneous Description**

$$(q_1, bbb, aaa\$)$$

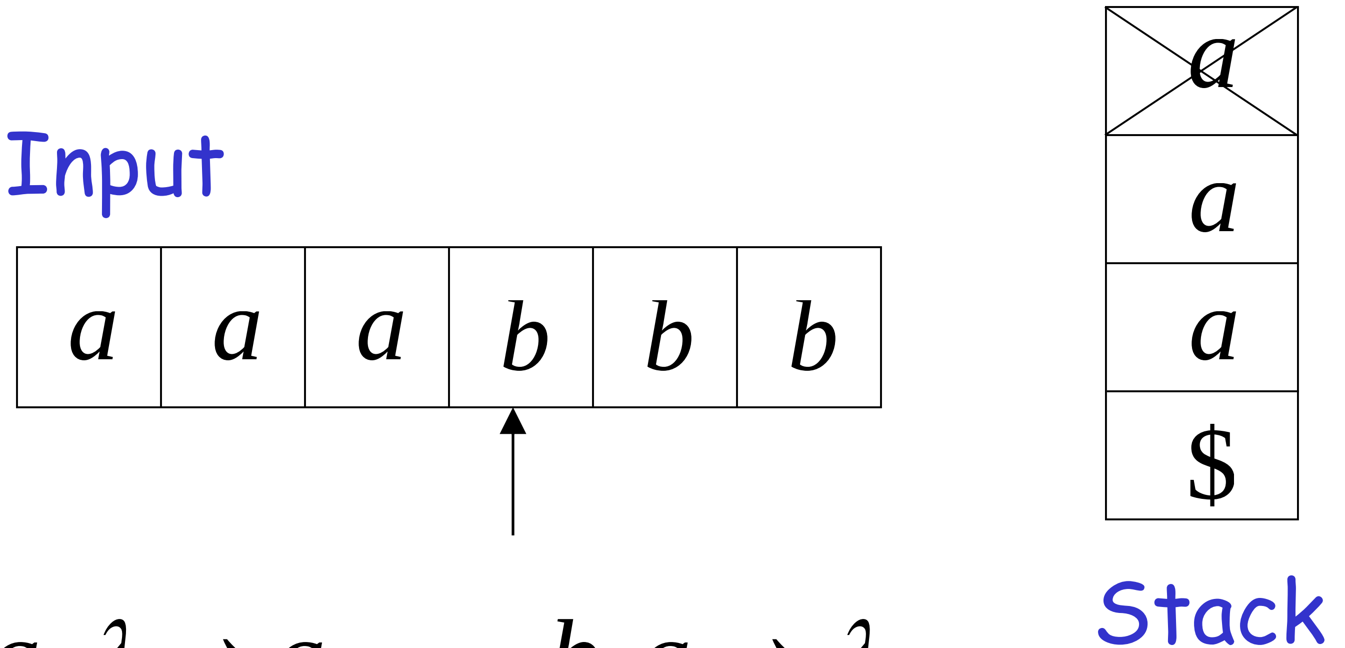
Time 4:

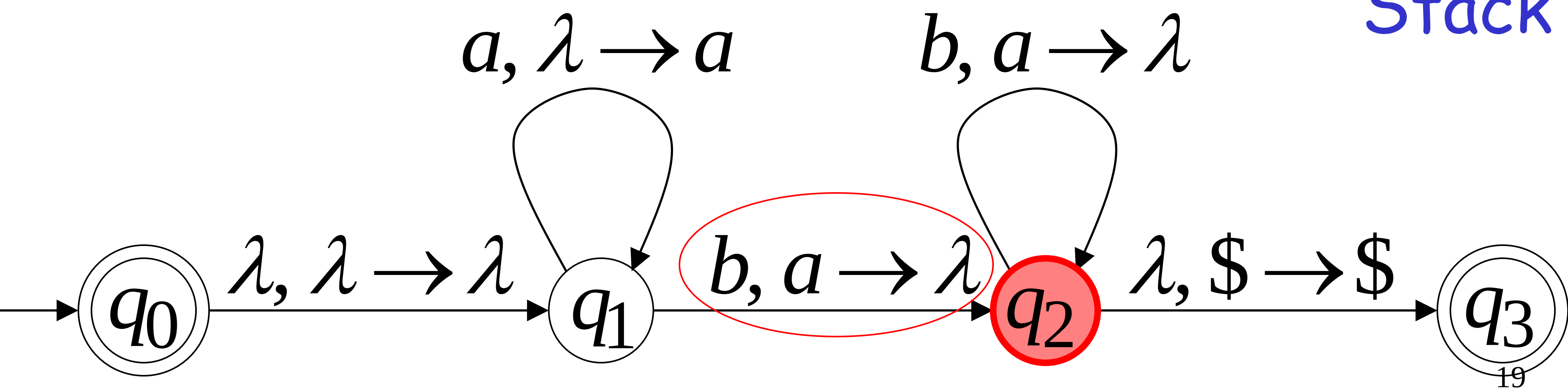


Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

Time 5:





We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

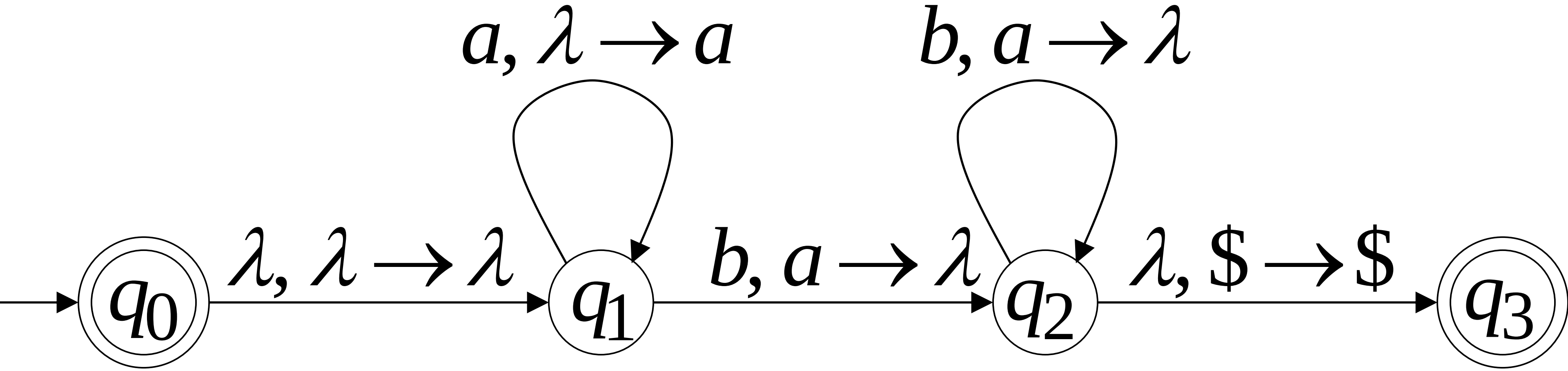
Time 4 Time 5

A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Formal Definition

Language of NPDA M:

$$L(M) = \{w: \ (q_0, w, s) \overset{*}{\succ} (q_f, \lambda, s')\}$$

Initial state

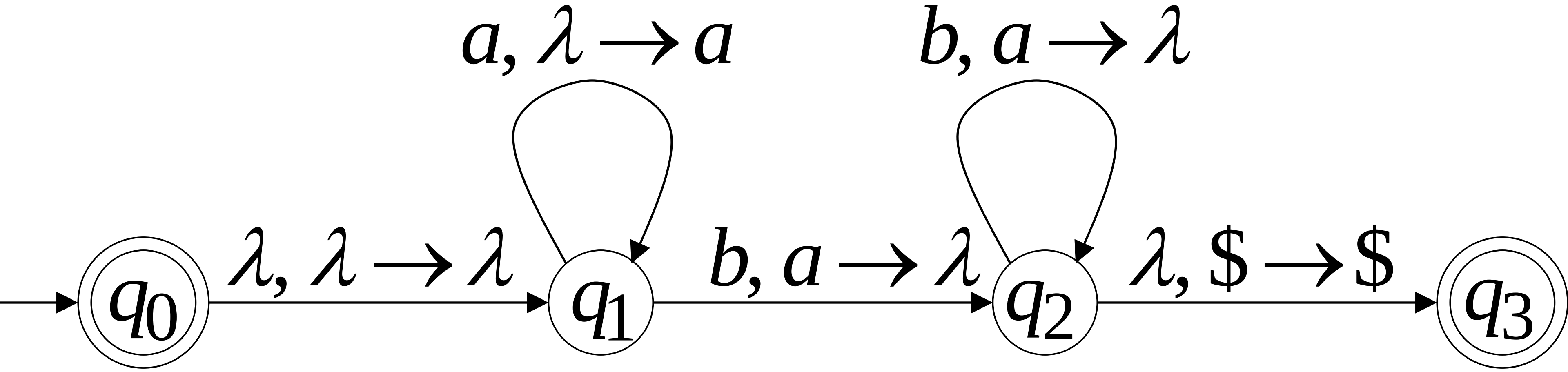
Final state

Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$aaabbb \in L(M)$$

NPDA M:

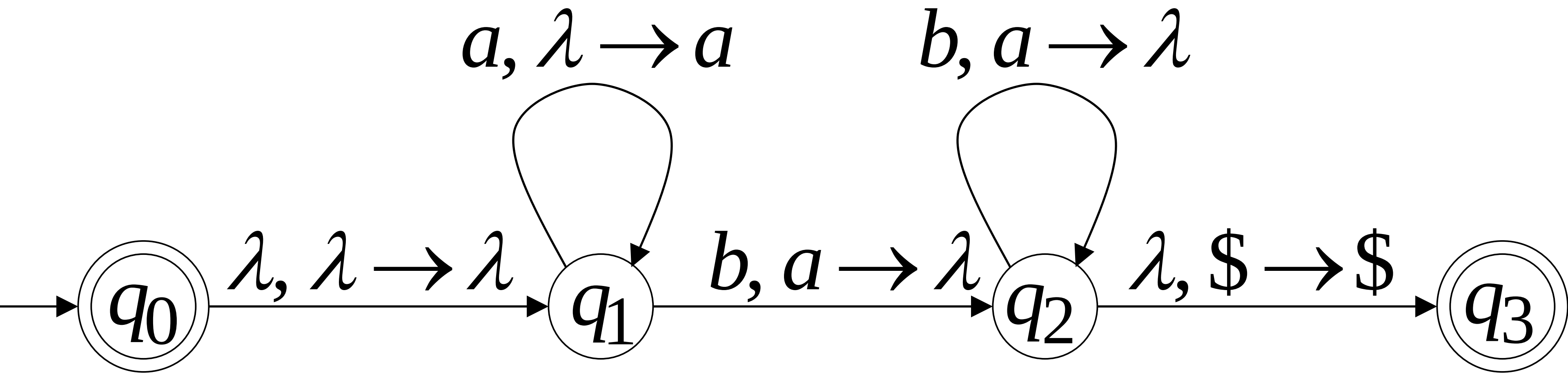


$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$\Downarrow$$

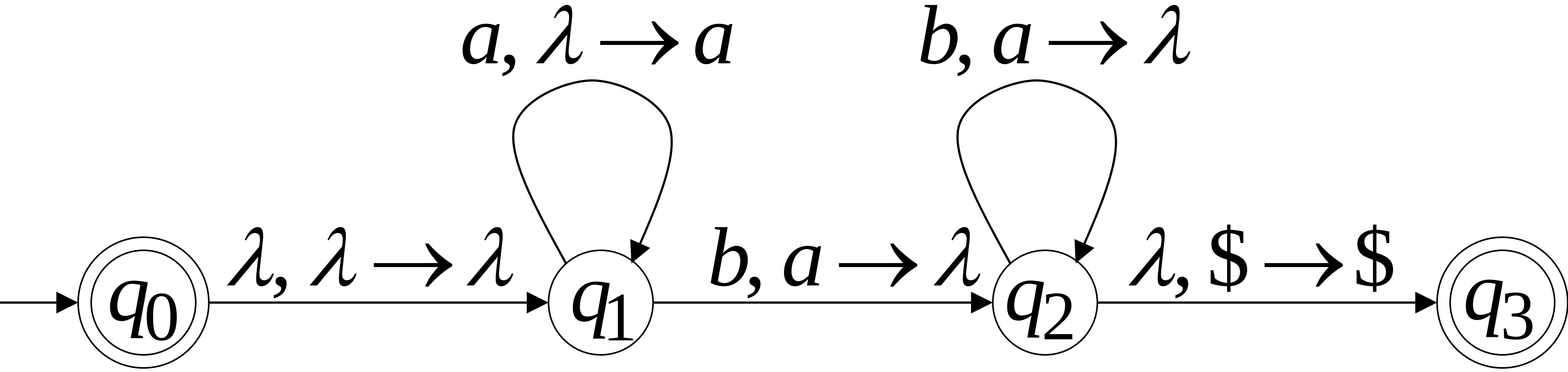
$$a^n b^n \in L(M)$$

NPDA M :



Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

NPDA M:



NPDAs Accept Context-Free Languages

Theorem:

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} = \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{NPDAs}\end{array}\right\}$$

Proof - Step 1:

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{NPDAs}\end{array}\right\}$$

Convert any context-free grammar G to a NPDA M with: $L(G) = L(M)$

Proof - Step 2:

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} \supseteq \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{NPDAs}\end{array}\right\}$$

Convert any NPDA M to a context-free grammar G with: $L(G) = L(M)$

Converting
Context-Free Grammars
to
NPDAs

An example grammar:

$$S \to aSTb$$

$$S \to b$$

$$T \to Ta$$

$$T \to \lambda$$

What is the equivalent NPDA?

Grammar:

$$S \rightarrow aSTb$$

$$S \rightarrow b$$

$$T \rightarrow Ta$$

$$T \rightarrow \lambda$$

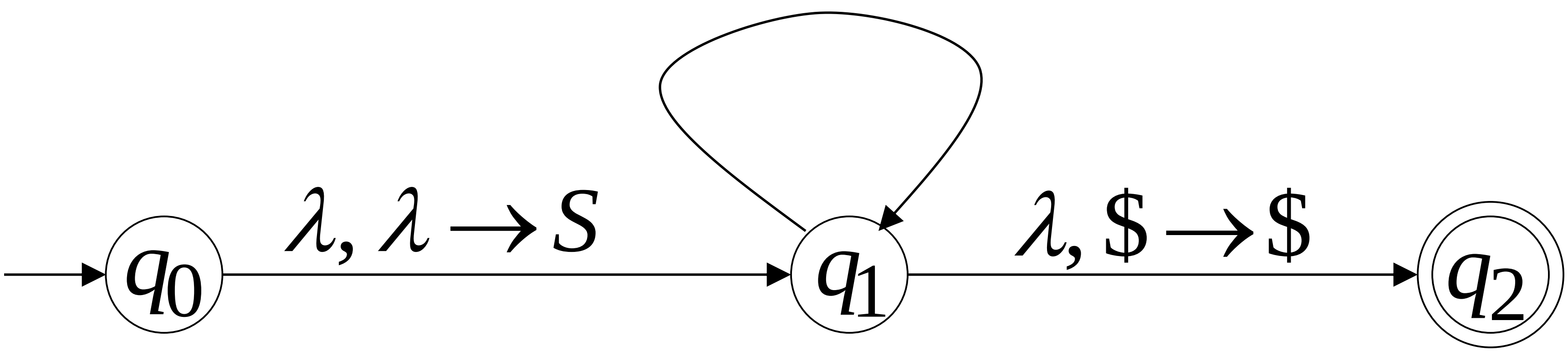
NPDA:

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$



The NPDA simulates
leftmost derivations of the grammar

L(Grammar) = L(NPDA)

Grammar:

$$S \to aSTb$$

$$S \to b$$

$$T \to Ta$$

$$T \to \lambda$$

A leftmost derivation:

$$S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$$

NPDA execution: Time 0

Input

a	b	a	b

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

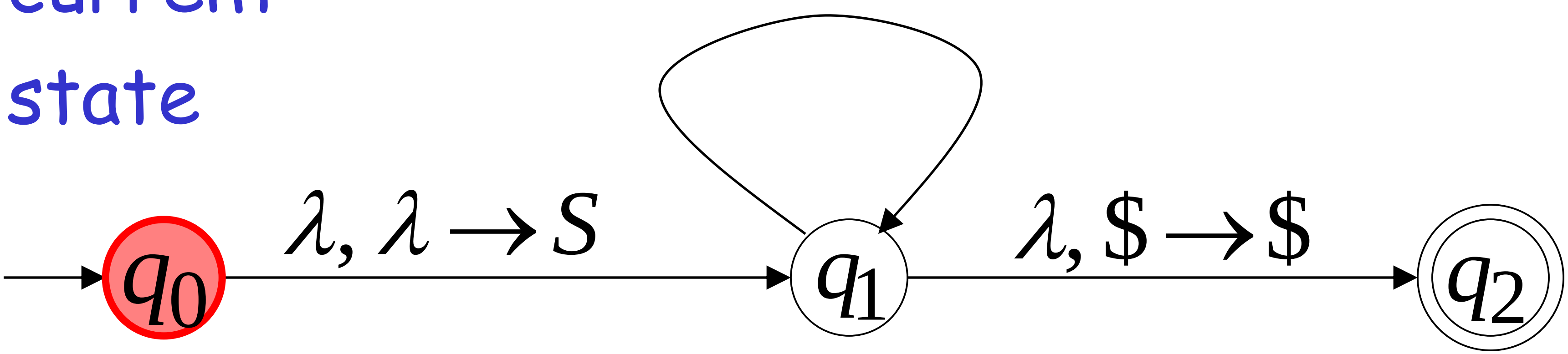
$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$

$\$$

Stack

current state



Time 1

Input

a	b	a	b

Stack

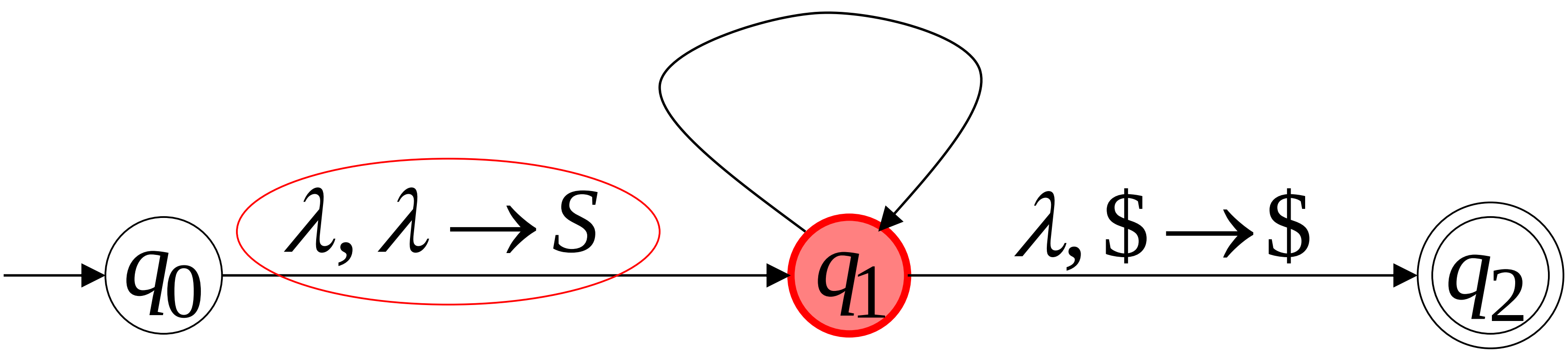
S
$\$$

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$



Time 2

Input

a	b	a	b

a
S
T
b
$\$$

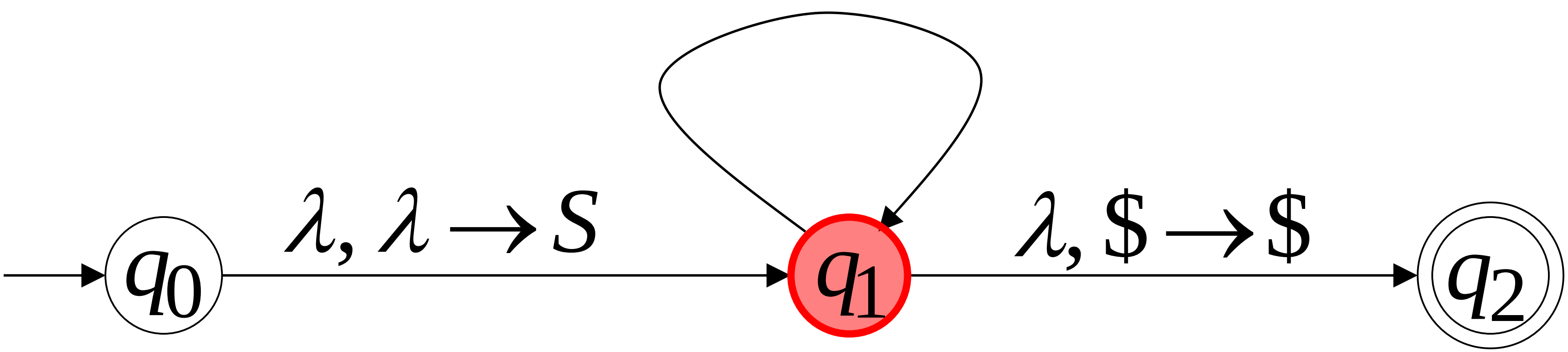
Stack

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$



Time 3

Input

a	b	a	b

$\uparrow$

Stack

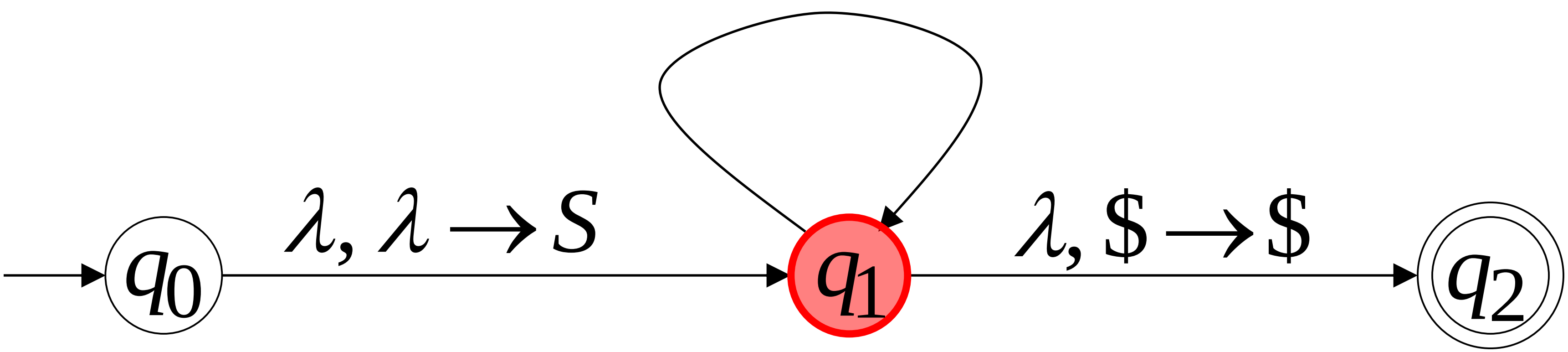
$\cancel{a}$
S
T
b
$\$$

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$



Time 4

Input

a	b	a	b

Stack

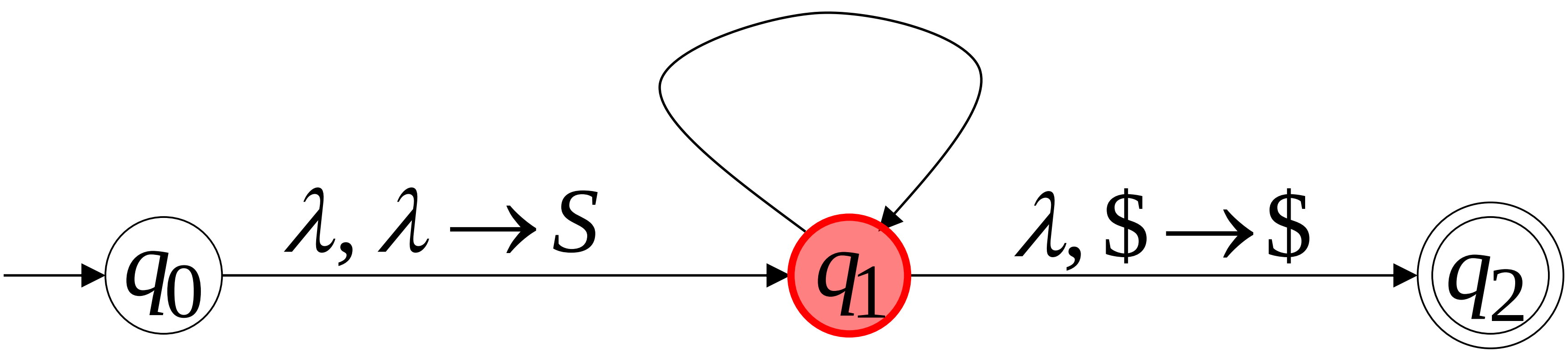
b
T
b
$\$$

$$\lambda, S \to aSTb$$

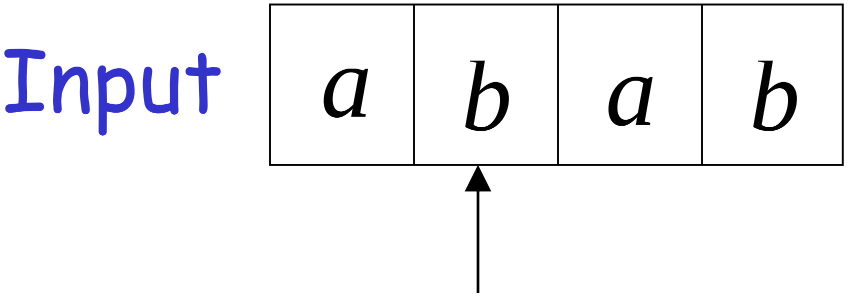
$$\lambda, S \to b$$

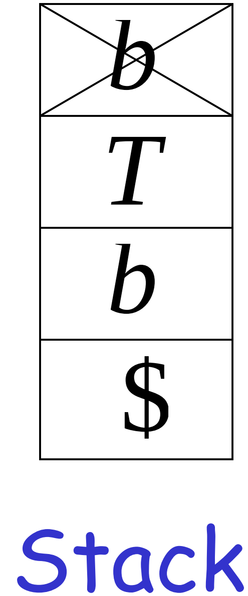
$$\lambda, T \to Ta \qquad a, a \to \lambda$$

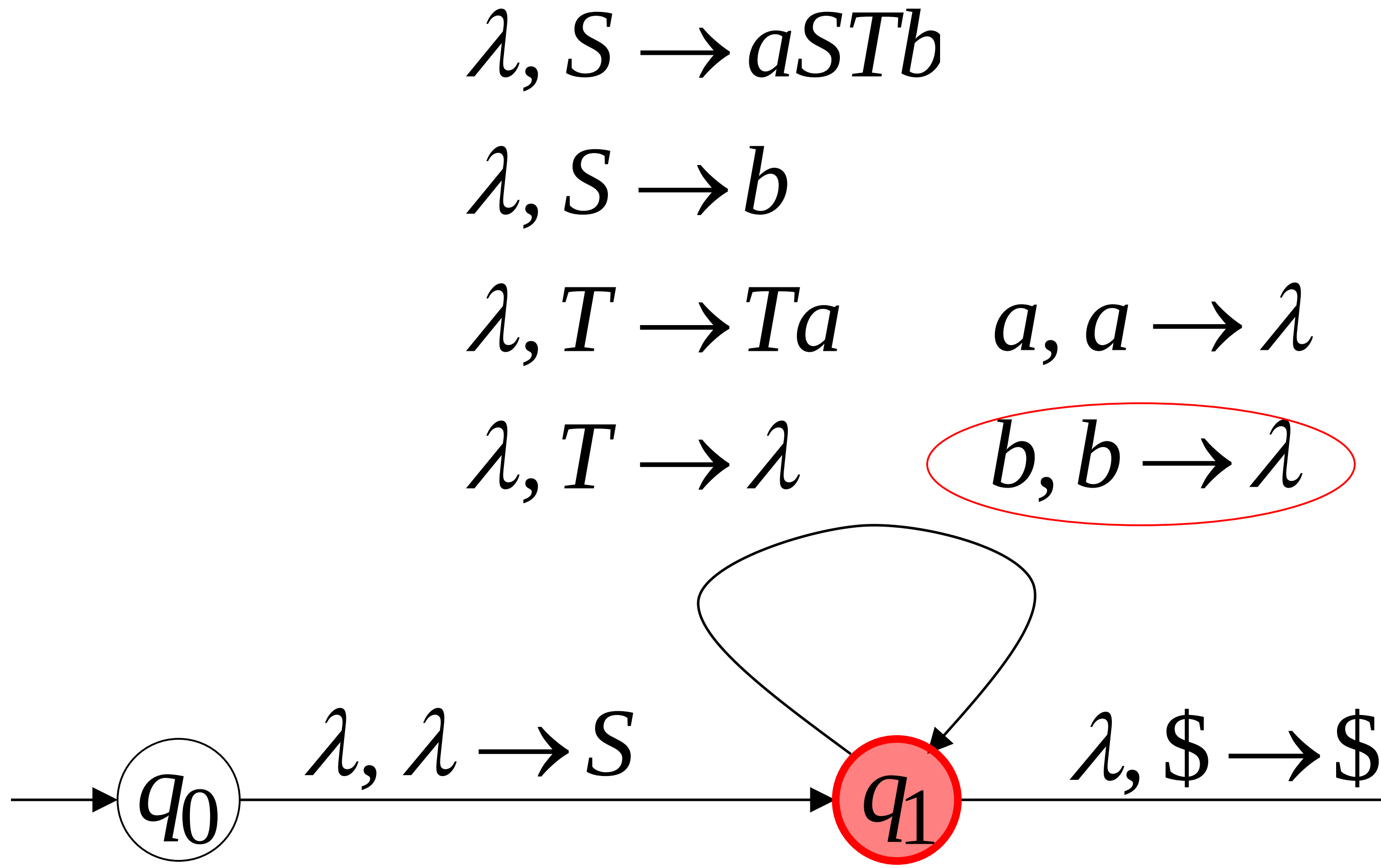
$$\lambda, T \to \lambda \qquad b, b \to \lambda$$



Time 5


Input
a
b
a
b


$\cancel{b}$
T
b
$
Stack


$\lambda, S \rightarrow aSTb$
$\lambda, S \rightarrow b$
$\lambda, T \rightarrow Ta$
$\lambda, T \rightarrow \lambda$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow S$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 6

Input

a	b	a	b

$$\lambda, S \rightarrow aSTb$$

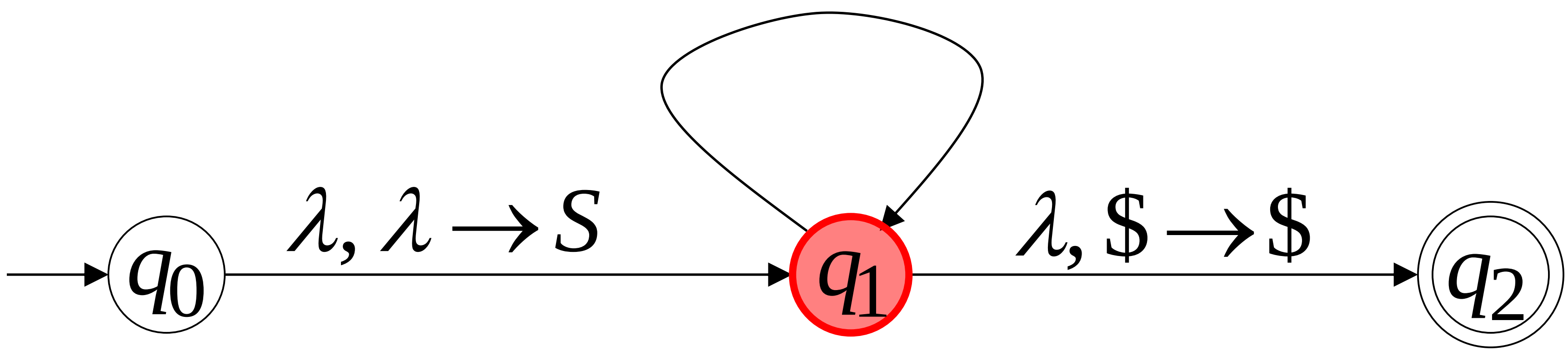
$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$

T
a
b
$\$$

Stack



Time 7

Input

a	b	a	b

Stack

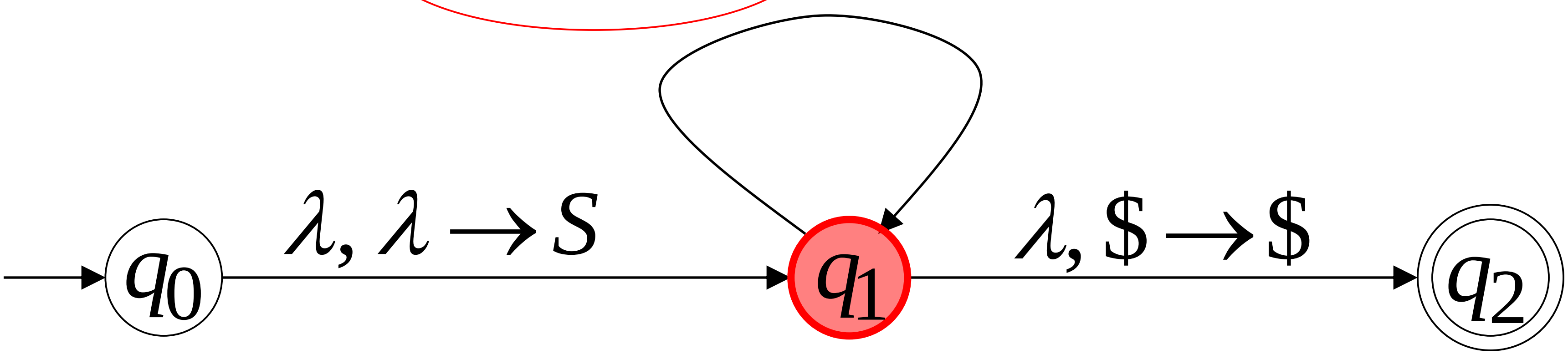
$\cancel{T}$
a
b
$\$$

$$\lambda, S \to aSTb$$

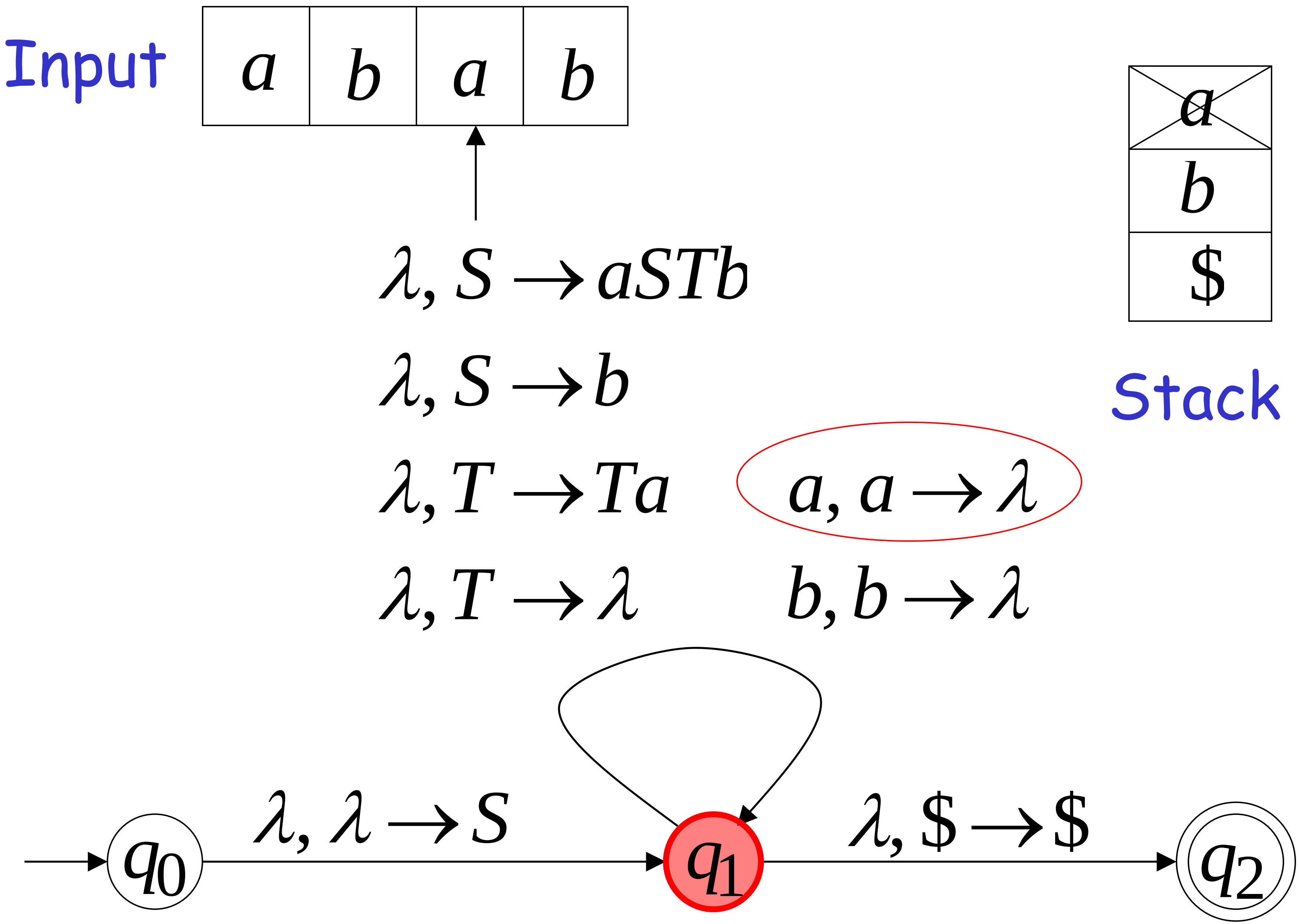
$$\lambda, S \to b$$

$$\lambda, T \to Ta \qquad a, a \to \lambda$$

$$\lambda, T \to \lambda \qquad b, b \to \lambda$$



Time 8



Input
a b a b
a
b
$
Stack
$\lambda, S \rightarrow aSTb$
$\lambda, S \rightarrow b$
$\lambda, T \rightarrow Ta$
$a, a \rightarrow \lambda$
$\lambda, T \rightarrow \lambda$
$b, b \rightarrow \lambda$
q_0
$\lambda, \lambda \rightarrow S$
q_1
$\lambda, \$ \rightarrow \$$
q_2

Time 9

Input

a	b	a	b

$\cancel{b}$
$\$$

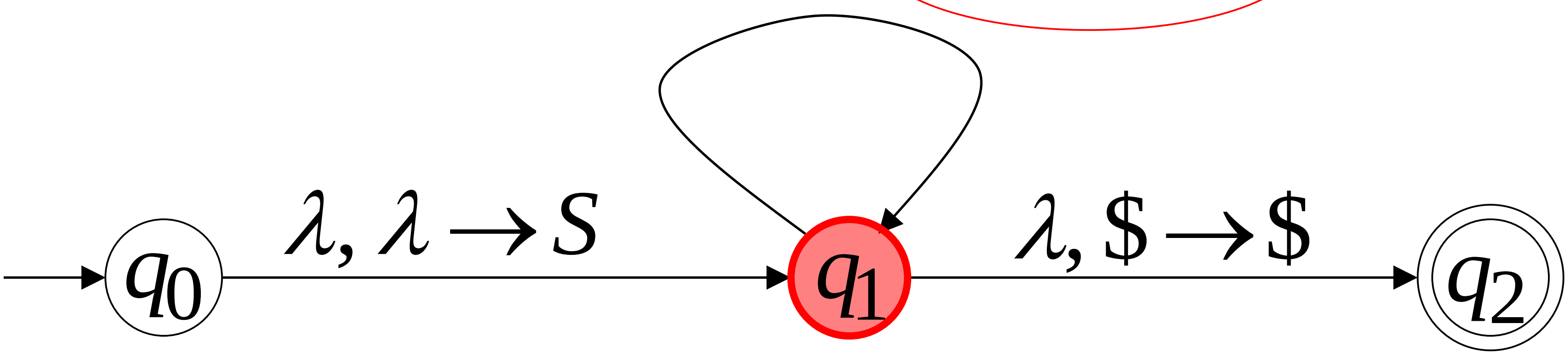
Stack

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$



Time 10

Input

a	b	a	b

$$\lambda, S \rightarrow aSTb$$

$$\lambda, S \rightarrow b$$

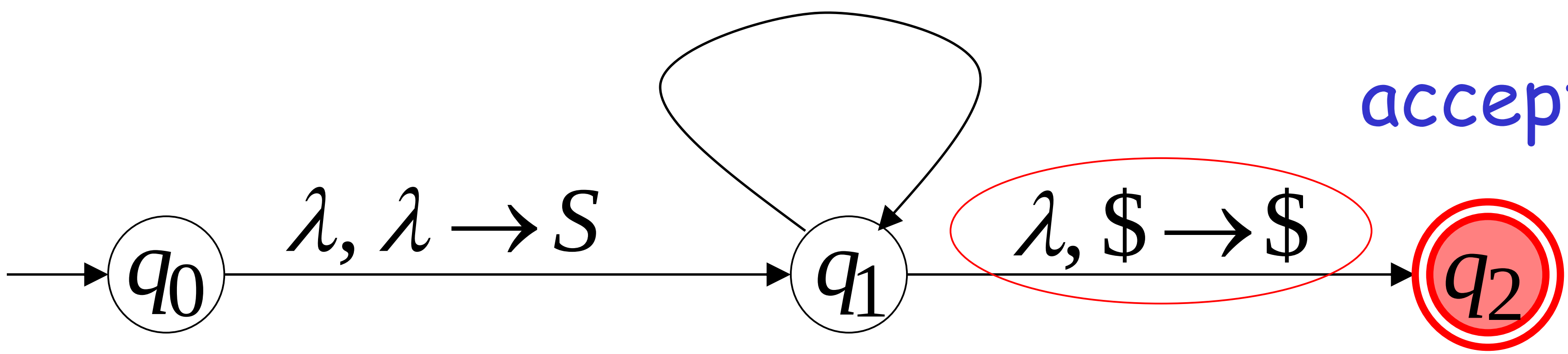
$$\lambda, T \rightarrow Ta \qquad a, a \rightarrow \lambda$$

$$\lambda, T \rightarrow \lambda \qquad b, b \rightarrow \lambda$$

$\$$

Stack

accept



In general:

Given any grammar G

We can construct a NPDA M

With $L(G) = L(M)$

Constructing NPDA M from grammar G:

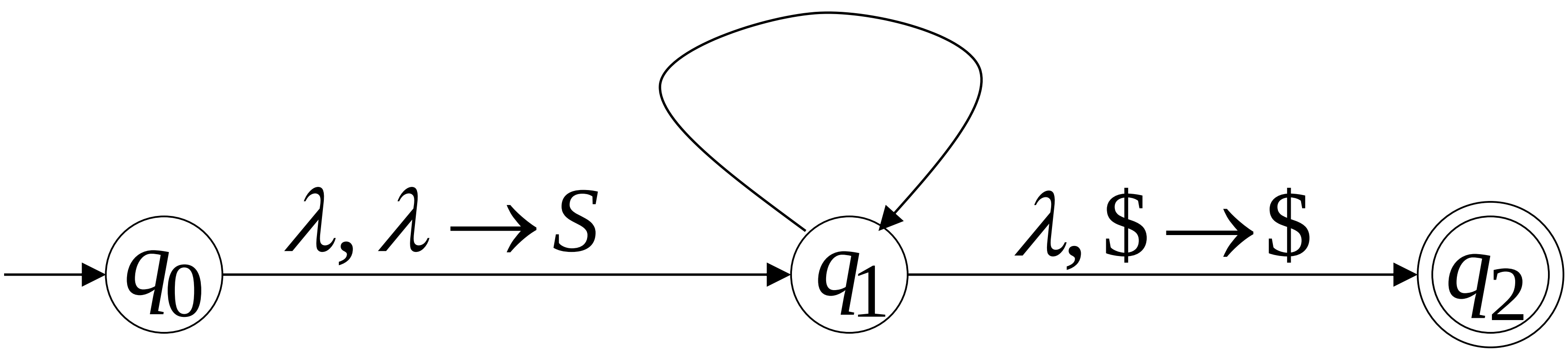
For any production

$$A \rightarrow w$$

For any terminal

$$a$$

$$\lambda, A \rightarrow w \qquad a, a \rightarrow \lambda$$



Grammar G generates string w

if and only if

NPDA M accepts w

$$L(G) = L(M)$$

Therefore:

For any context-free language
there is an NPDA
that accepts the same language

Converting NPDAs to Context-Free Grammars

For any NPDA M

we will construct

a context-free grammar G with

$$L(M) = L(G)$$

Intuition: The grammar simulates the machine

A derivation in Grammar G :

$$S \Rightarrow \cdots \Rightarrow \underbrace{abc\ldots ABC\ldots} \Rightarrow \ldots \Rightarrow abc\ldots$$

Current configuration in NPDA M

A derivation in Grammar G :

terminals variables

$$S \Rightarrow \cdots \Rightarrow abc\ldots ABC\ldots \Rightarrow \ldots \Rightarrow abc\ldots$$

Input processed Stack contents

in NPDA M

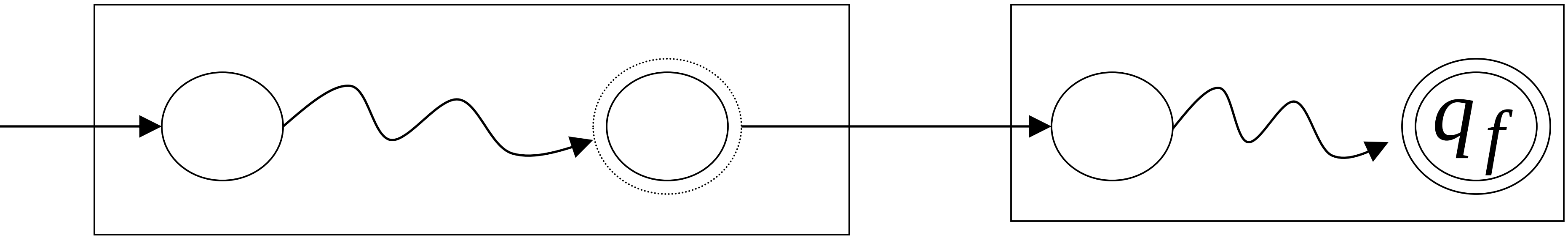
Some Necessary Modifications

First, we modify the NPDA:

- It has a single final state q_f
- It empties the stack
  when it accepts the input

Original NPDA

Empty Stack



Second, we modify the NPDA transitions:

all transitions will have form

$$q_i \xrightarrow{a,\ B \to \lambda} q_j$$

or

$$q_i \xrightarrow{a,\ B \to CD} q_j$$

B, C, D : stack symbols

Example of a NPDA in correct form:

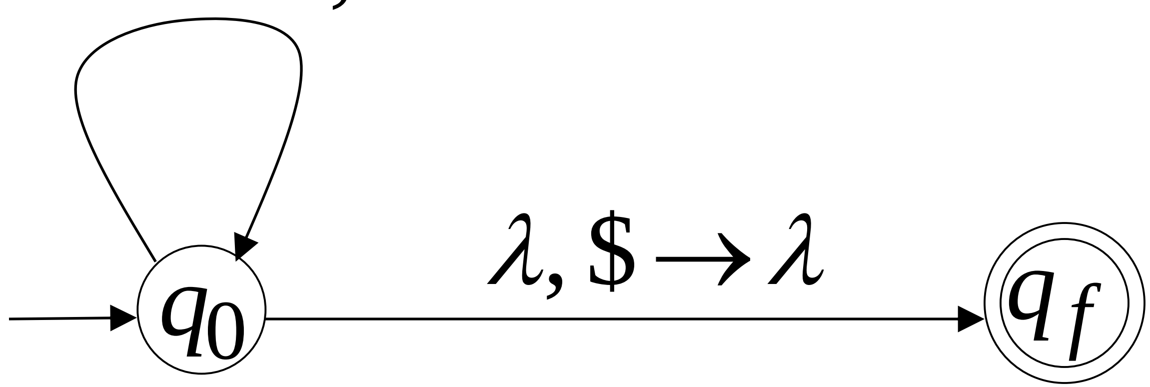
$$L(M) = \{w:\ \ n_a = n_b\}$$

$\$$: initial stack symbol

$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$



The Grammar Construction

In grammar G:

Stack symbol

Variables: $(q_i B q_j)$

states

Terminals:

Input symbols of NPDA

For each transition

$$q_i \xrightarrow{a,\ B \to \lambda} q_j$$

We add production

$$(q_i B q_j) \to a$$

For each transition $q_i \xrightarrow{a,\, B \rightarrow CD} q_j$

We add production $(q_i B q_k) \rightarrow a(q_j C q_l)(q_l D q_k)$

For all states q_k, q_l

Stack bottom symbol

Start Variable: $(q_o \$ q_f)$

Start state

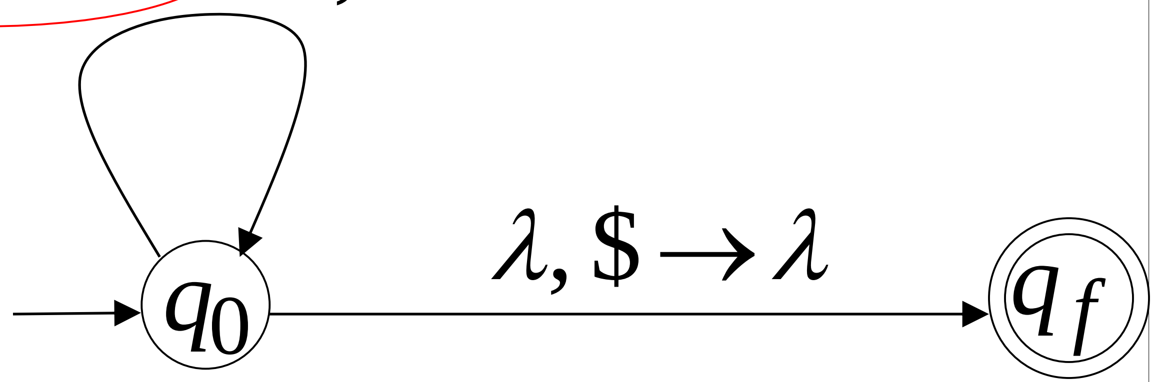
final state

Example:

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



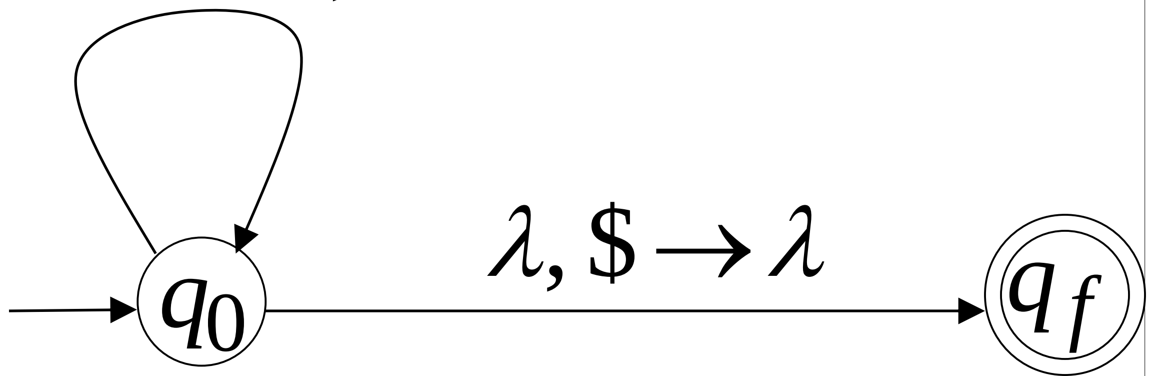
Grammar production: $(q_0 1 q_0) \rightarrow a$

Example:

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Grammar productions:

$$(q_0 \$ q_0) \rightarrow b(q_0 1 q_0)(q_0 \$ q_0) \mid b(q_0 1 q_f)(q_f \$ q_0)$$

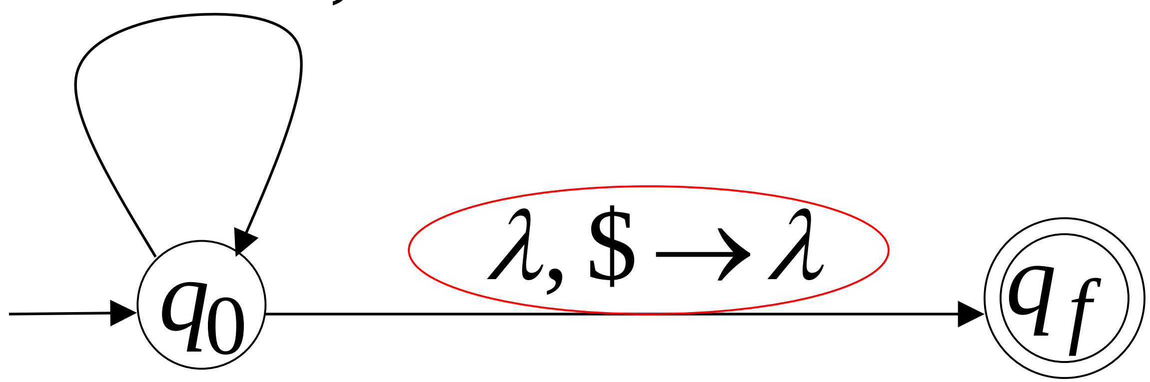
$$(q_0 \$ q_f) \rightarrow b(q_0 1 q_0)(q_0 \$ q_f) \mid b(q_0 1 q_f)(q_f \$ q_f)$$

Example:

$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$



Grammar production: $(q_0 \$ q_f) \rightarrow \lambda$

Resulting Grammar: $(q_0 \$ q_f)$: start variable

$$(q_0 \$ q_0) \rightarrow b(q_0 1 q_0)(q_0 \$ q_0) \mid b(q_0 1 q_f)(q_f \$ q_0)$$

$$(q_0 \$ q_f) \rightarrow b(q_0 1 q_0)(q_0 \$ q_f) \mid b(q_0 1 q_f)(q_f \$ q_f)$$

$$(q_0 1 q_0) \rightarrow b(q_0 1 q_0)(q_0 1 q_0) \mid b(q_0 1 q_f)(q_f 1 q_0)$$

$$(q_0 1 q_f) \rightarrow b(q_0 1 q_0)(q_0 1 q_f) \mid b(q_0 1 q_f)(q_f 1 q_f)$$

$$(q_0 \$ q_0) \rightarrow a(q_0 0 q_0)(q_0 \$ q_0) \mid a(q_0 0 q_f)(q_f \$ q_0)$$

$$(q_0 \$ q_f) \rightarrow a(q_0 0 q_0)(q_0 \$ q_f) \mid a(q_0 0 q_f)(q_f \$ q_f)$$

$$(q_0 0 q_0) \rightarrow a(q_0 0 q_0)(q_0 0 q_0) \mid a(q_0 0 q_f)(q_f 0 q_0)$$

$$(q_0 0 q_f) \rightarrow a(q_0 0 q_0)(q_0 0 q_f) \mid a(q_0 0 q_f)(q_f 0 q_f)$$

$$(q_0 1 q_0) \rightarrow a$$

$$(q_0 0 q_0) \rightarrow b$$

$$(q_0 \$ q_f) \rightarrow \lambda$$

Derivation of string $abba$

$$(q_0 \$ q_f) \Rightarrow \; a(q_0 0 q_0)(q_0 \$ q_f) \Rightarrow$$

$$ab(q_0 \$ q_f) \Rightarrow$$

$$abb(q_0 1 q_0)(q_0 \$ q_f) \Rightarrow$$

$$abba(q_0 \$ q_f) \Rightarrow \; abba$$

In general, in Grammar:

$$(q_0 \$ q_f) \overset{*}{\Rightarrow} w$$

if and only if

w is accepted by the NPDA

Explanation:

By construction of Grammar:

$$(q_i A q_j) \overset{*}{\Longrightarrow} w$$

if and only if

in the NPDA going from q_i to q_j
the stack doesn't change below A
and A is removed from stack