The Note of Graph Theory

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摘要

I hate definition!

1 definition

simple graph:

- a non-empty finite set $V(G)$ of elements **vertices**
- a finnite set $E(G)$ of distinct unordered pairs of deistinct elements **edges**
- at most **one edge** joining a given pair of vertices

general graph:

- multiple edges
- loops

graph

- a non-empty finite set $V(G)$ of elements **vertices**
- a finnite family $E(G)$ of distinct unordered pairs of not necessarily deistinct elements **edges**

- the use of family permits the existence of multiple edge
- vertex-set edge-family

$\{v, w\}$ is said to join the vertices v , w and is again abbreviated to vw. Example : vv

isomorphism

- one-one correspondence between the vertices of G_1 and those of G_2 such that the number of edges joining any two vertices of G_1 equals the number of edges joining the corresponding vertices of G_2
- the **unlabelled graphs** and **labelled graphs** is different

connected graphs

- A graph is connected of it cannot be expressed as a union of graghs and disconnected otherwise.

Adjacency and degrees（邻接）

- the v, w is **adjacency** if there is an edge vw joining them. and v, w are incident with such edge
- degree of a vetex is the number of edges incident with itself.
- A vetex of degree 0 is an isolated vertex
- A vetex of degree 1 is an end vertex

The **degree sequence** of a graph consists of degrees written in **increasing order**

Handshaking lemma In any graph the sum of all the vertex-degrees is an even number.

Corollary In any graph the number of vertices of odd degree is even.

Subgraphs

- A graph is a **subgraph** of graph G if each of its vertices belongs to $V(G)$ and each of its edges belongs to $E(G)$
- we can obtain subgraphs by deleting edges and vertices.
- $G - F$

By the way, $G \backslash e$ is the subgraph of G obtained by deleting the edge e and $G \backslash v$ is the subgraph of G obtained by deleting the vertex v and all edges incident with it.

The complement of a simple graph

- **complement** $\overline{G}$ is the simple graph with the same vertex-set as G and with two vertices adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Matrix representation of graphs

There are two matrix representations of graphs, the **adjacency matrix** and the **incidence matrix**.

- **adjacency matrix** $A(G)$ is the $n \times n$ matrix whose (i, j)-entry is 1 if v_i is adjacent to v_j and is 0 otherwise.
- **incidence matrix** $B(G)$ is the $n \times m$ matrix whose (i, j)-entry is 1 if v_i is incident with e_j and is 0 otherwise.

1.1 Examples

1.1.1 Null graphs

A graph whose edge-set is empty is called a **null graph** and is denoted by N_n or simply N if the number of vertices is clear from the context.

1.1.2 Complete graphs

A graph with n vertices in which each pair of distinct vertices is joined by an edge is called a **complete graph** and is denoted by K_n or simply K if the number of vertices is clear from the context. You can check that K_n has $\frac{n(n-1)}{2}$ edges.

1.1.3 Cycle graphs, path graphs and wheels

A **connected graph** in which each vertex has degree 2 is called a **cycle graph** and is denoted by C_n or simply C if the number of vertices is clear from the context. You can check that C_n has n vertices and n edges.

A graph obtained from C_n by removing an edge is the **path graph** P_n or simply P if the number of vertices is clear from the context.

The gragh obtained from C_{n-1} by joining each vertex to a new vertex v is the *wheel*,denoted by W_n or simply W if the number of vertices is clear from the context.

1.1.4 Regular graphs

A graph is **regular** if each of its vertices has the same degree. If each vertex has degree r, the graph is said to be **r-regular**. A **complete graph** is $r-regular$ for $r = n-1$. Note that **cubic graphs** are $3-regular$ graphs, and **null graphs** are $0-regular$ graphs.

1.1.5 Platonic graphs

A **Platonic graph** is a **regular** graph with the property that the number of vertices

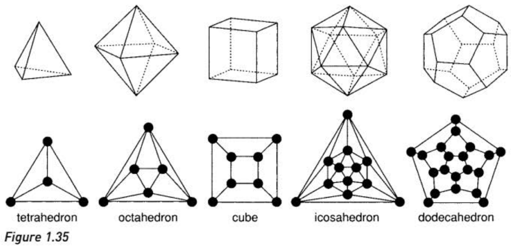

图 1: Platonic graphs

1.1.6 Bipartite graphs

If the vertex-set of G can be split into two disjoint sets X and Y such that each edge of G joins a vertex in X to a vertex in Y, then G is said to be **bipartite**.

A **complete bipartite graph** is a bipartite graph in which every vertex in X is adjacent to every vertex in Y and is denoted by $K_{X,Y}$.

1.2 Cubes

The **cube** Q_n is the graph whose vertices are the 2^n binary strings of length n

2 Paths and cycles

2.1 Connectivity

Walks

- A **walk** in a graph G is a sequence of vertices $v_0, v_1, ..., v_n$ such that $v_i v_{i+1}$ is an edge of G for $i = 0, 1, ..., n-1$.
- We call V_0 the **initial vertex** and v_n the **terminal vertex** of the walk.
- A **length** of a walk is the number of edges in it.
- A **closed walk** is a walk in which $v_0 = v_n$.

- A **trail** is a walk in which all edges are distinct.
- A **path** is a trail in which all vertices are distinct.
- A **cycle** is a closed trail in which all vertices are distinct except for $v_0 = v_n$.
    - A **triangle** is a cycle of length 3.

A graph G is bipartite if and only if every cycle of G has even length.