Topics in (Co)Homology

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Abstract

Contents

1 The Universal Coefficient Theorem for Homology

1.1 The Tor Functor

1.2 The Universal Coefficient Theorem

Theorem 1.2.1

Let $C_\bullet$ be a chain complex of free abelian groups. Let A be an abelian group. Then there exists a natural map $h : H_n(C_\bullet) \otimes A \to H_n(C_\bullet; A)$ such that $\text{coker}(h) \cong \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A)$ and a split exact sequence (that is not natural) of the form

$$0 \longrightarrow H_n(C_\bullet) \otimes A \xrightarrow{h} H_n(C_\bullet; A) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A) \longrightarrow 0$$

for any $n \in \mathbb{N}$. In particular, split exactness implies that there is an isomorphism

$$H_n(C_\bullet; A) \cong H_n(C_\bullet) \otimes A \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(C_\bullet), A)$$

for any $n \in \mathbb{N}$.

Corollary 1.2.2

Let (X, A) be a pair of space. Let T be an abelian group. Then there exists a natural map $h : H_n(X, A) \otimes T \to H_n(X, A; T)$ such that $\text{coker}(h) \cong \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T)$ and a split exact sequence (that is not natural) of the form

$$0 \longrightarrow H_n(X, A) \otimes T \xrightarrow{h} H_n(X, A; T) \longrightarrow \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T) \longrightarrow 0$$

for any $n \in \mathbb{N}$. In particular, split exactness implies that there is an isomorphism

$$H_n(X, A; T) \cong H_n(X, A) \otimes T \oplus \text{Tor}_1^{\mathbb{Z}}(H_{n-1}(X, A), T)$$

for any $n \in \mathbb{N}$.

1.3 The General Kunneth Theorem

Definition 1.3.1: The Homological Cross Product

Theorem 1.3.2

Let X and Y be CW-complexes. Let R be a principal ideal domain. Then there is a short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X; R) \otimes_R H_j(Y; R) \xrightarrow{\times} H_n(X \times Y; R) \longrightarrow \bigoplus_{i+j=n} \text{Tor}_1^R(H_i(X; R), H_{j-1}(Y; R)) \longrightarrow 0$$

induced by the cross product, that is natural in maps $f : X \to A$ and $g : Y \to B$. Moreover, this sequence splits.

2 The Cohomology of Some Topological Groups

3 Cohomology Operations

4 Spectral Sequences in Algebraic Topology

4.1 Spectral Sequences in Topology

Theorem 4.1.1

Let X be a space. Let the following be a sequence

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X$$

of subspaces. Let G be an abelian group. Then the following data

- $A_{p,q} = H_{p+q}(X_p; G)$
- $E_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$
- $i : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_{p+1}; G) = A_{p+1,q-1}$ (degree $(1, -1)$)
- $j : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_p, X_{p-1}; G) = E_{p,q}$ (degree $(0, 0)$)
- $k : H_{p+q}(X_p, X_{p-1}; G) = A_{p,q} \to H_{p+q-1}(X_{p-1}; G) = A_{p-1,q}$ (degree $(-1, 0)$)

defines an exact couple and hence a spectral sequence with E^1 page given by

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$$

where the differential $d : E^1_{p,q} \to E^1_{p-1,q}$ is given by the composition

$$H_{p+q}(X_p, X_{p-1}; G) \xrightarrow{k} H_{p+q-1}(X_{p-1}; G) \xrightarrow{j} H_{p+q-1}(X_{p-1}, X_{p-2}; G)$$

The E_1 page of such a spectral sequence is given by

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & \cdots \\
H_3(X_1, X_0; G) & \longleftarrow & H_4(X_2, X_1; G) & \longleftarrow & H_4(X_3, X_2; G) & \longleftarrow & \cdots \\
H_2(X_1, X_0; G) & \longleftarrow & H_3(X_2, X_1; G) & \longleftarrow & H_4(X_3, X_2; G) & \longleftarrow & \cdots \\
H_1(X_1, X_0; G) & \longleftarrow & H_2(X_2, X_1; G) & \longleftarrow & H_3(X_3, X_2; G) & \longleftarrow & \cdots
\end{array}$$

Things get interesting when we choose X to be a CW complex and we choose the filtration of X by the skeleton of X. Recall that we have the formula

$$H_{p+q}(X_p, X_{p-1}; G) \cong \begin{cases} C_p^{\text{CW}}(X; G) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the E^1 page is only left with a chain complex at $q = 0$.

Let us also compute the derived couple of this exact couple or in other words, the E^2 page of the spectral sequence. This is more intuitive then the one thinks about on the definition of the derived couple. The $E^2_{p,q}$ slot is simply the homology of the chain complex at the (p, q)th slot. The direction of the maps of the E^2 page depends not on the choice of spectral sequence at all (In fact, the direction only depends on the page). Now in our case, the homology can be given by a known construct:

$$E^2_{p,q} = \frac{\ker(d : H_{p+q}(X_p, X_{p-1}; G) \to H_{p+q-1}(X_{p-1}, X_{p-2}; G))}{\text{im}(d : H_{p+q+1}(X_{p+1}, X_p; G) \to H_{p+q}(X_p, X_{p-1}; G))} = H^{\text{CW}}_{p+q}(X; G)$$

Since the direction of the maps are now diagonal and when $q \neq 0$ we have $E^2_{p,q} = 0$, all maps in E^2 are 0 and we are left with

$$H_0^{\text{CW}}(X; G) \qquad H_1^{\text{CW}}(X; G) \qquad H_2^{\text{CW}}(X; G) \qquad H_3^{\text{CW}}(X; G) \qquad \cdots$$

Theorem 4.1.2: Leray-Serre Spectral Sequence

Let $p : E \to B$ be a Serre fibration with fibre F and path connected B. Suppose that the action of $\pi_1(B)$ on $H_*(F;G)$ is trivial. Then there is a first quadrant homological spectral sequence starting with E^2 and weakly converging to $H_\bullet(E;\mathbb{Z})$. Explicitly, there is a convergence

$$E^2_{p,q} = H_p(B, H_1(F)) \Rightarrow H_{p+q}(E;\mathbb{Z})$$

4.2 Spectral Kunneth Theorem