A Non Context-Free Language

(We will prove later)

Non Context-free languages

$$a^n b^n c^n$$

Context-free languages

$$a^n b^n$$

Regular languages

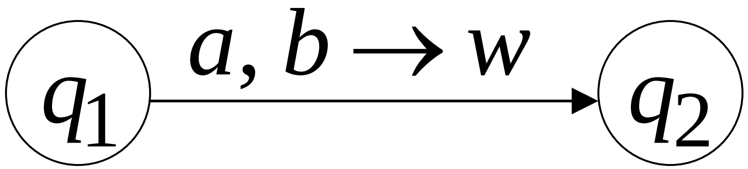
$$a^* b^*$$

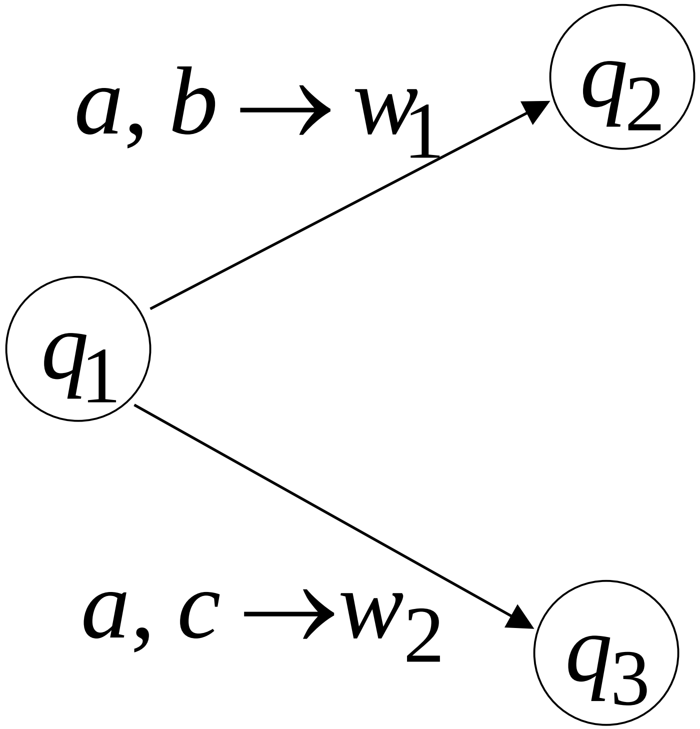
Deterministic PDAs

DPDAs

DPDAs

Allowed:





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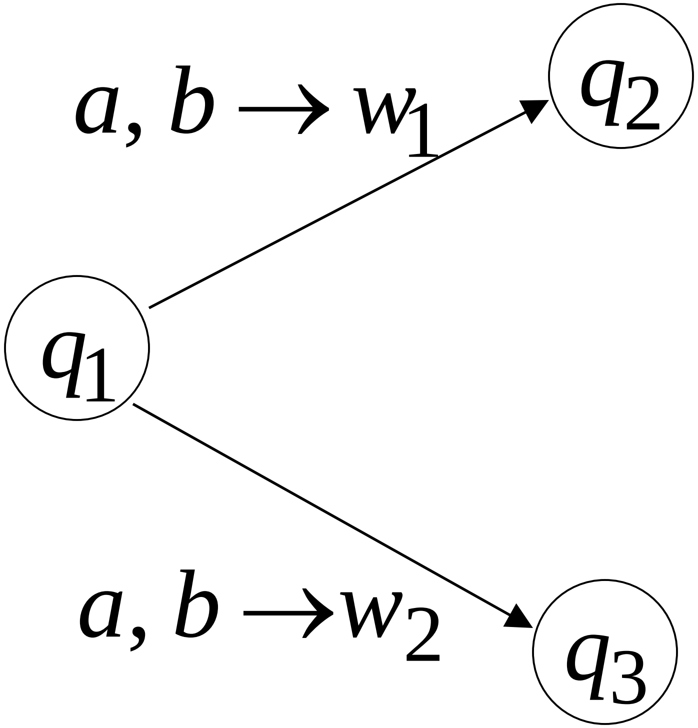
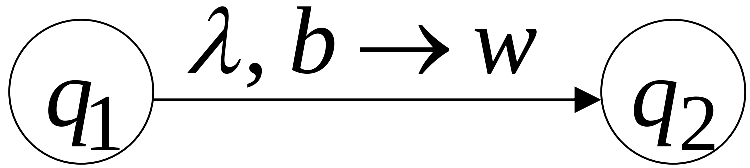
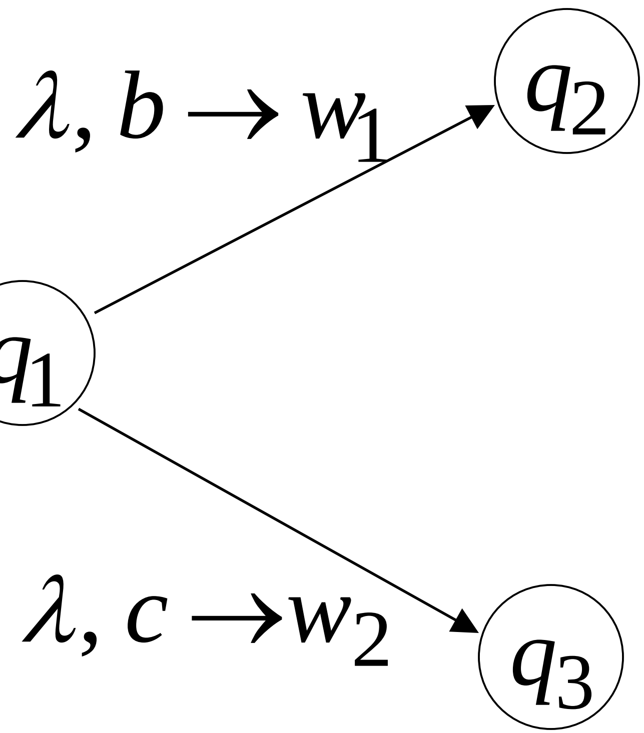


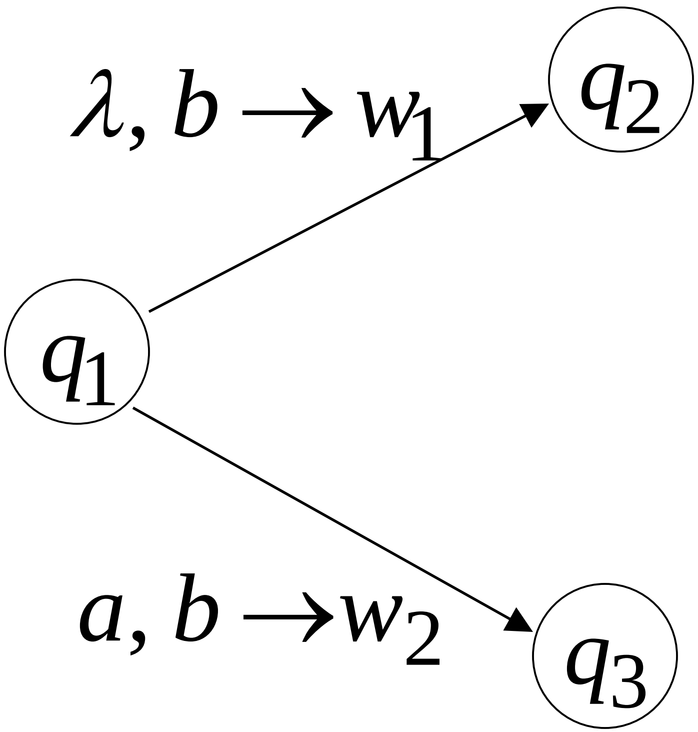
Allowed:





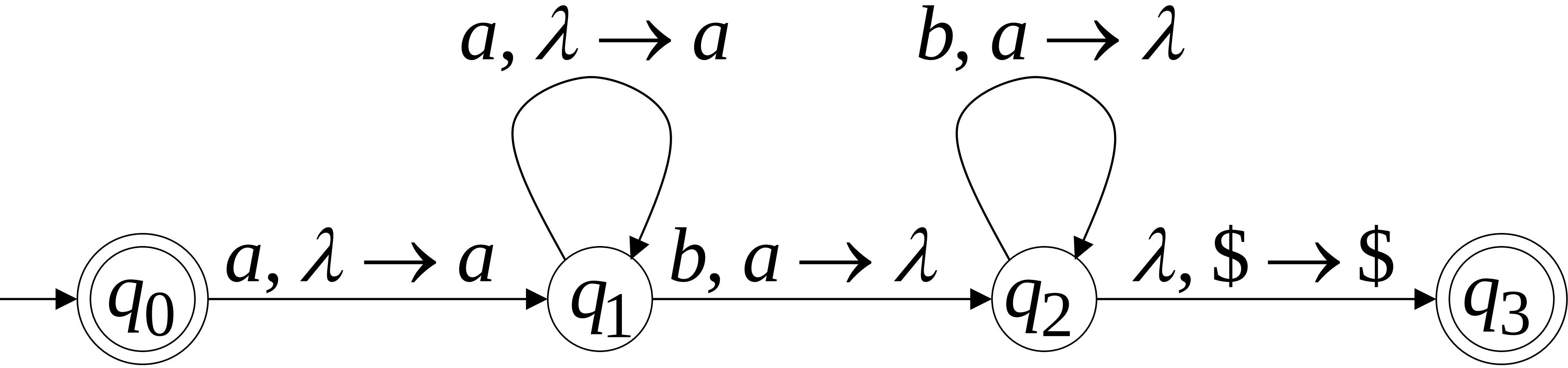
Something must be matched from the stack

Not allowed:



DPDA example

$$L(M) = \{a^n b^n : n \geq 0\}$$



The language $L(M) = \{a^n b^n : n \geq 0\}$

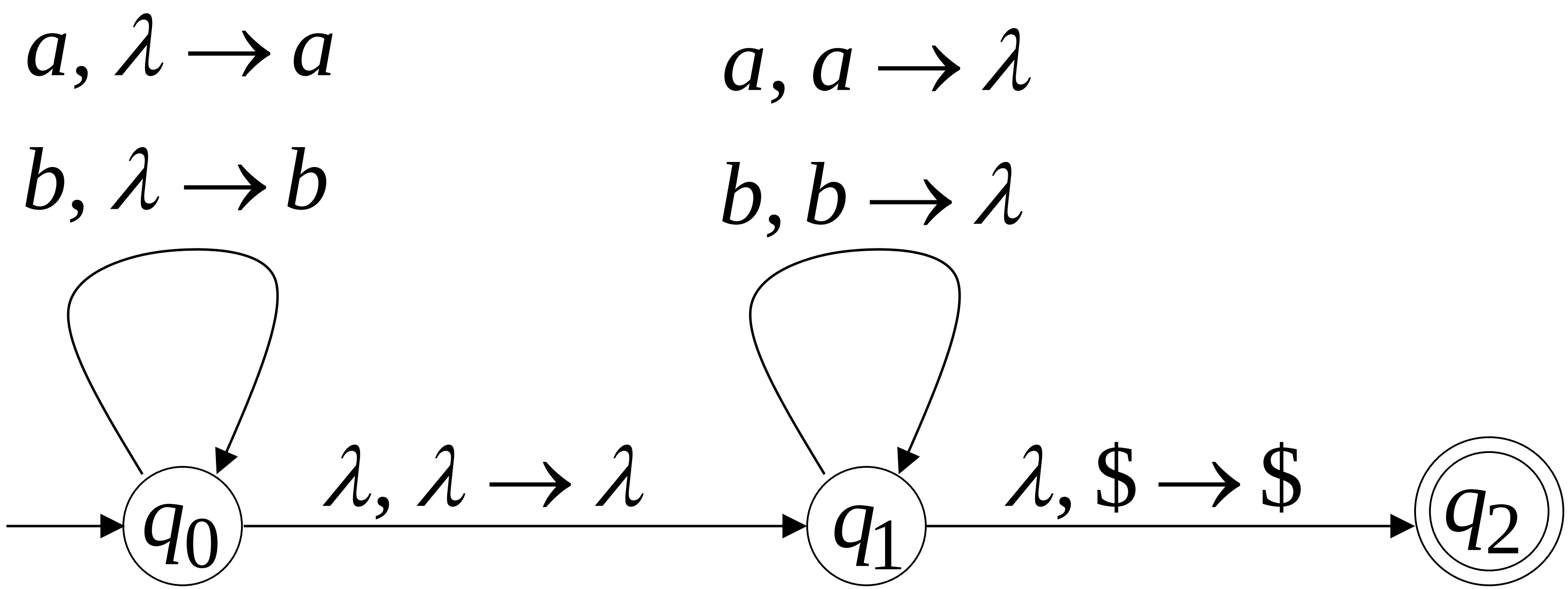
is **deterministic context-free**

Definition:

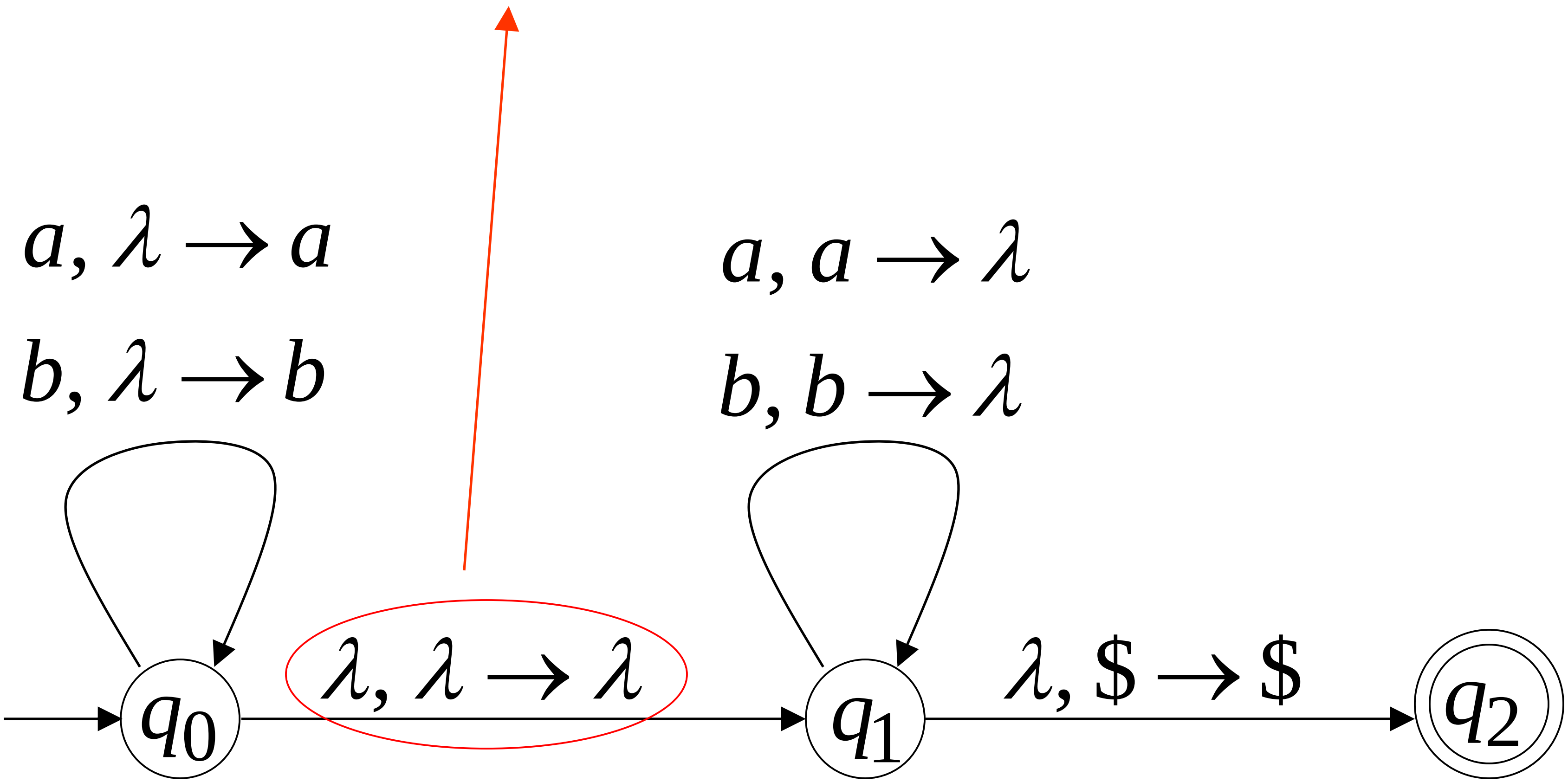
A language is **deterministic context-free** if some DPDA accepts it

Example of Non-DPDA (NPDA)

$$L(M) = \{ww^R\}$$



Not allowed in DPDAs



NPDAs

Have More Power than

DPDAs

We will show:

there is a context-free language L
(accepted by a NPDA)

which is **not** deterministic context-free

(**not** accepted by a DPDA)

The language is:

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

$$n \geq 0$$

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

The language L is context-free

Context-free grammar for L :

$$S \to S_1 \mid S_2$$

$$S_1 \to aS_1b \mid \lambda$$

$$S_2 \to aS_2bb \mid \lambda$$

there is an NPDA that accepts L

Theorem:

The language $L = \{a^n b^n\} \cup \{a^n b^{2n}\}$

is **not** deterministic context-free

(there is **no** DPDA that accepts L)

Proof: Assume for contradiction that

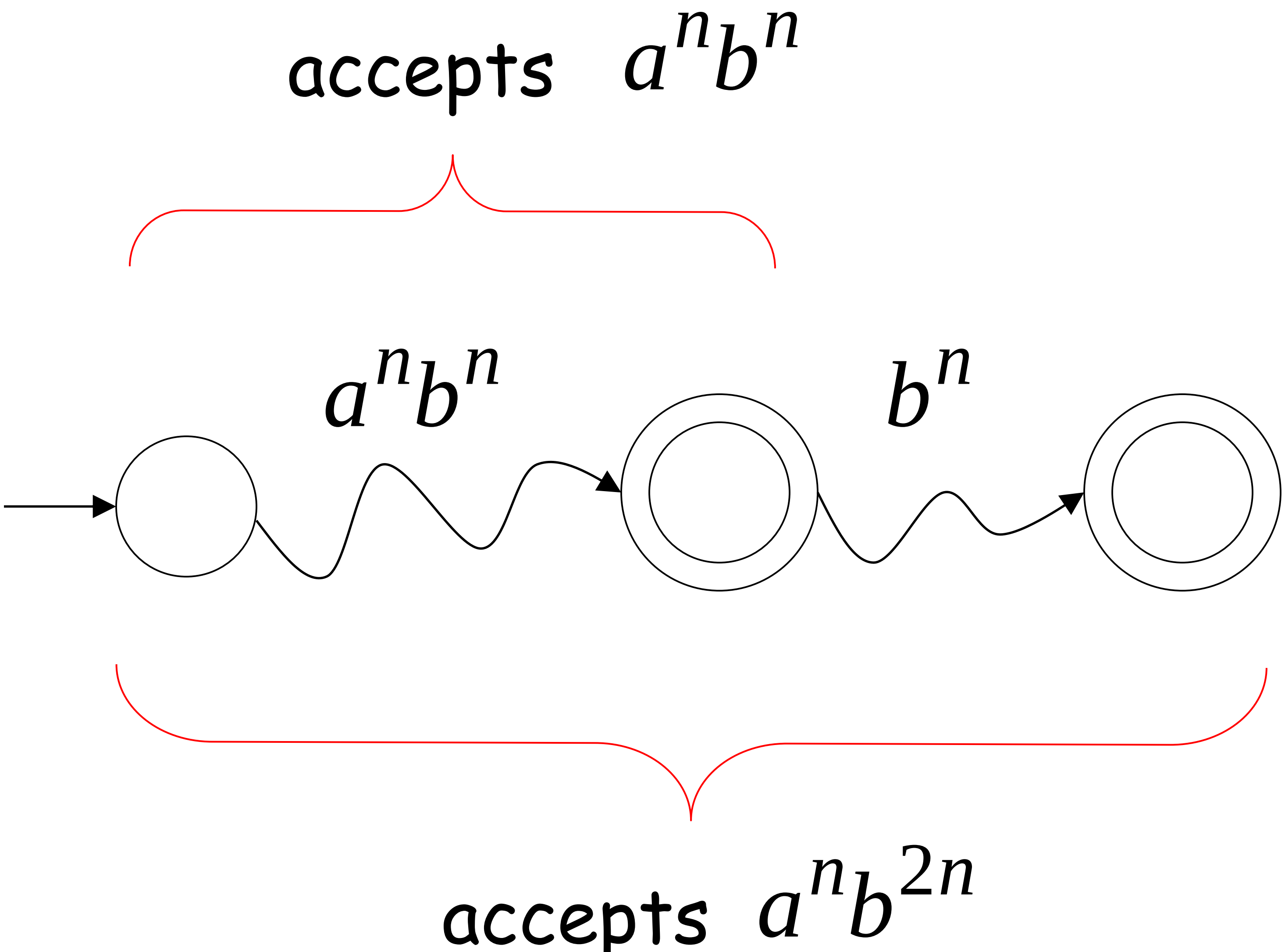
$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

is deterministic context free

Therefore:

there is a DPDA M that accepts L

DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$



Fact 1: The language $\{a^n b^n c^n\}$ is **not** context-free

(we will prove it at the next class)

Fact 2: The language $L \cup \{a^n b^n c^n\}$

is **not** context-free

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

Use pumping lemma. Example 8.1, page 217

(a consequence of Fact 1)

We will construct a NPDA that accepts:

$$L \cup \{a^n b^n c^n\}$$

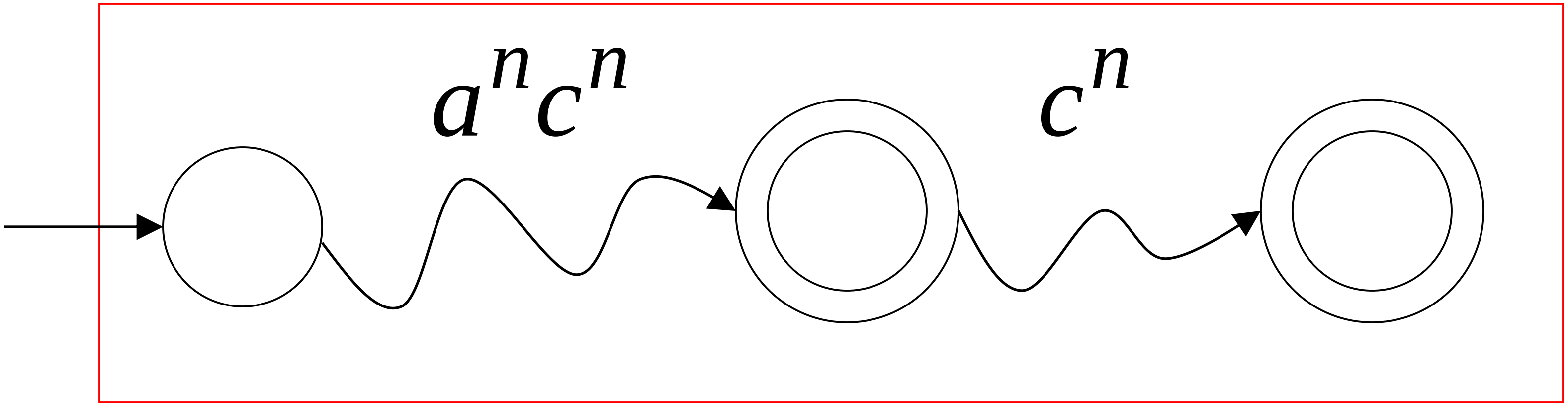
Contradiction!

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

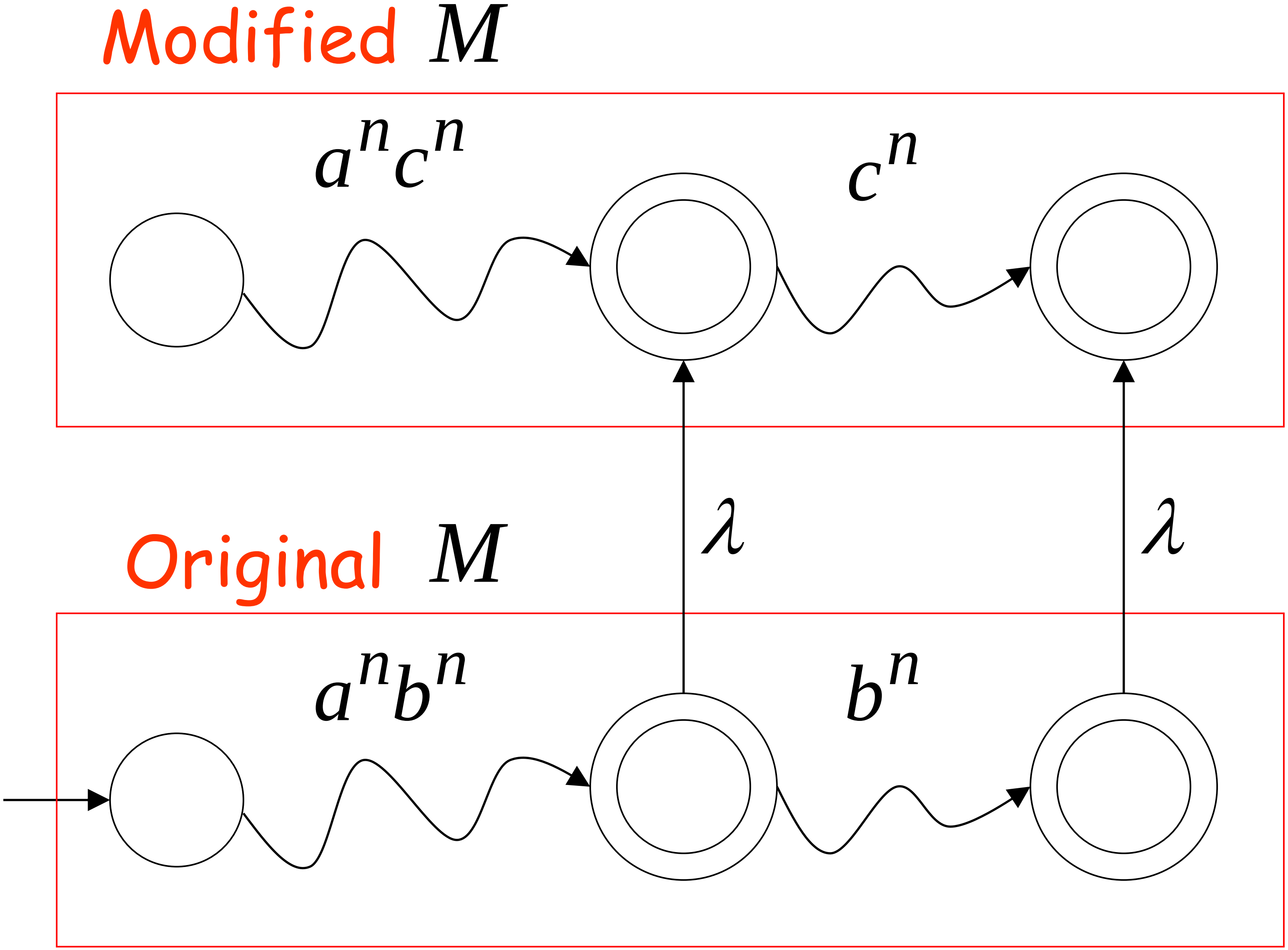
We modify M $\quad (L = \{a^n b^n\} \cup \{a^n b^{2n}\})$

Replace b with c

Modified M $\quad (L' = \{a^n c^n\} \cup \{a^n c^{2n}\})$



The NPDA that accepts $L \cup \{a^n b^n c^n\}$



Since $L \cup \{a^n b^n c^n\}$ is accepted by a NPDA

it is context-free

Contradiction!

(since $L \cup \{a^n b^n c^n\}$ is not context-free)

Therefore:

There is **no** DPDA that accepts

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

Not deterministic context free

End of Proof

Positive Properties of Context-Free languages

Union

Context-free languages
are closed under: **Union**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cup L_2$ is context-free

Example

Language | **Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \to aS_1b \mid \lambda$$

$$L_2 = \{ww^R\}$$

$$S_2 \to aS_2a \mid bS_2b \mid \lambda$$

Union

$$L = \{a^n b^n\} \cup \{ww^R\}$$

$$S \to S_1 \mid S_2$$

In general:

For context-free languages L_1, L_2

with context-free grammars G_1, G_2

and start variables S_1, S_2

The grammar of the **union** $L_1 \cup L_2$

has new start variable S

and additional production $S \rightarrow S_1 \mid S_2$

Concatenation

Context-free languages are closed under: **Concatenation**

L_1 is context free

L_2 is context free

$\Rightarrow$

$L_1 L_2$

is context-free

Example

Language | **Grammar**

$$L_1 = \{a^n b^n\}$$

$$S_1 \to aS_1b \mid \lambda$$

$$L_2 = \{ww^R\}$$

$$S_2 \to aS_2a \mid bS_2b \mid \lambda$$

Concatenation

$$L = \{a^n b^n\}\{ww^R\}$$

$$S \to S_1S_2$$

In general:

For context-free languages L_1, L_2

with context-free grammars G_1, G_2

and start variables S_1, S_2

The grammar of the **concatenation** $L_1 L_2$

has new start variable S

and additional production $S \to S_1 S_2$

Star Operation

Context-free languages
are closed under: **Star-operation**

L is context free ⇒ L^* is context-free

Example

Language

$$L = \{a^n b^n\}$$

Grammar

$$S \to aSb \mid \lambda$$

Star Operation

$$L = \{a^n b^n\}^*$$

$$S_1 \to SS_1 \mid \lambda$$

In general:

For context-free language L
with context-free grammar G
and start variable S

The grammar of the **star operation** L^*
has new start variable S_1
and additional production $S_1 \to SS_1 \mid \lambda$

Negative Properties of Context-Free Languages

Intersection

Context-free languages
are **not** closed under: **intersection**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$

not necessarily
context-free

Example

$L_1 = \{a^n b^n c^m\}$

Context-free:

$S \rightarrow AC$

$A \rightarrow aAb \mid \lambda$

$C \rightarrow cC \mid \lambda$

$L_2 = \{a^n b^m c^m\}$

Context-free:

$S \rightarrow AB$

$A \rightarrow aA \mid \lambda$

$B \rightarrow bBc \mid \lambda$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Complement

Context-free languages
are **not** closed under: **complement**

L is context free $\Rightarrow$ $\overline{L}$ **not** necessarily context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \lambda$$

$$C \to cC \mid \lambda$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \lambda$$

$$B \to bBc \mid \lambda$$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

Intersection of Context-free languages and Regular Languages

The intersection of

a context-free language and

a regular language

is a context-free language

L_1 context free

L_2 regular

$\Longrightarrow$ $L_1 \cap L_2$

context-free

Machine M_1

NPDA for L_1

context-free

Machine M_2

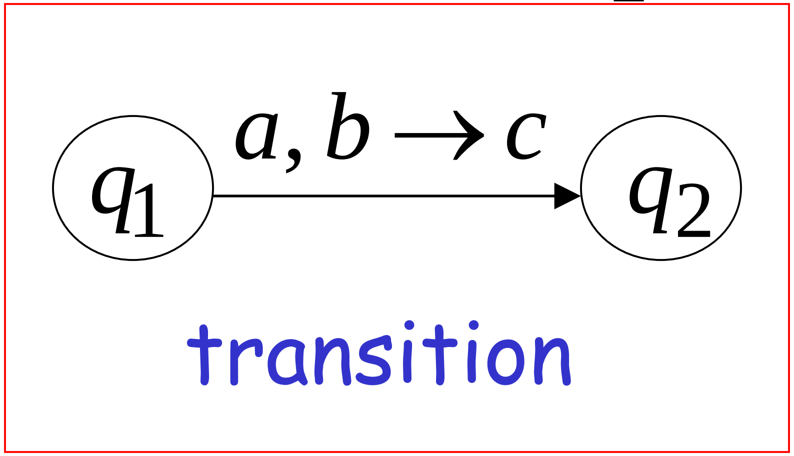
DFA for L_2

regular

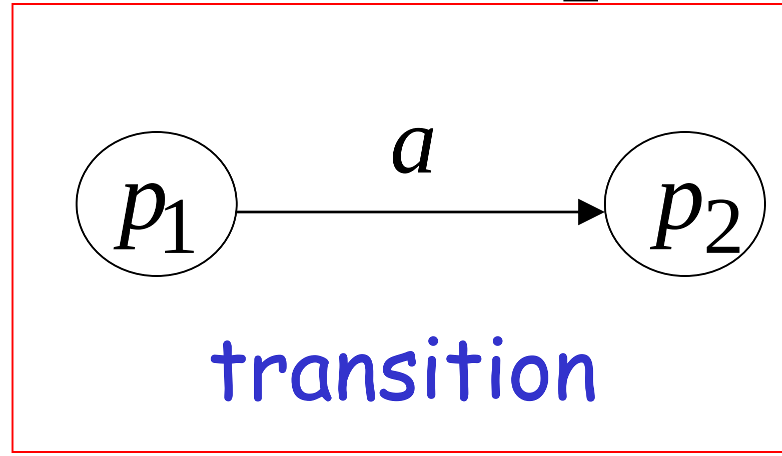
Construct a new NPDA machine M that accepts $L_1 \cap L_2$

M simulates in parallel M_1 and M_2

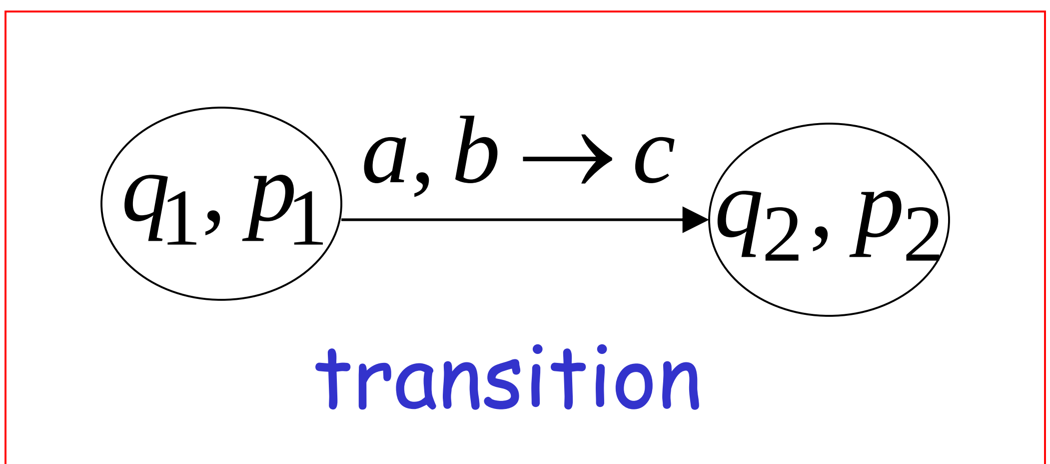
NPDA M_1

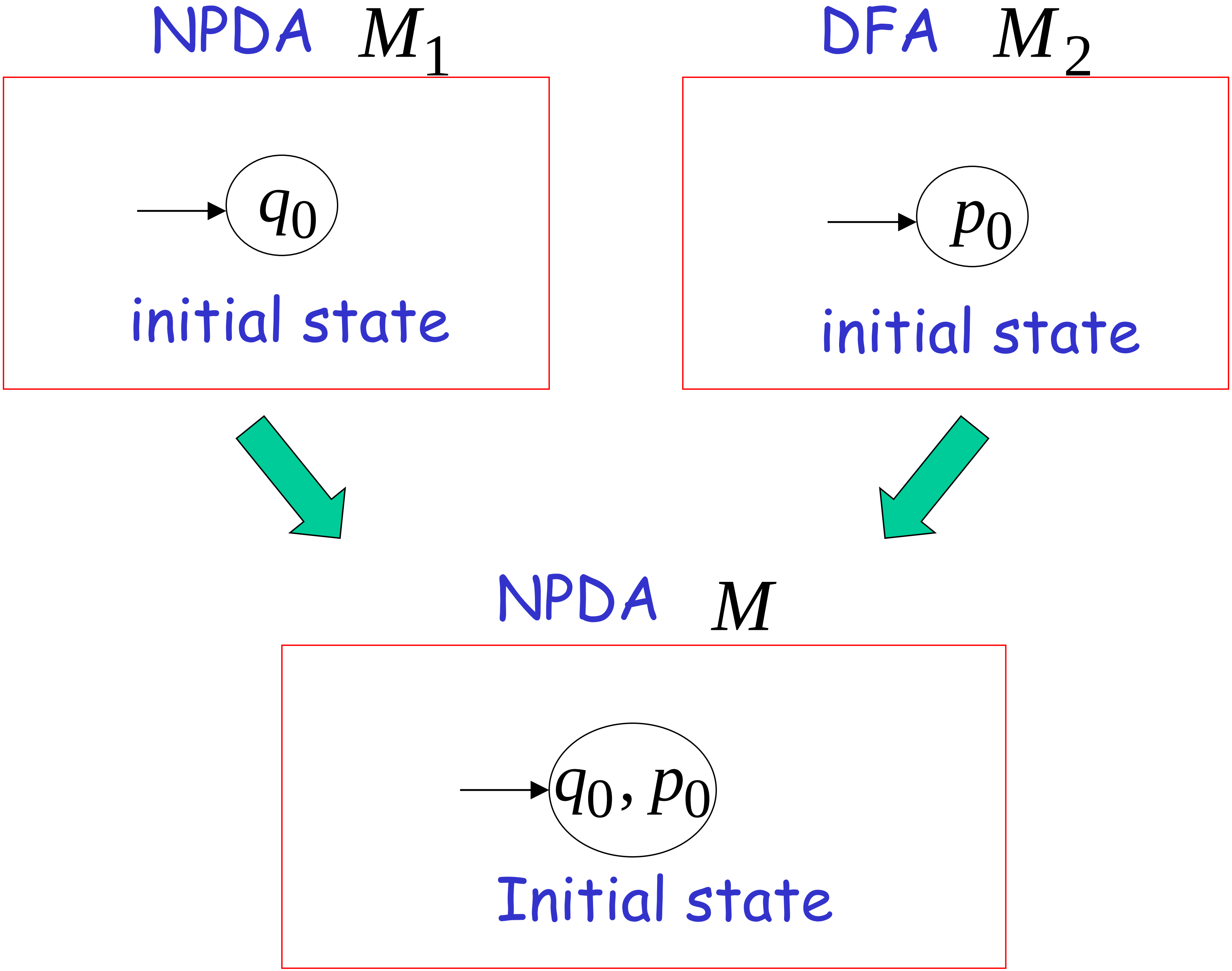


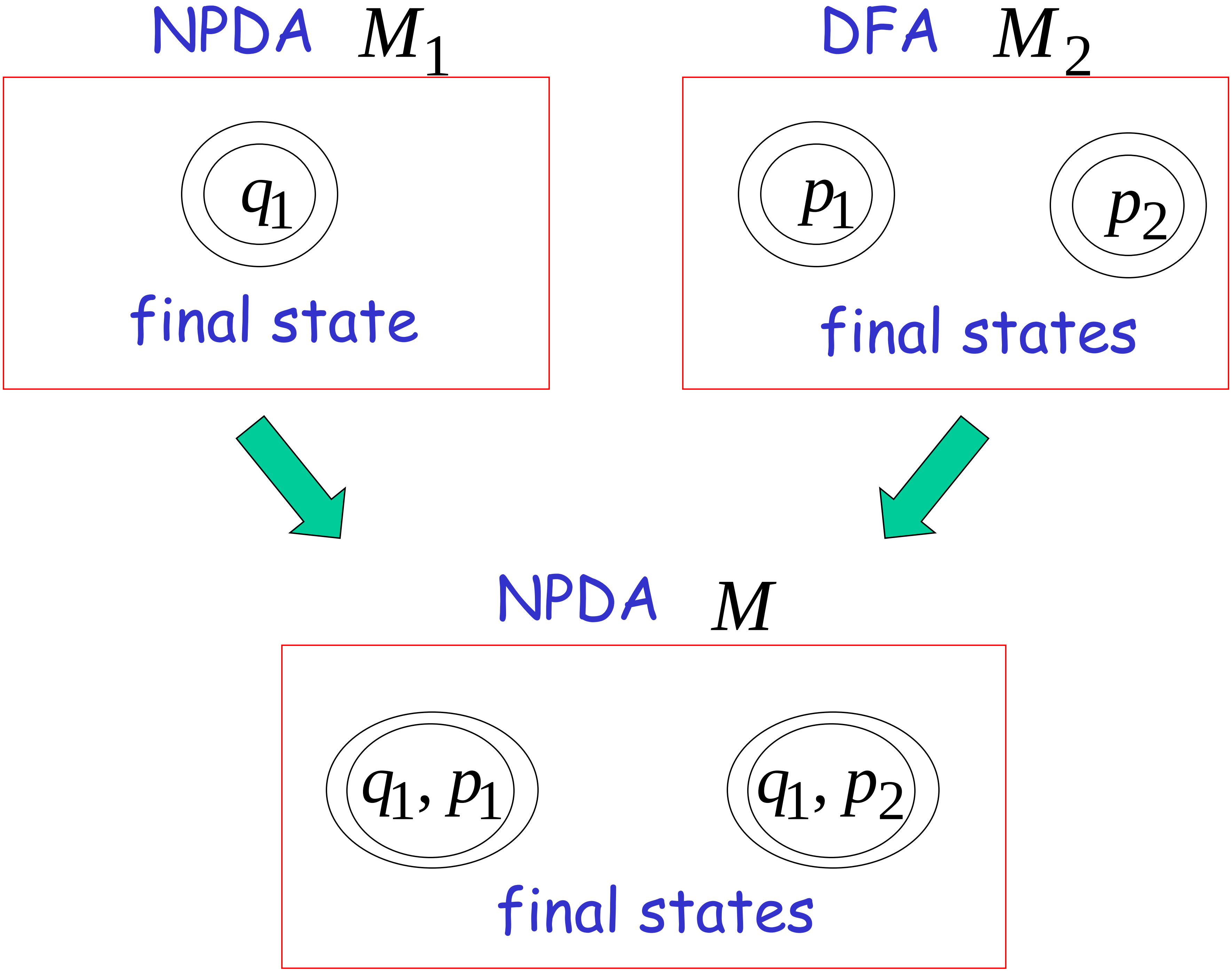
DFA M_2



NPDA M




NPDA M_1
q_0
initial state
DFA M_2
p_0
initial state
NPDA M
q_0, p_0
Initial state


NPDA M_1
q_1
final state
DFA M_2
p_1
p_2
final states
NPDA M
q_1, p_1
q_1, p_2
final states

M simulates in parallel M_1 and M_2

M accepts string w if and only if

M_1 accepts string w and

M_2 accepts string w

$$L(M) = L(M_1) \cap L(M_2)$$

Therefore: $L(M_1) \cap L(M_2)$ is context-free

(since M is NPDA)

$\Downarrow$

$L_1 \cap L_2$ is context-free