Derivation of Lucas-Kanade Tracker

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Below follows a short version of the derivation of the Lucas-Kanade tracker in [1]. A derivation of a symmetric version can also be found in [2] (the derivation here is very much inspired from [2], with a few iterative and practical issues added).

Define the dissimilarity between two local regions, one in image I and one in image J:

$$\epsilon = \iint_W [J(\mathbf{x}+\mathbf{d}) - I(\mathbf{x})]^2 w(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $\mathbf{x} = [x, y]^T$, the displacement $\mathbf{d} = [d_x, d_y]^T$. The integration region W is a local region around a pixel. The weighting function $w(\mathbf{x})$ is usually set to the constant 1, and we will for simplicity ignore the weight in the derivation from now on. Equation 4 is identical to the equation given in [1]. Now the Taylor series expansion of $J(\mathbf{x}+\mathbf{d})$ about the point $\mathbf{x}$, truncated to the linear term, is

$$J(\mathbf{x}+\mathbf{d}) \approx J(\mathbf{x}) + d_x \frac{\partial J}{\partial x}(\mathbf{x}) + d_y \frac{\partial J}{\partial y}(\mathbf{x}) = J(\mathbf{x}) + \mathbf{d}^T \nabla J(\mathbf{x}) \ ,$$

where $\nabla J = [\frac{\partial J}{\partial x}, \frac{\partial J}{\partial x}]^T$. Therefore (ignoring w),

$$\epsilon \approx \iint_W [J(\mathbf{x}) - I(\mathbf{x}) + \mathbf{d}^T \nabla J(\mathbf{x})]^2 d\mathbf{x}$$

and

$$\frac{\partial \epsilon}{\partial \mathbf{d}} \approx 2 \iint_W [J(\mathbf{x}) - I(\mathbf{x}) + \mathbf{d}^T \nabla J(\mathbf{x})] \nabla J(\mathbf{x}) d\mathbf{x} \, .$$

To find the displacement $\mathbf{d}$, we set the derivative to zero

$$\iint_W [J(\mathbf{x}) - I(\mathbf{x}) + \mathbf{d}^T \nabla J(\mathbf{x})] \nabla J(\mathbf{x}) d\mathbf{x} = 0 \, .$$

Rearranging terms, we get

$$\begin{aligned} \iint_W [J(\mathbf{x}) - I(\mathbf{x})] \nabla J(\mathbf{x}) d\mathbf{x} &= -\iint_W \nabla J^T(\mathbf{x}) \mathbf{d} \nabla J(\mathbf{x}) d\mathbf{x} \\ &= -\left[\iint_W \nabla J(\mathbf{x}) \nabla J^T(\mathbf{x}) d\mathbf{x} \right] \mathbf{d} \, . \end{aligned}$$

In other words, we must solve the equation

$$\mathbf{Z}\mathbf{d} = \mathbf{e}\,, \tag{2}$$

where $\mathbf{Z}$ is the 2×2 matrix

$$\mathbf{Z} = \iint_W \nabla J(\mathbf{x}) \nabla J^T(\mathbf{x}) d\mathbf{x} \tag{3}$$

and $\mathbf{e}$ is the 2×1 vector

$$\mathbf{e} = \iint_W [I(\mathbf{x}) - J(\mathbf{x})] \nabla J(\mathbf{x}) d\mathbf{x}\,. \tag{4}$$

Iterating

The derivation above only approximately minimizes the dissimilarity (1), since we are using a truncated Taylor expansion. The solution can be improved by iterating the procedure above in the following way:

1. Set $\mathbf{d}_{tot} = \mathbf{0}$.
2. Compute $\mathbf{Z}$ and $\mathbf{e}$ in (3) and (4) respectively, and solve (2) to get $\mathbf{d}$.
3. Update $\mathbf{d}_{tot} \leftarrow \mathbf{d}_{tot} + \mathbf{d}$. Compute a new image $J(\mathbf{x} + \mathbf{d}_{tot})$ and gradients $\nabla J(\mathbf{x} + \mathbf{d}_{tot})$ by interpolating the original data $J(\mathbf{x})$ and $\nabla J(\mathbf{x})$.
4. Go back to step 2, using the new data from step 3 instead of the original J and ∇J.

Iterate until some stop criteria is fulfilled, e.g. maximum number of iterations or if $\|d\|$ is below a certain value.

Practical issues

A true derivative cannot be computed in practise on pixel-discretized images. It is however possible to compute a *regularized derivative*, i.e. the derivative of a smoothed signal. For example, let $g(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$ be a 2D Gaussian with standard deviation σ, and compute the regularized derivative with respect to x as

$$\frac{\partial}{\partial x}(J * g) = \frac{\partial}{\partial x} J * g = J * \frac{\partial}{\partial x} g = J * \frac{-x}{\sigma^2} g\,. \tag{5}$$

In other words, if we use the filter $\frac{-x}{\sigma^2} g$ to compute the derivative of J with respect to x, we are actually computing the derivative of $J * g$ with respect to x. Therefore, the difference $I - J$ in (4) should in practise be replaced by $I * g - J * g$.

References

[1] B.D. Lucas and T. Kanade, *An Iterative Image Registration Technique with an Application to Stereo Vision*, In Proceedings of Imaging Understanding Workshop, 1981. The original article for KLT, http://www-cse.ucsd.edu/classes/sp02/cse252/lucaskanade81.pdf

[2] S. Birchfield, *Deriviation of Kanade-Lucas-Tomasi Tracking Equation*, http://www.ces.clemson.edu/~stb/klt/birchfield-klt-derivation.pdf