Operad Theory

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Abstract

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Definition 1.0.1: Operads

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An operad in $\mathcal{C}$ consists of the following data

- A sequence $P = \{P(n) \mid n \in \mathbb{N}\}$ of objects in $\mathcal{C}$
- A composition function

$$\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$$

- An unit $\mu : I \to P(1)$

such that the following compatibility conditions are satisfied.

- Associativity in the first symmetric product:

$$\begin{tikzcd}
P(n) \otimes \left(\bigotimes_{k=1}^{n} \left(P(i_k) \otimes \bigotimes_{t=1}^{i_k} P(j_{k,t})\right)\right) \arrow[r, "\mathrm{id}_{P(n)} \otimes \gamma"'] \arrow[dd, "\cong"] & P(n) \otimes \left(\bigotimes_{k=1}^{n} P\left(\sum_{t=1}^{i_k} j_{k,t}\right)\right) \arrow[dr, "\gamma"] & \\
& & P\left(\sum_{k=1}^{n} \sum_{t=1}^{i_k} j_{k,t}\right) \\
P(n) \otimes (\bigotimes_{k=1}^{n} P(j_k)) \otimes \left(\bigotimes_{k=1}^{n} \bigotimes_{t=1}^{t_k} P(j_{k,t})\right) \arrow[r, "\gamma \otimes \mathrm{id}_{\otimes\otimes}"] & P\left(\sum_{u=1}^{k} i_u\right) \otimes \left(\bigotimes_{k=1}^{n} \bigotimes_{t=1}^{i_k} P(j_{k,t})\right) \arrow[ur, "\gamma"] &
\end{tikzcd}$$

- Unitality:

$$\begin{tikzcd}
P(n) \otimes P(1)^{\otimes n} \arrow[dr, "\gamma"'] & P(n) \otimes I^{\otimes n} \arrow[l, "\mathrm{id}_{P(n)} \otimes \mu^{\otimes n}"'] \arrow[d, "\cong"] \\
& P(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes P(n) \arrow[r, "\mu \otimes \mathrm{id}_{P(n)}"] \arrow[d, "\cong"'] & P(1) \otimes P(n) \arrow[dl, "\gamma"] \\
P(n) &
\end{tikzcd}$$

TBA: An operad is a monoid in the monoidal category $(\mathrm{Func}(\mathbf{S}, \mathcal{C}), \circ, I)$

Definition 1.0.2: Symmetric Operads

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. A symmetric operad is an operad (P, γ, μ) on $\mathcal{C}$ such that the following are true.

- Each $P(n)$ is an S_n-module for each $n \in \mathbb{N}$.
- $\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$ is equivariant in the following sense:

$$\gamma(c \cdot \sigma, d_1, \dots, d_n) = \gamma(c, d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(n)}) \cdot \sigma(k_1, \dots, k_n)$$
$$\gamma(c, d_1 \cdot \tau_1, \dots, d_n \cdot \tau_n) = \gamma(c, d_1, \dots, d_n) \cdot (\tau_1 \oplus \cdots \oplus \tau_n)$$