Transcendental Algebraic Geometry

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Abstract

Contents

1 Analytification of a Variety

1.1 The Set of Closed Points of a Scheme

Recall that a point $x \in X$ of a space is said to be closed $\{x\}$ is a closed set.

Definition 1.1.1: Closed Points of a Variety

Let X be a variety over $\mathbb{C}$. Denote its set of closed points by

$$X(\mathbb{C}) = \{x \in X \mid x \text{ is a closed point}\}$$

Definition 1.1.2: Subspace Topology on Closed Points

Let X be a variety over $\mathbb{C}$. Denote

$$\text{Max}(X)$$

the set $X(\mathbb{C})$ together with the subspace topology inherited from X. If $X = \text{Spec}(R)$ for some ring R, then we simply write $\text{maxSpec}(R) = \text{Max}(\text{Spec}(R))$.

Note: For a ring R, $X = \text{Spec}(R)$, then $\text{Max}(X) = \text{maxSpec}(R)$ because the closed points are precisely the maximal ideals. Moreover, the Zariski topology of $\text{maxSpec}(R)$ coincides with the subspace topology of $\text{Max}(X)$.

We will first investigate for when X is affine, before moving on to the general theory of schemes. Therefore much of the following section, we will be working with $X = \text{Spec}(R)$ for some R a finitely generated $\mathbb{C}$-algebra.

Theorem 1.1.3

Let R be a finitely generated $\mathbb{C}$-algebra. Then there is a natural bijection

$$\text{maxSpec}(R) = \left\{ \begin{matrix} \text{Closed points} \\ \text{in Spec}(R) \end{matrix} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} \mathbb{C}\text{-algebra homomorphisms} \\ \varphi : R \to \mathbb{C} \end{matrix} \right\}$$

The forward map sends $x \in \text{Spec}(R)$ to the unique φ whose kernel is (x). The backward map sends $\varphi : R \to \mathbb{C}$ to the image of $\varphi^* : \text{Spec}(\mathbb{C}) \to \text{Spec}(R)$.

Now we pair it up with the natural bijection between $\mathbb{C}$-algebra homomorphisms $\varphi : R \to \mathbb{C}$ and morphisms of locally ringed spaces

$$(\varphi^*, \varphi^\#) : (\text{Spec}(\mathbb{C}), \mathcal{O}_{\text{Spec}(\mathbb{C})}) \to (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$$

In fact, we can do one step further by starting with an arbitrary scheme $(X, \mathcal{O}_X)$ locally of finite type over $\mathbb{C}$.

Proposition 1.1.4

Proposition 1.1.5

Let $(\Psi, \Psi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphisms of schemes that is locally of finite type over $\mathbb{C}$. Then the continuous map $\Psi : X \to Y$ takes the subspace $\text{Max}(X)$ to the subspace $\text{Max}(Y)$.

Definition 1.1.6: Max Map

Let $(\Psi, \Psi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphisms of schemes that is locally of finite type over

$\mathbb{C}$. Define the induced map of closed points by

$$\text{Max}(\Psi) : \text{Max}(X) \to \text{Max}(Y)$$

Proposition 1.1.7

Let $\theta : R \to S$ be a surjective map of finitely generated $\mathbb{C}$-algerbas. Then the map

$$\text{maxSpec}(\theta) : \text{maxSpec}(S) \to \text{maxSpec}(R)$$

embeds $\text{maxSpec}(S)$ homeomorphically into a subspace of $\text{maxSpec}(R)$. The image is identified with the set of all $\varphi : R \to \mathbb{C}$ such that $\varphi(\ker(\theta)) = 0$

1.2 Complex Topology on Spec

Lemma 1.2.1

There is a bijective correspondence

$$\mathbb{C}^n \quad \overset{1:1}{\longleftrightarrow} \quad \begin{Bmatrix} \mathbb{C}\text{-algebra homomorphisms} \\ \varphi:\mathbb{C}[x_1,\dots,x_n]\to\mathbb{C} \end{Bmatrix}$$

The forward map sends $a = (a_1, \dots, a_n)$ to the map $\varphi_a : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$ defined by $f \mapsto f(a_1, \dots, a_n)$. The backward map sends $\varphi : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{C}$ to $(\varphi(x_1), \dots, \varphi(x_n))$.

For the finitely generated $\mathbb{C}$-algebra $\mathbb{C}[x_1, \dots, x_n]$, we now have a series of correspondences

$$\text{maxSpec}(\mathbb{C}[x_1, \dots, x_n]) = \begin{Bmatrix} \text{Closed points} \\ \text{in Spec}(\mathbb{C}[x_1,\dots,x_n]) \end{Bmatrix} \quad \overset{1:1}{\longleftrightarrow} \quad \begin{Bmatrix} \mathbb{C}\text{-algebra homomorphisms} \\ \varphi:\mathbb{C}[x_1,\dots,x_n]\to\mathbb{C} \end{Bmatrix} \quad \overset{1:1}{\longleftrightarrow} \quad \mathbb{C}^n$$

Definition 1.2.2: Complex Topology on Spec

Let S be a finitely generated $\mathbb{C}$-algebra. Let $a_1, \dots, a_n$ be generators of S. Consider the surjection

$$\theta : \mathbb{C}[x_1, \dots, x_n] \to S$$

defined by $x_i \mapsto a_i$. Define the complex topology of $X = \text{Spec}(S)$ to be the subspace topology of $\mathbb{C}^n$ via the injective map

$$\text{maxSpec}(\theta) : \text{maxSpec}(S) \to \text{maxSpec}(\mathbb{C}^n) \cong \mathbb{C}^n$$

Denote X^{an} to be the set $X = \text{maxSpec}(S)$ together with the complex topology.

Lemma 1.2.3

Let S be a finitely generated $\mathbb{C}$-algebra. Then the complex topology on $\text{maxSpec}(S)$ is independent of the choice of generators of S.

Proposition 1.2.4

Let S be a finitely generated $\mathbb{C}$-algebra. Then the natural inclusion

$$\text{maxSpec}(S) \hookrightarrow \text{Spec}(S)$$

is continuous if we give $\text{maxSpec}(S)$ the complex topology and $\text{Spec}(S)$ the Zariski topology.

Proposition 1.2.5

Let $\varphi : R \to S$ be a homomorphism of finitely generated $\mathbb{C}$-algebras. Then the natural map

$$\text{maxSpec}(\varphi) : (\text{Spec}(S))^{\text{an}} \to (\text{Spec}(R))^{\text{an}}$$

is continuous.

This marks the fact that the passage from affine varieties to topological spaces defined by sending $X = \text{Spec}(R)$ to X^{an} is functorial. The following corollary should be of no surprise.

Corollary 1.2.6

Let $\varphi : R \to S$ be an isomorphism of finitely generated $\mathbb{C}$-algebras. Then the natural map

$$\text{maxSpec}(\varphi) : (\text{Spec}(S))^{\text{an}} \to (\text{Spec}(R))^{\text{an}}$$

is a homeomorphism.

Lemma 1.2.7

Let $\varphi : R \to S$ be a surjective homomorphism of finitely generated $\mathbb{C}$-algebras. Then the natural map

$$\text{maxSpec}(\varphi) : (\text{Spec}(S))^{\text{an}} \to (\text{Spec}(R))^{\text{an}}$$

an embedding.

1.3 Complex Topology for Schemes Locally of Finite Type

Recall that a scheme is locally of finite type over $\mathbb{C}$ if it has an open cover $X = \bigcup_{i \in I} U_i$ for which $U_i \cong \text{Spec}(R_i)$ for some R_i a finitely generated $\mathbb{C}$-algebra. Every scheme of finite type is necessarily a scheme that is locally of finite type. And it follows that when we discuss schemes that is locally of finite type, this includes the general theory of varieties.

Lemma 1.3.1

Let $(Y, \mathcal{O}_Y)$ be a scheme locally of finite type over $\mathbb{C}$. Let $X \subseteq Y$ be an open set. Then the inclusion map

$$\Psi : X \to Y$$

embeds $\text{Max}(X)$ homeomorphically onto the open subset $\Psi(X) \cap \text{Max}(Y)$.

Corollary 1.3.2

Let X be a scheme locally of finite type over $\mathbb{C}$. If $X = \bigcup_{i \in I} U_i$ is an open cover, then $\text{Max}(U_i)$ is an open cover for $\text{Max}(X)$.

This does not help much with respect to the complex topology unfortunately. Therefore we need a technical lemma.

Lemma 1.3.3

Let $(Z, \mathcal{O}_Z)$ be a scheme locally of finite type over $\mathbb{C}$. Let U and V be open subsets of Z. Suppose that $(U, \mathcal{O}_X|_U) \cong (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ and $(V, \mathcal{O}_X|_V) \cong (\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)})$. Then $\text{Max}(U) \cap \text{Max}(V)$ is open in both $(\text{Spec}(R))^{\text{an}}$ and $(\text{Spec}(S))^{\text{an}}$. Moreover, the subspaces topologies induced with respect to both embeddings agree with each other.

The final ingredient would be the weak topology. Let X be a set. Let $\varphi_i : U_i \to X$ for $i \in I$ be functions from a topological space U_i to X. Then the weak topology of X with respect to φ_i is the finest topology such that all φ_i are continuous. This means that a subset $V \subseteq X$ is open if and only if $\varphi_i^{-1}(V)$ is open in U_i for all $i \in I$.

Definition 1.3.4: Complex Topology

Let X be a scheme locally of finite type over $\mathbb{C}$. Let V be the set of all open immersions

$$(\Psi_i, \Psi_i^\#) : (\text{Spec}(R_i), \mathcal{O}_{\text{Spec}(R_i)})$$

of ringed spaces over $\mathbb{C}$ with each R_i a finitely generated $\mathbb{C}$-algebra. Define the complex topology on $\text{Max}(X)$ to be the weak topology with respect to the maps

$$\text{Max}(\Psi_i) : (\text{Spec}(R_i))^{\text{an}} \to \text{Max}(X)$$

In this case we denote $\text{Max}(X)$ together with the complex topology by X^{an}.

Lemma 1.3.5

Let X be a scheme locally of finite type over $\mathbb{C}$. Suppose that there is an open immersion

$$(\Psi, \Psi^\#) : (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$$

Then the map $\text{Max}(\Psi) : (\text{Spec}(R))^{\text{an}} \to X^{\text{an}}$ is a homeomorphism onto its image, and the image is open in X.

Lemma 1.3.6

Let X be a scheme locally of finite type over $\mathbb{C}$. Then the inclusion

$$X^{\text{an}} \hookrightarrow X$$

is a continuous map where X has the Zariski topology and X^{an} has the complex topology.

Corollary 1.3.7

Let X, Y, Z be schemes locally of finite type over $\mathbb{C}$. Suppose that there are morphisms of schemes $\Phi : X \to Y$ and $\Psi : Y \to Z$. Then

$$\Psi^{\text{an}} \circ \Phi^{\text{an}} = (\Psi \circ \Phi)^{\text{an}}$$

Corollary 1.3.8

Let $(\Psi, \Psi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes locally of finite type over $\mathbb{C}$. Then the following square commutes:

$$\begin{tikzcd} X^{\text{an}} \arrow[r, "\Psi^{\text{an}}"] \arrow[d, "\lambda_X"'] & Y^{\text{an}} \arrow[d, "\lambda_Y"] \\ X \arrow[r, "\Psi"'] & Y \end{tikzcd}$$

where $\lambda_X : X^{\text{an}} \to X$ is the inclusion.

We are almost done with complex analytification. We even showed that analytification is functorial, and more over there is a natural transformation from the analytification functor to the forgetful functor. Given a scheme locally of finite type, we constructed a topological space that is a subspace of $\mathbb{C}^n$. We also want to produce a sheaf on the subspace so that the resulting construct is an analytic space.

1.4 The Analytic Sheaf

Once again, we first work with the affine case.

1.5 The Functorial Conclusion

Definition 1.5.1: Complex Analytification Functor

Define the complex analytification functor $(\,\cdot\,)^{\text{an}} : \text{Var}_{\mathbb{C}} \to \text{ASpace}$ as follows.

- For each variety $(X, \mathcal{O}_X)$ over $\mathbb{C}$, it is sent to $(X^{\text{an}}, \mathcal{O}_X^{\text{an}})$
- For each morphism $(\Psi, \Psi^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, it is sent to the morphism

$$(\Psi^{\text{an}}, (\Psi^{\#})^{\text{an}}) : (X^{\text{an}}, \mathcal{O}_X^{\text{an}}) \to (Y^{\text{an}}, \mathcal{O}_Y^{\text{an}})$$

Proposition 1.5.2

Let $I_V : \text{Var}_{\mathbb{C}} \to \text{RSpace}$ and $I_A : \text{ASpace} \to \text{RSpace}$ be inclusion functors. Then there is a natural transformation $\lambda : I_A \circ (\,\cdot\,)^{\text{an}} \to I_V$

Theorem 1.5.3: GAGA I

Let X be a projective complex algebraic variety. The restricted complex analytification functor from the category of coherent sheaves on X to the category of coherent analytic sheaves on X defines an equivalence of categories.