Algebraic Topology 3

Labix

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Abstract

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1 Homotopy Theory

1.1 The nth Homotopy Groups

Definition 1.1.1: Pairs of Space

Let X be a topological space. A pair of space is a pair (X, A) where $A \subseteq X$ is a subspace of X. A map of pairs $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

Definition 1.1.2: Homotopy between Maps of Pairs

Let $f, g : (X, A) \to (Y, B)$ be maps of pairs. A homotopy between f and g is a homotopy $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subseteq B$.

Definition 1.1.3: The nth Homotopy Groups

Let (X, x_0) be a pointed space. Define the nth homotopy group $\pi_n(X, x_0)$ to be

$$\pi_n(X, x_0) = \frac{\left\{ f : (I^n, \partial I^n) \to (X, \{x_0\}) \;\middle|\; f \text{ is continuous} \right\}}{\simeq}$$

where we say that $f \simeq g$ if there exists a homotopy between f and g.

Definition 1.1.4: Concatenation

For $n \geq 1$, define a composition law on $\pi_n(X, x_0)$ for a pointed space (X, x_0) by the formula

$$(f \cdot g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $f, g \in \pi_n(X, x_0)$.

Theorem 1.1.5

Let (X, x_0) be a pointed space and $n \geq 1$. The operation $\cdot$ on $\pi_n(X, x_0)$ is well defined and endows it with the structure of a group.

Proposition 1.1.6

Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

1.2 Properties of Homotopy

Definition 1.2.1: Category of Pointed Spaces

The Category of Pointed spaces Top_* is defined where

- The objects are pointed topological spaces (X, x_0) for $x_0 \in X$.
- The morphisms are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$ for two pointed spaces (X, x_0) and (Y, y_0).
- Composition is defined as the composition of continuous maps that preserve the base point.

Proposition 1.2.2: Functoriality

For each $n \geq 1$, $\pi_n(-) : \text{Top}_* \to \text{Grp}$ is a functor where

- On objects, it sends (X, x_0) to the nth homotopy group $\pi_n(X, x_0)$
- On morphisms, it sends $f : (X, x_0) \to (Y, y_0)$ to the induced map

$$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

  defined as $[\varphi] \mapsto [f \circ \varphi]$

Proposition 1.2.3

Let $(X, x_0), (Y, y_0)$ be pointed spaces and $f, g : (X, x_0) \to (Y, y_0)$ be pointed maps. If f and g are homotopic, then the induced maps

$$\pi_n(f) = \pi_n(g) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then $\pi_n(f)$ is an isomorphism.

Theorem 1.2.4

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. If $u : I \to X$ is a path from x_0 to x_1, then u induces a map

$$u_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

satisfying the following functorial properties:

- $u_\#$ is a group homomorphism
- If $v : I \to X$ is a path from x_1 to x_2 and $u \cdot v$ is the concatenation of these paths, then

$$(u \cdot v)_\# = u_\# \circ v_\#$$

- If c_{x_0} is the constant path from x_0 to x_0 then $(c_{x_0})_\#$ is the identity

Proposition 1.2.5

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. Let $u, v : I \to X$ be paths from x_0 to x_1. If u and v are homotopic relative to end points then the induced maps

$$u_\# = v_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

are equal.

Corollary 1.2.6

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. If x_0 and x_1 are path connected, then

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

where the isomorphism depends on the choice of path from x_0 to x_1.

Proposition 1.2.7

Let (X, x_0) be a pointed space and $f \in \pi_n(X, x_0)$. Let $u : I \to X$ be a loop on x_0. Then u induces a left action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ by the map

$$(u, f) \mapsto u_\#(f) = u \cdot f \cdot u^{-1}$$

In particular, for $n \geq 2$, $\pi_n(X, x_0)$ is a $\mathbb{Z}\pi_1(X, x_0)$-module.

1.3 Relative Homotopy Groups

Definition 1.3.1: Triplets of Spaces

Let X be a topological space. A pointed pair of space is a triple (X, A_1, A_2) where $A_2 \subseteq A_1 \subseteq X$ are subspaces of X. A map between triplets of spaces $f : (X, A_1, A_2) \to (Y, B_1, B_2)$ is a map $f : X \to Y$ such that $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$.

If $A_2 = \{x_0\}$ is a single point we say that (X, A, x_0) is a pointed pair of spaces.

Definition 1.3.2: Homotopy between Maps of Triplets

Let $f, g : (X, A_1, A_2) \to (Y, B_1, B_2)$ be maps triplets of spaces. A homotopy between f and g is a homotopy between $f : X \to Y$ and $g : X \to Y$, namely $H : X \times [0, 1] \to Y$ such that $H(A_1 \times [0, 1]) \subseteq B_1$ and $H(A_2 \times [0, 1]) \subseteq B_2$.

For a pointed pair of spaces (X, A, x_0), the inclusion $\iota : (A, x_0) \to (X, x_0)$ induces a map on homotopy

$$\pi_n(\iota) = \pi_n(A, x_0) \to \pi_n(X, x_0)$$

which is in general not injective. For $[\alpha] \in \pi_n(A, x_0)$ to lie in the kernel, it must satisfy that for any map $f : (I, \partial I^n) \to (A, x_0)$ representing $[\alpha]$, $\iota \circ f$ is homotopic to the constant map c_{x_0} on x_0. Such a homotopy is a map $H : I^n \times I \to X$ satisfying the following conditions:

- $H(-, 1) = f$
- $H(-, 0) = c_{x_0}$
- $H|_{\partial I^n \times I} = c_{x_0}$

Thus if we denote

$$J^n = I^n \times \{0\} \cup \partial I^n \times I$$

which is a subspace of the boundary ∂I^{n+1}, such a homotopy H is a map of triplets of spaces

$$H : (I^{n+1}, \partial I^n, J^n) \to (X, A, x_0)$$

Definition 1.3.3: The nth Relative Homotopy Groups

Let (X, A, x_0) be a pointed pair of space. Define the relative homotopy groups of the triple by

$$\pi_n(X, A, x_0) = \frac{\left\{ f : (I^n, \partial I^n, J^{n-1}) \to (X, A, \{x_0\}) \;\middle|\; f \text{ is continuous} \right\}}{\simeq}$$

for $n \geq 2$, where $J^n = I^n \times \{0\} \cup \partial I^n \times I$ and we say that $f \simeq g$ if there exists a homotopy between f and g.

Theorem 1.3.4

Let (X,A,x_0) be a pointed pair of space. The composition law on definition 1.1.4 defines a group structure on $\pi_n(X,A,x_0)$ for $n\geq 2$.

Corollary 1.3.5

Let (X,A,x_0) be a pointed pair of space. For $n\geq 3$, $\pi_n(X,A,x_0)$ is abelian.

1.4 Induced Maps of Relative Homotopy Groups

Theorem 1.4.1

Let (X,A,x_0) and (Y,B,y_0) be pointed pairs of spaces and $f:(X,A,x_0)\to(Y,B,y_0)$ a map. Then f induces a map on the relative homotopy groups

$$f_*:\pi_n(X,A,x_0)\to\pi_n(Y,B,y_0)$$

for $n\geq 2$ satisfying the following functorial properties:

- f_* is a group homomorphism
- If $g:(Y,B,y_0)\to(Z,C,z_0)$ is a map, then
$$(g\circ f)_*=g_*\circ f_*$$
- If $\text{id}_{(X,A,x_0)}$ is the identity map on (X,A,x_0), then
$$(\text{id}_{(X,A,x_0)})_*=\text{id}_{\pi_n(X,A,x_0)}$$

Proposition 1.4.2

Let $(X,A,x_0),(Y,B,y_0)$ be pointed pairs of spaces and $f,g:(X,A,x_0)\to(Y,B,y_0)$ be pointed maps. If f and g are homotopic, then the induced maps

$$f_*=g_*:\pi_n(X,A,x_0)\to\pi_n(Y,B,y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then f_* is an isomorphism.

Proposition 1.4.3

The relative homotopy groups of (X,A,x_0) fit into a long exact sequence

$$\cdots\longrightarrow\pi_{n+1}(X,A,x_0)\xrightarrow{\partial_{n+1}}\pi_n(A,x_0)\xrightarrow{i_*}\pi_n(X,x_0)\xrightarrow{j_*}\pi_n(X,A,x_0)\xrightarrow{\partial_n}\pi_{n-1}(A,x_0)\longrightarrow\cdots\longrightarrow\pi_0(X,x_0)\longrightarrow 0$$

where ∂_n is defined by $[f]\mapsto[f|_{I^{n-1}}]$ and i_* and j_* are induced by inclusions.

Note that even though at the end of the sequence group structures are not defined, exactness still makes sense: kernels in this case consists of elements that map to the homotopy class of the constant map.

Theorem 1.4.4: The Hurewicz Homomorphism

Let (X,A,x_0) be a pointed pair of space. Let u_n be a generator of $H_n(S^n)\cong\mathbb{Z}$. Then the map

$$h:\pi_n(X,A,x_0)\to H_n(X,A)$$

defined by $[f] \mapsto f_*(u_n)$ is a group homomorphism.

1.5 n-Connectedness

Definition 1.5.1: n-Connected Space

We say that the pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for $i \leq n$ and X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.

1.6 Weakly Contractible Space

Definition 1.6.1: Weakly Contractible

Let X be a space. We say that X is weakly contractible if

$$\pi_n(X) = 0$$

for all $n \geq 0$.

2 Homotopy and CW-Complexes

2.1 Whitehead's Theorem

Definition 2.1.1: Weak Homotopy Equivalence

We say that a map $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups π_n on any choice of base point.

Theorem 2.1.2: Whitehead's Theorem

If X and Y are CW-complexes and $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

Corollary 2.1.3

If X and Y are CW-complexes with $\pi_1(X) = \pi_1(Y) = 0$ and $f : X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.

2.2 Cellular Approximations

Definition 2.2.1: Cellular Maps

Let X and Y be CW-complexes. A map $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n$ for all n, where X_n is the n-skeleton of X.

Definition 2.2.2: Cellular Approximations

Let X and Y be CW-complexes. We say that $f : X \to Y$ has a cellular approximations if f is homotopic to a cellular map $f' : X \to Y$.

Theorem 2.2.3: Cellular Approximation Theorem

Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subseteq X$, then we can take $f'|_A = f|_A$.

Theorem 2.2.4: Relative Cellular Approximation

Any map $f : (X, A) \to (Y, B)$ between pairs of CW-complexes has a cellular approximation.

Corollary 2.2.5

Let $A \subset X$ be CW-complexes and suppose that all cells $X \setminus A$ have dimension larger than n. Then $\pi_i(X, A) = 0$ for all $i \leq n$.

Corollary 2.2.6

If X is a CW-complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.

Corollary 2.2.7

Let X be a CW-complex. Then

$$\pi(X) \cong \pi(X_n)$$

for $i < n$.

2.3 CW Approximations

Definition 2.3.1: CW Approximation

A CW approximation of X is a weak homotopy equivalence $f : Z \to X$ where Z is a CW approximation.

Definition 2.3.2: CW Model

Let (X, A) be a non-empty pair of CW-complexes. An n-connected CW model of (X, A) is an n-connected CW pair (Z, A) together with a map $f : Z \to X$ with $f|_A = \text{id}_A$ such that

$$f_* : \pi_i(Z) \to \pi_i(X)$$

is an isomorphism for $i > n$ and an injection for $i = n$ for any choice of base point.

Theorem 2.3.3

For any non-empty pair (X, A) of CW-complexes, there exists an n-connected model (Z, A). Moreover, Z can be built from A by attaching cells of dimension greater than n.

Corollary 2.3.4

Every pair of CW-complex (X, A) has a CW approximation (Z, B).

Thus we have shown existence of CW approximations, it remains to show uniqueness.

Corollary 2.3.5

CW-approximations are unique up to homotopy equivalence.

3 Main Results of Homotopy Theory on CW-Complexes

3.1 Excision for Homotopy Groups

Theorem 3.1.1: The Homotopy Excision Theorem

Let X be a CW-complex decomposed as the union of subcomplexes A and B with non-empty connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected for $m, n \geq 0$, then the map

$$\iota_* : \pi_i(A, C) \to (X, B)$$

induced by the inclusion $\iota : (A, C) \to (X, B)$ is an isomorphism for $i < m+n$ and a surjection for $i = m + n$.

3.2 Freudenthal Suspension Theorem

Definition 3.2.1: Reduced Suspension

Let (X, x_0) be a pointed space. Define the reduced suspension of X to be the space

$$\Sigma X = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)}$$

The reduced suspension defines a continuous map sending a space X to its reduced suspension ΣX.

Theorem 3.2.2: Freudenthal Suspension Theorem

Let X be an $(n-1)$-connected CW complex. Then for $0 \leq k \leq 2n-2$, the induced map

$$\Sigma_* : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism. For $k = 2n-1$, Σ_* is a surjection.

Corollary 3.2.3

Let $n \geq 0$. Then there is an isomorphism

$$\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$$

Definition 3.2.4: Stable Homotopy Groups

Let X be a space. Let $n \in \mathbb{N}$. Define the nth stable homotopy groups of X to be

$$\pi_n^s(X) = \operatorname*{colim}_{k \to \infty} \pi_{n+k}(\Sigma^k X)$$

3.3 Hurewicz's Theorem

Theorem 3.3.1: Hurewicz's Theorem

Let X be a $(n-1)$-connected space and $n \geq 2$. Then $\widetilde{H}_i(X) = 0$ for all $i < n$ and $\pi_n(X) \cong H_n(X)$.

Moreover, if a pair (X, A) is $(n-1)$-connected with $n \geq 2$ and $\pi_1(A) = 0$, then $H_i(X, A) = 0$ for all $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$

3.4 Eilenberg-MacLane Spaces

Definition 3.4.1: Eilenberg-MacLane Space

Let G be a group and $n \in \mathbb{N}$. We say that a space X is an Eilenberg-MacLane space of type $K(G,n)$ if $\pi_n(X) = G$ and $\pi_k(X) = 0$ for all $k \neq n$.

Proposition 3.4.2

Let G be a group. Then there exists a $K(G,1)$-CW complex.

Theorem 3.4.3

Let G be an abelian group and $n \geq 2$. Then there exists a $K(G,n)$-CW complex.

Theorem 3.4.4

Let X be a space. Then there is a natural bijection

$$H^n(X;G) \cong [X, K(G,n)]$$

Theorem 3.4.5

Let X be a CW complex. Let G be an abelian group. Let $n \in \mathbb{N}$. Then there is a natural isomorphism

$$\widetilde{H}_k(X;G) \cong \operatorname*{colim}_{n\to\infty} \pi_{k+n}(X \wedge K(G,n))$$

4 Generalized Homology and Cohomology Theories

4.1 Homology Theories

Definition 4.1.1: The Category of Pairs of CW Complexes

Define the category of CW pairs $\mathbf{CW}^2$ to be the category where objects are pairs (X, A) of CW complexes where $A \subseteq X$, and the morphisms are maps of pairs.

Definition 4.1.2: Generalized Homology Theory for CW Pairs

A Generalized Homology Theory is a collection of functors and natural transformations

$$h_n : \mathbf{CW}^2 \to \mathbf{Ab} \quad \text{and} \quad \delta_n : h_n(X, Y) \to h_{n-1}(Y, \emptyset)$$

satisfying the following.

- Homotopy Invariance: If $f \simeq g : (X, A) \to (Y, B)$ then
$$h_n(f) = h_n(g) : h_n(X, A) \to h_n(Y, B)$$
- Exactness: There exists a short exact sequence
$$\cdots \longrightarrow h_{n+1}(X, A) \xrightarrow{\delta_{n+1}} h_n(A) \longrightarrow h_n(X) \longrightarrow h_n(X, A) \xrightarrow{\delta_n} h_{n-1}(A) \longrightarrow \cdots$$
- Additivity: If $(X, A) = \coprod_{i \in I}(X_i, A_i)$, then the direct sum of the inclusion maps
$$\bigoplus_{i \in I} h_n(X_i, A_i) \cong h_n(X, A)$$
is an isomorphism
- Excision: If $\overline{E} \subseteq A^\circ \subseteq X$, then
$$h_n(X \setminus E, A \setminus E) \cong h_n(X, A)$$
induced by the inclusion map

We mention for once and for all that the additivity axiom is required only when the CW complexes are finite. In particular, in order for the homology theory to be meaningful, we must restrict the underlying category of spaces to be finite CW complexes if one drops the additivity axiom.

Lemma 4.1.3

The excision axiom is equivalent to saying that $X = A^\circ \cup B^\circ$ with inclusion map $\iota : (B, A \cap B) \to (X, A)$ implies $h_n(\iota) : h_n(B, A \cap B) \to h_n(X, A)$ is an isomorphism.

Definition 4.1.4: The Category of Pairs of Spaces

Define the category of CW pairs $\mathbf{Top}^2$ to be the category where objects are pairs (X, Y) of spaces, and the morphisms are maps of pairs.

Definition 4.1.5: Generalized Homology Theory

A Generalized Homology Theory is a collection of functors

$$h_n : \mathbf{Top}^2 \to \mathbf{Ab} \quad \text{and} \quad \delta_n : h_n(X, Y) \to h_{n-1}(Y, \emptyset)$$

satisfying the firs four axioms together with the following.

- Weak Equivalence: If $f : (X, A) \to (Y, B)$ is a weak equivalence, then
$$f_* : h_n(X, A) \to h_n(Y, B)$$
is an isomorphism.

By adding on the axiom of weak equivalence and the fact that every space admits a weak equivalence to a CW complex, we can see that the two theories are the same. However, note that in this case some of the working homology theories are not a generalized homology theory in this sense (when we encounter the dual notion, sheaf cohomology is not a generalized cohomology theory).

Theorem 4.1.6

Any generalized homology theory on $\mathbf{Top}^2$ determines and is determined by a generalized homology theory on $\mathbf{CW}^2$.

Definition 4.1.7: Homology Theory

If a generalized homology theory (h_n, δ_n) in addition satisfies

- Dimension:
$$h_n(*) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then h_n is called a homology theory.

The same can be said for reduced homology theories. This means that any generalized homology theory determines and is determined by a reduced homology theory.

Theorem 4.1.8: Eilenberg-Steenrod Uniqueness Theorem

Let $T : (h_n, \delta_n) \to (h'_n, \delta'_n)$ be a natural transformation of generalized homology theories defined on $\mathbf{CW}^2$ such that $h_n(*) \cong h'_n(*)$, then T is a natural isomorphism
$$(h_n, \delta_n) \cong (h'_n, \delta'_n)$$

4.2 Cohomology Theories

Definition 4.2.1: Generalized Cohomology Theory for CW Pairs

A Generalized cohomology theory is a collection of contravariant functors
$$h^n : \mathbf{CW}_2 \to \mathbf{Ab} \quad \text{and} \quad \delta^n : h^n(A, \emptyset) \to h^{n+1}(X, A)$$
satisfying the following.

- Homotopy Invariance: If $f \simeq g : (X, A) \to (Y, B)$ then
$$h^n(f) = h^n(g) : h^n(X, A) \to h^n(Y, B)$$
- Exactness: There exists a short exact sequence
$$\cdots \longrightarrow h^n(X, A) \longrightarrow h^n(X) \longrightarrow h^n(A) \xrightarrow{\partial_n} h^{n+1}(X, A) \longrightarrow h^{n+1}(X) \longrightarrow \cdots$$
- Additivity: If $(X, A) = \coprod_{i \in I}(X_i, A_i)$, then the direct sum of the inclusion maps
$$\bigoplus_{i \in I} h^n(X_i, A_i) \cong h^n(X, A)$$

is an isomorphism

- Excision: If $\overline{E} \subseteq A^\circ \subseteq X$, then

$$h^n(X \setminus E, A \setminus E) \cong h^n(X, A)$$

induced by the inclusion map

Definition 4.2.2: Generalized Cohomology Theory

A Generalized cohomology theory is a collection of contravariant functors

$$h^n : \mathbf{Top_2} \to \mathbf{Ab} \qquad \text{and} \qquad \delta^n : h^n(A, \emptyset) \to h^{n+1}(X, A)$$

satisfying the above first four axioms and the following.

- Weak Equivalence: If $f : (X, A) \to (Y, B)$ is a weak equivalence, then

$$f_* : h^n(Y, B) \to h^n(X, A)$$

is an isomorphism.

5 The Fundamental Groupoid

5.1 The Fundamental Groupoid

Definition 5.1.1: The Fundamental Groupoid

Let X be a space. Define the fundamental groupoid $\Pi_1 X$ of X to be the category with the following data.

- The objects are the points of X.
- Let $x, y \in X$. The morphisms of $\Pi_1 X$ are given by

$$\text{Hom}_{\Pi_1 X}(x, y) = \{\gamma : I \to X \mid \gamma(0) = x \text{ and } \gamma(1) = y \text{ is a path } \}/\sim$$

  where we say that two paths are equivalent if they are homotopic.
- Composition is defined by the concatenation of paths.

We have seen in Algebraic Topology 1 that composition of homotopy classes of paths are well defined.

Lemma 5.1.2

Let X be a space. Then $\Pi_1 X$ is a groupoid.

Proof. Every path in X has an inverse that lies in $\Pi_1 X$ given by reversing traversal of the path. □

Lemma 5.1.3

Let X be a space and $x_0 \in X$. Then there is a group isomorphism

$$\text{Hom}_{\Pi_1 X}(x_0, x_0) \cong \pi_1(X, x_0)$$

Proposition 5.1.4

Let $f : X \to Y$ be a continuous map. Then f induces a functor $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$ defined by

$$\Pi_1 f([\alpha]) = [f \circ \alpha]$$

on morphisms.

Proof. Direct from Algebraic Topology 1 due to the above group isomorphism. We have also seen that it is functorial in Algebraic Topology 1. □

Theorem 5.1.5

The fundamental groupoid defines a functor $\Pi_1 : \textbf{Top} \to \textbf{Grps}\}$ from the category of spaces to the category of groupoids with the following data.

- Π_1 sends each space X to $\Pi_1 X$
- Π_1 sends each continuous map $f : X \to Y$ to the functor $\Pi_1 f$

5.2 The Seifert-Van Kampen Theorem on Fundamental Groupoids

Definition 5.2.1: The Fundamental Groupoid of Subspaces

Let X be a space and $A \subseteq X$ a subspace. Define $\Pi_1 X[A]$ to be the full subcategory of $\Pi_1 X$ where the objects are A. Explicitly, $\Pi_1 X[A]$ consists of the following data.

- The objects of $\Pi_1 X[A]$ are the points of A.
- The morphisms are given by
$$\text{Hom}_{\Pi_1 X[A]}(x, y) = \text{Hom}_{\Pi_1 X}(x, y)$$
for any $x, y \in X$.
- Composition is inherited from $\Pi_1 X$.

Lemma 5.2.2

Let X be a space and $A \subseteq X$ a subspace of X such that every path component of X contains a point of A. Then the inclusion
$$\Pi_1 X[A] \to \Pi_1 X$$
of groupoids is an equivalence of categories.

Proof. The inclusion is already fully faithful since $\Pi_1 X[A]$ is a full subcategory. Now let $x \in X$. Let $a \in A$ lie in the same path component as x. Let $\alpha : I \to X$ be a path from x to a. Then the morphism $[\alpha] : x \to a$ of $\Pi_1 X$ is an isomorphism since $\Pi_1 X$ is a groupoid. Thus we conclude. □

Corollary 5.2.3

Let X be a space. Then there is an equivalence of categories
$$\coprod_{[x_0] \in \pi_0(X)} B\pi_0(X, x_0) \cong \Pi_1 X$$

Proof. This is done by choosing A to contain exactly one point of each path component, and then by applying the isomorphism
$$\Pi_1 X[x_0] = B\text{Aut}_{\Pi_1 X}(x_0) = B\pi_1(X, x_0)$$
and the above lemma. □

If X is path connected, then this shows that any choice of base point $x_0 \in X$ gives an equivalence of categories
$$B\pi_0(X, x_0) \cong \Pi_1 X$$
This translates roughly to the standard fact in Algebraic Topology that the fundamental group of a path connected space for any two base points are isomorphic. Indeed in the equivalence of categories exhibited, the former depends on the base point while the latter does not.

We need a lemma.

Lemma 5.2.4

Let $\mathcal{J}$ be the following category.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "i"] \\ 2 \arrow[r, "j"'] & 3 \end{tikzcd}$$

Let $X, Y : \mathcal{J} \to \mathcal{C}$ be two squares and let $s : X \Rightarrow Y$ and $p : Y \Rightarrow X$ be natural transformations such that $p \circ s = \text{id}_X$. Then if Y is a pushout square, so is X.

Proof. Consider the following diagram:

$$\begin{tikzcd} \text{Hom}_{\mathcal{C}}(X_3, Z) \arrow[r, "\varphi_x"'] \arrow[d, shift left, "p_3^*"] & \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(X, \Delta Z) \arrow[d, shift left, "p^*"] \\ \text{Hom}_{\mathcal{C}}(Y_3, Z) \arrow[r, "\varphi_y, \cong"'] \arrow[u, shift left, "s_3^*"] & \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(Y, \Delta Z) \arrow[u, shift left, "s^*"] \end{tikzcd}$$

It is easy to see that $p^*\varphi_x = \varphi_y p_3^*$ and $\varphi_x s_3^* = s^*\varphi_y$. Since $s^* \circ p^* = (p \circ s)^* = \text{id}$, we have that p^* and p_3^* are injective and s^* and s_3^* are surjective. Since φ_y is bijective, $p^*\varphi_x = \varphi_y p_3^*$ implies that φ_x is injective. Also the bijection of φ_y and $\varphi_x s_3^* = s^*\varphi_y$ implies that φ_x is surjective. $\square$

Theorem 5.2.5: The Seifert-Van Kampen Theorem on Fundamental Groupoids

Let X be a space and $U, V \subseteq X$ an open cover of X. Let $A \subseteq X$ be a subspace such that every path connected component of U, V, X contains a point in A. Then the inclusions

$$\Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 U[U \cap A] \quad \text{and} \quad \Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 V[V \cap A]$$

give a pushout diagram to $\Pi_1 X[A]$. This means that the following diagram is a pushout:

$$\begin{tikzcd} \Pi_1(U \cap V)[U \cap V \cap A] \arrow[r] \arrow[d] & \Pi_1 U[U \cap A] \arrow[d] \\ \Pi_1 V[V \cap A] \arrow[r] & \Pi_1 X[A] \end{tikzcd}$$

where each arrow is an inclusions.

Proof. First assume that $X = A$. We want to show that for any groupoid $\mathcal{G} \in \mathbf{Grp}$ with maps $\Pi_1 U, \Pi_1 V \to \mathcal{G}$, there exists a unique map $\Pi_1 X \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \Pi_1(U \cap V) \arrow[r] \arrow[d] & \Pi_1 U \arrow[d] \arrow[ddr, bend left, "f"] & \\ \Pi_1 V \arrow[r] \arrow[drr, bend right, "g"'] & \Pi_1 X \arrow[dr, dashed, "\exists! u"] & \\ & & \mathcal{G} \end{tikzcd}$$

Define the functor $u : \Pi_1 X \to \mathcal{G}$ as follows. For each $x \in \Pi_1 X$, define

$$u(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$

This is well defined on $U \cap V$ since the outer square of the above diagram commutes. Depending on the path in X, there will be different constructions. Let $[\alpha]$ be a morphism in

$\Pi_1 X$. If $\alpha : I \to X$ has image in U, then define $u([\alpha]) = f([\alpha])$. Similarly, define $u([\alpha]) = g([\alpha])$ if α has image in V.

Otherwise, by the Lebesgue covering theorem, there is a finite sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\alpha([a_i, a_{i+1}]) \subseteq U$ or V. Define $\alpha_i = \alpha \mid_{a_i, a_{i+1}}$. It is easy to see that

$$\begin{aligned} [\alpha] &= [\alpha \mid_{0,a_1}] \cdot [\alpha \mid_{a_1,a_2}] \cdots [\alpha \mid_{a_{n-1},1}] && \text{(Viewed as paths)} \\ &= [\alpha_{n-1}] \circ \cdots [\alpha_1] \circ [\alpha_0] && \text{(Viewed as morphisms in } \Pi_1 X) \end{aligned}$$

Then we can define $u(\alpha)$ as

$$u([\alpha]) = u([\alpha_{n-1}]) \circ u(\cdots [\alpha_1]) \circ u([\alpha_0])$$

where we have that

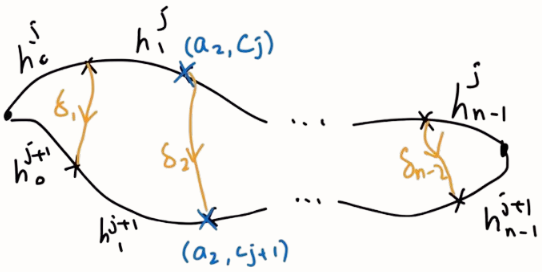
$$u([\alpha_i]) = \begin{cases} f([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq U \\ g([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq V \end{cases}$$

If u exists, then u must take the above form. Thus we have shown uniqueness.

For existence, we have to show that above construction of u is well defined. Let α, β be paths in X from x to y that are homotopic via the map $H : I \times I \to X$. We want to show that $u([\alpha]) = u([\beta])$. By the Lebesgue covering theorem, there is a grid in $I \times I$ where the x-axis is subdivided into $0 = a_0 < a_1 < \cdots < a_n = 1$ and the y-axis is subdivided into $0 = c_0 < c_1 < \cdots < c_k = 1$. such that H sends each rectangle with vertices $\{a_i, a_{i+1}, c_j, c_{j+1}\}$ to either U or V. Let $h^j = H(-, c_j) : I \to X$ so that $h^0 = \alpha$ and $h^k = \beta$. Define

$$\delta_i = H(\alpha_i, -) \mid_{[c_j, c_{j+1}]} : I \to X$$

which are paths from (a_i, c_j) to (a_i, c_{j+1}) in $I \times I$. Also define $h_i^j = h^j \mid_{[\alpha_i, \alpha_{i+1}]}$. Now we have the following which lies entirely in X:



Now we have that

$$\begin{aligned} u([h^j]) &= u([h^j_{n-1}]) \circ \cdots \circ u([h^j_0]) \\ &= u([h^{j+1}_{n-1} \circ \delta_{n-2}]) \circ u([\overline{\delta_{n-2}} \circ h^{j+1}_{n-2} \circ \delta_{n-3}]) \circ \cdots \circ u([\overline{\delta_1} \circ h_0^{j+1}]) \\ &= u([h^{j+1}_{n-1}]) \circ \cdots \circ u([h^{j+1}_0]) \\ &= u([h^{j+1}]) \end{aligned}$$

By induction, we conclude that

$$u([\alpha]) = u([h^0]) = u([h^1]) = \cdots = u([h^k]) = u([\beta])$$

Now suppose that $A \subseteq X$. By the above lemma, it is sufficient to show that the square for A is a retract of the square for X. Let $x \in U \cap V$ and $a_x \in A \cap U \cap V$ lying in the same path component as x. Choose a path $\alpha_x : I \to X$ from a_x to x with α_x being constant if $x \in A$.

Do a similar choice for $x \in U \setminus (U \cap V)$ and $x \in V \setminus (U \cap V)$. Define $p_{U \cap V} : \Pi_1(U \cap V) \to \Pi_1(U \cap V)[U \cap V \cap A]$ defined by $x \mapsto a_x$ on objects and

$$[x \overset{\alpha}{\to} y] \mapsto \left(a_x \overset{[\alpha_x]}{\to} x \overset{[\alpha]}{\to} y \overset{[\alpha_y]}{\to} a_y \right)$$

and similarly for p_U and p_V. This defines the natural transformation p in lemma 5.3.4. We conclude by lemma 5.3.4. $\square$

Take $A = \{x_0\}$ be a single point in $U \cap V$. Then this theorem shows that there is a pushout diagram

$$\begin{tikzcd}
\pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\
\pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

in **Grp**, provided that A contains every path connected component of U, V, X. But A is just one point so the condition becomes that U, V, X and $U \cap V$ being path connected. Hence we recover the usual Seifert-Van Kampen theorem in Algebraic Topology 1.