Cohomology of Schemes

Labix

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Abstract

References:

Contents

1 Symmetric Polynomials

1.1 Symmetric Polynomials

The theory of symmetric functions are important in combinatorics, representation theory, Galois theory and the theory of λ-rings.

Requirements: Groups and Rings
Books: Donald Yau: Lambda Rings

Definition 1.1.1: Symmetric Group Action on Polynomial Rings

Let R be a ring. Define a group action of S_n on $R[x_1, \dots, x_n]$ by

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

It is easy to check that this defines a group action.

Definition 1.1.2: Symmetric Polynomials

Let R be a ring. We say that a polynomial $f \in R[x_1, \dots, x_n]$ is symmetric if

$$\sigma \cdot f = f$$

for all $\sigma \in S_n$.

Definition 1.1.3: The Ring of Symmetric Polynomials

Let R be a ring. Define the ring of symmetric polynomials in n variables over R to be the set

$$\Sigma = \{f \in R[x_1, \dots, x_n] \mid \sigma \text{ is a symmetric polynomial } \}$$

Definition 1.1.4: Elementary Symmetric Polynomials

Let R be a ring. Define the elementary symmetric polynomials to be the elements $s_1, \dots, s_n \in R[x_1, \dots, x_n]$ given by the formula

$$s_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

Theorem 1.1.5: The Fundamental Theorem of Symmetric Polynomials

Let R be a ring. Then $s_1, \dots, s_n$ are algebraically independent over R. Moreover,

$$\Sigma = R[s_1, \dots, s_n]$$

2 λ-Rings

2.1 λ-Rings

Complex representation of a group is a λ-ring. Topological K theory is a λ-ring.

Requirements: Category Theory, Groups and Rings, Symmetric Functions
Books: Donald Yau: Lambda Rings

We need the theory of symmetric polynomials before defining λ-structures.

Definition 2.1.1: λ-Structures

Let R be a commutative ring. A λ-structure on R consists of a sequence of maps $\lambda^n : R \to R$ for $n \geq 0$ such that the following are true.

- $\lambda^0(r) = 1$ for all $r \in R$
- $\lambda^1 = \text{id}_R$
- $\lambda^n(1) = 0$ for all $n \geq 2$
- $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$ for all $r, s \in R$
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \dots, \lambda^{mn}(r))$ for all $r \in R$

Here P_n and $P_{m,n}$ are defined as follows.

- The coefficient of t^n in the polynomial
$$h(t) = \prod_{i,j=1}^n (1 + x_i y_j t)$$
is a symmetric polynomial in x_i and y_j with coefficients in $\mathbb{Z}$. P_n is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of x_i and y_j respectively.
- The coefficient of t^n in the polynomial
$$g(t) = \prod_{1 \leq i_1 \leq \dots \leq i_m \leq nm} (1 + x_{i_1} \cdots x_{i_m} t)$$
is a symmetric polynomial in x_i with coefficients in $\mathbb{Z}$. $P_{m,n}$ is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ of x_i.

In this case, we call R a λ-ring.

Note that we do not require that the λ^n are ring homomorphisms.

Definition 2.1.2: Associated Formal Power Series

Let R be a λ-ring. Define the associated formal power series to be the function $\lambda_t : R \to R[[t]]$ given by
$$\lambda_t(r) = \sum_{k=0}^\infty \lambda^k(r) t^k$$
for all $r \in R$

Proposition 2.1.3

Let R be a λ-ring. Then the following are true regarding $\lambda_t(r)$.

- $\lambda_t(1) = 1 + t$
- $\lambda_t(0) = 1$
- $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$
- $\lambda_t(-r) = \lambda(r)^{-1}$

Proposition 2.1.4

The ring $\mathbb{Z}$ has a unique λ-structure given by

$$\lambda_t(n) = (1+t)^n$$

Proposition 2.1.5

Let R be a λ-ring. Then R has characteristic 0.

Definition 2.1.6: Dimension of an Element

Let R be a λ-ring and let $r \in R$. We say that r has dimension n if $\deg(\lambda_t(r)) = n$. In this case, we write $\dim(r) = n$.

Proposition 2.1.7

Let R be a λ-ring. Then the following are true regarding the dimension of n.
- $\dim(r+s) \leq \dim(r) + \dim(s)$ for all $r, s \in R$
- If r and s both has dimension 1, then so is rs.

2.2 λ-Ring Homomorphisms and Ideals

Definition 2.2.1: λ-Ring Homomorphisms

Let R and S be λ-rings. A λ-ring homomorphism from R to S is a ring homomorphism $f : R \to S$ such that

$$\lambda^n \circ f = f \circ \lambda^n$$

for all $n \in \mathbb{N}$.

Definition 2.2.2: λ-Ideals

Let R be a λ-ring. A λ-ideal of R is an ideal I of R such that

$$\lambda^n(i) \in I$$

for all $i \in I$ and $n \geq 1$.

TBA:λ-ideal and subring. Ker, Im, Quotient Product, Tensor, Inverse Limit are λ-rings

Proposition 2.2.3

Let R be a λ-ring. Let $I = \langle z_i \mid i \in I \rangle$ be an ideal in R. Then I is a λ-ideal if and only if $\lambda^n(z_i) \in I$ for all $n \geq 1$ and $i \in I$.

Proposition 2.2.4

Every λ-ring R contains a λ-subring isomorphic to $\mathbb{Z}$.

2.3 Augmented λ-Rings

Definition 2.3.1: Augmented λ-Rings

Let R be a λ-ring. We say that R is an augmented λ-ring if it comes with a λ-homomorphism

$$\varepsilon : R \to \mathbb{Z}$$

called the augmentation map.

TBA: tensor of augmented is augmented

Proposition 2.3.2

Let R a λ-ring. Then R is augmented if and only if there exists a λ-ideal I such that

$$R = \mathbb{Z} \oplus I$$

as abelian groups.

2.4 Extending λ-Structures

Proposition 2.4.1

Let R be a λ-ring. Then there exists a unique λ-structure on $R[x]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[x]$ and $\varepsilon(r) = 0$ or 1.

Proposition 2.4.2

Let R be a λ-ring. Then there exists a unique λ-structure on $R[[x]]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[[x]]$ and $\varepsilon(r) = 0$ or 1.

2.5 Free λ-Rings

2.6 The Universal λ-Ring

2.7 Adams Operations

3 Witt Vectors

3.1 Fundamentals of the Ring of Big Witt Vectors

Prelim: Symm Functions, Lambda Rings, Category theory, Frobenius endomorphism (Galois), Rings and Modules
Leads to: K theory
Books: Donald Yau: Lambda Rings

Definition 3.1.1: The Witt Polynomials

Let R be a ring. For $n \geq 1$, define the nth Witt polynomial $w_n : R^n \to R$ to be given by

$$w_n(x_1, \dots, x_n) = \sum d|ndx_d^{n/d}$$

Theorem 3.1.2: Dwork's Theorem

Let R be a $\mathbb{Z}$-torsion-free ring. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i\in\mathbb{N}} \in \prod_{i=1}^\infty R$ has the form

$$(b_i)_{i\in\mathbb{N}} = (w_i(a))_{i\in\mathbb{N}}$$

  for some $a \in R$
- For all p prime and $\gcd(p, m) = 1$,

$$b_{p^n m} \equiv \sigma_p(b_{p^{n-1}m}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on $b_1, \dots, b_n$).

We wish to equip $\prod_{i=1}^\infty$ with a non-standard addition and multiplication to make it into a ring.

Proposition 3.1.3

Consider the ring $R = \mathbb{Z}[x_1, x_2, \dots, y_1, y_2, \dots]$. There exists unique polynomials

$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$

for $n \geq 1$ such that

- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i\in\mathbb{N}}) + w_n((y_i)_{i\in\mathbb{N}})$
- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i\in\mathbb{N}}) \cdot w_n((y_i)_{i\in\mathbb{N}})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i\in\mathbb{N}})$

for all $n \geq 1$.

Definition 3.1.4: The Ring of Big Witt Vector

Let R be a ring. Define the ring of big Witt vectors $W(R)$ of R to consist of the following.

- The underlying set $\prod_{i=1}^\infty R$
- Addition defined by $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n\in\mathbb{N}}$
- Multiplication defined by $(a_n)_{n\in\mathbb{N}} \times (b_n)_{n\in\mathbb{N}} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n\in\mathbb{N}}$

Theorem 3.1.5

Let R be a ring. Then the ring of big Witt vectors $W(R)$ of R is a ring with additive identity $(0, 0, \dots)$ and multiplicative identity $(1, 0, 0, \dots)$. Moreover, for $(a_n)_{n\in\mathbb{N}} \in W(R)$, its additive inverse is given by $(\iota_n(a_1, \dots, a_n))_{n\in\mathbb{N}}$.

Proposition 3.1.6

Let $\phi : R \to S$ be a ring homomorphism. Then the induced map $W(\phi) : W(R) \to W(S)$ defined by
$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$
is a ring homomorphism.

Definition 3.1.7: The Witt Functor

Define the Witt functor $W : \textbf{Ring} \to \textbf{Ring}$ to consist of the following data.

- For each ring R, $W(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by
$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$

Proposition 3.1.8

The Witt functor is indeed a functor. Moreover, for each $n \in \mathbb{N}$, $w_n : W(R) \to R$ defines a natural transformation $w_n : W \to \text{id}$

Theorem 3.1.9

The Witt functor $W : \textbf{Ring} \to \textbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W(R)$ is given by $\prod_{k=1}^{\infty} R$
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by
$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$
- For each $n \in \mathbb{N}$, $w_n : W(R) \to R$ defines a natural transformation $w_n : W \to \text{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k=1}^{\infty} R$ is unique under the above conditions.

3.2 Fundamentals of the Ring of Big Witt Vectors

Definition 3.2.1: Truncation Sets

Let $S \subseteq \mathbb{N}$. We say that S is a truncation set if for all $n \in S$ and $d|n$, then $d \in S$. For $n \in \mathbb{N}$ and S a truncation set, define
$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

For instance, $\mathbb{N} \setminus \{0\}$ is a truncation set. We will also use $\{1, \dots, n\}$.

Theorem 3.2.2: Dwork's Theorem

Let R be a ring and let S be a truncation set. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i\in S} \in \prod_{i\in S} R$ has the form
$$(b_i)_{i\in S} = (w_i(a))_{i\in S}$$
for some $a \in R$

- For all primes p and all $n \in S$ such that $p|n$, we have

$$b_n \equiv \sigma_p(b_{n/p}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on all the b_k for $1 \leq k \leq n$ and $k \in S$.

We wish to equip $\prod_{i \in S} R$ with a non-standard addition and multiplication to make it into a ring.

Proposition 3.2.3

Consider the ring $R = \mathbb{Z}[x_i, y_i \mid i \in S]$. There exists unique polynomials

$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$

for $n \in S$ such that
- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i \in S}) + w_n((y_i)_{i \in S})$
- $w_n(\pi_1, \dots, \pi_n) = w_n((x_i)_{i \in S}) \cdot w_n((y_i)_{i \in S})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i \in S})$

for all $n \in S$.

Note that the polynomials ξ_n, π_n have variables x_k and y_k for $k \leq n$ and $k \in S$. This is similar for the variables of ι. From now on, this will be the convention: For S a truncation set, the sequence $a_1, \dots, a_n$ actually refers to the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \cdots \leq d_k \leq n$ and $d_1, \dots, d_k$ are all divisors of n. The result of this is that sequences in $\mathbb{N}$ are now restricted to S.

Definition 3.2.4: The Ring of Truncated Witt Vector

Let R be a ring. Let S be a truncation set. Define the ring of big Witt vectors $W_S(R)$ of R to consist of the following.
- The underlying set $\prod_{i \in S} R$
- Addition defined by $(a_n)_{n \in S} + (b_n)_{n \in S} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$
- Multiplication defined by $(a_n)_{n \in S} \times (b_n)_{n \in S} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$

Theorem 3.2.5

Let R be a ring. Let S be a truncation set. Then the ring of big Witt vectors $W_S(R)$ of R is a ring with additive identity $(0, 0, \dots)$ and multiplicative identity $(1, 0, 0, \dots)$. Moreover, for $(a_n)_{n \in S} \in W(R)$, its additive inverse is given by $(\iota_n(a_1, \dots, a_n))_{n \in \mathbb{N}}$.

Proposition 3.2.6

Let $\phi : R \to R'$ be a ring homomorphism. Then the induced map $W_S(\phi) : W_S(R) \to W_S(R')$ defined by

$$W(\phi)((a_n)_{n \in S}) = (\phi(a_n))_{n \in S}$$

is a ring homomorphism.

Definition 3.2.7: The Witt Functor

Define the Witt functor $W_S : \mathbf{Ring} \to \mathbf{Ring}$ to consist of the following data.
- For each ring R, $W_S(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi : R \to R'$, $W_S(\phi) : W_S(R) \to W_S(R')$ is the induced ring homomorphism defined by

$$W_S(\phi)((a_n)_{n \in S}) = (\phi(a_n))_{n \in S}$$

Proposition 3.2.8

Let S be a truncation set. The Witt functor is indeed a functor.

Definition 3.2.9: The Ghost Map

Let R be a ring. Let S be a truncation set. Define the ghost map to be the map

$$w : W_S(R) \to \prod_{k\in S} R$$

by the formula

$$w((a_n)_{n\in S}) = (w_n(a_1, \dots, a_n))_{n\in S}$$

Remember, by the sequence $a_1, \dots, a_n$ we mean the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \cdots \leq d_k \leq n$ and $d_1, \dots, d_k$ the complete collection of divisors of n.

Proposition 3.2.10

Let S be a truncation set. Then the following are true.

- For each $n \in S$, the collection of maps $w_n : W_S(R) \to R$ for a ring R defines a natural transformation $w_n : W_S \to \mathrm{id}$.
- The collection of ghost maps $w_R : W_S(R) \to \prod_{k\in S} R$ for R a ring defines a natural transformation $w : W_S \to (-)^S$.

Proposition 3.2.11

Let S be a truncation set. The truncated Witt functor $W_S : \mathbf{Ring} \to \mathbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W_S(R)$ is given by $\prod_{k\in S} R$
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by

$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$

- For each $n \in S$, $w_n : W_S(R) \to R$ defines a natural transformation $w_n : W \to \mathrm{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k\in S} R$ is unique under the above conditions.

3.3 Important Maps of Witt Vectors

Definition 3.3.1: The Forgetful Map

Let R be a ring. Let $T \subseteq S$ be truncation sets. Define the forgetful map $R^S_T : W_S(R) \to W_T(R)$ to be the ring homomorphism given by forgetting all elements $s \in S$ but $s \notin T$.

Definition 3.3.2: The nth Verschiebung Map

Let R be a ring. Let S be a truncation set. For $n \in \mathbb{N}$, define the nth Verschiebung map $V_n : W_{S/n}(R) \to W_S(R)$ by

$$V_n((a_d)_{d\in S/n})_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise} \end{cases}$$

Note that this is not a ring homomorphism. However, it is additive.

Lemma 3.3.3

Let R be a ring. Let S be a truncation set. Then for all $a, b \in W_{S/n}(R)$, we have that

$$V_n(a+b) = V_n(a) + V_n(b)$$

Definition 3.3.4: Frobenius Map

Let S be a truncation set. Let R be a ring. Define the Frobenius map to be a natural ring homomorphism $F_n : W_S(R) \to W_{S/n}(R)$ such that the following diagram commutes:

$$\begin{tikzcd} W_S(R) \arrow[r, "w"] \arrow[d, "F_n"'] & \prod_{k\in S} R \arrow[d, "F_n^w"] \\ W_{S/n}(R) \arrow[r, "w"'] & \prod_{k\in S/n} R \end{tikzcd}$$

if it exists.

Lemma 3.3.5

Let S be a truncation set. Let R be a ring. Then the Frobenius map exists and is unique.

The following lemma relates this notion of Frobenius map to that in ring theory.

Lemma 3.3.6

Let A be an F_p algebra. Let S be a truncation set. Let $\varphi_p : A \to A$ denote the Frobenius homomorphism given by $a \mapsto a^p$. Then

$$F_p = R^S_{S/p} \circ W_S(\varphi) : W_S(A) \to W_{S/p}(A)$$

Definition 3.3.7: The Teichmuller Representative

Let R be a ring. Let S be a truncation set. Define the Teichmuller representative to be the map $[-]_S : R \to W_S(R)$ defined by

$$([a]_S)_n = \begin{cases} a & \text{if } n = 1 \\ 0 & b \text{ otherwise} \end{cases}$$

The Teichmuller representative is in general not a ring homomorphism, but it is still multiplicative.

Lemma 3.3.8

Let R be a ring. Let S be a truncation set. The for all $a, b \in R$, we have that

$$[ab]_S = [a]_S \cdot [b]_S$$

The three maps introduced are related as follows.

Proposition 3.3.9

Let R be a ring. Let S be a truncated set. Then the following are true.

- $r = \sum_{n\in S} V_n([r_n]_{S/n})$ for all $r \in W_S(R)$
- $F_n(V_n(a)) = na$ for all $a \in W_{S/n}(R)$
- $r \cdot V_n(a) = V_n(F_n(r) \cdot a)$ for all $r \in W_S(R)$ and all $a \in W_{S/n}(R)$
- $F_m \circ V_n = V_n \circ F_m$ if $\gcd(m, n) = 1$

The remaining section is dedicated to the example of $R = \mathbb{Z}$.

Proposition 3.3.10

Let S be a truncation set. Then the ring of big Witt vectors of $\mathbb{Z}$ is given by

$$W_S(\mathbb{Z}) = \prod_{n\in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = \gcd(m,n) \cdot V_d([1]_{S/d})$$

and $d = \operatorname{lcm}(m,n)$.

3.4 The Ring of p-Typical Witt Vectors

For the ring of p-typical Witt vectors, we consider the truncation set $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$ for a prime p.

Definition 3.4.1: The Ring of p-Typical Witt Vectors

Let R be a ring. Let p be a prime. Let $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$. Define the ring of p-typical Witt vectors to be

$$W_p(R) = W_P(R)$$

Define the ring of p-typical Witt vectors of length n to be

$$W_n(R) = W_{\{1,p,\dots,p^{n-1}\}}(R)$$

when the prime p is understood.

3.5 The λ-structure on $\mathbf{W(R)}$

Lemma 3.5.1

Let R be a ring. Then every $f \in \Lambda(R)$ can be written uniquely as

$$f = \prod_{k=1}^{\infty}(1 - (-1)^n a_n t^n)$$

Theorem 3.5.2: The Artin-Hasse Exponential

There is a natural isomorphism $E : \Lambda \to W$ given as follows. For a ring R, $E_R : \Lambda(R) \to W(R)$ is defined by

$$E_R\left(\prod_{k=1}^{\infty}(1 - (-1)^n a_n t^n)\right) = (a_n)_{n\in\mathbb{N}}$$

Corollary 3.5.3

Let R be a ring. Then $W(R)$ has a canonical λ-structure inherited from $\Lambda(R)$.

TBA: The forgetful functor $U : \Lambda\mathbf{Ring} \to \mathbf{CRing}$ has a left adjoint Symm and has a right adjoint W.