Selected Topics

Labix

November 26, 2024

Abstract

References:

Contents

1 Symmetric Polynomials

1.1 Symmetric Polynomials

The theory of symmetric functions are important in combinatorics, representation theory, Galois theory and the theory of λ-rings.

Requirements: Groups and Rings
Books: Donald Yau: Lambda Rings

Definition 1.1.1: Symmetric Group Action on Polynomial Rings

Let R be a ring. Define a group action of S_n on $R[x_1, \dots, x_n]$ by

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

It is easy to check that this defines a group action.

Definition 1.1.2: Symmetric Polynomials

Let R be a ring. We say that a polynomial $f \in R[x_1, \dots, x_n]$ is symmetric if

$$\sigma \cdot f = f$$

for all $\sigma \in S_n$.

Definition 1.1.3: The Ring of Symmetric Polynomials

Let R be a ring. Define the ring of symmetric polynomials in n variables over R to be the set

$$\Sigma = \{f \in R[x_1, \dots, x_n] \mid \sigma \text{ is a symmetric polynomial } \}$$

Definition 1.1.4: Elementary Symmetric Polynomials

Let R be a ring. Define the elementary symmetric polynomials to be the elements $s_1, \dots, s_n \in R[x_1, \dots, x_n]$ given by the formula

$$s_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

Theorem 1.1.5: The Fundamental Theorem of Symmetric Polynomials

Let R be a ring. Then $s_1, \dots, s_n$ are algebraically independent over R. Moreover,

$$\Sigma = R[s_1, \dots, s_n]$$

2 λ-Rings

2.1 λ-Rings

Complex representation of a group is a λ-ring. Topological K theory is a λ-ring.

Requirements: Category Theory, Groups and Rings, Symmetric Functions
Books: Donald Yau: Lambda Rings

We need the theory of symmetric polynomials before defining λ-structures.

Definition 2.1.1: λ-Structures

Let R be a commutative ring. A λ-structure on R consists of a sequence of maps $\lambda^n : R \to R$ for $n \geq 0$ such that the following are true.

- $\lambda^0(r) = 1$ for all $r \in R$
- $\lambda^1 = \text{id}_R$
- $\lambda^n(1) = 0$ for all $n \geq 2$
- $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$ for all $r, s \in R$
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \dots, \lambda^{mn}(r))$ for all $r \in R$

Here P_n and $P_{m,n}$ are defined as follows.

- The coefficient of t^n in the polynomial
$$h(t) = \prod_{i,j=1}^n (1 + x_i y_j t)$$
is a symmetric polynomial in x_i and y_j with coefficients in $\mathbb{Z}$. P_n is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of x_i and y_j respectively.
- The coefficient of t^n in the polynomial
$$g(t) = \prod_{1 \leq i_1 \leq \dots \leq i_m \leq nm} (1 + x_{i_1} \cdots x_{i_m} t)$$
is a symmetric polynomial in x_i with coefficients in $\mathbb{Z}$. $P_{m,n}$ is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ of x_i.

In this case, we call R a λ-ring.

Note that we do not require that the λ^n are ring homomorphisms.

Definition 2.1.2: Associated Formal Power Series

Let R be a λ-ring. Define the associated formal power series to be the function $\lambda_t : R \to R[[t]]$ given by
$$\lambda_t(r) = \sum_{k=0}^{\infty} \lambda^k(r) t^k$$
for all $r \in R$

Proposition 2.1.3

Let R be a λ-ring. Then the following are true regarding $\lambda_t(r)$.

- $\lambda_t(1) = 1 + t$
- $\lambda_t(0) = 1$
- $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$
- $\lambda_t(-r) = \lambda(r)^{-1}$

Proposition 2.1.4

The ring $\mathbb{Z}$ has a unique λ-structure given by

$$\lambda_t(n) = (1+t)^n$$

Proposition 2.1.5

Let R be a λ-ring. Then R has characteristic 0.

Definition 2.1.6: Dimension of an Element

Let R be a λ-ring and let $r \in R$. We say that r has dimension n if $\deg(\lambda_t(r)) = n$. In this case, we write $\dim(r) = n$.

Proposition 2.1.7

Let R be a λ-ring. Then the following are true regarding the dimension of n.
- $\dim(r+s) \leq \dim(r) + \dim(s)$ for all $r, s \in R$
- If r and s both has dimension 1, then so is rs.

2.2 λ-Ring Homomorphisms and Ideals

Definition 2.2.1: λ-Ring Homomorphisms

Let R and S be λ-rings. A λ-ring homomorphism from R to S is a ring homomorphism $f : R \to S$ such that

$$\lambda^n \circ f = f \circ \lambda^n$$

for all $n \in \mathbb{N}$.

Definition 2.2.2: λ-Ideals

Let R be a λ-ring. A λ-ideal of R is an ideal I of R such that

$$\lambda^n(i) \in I$$

for all $i \in I$ and $n \geq 1$.

TBA:λ-ideal and subring. Ker, Im, Quotient Product, Tensor, Inverse Limit are λ-rings

Proposition 2.2.3

Let R be a λ-ring. Let $I = \langle z_i \mid i \in I \rangle$ be an ideal in R. Then I is a λ-ideal if and only if $\lambda^n(z_i) \in I$ for all $n \geq 1$ and $i \in I$.

Proposition 2.2.4

Every λ-ring R contains a λ-subring isomorphic to $\mathbb{Z}$.

2.3 Augmented λ-Rings

Definition 2.3.1: Augmented λ-Rings

Let R be a λ-ring. We say that R is an augmented λ-ring if it comes with a λ-homomorphism
$$\varepsilon : R \to \mathbb{Z}$$
called the augmentation map.

TBA: tensor of augmented is augmented

Proposition 2.3.2

Let R a λ-ring. Then R is augmented if and only if there exists a λ-ideal I such that
$$R = \mathbb{Z} \oplus I$$
as abelian groups.

2.4 Extending λ-Structures

Proposition 2.4.1

Let R be a λ-ring. Then there exists a unique λ-structure on $R[x]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[x]$ and $\varepsilon(r) = 0$ or 1.

Proposition 2.4.2

Let R be a λ-ring. Then there exists a unique λ-structure on $R[[x]]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[[x]]$ and $\varepsilon(r) = 0$ or 1.

2.5 Free λ-Rings

2.6 The Universal λ-Ring

2.7 Adams Operations

3 Witt Vectors

3.1 Fundamentals of the Ring of Big Witt Vectors

Prelim: Symm Functions, Lambda Rings, Category theory, Frobenius endomorphism (Galois), Rings and Modules, Kaehler differentials (commutative algebra 2)
Leads to: K theory
Books: Donald Yau: Lambda Rings

Definition 3.1.1: Truncation Sets

Let $S \subseteq \mathbb{N}$. We say that S is a truncation set if for all $n \in S$ and $d|n$, then $d \in S$. For $n \in \mathbb{N}$ and S a truncation set, define
$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

For instance, $\mathbb{N} \setminus \{0\}$ is a truncation set. We will also use $\{1, \dots, n\}$.

Theorem 3.1.2: Dwork's Theorem

Let R be a ring and let S be a truncation set. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i \in S} \in \prod_{i \in S} R$ has the form
$$(b_i)_{i \in S} = (w_i(a))_{i \in S}$$
for some $a \in R$
- For all primes p and all $n \in S$ such that $p|n$, we have
$$b_n \equiv \sigma_p(b_{n/p}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on all the b_k for $1 \leq k \leq n$ and $k \in S$.

We wish to equip $\prod_{i \in S} R$ with a non-standard addition and multiplication to make it into a ring.

Proposition 3.1.3

Consider the ring $R = \mathbb{Z}[x_i, y_i \mid i \in S]$. There exists unique polynomials
$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$
for $n \in S$ such that

- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i \in S}) + w_n((y_i)_{i \in S})$
- $w_n(\pi_1, \dots, \pi_n) = w_n((x_i)_{i \in S}) \cdot w_n((y_i)_{i \in S})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i \in S})$

for all $n \in S$.

Note that the polynomials ξ_n, π_n have variables x_k and y_k for $k \leq n$ and $k \in S$. This is similar for the variables of ι. From now on, this will be the convention: For S a truncation set, the sequence $a_1, \dots, a_n$ actually refers to the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \dots \leq d_k \leq n$ and $d_1, \dots, d_k$ are all divisors of n. The result of this is that sequences in $\mathbb{N}$ are now restricted to S.

Definition 3.1.4: The Ring of Truncated Witt Vector

Let R be a ring. Let S be a truncation set. Define the ring of big Witt vectors $W_S(R)$ of R to consist of the following.

- The underlying set $\prod_{i \in S} R$
- Addition defined by $(a_n)_{n \in S} + (b_n)_{n \in S} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$
- Multiplication defined by $(a_n)_{n \in S} \times (b_n)_{n \in S} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$

Theorem 3.1.5

Let R be a ring. Let S be a truncation set. Then the ring of big Witt vectors $W_S(R)$ of R is a ring with additive identity $(0, 0, \dots)$ and multiplicative identity $(1, 0, 0, \dots)$. Moreover, for $(a_n)_{n\in S} \in W(R)$, its additive inverse is given by $(\iota_n(a_1, \dots, a_n))_{n\in\mathbb{N}}$.

Proposition 3.1.6

Let $\phi : R \to R'$ be a ring homomorphism. Then the induced map $W_S(\phi) : W_S(R) \to W_S(R')$ defined by

$$W(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

is a ring homomorphism.

Definition 3.1.7: The Witt Functor

Define the Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ to consist of the following data.

- For each ring R, $W_S(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi : R \to R'$, $W_S(\phi) : W_S(R) \to W_S(R')$ is the induced ring homomorphism defined by

$$W_S(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

Proposition 3.1.8

Let S be a truncation set. The Witt functor is indeed a functor.

Definition 3.1.9: The Ghost Map

Let R be a ring. Let S be a truncation set. Define the ghost map to be the map

$$w : W_S(R) \to \prod_{k\in S} R$$

by the formula

$$w((a_n)_{n\in S}) = (w_n(a_1, \dots, a_n))_{n\in S}$$

Remember, by the sequence $a_1, \dots, a_n$ we mean the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \cdots \leq d_k \leq n$ and $d_1, \dots, d_k$ the complete collection of divisors of n.

Proposition 3.1.10

Let S be a truncation set. Then the following are true.

- For each $n \in S$, the collection of maps $w_n : W_S(R) \to R$ for a ring R defines a natural transformation $w_n : W_S \to \text{id}$.
- The collection of ghost maps $w_R : W_S(R) \to \prod_{k\in S} R$ for R a ring defines a natural transformation $w : W_S \to (-)^S$.

Proposition 3.1.11

Let S be a truncation set. The truncated Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W_S(R)$ is given by $\prod_{k\in S} R$
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by

$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$

- For each $n \in S$, $w_n : W_S(R) \to R$ defines a natural transformation $w_n : W \to \mathrm{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k \in S} R$ is unique under the above conditions.

3.2 Important Maps of Witt Vectors

Definition 3.2.1: The Forgetful Map

Let R be a ring. Let $T \subseteq S$ be truncation sets. Define the forgetful map $R_T^S : W_S(R) \to W_T(R)$ to be the ring homomorphism given by forgetting all elements $s \in S$ but $s \notin T$.

Definition 3.2.2: The nth Verschiebung Map

Let R be a ring. Let S be a truncation set. For $n \in \mathbb{N}$, define the nth Verschiebung map $V_n : W_{S/n}(R) \to W_S(R)$ by

$$V_n((a_d)_{d \in S/n})_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise} \end{cases}$$

Note that this is not a ring homomorphism. However, it is additive.

Lemma 3.2.3

Let R be a ring. Let S be a truncation set. Then for all $a, b \in W_{S/n}(R)$, we have that

$$V_n(a+b) = V_n(a) + V_n(b)$$

Definition 3.2.4: Frobenius Map

Let S be a truncation set. Let R be a ring. Define the Frobenius map to be a natural ring homomorphism $F_n : W_S(R) \to W_{S/n}(R)$ such that the following diagram commutes:

$$\begin{tikzcd} W_S(R) \arrow[r, "w"] \arrow[d, "F_n"'] & \prod_{k \in S} R \arrow[d, "F_n^w"] \\ W_{S/n}(R) \arrow[r, "w"'] & \prod_{k \in S/n} R \end{tikzcd}$$

if it exists.

Lemma 3.2.5

Let S be a truncation set. Let R be a ring. Then the Frobenius map exists and is unique.

The following lemma relates this notion of Frobenius map to that in ring theory.

Lemma 3.2.6

Let A be an F_p algebra. Let S be a truncation set. Let $\varphi_p : A \to A$ denote the Frobenius homomorphism given by $a \mapsto a^p$. Then

$$F_p = R_{S/p}^S \circ W_S(\varphi) : W_S(A) \to W_{S/p}(A)$$

Definition 3.2.7: The Teichmuller Representative

Let R be a ring. Let S be a truncation set. Define the Teichmuller representative to be the map $[-]_S : R \to W_S(R)$ defined by

$$([a]_S)_n = \begin{cases} a & \text{if } n = 1 \\ 0 & b \text{ otherwise} \end{cases}$$

The Teichmuller representative is in general not a ring homomorphism, but it is still multiplicative.

Lemma 3.2.8

Let R be a ring. Let S be a truncation set. The for all $a, b \in R$, we have that

$$[ab]_S = [a]_S \cdot [b]_S$$

The three maps introduced are related as follows.

Proposition 3.2.9

Let R be a ring. Let S be a truncated set. Then the following are true.

- $r = \sum_{n \in S} V_n([r_n]_{S/n})$ for all $r \in W_S(R)$
- $F_n(V_n(a)) = na$ for all $a \in W_{S/n}(R)$
- $r \cdot V_n(a) = V_n(F_n(r) \cdot a)$ for all $r \in W_S(R)$ and all $a \in W_{S/n}(R)$
- $F_m \circ V_n = V_n \circ F_m$ if $\gcd(m, n) = 1$

The remaining section is dedicated to the example of $R = \mathbb{Z}$.

Proposition 3.2.10

Let S be a truncation set. Then the ring of big Witt vectors of $\mathbb{Z}$ is given by

$$W_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = \gcd(m, n) \cdot V_d([1]_{S/d})$$

and $d = \mathrm{lcm}(m, n)$.

3.3 The Ring of p-Typical Witt Vectors

For the ring of p-typical Witt vectors, we consider the truncation set $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$ for a prime p.

Definition 3.3.1: The Ring of p-Typical Witt Vectors

Let R be a ring. Let p be a prime. Let $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$. Define the ring of p-typical Witt vectors to be

$$W_p(R) = W_P(R)$$

Define the ring of p-typical Witt vectors of length n to be

$$W_n(R) = W_{\{1, p, \dots, p^{n-1}\}}(R)$$

when the prime p is understood.

Theorem 3.3.2

Let R be a ring. Let p be a prime number. Let S be a truncation set. Write $I(S) = \{k \in S \mid k \text{ does not divide } p\}$. Suppose that all $k \in I(S)$ are invertible in R. Then there is a decomposition

$$W_S(R) = \prod_{k \in I(S)} W_S(R) \cdot e_k$$

where

$$e_k = \prod_{t \in I(S) \setminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kt} V - kt([1]_{S/kt}) \right)$$

Moreover, the composite map given by

$$W_S(R) \cdot e_k \hookrightarrow W_S(R) \xrightarrow{F_k} W_{S/k} R \xrightarrow{R^{S/k}_{S/k \cap P}} W_{S/k \cap P}(R)$$

is an isomorphism.

3.4 The λ-structure on W(R)

Lemma 3.4.1

Let R be a ring. Then every $f \in \Lambda(R)$ can be written uniquely as

$$f = \prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n)$$

Theorem 3.4.2: The Artin-Hasse Exponential

There is a natural isomorphism $E : \Lambda \to W$ given as follows. For a ring R, $E_R : \Lambda(R) \to W(R)$ is defined by

$$E_R \left(\prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n) \right) = (a_n)_{n \in \mathbb{N}}$$

Corollary 3.4.3

Let R be a ring. Then $W(R)$ has a canonical λ-structure inherited from $\Lambda(R)$.

TBA: The forgetful functor $U : \Lambda\mathbf{Ring} \to \mathbf{CRing}$ has a left adjoint Symm and has a right adjoint W.

4 Formal Group Laws

Definition 4.0.1: Formal Group Laws

Let R be a ring. A formal group law over R is a power series

$$f(x,y)\in R[[x,y]]$$

such that the following are true.

- $f(x,0)=f(0,x)=x$
- $f(x,y)=f(y,x)$
- $f(x,f(y,z))=f(f(x,y),z)$

Definition 4.0.2: The Formal Group Law Functor

Define the formal group law functor

$$FGL:\mathbf{Ring}\to\mathbf{Set}$$

by the following data.

- For each ring R, $FGL(R)$ is the set of all formal group laws over R
- For each ring homomorphism $f:R\to S$, $FGL(f)$ sends each formal group law $\sum_{i,j=0}^{\infty}c_{i,j}x^iy^j$ over R to the formal group law $\sum_{i,j=0}^{\infty}f(c_{i,j})x^iy^j$ over S.

Definition 4.0.3: The Lazard Ring of a Formal Group Law

Define the lazard ring by

$$L=\frac{\mathbb{Z}[c_{i,j}]}{Q}$$

where Q is the ideal generated as follows. Write $f=\sum_{i,j=0}^{\infty}c_{i,j}x^iy^j$. Then Q is generated by the constraints on $c_{i,j}$ for which f becomes a formal group law.

Lemma 4.0.4

The Lazard ring $L=\mathbb{Z}[c_{i,j}]/Q$ has the structure of a graded ring where $c_{i,j}$ has degree $2(i+j-1)$.

Theorem 4.0.5

The formal group law functor $FGL:\mathbf{Ring}\to\mathbf{Set}$ is representable

$$FGL(R)\cong\mathrm{Hom}_{\mathbf{Ring}}(L,R)$$

There exists a universal element $f\in L$ such that the map $\mathrm{Hom}_{\mathbf{Ring}}(L,R)\to FGL(R)$ given by evaluation on f is a bijection for any ring R.

Theorem 4.0.6

There is an isomorphism of the Lazard ring

$$L\cong\mathbb{Z}[t_1,t_2,\dots]$$

where each t_k has degree $2k$.

5 Calculus of Functors

5.1 Excisive Functors

Definition 5.1.1: Homotopy Functors

Let $\mathcal{C}, \mathcal{D}$ be categories with a notion of weak equivalence. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy functor if F preserves weak equivalences.

Definition 5.1.2: n-Excisive Functors

Let F be a homotopy functor. We say that F is n-excisive if it takes strongly homotopy cocartesian $(n+1)$-cubes to homotopy cartesian $(n+1)$-cubes.

5.2 The Taylor Tower

Definition 5.2.1: Fiberwise Join

Let X, Y, U be spaces. Let $f : X \to Y$ be a map. Define the fiberwise join of X and U along f to be the space

$$X *_Y U = \text{hocolim}(X \longleftarrow X \times U \longrightarrow Y \times U)$$

Lemma 5.2.2

Let X, Y, U, V be spaces. Let $f : X \to Y$ be a map. Then there is a natural isomorphism

$$(X *_Y U) *_Y V \cong X *_Y (U * V)$$

Proposition 5.2.3

Let $\mathcal{P}(n)$ denote the category of posets. Let X be a space over Y. Then the assignment

$$U \mapsto X *_Y U$$

defines an n-dimensional cubical diagram in **Top**. Moreover, it is strongly cocartesian.

Definition 5.2.4

Let Y be a space. Let $F : \mathbf{Top}_Y \to \mathbf{Top}$ be a homotopy functor. Define the functor

$$T_n F : \mathbf{Top}_Y \to \mathbf{Top}$$

to consist of the following data.

- For each $X \in \mathbf{Top}$, consider the functor $\mathcal{X} : \mathcal{P}(n+1) \to \mathbf{Top}$ given by $U \mapsto F(X *_Y U)$. Define
$$T_n F(X) = \text{holim}(\mathcal{X}) = \underset{U \in \mathcal{P}(n+1)}{\text{holim}} (F(X *_Y U))$$
- For each $f : X \to Z$ a morphism of spaces over Y, define a map $T_n F(X) \to T_n F(Y)$ to be the map
$$F(f *_Y \text{id}) \circ \mathcal{X}$$

Lemma 5.2.5

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then $T_n F$ is a homotopy functor.

Proposition 5.2.6

Let F be a homotopy functor. Then there exists a natural map $t_nF : F \Rightarrow T_nF$ given by the canonical map of homotopy limits. Moreover, t_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "t_nF"'] & G \arrow[d, "t_nG"] \\ T_nF \arrow[r, "T_n\lambda"'] & T_nG \end{tikzcd}$$

Definition 5.2.7

Let Y be a space. Let $F : \mathbf{Top}_Y \to \mathbf{Top}$ be a homotopy functor. Define the functor

$$P_nF : \mathbf{Top}_Y \to \mathbf{Top}$$

to consist of the following data.

- For each space X over Y, define $P_nF(X)$ to be the homotopy limit
$$P_nF(X) = \text{holim}(F(X) \to T_nF(X) \to (T_n(T_nF))(X) \to \dots)$$
- For each morphism $f : X \to Z$ of spaces over Y, define $P_nF(f) : P_nF(X) \to P_nF(Z)$ to be the map ????

Lemma 5.2.8

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then P_nF is a homotopy functor.

Proposition 5.2.9

Let F be a homotopy functor. Then there exists a natural map $p_nF : F \Rightarrow P_nF$ given by the canonical map of homotopy limits. Moreover, p_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "p_nF"'] & G \arrow[d, "p_nG"] \\ P_nF \arrow[r, "P_n\lambda"'] & P_nG \end{tikzcd}$$

Definition 5.2.10: n-Reduced Functors

Let F be a homotopy functor. We say that F is n-reduced if $P_{n-1}F \simeq *$.

Definition 5.2.11: n-Homogenous Functor

Let F be a homotopy functor. We say that F is n-homogenous if F is n-excisive and n-reduced.

5.3 Linear Functors

Definition 5.3.1: Linear Functors

Let F be a homotopy functor. We say that F is linear if F is 1-homogenous. Explicitly, this means that

- F sends homotopy pushouts to homotopy pullbacks
- $F(X)$ is homotopy equivalent to $*$

Let us consider the case $n = 1$ and $Y = *$. Now $\mathcal{P}_0(2)$ is the small category given in a diagram as follows:

$$\begin{tikzcd} & \{1\} \arrow[d] \\ \{0\} \arrow[r] & \{0,1\} \end{tikzcd}$$

Now T_1F sends every space X to the homotopy limit of the following diagram:

$$\begin{tikzcd} & F(X * \{1\}) \arrow[d] \\ F(X * \{0\}) \arrow[r] & F(X * \{0,1\}) \end{tikzcd}$$

But we know that $X * \{0\}$ is the cone CX and $X * \{0, 1\}$ is the reduced suspension. This means that we can simplify the above diagram into

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

Now $CX \simeq *$ and F is a reduced functor. Thus we can further simplify the diagram into

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

We recognize this as the homotopy pullback, and so $T_1F(X) \simeq \Omega F(\Sigma X)$. Now recall that

$$P_1F(X) = \text{hocolim}(F(X) \overset{t_1F(X)}{\longrightarrow} T_1F(X) \overset{t_1(T_1F)}{\longrightarrow} (T_1(T_1F))(X) \longrightarrow)$$

Again because we know that $T_1F(X) \simeq \Omega F(\Sigma X)$ and we care about everything only up to homotopy, we can write P_1F as

$$P_1F(X) = \text{hocolim}(F(X) \overset{t_1F(X)}{\longrightarrow} \Omega F(\Sigma X) \overset{t_1(T_1F)}{\longrightarrow} \Omega(T_1F)(\Sigma X) \longrightarrow)$$

which further simplifies to

$$P_1F(X) = \text{hocolim}(F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \longrightarrow)$$

Thus in general,

$$P_1F(X) = \underset{n\to\infty}{\text{hocolim}}(\Omega^n F(\Sigma^n X))$$

We are considering the case $n = 2$. Now $\mathcal{P}_0(3)$ is the small category given in a diagram as follows:

$$
\begin{tikzcd}
 & \{2\} \arrow[ddl] \arrow[drr] & & \\
 & & & \{1,2\} \arrow[ddl] \\
\{0,2\} \arrow[drr] & & & \{1\} \arrow[u] \arrow[ddl] \\
\{0\} \arrow[u] \arrow[drr] & & \{0,1,2\} & \\
 & & \{0,1\} \arrow[u] &
\end{tikzcd}
$$

If we plug it into the definition of T_nF and choose $Y = *$, we obtain a functor

$$T_2F : \mathbf{Top}_* \to \mathbf{Top}$$

that consists of the following data. For each $X \in \mathbf{Top}$, $T_2F(X)$ is precisely the homotopy limit of the diagram

$$
\begin{tikzcd}
 & F(X * \{2\}) \arrow[ddl] \arrow[drr] & & \\
 & & & F(X * \{1,2\}) \arrow[ddl] \\
F(X * \{0,2\}) \arrow[drr] & & & F(X * \{1\}) \arrow[u] \arrow[ddl] \\
F(X * \{0\}) \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
 & & F(X * \{0,1\}) \arrow[u] &
\end{tikzcd}
$$

which simplifies to the diagram:

$$
\begin{tikzcd}
 & F(CX) \arrow[ddl] \arrow[drr] & & \\
 & & & F(\Sigma X) \arrow[ddl] \\
F(\Sigma X) \arrow[drr] & & & F(CX) \arrow[u] \arrow[ddl] \\
F(CX) \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
 & & F(\Sigma X) \arrow[u] &
\end{tikzcd}
$$

Now since F is reduced and $CX \simeq *$, we can further simplify it into

$$\begin{tikzcd}
& * \arrow[ddl] \arrow[drr] & & \\
& & & F(\Sigma X) \arrow[ddl] \\
F(\Sigma X) \arrow[drr] & & & * \arrow[u] \arrow[ddl] \\
* \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
& & F(\Sigma X) \arrow[u] &
\end{tikzcd}$$

(what does the maps look like?)

Definition 5.3.2: The Category of Linear Functors

Define the category

$$\mathcal{H}_1(\mathcal{C}, \mathcal{D})$$

of linear functors to be the full subcategory of $\mathcal{D}^{\mathcal{C}}$ consisting of linear functors.

Theorem 5.3.3

There is an equivalence of categories

$$\mathcal{H}_1(\mathbf{CGWH}_*, \mathbf{CGWH}_*) \cong \Omega\mathrm{Sp}^{\mathbb{N}}(\mathbf{CGWH}_*)$$

given as follows. For a linear functor F, we associate to it the sequence of spaces $\{F(S^n) \mid n \in \mathbb{N}\}$, and this defines a spectra.

5.4 Catalogue of Construction Needed

Example 5.4.1

$\mathcal{P}_0(n+1)$ for small values of n is given as follows:

- When $n = 0$, $\mathcal{P}_0(1)$ consists of only one object

$$\{1\}$$

- When $n = 1$, $\mathcal{P}_0(2)$ is given by the following diagram:

$$\begin{tikzcd}
& \{1\} \arrow[d] \\
\{0\} \arrow[r] & \{0,1\}
\end{tikzcd}$$

Example 5.4.2: Joins

We consider the join of a space and some finite space with discrete topology.

- When $n = 1$, the join of X and $\{1\}$ is given by

$$CX = X * \{1\}$$

- When $n = 2$, the join of X and $\{0, 1\}$ is given by

$$\Sigma X = X * \{0, 1\}$$

Example 5.4.3

Let F be a homotopy functor. We consider the intermediate functors T_nF for a homotopy functor F.

- When $n = 0$, $T_0F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$T_0F(X) = F(CX) \simeq F(*)$$
because F is a homotopy functor. If F is reduced then $T_0F(X) \simeq *$.
- When $n = 1$, $T_1F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$T_1F(X) \simeq \Omega F(\Sigma X)$$
because F is a homotopy functor.

Example 5.4.4

Let F be a homotopy functor. We consider the intermediate functors P_nF for a homotopy functor F.

- When $n = 0$, $P_0F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$P_0F(X) = \operatorname{hocolim}\left(F(X) \to (T_0F)(X) \to (T_0(T_0F))(X) \to \dots\right) \simeq P_0F(*)$$
because F is a homotopy functor. If F is reduced then $P_0F(X) \simeq *$.
- When $n = 1$, $P_1F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$P_1F(X) \simeq \operatorname*{hocolim}_{n\to\infty} \Omega^n F(\Sigma^n X)$$
because F is a homotopy functor.

5.5 Important Theorems

Denote Sp by the category of spectra. Define a map $\mathcal{L}(\mathbf{Top}_*, \mathrm{Sp}) \to \mathrm{Sp}$ that sends $F : \mathbf{Top}_* \to \mathrm{Sp}$ to the spectra $F(S^0)$. Conversely, define a map $\mathrm{Sp} \to \mathcal{L}(\mathbf{Top}_*, \mathrm{Sp})$ by sending each spectra X to the functor $X \wedge -$.

Now define a map $\mathcal{L}(\mathbf{Top}_*) \to \mathrm{Sp}$ as follows. For each $F : \mathbf{Top}_* \to \mathbf{Top}_*$, $F(S^n)$ is a collection of spaces indexed by $\mathbb{N}$. As for the bonding maps $F(S^n) \wedge S^1 \to F(S^{n+1})$, this is defined as follows:

1. Consider the identity map $\mathrm{id} : X \wedge Y \to X \wedge Y$.
2. By the smash-hom adjunction, this corresponds to a map $Y \to \mathrm{Map}(X, X \wedge Y)$.
3. Now composing with F gives a map
$$Y \to \mathrm{Map}(X, X \wedge Y) \to \mathrm{Map}(F(X), F(X \wedge Y))$$
(Why is the latter map continuous?)
4. By the smash-hom adjunction, this corresponds to a map $F(X) \wedge Y \to F(X \wedge Y)$
5. Taking $X = S^n$ and $Y = S^1$ gives the desired results.

At the same time, we can do the following:

1. We begin by noticing that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

is a homotopy pushout.

2. Applying F sends the homotopy pushout to a homotopy pullback:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(*) \arrow[d] \\ F(*) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

3. Since F is reduced, the diagram can be simplified into

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

4. Now recall that $\Omega(F(\Sigma X))$ is the homotopy pullback of $* \to F(\Sigma X) \leftarrow *$.
5. We obtain maps $F(X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ and $\Omega F(\Sigma X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ which are both weak

Now take the first map constructed $f : F(X) \wedge Y \to F(X \wedge Y)$ and substitute X and Y with our wanted values to get a map $f : F(S^n) \wedge S^1 \to F(S^{n+1})$. Adjunct it to the map $f : F(S^n) \to \Omega(F(S^{n+1}))$. Using the weak equivalences we obtained, we conclude that there is a diagram

$$\begin{tikzcd} F(S^n) \arrow[rr, "f"] \arrow[dr, "\simeq"'] & & \Omega F(S^{n+1}) \arrow[dl, "\simeq"] \\ & \text{Holim} & \end{tikzcd}$$

which we can prove to be commutative. By the two out of three property we easily conclude that f is a weak equivalences. This is exactly where the bonding maps come from.

We now have maps $\mathcal{L}(\textbf{Top}_*) \rightleftarrows \text{Sp}$. This actually gives an equivalence of categories. In fact, one can find out that it is a two step process:

$$\mathcal{L}(\textbf{Top}_*) \rightleftarrows \mathcal{L}(\textbf{Top}_*, \text{Sp}) \rightleftarrows \text{Sp}$$

Theorem 5.5.1

There is an equivalence of categories

$$\mathcal{L}(\textbf{Top}_*, \textbf{Sp}) \to \textbf{Sp}$$

given by $F \mapsto F(S^0)$.

Proof. Firstly, note that the above assignment defines a functor. Let $\lambda : F \Rightarrow G$ be a morphism in $\mathcal{L}(\textbf{Top}_*, \textbf{Sp})$. This means that for any $X \in \textbf{Top}_*$, we have a map of spectra $\lambda_X : F(X) \to G(X)$. Applying $X = S^0$ gives our map of spectra $F(S^0) \to G(S^0)$. Composition is preserved in this construction, and if $F = G$ then the identity natural transformation $\lambda : F \Rightarrow F$ gives the identity map $F(S^0) \to F(S^0)$ of spectra.

Now define a functor $\textbf{Sp} \to \mathcal{L}(\textbf{Top}_*, \textbf{Sp})$ by sending each spectra X to the functor $X \wedge -$. We want to show that $X \wedge -$ sends homotopy pushouts to homotopy pullbacks. Cubical 10.1.9. So let

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_2 \arrow[r] & X_{12}
\end{tikzcd}$$

be a homotopy pushout in $\mathbf{Top}_*$. $\square$

6 Stable Infinity Categories

6.1 Stable Infinity Categories

Definition 6.1.1: Zero Objects

Let $\mathcal{C}$ be an infinity category. A zero object of $\mathcal{C}$ is an object 0 of $\mathcal{C}$ such that 0 is both initial and final. We say that $\mathcal{C}$ is pointed if it contains a zero object.

Lemma 6.1.2

Let $\mathcal{C}$ be an infinity category. Then $\mathcal{C}$ is pointed if and only if the following are true.

- $\mathcal{C}$ has an initial object $\emptyset$
- $\mathcal{C}$ has a final object $*$
- There exists a morphism $* \to \emptyset$ in $\mathcal{C}$

Definition 6.1.3: Triangles

Let $\mathcal{C}$ be a pointed infinity category. A triangle in $\mathcal{C}$ consists of a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\exists!"'] & Y \arrow[d, "g"] \\ 0 \arrow[r, "\exists!"'] & Z \end{tikzcd}$$

where X, Y, Z are objects and f, g are morphisms.

Definition 6.1.4: Fiber and Cofiber Sequences

Let $\mathcal{C}$ be a pointed infinity category.

- A triangle in $\mathcal{C}$ is called a fiber sequence if it is a pullback square
- A triangle in $\mathcal{C}$ is called a cofiber sequence if it is a pushout square.

Definition 6.1.5: Stable Infinity Categories

Let $\mathcal{C}$ be an infinity category. We say that $\mathcal{C}$ is stable if the following are true.

- $\mathcal{C}$ has a zero object 0
- Every morphism in $\mathcal{C}$ admits a fiber and a cofiber
- A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence

6.2

Recall that Lurie defined the infinity category of spaces as $\mathcal{S} = N_\bullet^{\text{hc}}(\mathbf{Kan})$.

6.3

7 Algebras and Coalgebras

7.1 Coalgebras

There is a need to revisit the definition of an algebra (over a field)

Proposition 7.1.1

A vector space V over a field k is an algebra if and only if there is a following collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map

such that the following two diagrams are commutative:

$$
\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\cong"'] & V \otimes V \arrow[dl, "m"'] \\
V & V \otimes k \arrow[l, "\cong"] \arrow[u, "\mathrm{id}\otimes u"']
\end{tikzcd}
$$

where the unnamed maps is the canonical isomorphisms.

Evidently, the map μ gives a multiplicative structure for V and Δ gives the unitary structure of an algebra. The diagram on the left then represent associativity of multiplication. Notice that such additional structure on V formally lives in the category $\mathbf{Vect}_k$ of vector spaces over a fixed field k.

Therefore we can formally dualize all arrows to obtain a new object.

Definition 7.1.2: Coalgebra

Let V be a vector space over a field k. We say that V is a coalgebra over k if there is a collection of data:

- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that the following diagrams are commutative:

$$
\begin{tikzcd}
V \otimes V \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\Delta"'] \\
V \otimes V \arrow[u, "\Delta\otimes\varepsilon"] & V \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
V \arrow[u, "\cong"] \arrow[ur, "\Delta"] \arrow[r, "\cong"'] & V \otimes k
\end{tikzcd}
$$

where the unnamed maps is the canonical isomorphisms.

Lemma 7.1.3

Every vector space V over a field k can be given the structure of a coalgebra where

- $\Delta : V \to V \otimes V$ is defined by $\Delta(v) = v \otimes v$
- $\varepsilon : V \to k$ is defined by $\varepsilon(v) = 1_k$

We would like to formally invert the definitions of algebra homomorphisms in order to define coalgebra homomorphisms.

7.2 Bialgebras

Definition 7.2.1: Bialgebras

Let V be a vector space over a field k. We say that V is a bialgebra if there is a collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map
- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that (V, m, u) is an algebra over k and (V, Δ, ε) is a coalgebra over k and that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V \otimes V \otimes V \otimes V \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & V \otimes V \otimes V \otimes V \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & k \\
& V \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V \otimes V \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & V \arrow[dl, "\varepsilon"] \\
& k \otimes k \cong k &
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \otimes k \cong k \arrow[dl, "u\otimes u"'] \arrow[dr, "u"] & \\
V \otimes V & & V \arrow[ll, "\Delta"]
\end{tikzcd}$$

where $\tau : V \otimes V \to V \otimes V$ is the commutativity map defined by $\tau(x \otimes y) = y \otimes x$.

Theorem 7.2.2

Let V be a vector space over k. Suppose that (V, m, u) is an algebra and (V, Δ, ε) is a coalgebra. Then the following conditions are equivalent.

- $(V, m, u, \Delta, \varepsilon)$ is a bialgebra
- $m : V \otimes V \to V$ and $u : k \to V$ are coalgebra homomorphisms
- $\Delta : V \to V \otimes V$ and $\varepsilon : V \to k$ are algebra homomorphisms

8 Hopf Algebras

8.1 Hopf Algebras

Definition 8.1.1: Hopf Algebra

Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. We say that H is a Hopf algebra if there is a k-linear map $S : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{tikzcd}
& H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & H \\
& H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

9 Differential Graded Algebra

9.1 Basic Definitions

Similar to how chain complexes and cochain complexes are two names of the same object, we can define differential graded algebra using either the chain complex notation or cochain complex notation. For our purposes, we will use the cochain version. This means that differentials will go up in index.

A differential graded algebra equips a graded algebra with a differential so that the algebra in the grading form a cochain complex.

Definition 9.1.1: Differential Graded Algebra

A differential graded algebra is a graded algebra $A_\bullet$ together with a map $d : A \to A$ that has degree 1 such that the following are true.

- $d \circ d = 0$
- For $a \in A_n$ and $b \in A_m$, we have $d(ab) = (da)b + (-1)^n a(db)$

Lemma 9.1.2

Let (A, d) be a differential graded algebra. Then (A, d) is also a cochain complex.

Recall that a graded commutative algebra A is a collection of algebra over some ring A_0, graded in $\mathbb{N}$ together with a multiplication $A_n \times A_m \to A_{m+n}$ such that

$$a \cdot b = (-1)^{nm} b \cdot a$$

Such a multiplication rule is said to be graded commutative.

Definition 9.1.3: Commutative Differential Graded Algebra

A differential graded algebra A is said to be a commutative differential graded algebra (CDGA) if A is also graded commutative.

We will often be concerned of differential graded algebra over a field $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In particular this means that the algebra has the structure of a vector space.

10 Introduction to Group Homology and Cohomology

10.1 G-Modules

Definition 10.1.1: G-Modules

Let G be a group. A G-module is an abelian group A together with a group action of G on A.

Definition 10.1.2: Morphisms of G-Modules

Let G be a group. Let M and N be G-modules. A function $f : M \to N$ is said to be a G-module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all $m \in M$ and $g \in G$.

10.2 Invariants and Coinvariants

Definition 10.2.1: The Group of Invariants

Let G be a group and let M be a G-module. Define the group of invariants of G in M to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of M for which G acts trivially.

Definition 10.2.2: Functor of Invariants

Let G be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$$

as follows.

- For each G-module M, M^G is the group of invariants
- For each morphism $f : M \to N$ of G-modules, $f^G : M^G \to N^G$ is the restriction of f to M^G.

Theorem 10.2.3

Let G be a group. The functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

Definition 10.2.4: The Group of Coinvariants

Let G be a group and let M be a G-module. Define the group of coinvariants of G in M to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M\rangle}$$

This is the largest quotient of M for which G acts trivially.

10.3 Group Cohomology and its Equivalent Forms

Definition 10.3.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H_n(G;M) = (L_n(-)_G)(M)$$

the nth left derived functor of $(-)_G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 10.3.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H^n(G;M) \cong \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

Recall that there are two descriptions of Ext by considering it as a functor of the first or second variable. Since the above theorem exhibits an isomorphism that is natural in the second variable, let us consider Ext as the right derived functor of the functor $\text{Hom}_{\mathbb{Z}[G]}(-, M)$ applied to $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

Proposition 10.3.3

Let G be a group and let M be a G-module. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules. Then there is an isomorphism

$$H^n(G;M) \cong H^n(\text{Hom}_{\mathbb{Z}[G]}(P_\bullet, M))$$

that is natural in M.

For any group G, there is always the trivial choice of projective resolution. In the following lemma, we use the notation $(g_0, \dots, \hat{g}_i, \dots, g_n)$ as a shorthand for writing the element in G^n but with the ith term omitted.

Lemma 10.3.4

Let G be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $f_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^n (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of $\mathbb{Z}$ lying in ${}_{\mathbb{Z}[G]}\mathbf{Mod}$.

Let A be an R-algebra and let M be an A-module. Recall that the bar resolution is defined to be the chain complex consisting of $M \otimes A^{\otimes n}$ for each $n \in \mathbb{N}$ together with the boundary maps defined by multiplying the $i$the element to the $i+1$th element. Now let G be a group. By considering $\mathbb{Z}[G]$ as a $\mathbb{Z}$-algebra and that and ring is a module over itself, it makes sense to talk about the bar resolution of $\mathbb{Z}[G]$.

Theorem 10.3.5

Let G be a group. Consider the bar resolution

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \longrightarrow \mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}[G]$. Then it is a free resolution, and hence a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules.

Thus, given a group G and a G-module M, the group cohomology of G with coefficients in M can be thought of in the following way:

- It is the right derived functor of the functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$
- It is the extension group $\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ (which is computable by the obvious projective resolution $\mathbb{Z}[G^\bullet]$, or the bar resolution)

10.4 Group Homology and its Equivalent Forms

Definition 10.4.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H^n(G; M) = (R^n(-)^G)(M)$$

the nth right derived functor of $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 10.4.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_n(G; M) \cong \text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

10.5 Low Degree Interpretations

Theorem 10.5.1

Let G be a group and let M be a G-module. Then there are natural isomorphisms

$$H^0(G, M) = M^G \quad \text{and} \quad H_0(G; M) = M_G$$

Theorem 10.5.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_1(G, M) \cong \frac{G}{[G, G]} = G_{\text{ab}}$$

Theorem 10.5.3

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) = \frac{(\{f : G \to M \mid f(ab) = f(a) + af(b)\}, +)}{\langle f : G \to M \mid f(g) = gm - m \text{ for some fixed } m\rangle}$$

Corollary 10.5.4

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) \cong \text{Hom}_{\mathbf{Grp}}(G, M)$$

11 Hochschild Homology

11.1 Hochschild Homology

Definition 11.1.1: Hochschild Complex

Let M be an R-module. Define the Hoschild complex to be the chain complex $C(R,M)$ given as follows.

$$\cdots \longrightarrow M\otimes R^{\otimes n+1} \xrightarrow{d} M\otimes R^{\otimes n} \xrightarrow{d} M\otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M\otimes R \longrightarrow M \longrightarrow 0$$

The map d is defined by $d=\sum_{i=0}^n(-1)^i d_i$ where $d_i : M\otimes R^{\otimes n}\to M\otimes R^{\otimes n-1}$ is given by the following formula.

- If $i=0$, then $d_0(m\otimes r_1\otimes\cdots\otimes r_n)=mr_1\otimes r_2\otimes\cdots\otimes r_n$
- If $i=n$, then $d_n(m\otimes r_1\otimes\cdots\otimes r_n)=r_nm\otimes r_1\otimes\cdots\otimes r_{n-1}$
- Otherwise, then $d_i(m\otimes r_1\otimes\cdots\otimes r_n)=m\otimes r_1\otimes\cdots\otimes r_ir_{i+1}\otimes\cdots\otimes r_{n-1}$

Definition 11.1.2: Hochschild Homology

Let M be an R-module. Define the Hochschild homology of M to be the homology groups of the Hochschild complex $C(R,M)$:

$$H_n(R,M)=\frac{\ker(d:M\otimes R^{\otimes n}\to M\otimes R^{\otimes n-1})}{\operatorname{im}(d:M\otimes R^{\otimes n+1}\to M\otimes R^{\otimes n})}=H_n(C(R,M))$$

If $M=R$ then we simply write

$$HH_n(R)=H_n(R,R)=H_n(C(R,R))$$

TBA: Functoriality.

Proposition 11.1.3

Let A be an R-algebra. Then $HH_n(A)$ is a $Z(A)$-module.

Proposition 11.1.4

Let A be an R-algebra. Then the following are true regarding the 0th Hochschild homology.

- Let M be an A-module. Then $H_0(A,M)=\frac{M}{\{am-ma\mid a\in A,m\in M\}}$
- The 0th Hochschild homology of A is given by $HH_0(A)=\frac{A}{[A,A]}$
- If A is commutative, then the 0th Hochschild homology is given by $HH_0(A)=A$.

Theorem 11.1.5

Let A be a commutative R-algebra. Then there is a canonical isomorphism

$$HH_1(A)\cong\Omega^1_{A/R}$$

11.2 Bar Complex

Definition 11.2.1: Enveloping Algebra

Let A be an R-algebra. Define the enveloping algebra of A to be

$$A^e=A\otimes A^{\mathrm{op}}$$

Proposition 11.2.2

Let A be an R-algebra. Then any A, A-bimodule M equal to a left (right) A^e-module.

Definition 11.2.3: Bar Complex

Proposition 11.2.4

Let A be an R-algebra. The bar complex of A is a resolution of the A viewed as an A^e-module.

Theorem 11.2.5

Let A be an R-algebra that is projective as an R-module. If M is an A-bimodule, then there is an isomorphism
$$H_n(A, M) = \text{Tor}_n^{A^e}(M, A)$$

11.3 Relative Hochschild Homology

11.4 The Trace Map

Definition 11.4.1: The Generalized Trace Map

Let R be a ring and let M be an R-module. Define the generalized trace map
$$\text{tr} : M_r(M) \otimes M_r(A)^{\oplus n} \to M \otimes A^{\otimes n}$$
by the formula
$$\text{tr}((m_{i,j}) \otimes (a_{i,j})_1 \otimes \cdots \otimes (a_{i,j})_n) = \sum_{0 \leq i_0, \dots, i_n \leq r} m_{i_0,i_1} \otimes (a_{i_1,i_2})_1 \otimes \cdots \otimes (a_{i_n,i_0})_n$$

Theorem 11.4.2

The trace map defines a morphism of chain complex
$$\text{tr} : C_\bullet(M_r(A), M_r(M)) \to C_\bullet(A, M)$$

11.5 Morita Equivalence and Morita Invariance

Definition 11.5.1

Let R and S be rings. We say that R and S are Morita equivalent if there is an equivalence of categories
$$\textbf{Mod}_R \cong \textbf{Mod}_S$$

Theorem 11.5.2: Morita Invariance for Matrices

12 Group Structures on Maps of Spaces

Req: AT3

H-spaces is a natural generalization of topological groups in the direction of homotopy theory.

Definition 12.0.1: H-Spaces

Let (X, x_0) be a pointed space. Let $\mu : (X, x_0) \times (X, x_0) \to (X, x_0)$ be a map. Let $e : (X, x_0) \to (X, x_0)$ be the constant map $x \mapsto x_0$. We say that (X, x_0, μ) is an H-space if the following diagram:

$$\begin{tikzcd} X \arrow[r, "(e,\mathrm{id}_X)"] \arrow[d, "(\mathrm{id}_X, e)"'] \arrow[dr, "\mathrm{id}_X"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map μ is called H-multiplication.

Definition 12.0.2: H-Associative Spaces

Let (X, x_0, μ) be an H-space. We say that (X, x_0, μ) is an H-associative space if the following diagram:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times \mu"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 12.0.3: H-Group

Let (X, x_0, μ) be an H-space. Let $j : (X, x_0) \to (X, x_0)$ be a map. We say that (X, x_0, μ, j) is an H-group if the following diagram:

$$\begin{tikzcd} X \arrow[r, "(j,\mathrm{id}_X)"] \arrow[d, "(\mathrm{id}_X, j)"'] \arrow[dr, "e"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map j is called H-inverse.

Example 12.0.4

Let X be a pointed space. Then the loopspace ΩX is an H-group.

Definition 12.0.5: H-Abelian

Let (X, x_0, μ, j) be an H-group. Let $T : (X, x_0) \times (X, x_0) \to (X, x_0)$ be the map $T(x, y) = T(y, x)$. We say that (X, x_0, μ, j) is an H-abelian if the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, "T"] \arrow[dr, "\mu"'] & X \times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 12.0.6: Natural Group Structure

Let (X, x_0) be pointed spaces. We say that $[Z, X]_*$ has a natural group structure for all spaces (Z, z_0) if the following are true.

- $[Z, X]_*$ has a group structure such that the constant map $[e]$ is the identity of the group.
- For every map $f : A \to B$, the induced function

$$f^* : [B, X]_* \to [A, X]_*$$

is a group homomorphism.

13 Homological Algebra

13.1 Koszul Complexes

The following definitions requires the use of central elements. Recall that when R is commutative, this condition is null and so we can choose any element in R.

Definition 13.1.1: Koszul Complexes

Let R be a ring. Let $x \in R$ be a central element. Define the Koszul complex $K(x)$ of x in R to be the chain complex

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

Definition 13.1.2: Generalized Koszul Complexes

Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Define the generalized Koszul complex $K(x_1, \dots, x_n)$ of $x_1, \dots, x_n$ in R to be the chain complex given by

$$K(x_1, \dots, x_n) = \text{Tot}^{\oplus}\left(K(x_1) \oplus_R \cdots \oplus_R K(x_n)\right)$$

If M is an R-module, define the generalized Koszul complex of M to be

$$K(x_1, \dots, x_n; M) = K(x_1, \dots, x_n) \otimes_R M$$

Theorem 13.1.3

Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Then the Koszul complex $K(x_1, \dots, x_n)$ is given explicitly as

$$0 \longrightarrow \bigwedge_{i=1}^n R^n \xrightarrow{d_n} \bigwedge_{i=1}^{n-} R^n \longrightarrow \cdots \longrightarrow R^n \xrightarrow{d_1} R \longrightarrow 0$$

where the differential $d_k : \bigwedge_{i=1}^k R^n \to \bigwedge_{i=1}^{k-1} R^n$ is given on basis elements by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=0}^{k} (-1)^{j+1} x_{i_j} e_{i_0} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$$

Definition 13.1.4: Koszul (Co)Homology

Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Let M be an R-module. Define the Koszul homology of M with respect to $x_1, \dots, x_n$ by

$$H_k^{\text{Kos}}(x_1, \dots, x_n; M) = H_k(K(x_1, \dots, x_n; M))$$

Define the Koszul cohomology of M with respect to the central elements by

$$H^k_{\text{Kos}}(x_1, \dots, x_n; M) = H^k(\text{Hom}_R(K(x_1, \dots, x_n)), M)$$

Lemma 13.1.5

Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Let M be an R-module. Then the following are true.

- $H_0(x_1, \dots, x_n; M) = \frac{M}{(x_1, \dots, x_n)M}$
- $H^0(x_1, \dots, x_n; M) = \text{Ann}_M(\{x_1, \dots, x_n\})$
- $H_p(x_1, \dots, x_n; M) \cong H^{n-p}(x_1, \dots, x_n; M)$

Theorem 13.1.6: Kunneth Theorem

Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Let $C_\bullet$ be a chain complex of R-modules. Then there is an exact sequence given by

$$0 \longrightarrow H_0(x_1, \dots, x_n; H_q(C_\bullet)) \longrightarrow H_q^{\text{Tot}}(K(x_1, \dots, x_n) \otimes_R C_\bullet) \longrightarrow H_1(x_1, \dots, x_n; H_{q-1}(C_\bullet)) \longrightarrow 0$$