The Topology of Fiber Bundles

Labix

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Abstract

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1 Fibrations

1.1 Fiber Bundles

Definition 1.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and $p : E \to B$ a trivial map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$ for all $b \in B$
- $p : E \to B$ is surjective
- For every $x \in B$, there is an open neighbourhood $U \subset B$ of x and a fiber preserving homomorphism $\Psi_U : p^{-1}(U) \to U \times F$ that is a homeomormorphism such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Psi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  where π is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

Definition 1.1.2: Map of Fiber Bundles

Let (F_1, E_1, B_1) and (F_2, E_2, B_2) be fiber bundles. A morphism of fiber bundles is a pair of basepoint preserving continuous maps $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

Such a map of fibrations determine a continuous of the fibers $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$.

A map of fibrations $(\tilde{f}, f)$ is said to be an isomorphism if there is a map $(\tilde{g} : E_2 \to E_1, g : B_2 \to B_1)$ such that $\tilde{g}$ is the inverse of $\tilde{f}$ and g is the inverse of f.

Definition 1.1.3: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration $B \times F \to B$.

Definition 1.1.4: Sections

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$. Let $U \subset B$ be an open set. A local section of the fiber bundle on U is a map $s : U \to B$ such that $p \circ s = \mathrm{id}_U$.

1.2 G-Bundles and the Structure Groups

Notice that for non empty intersections $U_i \cap U_j$ for U_i, U_j open sets in B, there is a well defined homeomorphism

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

This is reminiscent of properties of an atlas on M.

Definition 1.2.1: G-Atlas

Let (F, E, B) be a fiber bundle. Let G be topological group with a continuous faithful action on F. A G-atlas on (F, E, B) is a set of local trivalization charts $\{(U_k, \varphi_k) \mid k \in I\}$ such that the following are true.

- For (U_k, φ_k) a chart, define $\varphi_{i,x} : F \to F$ by $f \mapsto \varphi_i(x, f)$. Then the homeomorphism
$$\varphi_{j,x} \circ \varphi_{i,x}^{-1} : F \to F$$
for $x \in U_i \cap U_j \neq \emptyset$ is an element of G.
- For $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to G$ defined by
$$g_{ij}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$$
is continuous.

If the (F, E, B) is a fiber bundle with $F = \mathbb{R}$, then it is often seen that $G = GL(n, \mathbb{R})$. Similarly, if $F = \mathbb{C}$ then the structure group is $G = GL(n, \mathbb{C})$.

Definition 1.2.2: Equivalent G-Atlas

Two G-atlases on a fiber bundle (F, E, B) is said to be equivalent if their union is a G-atlas.

Definition 1.2.3: G-Bundle

Let G be a topological group. A G-bundle is a fiber bundle (F, E, B) together with an equivalence class of G-atlas. In this case, G is said to be the structure group of the fiber bundle.

The structure group and the trivialization charts completely determine the isomorphism type of the fiber bundle.

1.3 Morphisms of G-Bundles

Definition 1.3.1: Morphisms of G-Bundles

Let G be a topological group. A morphism of G-bundles is a morphism of fiber bundles $(\tilde{h}, h) : (F, E_1, B_1) \to (F, E_2, B_2)$ where the two are G-bundles, such that the following are true.

- Let U_i be open in B_1 and V_j be open in B_2. Let $x \in U_u \cap h^{-1}(V_j)$. Let $\widetilde{h_{(E_1)_x}} : (E_1)_x \to (E_2)_{f(x)}$ be the map induced by $\tilde{h} : E_1 \to E_2$. Then the map
$$\varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1} : F \to F$$
is an element of G.
- The map $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ defined by
$$\widetilde{g_{ij}}(x) = \varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1}$$
is continuous.

It is easy to see that the mapping transformations $\widetilde{g_{ij}}$ satisfy the following two relations:

- $\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap U_j \cap h^{-1}(V_k)$
- $g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap h^{-1}(V_j \cap V_k)$

g'_{jk} here refers to the transition charts in (F, E_2, B_2).

Just as the structure groups and trivialization charts determine the isomorphism type of a fiber bundle, the $\widetilde{g_{ij}}$ and a map of base space $h : B_1 \to B_2$ completes determines a morphism of G-bundle.

Lemma 1.3.2

Let (F, E_1, B_1) and (F, E_2, B_2) be two G-bundles for a topological group G with the same fiber F. Suppose that we have the following.

- A map $h : B_1 \to B_2$ of base space
- $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ a set of continuous maps such that

$$\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x) \quad \text{for all} \quad x \in U_i \cap U_j \cap h^{-1}(V_k)$$
$$g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x) \quad \text{for all} \quad x \in U_i \cap h^{-1}(V_j \cap V_k)$$

Then there exists a unique G-bundle morphism having h as the map of base space and having $\{\widetilde{g_{ij}} \mid i, j \in I\}$ as its mapping transformations.

1.4 Principal G-Bundles

Definition 1.4.1: Principal Bundles

Let G be a topological group. A principal G-bundle is a G-bundle (F, E, B) together with a continuous group action G on E such that the following are true.

- The action of G preserves fibers. This means that $g \cdot x \in E_b$ if $x \in E_b$. (This also means that G is a group action on each fiber)
- The action of G on each fiber is free and transitive
- For each $x \in E_b$, the map $G \to E_b$ defined by $g \mapsto g \cdot x$ is homeomorphism.
- Local triviality condition: If $\Psi_U : p^{-1}(U) \to U \times F$ are the local triviality maps, then each Ψ_U are G-equivariant maps.

Note that since G is homeomorphic to each fiber E_b of the total space, we can think of the action of G on the fiber simply becomes left multiplication.

For those who know what homogenous spaces are, principal bundles are G-bundles such that F is a principal homogenous space for the left action of G itself.

Conversely, given a continuous group action on a space, we can ask in what conditions will the space be a principal bundle over the orbit space.

Proposition 1.4.2

Let E be a space with a free G action. Let $p : E \to E/G$ be the projection map to the orbit space. If for all $x \in E/G$, there is a neighbourhood U of x and a continuous map $s : U \to E$ such that $p \circ s = \text{id}_U$, then $(G, E, E/G)$ is a principal G-bundle.

This proposition essentially means that if for each point in E/G, there is a local section, then it is sufficient for E to be a principal G bundle over E/G.

Theorem 1.4.3

A principal G-bundle is trivial if and only if it admits a global section.

This is entirely untrue for general bundles. For examples, the zero section of a fiber bundle is a global section.

1.5 Classifying Space

Definition 1.5.1: Universal G-Bundles

Let G be a topological group. A principal G-bundle (F, E, B) is said to be universal if for any space X, the induced pullback map

$$\psi : [X, B] \to \text{Prin}_G(X)$$

defined by $f \mapsto f^*(E)$ is a bijective correspondence.

Theorem 1.5.2

Let (F, E, B) be a principal G-bundle. If E is contractible then (F, E, B) is a universal G-bundle.

Theorem 1.5.3

Let (F, E_1, B_1) and (F, E_2, B_2) be universal principal G-bundles. Then there exists a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

such that f is a homotopy equivalence.

Definition 1.5.4: Classifying Space

Let G be a topological group. The classifying space BG of G is the homotopy type of the universal principal G-bundle. Also denote EG as the total space of the universal G-bundle.

2 Fibrations and Cofibrations

2.1 Fibrations

Definition 2.1.1: Fibrations

We say that a map $p : E \to B$ is a fibration if it has the homotopy lifting property with respect to all topological spaces X.

In other words, for any space X together with a homotopy $H : X \times I \to B$ and a lift $\widetilde{H(-,0)} : X \to E$ of $H(-,0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ lifting $\widetilde{H}$ and extending $\widetilde{H(-,0)}$:

$$\begin{tikzcd} X \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \widetilde{H}" description] & B \end{tikzcd}$$

We call B the base space and E the total space. Define the fiber over $b \in B$ to be the subspace

$$F_b = p^{-1}(b) \subseteq E$$

Definition 2.1.2: Fibration Homomorphism

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two fibrations. We say that a map $f : E_1 \to E_2$ is a fibration homomorphism if

$$p_2 \circ f = p_1$$

In other words, the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Definition 2.1.3: Fiber Homotopy Equivalence

We say that a fiber homomorphism $f : E_1 \to E_2$ is a fiber homotopy equivalence if there exists a fiber homomorphism $g : E_2 \to E_1$ such that $f \circ g$ and $g \circ f$ are homotopic by fibration homomorphisms to the identities id_{E_2} and id_{E_1} respectively.

Definition 2.1.4: Serre Fibration

We say that a map $p : E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

It is clear that every (Hurewicz) fibration is a Serre fibration. Moreover, every fiber bundle is also a Serre fibration.

Proposition 2.1.5

Every (Hurewicz) fibration is a Serre fibration. Every fiber bundle is a Serre fibration.