Riemannian Manifolds

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Abstract

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1 Riemannian Metrics

1.1 The Riemannian Metric

Definition 1.1.1: Riemannian Metric

Let M be a smooth manifold. A Riemannian metric on M is a function $g : TM \times TM \to \mathbb{R}$ such that for each $p \in M$, the restriction of g to $T_pM \times T_pM$, denoted g_p has the following properties.

- Symmetric: $g_p(X_p, Y_p) = g_p(Y_p, X_p)$ for all $X_p, Y_p \in T_pM$
- Positive Definite: $g_p(X_p, X_p) > 0$ for all $X_p \in T_pM$ with $X_p \neq 0$
- Bilinearity: $g_p(aX_p + bY_p, Z_p) = ag_p(X_p, Z_p) + bg_p(Y_p, Z_p)$ and $g_p(X_p, aY_p + bZ_p) = ag_p(X_p, Y_p) + bg_p(X_p, Z_p)$

Definition 1.1.2: Riemannian Manifold

A Riemannian manifold (M, g) is a manifold M together with a Riemannian metric g on M.

1.2 Bundle Metric

Definition 1.2.1: Bundle Metric

Let M be a topological manifold and $p : E \to M$ a vector bundle on M. Then a bundle metric on E is a section of $E^* \otimes E^*$ such that it is nondegenerate and symmetric.

In other words, a bundle metric is an assignment to each fibre, an inner product. Bilinearity is seen from $E^* \otimes E^*$, which is exactly the set of all bilinear forms $E \times E \to \mathbb{R}$.

Proposition 1.2.2

Let M be a smooth manifold. Then a Riemannian metric give rise to a bundle metric on TM. A bundle metric on TM gives rise to a Riemannian metric.

2 Levi-Civita Connection

3 Geodesics

Definition 3.0.1: Geodesics

A curve $\gamma : (a, b) \to M$ is called a geodesic if $D_t(\gamma'(t)) = 0$ for all $t \in (a, b)$.