Topological Groups

Labix

June 25, 2024

Abstract

Contents

1 Topological Groups and Actions

1.1 Basic Definitions

Definition 1.1.1: Topological Groups

Let G be a group. We say that G is a topological group if G is also a topological space and that the following are true.

- The map $l_h : G \to G$ defined by $g \mapsto hg$ is continuous for all $h \in G$
- The map $i : G \to G$ defined by $g \mapsto g^{-1}$ is continuous

1.2 Continuous Group Actions

In algebraic topology, we have the results of considering groups acting on spaces. We can in fact consider topological groups acting on spaces.

Definition 1.2.1: Continuous Group Actions

Let G be a topological group and X a space. We say that G is a continuous group action if G is a group acting on X such that the group action map

$$\cdot : G \times X \to X$$

is continuous.

Frequently a continuous group action is also called a (topological) transformation group, for example in Milnor's Topology of Fiber Bundles.

Proposition 1.2.2

Let G be a continuous group action of X. Then for each $g \in G$, the map $\mathrm{A}_g : X \to X$ defined by $x \mapsto g \cdot x$ is a homeomorphism.

Proof. Every element of g has an inverse g^{-1} which are both continuous and are bijections on X. □

Proposition 1.2.3

Let G be a topological group and $(X, \mathcal{T})$ a topological space. Then G is a continuous group action on X if and only if G acts on $\mathcal{T}$.

Proof. Suppose that G is a continuous group action on X. Then for each $g \in G$, $g \cdot U = \{g \cdot x \mid x \in U\}$ for $U \in \mathcal{T}$ is open since A_g as above is a homeomorphism. Now suppose that G acts on $\mathcal{T}$. Then for each open set U of X, $g^{-1} \cdot U$ is open. Thus G is a continuous group action. □

In particular, some authors would assume one knows this fact, so it is always nice to see it spelled out. It is also standard to denote this action just by the element g instead of A_g. Notice that in particular, if G is a continuous group action, then there is a homomorphism $G \to \text{Homeo}(X)$. If this homomorphism is injective, then G includes into $\text{Homeo}(X)$ so that G is a subgroup of homeomophisms.

Definition 1.2.4: Group of Diffeomorphisms

Let G be a continuous group action on a space X. We say that G is a group of homeomorphisms of X if for every $g \in G$, the map

$$x \mapsto g \cdot x$$

is a homeomorphism.

1.3 Properly Discontinuous Group Actions

Definition 1.3.1: Proper Group Actions

Let G be a topological group acting continuously on a topological space X. The action is said to be proper if the map $G \times X \to X \times X$ defined by

$$(g, x) \mapsto (x, g \cdot x)$$

is a proper map.

Definition 1.3.2: Properly Discontinuous Group Actions

Let G be a group acting on a space X. Then we say that G is a properly discontinuous group action if for every compact set $K \subseteq X$, we have

$$(g \cdot K) \cap K \neq \emptyset$$

for finitely many $g \in G$.

Proposition 1.3.3

Every properly discontinuous group action is a wandering action.

Proposition 1.3.4

If G is a proper group action on a space X, then the action is properly discontinuous.

The converse is not true in general, unless we assume that X is locally compact.

Recall the notion of a covering space action. G is a covering space action on X if $g \cdot U \cap U \neq \emptyset$ implies $g = 1$. This is also related to properly discontinuous group actions. In fact, properly discontinuous group actions are in general stronger than covering space actions.

Proposition 1.3.5

Let G be a covering space action on X. If X is locally compact and Hausdorff, then G is a properly discontinuous group action on X.

2 The Coset Space

2.1 The Topology of Coset Spaces

Definition 2.1.1: Coset Space

Let B be a topological group and G a closed subgroup of B. The coset space of B by G is the set
$$B/G = \{bG \mid b \in B\}$$
together with the topology in which $U \subseteq B/G$ is open $p^{-1}(U)$ is open, where $p : B \to B/G$ is the quotient homomorphism.

Note that there is also a definition of the coset space by right cosets instead of left. However it is easy to show that they are homeomorphic through the inverse map $b \mapsto b^{-1}$ for each $b \in B$.

Proposition 2.1.2

Let B be a topological group and G a closed subgroup of B. Then the quotient map $p : B \to B/G$ is an open map.

Proposition 2.1.3

Let B be a topological group and G a closed subgroup of B. Then B/G is a Hausdorff space.

2.2 Coset Spaces of Transitive Actions

Proposition 2.2.1

Let G be a topological group acting continuously on a space X. Let $x_0 \in X$. Then the map
$$p : \frac{G}{\text{Stab}_G(x_0)} \to Gx_0 \subseteq X$$
given by $g \mapsto g \cdot x_0$ is well defined, and moreover, a homeomorphism.

Corollary 2.2.2

Let G be a topological group acting continuously on a space X. Then there exists a normal subgroup H of G such that
$$\frac{G}{H} \cong X$$

2.3 Coset Spaces with the Group Action of the Base Space

Definition 2.3.1: The Translation Map

Let B be a topological group and G a closed subgroup of B. Let $b \in B$ and suppose that $p : B \to B/G$ is the quotient map. Define the translation map $B \times B/G \to B/G$ defined by
$$(b, x) \mapsto p(bp^{-1}(x))$$

Proposition 2.3.2

Let B be a topological group and G a closed subgroup of B. Then the translation map is a continuous group action of B on B/G. Moreover, B is a group of homeomorphisms of B/G.

Proposition 2.3.3

Let B be a topological group and G a closed subgroup of B. Let

$$G_0 = \bigcap_{b\in B} bGb^{-1}$$

Then B/G_0 acts faithfully on B/G. Moreover, B/G_0 is a group acting continuously on B/G.