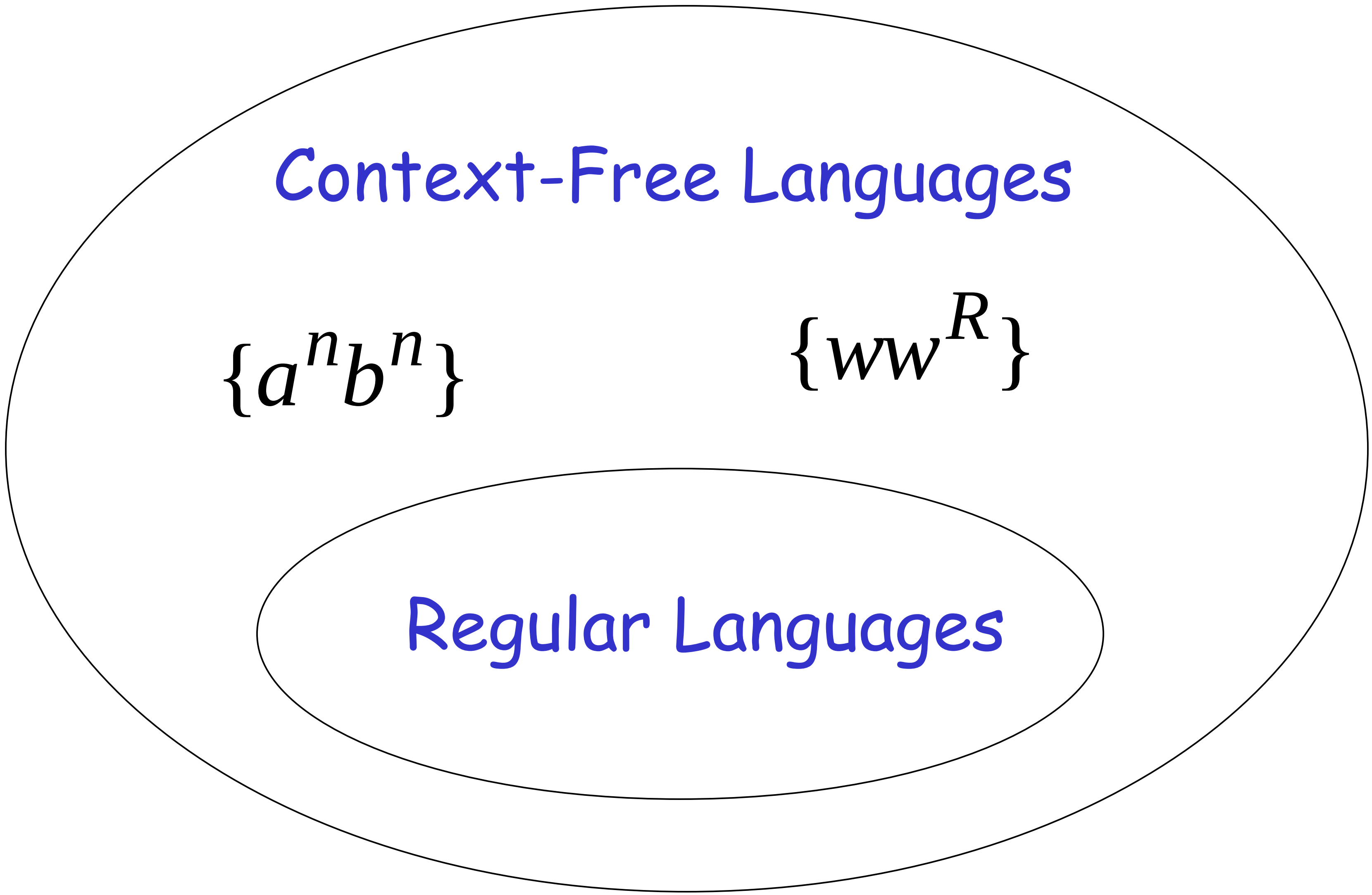
Context-Free Languages

$\{a^n b^n\}$ $\{ww^R\}$

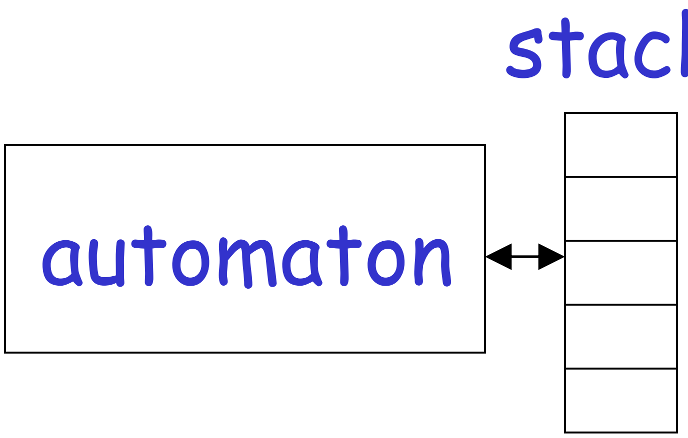
Regular Languages


Context-Free Languages
$\{a^n b^n\}$
$\{ww^R\}$
Regular Languages

Context-Free Languages

Context-Free Grammars

Pushdown Automata



Context-Free Grammars

Example

A context-free grammar G:

$$S \to aSb$$

$$S \to \lambda$$

A derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

A context-free grammar G:

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

Another derivation:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

(((( ))))

Example

A context-free grammar G:

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

A derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

A context-free grammar G:

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

Another derivation:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

$$L(G) = \{ww^R : \quad w \in \{a,b\}^*\}$$

Example

A context-free grammar G:

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

A context-free grammar G:

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

A derivation:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

$$L(G) = \{w \;\; : n_a(w) = n_b(w),$$
$$\text{and } \; n_a(v) \geq n_b(v)$$
$$\text{in any prefix } \; v\}$$

( ) ((( ))) (( ))

Definition: Context-Free Grammars

Grammar $G = (V, T, S, P)$

Variables

Terminal symbols

Start variable

Productions of the form:

$$A \rightarrow x$$

x is string of variables and terminals

Definition: Context-Free Languages

A language L is context-free

if and only if

there is a grammar G with $L = L(G)$

Derivation Order

1. $S \to AB$
2. $A \to aaA$
3. $A \to \lambda$
4. $B \to Bb$
5. $B \to \lambda$

Leftmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

Rightmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

$$S \to aAB$$

$$A \to bBb$$

$$B \to A \mid \lambda$$

Leftmost derivation:

$$S \Rightarrow aAB \Rightarrow abBbB \Rightarrow abAbB \Rightarrow abbBbbB$$

$$\Rightarrow abbbbB \Rightarrow abbbb$$

Rightmost derivation:

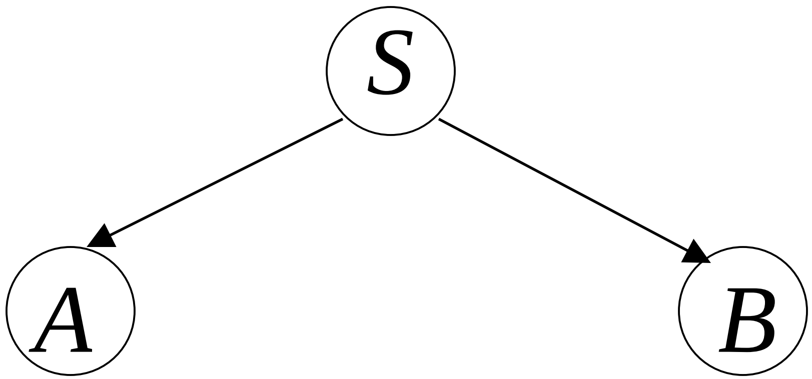
$$S \Rightarrow aAB \Rightarrow aA \Rightarrow abBb \Rightarrow abAb$$

$$\Rightarrow abbBbb \Rightarrow abbbb$$

Derivation Trees

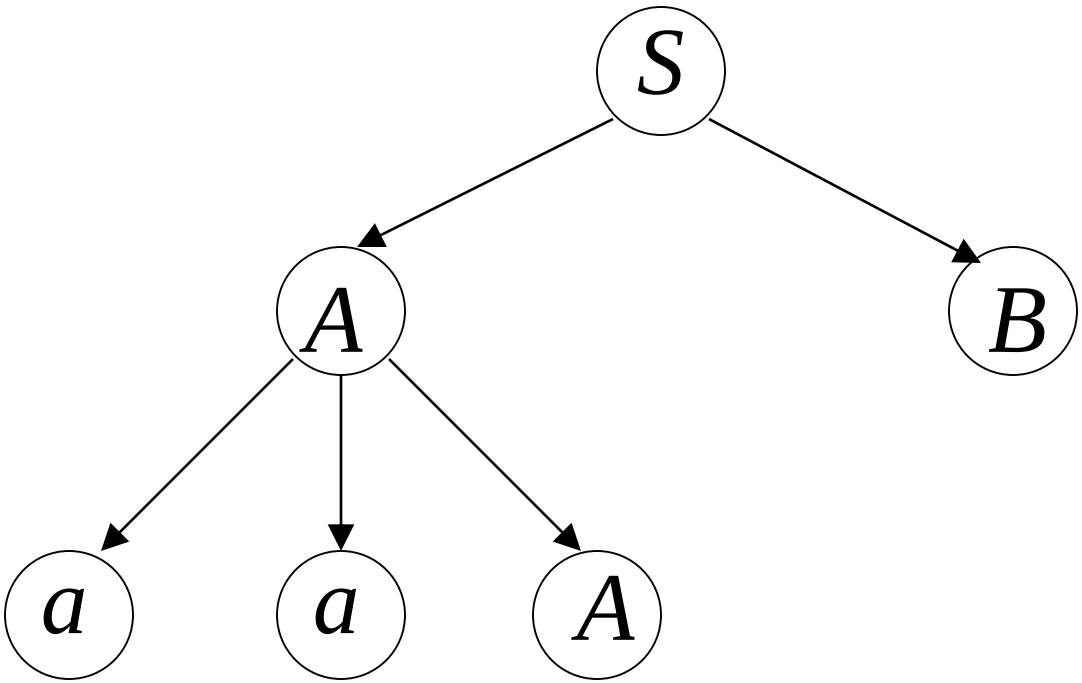
$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$

$$S \Rightarrow AB$$



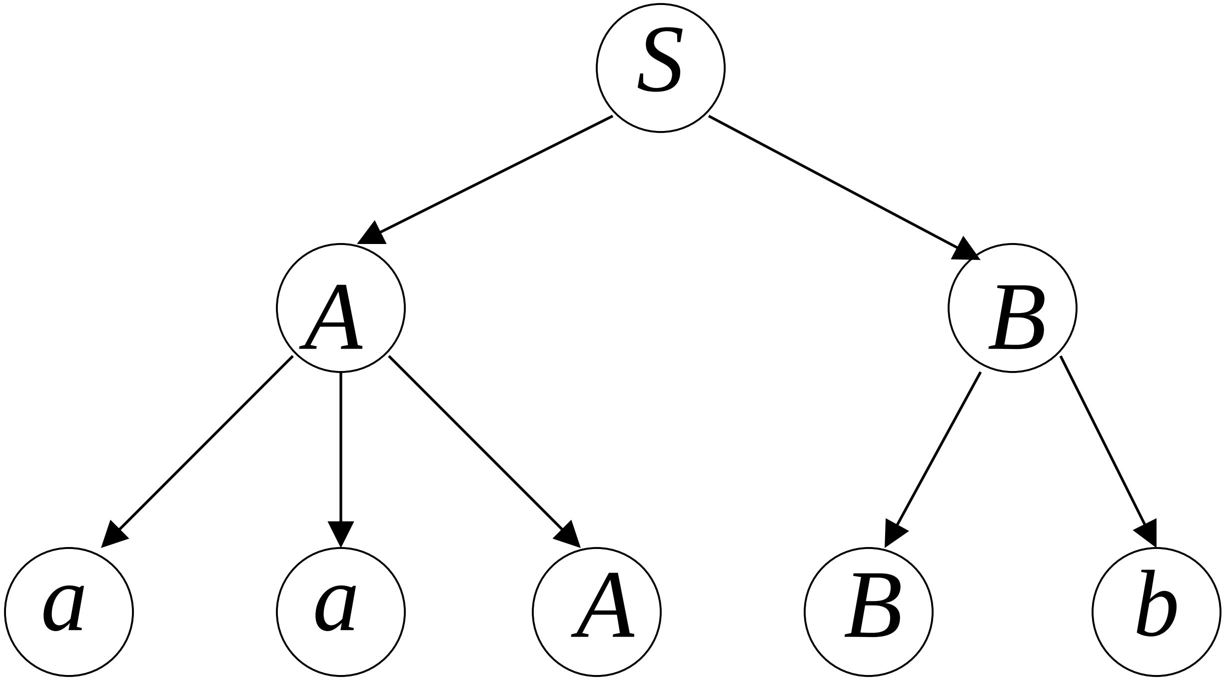
$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB$$



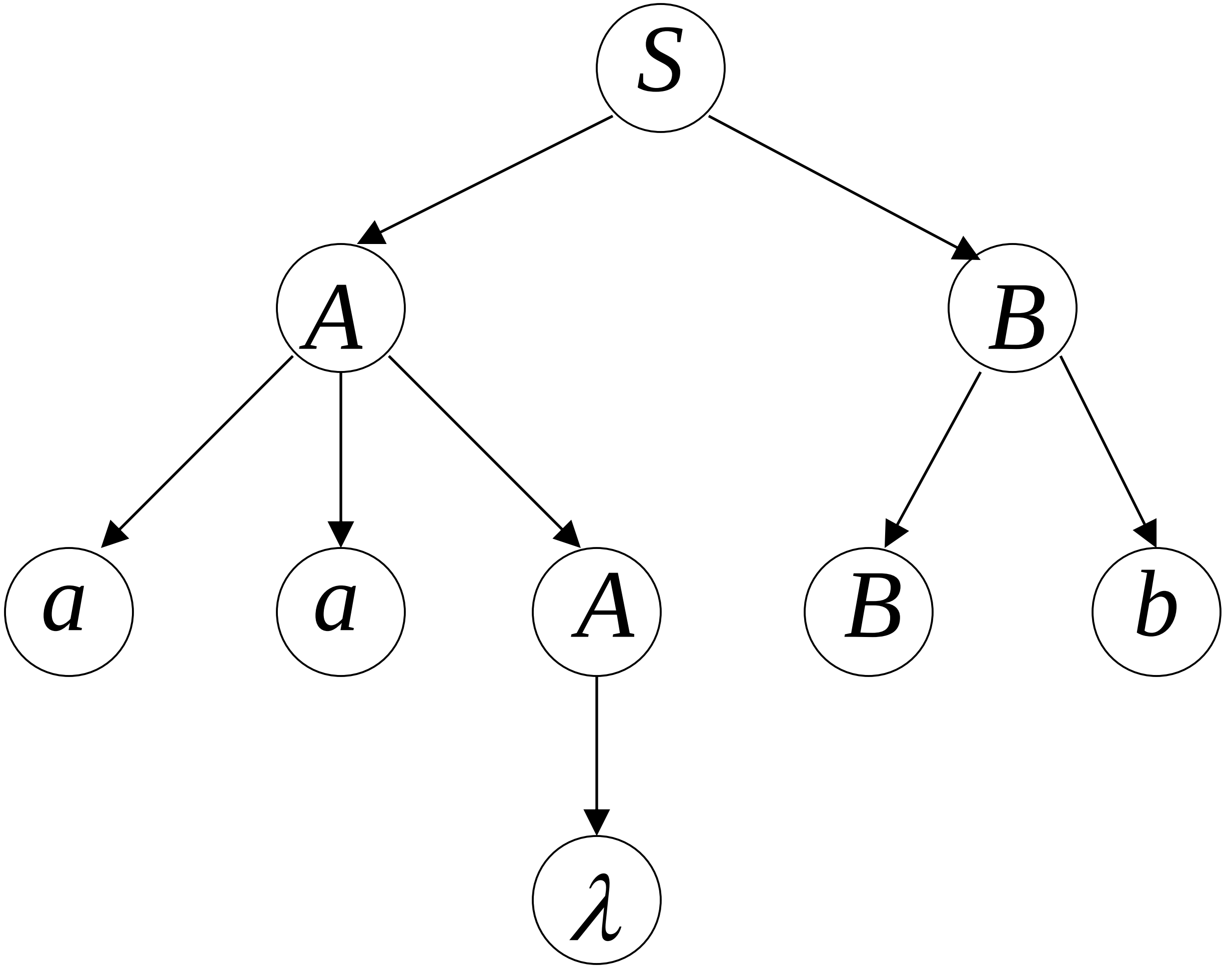
$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

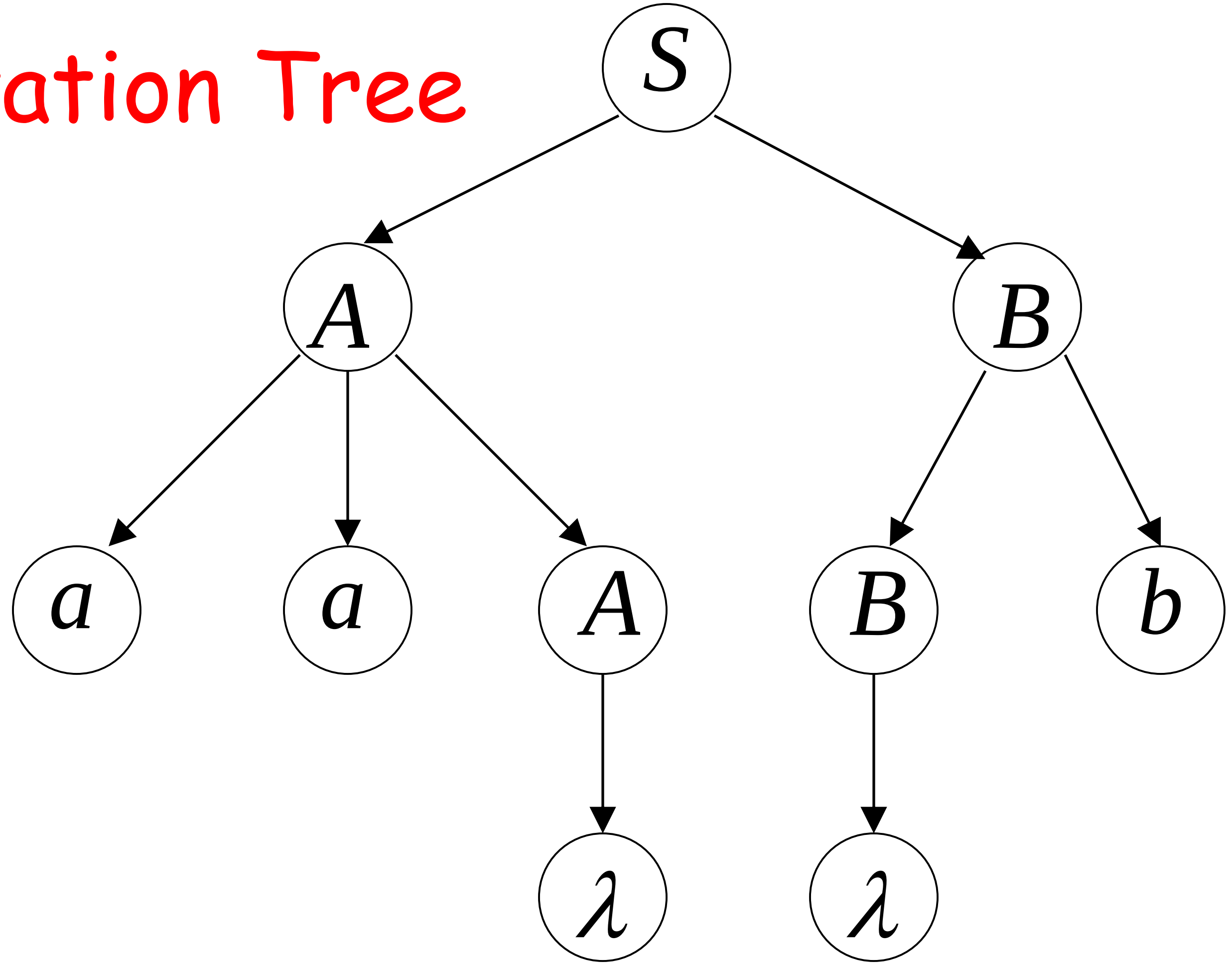
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

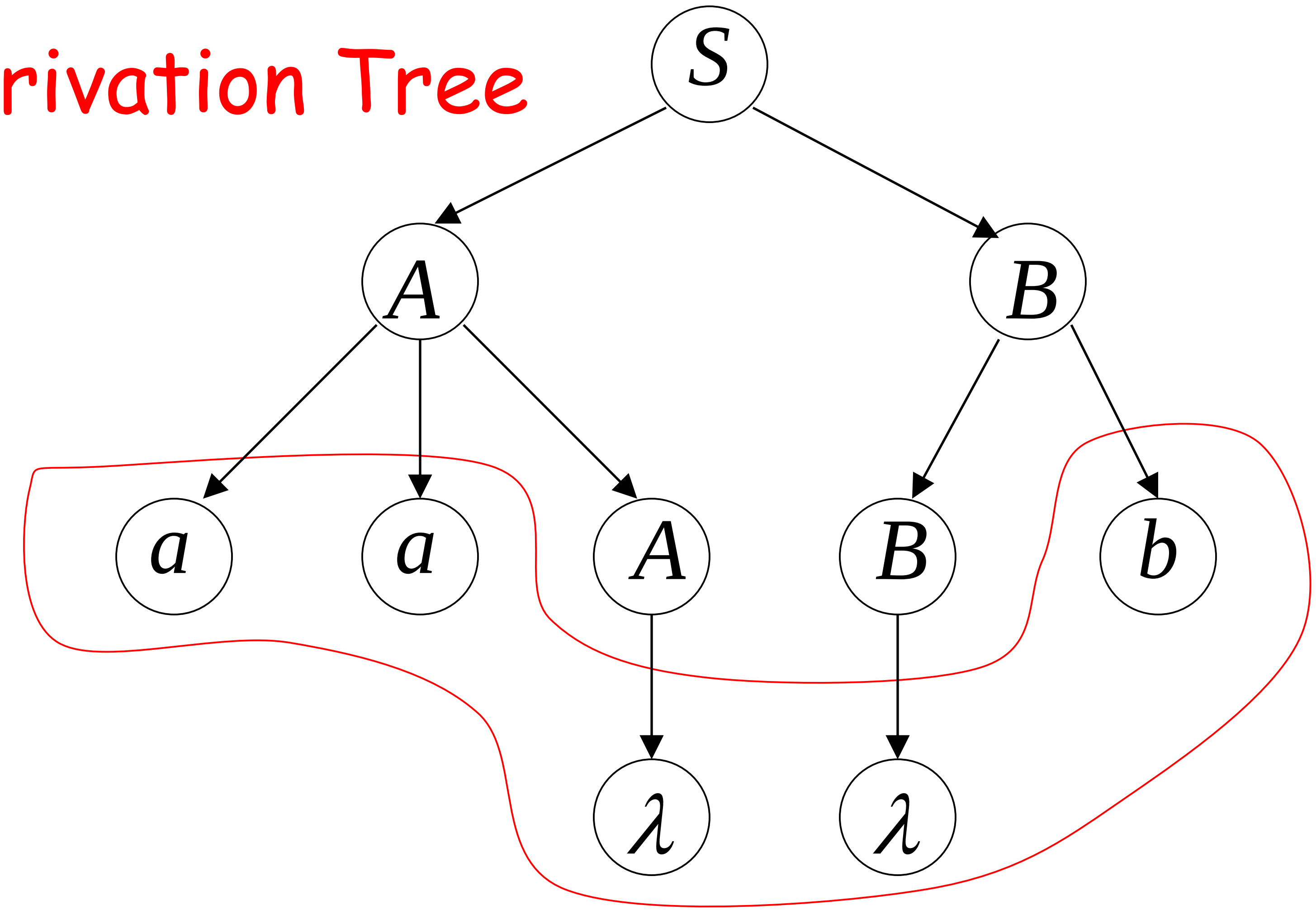
Derivation Tree



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree



yield

$aa\lambda\lambda b$

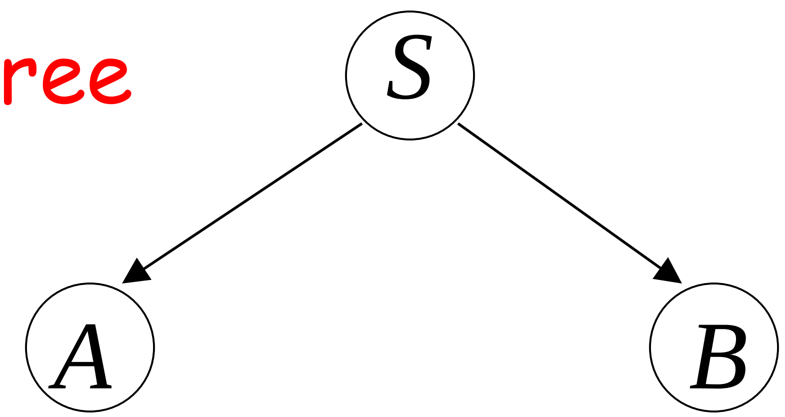
$= aab$

Partial Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

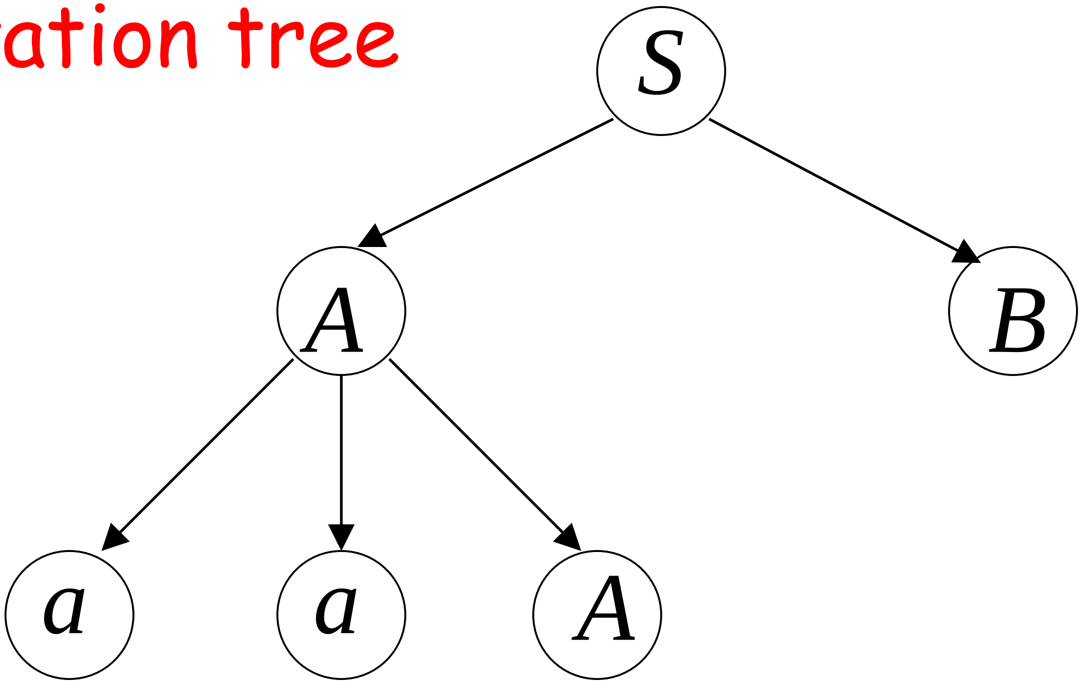
$$S \Rightarrow AB$$

Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

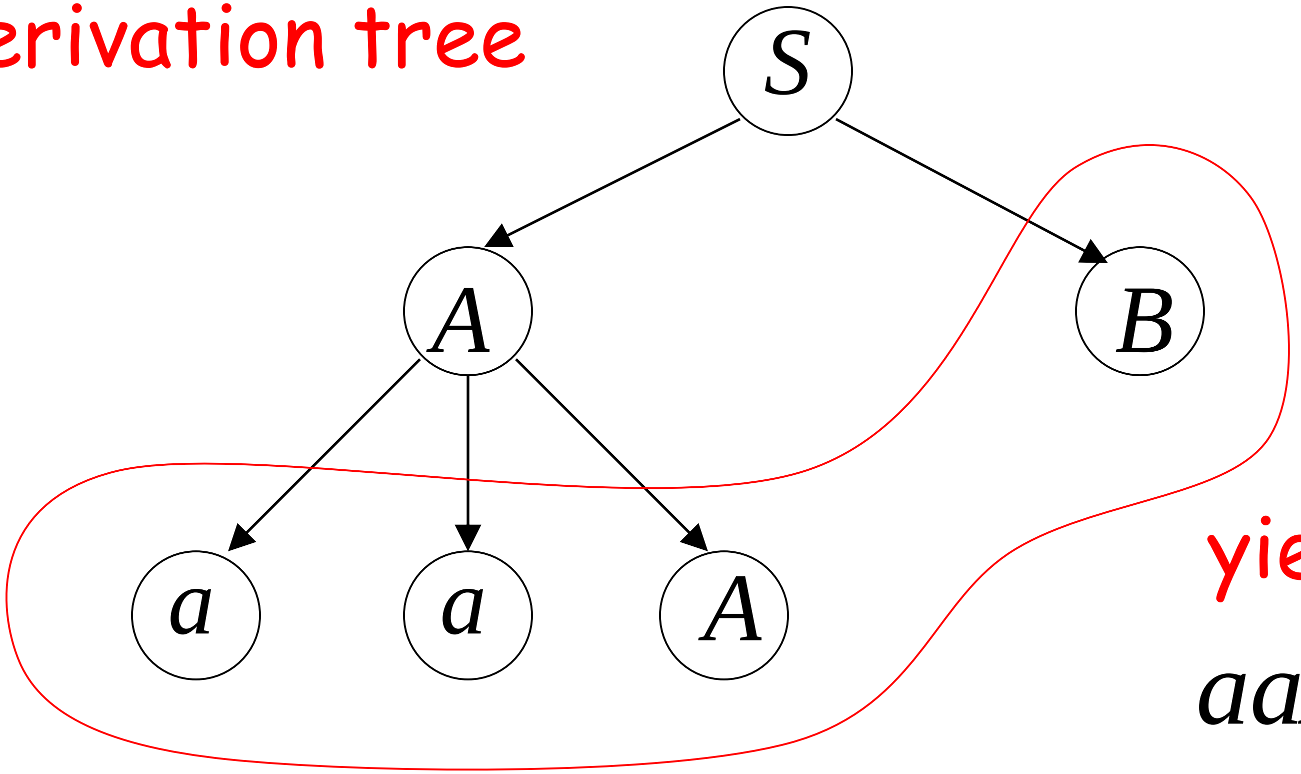
Partial derivation tree



$$S \Rightarrow AB \Rightarrow aaAB$$

sentential form

Partial derivation tree



yield

$aaAB$

Sometimes, derivation order doesn't matter

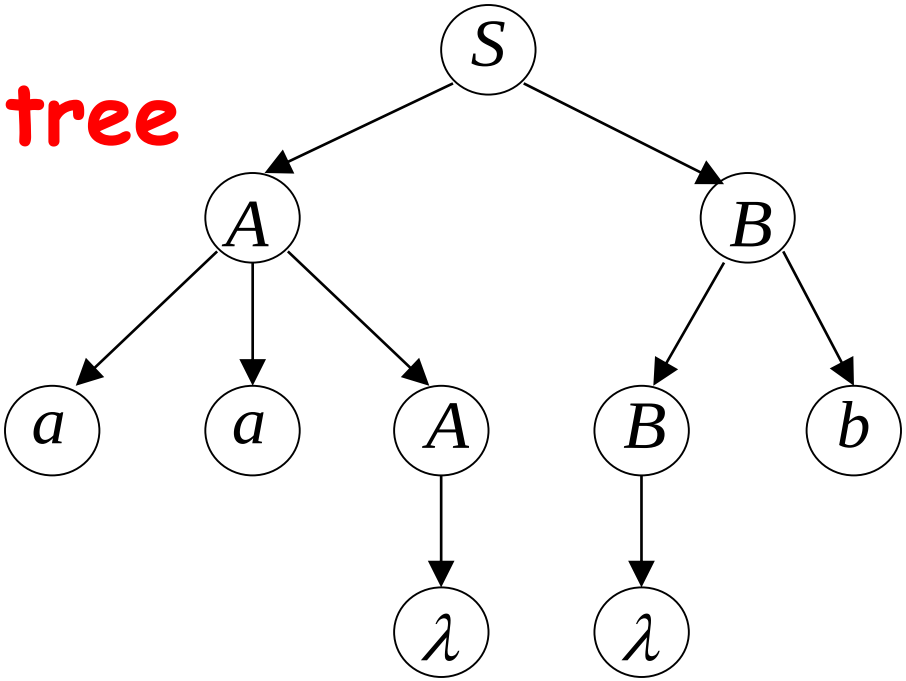
Leftmost:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

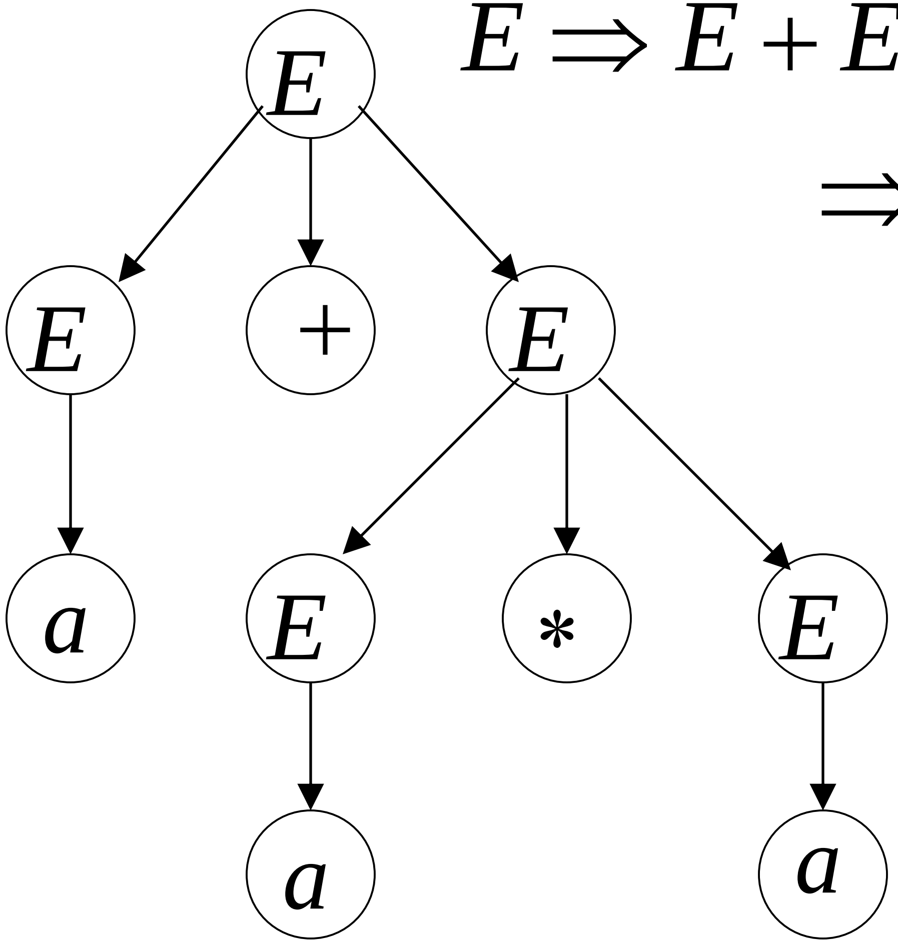
Same derivation tree



Ambiguity

$$E \to E + E \;\mid\; E * E \;\mid\; (E) \;\mid\; a$$

$$a + a * a$$



$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

leftmost derivation

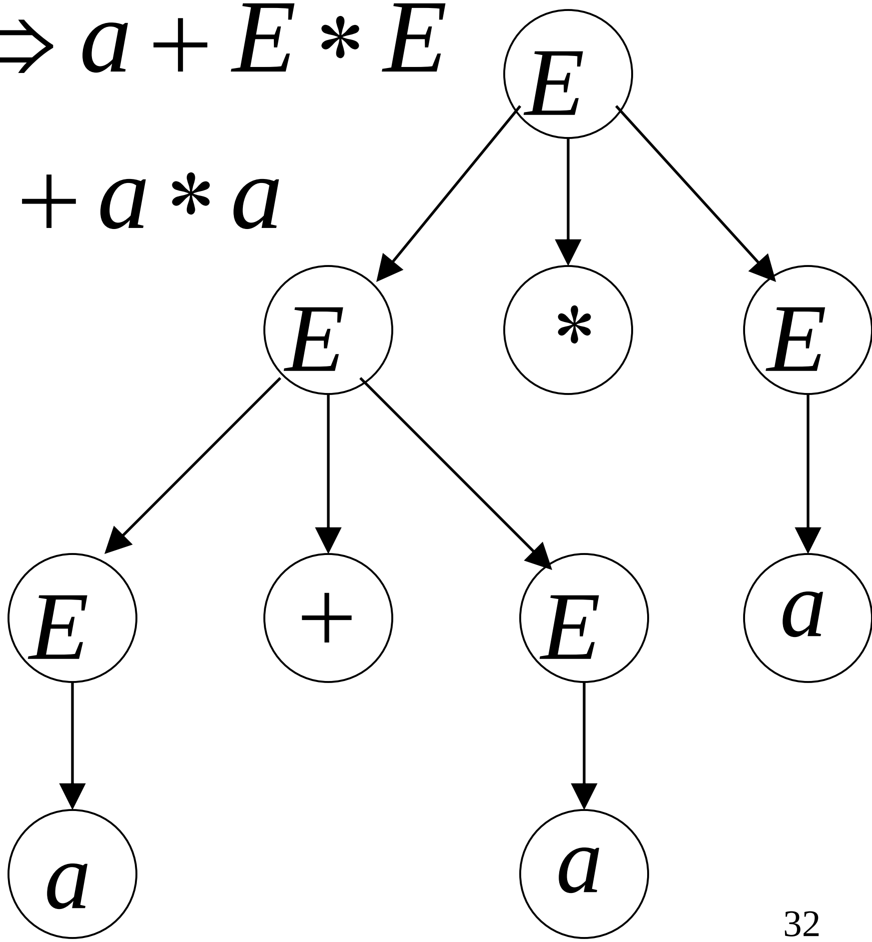
$$E \to E + E \quad | \; E * E \; | \; (E) \; | \; a$$

$$a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$

$$\Rightarrow a + a * E \Rightarrow a + a * a$$

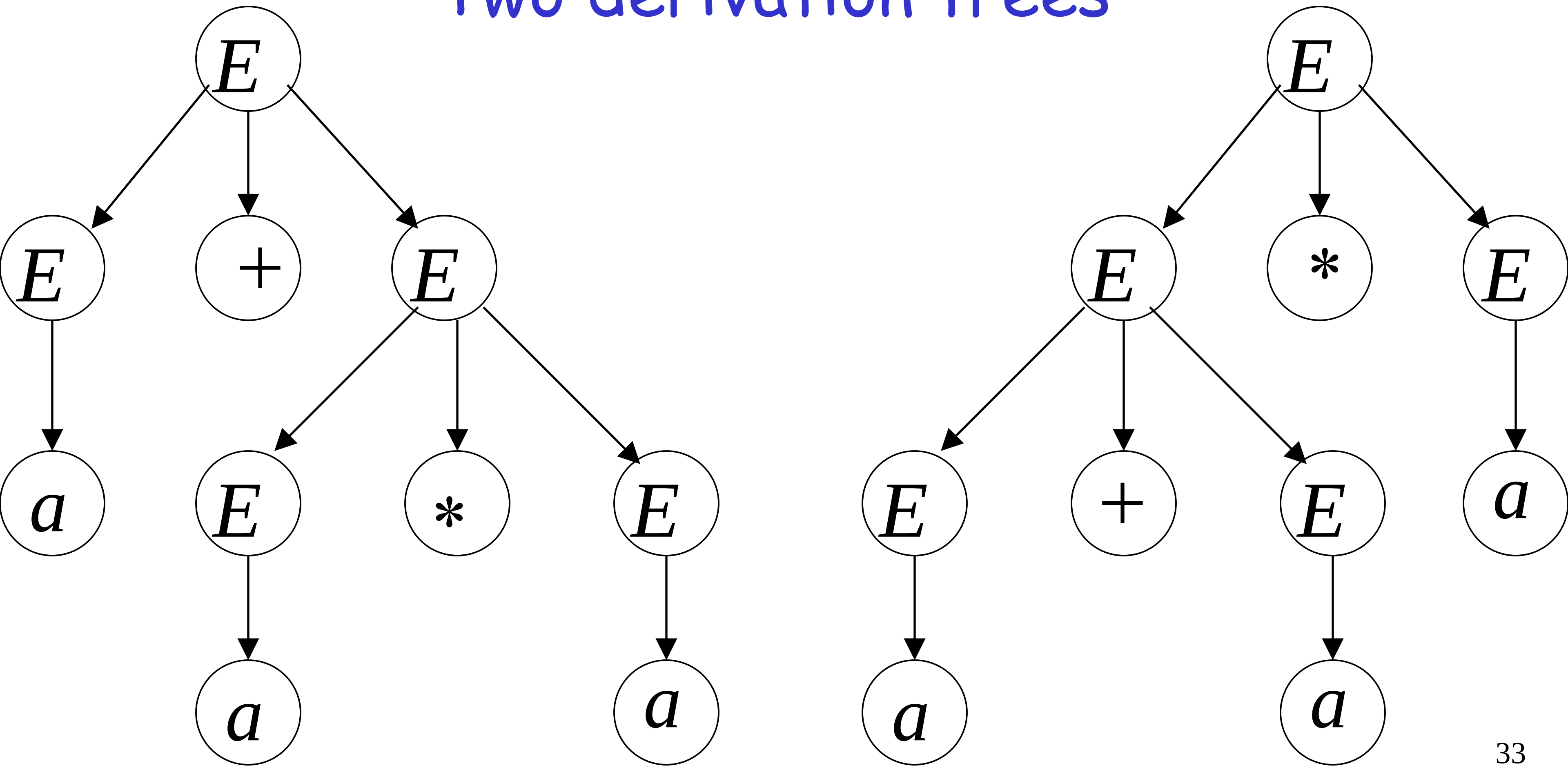
leftmost derivation



$$E \to E + E \mid E * E \mid (E) \mid a$$

$$a + a * a$$

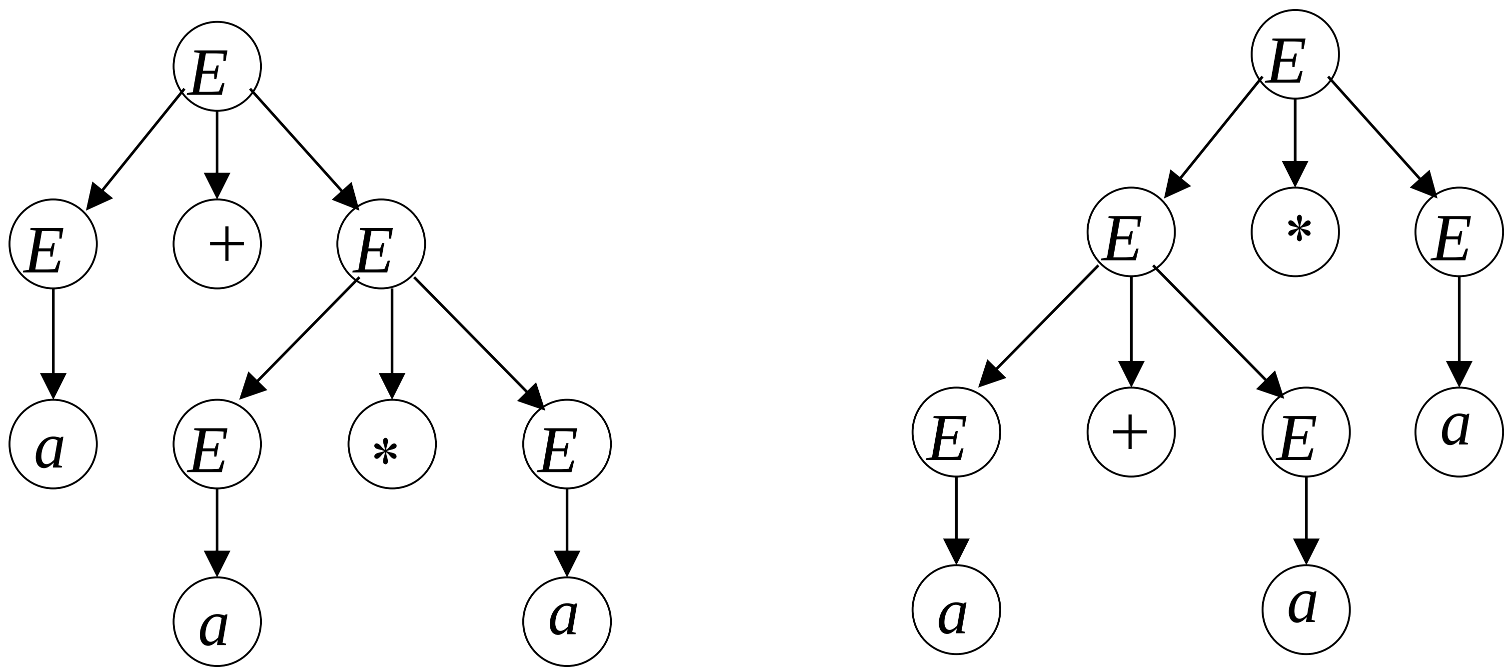
Two derivation trees



The grammar $E \to E + E \mid E * E \mid (E) \mid a$

is ambiguous:

string $a + a * a$ has two derivation trees



The grammar $E \rightarrow E+E \;\; | \; E*E \; | \; (E) \; | \; a$

is ambiguous:

string $a+a*a$ has two leftmost derivations

$$E \Rightarrow E+E \Rightarrow a+E \Rightarrow a+E*E$$
$$\Rightarrow a+a*E \Rightarrow a+a*a$$

$$E \Rightarrow E*E \Rightarrow E+E*E \Rightarrow a+E*E$$
$$\Rightarrow a+a*E \Rightarrow a+a*a$$

Definition:

A context-free grammar G is **ambiguous**

if some string $w \in L(G)$ has:

two or more derivation trees

In other words:

A context-free grammar G is **ambiguous**

if some string $w \in L(G)$ has:

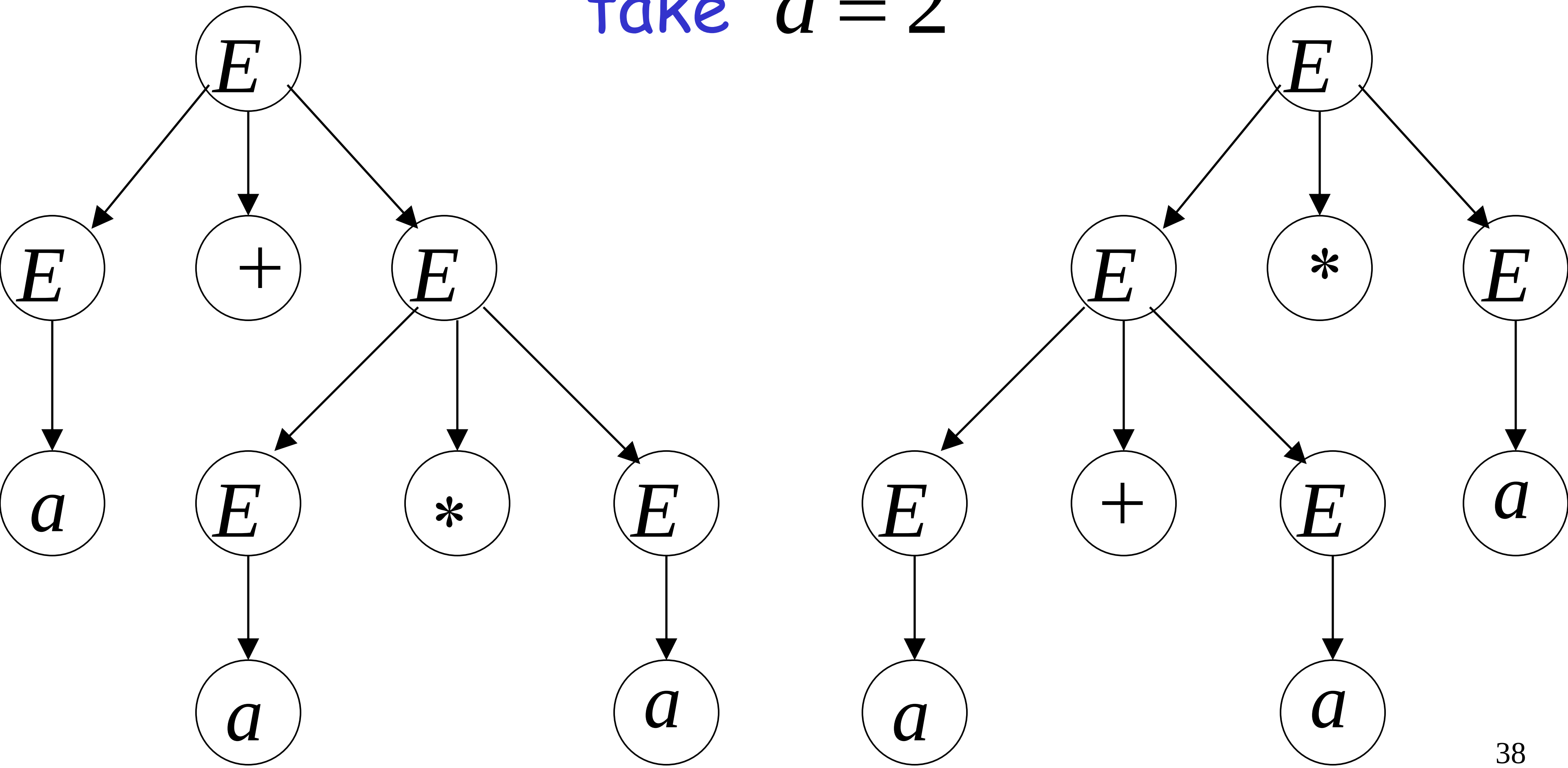
two or more leftmost derivations

(or rightmost)

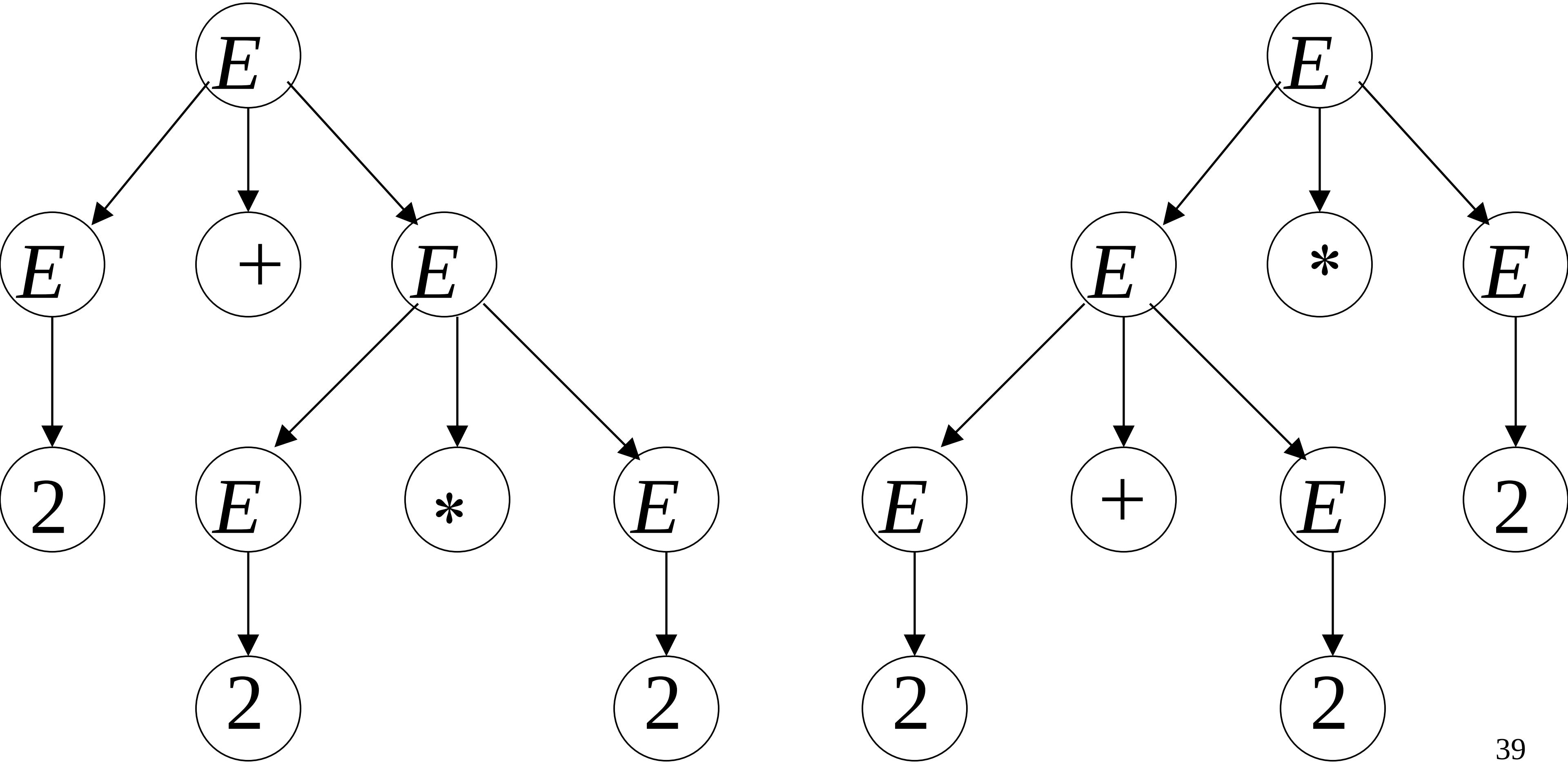
Why do we care about ambiguity?

$$a + a * a$$

take $a = 2$

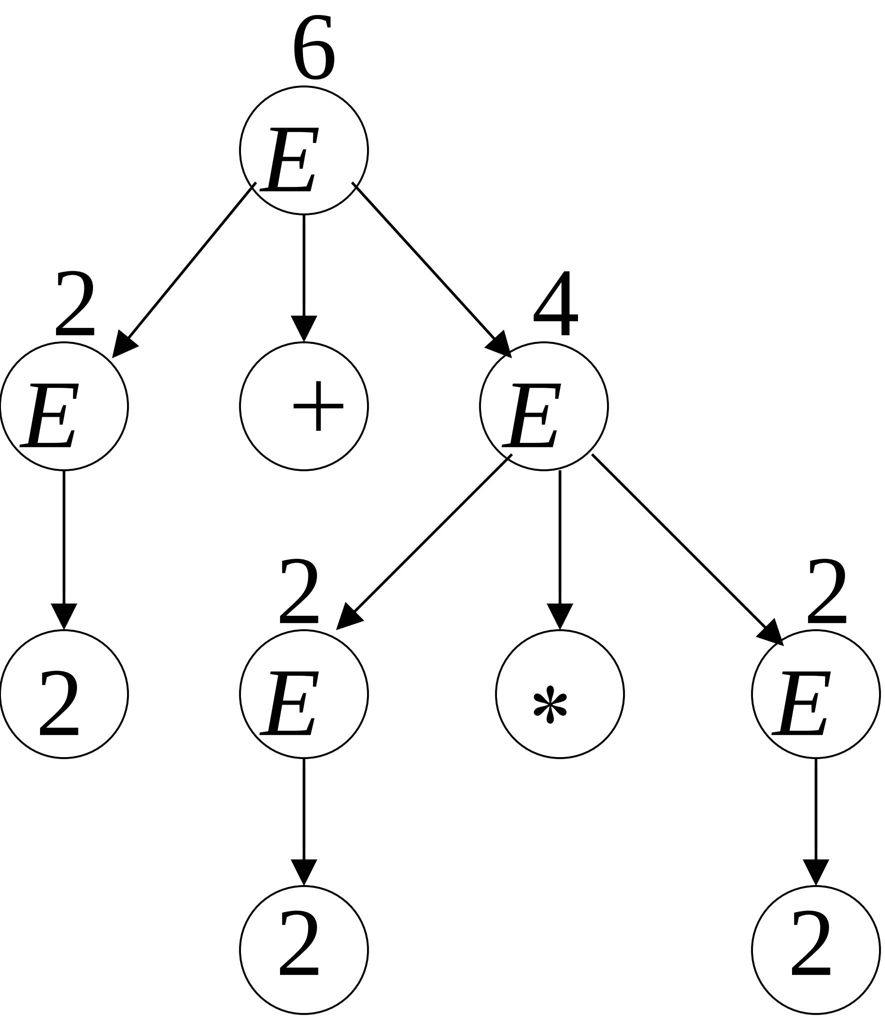


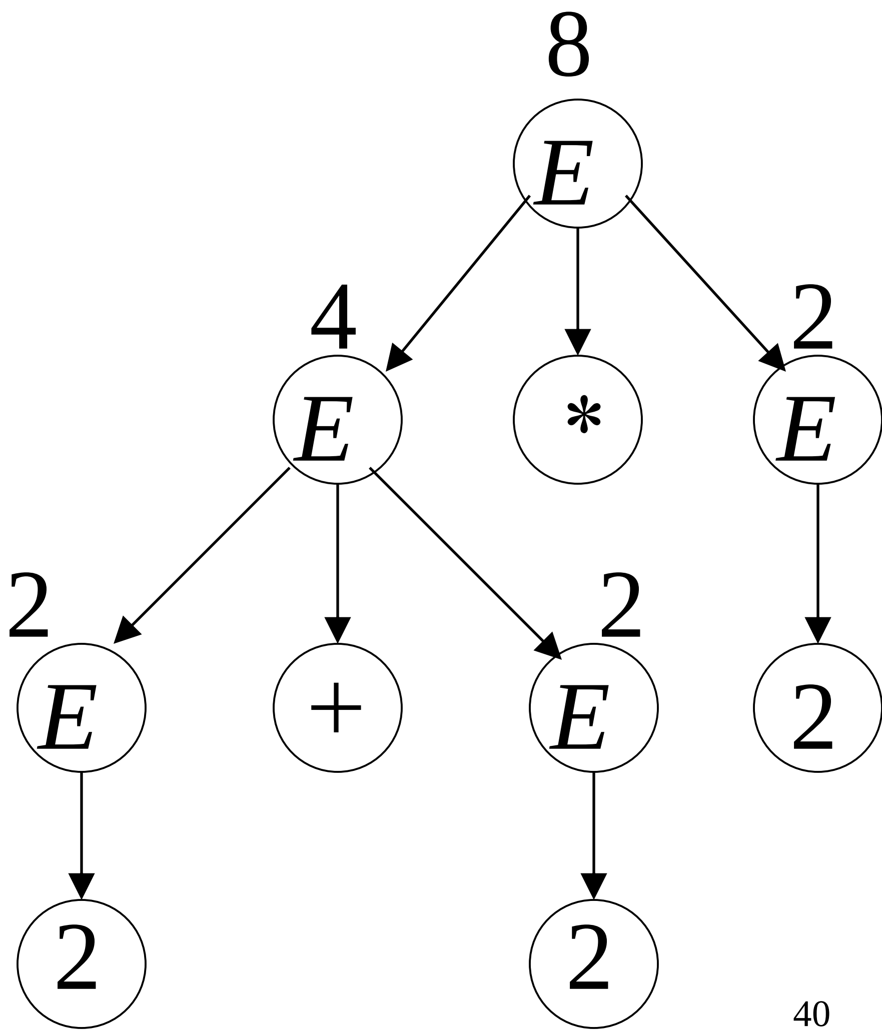
$$2 + 2 * 2$$



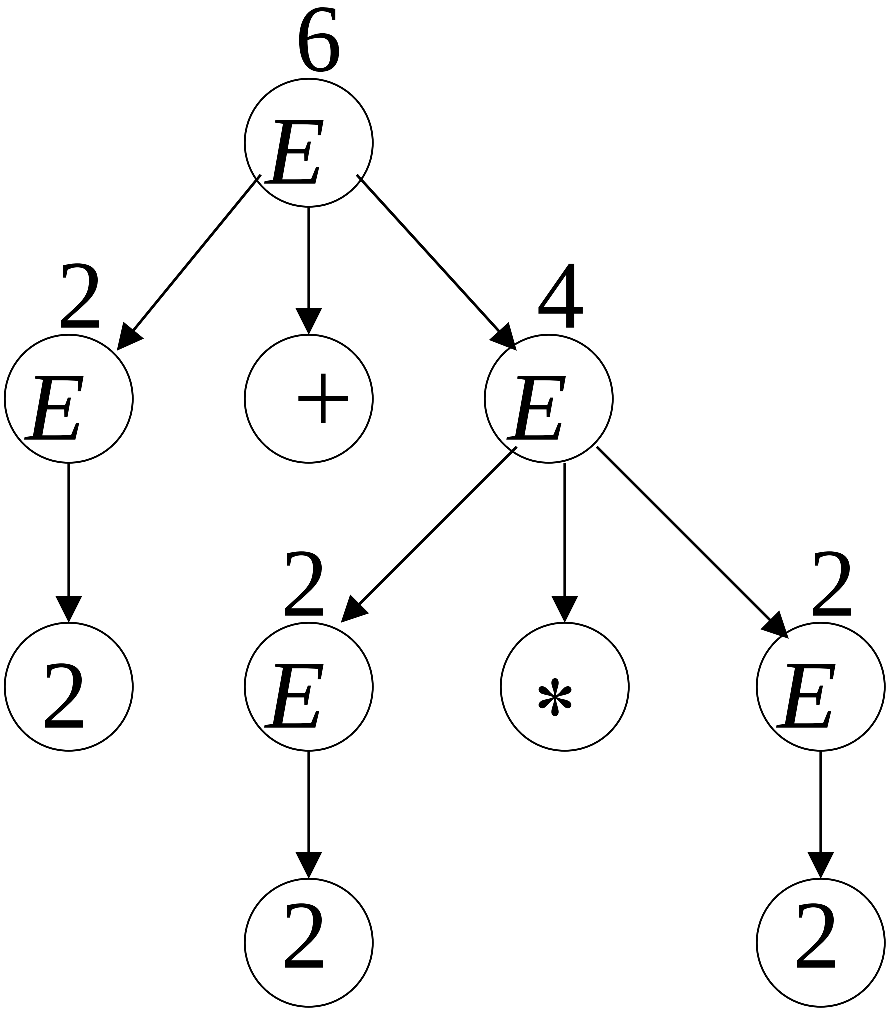
$$2+2*2=6$$

$$2+2*2=8$$





Correct result: $2+2*2=6$



- Ambiguity is **bad** for programming languages

- We want to remove ambiguity

We fix the ambiguous grammar:

$$E \to E + E \ \mid E * E \mid (E) \mid a$$

New non-ambiguous grammar:

$$E \to E + T$$

$$E \to T$$

$$T \to T * F$$

$$T \to F$$

$$F \to (E)$$

$$F \to a$$

$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow a+T \Rightarrow a+T*F$$

$$\Rightarrow a+F*F \Rightarrow a+a*F \Rightarrow a+a*a$$

$$a+a*a$$

$$E \to E+T$$

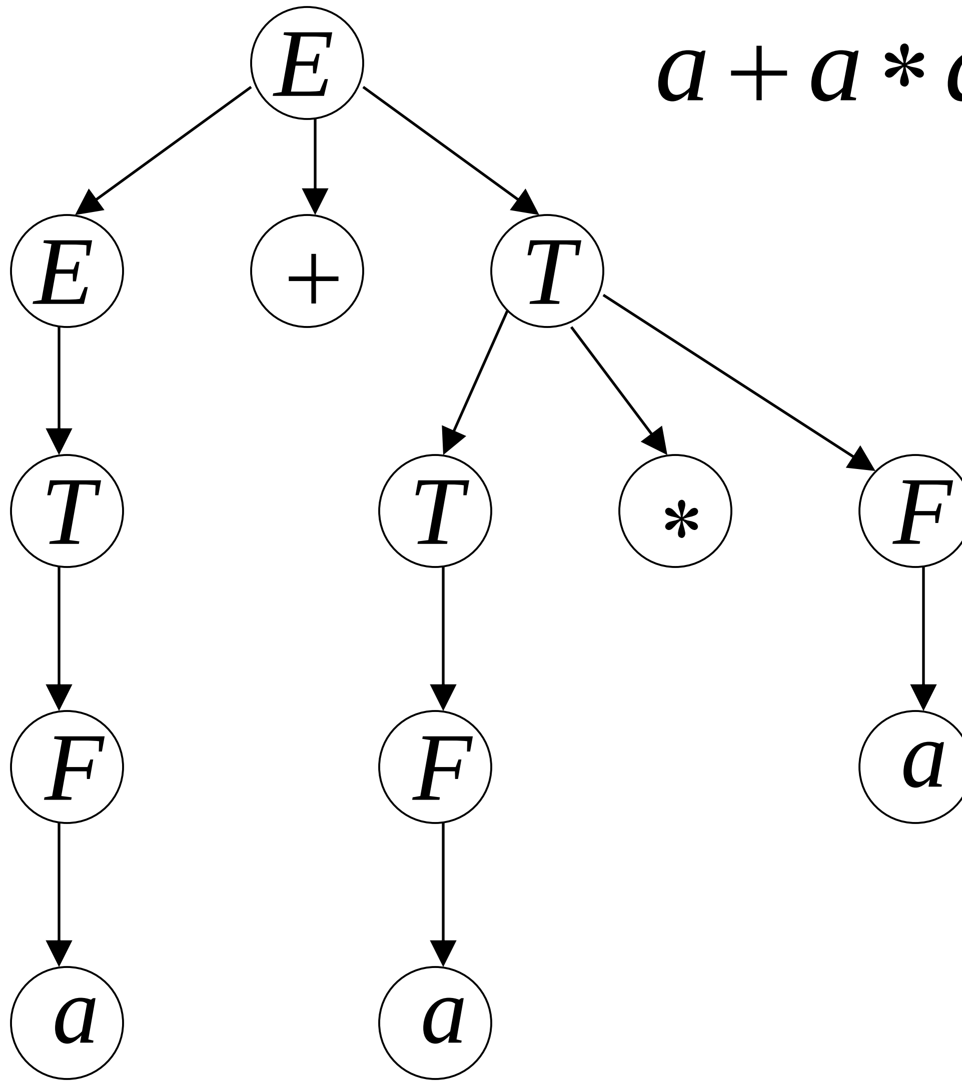
$$E \to T$$

$$T \to T*F$$

$$T \to F$$

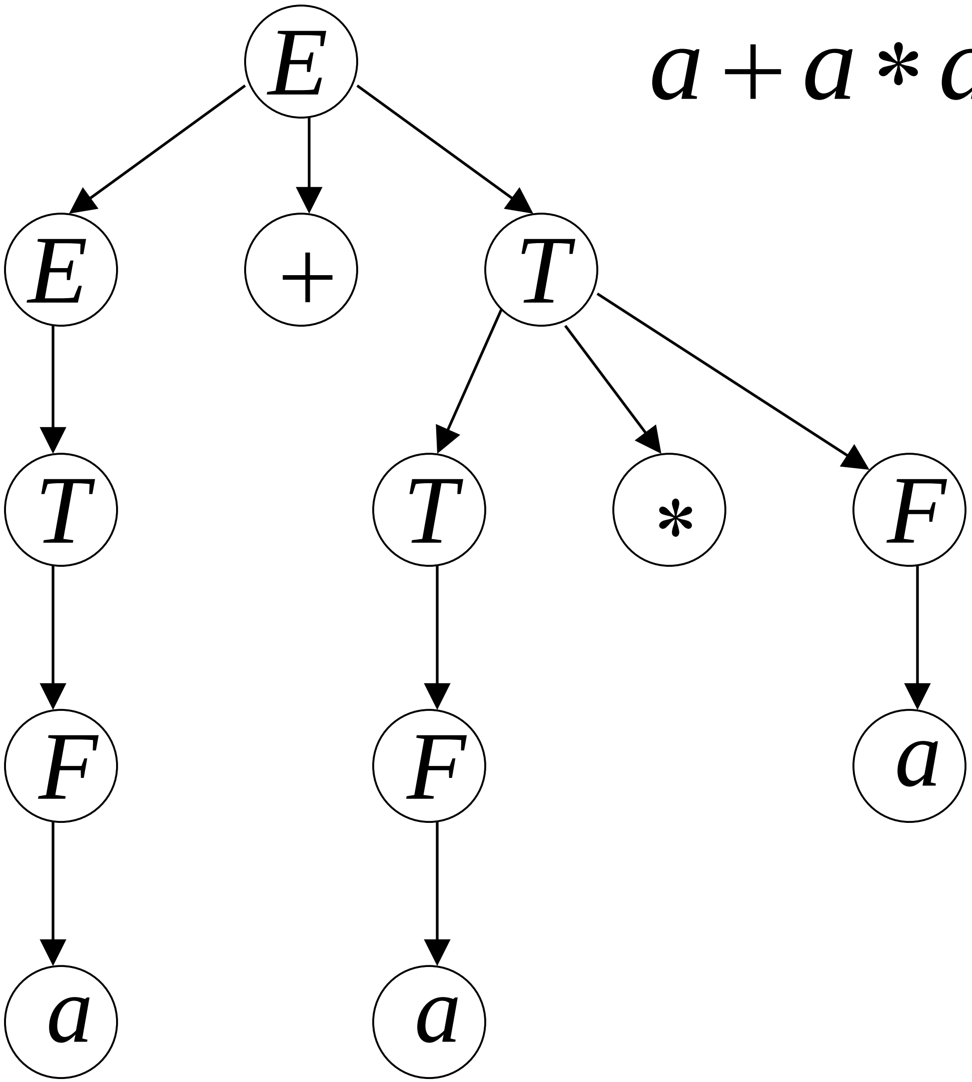
$$F \to (E)$$

$$F \to a$$



Unique derivation tree

$$a + a * a$$



The grammar G:

$$E \to E + T$$

$$E \to T$$

$$T \to T * F$$

$$T \to F$$

$$F \to (E)$$

$$F \to a$$

is non-ambiguous:

Every string $w \in L(G)$ has a unique derivation tree

Inherent Ambiguity

Some context free languages
have only ambiguous grammars

Example: $L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$

$S \to S_1 \mid S_2$

$S_1 \to S_1 c \mid A$

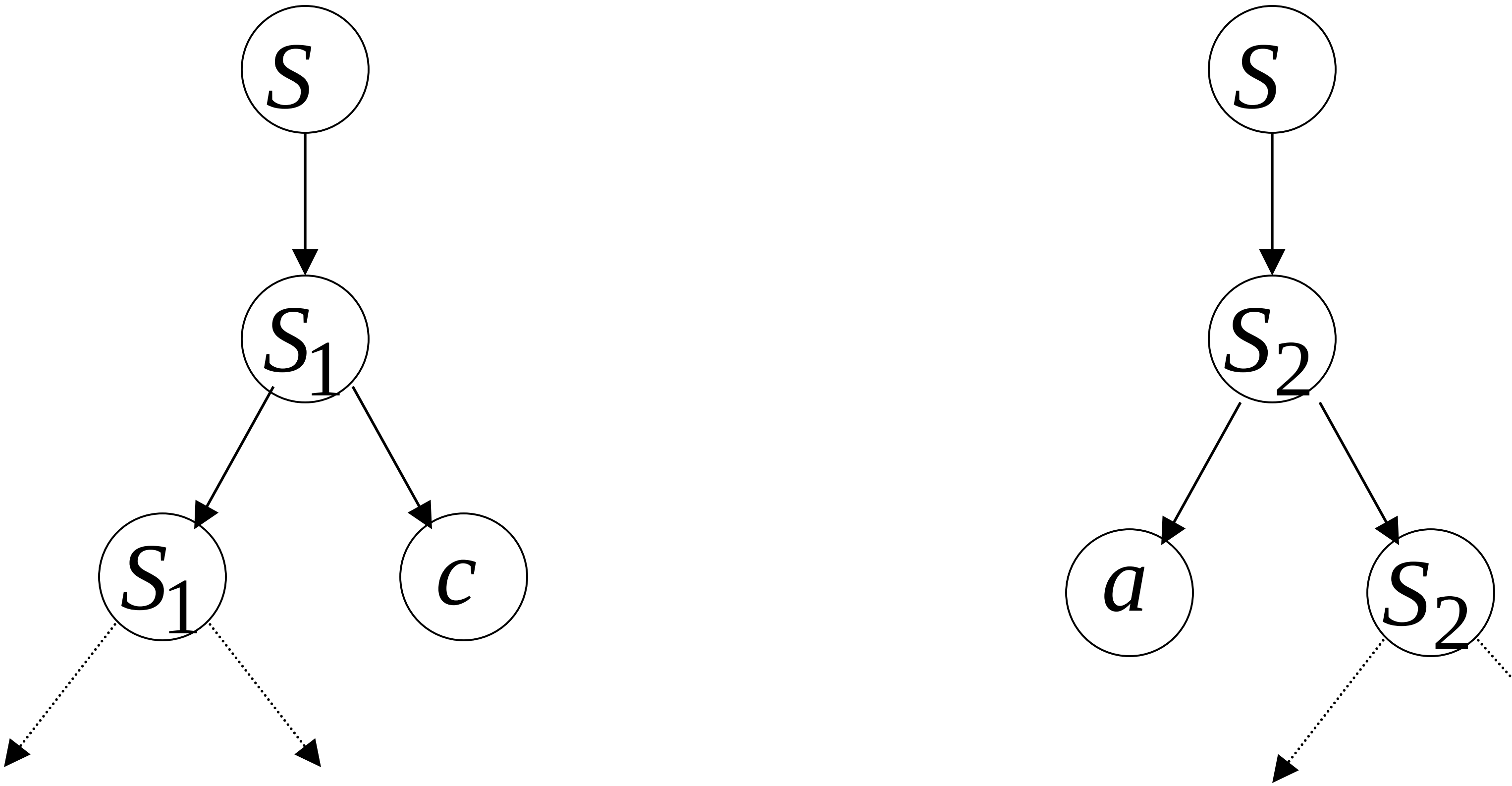
$A \to aAb \mid \lambda$

$S_2 \to aS_2 \mid B$

$B \to bBc \mid \lambda$

The string $a^n b^n c^n$

has two derivation trees



It does not, of course, follow from this that L is inherently ambiguous as there might exist some other unambiguous grammars for it. But in some way L1 and L2 have conflicting requirements. A rigorous argument, though, is quite technical. One proof can be found in Harrison 1978.

Linz, 6th, page 149