# The Laplace Transform



**Defn**

Let $f(t)$ be a given function which is defined for all $t \geq 0$. We multiply $f(t)$ by $e^{-st}$ and integrate w.r.t 't' from $0 \to \infty$. Then if the resulting integral exists, it is a function of s, say $F(s)$

$$\therefore \quad F(s) = \int_0^\infty e^{-st} f(t)\, dt \quad \text{- - - - (1)}$$

The function $F(s)$ is called Laplace Transform
The original function $f(t)$ in eqn (1) is called inverse transform.

In this discussion ie for signal, $F(s)$ is denoted by $X(s)$ and $f(t)$ by $x(t)$.

## Laplace Transform Evaluations and theorems :-

Ex. (1). Find the Laplace transform and abscissa of convergence for the signal $x(t) = e^{-\alpha t} u(t)$.

Ans:⇒ Given $x(t) = e^{-\alpha t} u(t)$

here $u(t)$ is unit step signal.

$$\therefore \quad L\{x(t)\} = X(s) = \int_0^\infty e^{-\alpha t} . u(t)\ e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s+\alpha)t}\, dt = \left[ -\frac{e^{-(s+\alpha)t}}{s+\alpha} \right]_0^\infty$$

$$\therefore \quad X(s) = \frac{1}{s+\alpha}$$

Ex-(2) Find the Laplace Transform of u(t).

Ans:⇒ The unit step signal $u(t)$ is the special case of the signal $e^{-at}\,u(t)$, that occurs when $\alpha = 0$. Therefore $x(t) = u(t)$.

$$\therefore\ X(s) = \frac{1}{s} \text{ as } \alpha = 0$$

$$X(s) = \int_0^{\alpha} u(t)e^{-st}\,dt = \int_0^{\alpha} e^{-st}\,dt = -\frac{1}{s}\left[e^{-st}\right]_0^{\alpha} = \frac{1}{s}$$

Ex. (3) J-97 Find the Laplace Transform of the unit ramp signal.

Ans:⇒ $x(t) = r(t) = t\,u(t)$

$$\therefore\ L\{x(t)\} = X(s) = \int_0^{\alpha} t\,u(t)\,e^{-st}\,dt$$

$$= \int_0^{\alpha} t\,e^{-st}\,dt = \left[\frac{e^{-st}}{s^2}(-st-1)\right]_0^{\alpha}$$

$$= \left[0 - \left(-\frac{1}{s^2}\right)\right] = \frac{1}{s^2}$$

$$\int t\,e^{-st}\,dt = \frac{t\,e^{-st}}{-s} - \int \frac{e^{-st}}{-s}\,dt = \left[\frac{t\,e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\alpha} = -\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$$

Theorem (1) Linearity,

If $x(t) \longleftrightarrow X(s)$ and $y(t) \longleftrightarrow Y(s)$

Then $a\,x(t) + b\,y(t) \longleftrightarrow a\,X(s) + b\,Y(s)$.

Problem (4) Find the Laplace Transform of the signal $x(t) = \sin(\omega_0 t)\cdot u(t)$.

Ans:⇒ Given $x(t) = \sin(\omega_0 t)\,u(t)$

$$\therefore\ x(t) = \frac{1}{2j}e^{j\omega_0 t}u(t) - \frac{1}{2j}e^{-j\omega_0 t}u(t)$$

Now we know $e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{s+\alpha}$ (3)

$$\therefore L\{x(t)\} = X(s) = \frac{1}{2j}\left[\frac{1}{s-j\omega_0}\right] - \frac{1}{2j}\left[\frac{1}{s+j\omega_0}\right]$$

$$= \frac{\omega_0}{s^2+\omega_0^2}.$$

Theorem (2) Scale change

If $x(t) \longleftrightarrow X(s)$ then $x(mt) \longleftrightarrow \frac{1}{m} X\left(\frac{s}{m}\right)$

for $m > 0$ ; $m \nless 0$ or $m \neq 0$.

J-97 7(d) Prove that if the LT of $x(t)$ is $X(s)$, then LT of $x(mt)$ is $\frac{1}{m} X\left(\frac{s}{m}\right)$ for $m>0$. How is the result modified if $m<0$?

Ans:⇒ We have $L\{x(t)\} = X(s)$

Now $L\{x(mt)\} = \int_0^\infty e^{-st} x(mt)\, dt$

$$= \frac{1}{m}\int_0^\infty e^{-s\tau/m} \cdot x(\tau)\, d\tau ; \text{ Putting } mt = \tau.$$

$$= \frac{1}{m}\int_0^\infty e^{-s'\tau} x(\tau)\, d\tau \text{ where } s' = \frac{s}{m}$$

$$= \frac{1}{m} X(s') = \frac{1}{m} X\left(\frac{s}{m}\right). \text{ Proved.}$$

Now when $m<0$ ie $m$ is -ve, it would result in a reversal of $x(t)$, meaning $x(mt)$ would be non-zero only for $t<0$, since $x(t) = 0$ for $t<0$, Thus the Laplace Transform of $x(mt)$ is $X(s)=0$ when $m<0$ and the scaling theorem is not valid.

Ex. 5. Find the Laplace transform of the signal $y(t) = r(3t)$.

Soln:

Let $z(t) = r(t)$

$$Z(s) = \frac{1}{s^2}$$

Thus the scale change theorem gives

$$X(s) = \frac{1}{3} Z\left(\frac{s}{3}\right) = \frac{1}{3}\left[\frac{1}{(s/3)^2}\right] = \frac{3}{s^2}$$

2nd process $\Rightarrow$ $y(t) = r\,3(t) = 3t\; u(3t)$

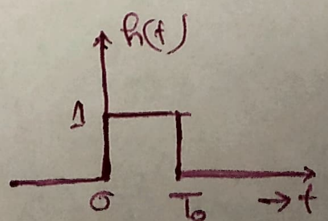
$$= 3t\; u(t) = 3\,r(t) \longleftrightarrow 3/s^2$$

Theorem-3 Time Delay.

If $x(t) \longleftrightarrow X(s)$ and $x(t) = 0$ for $t<0$

then $x(t-t_0) \longleftrightarrow X(s)\cdot e^{-st_0}$ for $t_0>0$

Ex. 6. A zero-order hold system has the impulse response shown in fig. Find the LT of this impulse response.




Soln: From fig.

$$h(t) = u(t) - u(t-T_0)$$

We use the time delay and linearity

theorems and the transform $u(t) \longleftrightarrow \frac{1}{s}$ to given

$$L\{h(t)\} = H(s) = \frac{1}{s} - \frac{1}{s} \cdot e^{-sT_0}$$

$$= \frac{1-e^{-sT_0}}{s}$$

This system is called hold system. It holds the value of the strength of an impulse for time interval $T_0$.

Theorem ④ s-shift, If $x(t) \longleftrightarrow X(s)$. then

$e^{-kt}x(t) \longleftrightarrow X(s+k)$ ; k is any real or complex number.

Proof ⇒

$$L\{e^{-kt}x(t)\} = \int_0^{\infty} e^{-st} \cdot e^{-kt}\, x(t)\, dt$$

$$= \int_0^{\infty} e^{-(s+k)t}\, x(t)\, dt = X(s+k)$$

Theorem ⑤ t-Multiplication, if $x(t) \longleftrightarrow X(s)$

then $t\, x(t) \longleftrightarrow -\frac{dX(s)}{ds}$

Ex.7. Find the Laplace Transform of the signal

$x(t) = t\, e^{-at}\, u(t)$.

Soln let $y(t) = e^{-at}\, u(t)$

$$\therefore\ Y(s) = \frac{1}{s+a}$$

$$\therefore\ L\{x(t)\} = X(s) = -\frac{d}{ds}\left[\frac{1}{s+a}\right] = +\frac{1}{(s+a)^2}$$

Note: Extending theorem ⑤ by induction gives

$$t^n x(t) \longleftrightarrow (-1)^n \cdot \frac{d^n [X(s)]}{ds^n}$$

Theorem-6 Time Differentiation;

If $x(t) \longleftrightarrow X(s)$

then $$\frac{d^n x(t)}{dt^n} \longleftrightarrow s^n X(s) - \sum_{i=0}^{n-1} s^{n-1-i} x^{(i)}(0^-)$$

where $$x^{(i)}(0^-) = \left.\frac{d^i x(t)}{dt^i}\right|_{t=0^-}$$

Thus $$L\{x^{(1)}(t)\} = sX(s) - x(0^-)$$

$$L\{x^{(2)}(t)\} = s^2X(s) - sx(0^-) - x^{(1)}(0^-)$$

Ex.8. Find the Laplace Transform of the derivative of the signal $x(t) = e^{-t}u(t)$. Invert the transform to find the signal derivative.

Soln. By the time differentiation theorem,

$$L\{x^{(1)}(t)\} = sX(s) - x(0^-)$$

Now $x(t) = e^{-t}u(t)$

$$\therefore X(s) = \frac{1}{s+1}$$

and $x(0^-) = 0$ because $u(t) = 0,\ t<0$

Therefore, $$L\{x^{(1)}(t)\} = \frac{s}{s+1} - 0$$

$$= \frac{s+1-1}{s+1} = 1 - \frac{1}{s+1}$$

By using transform ——, we find that

the derivative of the signal

$$x^{(1)}(t) = \delta(t) - e^{-t}u(t).$$

Theorem 7 Time integration:-

If $x(t) \longleftrightarrow X(s)$

then $y(t) = \int_{0^-}^{t} x(\lambda)\,d\lambda + y(0^-) \longleftrightarrow \frac{X(s)}{s} + \frac{y(0^-)}{s}$

* Find the Laplace Transform of $y(t) = \int_{0^-}^{t} x(\lambda)\,d\lambda + y(0^-)$

when $x(t) = r(t)$ and $y(0^-) = 2$.

Soln Given $x(t) = r(t)$

$$\therefore X(s) = \frac{1}{s^2}$$

Thus time integration theorem gives

$$Y(s) = \frac{1/s^2}{s} + \frac{2}{s} = \frac{1}{s^3} + \frac{2}{s} = \frac{2s^2+1}{s^3}$$

Inverse Laplace Transform Evaluations:-

Ex. 1 Find the signal $y(t)$, the Laplace transform of which is $Y(s) = \frac{s^3+7s^2+18s+20}{s^2+5s+6}$

Soln We have $Y(s) = \frac{s^3+7s^2+18s+20}{s^2+5s+6}$

or, $Y(s) = s+2+\frac{2s+8}{(s+3)(s+2)}$

or, $Y(s) = s+2+4\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+3}\right)$

$$\boxed{\therefore y(t) = \delta'(t) + 2\delta(t) + 4e^{-2t}u(t) - 2e^{-3t}u(t)}$$

## Different Types :-

v.v.i Case I Exponential in the Numerator.

Ex. (2) Find $x(t)$ that corresponds to

$$X(s) = \frac{s^2 e^{-2s} + e^{-3s}}{s(s^2+3s+2)}$$

Soln First we separate the transform into a sum of rational functions multiplied by exponentials in s

$$X(s) = \left[\frac{s}{s^2+3s+2}\right] e^{-2s} + \left[\frac{1}{s(s^2+3s+2)}\right] e^{-3s}$$

$$= \left[2\left(\frac{1}{s+2}\right) - \left(\frac{1}{s+1}\right)\right] e^{-2s} + 0.5\left(\frac{1}{s}\right) e^{-3s}$$

$$+ 0.5\left[\frac{1}{s+2}\right] e^{-3s} - \left[\frac{1}{s+1}\right] e^{-3s}$$

$$\therefore x(t) = 2e^{-2(t-2)} u(t-2) - e^{-(t-2)} u(t-2)$$

$$+ 0.5\, u(t-3) + 0.5\, e^{-2(t-3)} u(t-3) - e^{-(t-3)} u(t-3)$$

Here we have applied time shift and linearity theorems.

# PERIODIC FUNCTION

Dec' 97

9(a)

Given a periodic function $f(t)$ with period $T$ and $\int_0^T f(t)\, e^{-st}\, dt = F_1(s)$. Show that the Laplace Transform of $f(t)$ is $F(s) = \frac{F_1(s)}{1-e^{-sT}}$.

Soln:

Let $f(t)$ be a periodic function, defined for all +ve '$t$' and has the period $T$.

$$\therefore \quad f(t+nT) = f(t) \quad \text{for all } t \geq 0.$$

$n$ = any integer

If $f(t)$ is piecewise continuous on an interval of length $T$, then its Laplace transform exists, and we can write the integral from zero to infinity as the series of integrals over successive periods.

$$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt$$

$$+ \int_T^{2T} e^{-st} f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + \cdots$$

If we substitute $t = \tau + T$ in the second integral, $t = \tau + 2T$ in the 3rd integral ----, $t = \tau + (n-1)T$ in the nth integral..., then the new limits in every integral are zero and $T$.

Since $f(\tau+T) = f(\tau)$, $f(\tau+2T) = f(\tau)$ etc.

we can write —

$$F(s) = \int_0^T e^{-s\tau} f(\tau)\, d\tau + \int_0^T e^{-s(\tau+T)} f(\tau)\, d\tau + \cdots$$

$$+ \int_0^{\tau} e^{-s(\tau+2T)} f(\tau)\, d\tau + \cdots$$

The factors which do not depend on $\tau$ can be taken out from under the integral signs; this gives

$$F(s) = \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right] \int_0^{T} e^{-s\tau} f(\tau)\, d\tau$$

$$\therefore F(s) = \left[\frac{1}{1-e^{-sT}}\right] \int_0^{T} e^{-st} f(t)\, dt$$

$$\boxed{\therefore F(s) = \frac{F_1(s)}{1-e^{-sT}}}$$ Proved.

Laplace Transform solutions of linear integro-differential equations:-

Ex. 1 A second order control system is characterised by the differential equation

$$y^{(2)}(t) + 5y^{(1)}(t) + 6y(t) = x(t).$$

Solve for $y(t)$ for $t \geq 0$ when $x(t) = u(t)$ and the initial conditions are $y(0^-) = 2$ and $y^{(1)}(0^-) = -12$.

Soln: Using the time differentiation and linearity theorems, we find that the Laplace transform of the differential equation is

$$s^2 Y(s) - s\,y(0^-) - y^{(1)}(0^-) + 5[s\,Y(s) - Y(0^-)] + 6Y(s) = \frac{1}{s}$$

Since $u(t) \longleftrightarrow \frac{1}{s}$, substituting the initial condition and rearranging yields.

$$(s^2+5s+6)\,Y(s) = (2s^2-2s+1)/s.$$

$$\therefore\ Y(s) = \frac{(2s^2-2s+1)}{s(s+3)(s+2)}$$

$$\therefore\ Y(s) = \frac{1}{6}\left[\frac{1}{s}\right] - \frac{13}{2}\left[\frac{1}{s+2}\right] + \frac{25}{3}\left[\frac{1}{s+3}\right]$$

$\therefore$ The system response is

$$\boxed{y(t) = \frac{1}{6}u(t) - \frac{13}{2}e^{-2t}u(t) + \frac{25}{3}e^{-3t}u(t)}$$