Algebraic Differential Forms

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Motivation

Let M be a smooth manifold.

For each point $p \in M$, we have the cotangent space $T_p^* M$.

They organize into a vector bundle $T^* M$.

Smooth sections of $T^* M$ are called smooth differential 1-forms. Its collection organizes into a $\mathcal{C}^\infty(M)$-module denoted by $\Omega^1(M)$.

$$0 \longrightarrow \mathcal{C}^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots$$

We would like to have a similar notion for varieties in algebraic geometry.

Definitions

Derivations

Let A be a ring and B an A-algebra. Let M be a B-module. An A-derivation of B into M is an A-module homomorphism $d : B \to M$ such that the Leibniz rule holds:

$$d(b_1 b_2) = b_1 d(b_2) + d(b_1) b_2$$

for $b_1, b_2 \in B$.

Denote the set of all A-derivations from B to M by

$$\mathrm{Der}_A(B, M) = \{d : B \to M \mid d \text{ is an } A \text{ derivation } \}$$

Examples

Let M be a smooth manifold. Then

$$T_p(M) = \mathrm{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{M,p}, \mathbb{R})$$

where $\mathcal{C}^\infty_{M,p}$ is the germ of smooth functions at p.

Definitions

Kähler Differentials

Let A be a ring and let B be an A-algebra. A B-module $\Omega^1_{B/A}$ together with an A-derivation $d : B \to \Omega^1_{B/A}$ is said to be a module Kähler Differentials of B over A if it satisfies the following universal property:

For any B-module M, and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega^1_{B/A} \to M$ such that $d' = f \circ d$. In other words, the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega^1_{B/A} \arrow[d, dashed, "\exists! f"] \\ & M \end{tikzcd}$$

Constructions

Kähler Differentials as the Quotient of a Free Module

Let A be a ring and B be an A-algebra. Let F be the free B-module generated by the symbols $\{d(b) \mid b \in B\}$. Let R be the submodule of F generated by the following relations:

- $d(a_1 b_1 + a_2 b_2) - a_1 d(b_1) - a_2 d(b_2)$ for all $b_1, b_2 \in B$ and $a_1, a_2 \in A$
- $d(b_1 b_2) - b_1 d(b_2) - b_2 d(b_1)$ for all $b_1, b_2 \in B$

Then F/R is a module of Kähler Differentials for B over A.

Kähler Differentials as a Kernel

Let A be a ring and B be an A-algebra. Let $f : B \otimes_A B \to B$ be a function defined to be $f(b_1 \otimes_A b_2) = b_1 b_2$. Let I be the kernel of f. Then $(I/I^2, d)$ is a module of Kähler Differentials of B over A, where the derivation is the homomorphism $d : B \to I/I^2$ defined by $db = 1 \otimes b - b \otimes 1 \pmod{I^2}$.

Two Exact Sequences

First Exact Sequence

Let B, C be A-algebras and let $\phi : B \to C$ be an A-algebra homomorphism. Then the following sequence is an exact sequence of C-modules:

$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0$$

Second Exact Sequence

Let A be a ring and B an A-algebra. Let I be an ideal of B and $C = B/I$. Then the following sequence is an exact sequence of C-modules:

$$I/I^2 \longrightarrow \Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow 0$$

Some properties of the module

Commutes with localization

Let B be an algebra over A. Let S be a multiplicative subset of B. Then

$$S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$$

Computing using the Jacobian

Let A be a field. Let $C = \frac{A[x_1,\dots,x_n]}{(f_1,\dots,f_r)}$. Let J be the Jacobian of $F = (f_1, \dots, f_r)$. Then

$$\Omega^1_{C/A} \cong \mathrm{coker}(J)$$

Computing some examples

Examples

Let A be a ring and $B = A[x_1, \ldots, x_n]$ so that B is an A-algebra. Then

$$\Omega^1_{B/A} \cong \bigoplus_{i=1}^{n} Bd(x_i)$$

In particular, the module $\Omega^1_{B/A}$ is a finitely generated B-module.

Examples

Let $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ be the vanishing locus of the cuspidal cubic. Then

$$\Omega^1_{\mathbb{C}[V]/\mathbb{C}} \cong \frac{\mathbb{C}[V]dx \oplus \mathbb{C}[V]dy}{((-3x^2)dx \oplus (2y)dy)}$$

Computing some examples

Examples

Let $W = \mathbb{V}(4x^2 + 9y^2 - 36) \subseteq \mathbb{A}^2_{\mathbb{C}}$ be the vanishing locus of an ellipse. Then

$$\Omega^1_{(\mathbb{C}[W])/\mathbb{C}} \cong \frac{\mathbb{C}[W]dx \oplus \mathbb{C}[W]dy}{((8x)dx \oplus (18y)dy)}$$

Examples

Let $U = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{A}^3_{\mathbb{C}}$ be the vanishing locus of the double cone. Then

$$\Omega^1_{\mathbb{C}[U]/\mathbb{C}} \cong \frac{\mathbb{C}[U]dx \oplus \mathbb{C}[U]dy \oplus \mathbb{C}[U]dz}{(2xdx \oplus 2ydy \oplus -2zdz)}$$

Cotangent space from the module

Recovering the Cotangent Space

Let (B, m) be a local ring which contains a field K that is isomorphic to B/m the residue field. Then the second exact sequence induces a vector space isomorphism

$$\frac{m}{m^2} \cong \Omega^1_{B/K} \otimes_B K$$

In particular, if (B, m) is the local ring of a variety at a point, the module is just the cotangent space, up to a change of base ring to the residue field.

Computing some examples

Recall that

$$\Omega^1_{\mathbb{C}[V]/\mathbb{C}} \cong \frac{\mathbb{C}[V]dx \oplus \mathbb{C}[V]dy}{(-3x^2dx, 2ydy)}$$

When $p = (p_1, p_2) \neq (0,0)$, $x, y \notin m_p$ and so are in invertible in the localization. Thus within this localization, we can write the in the quotient as $dy = \frac{3x^2}{2y}dx$. And so we are left with

$$\left(\Omega^1_{\mathbb{C}[V]/\mathbb{C}}\right)_{m_p} \cong \mathbb{C}[V]_{m_p}dx$$

Clearly this is a free $\mathbb{C}[V]_{m_p}$-module of rank 1. Then

$$\frac{m_p}{m_p^2} \cong \left(\Omega^1_{\mathbb{C}[V]_{m_p}/\mathbb{C}}\right) \otimes_{\mathbb{C}[V]_{m_p}} \frac{\mathbb{C}[V]_{m_p}}{m_p} \cong \mathbb{C}dx$$

which shows that $\frac{m_p}{m_p^2}$ is a 1-dimensional vector space over $\mathbb{C}$.

Computing some examples

Consider the point $(0,0)$. There is a surjection $\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) \to \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dx \oplus \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)}dy$ with kernel precisely

$$\left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) x \oplus \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) y$$

Then

$$\frac{m_{(0,0)}}{m^2_{(0,0)}} \cong \left(\Omega^1_{\mathbb{C}[V]_{(x,y)}/\mathbb{C}}\right) \otimes_{\mathbb{C}[V]_{(x,y)}} \frac{\mathbb{C}[V]_{(x,y)}}{(x,y)} \cong \mathbb{C}dx \oplus \mathbb{C}dy$$

which shows that $\frac{m_{(0,0)}}{m^2_{(0,0)}}$ is a vector space of dimension 2 over $\mathbb{C}$.

Two notions of differentials

From a differential geometry perspective, we may ask whether $\Omega^1_{C^\infty(M)/\mathbb{R}}$ and $\Omega^1(M)$ are the same thing.

The Two Modules Are Not Isomorphic In General

Consider $\mathbb{R}$ as a smooth manifold. Then $\Omega^1(\mathbb{R})$ is not isomorphic to $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$. In particular, for $f(x) = x$ and $g(x) = e^x$, $d(e^x) = e^x d(x)$ in $\Omega^1(\mathbb{R})$ but $d(e^x)$ and $d(x)$ are linearly independent in $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$.

The Leibniz rule and linearity of d can only be applied finitely many times. For $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ there is no reason for its derivative to be the same as its term by term derivative.