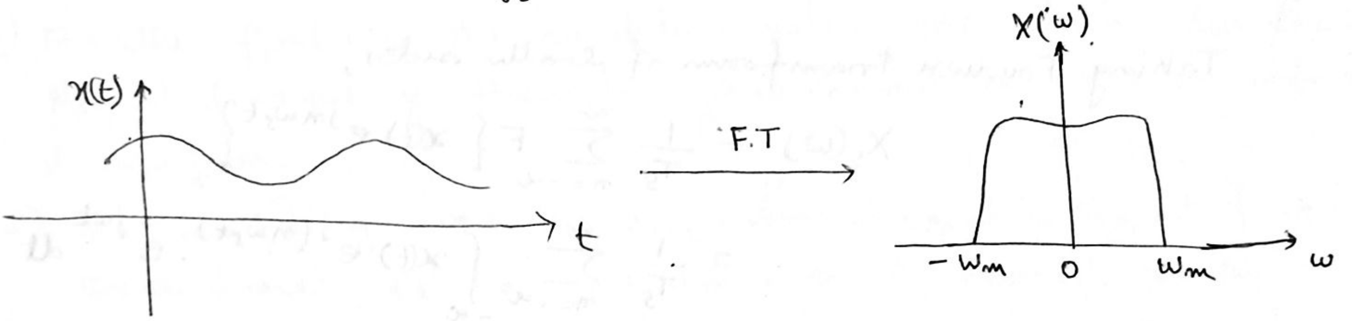
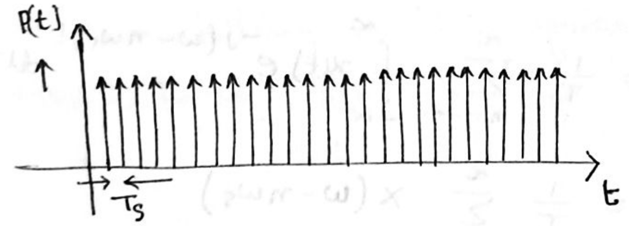
Sampling theorem :-

A band limited signal of finite energy, which has no frequency components higher than W HZ, is completely described by specifying the values of the signal at instants of time seperated by $\frac{1}{2W}$ seconds.

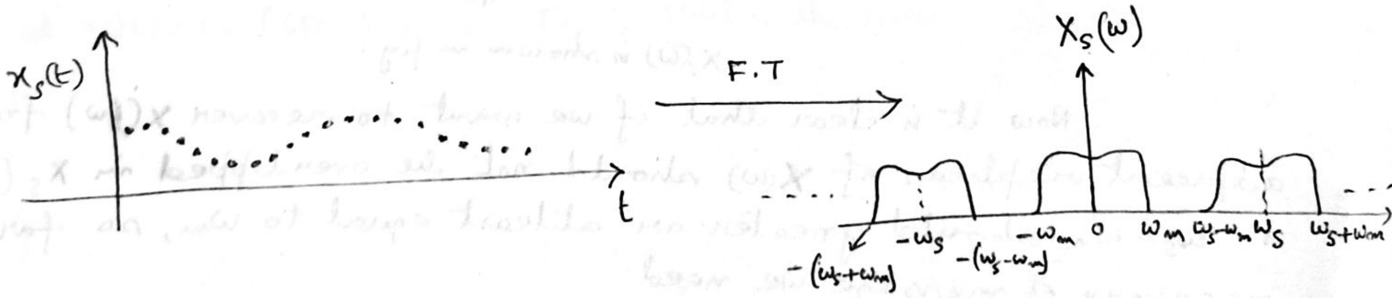
In other words, a band limited signal of finite energy, which has no frequency components higher than W HZ, may be completely recovered from a knowledge of its samples taken at the rate of 2W sec^{-1}.

ie $f_s = 2W$ where f_s = Sampling rate.





$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



Consider an arbitrary signal x(t) of finite energy and band width W_m HZ shown in above fig. The fourier tranform of x(t) is X(ω) is shown (assuming).

Let P(t) is the impulse train which can be expressed as $P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ shown in fig. where T_s is time period.

$\therefore$ Fourier series expansion of $\sum_{n=-\infty}^{\infty} \delta(t-nT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$

where $\omega_s = \frac{2\pi}{T_s}$

Now we sample the signal $x(t)$ instantaneously at sampling rate f_s with sampling period T_s by multiplying $x(t)$ with $P(t)$.

Therefore sampled signal $x_s(t) = x(t) \cdot P(t)$

$$\therefore x_s(t) = x(t) \cdot P(t)$$
$$= x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$
$$= x(t) \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$
$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(t) e^{jn\omega_s t}$$

Taking Fourier transform of both sides,

$$X_s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} F\{x(t) e^{jn\omega_s t}\}$$
$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{j(n\omega_s t)} \cdot e^{-j\omega t} dt$$
$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-j(\omega - n\omega_s)t} dt$$
$$= \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$X_s(\omega)$ is shown in fig.

Now it is clear that if we want to recover $X(j\omega)$ from $X_s(j\omega)$ adjacent replicas of $X(\omega)$ should not be overlapped in $X_s(\omega)$. That is $\omega_s - \omega_m$ should greater or atleast equal to ω_m, so for faithfull recovery of message we need

$$\omega_s - \omega_m \geq \omega_m$$
$$\text{or, } \omega_s \geq 2\omega_m$$
$$\text{or, } 2\pi f_s \geq 2\pi f_m \times 2$$
$$\text{or, } \boxed{f_s \geq 2 f_m}$$

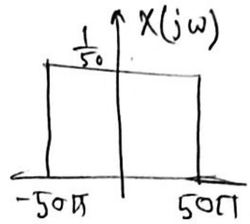
Proved

$$\therefore \frac{1}{f_s} \leq \frac{1}{2f_m}$$
$$\text{or, } \boxed{T_s \leq \frac{1}{2f_m}}$$

Proved

Prob. 1:- A signal $x(t) = \text{sinc}(50\pi t)$ is sampled at a rate of (a) 20 Hz (b) 50 Hz (c) 75 Hz. For each of these three cases, Explain if you can recover the signal $x(t)$ from the sampled signal.

Sol^n



Given, $x(t) = \text{sinc}(50\pi t)$

$\therefore 2\pi f_m = 50\pi$

$f_m = 25$ Hz

$\therefore$ Nyquist frequency $= 50$ Hz.

a) For the first case the sampling rate is 20 Hz, which is less than Nyquist frequency. Therefore $x(t)$ cannot be recovered from its samples.

b) The sampling rate is 50 Hz, which is equal to Nyquist frequency. Therefore $x(t)$ can be recovered from its samples.

c) The sampling rate is greater than Nyquist frequency. Therefore $x(t)$ can be recovered from its samples.

Prob 2:-

Let $x_a(t) = \cos(650\pi t) + 2\sin(720\pi t)$. What is the Nyquist rate for $x_a(t)$?

Sol^n