Fiber Bundles

Labix

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Abstract

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Contents

1 The Topology of Fiber Bundles

1.1 Fiber Bundles

Fiber bundles serve as somewhat of a generalization of both vector bundles and covering spaces, while being a special case of a fibration. It therefore has the properties of a fibration.

Definition 1.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and $p : E \to B$ a continuous map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$ for all $b \in B$
- $p : E \to B$ is surjective
- Local Triviality: For every $x \in B$, there is an open neighbourhood $U \subset B$ of x and a homeomorphism $\phi_U : p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  where π is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

Intuitively, we would like a fiber bundle to locally look like the product $B \times F$. The condition is also equivalent to the following form: There exists an open cover $\{U_i \mid i \in I\}$ and a collection of homeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times F$ for which the same diagram commutes.

Vector bundles generalizes vector bundles in the sense that the fibers are no longer vector spaces but instead arbitrary spaces.

Lemma 1.1.2

Every vector bundle is a fiber bundle.

Proof. Indeed if $p : E \to B$ is a vector bundle, then each fiber $p^{-1}(b)$ is an n-dimensional vector spaces over a field F. Moreover, by definition the local triviality condition is also satisfied. □

A lot of examples of fiber bundles therefore come from vector bundles. Another familiar collection of examples come from covering space theory.

Lemma 1.1.3

Every covering space is a fiber bundle.

Proof. If $p : \tilde{X} \to X$ is a covering space, then we have seen that $p^{-1}(x)$ remains constant as $x \in X$ varies. Moreover, $p^{-1}(x)$ has the discrete topology with countable fiber since each $p^{-1}(U)$ is a disjoint union for $U \subseteq X$ open. Thus they must all be homeomorphic.

Finally, for any $U \subseteq X$, recall that

$$p^{-1}(U) = \coprod_{i \in I} V_i$$

where each $V_i \cong U$. It is clear by definition that $|p^{-1}(x)||I|$ for any $x \in X$. By giving I the discrete topology, we obtain a homeomorphism $p^{-1}(x) \cong I$. The homeomorphism $p^{-1}(U) = \coprod_{i\in I} V_i$ translates to

$$p^{-1}(U) = \coprod_{i\in I} V_i \cong \coprod_{i\in I} U \cong U \times I$$

defined by $\tilde{x} \in V_i \mapsto (p(\tilde{x}) = x, i)$. It is thus clear that the local triviality condition is satisfied. $\square$

Proposition 1.1.4

Every fiber bundle is a Serre fibration.

We can provide a partial converse for the fact that every fiber bundle is a Serre fibration.

Proposition 1.1.5

Let $p : E \to B$ be a fiber bundle. If B is paracompact, then p is a (Hurewicz) fibration.

We there fore have inclusions

$$\begin{matrix}\text{Fiber}\\\text{Bundles}\end{matrix} \subset \begin{matrix}\text{Serre}\\\text{Fibrations}\end{matrix} \subset \begin{matrix}\text{(Hurewicz)}\\\text{Fibrations}\end{matrix}$$

Definition 1.1.6: Map of Fiber Bundles

Let (F_1, E_1, B_1) and (F_2, E_2, B_2) be fiber bundles. A map of fiber bundles is a pair of base-point preserving continuous maps $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

Such a map of fiber bundles determine a continuous of the fibers $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$.

A map of fiber bundles $(\tilde{f}, f)$ is said to be an isomorphism if there is a map $(\tilde{g} : E_2 \to E_1, g : B_2 \to B_1)$ such that $\tilde{g}$ is the inverse of $\tilde{f}$ and g is the inverse of f.

Notice that a morphism of fiber bundles preserves fibers. Indeed, If $p_1^{-1}(b)$ is a fiber of B, then using the commutativity of the diagram we have that

$$p_2(\overline{f}(p_1^{-1}(b))) = f(p_1(p_1^{-1}(b))) = f(b)$$

which implies that

$$p_2^{-1}(f(b)) = \overline{f}(p^{-1}(b))$$

or in other words, the fiber at $f(b)$ is the same as the fiber at b applied with $\overline{f}$.

Definition 1.1.7: Equivalent Fiber Bundles

Let $p : E_1 \to B_1$ and $p : E_2 \to B_2$ be two fiber bundles. We say that they are equivalent if there exists an isomorphism $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ of fiber bundles.

There are two important special cases of fiber bundles that will appear time and time again.

Definition 1.1.8: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration $B \times F \to B$.

Definition 1.1.9: The Pullback Bundle

Let $p : E \to B$ be a fiber bundle with fiber F. Let $f : B' \to B$ be a continuous function. Define the pullback of p by f to be the space

$$f^*(E) = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$$

1.2 Sections of a Bundle

Definition 1.2.1: Sections

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map $s : B \to E$ such that

$$p \circ s = \mathrm{id}_B$$

Definition 1.2.2: Local Sections

Let (F, E, B) be a fiber bundle. Let $U \subset B$ be an open set. A local section of the fiber bundle on U is a map $s : U \to B$ such that

$$p \circ s = \mathrm{id}_U$$

1.3 Homotopy of Fiber Bundles

Theorem 1.3.1

Let $p : E \to B$ be a fiber bundle. Suppose that $f, g : X \to B$ are homotopic maps. Then the pull back bundles

$$f^*(E) \cong g^*(E)$$

are equivalent.

Theorem 1.3.2

Let $p : E \to B$ be a fiber bundle. Let $A \subseteq B$. Let $y_0 \in E$ and $p(y_0) = x_0$. Then there is an isomorphism

$$\pi_n(E, p^{-1}(A), y_0) \cong \pi_n(B, A, x_0)$$

given by the induced map p_* for all $n \geq 2$.

2 Fiber Bundles with a Group Structure

2.1 G-Bundles and the Structure Groups

We would now like to enrich the structure of a fiber bundle with a group action on the fibers. Recall that a topological group is a group with the structure of a topology such that multiplication $(g_1, g_2) \mapsto g_1 g_2$ and the inverse $g \mapsto g^{-1}$ are both continuous maps.

Also, recall that a group action is faithful if for all $g \in G$, $g \cdot x \neq x$ for all $x \in X$.

Definition 2.1.1: G-Atlas

Let (F, E, B) be a fiber bundle. Let G be topological group with a continuous faithful action on F. A G-atlas on (F, E, B) is a set of local trivalization charts $\{(U_k, \varphi_k) \mid k \in I\}$ such that the following are true.

- For (U_k, φ_k) a chart, define $\varphi_{i,x} : F \to F$ by $f \mapsto \varphi_i(x, f)$. Then the homeomorphism
$$\varphi_{j,x} \circ \varphi_{i,x}^{-1} : F \to F$$
for $x \in U_i \cap U_j \neq \emptyset$ is an element of G.
- For $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to G$ defined by
$$g_{ij}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$$
is continuous. These g_{ji} are called coordinate transformations.

Notice that for non empty intersections $U_i \cap U_j$ for U_i, U_j open sets in B, there is a well defined homeomorphism
$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$
This is reminiscent of properties of an atlas on M.

Lemma 2.1.2

Let (F, E, B) be a fiber bundle and let G be a topological group. Let $\{(U_k, \varphi_k) \mid k \in I\}$ be a G-atlas on (F, E, B). Then the following are true regarding the coordinate transformations $g_{ji}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$

- Cocycle condition: $g_{ki}(x) = g_{kj}(x) g_{ji}(x)$ for all $x \in U_i \cap U_j \cap U_k$.
- $g_{ji}(x) = (g_{ij}(x))^{-1}$ for all $x \in U_i \cap U_j$
- $g_{ii}(x) = 1$ for all $x \in U_i$

Definition 2.1.3: Equivalent G-Atlas

Let G be a topological group. Let (F, E, B) be a fiber bundle and let $\{(U_i, \varphi_i \mid i \in I\}$ and $\{(V_j, \phi_j) \mid j \in J\}$ be two G-atlas on the fiber bundle. We say that they are equivalent if for all $x \in U_i \cap V_j$,
$$\overline{g}_{ji}(x) = \phi_{j,x} \circ \phi_{i,x}^{-1}$$
is an element of G and the map
$$\overline{g}_{ji} : U_i \cap V_j \to G$$
defined by $x \mapsto \overline{g}_{ji}(x)$ is continuous.

It is clear that this is an equivalent relation. It is reflexive directly from definitions of a G-atlas. ???

Definition 2.1.4: G-Bundle

Let G be a topological group. A G-bundle is a fiber bundle (F, E, B) together with an equivalence class of G-atlas. In this case, G is said to be the structure group of the fiber bundle.

Similar to the case of manifolds, an maximal atlas is exactly such an equivalence class of atlases. Therefore one can also think of a G-bundle as a fiber bundle equipped with a maximal G-atlas.

A morphism of G-bundles must now take into account of the group G.

Definition 2.1.5: Morphisms of G-Bundles

Let G be a topological group. Let (F, E_1, B_1) and (F, E_2, B_2) be two G-bundles with common fiber. A morphism of G-bundles is a morphism of bundles

$$(\overline{h} : E_1 \to E_2, h : B_1 \to B_2)$$

such that the following are true.

Let $\{(U_i, \varphi_i) \mid i \in I\}$ and $\{(V_j, \phi_j) \mid j \in J\}$ be local trivialities.

- For all $x \in U_i \cap h^{-1}(V_j)$, the map
$$\overline{g_{ji}}(x) = \phi_{j,x} \circ h|_{F_x} \circ \varphi_{i,x}^{-1} : F \to F$$
is an element of G.
- The map
$$\overline{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$$
defined by $x \mapsto \overline{g_{ji}}(x)$ is continuous.

Since morphisms of fiber bundles preserve fibers, morphisms of G-bundles also preserve fibers. It is easy to see that the mapping transformations $\widetilde{g_{ij}}$ satisfy the following two relations:

- $\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap U_j \cap h^{-1}(V_k)$
- $g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap h^{-1}(V_j \cap V_k)$

g'_{jk} here refers to the transition charts in (F, E_2, B_2).

Just as the structure groups and trivialization charts determine the isomorphism type of a fiber bundle, the $\widetilde{g_{ij}}$ and a map of base space $h : B_1 \to B_2$ completes determines a morphism of G-bundle.

Theorem 2.1.6

Let (F, E_1, B_1) and (F, E_2, B_2) be two G-bundles for a topological group G with the same fiber F. Suppose that we have the following.

- A map $h : B_1 \to B_2$ of base space
- A collection $\{\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G \mid i \in I, j \in J\}$ of continuous maps such that
$$\begin{aligned} \widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap U_j \cap h^{-1}(V_k) \\ g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap h^{-1}(V_j \cap V_k) \end{aligned}$$

Then there exists a unique G-bundle morphism having h as the map of base space and having $\{\widetilde{g_{ij}} \mid i, j \in I\}$ as its mapping transformations.

We end the section by noting that a collection of coordinate transformations satisfying the cocyle condition uniquely determines the structure of a fiber bundle.

Theorem 2.1.7

Let X be a space. Let G be a topological group. Then the coordinate transformations uniquely determines and is determined by a fiber bundle in the following sense:

$$\left\{\substack{G\text{-bundles over } B\\ \text{with fiber } F}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{\text{An open cover } \{U_i \mid i\in I\} \text{ of } B\\ \text{Maps } \{\phi_{i,j}:U_i\cap U_j\to G \mid i,j\in I\} \text{ satisfying the cocyle condition}\\ \text{A space } F \text{ such that } G \text{ acts continuously on}}\right\}$$

For each G-bundle, the coordinate transformations satisfy the cocyle condition. For a collection of coordinate transformations $\{\phi_{i,j} : U_i \cap U_j \to G \mid i, j \in I\}$, consider the space

$$E = \frac{\bigcup_{i\in I} U_i \times F}{\sim}$$

where $(x, k) \sim (x, k\phi_{j,i}(x))$ for $x \in U_i \cap U_j$. Then E defines a fiber bundle with an appropriate ???? continuous map.

2.2 Associated Bundle

Definition 2.2.1: Associated Bundles with Specified Fiber

Let G be a topological group. Let (F, E, B) be a fiber bundle with structure group G and open cover $\{U_i \mid i \in I\}$ and coordinate transformations $\{g_{i,j} \mid i, j \in I\}$. If F is a G-space, then define the associated bundle with fiber F to be given by theorem 2.1.7 using the following data:

- The open cover $\{U_i \mid i \in I\}$
- The coordinate transformations $g_{ij} : U_i \cap U_j \to G$
- The G-space F

2.3 The Bundle Structure Theorem

Theorem 2.3.1: The Bundle Structure Theorem

Let B be a topological group and let G be a closed subgroup of B. Let H be a closed subgroup of G. Suppose that B/G has a local cross section at $G \in B/G$. Then the induced map of cosets

$$p : B/H \to B/G$$

give a G/H_0-bundle with fiber G/H, where

$$H_0 = \bigcap_{g\in G} gHg^{-1}$$

Theorem 2.3.2

Let B be a topological group and let G be a closed subgroup of B. Let H be a closed subgroup of G. Then any two local cross sections of B/G at $G \in B/G$ induces equivalent bundles.

Corollary 2.3.3

Let B be a topological group and let G be a closed subgroup of B. Then the projection map $p : B \to B/G$ is a G-bundle with fiber G.

2.4 Reduction of the Structure Group

3 Principal Bundles and Classifying Spaces

3.1 Point Set Topology Principal G-Bundles

Let us recall some definitions from Groups and Rings. Let G be a group acting on a set X. We say that a group action is free if $g \cdot x = x$ for all $x \in X$ implies that $g = 1_G$. A group action is transitive if for any $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

Definition 3.1.1: Principal Bundles

Let G be a topological group. A principal G-bundle is a G-bundle (F, E, B) together with a continuous group action G on E such that the following are true.

- The action of G preserves fibers. This means that $g \cdot x \in E_b$ if $x \in E_b$. (This also means that G is a group action on each fiber)
- The action of G on each fiber is free and transitive
- The map $G \to F$ defined by sending $g \mapsto g \cdot x$ for any $x \in X$ is a homeomorphism.
- Local triviality condition: Each local triviality map $\varphi_U : p^{-1}(U) \to U \times F$ are G-equivariant maps.

Recall that even if G acts freely and transitively on X, G is still not homeomorphic to X. This is because the map $G \to X$ defined by $g \mapsto g \cdot x$ for any $x \in X$ is continuous and bijective. One usually requires that either G is compact or more in general, the above map is an open map then G will be homeomoprhic to X.

For those who know what homogenous spaces are, principal bundles are G-bundles such that F is a principal homogenous space for the left action of G itself.

Proposition 3.1.2

Let G be a topological group. Let $p : E \to B$ be a principal G-bundle. Then there is a homeomorphism

$$\frac{E}{G} \cong B$$

where E/G is the orbit space.

Conversely, given a continuous group action on a space, we can ask in what conditions will the space be a principal bundle over the orbit space.

Proposition 3.1.3

Let X be a space with a free G action. Let $p : X \to X/G$ be the projection map to the orbit space. If for all $x \in X/G$, there is a neighbourhood U of x and a continuous map $s : U \to X$ such that $p \circ s = \text{id}_U$, then $(G, X, X/G)$ is a principal G-bundle.

This proposition essentially means that if for each point in X/G, there is a local section, then it is sufficient for X to be a principal G bundle over X/G.

Proposition 3.1.4

Let X be a space. Let $p : \tilde{X} \to X$ be a covering space. If p is a normal covering space, then p is a principal G-bundle for some group G.

3.2 Morphisms of Principal G-Bundles

Definition 3.2.1: Morphism of Principal Bundles

Let G be a topological group. Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two prinicipal G-bundle. A morphism of principal G-bundles is a morphism of fiber bundles $(\tilde{f}, f)$ such that $\tilde{f}$ is a G-equivariant map.

Definition 3.2.2: Isomorphism of Principal Bundles

Let G be a topological group. Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be two prinicipal G-bundle. We say that they are isomorphic if there exists a morphism of principal bundles $(\tilde{f}, f)$ such that it is an isomorphism in the sense of 1.1.6.

Principal bundles enjoy a very unique property. If one can construct a morphism between two principal bundles then the structure of a fiber as the structure group ensures that the two bundles are isomorphic.

Proposition 3.2.3

Every morphism of principal bundles is an isomorphism of principal bundles.

Theorem 3.2.4

Let G be topological group. A principal G-bundle is trivial if and only if it admits a global section.

This is entirely untrue for general bundles. For examples, the zero section of a fiber bundle is a global section.

3.3 Associated Principal Bundles

Principal bundles are easier to understand than fiber bundles with structure group because their fibers are isomorphic to the group G and its action is just left translations. If we are able to translate some properties of fiber bundles to principal bundles we can simplify proofs.

Definition 3.3.1: Associated Principal Bundles

Let G be a topological group. Let (F, E, B) be a fiber bundle with structure group G. The associated principal G-bundle is defined to be the associated bundle with fiber G, and G acts on itself by left translations.

Proposition 3.3.2

Let G be a topological group. Let (F, E, B) be a fiber bundle with structure group G. Then the associated principal bundle of the bundle is a principal bundle in its own right.

Theorem 3.3.3

Two fiber bundles are equivalent if and only if their associated principal bundles are equivalent.

Corollary 3.3.4

A fiber bundle is equivalent to the trivial bundle if and only if its associated principal bundle admits a cross section.

3.4 Classifying Space

Recall that homotopic maps give isomorphic fiber bundles. (???: Descends to isomorphic principal G-bundles)

Definition 3.4.1: Principal Bundle Functor

Let G be a topological group and X a space. Define a contravariant functor $\text{Prin}_G : \textbf{hTop} \to \textbf{Set}$ as follows.

- For X a topological space, $\text{Prin}_G(X)$ is the set of isomorphism classes of principal G-bundles over X.
- If $[f : X \to Y]$ is a homotopy class of continuous maps,
$$\text{Prin}_G([f]) : \text{Prin}_G(Y) \to \text{Prin}_G(X)$$
is defined as follows. If $[p : E \to Y]$ is an isomorphism class of principal G-bundles over Y, then it is sent to $[f^*(E)]$ the isomorphism class of the pullback of p.

Theorem 3.4.2

Let G be ta topological group. Then the principal bundle functor is representable. Explicitly, this means that there exists a principal G-bundle $EG \to BG$ together with a natural isomorphism
$$\psi : [X, BG] \to \text{Prin}_G(X)$$
This natural isomorphism is defined by $f \mapsto [f^*(EG)]$.

Definition 3.4.3: Universal G-Bundles

Let G be a topological group. A principal G-bundle (F, E, B) is said to be universal if it represents the principal bundle functor.

Theorem 3.4.4

Let (F, E, B) be a principal G-bundle. If E is contractible then (F, E, B) is a universal G-bundle.

A surprising thing is that BG is not determined by its isomorphism type but instead by the weaker condition of its homotopy type.

Theorem 3.4.5

Let (F, E_1, B_1) and (F, E_2, B_2) be universal principal G-bundles. Then there exists a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

such that f is a homotopy equivalence. In particular, this means that any two universal principal G-bundles are homotopy equivalent.

Definition 3.4.6: Classifying Space

Let G be a topological group. The classifying space BG of G is the homotopy type of the universal principal G-bundle. Denote the total space of BG by EG. For a principal G-bundle

$f : Y \to X \in \mathrm{Prin}_G(X)$, define the classifying map to be the associated map $X \to BG$ given in 1.5.3.

TBA: Functoriality of $B : \mathbf{Grp} \to \mathbf{Top}$.

4 Vector Bundles and K_0

4.1 Relation to Principal Bundles

Definition 4.1.1: Frame Bundle

Let $p : E \to B$ be a vector bundle. Define the frame bundle to be the associated principal bundle of $p : E \to B$ viewed as a fiber bundle.

Explicitly, the total space is given by

$$E = \frac{\bigcup_{i\in I} U_i \times \text{GL}(n,\mathbb{R})}{\sim}$$

where $(x, A) \sim (x, A\phi_{j,i}(x))$ for $x \in U_i \cap U_j$.

Definition 4.1.2: Set of All Vector Bundles

Let X be a space. Denote the isomorphism classes of all $\mathbb{R}$-vector bundles of dimension n and $\mathbb{C}$-vector bundles of dimension n respectively by

$$\text{Vect}_n^{\mathbb{R}}(X) \quad \text{and} \quad \text{Vect}_n^{\mathbb{C}}(X)$$

Denote the isomorphism classes of all $\mathbb{R}$ (respectively $\mathbb{C}$) vector bundles regardless of rank by $\text{Vect}^{\mathbb{R}}(X)$ (respectively $\text{Vect}^{\mathbb{C}}(X)$).

Theorem 4.1.3

Let X be a space. Then there is a natural bijection

$$\phi : \text{Prin}_{\text{GL}(n,\mathbb{R})}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{R}}(X)$$

given by mapping $p : E \to B$ to the frame bundle $F(E)$. Similarly, there is a natural bijection

$$\phi : \text{Prin}_{\text{GL}(n,\mathbb{C})}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{C}}(X)$$

4.2 The Orthogonal Group as the Structure Group

Theorem 4.2.1

Let $n \in \mathbb{N}$, then there is an isomorphism in the classifying spaces

$$B\text{GL}(n,\mathbb{R}) \cong BO(n) \cong \text{GL}_n(\mathbb{R}^\infty)$$

Theorem 4.2.2

Let $n \in \mathbb{N}$, then there is an isomorphism in the classifying spaces

$$B\text{GL}(n,\mathbb{C}) \cong BU(n)$$

Theorem 4.2.3

Let X be a paracompact space. Then there is a natural bijection

$$\phi : \text{Prin}_{O(n)}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{R}}(X)$$

given by mapping $p : E \to B$ to the frame bundle $F(E)$. Similarly, there is a natural bijection

$$\phi : \text{Prin}_{U(n)}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{C}}(X)$$

4.3 The Tautological Bundle

4.4 The Thom Isomorphism

Definition 4.4.1: Unit Sphere and Unit Disc Bundle

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$. Let $\langle -, - \rangle : E \times E \to \mathbb{R}$ be a smoothly varying inner product on E. Define the disc bundle to be

$$D(E) = \{e \in E \mid \langle e, e \rangle \leq 1\}$$

together with the map $p|_{D(E)} : D(E) \to B$. Define the sphere bundle to be

$$S(E) = \{e \in E \mid \langle e, e \rangle = 1\}$$

together with the map $p|_{S(E)} : S(E) \to B$.

Definition 4.4.2: Thom Space

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$ such that B is paracompact. Define the Thom space of E to be

$$T(E) = \frac{D(E)}{S(E)}$$

The base point is taken as the equivalent class $S(E)$ if needed.

Lemma 4.4.3

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$. Let E_0 denote the zero section of E. Then there is a natural isomorphism

$$\widetilde{H}^n(T(E); G) = H^n(E, E \setminus E_0)$$

for any abelian group G.

Theorem 4.4.4: The Thom Isomorphism

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$. Let E_0 denote the zero section of E. Then there exists a unique $u \in H^n(E, E \setminus E_0; \mathbb{Z}/2\mathbb{Z})$ such that

$$u|_{(F_b, F_b \setminus \{0\})} \in H^n(F_b, F_b \setminus \{0\}; \mathbb{Z}/2\mathbb{Z})$$

is non-zero for all $b \in B$. Moreover, there is an isomorphism

$$\Phi : H^k(E; \mathbb{Z}/2\mathbb{Z}) \to \widetilde{H}^{k+n}(E, E \setminus E_0; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H^n}(T(E); \mathbb{Z}/2\mathbb{Z})$$

given by $y \mapsto y \smile u$ for all $k \in \mathbb{Z}$.

Ref: Milnor

4.5 Orientation of a Bundle

Definition 4.5.1: Orientation of a Vector Space

Let V be a finite dimensional vector space over F. An orientation on V is an equivalence class of bases, where we say that two ordered bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are equivalent if the matrix defined by the equations

$$w_i = \sum_{k=0}^{n} a_k v_k$$

has positive determinant.

Lemma 4.5.2

Let V be a finite dimensional vector space. Then there are only two possible orientations on V.

Definition 4.5.3

Let $p : E \to B$ be a vector bundle with fiber F. An orientation on E is an assignment of an orientation to each fiber of E such that the following local compatibility condition is satisfied.

For every $b \in B$, there exists a local coordinate system (U, φ) of b and $\varphi : U \times \mathbb{R}^n \to p^{-1}(U)$ such that for all $x \in U$, the homomorphism $\varphi(b, -) : \mathbb{R}^n \to F$ is orientation preserving.

Theorem 4.5.4

Let $p : E \to B$ be a vector bundle with fiber F. An orientation on E is equivalent to the following data. To each $b \in B$ there is assignment

$$u_b \in H^n(F_b, F_b \setminus \{0\}; \mathbb{Z})$$

called the orientation class of F_b, such that for every $b \in B$, there exists a neighbourhood U of b and a cohomology class

$$u \in H^n(p^{-1}(U), p^{-1}(U) \setminus 0;)$$

where 0 is the zero section such that for every $x \in U$,

$$u|_{(F_x, F_x \setminus \{0\})} \in H^n(F_x, F_x \setminus \{0\}\ \mathbb{Z})$$

is equal to u_b.

Theorem 4.5.5: The Thom Isomorphism

Let $p : E \to B$ be an orientable n-dimensional vector bundle over $\mathbb{R}$. Let R be a ring. Let E_0 denote the zero section of E. Then there exists a unique $u \in H^n(E, E \setminus E_0; R)$ such that

$$u|_{(F_b, F_b \setminus \{0\})} \in H^n(F_b, F_b \setminus \{0\}; R)$$

gives precisely the orientation class on F_b for all $b \in B$. Moreover, there is an isomorphism

$$\Phi : H^k(E; R) \to \widetilde{H}^{k+n}(E, E \setminus E_0; R)$$

given by $y \mapsto y \smile u$ for all $k \in \mathbb{Z}$.

5 Characteristic Classes

5.1 Characteristic Classes as a Ring

Definition 5.1.1: Characteristic Classes

Let G be a topological group and X a space. Denote $\text{Prin}_G(X)$ the isomorphism classes of principal G-bundles over X. Let $H^*(-)$ be a cohomology functor. A characteristic class for G is a natural transformation

$$c : \text{Prin}_G(-) \Rightarrow H^*(-)$$

Explicitly, if $p : E \to X$ is a principal G-bundle, then c assigns p to the collection of cohomology groups $c(p) \in H^*(X)$.

Here cohomology can be taken for example singular cohomology with coefficients in a fixed group.

Lemma 5.1.2

Let G be a topological group. Let c be a characteristic class for G. If e is the trivial G-bundle, then $c(e) = 0$.

Definition 5.1.3: Ring of Characteristic Classes

Let G be a topological group. Let R be a commutative ring. Define $\text{Char}_G(R)$ to be the set of all characteristic classes for principal G-bundles that take values in $H^*(-; R)$.

Proposition 5.1.4

Let G be a topological group. Let R be a commutative ring. Then $\text{Char}_G(R)$ is a ring with unit the constant characteristic class.

Theorem 5.1.5

Let G be a topological group and let R be a commutative ring. Then there is an isomorphism

$$\text{Char}_G(R) \cong H^*(BG; R)$$

5.2 The Stiefel-Whitney Class

Definition 5.2.1: The Stiefel-Whitney Class

Consider the group $O(n)$. Let

$$w_i : \text{Prin}_{O(n)}(-) \to H^i(-, \mathbb{Z}/2\mathbb{Z})$$

be a collection of natural transformations. We say that they form a Stiefel-Whitney class if the following are satisfied.

1. Rank: If E is a principal $O(n)$-bundle, then $w_0(E) = 1$ and $w_i(E) = 0$ for $i > \text{rank}(E)$.
2. Naturality: Let $p : E \to X$ be a principal $O(n)$-bundle and let $f : Y \to X$ be a map. Then
$$w_i(f^*(E)) = f^*(w_i(E))$$
3. Whitney Product Formula: If E_1, E_2 are principal $O(n)$-bundles, then
$$w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \smile w_{k-i}(E_2)$$

4. Normalization: If γ is the tautological line bundle over $\mathbb{P}^1(\mathbb{R})$, then $w_1(\gamma)$ is non-zero.

The total Stiefel-Whitney class

$$w = \sum_{k=0}^{\infty} w_k : \text{Prin}_{O(n)}(-) \to H^*(-, \mathbb{Z}/2\mathbb{Z})$$

is well defined by the rank axiom. The Whitney product formula then translates to $w(E_1 \oplus E_2) = w(E_1)w(E_2)$.

Theorem 5.2.2

The Stiefel-Whitney class exists and is unique.

Proposition 5.2.3

The following are true regarding the Stiefel-Whitney class.

- If $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ are isomorphic principal $O(n)$-bundles, then $w(E_1) = w(E_2)$
- If $e = B \otimes \mathbb{R}^n$ is the trivial bundle, then $w(e \oplus E) = w(E)$ for any principal $O(n)$-bundle E.
-

Theorem 5.2.4

Let $n \in \mathbb{N}$, then the ring of characteristic classes of $O(n)$ is isomorphic to

$$\text{Char}_{O(n)}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1, \dots, w_n]$$

a polynomial ring in n variables for $w_i \in H^i(BO(n), \mathbb{Z}/2\mathbb{Z})$.

Proposition 5.2.5

Let X be a space. Then the function

$$w_1 : \text{Vect}_1^{\mathbb{R}}(X) \to H^1(X; \mathbb{Z}/2\mathbb{Z})$$

is a homomorphism. It is an isomorphism if X is homotopy equivalent to a CW complex.

Corollary 5.2.6

Let X be homotopy equivalent to a CW complex. Then any vector bundle $E \to X$ is orientable if and only if $w_1(E) = 0$.

5.3 The Chern Class

Definition 5.3.1: The Chern Class

Consider the group $U(n)$. Let

$$c_i : \text{Prin}_{U(n)}(-) \to H^{2i}(-, \mathbb{Z}/2\mathbb{Z})$$

be a collection of natural transformations. We say that they form a Chern class if the following are satisfied.

1. Rank: If E is a principal $U(n)$-bundle, then $c_0(E) = 1$ and $c_i(E) = 0$ for $i > \text{rank}(E)$.
2. Naturality: Let $p : E \to X$ be a principal $O(n)$-bundle and let $f : Y \to X$ be a map. Then
$$c_i(f^*(E)) = f^*(c_i(E))$$
3. Whitney Product Formula: If E_1, E_2 are principal $O(n)$-bundles, then
$$c_k(E_1 \oplus E_2) = \sum_{i=0}^{k} c_i(E_1) \smile c_{k-i}(E_2)$$
4. Normalization: If γ is the tautological line bundle over $\mathbb{P}^1(\mathbb{R})$, then $c_1(\gamma)$ is non-zero.

The total Chern class
$$c = \sum_{k=0}^{\infty} c_k : \text{Prin}_{O(n)}(-) \to H^*(-, \mathbb{Z}/2\mathbb{Z})$$
is well defined by the rank axiom. The Whitney product formula then translates to $c(E_1 \oplus E_2) = c(E_1)c(E_2)$.

Theorem 5.3.2

The Chern class exists and is unique.

Theorem 5.3.3

Let E be an n-dimensional complex vector bundle over X. Then $c_1(E) = 0$ if and only if E has an $SU(n)$-structure.

TBA: First chern class is complete invariant of complex line bundles. First Stiefel-Whitney class is a complete invariant of real line bundle.

Theorem 5.3.4

Let $n \in \mathbb{N}$, then the ring of characteristic classes of $U(n)$ is isomorphic to
$$\text{Char}_{U(n)}(\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$$
a polynomial ring in n variables for $c_i \in H^{2i}(BU(n), \mathbb{Z})$.

Proposition 5.3.5

Let X be a space. Then the function
$$c_1 : \text{Vect}_1^{\mathbb{C}}(X) \to H^2(X; \mathbb{Z})$$
is a homomorphism. It is an isomorphism if X is homotopy equivalent to a CW complex.

5.4 The Euler Class

The Euler class can be thought of as a refinement of the Stiefel-Whitney class in the orientable case.

Definition 5.4.1: The Euler Class

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. Let $E_0 \subseteq E$ denote the zero section. Consider the inclusion $B \hookrightarrow E$ as E_0. Let $u \in H^n(E, E \setminus E_0; \mathbb{Z})$ be the orientation class. Define the euler class of E

$$e(E) \in H^n(B;\mathbb{Z})$$

to be the image of u under the compositions

$$H^n(E, E \setminus E_0; \mathbb{Z}) \longrightarrow H^n(E, \mathbb{Z}) \longrightarrow H^n(B; \mathbb{Z})$$

that is induced by the sequence of inclusions $(B, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E \setminus E_0)$.

Proposition 5.4.2

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. Then the following are true regarding the Euler class.

- If $f : C \to B$ is a map, then $e(f^*(E)) = f^*(e(E))$
- If the orientation of E is reversed, then $e(E)$ changes sign.
- If F is another orientable vector bundle, then $e(E \oplus F) = e(E) \smile e(F)$.

Proposition 5.4.3

Let $p : E \to B$ be an orientable vector bundle over $\mathbb{R}$. If the dimension of the bundle is odd, then $2e(E) = 0$.

Proposition 5.4.4

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. The natural homomorphism

$$H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2\mathbb{Z})$$

sends the Euler class $e(E)$ to the top Stiefel-Whitney class $w_n(E)$.

Proposition 5.4.5

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. If E possess a nowhere 0 section, then $e(E) = 0$.

5.5 The Pontrjagin Class

6 Obstruction Theory