Rational Homotopy Theory

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Abstract

References

- Rational Homotopy Theory and Differential Forms

Contents

1 Homotopy with Coefficients

1.1 Rational Spaces

Definition 1.1.1: Rational Spaces

Let X be a space. We say that X is a rational space if the following are true.

- X is homotopy equivalent to a CW complex
- $\pi_1(X) = 0$
- $\pi_n(X)$ is a $\mathbb{Q}$-vector space for each $n \in \mathbb{N}$

Lemma 1.1.2

The following are true regarding the Eilenberg-maclane spaces of $\mathbb{Q}$.

- $\widetilde{H}_k(K(\mathbb{Q}, n); \mathbb{Z})$ is a $\mathbb{Q}$-vector space for each $k \in \mathbb{N}$.
- $H^k(K(\mathbb{Q}, 2n); \mathbb{Q})$ is a $\mathbb{Q}$-polynomial algebra on one generator, and the generator has degree $2n$.
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Corollary 1.1.3

The induced map $K(\mathbb{Z}, n) \to K(\mathbb{Q}, n)$ given by the map $\mathbb{Z} \to \mathbb{Q}$ induces an isomorphism of rational cohomology and rational homology.

Theorem 1.1.4

Let X be a space. Then X is a rational space if and only if the following conditions are satisfied:

- X is homotopy equivalent to a CW complex
- $\pi_1(X) = 0$
- $\widetilde{H}_n(X, \mathbb{Z})$ is a $\mathbb{Q}$-vector space

1.2 Rational Homotopy Type

Definition 1.2.1: Rational Homotopy Equivalence

Let $f : X \to Y$ be a map of spaces. We say that f is a rational homotopy equivalence if the induced map gives isomorphisms

$$f_* : \pi_n(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_n(Y) \otimes \mathbb{Q}$$

of $\mathbb{Q}$-vector spaces.

Intuitively, tensoring the homotopy groups with $\mathbb{Q}$ forgets about the torsion subgroups, thus leading to an even more crude algebraic invariant for (simply connected) spaces. In other words, we have the following relation:

$$\text{Homeomorphisms} \subset \substack{\text{Homotopy}\\\text{Equivalences}} \subset \substack{\text{Weak Homotopy}\\\text{Equivalences}} \subset \substack{\text{Rational Homotopy}\\\text{Equivalences}}$$

The hope is that while we are losing some information, it makes the rational homotopy groups more computable.

Theorem 1.2.2

Let X, Y be simply connected CW complexes and let $f : X \to Y$ be a map. If Y is a rational space, then the following conditions are equivalent.

- The induced map $f_* : \pi_n(X) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \cong \pi_n(Y)$ is a rational homotopy equivalence
- The induced map $f_* : \widetilde{H}_n(X; \mathbb{Q}) \to \widetilde{H}_n(Y; \mathbb{Q}) \cong \widetilde{H}_n(Y; \mathbb{Z})$ is an isomorphism for all $n \in \mathbb{N}$
- f is universal for maps of X into $\mathbb{Q}$-space. This means that if $g : X \to Z$ is another map into a $\mathbb{Q}$-space Z, then there exists a map $h : Y \to Z$ unique up to homotopy such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

Lemma 1.2.3

Let X be a simply connected CW complex. Then $\pi_n(X) \otimes \mathbb{Q} = 0$ for all $n \in \mathbb{N}$ if and only if $\widetilde{H}_n(X; \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$.

Definition 1.2.4: Rationalization and Rational Homotopy Type

Let X be a CW complex. The rationalization of X is a rational space $X_{(0)}$ together with a rational homotopy equivalence $f : X \to X_{(0)}$. In this case we say that $X_{(0)}$ is the rational homotopy type of X.

Prereq: postinokov towers

Theorem 1.2.5

Let X be a CW complex. The rationalization of X exists and is unique up to homotopy equivalence.

Explicitly, if $X_{(0)}$ and Y are both rationalizations of X with maps $f : X \to X_{(0)}$ and $g : X \to Y$, then there exists a homotopy equivalence $h : X_{(0)} \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Y \\ X_{(0)} \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$