Hopf Algebra

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Abstract

Contents

1 Algebras and Coalgebras

1.1 Coalgebras

There is a need to revisit the definition of an algebra (over a field)

Proposition 1.1.1

A vector space V over a field k is an algebra if and only if there is a following collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map

such that the following two diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\cong"'] & V \otimes V \arrow[dl, "m"'] \\
V & V \otimes k \arrow[l, "\cong"] \arrow[u, "\mathrm{id}\otimes u"']
\end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Evidently, the map μ gives a multiplicative structure for V and Δ gives the unitary structure of an algebra. The diagram on the left then represent associativity of multiplication. Notice that such additional structure on V formally lives in the category $\mathbf{Vect}_k$ of vector spaces over a fixed field k.

Therefore we can formally dualize all arrows to obtain a new object.

Definition 1.1.2: Coalgebra

Let V be a vector space over a field k. We say that V is a coalgebra over k if there is a collection of data:

- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\Delta"'] \\
V \otimes V \arrow[u, "\Delta\otimes\varepsilon"] & V \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
V \arrow[u, "\cong"] \arrow[ur, "\Delta"] \arrow[r, "\cong"'] & V \otimes k
\end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Lemma 1.1.3

Every vector space V over a field k can be given the structure of a coalgebra where

- $\Delta : V \to V \otimes V$ is defined by $\Delta(v) = v \otimes v$
- $\varepsilon : V \to k$ is defined by $\varepsilon(v) = 1_k$

We would like to formally invert the definitions of algebra homomorphisms in order to define coalgebra homomorphisms.

1.2

Every coalgebra gives rise to an algebra, but not the other way. Such an assignment is moreover functorial.

1.3 Bialgebras

Definition 1.3.1: Bialgebras

Let V be a vector space over a field k. We say that V is a bialgebra if there is a collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map
- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that (V, m, u) is an algebra over k and (V, Δ, ε) is a coalgebra over k and that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V \otimes V \otimes V \otimes V \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & V \otimes V \otimes V \otimes V \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & k \\
& V \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V \otimes V \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & V \arrow[dl, "\varepsilon"] \\
& k \otimes k \cong k &
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \otimes k \cong k \arrow[dl, "u\otimes u"'] \arrow[dr, "u"] & \\
V \otimes V & & V \arrow[ll, "\Delta"]
\end{tikzcd}$$

where $\tau : V \otimes V \to V \otimes V$ is the commutativity map defined by $\tau(x \otimes y) = y \otimes x$.

Theorem 1.3.2

Let V be a vector space over k. Suppose that (V, m, u) is an algebra and (V, Δ, ε) is a coalgebra. Then the following conditions are equivalent.

- $(V, m, u, \Delta, \varepsilon)$ is a bialgebra
- $m : V \otimes V \to V$ and $u : k \to V$ are coalgebra homomorphisms
- $\Delta : V \to V \otimes V$ and $\varepsilon : V \to k$ are algebra homomorphisms

2 Hopf Algebras

2.1

Definition 2.1.1: Hopf Algebra

Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. We say that H is a Hopf algebra if there is a k-linear map $S : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$