# Algebraic Curves

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**Abstract**

# Contents

# 1 Algebraic Curves in Classical Algebraic Geometry

**Definition 1.0.1: Algebraic Curves**

An algebraic curve is an irreducible variety of dimension 1.

**Theorem 1.0.2**

Any irreducible curve is birationally equivalent to a unique non-singular projective curve.

# 2 Algebraic Curves in the Context of Schemes

**Definition 2.0.1: Algebraic Curves**

Let $k$ be an algebraically closed field. A curve over $k$ is an integral separated scheme $X$ of finite type over $k$ that has dimension 1.

**Proposition 2.0.2**

Let $X$ be an algebraic curve. Then the arithmetic and geometric genus coincide. In particular,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

We will simply call the genus of a curve $g$ from now on since the arithmetic genus is the same as the geometric genus.

## 2.1 Riemann-Roch Theorem

**Definition 2.1.1: Canonical Divisor**

Let $X$ be an algebraic curve. The canonical divisor $K$ of $X$ is a divisor in the linear equivalence class of

$$\Omega^1_{X/k} = \omega_X$$

**Theorem 2.1.2: Riemann-Roch Theorem**

Let $X$ be an algebraic curve. Let $D$ be a divisor on $X$ and let $K$ be the canonical divisor of $X$. Let $\mathcal{L}(D)$ be the associated sheaf of the divisor $D$. Denote $l(D) = \dim_k(H^0(X, \mathcal{L}(D)))$ and let $g$ be the genus of $X$. Then

$$l(D) - l(K - D) = \deg(D) + 1 - g$$

## 2.2 Classification of Curves in $\mathbb{P}^3$