Geometric Group Theory

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Abstract

Potentially good books: Humphreys, Erdmann and Wildson

Contents

1 The Geometry of Presentations

1.1 The Cayley Graph of a Group

Definition 1.1.1: The Cayley Graph of a Group

Let G be a group. Let S be a generating set of G. Define the Cayley graph $\text{Cay}(G, S)$ of G with respect to S to consist of the following data.
- The vertices are given by $V(\text{Cay}(G, S)) = G$
- The edges are given by $E(\text{Cay}(G, S)) = \{(g, gs) \mid g \in G, s \in S\}$

Let (V, E) be a graph. Recall that a graph automorphism consists of a bijective map of vertices and a bjective map of edges such that
$$\{\phi(v), \phi(w)\} \in E$$
for all $\{v, w\} \in E$. They form a group by composition.

Lemma 1.1.2: The Action Lemma

Let G be a group. Let S be a generating set of G. Then G acts on the Cayley graph $\text{Cay}(G, S)$ of G with respect to S via tha map
$$\cdot : G \times \text{Cay}(G, S) \to \text{Cay}(G, S)$$
defined by $h \cdot g = hg$ and $h \cdot (g, gs) = (hg, hgs)$. Moreover, the action is faithful.

Proposition 1.1.3

Let G be a group. Let S be a generating set of G. Then the following are true regarding $\text{Cay}(G, S)$.
- $\text{Cay}(G, S)$ has no embedded cycles.
- $\text{Cay}(G, S)$ is connected.

Proposition 1.1.4

Let S be a set. Then $\text{Cay}(F_S, S)$ is a tree.

Proposition 1.1.5

Let G be a group. Let S be a generating set of G. Then $\text{Cay}(F_S, S)$ is a universal cover of $\text{Cay}(G, S)$.

1.2 Giving the Cayley Graph a Metric

Given a graph Γ, there are two ways to specify a path in Γ.

- We can define a path by a sequence $\gamma_V : [n] \to V(\Gamma)$ of adjacent vertices.
- We can also define a path by a sequence $\gamma_E : [n-1] \to E(\Gamma)$ of edges.

The above notation also indicates that any path is determined by either n vertices or $n-1$ edges.

Definition 1.2.1: The Word Metric

Let G be a group. Let S be a generating set of G. Define the word metric on $\text{Cay}(G, S)$ to be the map
$$d_S : V(\text{Cay}(G, S)) \times V(\text{Cay}(G, S)) \to \mathbb{N}$$

given by

$$d_S(g,h) = \min\{n \in \mathbb{N} \mid \gamma_V : [n] \to V(\text{Cay}(G,S)) \text{ is a path from } g \text{ to } h\}$$

Lemma 1.2.2

Let G be a group. Let S be a generating set of G. Then d_S is a metric on $\text{Cay}(G,S)$.

Proposition 1.2.3

Let G be a group. Let S be a generating set of G. Let $g \in G$ be fixed. Then the map

$$(h,k) \mapsto (gh, gk)$$

given by the action lemma is an isometry. In other words,

$$d_S(h,k) = d_S(gh, gk)$$

Let X be a metric space with two metrics d_1 and d_2. Recall that d_1 and d_2 are bilipschitz equivalent if there exists two constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$$

for all $x, y \in X$.

Lemma 1.2.4

Let G be a group. Let S, T be generating sets of G. Then d_S and d_T are bilipschitz equivalent.

Definition 1.2.5: The Word Norm

Let G be a group. Let S be a generating set of G. Let $\text{Cay}(G,S)$ be the Cayley complex of G and S. Define the word norm of $g \in G$ to be

$$\|g\|_S = d_S(1_G, g)$$

Lemma 1.2.6

Let G be a group. Let S be a generating set of G. Then the following are true.

- $d_S(g,h) = \|g^{-1}h\|_S$ for all $g, h \in G$.
- $\|g^{-1}\|_S = \|g\|_S$ for all $g \in G$.
- $\|gh\|_S \leq \|g\|_S + \|h\|_S$ for all $g, h \in G$.

1.3 Realizing the Cayley Graph as a Connected Space

We have proved that Cayley graphs are connected as graphs, in the sense that any two vertices are connected by a path. But a priori the graph is not connected as a topological space, whose topology is generated by the metric.

Definition 1.3.1: Geometric Realization of Cayley Graphs

Let G be a group. Let S be a generating set of G. Define the geometric realization $|\text{Cay}(G,S)|$ of the Cayley graph to be the space

$$|\text{Cay}(G,S)| = \frac{E(\text{Cay}(G,S)) \times I}{\sim}$$

where $((g_1, g_1 s_1), t_1) \sim ((g_2, g_2 s_2), t_2)$ if one of the following are true.

- They describe the same vertex (with different representations of elements in G): $((g_1, g_1 s_1), t_1) = ((g_2, g_2 s_2), t_2)$.
- They describe the same vertex but they lie on different edges: Either one of the following
  - $g_1 = g_2$ and $t_1 = t_2 = 0$
  - $g_1 = g_2 s_2$ and $t_1 = 0$, $t_2 = 1$
  - $g_1 s_1 = g_2 s_2$ and $t_1 = t_2 = 1$
  - $g_1 s_1 = g_2$ and $t_1 = 1$, $t_2 = 0$
- They describe the same point on an edge but different orientations: $(g_1, g_1 s_1) = (g_2, g_2 s_2^{-1})$ and $t_1 = 1 - t_2$.

In particular, this gives a 1-dimensional CW complex.

We can also give a metric on the realization so that its restriction to the actual Cayley graph recovers the word metric.

Definition 1.3.2: Metric on realization

Let G be a group. Let S be a generating set of G. Define a metric $d : |\text{Cay}(G, S)| \times |\text{Cay}(G, S)| \to \mathbb{R}$ as follows.

$$d([((g_1, g_1 s_1) t_1)], [((g_2, g_2 s_2), t_2)]) = \begin{cases} |t_1 - t_2| & \text{if } (g_1, g_1 s_1) = (g_2, g_2 s_2) \\ |t_1 - (1 - t_2)| & \text{if } (g_1, g_1 s_1) = (g_2 s_2, g_2) \\ \min \left\{ \begin{matrix} t_1 + d_S(g_1, g_2) + t_2 \\ t_1 + d_S(g_1, g_2 s_2) + 1 - t_2 \\ 1 - t_1 + d_S(g_1 s_1, g_2) + t_2 \\ 1 - t_1 + d_S(g_1 s_1, g_2 s_2) + 1 - t_2 \end{matrix} \right\} & \text{otherwise} \end{cases}$$

We abuse notation sometimes and freely interchange the use of the Cayley graph and its geometric realization when the context is clear.

1.4 Geodesics on Cayley Graphs

Definition 1.4.1: Geodesic Words

Let G be a group. Let S be a generating set. Let $\gamma_V : [n] \to V(\text{Cay}(G, S))$ be a path in $\text{Cay}(G, S)$. We say that γ_V is a geodesic word if

$$d_S(\gamma_V(0), \gamma_V(n)) = n$$

This is not the same definition as geodesics in metric spaces. (It doesn't make sense to talk about paths in $\text{Cay}(G, S)$ because it is a discrete topological space when we consider the topology generated by the metric).

Proposition 1.4.2

Let G be a group. Let S be a generating set of G. Then $\gamma_V : [n] \to V(\text{Cay}(G, S))$ is a geodesic word if and only if $|\gamma_V| : [0, n] \to |\text{Cay}(G, S)|$ is a geodesic in the sense of metric spaces.

Lemma 1.4.3

Let G be a group. Let S be a generating set. If $\gamma_V : [n] \to \text{Cay}(G, S)$ is a geodesic, then $\gamma_V(0) * \cdots * \gamma_V(n)$ is a reduced word.

Note: The converse is not true. Consider $G = \langle a, b \rangle a^3 = b^2$. Both a^3 and b^2 are reduced words but they have different lengths.

Note: geodesics are not the unique distance minimizing curve between two elements. Therefore we want to find a representative.

Definition 1.4.4: Short Lex Ordering

Let G be a group. Let S be a finite generating set of G. Let $u, v \in F(S)$. We say that

$$u <_{sl} v$$

if one of the following are true.

- $|u| < |v|$
- $|u| = |v|$ and there exists w such that $u = w * u'$, $v = w * v'$ and $u' <_{sl} v'$.

We call $<_{sl}$ the short lex ordering on $F(S)$.

Lemma 1.4.5

Let G be a group. Let S be a generating set. Then $<_{sl}$ is a total order on $F(S)$.

Definition 1.4.6: Short Lex Representative

Let G be a group. Let S be a generating set of G. Let $g \in G$. Define the short lex representative of g with respect to S to be

$$\min_{<_{sl}} \{s \in F(S) \mid s = g \text{ in G}\}$$

Lemma 1.4.7

Let G be a group. Let S be a generating set of G. Any subword of a short lex representative with respect to S is a short lex representative.

Corollary 1.4.8

Let G be a group. Let S be a generating set of G. Then the set of paths in Cay(G, S) consisting of short lex representatives form a spanning tree for Cay(G, S).

1.5 Growth Function

Definition 1.5.1: Ball Around an Element

Let G be a group. Let S be a finite generating set of G. Let $R > 0$. Define the ball around $g \in G$ with radius n to be

$$B_n^{G,S}(g) = \{h \in G \mid d_S(g, h) \leq n\}$$

Proposition 1.5.2

Let G be a group. Let S be a finite generating set. Let $g, h \in G$. Then

$$|B_n^G(g)| = |B_m^G(h)|$$

for any $n \in \mathbb{N}$.

Definition 1.5.3: Growth Function

Let G be a group. Let S be a finite generating set of G. Let $R > 0$. Define the growth function $\Gamma_{G,S}:\mathbb{N}\to\mathbb{N}$ of G with respect to S to be

$$\Gamma_{G,S}(n)=\left|B_n^{G,S}(1_G)\right|$$

for $n\in\mathbb{N}$.

Proposition 1.5.4

Let G be a group Let S be a finite generating set of G. Then the following are true.

- $\Gamma_{G,S}(m+n)\leq\Gamma_{G,S}(m)\Gamma_{G,S}(n)$ for all $m,n\in\mathbb{N}$
- $\Gamma_{G,S}(n)\leq(2|S|+1)^n$ for all $n\in\mathbb{N}$.

Proof. For any pair (h,k) of elements of G such that $d_S(1,h)=m$ and $d_S(1,k)=n$, we have that

$$d_S(1_G,hk)\leq d_S(1_G,h)+d_S(h,hk)=d_S(1_G,h)+d_S(1_G,k)=m+n$$

This means that for any unique pair of elements (h,k) with $h\in B_m^{G,S}(1_G)$ and $k\in B_n^{G,S}(1_G)$, there exists a possibly non-unique element $hk\in B_{m+n}^{G,S}(1_G)$. Hence

$$\left|B_{m+n}^{G,S}(1_G)\right|\leq\left|B_m^{G,S}(1_G)\right|\cdot\left|B_n^{G,S}(1_G)\right|$$

and so $\Gamma_{G,S}(m+n)\leq\Gamma_{G,S}(m)\Gamma_{G,S}(n)$.

Notice that $\Gamma_{G,S}(1)=(2|S|+1)$ since the paths of the Cayley graph is given by S and their inverses. Together with the identity element which has zero norm gives the formula. We can then recursively apply the above inequality to get

$$\Gamma_{G,S}(n)\leq(\Gamma_{G,S}(1))^n=(2|S|+1)^n$$

$\square$

Lemma 1.5.5

Let G be a group Let S be a finite generating set of G. Then the following are true.

- $\Gamma_{G,S}(n)\leq\Gamma_{F(S),S}(n)$ for all $n\in\mathbb{N}$.
- $\Gamma_{G,S}(n)=\Gamma_{F(S),S}(n)$ for all $n\in\mathbb{N}$ if and only if $G\cong F(S)$.

Proof. The induced homomorphism $\phi:F(S)\to G$ sends $B_n^{F(S),S}(1_{F(S)})$ surjectively to $B_n^{F(S),S}(1_{F(S)})$. Indeed if $\gamma_V:[n]\to F(S)$ is a geodesic, then $\phi\circ\gamma_V$ may not be a geodesic so that $d_S(1_G,\phi\circ\gamma_V(n))\leq n$. This means that $\phi\circ\gamma_V(n)\in B_n^{G,S}(1_G)$. Conversely, if $g\in B_n^{G,S}(1_G)$ then $g=w_1\cdots w_n$ is a reduced word in G for $w_1,\dots,w_n\in S$. Then $w_1\cdots w_n$ is also a reduced word in $F(S)$ and hence lie in $B_n^{F(S),S}(1_{F(S)})$. Moreover, $\phi(w_1\cdots w_n)=g$. Hence ϕ is surjective on the two balls. Then we have

$$\Gamma_{G,S}(n)=\left|B_n^{G,S}(1_G)\right|=\left|\phi\left(B_n^{F(S),S}(1_{F(S)})\right)\right|\leq\left|B_n^{F(S),S}(1_{F(S)})\right|=\Gamma_{F(S),S}(n)$$

$\square$

Lemma 1.5.6

Let S be a finite set. Then

$$\Gamma_{F(S),S}(n)=\frac{1-|S|(2|S|-1)^n}{1-|S|}$$

Proof. I claim that the number of reduced words of length n is $2|S|(2|S|-1)^{n-1}$ when $n \geq 1$. We induct on n. When $n = 1$, then any reduced word is just the choice of a letter. Hence there are $2|S|$ number of reduced words of length 1. Now suppose that the number of reduced words of length k is given by $2|S|(2|S|-1)^{k-1}$. Any reduced word of length $k+1$ is given by the concatenation of a reduced word of length k and a choice of letter that is not the inverse of the last element of the given word. Thus there are $2|S|(2|S|-1)^{k-1} \cdot (2|S|-1 = 2|S|(2|S|-1)^k)$ number of reduced words of length $k+1$. This completely the induction step.

Then we have

$$\begin{aligned}\Gamma_{F(S),S}(n) &= 1 + \sum_{i=1}^{n} 2|S|(2|S|-1)^{n-1} \\ &= 1 + 2|S|\sum_{i=0}^{n-1}(2|S|-1)^i \\ &= 1 + 2|S|\frac{1-(2|S|-1)^n}{1-2|S|+1} \\ &= 1 + |S|\frac{1-(2|S|-1)^n}{1-|S|} \\ &= \frac{1-|S|+|S|\left(1-(2|S|-1)^n\right)}{1-|S|} \\ &= \frac{1-|S|(2|S|-1)^n}{1-|S|}\end{aligned}$$

□

Proposition 1.5.7

Let G be a group Let S be a finite generating set of G. Then the following are equivalent.

- G is a finite group.
- $\Gamma_{G,S}$ is bounded.
- $\Gamma_{G,S}(n) = \Gamma_{G,S}(n+1)$ for some $n \in \mathbb{N}$.

Lemma 1.5.8

Let G be a group. Let S, T be finite generating sets of G. Then there exists $C, D > 0$ such that

$$\Gamma_{G,S}(n) \leq C\Gamma_{G,T}(n) \quad \text{and} \quad \Gamma_{G,T}(n) \leq D\Gamma_{G,S}(n)$$

for all $n \in \mathbb{N}$.

Theorem 1.5.9

There exists a finitely generated group G with finite generators S such that $\Gamma_{G,S}$ has super-polynomial growth but subexponential growth.

Theorem 1.5.10: [Hirsch 1958]

Let G be a finitely generated nilpotent group. Let $H \leq G$ be a subgroup of G. Then $[G : H]$ is finite and H is torsion-free.

Theorem 1.5.11: [Jennings 1955]

Let H be a finitely generated torsion-free and nilpotent group. Then H is isomorphic to a subgroup of $H_d(\mathbb{Z})$ for some $d \geq 1$.

Note: $H_d(\mathbb{Z})$ is the upper triangular matrices of $SL_d(\mathbb{Z})$.

Theorem 1.5.12: [Gromov 1981]

Let G be a finitely generated group such that $\Gamma_{G,S}$ has at most polynomial growth. Then there exists some subgroup $H \leq G$ such that $[G : H]$ is finite and H is nilpotent.

Theorem 1.5.13: [Bass 1972, Guivarch 1973]

Let G be a finitely generated nilpotent group. Then there exists $C, D, d \in \mathbb{N}$ such that

$$Cn^d \leq \Gamma_{G,S}(n) \leq Dp^d$$

($\Gamma_{G,S}$ has polynomial growth rate).

1.6 Distortion

Definition 1.6.1: Undistorted Subgroups

Let G be a group. Let S, T be generating sets of G. Let $H \leq G$ be a subgroup. We say that H is undistorted in G if there exists $C > 0$ such that

$$d_T(g, h) \leq Cd_S(g, h)$$

for all $g, h \in H$.

Intuitively, this means that when we restrict the metric to the subgroup, the shortest path when we had in H for two elements is still the shortest when we consider the two elements in G.

2 The Geometry of Boundaries

2.1 Ends of a Group via Geodesic Rays

Definition 2.1.1: Geodesic Ray

Let G be a group. Let S be a finite generating set of G. A geodesic ray in Cay(G, S) is a continuous map
$$\phi : [0, \infty) \to \text{Cay}(G, S)$$
such that if $B \subseteq \text{Cay}(G, S)$ is bounded then $\phi^{-1}(B)$ is bounded.

This is called proper rays in Loh.

Definition 2.1.2: Ends of a Group

Let G be a group. Let S be a finite generating set of G. Define
$$\text{Ends}(G, S) = \{\phi : [0, \infty) \to \text{Cay}(G, S) \mid \phi \text{ is a geodesic ray }\}/\sim$$
where $\phi_1 \sim \phi_2$ if for all $n \in \mathbb{N}$, there exists $t \in \mathbb{R}$ such that $\text{im}(\phi_1) \setminus B_n^{G,S}(1)$ and $\text{im}(\phi_2) \setminus B_n^{G,S}(1)$ lie in the same path component of Cay$(G, S) \setminus B_n^{G,S}(1)$.

2.2 Infinite Connected Components at Infinity

Definition 2.2.1: Infinite Connected Components

Let G be a group. Let S be a finite generating set of G. Define the set of infinite connected components of G with respect to S by
$$E_S(n) = \{[X] \in \pi_0(\text{Cay}(G, S) \setminus B_n^{G,S}(1)) \mid |X| = \infty\}$$

Definition 2.2.2

Let G be a group. Let S be a finite generating set of G. Define
$$C_S(n) = \text{Cay}(G, S) \setminus \bigcup_{A \in E_S(n)} A$$