Topological Manifolds

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Abstract

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1 Topological Manifolds and Singular Homology

1.1 Covering Spaces of Manifolds

Proposition 1.1.1

Any covering space of a manifold is also a manifold.

1.2 Orientability

Recall the notion of orientation in finite dimensional vector bases. We say that two bases of a vector space have the same orientation if the change of basis matrix has determinant greater than 0. Since topological manifolds locally look like finite-dimensional vector spaces, we expect that orientations can be generalized to manifolds.

The key observation in defining orientation through homology is the following proposition, which shows that the local homology groups on a manifold are isomorphic to $\mathbb{Z}$ on the top dimension.

Proposition 1.2.1

Let M be a k-dimensional topological manifold and $x \in M$ a point. Then

$$H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{*\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Proof. Let $x \in U$ be an open neighbourhood such that $U \cong \mathbb{R}^k$. Then by excision, we have that

$$H_n(M \setminus (M \setminus U), (M \setminus \{x\}) \setminus (M \setminus U)) \cong H_n(M, M \setminus \{x\})$$

This translates to $H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus *)$. By corollary 5.2.3 in Algebraic Topology 2, we are done. Alternatively, we have the following proof:

If $k = 0$, the results are clear. If $k \geq 1$, then the long exact sequence of the pair $(\mathbb{R}^k, \mathbb{R}^k \setminus *)$ together with the fact that $\mathbb{R}^k \setminus * \simeq S^{k-1}$ and $\mathbb{R}^k \simeq *$ gives

$$H_n(\mathbb{R}^k, \mathbb{R}^k \setminus *) = 0$$

for $n > k$ and $n < k$. When $n = k$, we have an exact sequence

$$0 \longrightarrow H_k(\mathbb{R}^k, \mathbb{R}^k \setminus *) \longrightarrow H_{k-1}(S^{k-1}) \longrightarrow H_{k-1}(\mathbb{R}^k)$$

when $k > 1$ since $H_{k-1}(\mathbb{R}^k) = 0$. Thus $H_k(\mathbb{R}^k, \mathbb{R}^k \setminus *) \cong \mathbb{Z}$. If $k = 1$, then the last map $H_0(S^0) \to H_0(\mathbb{R})$ is given by the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}$ thus also giving isomorphism. $\square$

Definition 1.2.2: Local Orientation

A local orientation of M at x is a choice of generator of $H_k(M, M \setminus \{x\})$.

One can think of local orientation as follows. Choose an open neighbourhood U of x that is homeomorphic to $\mathbb{R}^2$. Then the long exact sequence for relative homology gives an isomorphism

$$H_2(U, U \setminus \{x\}) \cong H_1(S^1)$$

since $U \setminus \{x\}$ deformation retracts onto a small circle around x, we can choose a local orientation ω_x for the circle which is the same as choosing in which direction to loop around the circle. It remains to patch them up into a global orientation. Of course, this does not necessarily work for every single manifold.

Let U be a chart on a topological manifold M and that $B \subseteq M$ is such that on the chart U, B is an open / closed ball $B_r(z)$. For convention, we give a name to subsets of these type.

Definition 1.2.3: Open and Closed Ball in Manifolds

Let M be a k-dimensional topological manifold and U a chart of M. We say that B is an open / closed ball if under the homeomorphism of the chart $U \cong \mathbb{R}^k$, the image of B is a ball $B_r(x) \subseteq \mathbb{R}^k$ for some $r \in \mathbb{R}^+$ and $x \in \mathbb{R}^k$.

The point of the definition is that we have the following sequence of isomorphisms

$$\begin{aligned} H_k(M, M \setminus B) &\cong H_k(U, U \setminus B) && \text{(Excise } M \setminus U) \\ &\cong H_k(\mathbb{R}^k, \mathbb{R}^k \setminus B_r(x)) \end{aligned}$$

and then using the long exact sequence in relative homology, we obtain an isomorphism

$$H_k(\mathbb{R}^k, \mathbb{R}^k \setminus B_r(x)) \cong H_{k-1}(\mathbb{R}^k \setminus B_r(x))$$

in which the latter space deformation retracts onto the boundary $\partial B_r(x) \cong S^{k-1}$. Thus

$$H_k(\mathbb{R}^k, \mathbb{R}^k \setminus B_r(x)) \cong H_{k-1}(\partial B_r(x)) \cong \mathbb{Z}$$

is infinite cyclic. This means that we can think of the choice of a local orientation as a choice of orientation on $\partial B_r(x)$.

Notice that the inclusion $(M, M \setminus B) \hookrightarrow (M, M \setminus \{y\})$ induces a map in homology:

$$H_k(M, M \setminus B) \stackrel{\cong}{\to} H_k(M, M \setminus \{y\})$$

It is an isomorphism since B is homeomorphic to a ball in $\mathbb{R}^k$ which is contractible. This leads to the following definition.

Definition 1.2.4: Consistent Local Orientations

Let $(\omega_y)_{y \in B}$ be a family of local orientations. We say that it is consistent if there is a generator $\omega_B \in H_k(M, M \setminus B)$ such that $\omega_B \mapsto \omega_y$ for each $y \in B$ under the isomorphism

$$H_k(M, M \setminus B) \cong H_k(M, M \setminus \{y\})$$

With this, we can now formally define orientations in a manifold.

Definition 1.2.5: Orientation of a Manifold

Let M be a k-dimensional topological manifold. An orientation of M is a function

$$x \mapsto \omega_x \in H_k(M, M \setminus \{x\})$$

assigning every point to a local orientation such that for every $x \in M$, there exists an open ball $x \in B$ such that $(\omega_x)_{x \in B}$ a consistent local orientation.

Since $H_k(M, M \setminus \{x\})$ is isomorphic to $\mathbb{Z}$, this means that there are only two possible choices of distinct orientation classes for each point $x \in M$.

Lemma 1.2.6

The k-sphere S^k is orientable for any $k \in \mathbb{N}$.

Proof. Choose a fundamental class in $H_k(S^k)$. It is clear that the long exact sequence in relative homology induces a map

$$H_k(S^k) \to H_k(S^k, S^k \setminus \{x\})$$

induces local orientation at each point $x \in S^k$. They are locally consistent since the map factors through $H_k(S^k, S^k \setminus B)$ for any open ball B in S^k. $\square$

In order to deduce orientability of a manifolds, we appeal to a vector bundle on the manifold.

Definition 1.2.7: Orientation Bundle

Let M be a topological manifold. Define the orientation bundle $\widetilde{M}$ to be the set of pairs

$$\widetilde{M} = \left\{ (x, \omega_x) \;\middle|\; x \in M, \omega_x \in H_k(M, M \setminus \{x\}) \right\}$$

together the projection map $\pi : \widetilde{M} \to M$ defined by $\pi(x, \omega_x) = x$ and with the topology defined as follows.

Let B be an open ball in M. Since there are exactly two distinct orientation classes on B we have that

$$\pi^{-1}(B) = B_+ \amalg B_-$$

where B_+ and B_- are homeomorphic to B. Define the topology of $\widetilde{M}$ to be generated by sets of the form B_+ and B_-.

Lemma 1.2.8

For any topological manifold M, $\widetilde{M}$ is a manifold and is a 2-sheeted covering.

Proof. Let (x, ω_x) and (y, ω_y) in $\widetilde{M}$ be distinct. If $x = y$ then $\omega_x = -\omega_y$. We know that there are two distinct orientation classes so π^{-1} is a disjoint union consisting of those with positive orientation and those with negative. Since ω_x and ω_y are opposite, they lie in the disjoint union separately so that they are disjoint. If $x \neq y$, then since M is Hausdorff then we can choose U_1 and U_2 disjoint neighbourhoods of x and y respectively. Then this means that $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint. Thus we have shown that M is Hausdorff.

Now let $(x, \omega_x) \in \widetilde{M}$. Then since M is manifold, there is an open ball B around x so that B is homeomorphic to $\mathbb{R}^k$. $\pi^{-1}(B)$ is then a disjoint union of two copies of B, one such copy contains (x, ω_x). Then we have found a neighbourhood for (x, ω_x) that is homeomorphic to $\mathbb{R}^k$. Thus we are done.

It is clear that it is a two sheeted covering because for any open set $B \subseteq M$, $\pi^{-1}(B) = B_+ \amalg B_-$. $\square$

Lemma 1.2.9

Let M be a topological manifold. Then the orientation bundle $\widetilde{M}$ is orientable.

Proof. To show orientability, it suffices to show that for every $(x, \omega_x) \in \tilde{M}$, there is a choice of orientation such that there exists an open set in $\tilde{M}$ for which the choice of orientation is locally consistent. So let (x, ω_x) be a point in $\tilde{M}$. Let B be a small open ball around x in M. Then $\pi^{-1}(B)$ is by definition a disjoint union of two copies of B, each with a locally consistent orientation. In other words, $\pi^{-1}(B) = A \amalg C$. Without loss of generality, take

$(x, \omega_x) \in A$. Now consider the following isomorphisms

$$H_k(\widetilde{M}, \widetilde{M} \setminus \{(x, \omega_x)\}) \cong H_k(A, A \setminus \{(x, \omega_x)\}) \cong H_k(B, B \setminus \{x\}) \cong H_k(M, M \setminus \{x\})$$

The first isomorphism is obtained by excising the piece $\widetilde{M} \setminus A$. The second isomorphism is obtained by considering the homeomorphism $\pi|_A$. The third isomorphism theorem is obtained by excising the piece $M \setminus B$ from the right.

Thus for any choice of orientation in $H_k(M, M \setminus \{x\})$ we obtain a choice of orientation in $H_k(\widetilde{M}, \widetilde{M} \setminus \{(x, \omega_x)\})$.
By two more excisions, we obtain

$$\begin{tikzcd} H_k(\widetilde{M}, \widetilde{M} \setminus A) \arrow[r, "\cong"] & H_k(\widetilde{M}, \widetilde{M} \setminus \{(x, \omega_x)\}) \\ H_k(M, M \setminus B) \arrow[r, "\cong"] & H_k(M, M \setminus \{x\}) \arrow[u, "\cong"'] \end{tikzcd}$$

Since A is a locally consistent choice of orientations, there exists a generator $\omega_B \in H_k(M, M \setminus B)$ such that $\omega_B \mapsto \omega_x$ under the bottom map of the diagram. This ω_x then gives $\widetilde{\omega}_x \in H_k(\widetilde{M}, \widetilde{M} \setminus \{(x, \omega_x)\})$. Which under the isomorphism of the top arrow in the diagram we obtain a generator of $H_k(\widetilde{M}, \widetilde{M} \setminus A)$. Since each operation described is independent of the choice of $x \in B$, the generator we obtained for $H_k(\widetilde{M}, \widetilde{M} \setminus A)$ is independent of the choice of $(x, \omega)_x$. Thus the consistent local orientation condition is satisfied so that $\widetilde{M}$ is orientable. $\square$

Lemma 1.2.10

The deck transformation of the orientation bundle of manifold is orientation reversing.

Proof. Any non-trivial deck transformation must permute the fibers of the covering space non-trivially. In this case, any (x, ω_x) can only be mapped to the other of the element of the fiber which is $(x, -\omega_x)$. $\square$

Lemma 1.2.11

Giving an orientation of M is equivalent to giving a continuous map $s : M \to \widetilde{M}$ such that $s \circ \pi = \text{id}$ (section of the orientation bundle).

Proof. Let $s : M \to \widetilde{M}$ be continuous and that $s \circ \pi = \text{id}$. Then s assigns a orientation ω_x to each $x \in M$. The map is continuous if and only if for each open ball in M and $\pi^{-1}(B) = B_+ \amalg B_-$, the preimages $s^{-1}(B_+)$ and $s^{-1}(B_-)$ are both open in B. Since these two preimages are disjoint and jointly cover B, this condition is equivalent $s(B) = B_+$ or $s(B) = B_-$. This precisely means that the local orientations are consistent. $\square$

Theorem 1.2.12

Let M be a connected topological manifold. Then exactly one of the following holds:

- $\widetilde{M} \to M$ is a non-trivial 2-sheeted cover and M is non-orientable
- $\widetilde{M} \cong M \amalg M$ and M admits precisely two orientations

Proof. Assume that $\pi : \widetilde{M} \to M$ is connected and $\omega : M \to \widetilde{M}$ is a continuous section to π. Let $x \in M$ and $\pi^{-1}(x) = \{(x, \omega_x), (x, -\omega_x)\}$. By assumption there is a path γ in $\widetilde{M}$ from (x, ω_x) to $(x, -\omega_x)$. Then γ and $\omega \circ \pi \circ \gamma$ are two paths in $\widetilde{M}$ lifting $\pi \circ \gamma$ and starting at (x, ω_x). But the first path ends at $(x, -\omega_x)$ and the second one ends at (x, ω_x). This is a contradiction to the uniqueness of lifting paths. Thus M is non-orientable.

If M is disconnected then $\widetilde{M} \cong M \amalg M$ since $\widetilde{M}$ is a covering space. Thus M admits two orientations. $\square$

Corollary 1.2.13

Any simply connected manifold is orientable.

Proof. By Galois theory of covering spaces, any 2-sheeted cover of a simply connected space is disconnected. $\square$

Proposition 1.2.14

Let $k \geq 1$. Then $\mathbb{RP}^k$ is orientable if and only if k is odd.

Proof. The quotient map $q : S^k \to \mathbb{RP}^k$ is the unique connected two-sheeted cover of $\mathbb{RP}^k$ by Galois theory for covering spaces. The non-trivial deck transformation is given by the antipodal map which has degree $(-1)^{k+1}$. If k is odd then this degree is 1 so that the deck transformation is orientation preserving. Since deck transformations of the orientation bundle must be orientation reversing, we conclude that $S^k \neq \widetilde{\mathbb{RP}^k}$. This means that the orientation bundle of $\mathbb{RP}^k$ is disconnected.

Now assume that k is even. By the lifting criterion, there exists a lift of q called $\tilde{q}$ such that

$$\begin{tikzcd} & \widetilde{\mathbb{RP}^k} \arrow[d, "p"] \\ S^k \arrow[ur, "\tilde{q}"] \arrow[r, "q"] & \mathbb{RP}^k \end{tikzcd}$$

where p is the covering map. Then $\tilde{q}$ must also be a covering space. Assume that q is not injective. This means that $\tilde{q} \circ (-\text{id}) = \tilde{q}$ since $-\text{id}$ is the only other deck transformation of S^k over $\mathbb{RP}^k$. This means that for any $x \in S^k$, we have that

$$H_k(S^k) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}) \longrightarrow H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

where the second map is given by the long exact sequence in relative homology. Denoting this entire map by α, we have that $\alpha \circ (-\text{id})_* = \alpha$ since $\tilde{q} \circ (-\text{id}) = \tilde{q}$. But α is a map from $\mathbb{Z}$ to $\mathbb{Z}$. Since $\alpha \circ (-\text{id})_* = \alpha$ this implies that $\alpha = 0$. But α also factors as

$$H_k(S^k) \xrightarrow{\cong} H_k(S^k, S^k \setminus \{x\}) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

by the long exact sequence in relative homology and naturality. But the second map is also an isomorphism since covering spaces of manifolds induces a an isomorphism in local homology groups.

Now S^k being compact and $\mathbb{RP}^k$ being Hausdorff together with $\tilde{q}$ being injective implies that $\tilde{q}$ is a homeomorphism onto an open and closed subspace of $\mathbb{RP}^k$. Assume that $\tilde{q}$ is not surjective, then we have that $\widetilde{\mathbb{RP}^k} \cong S^k \amalg X$ for some other space X. But this is impossible thus q is surjective and $\tilde{q}$ gives a homeomorphism between S^k and $\widetilde{\mathbb{RP}^k}$. Since S^k is connected, $\mathbb{RP}^k$ is thus non orientable. $\square$

One has to be careful that homotopy equivalence does not preserve orientability. For example, the Mobius strip is homotopy equivalent to S^1 but the former is non-orientable while the latter is.

1.3 Fundamental Class

Proposition 1.3.1

Let M be a connected compact smooth manifold of dimension n. If M is orientable then $H_n(M) \cong \mathbb{Z}$. Otherwise $H_n(M) = 0$.

Definition 1.3.2: Fundamental Class

Let M be a connected compact orientable smooth manifold of dimension n. A fundamental class for M is a generator for the top homology

$$H_n(M) \cong \mathbb{Z}$$

2 The Theory of Surfaces

2.1 Classification of Compact Surfaces

Recall that a compact surface is a connected topological manifold of dimension 2 that is compact.

Definition 2.1.1: Connected Sum

Let S_1 and S_2 be two compact surfaces. Let $D_i \subseteq S_i$ be two small closed disks for $i = 1, 2$. Define the connected sum to be

$$S_1 \# S_2 = \frac{(S_1 \setminus D_1^\circ) \amalg (S_2 \setminus D_2^\circ)}{\partial D_1 \cong \partial D_2}$$

Lemma 2.1.2

The connected sum of two compact surfaces is again a surface.

Proposition 2.1.3

The connected sum is invariant under the choice of homeomorphism and the location of the small discs.

We encounter our first family of compact surfaces by repeatedly applying connected sums to a number of toruses.

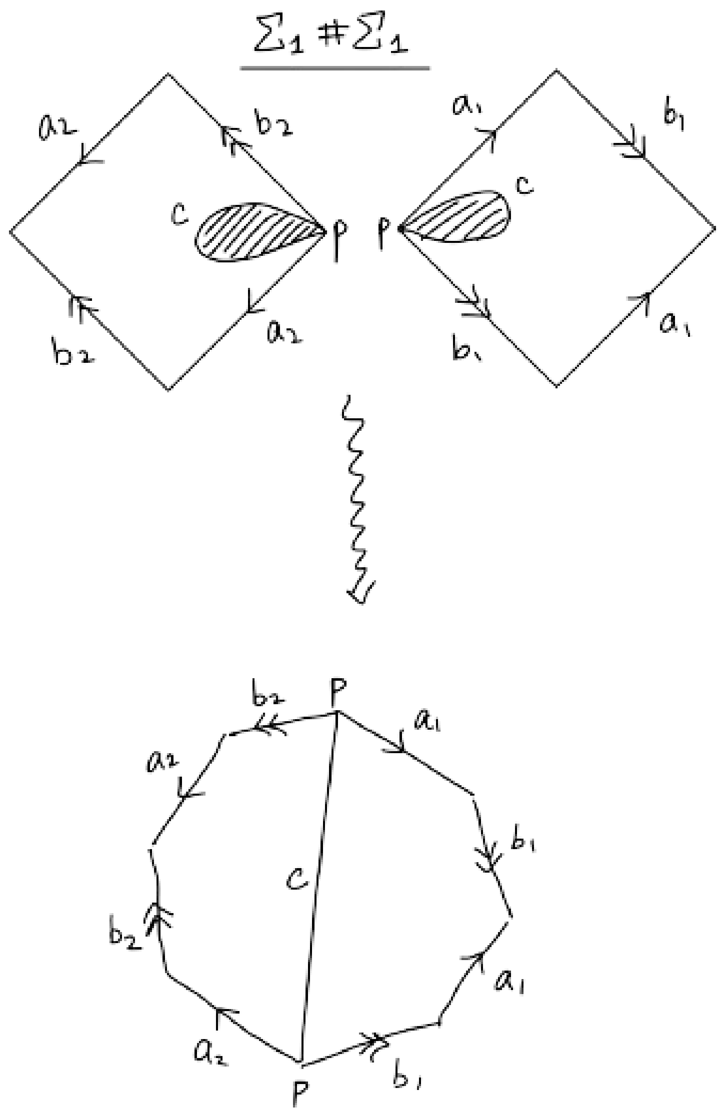
Definition 2.1.4: g-Holed Torus

For $g \geq 0$, define the g-holed torus to be

$$\Sigma_g = \mathbb{T} \# \cdots \# \mathbb{T}$$

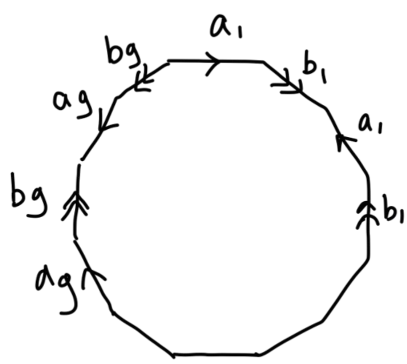
the connected sum of g toruses. By convention when $g = 0$, Σ_g is the 2-sphere.

Recall the CW complex of the torus. We can visualize the connected sum of two toruses using the CW complex.



This is done by cutting a hole at the CW complex at the point p, and the pushing the boundary c out, and then connecting them together. The cut-out hole is exactly a disc in the torus. By gluing the two toruses along the boundary c, we are effectively gluing the two toruses along the discs.

The new hectagon obtained is precisely then the CW complex of Σ_2. In general, we can perform the operation of connected sum on a $(4g-4)$-gon and a square. We then obtain the CW complex of the g-holed torus.



Another class of compact surfaces is the connected sum of projective spaces $\mathbb{RP}^2$.

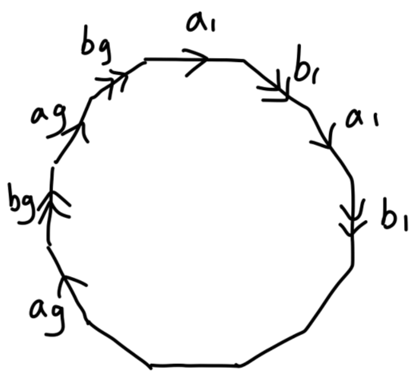
Definition 2.1.5: Non-Orientable Surface

For $h \geq 1$, define

$$N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$

the connected sum of h projective spaces.

We can do the same process of gluing the CW complexes just like that of the torus to obtain the $4h$-gon that represents N_h:



It is also meaningful to ask what would happen if we perform connected sums through the two class of compact surfaces. We obtain the following.

Proposition 2.1.6

Let N_3 denote the connected sum of three projective spaces $\mathbb{RP}^2$. Then we have that

$$T \# \mathbb{RP}^2 = N_3$$

The above two classes of compact surfaces together with the sphere exhausts all possible cases for compact surfaces.

Theorem 2.1.7

Every compact surface is homeomorphic to exactly one of the following.

- Σ_g for $g \geq 0$
- N_h for $h \geq 1$

2.2 The Homology of Surfaces

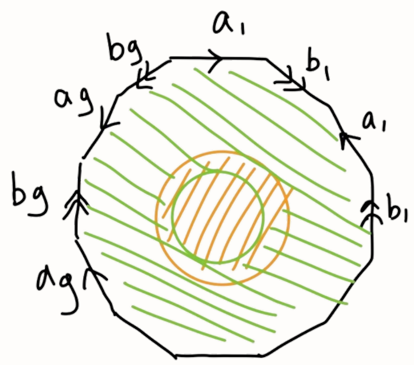
Recall that a compact surface is a connected topological manifold of dimension 2 that is compact. Moreover, every compact surface is homeomorphic to either $\Sigma_g = \mathbb{T} \# \cdots \# \mathbb{T}$ for $g \geq 0$ or $N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ for $h \geq 1$. We can now compute the homology groups of these surfaces and moreover, show that $\sum_g$ is orientable while N_h is not, using the CW complexes given above.

Theorem 2.2.1

Let $g \geq 0$. The homology of the g-holed torus Σ_g is given by

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Cut an open disc along the middle of the CW complex as follows



and label it V (the orange part). Label the green part as U. It is clear that $U \cap V \simeq S^1$, U is contractible and V deformation retracts to the boundary, which is actually just a wedge sum of $2g$ circles. By the formula for the homology of wedge sums we have that

$$H_n(V) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

By the reduced Mayer-Vietoris sequence, the only non-trivial homology groups in the sequence are

$$0 \longrightarrow \tilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \tilde{H}_1(\Sigma_g) \longrightarrow 0$$

and the exact sequence

$$0 \longrightarrow \tilde{H}_0(\Sigma_g) \longrightarrow 0$$

in which the latter immediately shows that $H_0(\Sigma_g) \cong \mathbb{Z}$. Now the map $\mathbb{Z} \to \mathbb{Z}^{2g}$ sends a generator of the first homology of $U \cap V \simeq S^1$ to the loop

$$a_1 + b_1 - a_1 - b_1 + \cdots + a_g + b_g - a_g - b_g$$

Since $\mathbb{Z}^{2g}$ is abelian, we conclude that this map is actually the zero map. It follows that $H_2(\Sigma_g) \cong \mathbb{Z}$ and $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. $\square$

We can immediate deduce the orientability of Σ_g using the machinery in section 1.

Corollary 2.2.2

The surfaces Σ_g for $g \geq 0$ is orientable.

Proof. By the above, we have that $H_2(\Sigma_g) \cong \mathbb{Z}$. The long exact sequence for relative homology groups give

$$\cdots \longrightarrow H_2(\Sigma_g \setminus \{x\}) \longrightarrow H_2(\Sigma_g) \longrightarrow H_2(\Sigma_g, \Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g) \longrightarrow \cdots$$

Let U be as the proof above. Then the inclusion from U to $\Sigma \setminus \{x\}$ is a homotopy equivalence. Moreover, $\Sigma \setminus \{x\}$ is a $2g$-fold wedge of circles labelled $a_1, b_1, \dots, a_g, b_g$ and $H_2(\Sigma_g \setminus \{x\}) = 0$. Also, we have that $H_1(U) \cong H_1(\Sigma_g)$ from above and hence

$H_1(\Sigma_g \setminus \{x\}) \cong H_1(\Sigma_g)$. The last map is invertible so that by exactness, the third map is the zero map. Then what remains is an isomorphism

$$H_2(\Sigma_g) \cong H_2(\Sigma_g, \Sigma_g \setminus \{x\})$$

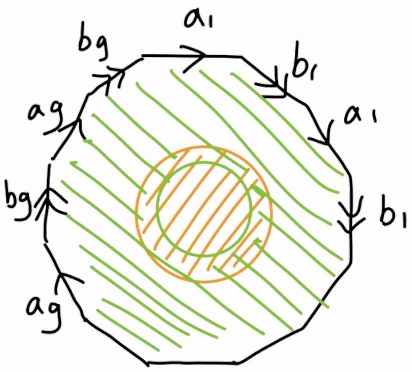
Now since this isomorphism factors through $H_2(\Sigma_g, \Sigma_g \setminus B)$ for any ball B containing x, we thus have a consistent local orientation throughout all of Σ_g. □

Theorem 2.2.3

Let $h \geq 1$. The homology of N_h is given by

$$H_n(N_h) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Similar to the proof in that of Σ_g, cut an open disc along the middle of the CW complex of N_h as follows



and again label the green part U and the orange part V. Then apply Mayer-Vietoris sequence to acquire a similar exact sequence

$$0 \longrightarrow \tilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \tilde{H}_1(\Sigma_g) \longrightarrow 0$$

together with $\tilde{H}_0(N_h) \cong 0$. Notice that the third non-zero term counting from the left is now $\mathbb{Z}^h$ instead of $\mathbb{Z}^{2g}$ as in the torus because the boundary circle is the wedge sum of h circles labelled $a_1 b_1, \dots, a_h b_h$. The map $\mathbb{Z}$ to $\mathbb{Z}^h$ is now given by sending the generator 1 to

$$2(a_1 + b_1 + \cdots + a_h + b_h)$$

The Smith Normal form of the matrix is an $h \times 1$ matrix with 2 at the first entry and 0 everywhere else. In particular, it means that this map is injective so that $\tilde{H}_2(N_h) \to \mathbb{Z}$ is the 0 map so that $\tilde{H}_2(N_h) \cong 0$. Now it remains an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \tilde{H}_1(N_h) \longrightarrow 0$$

The image of the matrix is $2\mathbb{Z}$ and by exactness this is the kernel of the map $\mathbb{Z}^h \to \tilde{H}_1(N_h)$. Thus we have an isomorphism

$$\tilde{H}_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}$$

and so we conclude. □

Corollary 2.2.4

The surfaces N_h for $h \geq 1$ is non-orientable.

Proof. Notice that removing a small closed disk from $\mathbb{RP}^2$ yields a space homeomorphic to the open Mobius strip. It follows that for $h > 0$, the space N_h contains the open Mobius strip as a subspace. Since the Mobius strip is non-orientable, N_h is also non-orientable. $\square$

2.3 The Euler Characteristic

Recall that if X is a CW complex such that $U \cap V = X$ and U and V are open subsets, then we have the formula

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Corollary 2.3.1

Let $S_1 \# S_2$ be the connected sum of two compact surfaces, then we have that

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

Proof. Let D_i be the gluing discs for S_i for $i = 1, 2$. Using the above formula, we have that

$$\chi(S_i) = \chi(D_i) + \chi(S_i \setminus D_i^\circ) - \chi(S^1)$$

since the intersection of the disc and S_i is S^1. It follows that

$$\begin{aligned} \chi(S_1 \# S_2) &= \chi(S_1 \setminus D_1^\circ) + \chi(S_2 \setminus D_2^\circ) - \chi(S^1) \\ &= \chi(S_1) + \chi(S_2) - 2 \end{aligned}$$

and so we conclude. $\square$

Corollary 2.3.2

For $g \geq 0$ and $h > 1$, the Euler characteristic of any compact surface is given by

$$\chi(\Sigma_g) = 2 - 2g \quad \text{and} \quad \chi(N_h) = 2 - h$$

Proof. It follows directly by repeated applications of the above corollary. $\square$

Recall that if $p : \tilde{X} \to X$ is a d-sheeted covering and X is a finite CW complex, then we have the formula

$$\chi(\tilde{X}) = d \cdot \chi(X)$$

3 Cohomology on Manifolds

3.1 de Rham Cohomology

Proposition 3.1.1

Let M be a smooth manifold. Then differential forms of M, $\Omega^0(M),\dots,\Omega^n(M),\dots$ together with the exterior derivative $d:\Omega^n(M)\to\Omega^{n+1}(M)$ form a cochain complex.

Definition 3.1.2

Let M be a smooth manifold. Define the de Rham cohomology groups of M to be the cohomology of the chain of differential forms:

$$H^n_{\text{dR}}(M;\mathbb{R})=H^n(\Omega^\bullet(M);\mathbb{R})$$

Proposition 3.1.3

Let M be a smooth manifold of dimension n. Then the following are true for the de Rham cohomology of M.

- $H^k_{\text{dR}}(M;\mathbb{R})$ is a vector space over $\mathbb{R}$ for all $k\in\mathbb{N}$.
- For $r>n$ we have $H^r_{\text{dR}}(M;\mathbb{R})=0$
- If M has m connected components then $H^0_{\text{dR}}(M;\mathbb{R})=\mathbb{R}^k$

Theorem 3.1.4

Let M be a smooth manifold of dimension n. Then the direct sum

$$H^*(M)=\bigoplus_{k=1}^n H^k_{\text{dR}}(M;\mathbb{R})$$

is an $\mathbb{R}$-algebra where multiplication defined by $a\wedge b\in H^{s+l}_{\text{dR}}(M;\mathbb{R})$ for $a\in H^s_{\text{dR}}(M;\mathbb{R})$ and $b\in H^l_{\text{dR}}(M;\mathbb{R})$. Moreover, this multiplication is anti-commutative, namely for $a\in H^s_{\text{dR}}(M;\mathbb{R})$ and $b\in H^l_{\text{dR}}(M;\mathbb{R})$, we have

$$a\wedge b=(-1)^{sl}b\wedge a$$

Proposition 3.1.5

Let M,N be smooth manifolds and $f:M\to N$ a smooth map. Then f induces an $\mathbb{R}$-linear map

$$f^*:H^*(N)\to H^*(M)$$

such that $f^*(a\wedge b)=f^*(a)\wedge f^*(b)$. Moreover, it is functorial:

- If $g:N\to K$ is another smooth map of manifolds, then $(g\circ f)^*=f^*\circ g^*$
- If $\text{id}:M\to M$ is the identity map on the manifold, then $\text{id}^*:H^*(M)\to H^*(M)$ is the trivial map on $\mathbb{R}$-algebras.

Theorem 3.1.6: Homotopy Invariance of de Rham Cohomology

Let $f:M\times I\to N$ be a smooth map of manifolds varying for each $t\in I=[0,1]$. Write $f_t(x)=f(x,t)$. Then the pull back maps $f_0^*,f_1^*:H^*(N)\to H^*(M)$ are equal:

$$f_0^*=f_1^*$$

3.2 de Rham Cohomology of Common Manifolds

Proposition 3.2.1

The real space $\mathbb{R}^n$ has the de Rham cohomology

$$H^k_{\text{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.2.2

The n-sphere S^n has the de Rham cohomology

$$H^k_{\text{dR}}(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.2.3

Let $p, q \geq 1$, the sphere S^{p+q} is not diffeomorphic to any $M \times N$ manifolds where $\dim(M) = p$ and $\dim(N) = q$.

Proposition 3.2.4

Every smooth vector fields on S^{2n} vanishes at some point of the sphere.

Proposition 3.2.5

The real projective space $\mathbb{RP}^n$ has the de Rham cohomology

$$H^k_{\text{dR}}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n \text{ where } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

4 Poincare Duality

4.1 The Cap Product

Definition 4.1.1: The Cap Product

Let $\sigma = [v_0, \dots, v_k] \in C_k(X)$ and $\phi \in C^l(X)$ where $k \geq l$ with coefficients in a ring R. Define the cap product to be

$$\sigma \frown \phi = \phi(\sigma|_{[v_0,\dots,v_l]})\sigma|_{[v_l,\dots,v_k]} \in C_{k-l}(X)$$

Lemma 4.1.2

The cap product $\frown: C_k(X) \times C^l(X) \to C_{k-l}(X)$ with coefficients in a ring R induces a cap product in homology $\frown: H_k(X) \times H^l(X, R) \to H_{k-l}(X)$ for $k \geq l$.

4.2 Cohomology with Compact Support

4.3 The Duality Theorem

Theorem 4.3.1: Poincare Duality

Let M be a compact and oriented topological n-manifold. Then the homomorphism

$$D : H^p(M) \to H_{n-p}(M)$$

is an isomorphism.