Simplicial Methods in Algebra

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Abstract

References:

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1 Simplicial Homological Algebra

1.1 Chain Complexes of Simplicial Objects

Definition 1.1.1: Associated Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a (semi)-simplicial object in $\mathcal{A}$. Define the associated chain complex of A to be

$$\cdots \longrightarrow C_{n+1}(A) \xrightarrow{\partial_{n+1}} C_n(A) \xrightarrow{\partial_n} C_{n-1}(A) \longrightarrow \cdots \longrightarrow C_0(A)$$

where $C_n(A) = A_n$ and the boundary operator given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n : A_n \to A_{n-1}$$

TBA: Functoriality of associated chain complex

Definition 1.1.2: Simplicial Homology

Let R be a ring. Let X be a (semi)-simplicial set. Define the simplicial homology of X with coefficients in R to be the homology groups

$$H_n^{\Delta}(X; R) = H_n(C_\bullet(R[X]))$$

Notice that this definition coincides with that in Algebraic Topology 2. Recall that in AT2 we defined the simplicial homology of a Δ-set, but in $\mathbb{Z}$ coefficients.

1.2 Normalized Chain Complexes

Definition 1.2.1: Normalized Chain Complexes

Let $\mathcal{A}$ be an abelian category or the category **Grp**. Let A be a simplicial object in $\mathcal{A}$. Define the normalized chain complex of A to be the chain complex:

$$\cdots \longrightarrow N_{k+1}(A) \xrightarrow{\partial_{k+1}} N_k(A) \xrightarrow{\partial_k} N_{k-1}(A) \longrightarrow \cdots \longrightarrow N_1(A)$$

where

$$N_k(A) = \bigcap_{i=1}^{k} \ker(d_i^k : A_k \to A_{k-1})$$

and the differential given by $\partial_k = d_0^K|_{N_k(A)}$. We denote the normalized chain complex by $(N_\bullet(G), \partial_\bullet)$

nLab: We may think of the elements of the complex in degree k as being k-dimensional disks in G all of whose boundary is captured by a single face.

Lemma 1.2.2

Let G be a simplicial group. Consider the normalized chain complex $(N_\bullet(G), \partial_\bullet)$. Then $\partial_n N_n(G)$ is a normal subgroup of $N - n - 1(G)$.

Because of this lemma, it now makes sense to take the homology group of the normalized chain complex even if we take a simplicial object in **Grp**.

Definition 1.2.3: Normalized Chain Complex Functor

Let $\mathcal{A}$ be an abelian category. Define the normalized chain complex functor N

Definition 1.2.4: Degenerate Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Define the degenerate chain complex $D_\bullet(A)$ of A to be the subcomplex of the associated chain complex $C_\bullet(A)$ defined by

$$D_n(A) = \langle s_i^n : A_n \to A_{n+1} \mid s_i \text{ is the degenerate maps} \rangle$$

Proposition 1.2.5

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Then there is a splitting

$$C_\bullet(A) \cong N_\bullet(A) \oplus D_\bullet(A)$$

in the abelian category of chain complexes of $\mathcal{A}$.

Theorem 1.2.6: Eilenberg-Maclane

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in A. Then the inclusion

$$N_\bullet(A) \hookrightarrow C_\bullet(A)$$

is a natural chain homotopy equivalence. In other words, $D_\bullet(A)$ is null homotopic.

Theorem 1.2.7: The Dold-Kan Correspondence

Consider the abelian category $\mathbf{Ab}$ of abelian groups. The normalized chain complex functor

$$N : \mathbf{sAb} \xrightarrow{\cong} \text{Ch}_{\geq 0}(\mathbf{Ab})$$

gives an equivalence of categories, with inverse as the simplicialization functor

$$\Gamma : \text{Ch}_{\geq 0}(\mathbf{Ab}) \to \mathbf{sAb}$$

1.3 Bar Resolutions

Definition 1.3.1: Bar Construction

Let A be an algebra over a ring R. Let M be an A-algebra. Define the maps $d_i^n : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ by the following formulas:

- If $i = 0$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
- If $0 < i < n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n$$
- If $i = n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}$$

Proposition 1.3.2

Let A be an algebra over a ring R. Let M be an A-algebra. Then $(M \otimes A^{\otimes n}, d_i^n)$ defines a simplicial object in ????

Definition 1.3.3: Bar Resolutions

Let A be an algebra over a ring R. Let M be an A-algebra. Define the bar resolution of M to be the associated chain complex of the simplicial object

$$(M \otimes A^{\otimes n}, d_i^n)$$

Explicitly, the chain complex is given in the form

$$\cdots \longrightarrow A^{\otimes n+1} \otimes M \longrightarrow A^{\otimes n} \otimes M \longrightarrow A^{\otimes n-1} \otimes M \longrightarrow \cdots \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0$$

with the boundary map $\partial : A^{\otimes n} \otimes M \to A^{\otimes n-1} \otimes M$ given by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i^n$$

2 (Co)Homology of Groups

2.1 G-Modules

Definition 2.1.1: G-Modules

Let G be a group. A G-module is an abelian group A together with a group action of G on A.

Definition 2.1.2: Morphisms of G-Modules

Let G be a group. Let M and N be G-modules. A function $f : M \to N$ is said to be a G-module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all $m \in M$ and $g \in G$.

2.2 Invariants and Coinvariants

Definition 2.2.1: The Group of Invariants

Let G be a group and let M be a G-module. Define the group of invariants of G in M to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of M for which G acts trivially.

Definition 2.2.2: Functor of Invariants

Let G be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$$

as follows.

- For each G-module M, M^G is the group of invariants
- For each morphism $f : M \to N$ of G-modules, $f^G : M^G \to N^G$ is the restriction of f to M^G.

Theorem 2.2.3

Let G be a group. The functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

Definition 2.2.4: The Group of Coinvariants

Let G be a group and let M be a G-module. Define the group of coinvariants of G in M to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M\rangle}$$

This is the largest quotient of M for which G acts trivially.

2.3 Group Cohomology and its Equivalent Forms

Definition 2.3.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H_n(G;M) = (L_n(-)_G)(M)$$

the nth left derived functor of $(-)_G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 2.3.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H^n(G;M) \cong \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

Recall that there are two descriptions of Ext by considering it as a functor of the first or second variable. Since the above theorem exhibits an isomorphism that is natural in the second variable, let us consider Ext as the right derived functor of the functor $\text{Hom}_{\mathbb{Z}[G]}(-, M)$ applied to $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

Proposition 2.3.3

Let G be a group and let M be a G-module. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules. Then there is an isomorphism

$$H^n(G;M) \cong H^n(\text{Hom}_{\mathbb{Z}[G]}(P_\bullet, M))$$

that is natural in M.

For any group G, there is always the trivial choice of projective resolution. In the following lemma, we use the notation $(g_0, \dots, \hat{g}_i, \dots, g_n)$ as a shorthand for writing the element in G^n but with the ith term omitted.

Lemma 2.3.4

Let G be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $f_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of $\mathbb{Z}$ lying in ${}_{\mathbb{Z}[G]}\mathbf{Mod}$.

Let A be an R-algebra and let M be an A-module. Recall that the bar resolution is defined to be the chain complex consisting of $M \otimes A^{\otimes n}$ for each $n \in \mathbb{N}$ together with the boundary maps defined by multiplying the $i$the element to the $i+1$th element. Now let G be a group. By considering $\mathbb{Z}[G]$ as a $\mathbb{Z}$-algebra and that and ring is a module over itself, it makes sense to talk about the bar resolution of $\mathbb{Z}[G]$.

Proposition 2.3.5

Let G be a group. Consider the bar resolution

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \longrightarrow \mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}[G]$. Then it is a free resolution, and hence a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules.

Thus, given a group G and a G-module M, the group cohomology of G with coefficients in M can be thought of in the following way:

- It is the right derived functor of the functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$
- It is the extension group $\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ (which is computable by the obvious projective resolution $\mathbb{Z}[G^\bullet]$)

2.4 Group Homology and its Equivalent Forms

Definition 2.4.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H^n(G; M) = (R^n(-)^G)(M)$$

the nth right derived functor of $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 2.4.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_n(G; M) \cong \text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

2.5 Low Degree Interpretations

Theorem 2.5.1

Let G be a group and let M be a G-module. Then there are natural isomorphisms

$$H^0(G, M) = M^G \quad \text{and} \quad H_0(G; M) = M_G$$

Theorem 2.5.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_1(G, M) \cong \frac{G}{[G, G]} = G_{\text{ab}}$$

Theorem 2.5.3

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) = \frac{(\{f : G \to M \mid f(ab) = f(a) + af(b)\}, +)}{\langle f : G \to M \mid f(g) = gm - m \text{ for some fixed } m\rangle}$$

Corollary 2.5.4

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) \cong \text{Hom}_{\mathbf{Grp}}(G, M)$$

3 Hochschild (Co)Homology for Rings

3.1 Hochschild Homology

Definition 3.1.1: Hochschild Complex

Let M be an R-module. Define the Hoschild complex to be the chain complex $C(R,M)$ given as follows.

$$\cdots \longrightarrow M\otimes R^{\otimes n+1} \xrightarrow{d} M\otimes R^{\otimes n} \xrightarrow{d} M\otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M\otimes R \longrightarrow M \longrightarrow 0$$

The map d is defined by $d=\sum_{i=0}^n(-1)^id_i$ where $d_i : M\otimes R^{\otimes n}\to M\otimes R^{\otimes n-1}$ is given by the following formula.

- If $i=0$, then $d_0(m\otimes r_1\otimes\cdots\otimes r_n)=mr_1\otimes r_2\otimes\cdots\otimes r_n$
- If $i=n$, then $d_n(m\otimes r_1\otimes\cdots\otimes r_n)=r_nm\otimes r_1\otimes\cdots\otimes r_{n-1}$
- Otherwise, then $d_i(m\otimes r_1\otimes\cdots\otimes r_n)=m\otimes r_1\otimes\cdots\otimes r_ir_{i+1}\otimes\cdots\otimes r_{n-1}$

Definition 3.1.2: Hochschild Homology

Let M be an R-module. Define the Hochschild homology of M to be the homology groups of the Hochschild complex $C(R,M)$:

$$H_n(R,M)=\frac{\ker(d: M\otimes R^{\otimes n}\to M\otimes R^{\otimes n-1})}{\text{im}(d: M\otimes R^{\otimes n+1}\to M\otimes R^{\otimes n})}=H_n(C(R,M))$$

If $M=R$ then we simply write

$$HH_n(R)=H_n(R,R)=H_n(C(R,R))$$

TBA: Functoriality.

Proposition 3.1.3

Let A be an R-algebra. Then $HH_n(A)$ is a $Z(A)$-module.

Proposition 3.1.4

Let A be an R-algebra. Then the following are true regarding the 0th Hochschild homology.

- Let M be an A-module. Then $H_0(A,M)=\frac{M}{\{am-ma\mid a\in A, m\in M\}}$
- The 0th Hochschild homology of A is given by $HH_0(A)=\frac{A}{[A,A]}$
- If A is commutative, then the 0th Hochschild homology is given by $HH_0(A)=A$.

Theorem 3.1.5

Let A be a commutative R-algebra. Then there is a canonical isomorphism

$$HH_1(A)\cong\Omega^1_{A/R}$$

3.2 Bar Complex

Definition 3.2.1: Enveloping Algebra

Let A be an R-algebra. Define the enveloping algebra of A to be

$$A^e=A\otimes A^{\text{op}}$$

Proposition 3.2.2

Let A be an R-algebra. Then any A, A-bimodule M equal to a left (right) A^e-module.

Definition 3.2.3: Bar Complex

Proposition 3.2.4

Let A be an R-algebra. The bar complex of A is a resolution of the A viewed as an A^e-module.

Theorem 3.2.5

Let A be an R-algebra that is projective as an R-module. If M is an A-bimodule, then there is an isomorphism

$$H_n(A, M) = \text{Tor}_n^{A^e}(M, A)$$

3.3 Relative Hochschild Homology

3.4 The Trace Map

Definition 3.4.1: The Generalized Trace Map

Let R be a ring and let M be an R-module. Define the generalized trace map

$$\text{tr} : M_r(M) \otimes M_r(A)^{\oplus n} \to M \otimes A^{\otimes n}$$

by the formula

$$\text{tr}((m_{i,j}) \otimes (a_{i,j})_1 \otimes \cdots \otimes (a_{i,j})_n) = \sum_{0 \leq i_0, \dots, i_n \leq r} m_{i_0,i_1} \otimes (a_{i_1,i_2})_1 \otimes \cdots \otimes (a_{i_n,i_0})_n$$

Theorem 3.4.2

The trace map defines a morphism of chain complex

$$\text{tr} : C_\bullet(M_r(A), M_r(M)) \to C_\bullet(A, M)$$

3.5 Morita Equivalence and Morita Invariance

Definition 3.5.1

Let R and S be rings. We say that R and S are Morita equivalent if there is an equivalence of categories

$$\mathbf{Mod}_R \cong \mathbf{Mod}_S$$

Theorem 3.5.2: Morita Invariance for Matrices

4 (Co)Homology for Lie Algebras