Topics in (Co)Homology

Labix

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Abstract

Contents

1 The Universal Coefficient Theorem for Homology

1.1 The Tor Functor

1.2 The Universal Coefficient Theorem

Theorem 1.2.1

Let $C_\bullet$ be a chain complex of free abelian groups. Let A be an abelian group. Then there exists a natural map $h : H_n(C_\bullet) \otimes A \to H_n(C_\bullet; A)$ such that $\text{coker}(h) \cong \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A)$ and a split exact sequence (that is not natural) of the form

$$0 \longrightarrow H_n(C_\bullet) \otimes A \xrightarrow{h} H_n(C_\bullet; A) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A) \longrightarrow 0$$

for any $n \in \mathbb{N}$. In particular, split exactness implies that there is an isomorphism

$$H_n(C_\bullet; A) \cong H_n(C_\bullet) \otimes A \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A)$$

for any $n \in \mathbb{N}$.

Corollary 1.2.2

Let (X, A) be a pair of space. Let T be an abelian group. Then there exists a natural map $h : H_n(X, A) \otimes T \to H_n(X, A; T)$ such that $\text{coker}(h) \cong \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T)$ and a split exact sequence (that is not natural) of the form

$$0 \longrightarrow H_n(X, A) \otimes T \xrightarrow{h} H_n(X, A; T) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T) \longrightarrow 0$$

for any $n \in \mathbb{N}$. In particular, split exactness implies that there is an isomorphism

$$H_n(X, A; T) \cong H_n(X, A) \otimes T \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T)$$

for any $n \in \mathbb{N}$.

1.3 The General Kunneth Theorem

Definition 1.3.1: The Homological Cross Product

Theorem 1.3.2

Let X and Y be CW-complexes. Let R be a principal ideal domain. Then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X; R) \otimes_R H_j(Y; R) \xrightarrow{\times} H_n(X \times Y; R) \longrightarrow \bigoplus_{i+j=n} \text{Tor}_1^R(H_i(X; R), H_{j-1}(Y; R)) \longrightarrow 0$$

induced by the cross product, that is natural in maps $f : X \to A$ and $g : Y \to B$. Moreover, this sequence splits.

2 The Cohomology of Some Topological Groups