Z-transformation

Shifting property :-

Shifting of f(t) to the right :-

① If $Z\{f(n)\} = F(z)$, then $Z\{f(n+1)\} = ?$

Solⁿ

From defination of Z-transform :-

$$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1)\, z^{-n} \rightarrow ①$$

let $n+1 = K$. or, $n = K-1$

when $n = 0$, $K = 1$ & $n = \infty$, $K = \infty$

Putting this in eqⁿ ①, we get

$$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(k)\, z^{-(k-1)}$$

$$= z \sum_{k=1}^{\infty} f(k)\, z^{-k}$$

$$= z \sum_{k=0}^{\infty} f(k)\, z^{-k} - z \cdot f(0)\, z^{-0}$$

$$= z \sum_{k=0}^{\infty} f(k)\, z^{-k} - z f(0)$$

$$= z\,[F(z) - f(0)]$$

If $Z\{f(n)\} = F(z)$, then find $Z\{f(n+2)\} = ?$

Solⁿ

From defination,

$$Z\{f(n+2)\} = \sum_{n=0}^{\infty} f(n+2)\, z^{-n}$$

let $n+2 = K$, $\therefore n = K-2$

at $n = 0$, $K = 2$; $n = \infty$, $K = \infty$

$$\therefore Z\{f(n+2)\} = \sum_{k=2}^{\alpha} f(k)\, z^{-(k-2)}$$ (2)

$$= z^2 \sum_{k=2}^{\alpha} f(k)\, z^{-k}$$

$$= z^2 \sum_{k=0}^{\alpha} f(k)\, z^{-k} - z^2 \sum_{k=0}^{1} f(k)\, z^{-n}$$

$$= z^2 \left[\sum_{k=0}^{\alpha} f(k)\, z^{-k} - f(0)\, z^{-0} - f(1)\, z^{-1} \right]$$

$$= z^2 \left[F(z) - f(0) - \frac{f(1)}{z} \right]$$

$$= z^2 F(z) - z^2 f(0) - z f(1).$$

If $Z\{f(n)\} = F(z)$ then find $Z\{f(n+3)\} = ?$

Sol^n

From definition,

$$Z\{f(n+3)\} = \sum_{n=0}^{\alpha} f(n+3)\, z^{-n}$$

Let $n+3 = k$, $\therefore n = k-3$

$n = 0,\ k = 3$

$n = \alpha,\ k = \alpha.$

$$\therefore Z\{f(n+3)\} = \sum_{k=3}^{\alpha} f(k)\, z^{-(k-3)}$$

$$= z^3 \sum_{k=3}^{\alpha} f(k)\, z^{-k}$$

$$= z^3 \left\{ \sum_{k=0}^{\alpha} f(k)\, z^{-k} - \sum_{k=0}^{2} f(k)\, z^{-k} \right\}$$

$$= z^3 \left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} \right]$$

$$= z^3 F(z) - z^3 f(0) - z^2 f(1) - z f(2) \quad (3)$$

If $Z\{f(t)\} = F(z)$, then prove that

$$Z\{f(n+k)\} = z^k \left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} \cdots - \frac{f(k-1)}{z^{k-1}}\right]$$

Sol^n:

From defination we have,

$$Z\{f(n+k)\} = \sum_{n=0}^{\infty} f(n+k)\, z^{-n}$$

let $n+k = m$ or, $n = m-k$.

$n=0$, $m=k$, $n=\infty$ $m=\infty$.

Putting this limit we get

$$Z\{f(n+k)\} = \sum_{m=k}^{\infty} f(m)\, z^{-(m-k)}$$

$$= z^k \left[\sum_{m=k}^{\infty} f(m)\, z^{-m}\right]$$

$$= z^k \left[\sum_{m=0}^{\infty} f(m)\, z^{-m} - \sum_{m=0}^{k-1} f(m)\, z^{-m}\right]$$

$$= z^k \left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2} \cdots\cdots - \frac{f(k-1)}{z^{k-1}}\right]$$

Proved

LEFT shift

If $Z\{f(n)\} = F(z)$, then find $Z\{f(n-1)\} = ?$

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Sol^n:

From defination, we have,

$$F(z) = Z\{f(n-1)\}$$

$$= \sum_{n=0}^{\infty} f(n-1)\, z^{-n}$$

let $n-1 = k$, $\therefore n = k+1$

$n=0$, $k=-1$, $n=\infty$ $k=\infty$

$$\therefore F(z) = \sum_{k=-1}^{\infty} f(k)\, z^{-(k+1)}$$

$$= z^{-1} \sum_{k=-1}^{\infty} f(k)\, z^{-k}$$

$$= z^{-1} \left\{ \sum_{k=0}^{\infty} f(k)\, z^{-k} + f(-1) \times z \right\}$$

$$= z^{-1} \left[F(z) + z f(-1) \right]$$

$$= z^{-1} F(z) + f(-1)$$

If $Z\{f(n)\} = F(z)$, then find $Z\{f(n-2)\} = ?$

Sol^n:

From defination,

$$F(z) = Z\{f(n-2)\}$$

$$= \sum_{n=0}^{\infty} f(n-2)\, z^{-n}$$

Let $n-2=k$ or, $n=k+2$

$n=0,\ k=-2,\ n=\infty,\ k=\infty.$

$$\therefore F(z) = \sum_{k=-2}^{\infty} f(k)\, z^{-(k+2)}$$

$$= z^{-2}\left[\sum_{k=-2}^{\infty} f(k)\, z^{-k}\right]$$

$$= z^{-2}\left[\sum_{k=0}^{\infty} f(k)\, z^{-k} + \sum_{k=-2}^{-1} f(k)\, z^{-k}\right]$$

$$= z^{-2}\left[F(z) + f(-2)\cdot z^{+2} + f(-1)\, z^{+1}\right]$$

$$= z^{-2}F(z) + f(-2) + z^{-1} f(-1)$$

Similarly we can prove that

$$Z[f(n-3)] = z^{-3}F(z) + z^{-2}f(-1) + z^{-1}f(-2) + f(-3)$$

\# If $Z\{f(n)\} = F(z)$, then find $Z\{f(n-k)\} = ?$

Sol^n:

From defination,

$$Z\{f(n-k)\} = F(z)$$

$$= \sum_{n=0}^{\infty} f(n-k)\, z^{-n}$$

Let $n-k=m$, or, $n=m+k$.

$n=0,\ m=-k,\ n=\infty,\ m=\infty$

$$\therefore F(z) = \sum_{m=-k}^{\alpha} f(k)\, z^{-(m+k)}$$

$$= z^{-m} \sum_{m=-k}^{\alpha} f(k)\, z^{-k}$$

$$= z^{-m} \left[\sum_{m=0}^{\alpha} f(k)\, z^{-k} + \sum_{m=-k}^{-1} f(k)\, z^{-k} \right]$$

$$= z^{-m} \left[F(z) + f(-k)\, z^{k} + f(-k+1)\, z^{k-1} + f(-k+2)\, z^{k-2} + \dots f(-1)\, z \right]$$

Ans

Find $F(z)$ for the difference eqn $f(n+1) - f(n) = 1$.
where $f(0) = 0$

Soln

$$f(n+1) - f(n) = 1.$$

Taking z-transformation on both side we get,

$$z F(z) - F f(0) - F(z) = \frac{z}{z-1}$$

$$\text{or, } z F(z) - F(z) = \frac{z}{z-1}$$

$$\text{or, } F(z) [z-1] = \frac{z}{z-1}$$

$$\therefore F(z) = \frac{z}{(z-1)^2}$$

~~If $F(z) = \frac{2z^2+5z+14}{(z-1)^4}$ Find $f(2)$ & $f(3)$.~~

~~Soln~~

Shifting with unit step:-

If $Z\{f(n)\} = F(z)$ then find $Z\{f(n-d)\,1(n-d)\} = ?$

Soln

By defination, we have,

$$Z\{f(n-d)\,1(n-d)\} = \sum_{n=0}^{\infty} f(n-d)\,1(n-d)\,z^{-n}$$

Let $n-d = P$. or, $n = P+d$.

$n=0 \quad P=-d$

$n=\infty \quad P=\infty$

$$\therefore F(z) = \sum_{P=-d}^{\infty} f(P)\,1(P).z^{-(P+d)}$$

$$= z^{-d}\sum_{P=-d}^{\infty} f(P)\,z^{-(P)}$$

$$= z^{-d}\left[\sum_{P=0}^{\infty} f(P)(1)^{P} z^{-P} + \sum_{P=-d}^{-1} f(P)(1)^{P} z^{-P}\right]$$

Since (1^P) exist $F(z)$ exist only $+0 \to \infty$. So,

Neglecting $P=-\infty \to 0$ then above eqn becomes

$$= z^{-d}\left[\sum_{P=0}^{\infty} f(P)(1)^{P} z^{-P}\right]$$

$$= z^{-d}F(z).$$

This above result also implies that

$$Z\{f(t-dT)\,1(t-dT)\} = z^{-d}F(z).$$

** Again $Z\{f(t+dT)\, I(t+dT)\} =$

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Initial value theorem & Final value theorem :-

Initial value theorem:-

Initial value of the $f(n)$ $f(t)$ or $f(n)$ can be obtained directly from $F(Z)$ by the relation $f(0) = \lim_{Z \to \infty} F(Z)$.

$$F(Z) = \sum_{n=0}^{\infty} f(n) Z^{-n}$$

$$= f(0) + f(1) Z^{-1} + \frac{f(2)}{Z^2} + \frac{f(3)}{Z^3} + \cdots \rightarrow (1)$$

Taking $\lim Z \to \infty$ on both side we get,

$$\lim_{Z \to \infty} F(Z) = f(0) + 0 + 0 \cdots$$

$$\therefore f(0) = \lim_{Z \to \infty} F(Z). \checkmark$$

From eqn (1),

$$\lim_{z\to\alpha} z F(z) = z f(0) + f(1) + 0 + 0 \ldots$$

$$\therefore f(1) = \lim_{z\to\alpha} z F(z) - z f(0).$$

$$= z\left[\lim_{z\to\alpha} F(z) - f(0)\right]$$

Similarly :- From eqn (1)

$$\lim_{z\to\alpha} z^2 F(z) = z^2 f(0) + z f(1) + f(2).$$

$$\therefore f(2) = z^2\left[\lim_{z\to\alpha} F(z) - f(0) - \frac{f(1)}{z}\right]$$

$$\therefore f(3) = \lim_{z\to\alpha} z^3\left[\lim_{z\to\alpha} F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2}\right]$$

If $F(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$ then find $f(2)$, & $f(3)$.

Soln

$$F(z) = \frac{2z^2+5z+14}{(z-1)^4}$$

$$= \frac{\frac{2}{z^2} + \frac{5}{z^3} + \frac{14}{z^4}}{(1-\frac{1}{z})^4}$$

We have,

$$f(0) = \lim_{z\to\alpha} F(z) = \frac{0+0+0}{(1-0)^4} = 0$$

$$f(1) = \lim_{z\to\alpha} z F(z) - z f(0)$$ (10)

$$= \lim_{z\to\alpha} z \times \frac{\frac{1}{z^2}\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{\left(1-\frac{1}{z}\right)^4}$$

$$= \lim_{z\to\alpha} \frac{\frac{1}{z}\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{\left(1-\frac{1}{z}\right)^4} = 0$$

$$f(2) = \lim_{z\to\alpha} z^2\left[F(z) - f(0) - \frac{f(1)}{z}\right]$$

$$= \lim_{z\to\alpha} z^2 F(z)$$

$$= \lim_{z\to\alpha} \frac{2+\frac{5}{z}+\frac{14}{z^2}}{\left(1-\frac{1}{z}\right)^4} = 2$$

$$f(3) = \lim_{z\to\alpha} z^3\left[F(z) - f(0) - \frac{f(1)}{z} - \frac{f(2)}{z^2}\right]$$

$$= \lim_{z\to\alpha} z^3\left[F(z) - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\alpha} z^3\left[\frac{\frac{1}{z^2}\left[2+\frac{5}{z}+\frac{14}{z^2}\right]}{\left(1-\frac{1}{z}\right)^4} - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\alpha}\left[\frac{z\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{\left(1-\frac{1}{z}\right)^4} - 2z\right]$$

$$\cancel{= \lim_{z\to\alpha}\left[\frac{\left(2+\frac{5}{z}+\frac{14}{z}\right)}{\frac{1}{z}\left(1-\frac{1}{z}\right)^4} - 2z\right]}$$

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$$= \underset{z\to\alpha}{\text{Lt}} \left[\frac{2 + \frac{5}{z} + \frac{14}{z} - 2z \times \frac{1}{z}\left(1-\frac{1}{z}\right)^4}{\frac{1}{z}\left(1-\frac{1}{z}\right)^4} \right]$$

$$= \underset{z\to\alpha}{\text{Lt}} \left[\frac{2 + \frac{5}{z} + \frac{14}{z} - 2\left(1-\frac{1}{z}\right)^4}{\frac{1}{z}\left(1-\frac{1}{z}\right)^4} \right]$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{\left(2 + \frac{5}{z} + \frac{14}{z^2}\right)z - 2z\left(1-\frac{1}{z}\right)^4}{\left(1-\frac{1}{z}\right)^4}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{\left(2z + 5 + \frac{14}{z}\right) - 2z\left(1-\frac{1}{z}\right)\left(1-\frac{1}{z}\right)^3}{\left(1-\frac{1}{z}\right)^4}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{\left(2z + 5 + \frac{14}{z}\right) - 2z\left(1-\frac{1}{z}\right)\left(1 - 3\cdot\frac{1}{z} + 3\cdot\frac{1}{z^2} - \frac{1}{z^3}\right)}{\left(1-\frac{1}{z}\right)^4}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{\left(2z + 5 + \frac{14}{z}\right) - 2z\left(1 - \frac{3}{z} + \frac{3}{z^2} - \frac{1}{z^3} + \frac{3}{z^2} - \frac{1}{z} - \frac{3}{z^3} + \frac{1}{z^4}\right)}{\left(1-\frac{1}{z}\right)^4}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{2z + 5 + \frac{14}{z} - \left[2z - 6 + \frac{6}{z} - \frac{2}{z^2} + \frac{6}{z} - 2 - \frac{6}{z^2} + \frac{2}{z^3}\right]}{\left(1-\frac{1}{z}\right)^4}$$

$$= \underset{z\to\alpha}{\text{Lt}} \frac{\cancel{2z} + 5 + \frac{14}{z} - \cancel{2z} + 6 - \frac{6}{z} + \frac{2}{z^2} - \frac{6}{z} + 2 + \frac{6}{z^2} - \frac{2}{z^3}}{\left(1-\frac{1}{z}\right)^4}$$

$$= \frac{5 + 6 + 2}{1} = 13 \quad \underline{\text{Ans}}$$

Complex Integration

Prove that $Z\{f(t)/t\} = \frac{1}{T}\int_z^{\infty} \frac{F(z)}{z}\,dz$

Sol^n

From the defination,

$$Z\{f(t)/t\} = \sum_{n=0}^{\infty} \frac{f(t)}{t} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{f(nT)}{nT} z^{-n}$$

$$= \frac{1}{T}\sum_{n=0}^{\infty} \frac{f(nT)}{n} z^{-n}$$

since $\int_z^{\infty} z^{-n-1}\,dz$

$$= \left[\frac{z^{-n}}{-n}\right]_z^{\infty}$$

$$= \frac{z^{-n}}{n}$$

$$= \frac{1}{T}\sum_{n=0}^{\infty} f(nT)\int_z^{\infty} z^{-n-1}\,dz$$

$$= \frac{1}{T}\int_z^{\infty}\sum_{n=0}^{\infty} f(nT)\, z^{-n}\cdot\frac{1}{z}\,dz$$

$$= \frac{1}{T}\int_z^{\infty}\frac{1}{z}\sum_{n=0}^{\infty} f(nT)\, z^{-n}\,dz$$

$$= \frac{1}{T}\int_z^{\infty}\frac{F(z)}{z}\,dz \quad \text{proved}$$

Also it implies that $Z\{f(n)/n\} = \int_z^{\infty}\frac{F(z)}{z}\,dz$

Evaluate $Z\left\{\frac{1}{t+T}\right\}$?

Sol^n

$$Z\left\{\frac{1}{t+T}\right\} = Z\left\{\frac{1}{T(n+1)}\right\}$$

$$= \frac{1}{T} Z\left\{\frac{1}{n+1}\right\}$$

(13)

$$= \frac{1}{T} z^{1} \int_z^{\alpha} \frac{F(z)}{z^{1+1}} dz \quad \text{where } F(z) = Z\{1(t)\}$$

$$= \frac{z}{z-1}$$

$$= \frac{1}{T} z \int_z^{\alpha} \frac{z/z-1}{z^2} dz$$

$$= \frac{1}{T} z \int_z^{\alpha} \frac{1}{z(z-1)} dz$$

$$= \frac{z}{T} \int_z^{\alpha} \frac{1}{(z-\frac{1}{2})^2 - (\frac{1}{2})^2} dz$$

$$= \frac{z}{T} \times \frac{1}{2 \cdot \frac{1}{2}} \left[\ln \left(\frac{z-\frac{1}{2}-\frac{1}{2}}{z-\frac{1}{2}+\frac{1}{2}} \right) \right]_z^{\alpha} \qquad \ln \alpha = 0$$

$$= \frac{z}{T} \left[\ln \frac{z-1}{z} \right]_z^{\alpha} = \frac{z}{T} \ln (z-1)^{-1}$$

If $Z\left\{ \frac{f(n)}{n} \right\} = \int_z^{\alpha} \frac{F(z)}{z} dz$ Prove that $Z\left\{ \frac{f(n)}{n+d} \right\} = z^{d} \int_z^{\alpha} \frac{F(z)}{z^{d+1}}$

Solⁿ: From the defination,

$$Z\left\{ \frac{f(n)}{n+d} \right\} = \sum_{n=0}^{\alpha} \frac{f(n)}{n+d} z^{-n}$$

$$= \sum_{n=0}^{\alpha} f(n) \times \frac{z^{-n}}{n+d}$$

Since $\int_z^{\alpha} z^{-n-1-d} dz$

$$= \left[\frac{z^{-n-d}}{-(n+d)} \right]_z^{\alpha}$$

$$= \frac{z^{-n-d}}{n+d} = \frac{z^{-n} \cdot z^{-d}}{n+d}$$

$$= \sum_{n=0}^{\alpha} f(n) \cdot z^{+d} \int_z^{\alpha} z^{-n-1-d} dz$$

$$= z^{d} \int_z^{\alpha} \sum_{n=0}^{\alpha} f(n) \cdot z^{-n} \cdot z^{-1-d} dz$$

$$= z^{d} \int_z^{\alpha} \sum_{n=0}^{\alpha} f(n) \cdot z^{-n} \times \frac{1}{z^{d+1}} dz$$

$$= z^{d} \int_z^{\alpha} \frac{F(z)}{z^{d+1}} dz$$

Final value theorem :-

The final value of f(t) or f(n) can be obtained from F(Z) as.

$$\lim_{t\to\infty} f(t) = \lim_{n\to\infty} f(n) = \lim_{z\to 1} (z-1) F(z).$$

Sol

$$Z\{ f(n+1) - f(n) \} = z[F(z) - f(0)] - F(z)$$

$$= F(z)(z-1) - z f(0)$$

Now taking the limit $z \to 1$.

$$\therefore \lim_{z\to 1} \left[F(z)(z-1) - z f(0) \right] = \lim_{z\to 1} \sum_{n=0}^{\infty} \{ f(n+1) - f(n) \} z^{-n}$$

Now

$$\lim_{z\to 1} \sum_{n=0}^{\infty} \{ f(n+1) - f(n) \} z^{-n}$$

$$= \sum_{n=0}^{\infty} \{ f(n+1) - f(n) \} \cdot 1^{-n}$$

$$= - f(0) + f(\infty)$$

Because

$$\sum_{n=0}^{\infty} [f(n+1) - f(n)] z^{-n}$$

$$= [\{ f(1) - f(0) \} + \{ f(2) - f(1) \} z^{-1} + (f(3) - f(2)) z^{-2} + \cdots + f(\infty)$$

$$= [f(1) \{ 1 - \frac{1}{z} \} + f(2) \{ 1 - \frac{1}{z} \} + \cdots + f(\infty) - f(0)]$$

$\therefore \lim_{z \to 1} (z-1) F(z) - f(0) = - f(0) + f(\infty)$ (15)

$\therefore \lim_{z \to 1} (z-1) . F(z) = f(\infty)$ Proved

Inverse z-transformation

Consider a system describe by a difference eqn $f(n+1) + 2f(n) = u(n)$, $f(0) = 1$, $n = 0, 1, 2 \ldots$

where $u(n)$ = unit step sequence.

Taking z-transform on both side we get

$$z F(z) - z f(0) + 2 F(z) = \frac{z}{z-1}$$

$$\therefore F(z) [z + 2] = \frac{z}{z-1} + z \quad \text{Since } f(0) = 1.$$

$$= \frac{z + z^2 - z}{z-1} = \frac{z^2}{z-1}$$

$$\therefore F(z) = \frac{z^2}{(z-1)(z+2)}$$

$$\therefore A = \frac{z^2}{z+2}\Big|_{z=1} = \frac{1}{1+2} = \frac{1}{3}$$

$$B = \frac{z^2}{z-1}\Big|_{z=2} = \frac{4}{2-1} = 4.$$

$$\therefore \frac{F(z)}{z} = \frac{z}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$\therefore A = \frac{z}{z+2}\Big|_{z=1} = \frac{1}{1+2} = \frac{1}{3} \qquad B = \frac{z}{z-1}\Big|_{z=2} = \frac{-2}{-3} = \frac{2}{3}$$

$$\therefore F(z) = \frac{\frac{2}{3} z}{z+2} + \frac{\frac{1}{3} z}{z-1}$$

$$\therefore f(n) = \frac{2}{3} (-2)^{+n} u(n) + \frac{1}{3} u(n).$$

Find $Z^{-1}\left\{ \frac{2z^2+3z}{(z+2)(z-4)} \right\}$

Sol^n:

Here,

$$F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

or, $$\frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

$$= \frac{A}{z+2} + \frac{B}{z-4}$$

$$\therefore A = \left.\frac{2z+3}{z-4}\right|_{z=-2} = \frac{-4+3}{-2-4} = \frac{-1}{-6} = \frac{1}{6}$$

$$B = \left.\frac{2z+3}{z+2}\right|_{z=4} = \frac{8+3}{4+2} = \frac{11}{6}$$

$$\frac{F(z)}{z} = \frac{1/6}{z+2} + \frac{11/6}{z-4}$$

or, $$F(z) = \frac{\frac{1}{6} z}{z+2} + \frac{\frac{11}{6} z}{z-4}$$

Taking inverse Z-transform

$$\boxed{f(n) = \frac{1}{6} (-2)^{+n} u(n) + \frac{11}{6} (4)^n}$$

Find the inverse transform of F(z).

where $F(z) = \dfrac{z^3 - 20z}{(z-2)^3(z-4)}$ (17)

Sol^n:

$$\frac{F(z)}{z} = \frac{z^2-20}{(z-2)^3(z-4)}$$

$$= \frac{A}{z-2} + \frac{B}{(z-2)^2} + \frac{C}{(z-2)^3} + \frac{D}{z-4}$$

$$\therefore z^2-20 = A(z-2)^2(z-4) + B(z-2)(z-4) + C(z-4)$$
$$D(z-2)^3$$

At $z = 2$

$$4-20 = C(2-4)$$

or, $-16 = -2C$

$\therefore C = 8$

At $z = 4$, $16-20 = D(4-2)^3$

or, $-4 = D \times 8$

$\therefore D = -\frac{1}{2}$

$$z^2-20 = A(z^2-2z+4)(z-4) + B(z^2-4z-2z+8)$$
$$+ 8z - 32$$
$$-\frac{1}{2}(z^3 - 6z^2 + 12z - 8)$$

$$= A(z^3 - 2z^2 + 4z - 4z^2 + 8z - 16) + \cdots$$

$0 = A - \frac{1}{2}$ $\therefore A = \frac{1}{2}$, $0 = 12A - 6B + 5 - 6$

or, $0 = \overset{6}{\cancel{12}} \times \frac{1}{\cancel{2}} - 6B - 1$

$\therefore 6B = 5$ $B = \frac{5}{6}$

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$$\frac{F(z)}{z} = \frac{1/2}{z-2} + \frac{5/6}{(z-2)^2} + \frac{8}{(z-2)^3} - \frac{1/2}{z-4}$$

$$F(z) = \frac{1/2\, z}{z-2} + \frac{5/6\, z}{(z-2)^2} + \frac{8z}{(z-2)^3} - \frac{1/2\, z}{z-4}$$

Taking inverse Laplace transform we get,

$$\boxed{f(n) = \frac{1}{2}(2)^n u(n) + \frac{5}{12} n(2)^n u(n) + 4n^2(2)^n u(n) - \frac{1}{2}(4)^n u(n)}$$

Ans

Find $Z^{-1}\left\{\log \frac{z}{z+1}\right\} = ?$

Soln

$$F(z) = \log \frac{z}{z+1}$$

$$= \log z - \log(z+1)$$

Put $z = \frac{1}{y}$

$$F(z) = \log \frac{1/y}{1+\frac{1}{y}}$$

$$= \log \frac{1}{1+y}$$

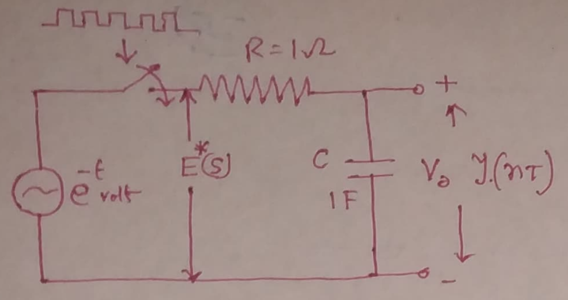
$$= \log 1 - \log(1+y)$$

$$= 0 - \log(1+y)$$

$$= -\left[y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \cdots\right]$$

$$\therefore F(z) = -\frac{1}{z} + \frac{1}{2}\times\frac{1}{z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \cdots$$

#



(19)

An RC circuit shown above is excited by an voltage source e^{-t} sampled periodically. Find and Polt the output at sampling instant using z-transform method.

Solⁿ

$$\frac{Y(s)}{E^*(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}}$$

$$= \frac{1}{1+RCs} \quad \text{since } R = 1\Omega,\ C = 1F$$

$$= \frac{1}{1+s}$$

$$\therefore Y(s) = \frac{1}{1+s} E^*(s)$$

Taking z-transform on both side

$$Y(z) = E^*(z)\ Z\left\{\frac{1}{1+s}\right\}$$

$$= E^*(z) \frac{z}{z - e^{-T}}$$

$$= \frac{z^2}{(z - e^{-T})^2}$$

$$= \frac{A}{z - e^{-T}} + \frac{B}{(z - e^{-T})^2}$$

$$\therefore z^2 = A(z - e^{-T}) + B$$

$z = e^{-T}$ $\therefore B = e^{-2T}$

$\therefore z^2 = Az - Ae^{-T} + e^{-2T}$ or, $A(z - e^{-T}) = z^2 - e^{-2T}$

$= (z + e^{-T})(z - e^{-T})$

$\therefore A = z + e^{-T}$

$$\therefore Y(z) = \frac{z+e^{-T}}{z-e^{-T}} + \frac{e^{-2T}}{(z-e^{-T})^2}$$

$$= \frac{z}{z-e^{-T}} + \frac{e^{-T}}{z-e^{-T}} + \frac{e^{-2T}}{(z-e^{-T})^2}$$

$$= \frac{z}{z-e^{-T}} + \frac{e^{-T}(z-e^{-T}) + e^{-2T}}{(z-e^{-T})^2}$$

$$= \frac{z}{z-e^{-T}} + \frac{z e^{-T} - e^{-2T} + e^{-2T}}{(z-e^{-T})^2}$$

$$= \frac{z}{z-e^{-T}} + \frac{z e^{-T}}{(z-e^{-T})^2}$$

Taking inverse z-transformation we get,

$$y(t) = e^{-t} + \frac{1}{T} \cdot t \cdot e^{-t}$$

$$y(nT) = e^{-nT} + \frac{1}{T} \times nT\, e^{-nT}$$

$$= e^{-nT} + n\, e^{-nT}$$

$$= (n+1)\, e^{-nT}$$

$$\boxed{\therefore\ y(nT) = (1+n)\, e^{-nT}}$$

Ans

Let T=1

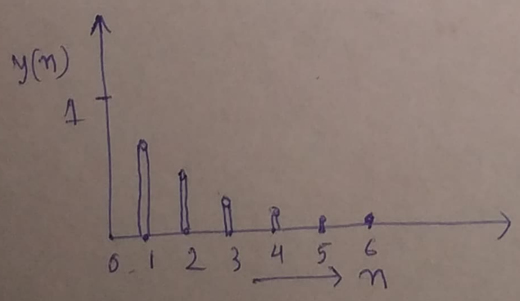


Fig:- Graphical representation of y(n)