Algebraic K Theory

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Abstract

Contents

1 The K_0-Group

1.1 K_0 of a Symmetric Monoidal Category

Definition 1.1.1: The K_0-Group of a Symmetric Monoidal Category

Let $(\mathcal{C}, I, \oplus)$ be a symmetric monoidal category. Let $\mathcal{C}^{\text{iso}}$ be the category consisting of isomorphism classes of objects, which is also an abelian monoid under the operation $\oplus$. Define the K_0 group of $\mathcal{C}$ by the Grothendieck completion

$$K_0(\mathcal{C}, I, \oplus) = \left(\mathcal{C}^{\text{iso}}\right)^{-1}\mathcal{C}^{\text{iso}}$$

1.2 K_0 of a Ring

Definition 1.2.1: The Category of Projective Modules over a Ring

Let R be a ring. Define the category $P(R)$ of projective modules over R as follows.

- The objects are projective modules M over R
- For two projective modules M, N over R, a morphism $M \to N$ is just an R-module homomorphism.
- Composition is given by the composition of functions.

Lemma 1.2.2

Let R be a ring. Then the category $P(R)$ is a symmetric monoidal category with the distinguished object R as an R-module and binary operator $\oplus : P(R) \times P(R) \to P(R)$ the direct sum.

Definition 1.2.3: The K_0-Group of a Ring

Let R be a ring. Define the K_0-group of R by the Grothendieck completion of the abelain monoid:

$$K_0(R) = P(R)^{-1}P(R) = K_0(P(R), R, \oplus)$$

1.3 K_0 of an Abelian Category

1.4 K_0 of a Waldhaussen Category

2 The K_1-Group

2.1 K_1 of a Ring

Definition 2.1.1: The K_1-Group of a Ring

Let R be a ring. Define the K_1-group of R to be the group

$$K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]}$$

Proposition 2.1.2

Let R and S be two rings. Then there is an isomorphism

$$K_1(R \times S) \cong K_1(R) \oplus K_1(S)$$

Proposition 2.1.3

Let R be a ring. Then there is an isomorphism

$$K_1(R) \cong K_1(M_n(R))$$

for any $n \in \mathbb{N}$.

2.2 The Fundamental Theorems for K_1 and K_0

3 The Negative K-Groups

4 The K_2-Group

4.1 The Steinberg Group

Definition 4.1.1: The nth Steinberg Group

Let R be a ring. For $n\geq 3$, define the nth Steinberg group by

$$\text{St}_n(R)=\frac{\langle x_{ij}(r)\text{ for }r\in R,1\leq i,j\leq n\rangle}{R}$$

where R is the relation generated by

- For $r,s\in R$, $x_{ij}(r)x_{ij}(s)=x_{ij}(rs)$ for $1\leq i,j\leq n$
- For $r,s\in R$,

$$[x_{ij}(r),x_{kl}(s)]=\begin{cases}1 & \text{if } j\neq k\text{ and } i\neq l\\ x_{il}(rs) & \text{if } j=k\text{ and } i\neq l\\ x_{kj}(-rs) & \text{if } j\neq k\text{ and } i=l\end{cases}$$

Lemma 4.1.2

Let R be a ring. For any $n\geq 3$, the nth Steinberg group $\text{St}_n(R)$ of R includes into the $(n+1)$th Steinberg group $\text{St}_{n+1}(R)$.

Proposition 4.1.3

Let R be a ring. Let $n\geq 3$. Then the universal property of free groups with relations induce a canonical group surjection

$$\phi_n:\text{St}_n(R)\to[GL(R),GL(R)]$$

that sends $x_{ij}(r)$ to $e_{ij}(r)$.

Definition 4.1.4: The Steinberg Group of a Ring

Let R be a ring. Define the Steinberg group of R by the direct limit

$$\text{St}(R)=\varinjlim_{n\in\mathbb{N}\setminus\{0,1,2\}}\text{St}_n(R)$$

Proposition 4.1.5

Let R be a ring. The universal property of the direct limit induces a canonical group surjection

$$\phi:\text{St}(R)\to[GL(R),GL(R)]$$

4.2 K_2 of a Ring

Definition 4.2.1: The K_2-Group of a Ring

Let R be a ring. Define the K_2-group of R to be the kernel

$$K_2(R)=\ker\left(\phi:\text{St}(R)\to[GL(R),GL(R)]\right)$$

Lemma 4.2.2

Let R be a ring. Then there is an exact sequence of groups

$$0 \longrightarrow K_2(R) \longrightarrow \mathsf{St}(R) \longrightarrow [GL(R), GL(R)] \longrightarrow K_1(R) \longrightarrow 0$$

Theorem 4.2.3: (Stein)

For any ring R, the K_2-group $K_2(R)$ is an abelian group. Moreover, we have

$$Z(\mathsf{St}(R)) = K_2(R)$$

5 The K_n-Group

5.1 Universal Definition

Definition 5.1.1: The Plus Construction

Let R be a ring. Define $BGL(R)^+$ to be any CW complex that has a distinguished map $BGL(R) \to BGL(R)^+$ such that the following are true.

- There is an isomorphism $\pi_1(BGL(R)^+) \cong K_1(R)$ given by the induced map $\pi_1(BGL(R)) \to \pi_1(BGL(R)^+)$, which is required to be surjective with kernel $[GL(R), GL(R)]$
- For each $n \in \mathbb{N}$, there are isomorphisms
$$H_n(BGL(R); M) \cong H_n(BGL(R)^+; M)$$
for any R-module M.

Intuitively, $BGL(R)^+$ is a modification of the classifying space of $GL(R)$ so that their homology remains the same while its fundamental group returns $K_1(R)$. The latter point is important because K_n will be defined as the nth homotopy group.

Definition 5.1.2: K_n of a Ring

Let R be a ring. Define the nth K-group of R to be
$$K_n(R) = \pi_n(BGL(R)^+)$$
for $n \geq 1$.

Notice that $BGL(R)^+$ for a ring R is not defined uniquely. However, we can prove that any two such plus constructions are homotopy equivalent so that $K_n(R)$ is well defined.

In order to accommodate the 0th K-group, we make the following amendments.

Definition 5.1.3: K-Theory of a Ring

Let R be a ring. Define the K-theory of R by
$$K(R) = K_0(R) \times BGL(R)^+$$
so that $\pi_n(BGL(R)^+) = K_n(R)$ for all $n \geq 0$.

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6.1

Theorem 6.1.1: Serre-Swan Theorem I

Let M be a smooth manifold. Let E be a smooth vector bundle over M. Then the space of smooth sections $\Gamma(E)$ of E is finitely generated and projective over $C^\infty(M)$.

If M is connected, then the space of smooth section is one-to-one with the finitely generated and projective modules over $C^\infty(M)$.

Theorem 6.1.2

Let M be a smooth and connected manifold. Then the category of smooth vector bundles $\text{SVect}(M)$ is equivalent to the category of finitely generated projective modules $\text{FinProj}_{C^\infty(M)}\text{Mod}$ via the global section functor

$$\Gamma : \text{SVect}(M) \to \text{FinProj}_{C^\infty(M)}\text{Mod}$$

defined by $E \mapsto \Gamma(E)$

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