Exercise Sheet for Week 9

Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f is discontinuous at all points $c \in \mathbb{R}$.

Question 2

Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that f is continuous at 0 and discontinuous at all points $c \in \mathbb{R} \setminus \{0\}$.

Question 3

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Show that there exists $x, y \in [0, 1]$ such that $|x - y| = \frac{1}{2}$ and $f(x) = f(y)$. (Hint: define a new function $g : [0, 1/2] \to \mathbb{R}$ and apply the intermediate value theorem)