Homological Algebra

Labix

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Abstract

Contents

1 A Second Course on Modules

1.1 Chain Complexes

Definition 1.1.1: Chain Complex

A chain complex (C, ∂) is a family of R-modules C_n for $n \in \mathbb{Z}$ and maps $\partial_n : C_n \to C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all n.

In other words, we have the diagram:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

Equivalently, we require that

$$\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$$

for each n.

Definition 1.1.2: Exact Sequence

A chain complex is said to be exact if $\operatorname{im}(\partial_{n+1}) = \ker(\partial_n)$ for all n.

Proposition 1.1.3

Let M, N be R-modules.

- $f : M \to N$ is surjective if and only if the following sequence is exact:

$$M \xrightarrow{f} N \longrightarrow 0$$

- $f : M \to N$ is injective if and only if the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{f} N$$

Definition 1.1.4: Short Exact Sequence

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Proposition 1.1.5

A chain complex

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact if and only if f is injective and g is surjective.

Definition 1.1.6: Split Exact Sequence

A short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is said to be split exact if $B \cong A \oplus C$

Proposition 1.1.7: Split Exact Sequence

The following are equivalent for a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

- The short exact sequence is split exact sequence
- There exists a homomorphism $p : B \to A$ such that $p \circ f$ is the identity
- There exists a homomorphism $s : C \to B$ such that $g \circ s$ is the identity

1.2 Projective and Injective Modules

Definition 1.2.1: Projective Modules

An R-module M is said to be projective if for every surjective homomorphism $f : N \twoheadrightarrow M$ and every module homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that $f \circ h = g$. In other words, the following diagram commutes:

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

Theorem 1.2.2

An R-module P is projective if and only if for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ we have that

$$0 \to \text{Hom}(P, A) \xrightarrow{f} \text{Hom}(P, B) \xrightarrow{g} \text{Hom}(P, C) \to 0$$

is exact.

Lemma 1.2.3

Every free module is projective.

Proposition 1.2.4

A direct sum $\oplus_{i \in I} P_i$ is projective if and only if each P_i is.

Proposition 1.2.5

Let P be a module. Then P is projective if and only if every exact sequence of the following form splits:

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

Definition 1.2.6: Injective Modules

An R-module M is said to be projective if for every injective homomorphism $f : N \rightarrowtail M$ and every module homomorphism $g : N \to I$, there exists a module homomorphism $h : M \to I$ such that $f \circ h = g$. In other words, the following diagram commutes:

$$\begin{tikzcd} N \arrow[d, tail, "f"'] \arrow[dr, "g"] & \\ M \arrow[r, dashed, "\exists h"'] & I \end{tikzcd}$$

Theorem 1.2.7

An R-module I is injective if and only if for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ we have that
$$0 \to \text{Hom}(A, I) \xrightarrow{f} \text{Hom}(B, I) \xrightarrow{g} \text{Hom}(C, I) \to 0$$
is exact.

Proposition 1.2.8

Let E be a module. Then E is projective if and only if every exact sequence of the following form splits:
$$0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$$

1.3 Flat Modules

Definition 1.3.1: Flat Modules

Let R be a ring. An R-module M is said to be flat if for every injective linear map $\phi : K \to L$ of R-modules, the map
$$\phi \otimes M : K \otimes_R M \to L \otimes_R M$$
is also injective.

Theorem 1.3.2

Let R be a ring and M an R-module. Let $0 \to K \to L \to J \to 0$ be an exact sequence, then the sequence
$$K \otimes_R M \to L \otimes_R M \to J \otimes_R M \to 0$$
is also exact.

Theorem 1.3.3

Let R be a ring and M an R-module. Then M is a flat module if and only if for every short exact sequence $0 \to K \to L \to J \to 0$, the sequence
$$0 \to K \otimes_R M \to L \otimes_R M \to J \otimes_R M \to 0$$
is also exact.

Theorem 1.3.4

Let R be a ring. Then the following are true.

- Product: If A and B are flat over R then $A \otimes_R B$ is flat over R
- Base Change: Let S be an R-algebra ($R \to S$ a ring hom). Then $M \otimes_R S$ is flat over S for any flat R-module M
- Transitivity: Let S be an R-algebra such that S is flat over R. If C is flat over S then C is flat over R.

We have the following inclusion of modules

$$\text{Free Modules} \subset \text{Projective Modules} \subset \text{Flat Modules} \subset \text{Torsion Free Modules}$$

2 Abelian Categories and its Properties

2.1 Category of Modules

Definition 2.1.1: Category of R-Modules

Define the category of R-modules to be ${}_R\mathcal{M}$ where objects are exactly modules and morphisms are morphisms between modules. Define $\text{Hom}_R(A, B)$ to be the set of R-modules homomorphisms between R-modules A and B.

Proposition 2.1.2

For any R-modules A and B, $\text{Hom}_R(A, B)$ is an R-module.

Proof. Trivially $\text{Hom}_R(A, B)$ is an abelian group by defining

$$(f + g)(x) = f(x) + g(x)$$

for $f, g \in \text{Hom}_R(A, B)$. For $r \in R$, define

$$(rf)(x) = rf(x)$$

Then clearly $\text{Hom}_R(A, B)$ is an R-module. □

2.2 Additive Categories

Definition 2.2.1: Pre-Additive Categories

A category $\mathcal{C}$ is pre-additive if it is a category that satisfies the fact that each $\text{Hom}_\mathcal{C}(X, Y)$ is given the structure of an abelian group where

$$\text{Hom}_\mathcal{C}(X, Y) \times \text{Hom}_\mathcal{C}(Y, Z) \to \text{Hom}_\mathcal{C}(X, Z)$$

are bilinear. This means that if $f : X \to Y$ and $g, h : Y \to Z$, then $g + h = h + g : Y \to Z$ and $f \circ (g + h) = (f \circ g) + (f \circ h)$ and the same distributive property for the first element.

Definition 2.2.2: Additive Categories

A category $\mathcal{A}$ is additive if in addition to being pre-additive, it also satisfies the following:

- $\mathcal{A}$ has a zero object, denoted 0
- $\mathcal{A}$ has finite products

Lemma 2.2.3

Let $\mathcal{A}$ be an additive category. Then coproducts and products coincide, meaning that

$$X \times Y \cong X \sqcup Y$$

for any $X, Y \in Obj(\mathcal{A})$.

2.3 Abelian Categories

Definition 2.3.1: Abelian Categories

An additive category $\mathcal{A}$ is said to be abelian if it satisfies the following:

- Every morphism in $\mathcal{A}$ has a kernel and a cokernel
- Every monic morphism is the kernel of its cokernel
- Every epic morphism is the cokernel of its kernel

Proposition 2.3.2

The category of R-modules is an abelian category.

Theorem 2.3.3

Let $\mathcal{A}$ be an abelian category whose objects form a set. Then there exists a ring R and an exact functor $F : \mathcal{A} \to R-\text{mod}$ which is an embedding on objects and an isomorphism on Hom sets.

Definition 2.3.4: Injectivity and Surjectivity

Let $f : X \to Y$ be a morphism in an abelian category.

- We say that f is injective if $\ker(f) = 0$
- We say that f is surjective if $\text{coker}(f) = 0$

In particular, these notions coincide that of epics and monics in an abelian category.

Proposition 2.3.5

Let $f : X \to Y$ be a morphism in an abelian category. Then the following are true.

- f is injective if and only if f is a monomorphism
- f is surjective if and only if f is epimorphism

Theorem 2.3.6

The category R-mod of R-modules is an abelian category.

2.4 Exact Functors

Definition 2.4.1: Additive Functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if for every $X, Y \in \mathcal{A}$, the map

$$\text{Hom}_{\mathcal{A}}(X, Y) \to \text{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Definition 2.4.2: Exact Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories.

- We say that F is exact if it preserves exact sequences

- We say that F is right exact if for every exact sequence $A \to B \to C \to 0$, the sequence
$$F(A) \to F(B) \to F(C) \to 0$$
is exact
- We say that F is left exact if for every exact sequence $0 \to A \to B \to C$, the sequence
$$0 \to F(A) \to F(B) \to F(C)$$
is exact

Proposition 2.4.3

Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F preserves split exact sequences.

Theorem 2.4.4: Freyd-Mitchell Embedding Theorem

Let $\mathcal{A}$ be a small abelianc category. Then there exists a ring R and an exact, fully faithful functor $F : \mathcal{A} \to R - \text{mod}$.

This means that
$$\text{Hom}_{\mathcal{A}}(M, N) \cong \text{Hom}_R(F(M), F(N))$$

Lemma 2.4.5

The Freyd-Mitchell embedding preserves kernels and cokernels. Moreover, it maps the zero object to the zero object.

2.5 Chain Complexes in an Abelian Category

Definition 2.5.1: Chain Complexes

Let $\mathcal{A}$ be an abelian category. Let $\{C_n | n \in \mathbb{N}\}$ be a collection of objects in $\mathcal{A}$ and $d_n : C_n \to C_{n-1}$ a collection of morphisms.
$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$
We say that this is a chain complex if $d_n \circ d_{n+1} = 0$ for each n, denoted $(C^\bullet, d)$

Definition 2.5.2: Cohomology of a Chain

Let $\mathcal{A}$ be an abelian category. Let $(C^\bullet, d)$ be a chain complex. Define the nth cohomology of $(C^\bullet, d)$ to be the object
$$H^n(C^\bullet) = \text{coker}(a^{n-1}) = \ker(b^n)$$
determined by the following commutative diagram:

$$
\begin{tikzcd}
 & \operatorname{coker}(d^{n-1}) \arrow[dr, "b^n"] & \\
X^{n-1} \arrow[r, "d^{n-1}"] \arrow[dr, "a^{n-1}"'] & X^n \arrow[u] \arrow[r, "d^n"] & X^{n+1} \\
 & \ker(d^n) \arrow[u] &
\end{tikzcd}
$$

Definition 2.5.3: Acyclic Object

Let $\mathcal{A}$ be an abelian category and $(C^\bullet, d)$ a chain complex in $\mathcal{A}$. The complex is said to be a cyclic at the nth term if $H^n(C^\bullet) = 0$.

Definition 2.5.4: Exact Sequences

Let $\mathcal{A}$ be an abelian category and $(C^\bullet, d)$ a chain complex in $\mathcal{A}$. The complex is said to be an exact sequence if it is acyclic at all terms.

Proposition 2.5.5

Let $\mathcal{A}$ be an abelian category. Let A, B, C be objects in $\mathcal{A}$ and $f : A \to B$, $g : B \to C$ be morphisms. Then the following are true.

- A sequence $0 \to A \to B$ is exact if and only if f is injective
- A sequence $B \to C \to 0$ is exact if and only if g is surjective
- A sequence $0 \to A \to B \to 0$ is exact if and only if f is an isomorphism
- A sequence $0 \to A \to B \to C \to 0$ is exact if and only if f is injective and g is surjective

Recall that coproducts and products coincide in an additive category. Denote this common product by $A \oplus B$.

Definition 2.5.6: Split Exact Sequence

Let $\mathcal{A}$ be an abelian category. We say that a sequence

$$
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0
$$

is split exact if $B \cong A \oplus C$.

Lemma 2.5.7: The Five Lemma

Suppose that the following diagram in an abelian category commutes:

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & X_4 \arrow[r] \arrow[d, "f_4"] & X_5 \arrow[d, "f_5"] \\
Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4 \arrow[r] & Y_5
\end{tikzcd}
$$

Suppose further that the rows are exact, and f_1 is an epimorphism and f_5 is a monomorphism and f_2, f_4 are isomorphisms. Then f_3 is also an isomorphism.

Lemma 2.5.8: The Snake Lemma

Suppose that the following diagram in an abelian category commutes:

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "f_1"] & X_2 \arrow[r] \arrow[d, "f_2"] & X_3 \arrow[r] \arrow[d, "f_3"] & 0 \\
0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & 0
\end{tikzcd}$$

Suppose further that the rows are exact. Then there exists two exact sequence

$$0 \longrightarrow \ker(f_1) \xrightarrow{a_1} \ker(f_2) \xrightarrow{a_2} \ker(f_3)$$

$$\operatorname{coker}(f_1) \xrightarrow{b_1} \operatorname{coker}(f_2) \xrightarrow{b_2} \operatorname{coker}(f_3) \longrightarrow 0$$

Moreover, there exists a natural morphism $\delta : \ker(f_3) \to \operatorname{coker}(f_1)$ that glues these two exact sequences into a long exact sequence.

3 Derived Functors

3.1 Injective and Projective Objects

Injectivity and Projectivity objects are created just for the sake of allowing the Hom functor to be exact. Therefore its definition is also direct.

Definition 3.1.1: Projective and Injective Objects

Let $\mathcal{A}$ be an abelian category.

- We say that an object Y of $\mathcal{A}$ is injective if the functor $X \mapsto \text{Hom}(X, Y)$ is exact.
- We say that an object Y of $\mathcal{A}$ is projective if the functor $X \mapsto \text{Hom}(Y, X)$ is exact.

Lemma 3.1.2

The projective objects in R-mod is precisely the projective R-modules. The injective objects in R-mod is precisely the injective R-modules.

Definition 3.1.3: Enough Injectives and Enough Projectives

Let $\mathcal{A}$ be an abelian category. $\mathcal{A}$ is said to have enough injectives if every object is subobject of an injective object. $\mathcal{A}$ is said to have enough projectives if every object is quotient of an projective object.

There are however equivalent definitions from the categorical point of view.

3.2 Resolutions of Objects

There are in general, four types of resolutions. Namely injective resolutions, projective resolutions, free resolutions and acyclic resolutions. We will study all four of them and their relations in this section.

Definition 3.2.1: Injective Resolution

Let $\mathcal{A}$ be an abelian category. An injective resolution of an object A is an exact sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

where each I^k is injective.

Theorem 3.2.2

Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ has enough injectives if and only if every object of $\mathcal{A}$ has an injective resolution.

Theorem 3.2.3

The category of R-mod has enough injectives.

Proposition 3.2.4

Let $\phi : A \to A'$ be a morphism in an abelian category $\mathcal{A}$. Let $d : A \to I$ and $d' : A' \to I'$ be injective resolutions of A and A' respectively. Then there exists a morphism of complexes $\phi' : I \to I'$ that extends ϕ. Moreover, any two such extensions are homotopic.

Lemma 3.2.5

Let $\mathcal{A}$ be an abelian category. Then any two injective resolutions of an object A are homotopically equivalent.

3.3 Derived Functors

Definition 3.3.1: Right Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose that $\mathcal{A}$ has enough injectives. Define the right derived functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ as follows.

- On objects, $R^iF(A) = H^i(F(I^\bullet))$ where $d : A \to I^\bullet$ is an injective resolution of A
- On Morphisms, $R^iF(\phi : A \to B) = H^i(F(\phi^\bullet : I^\bullet \to (I')^\bullet))$ where $\phi^\bullet : I^\bullet \to (I')^\bullet$ is an extension of ϕ to resolutions.

Lemma 3.3.2

Let A is an injective object, then $R^iF(A) = 0$ for $n \neq 0$.

Theorem 3.3.3

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. The nth right derived functor R^nF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

Corollary 3.3.4

If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, then $R^0F = F$.

Theorem 3.3.5

Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough injective. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. For any short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

there is a canonical long exact sequence

$$0 \longrightarrow R^0F(A') \longrightarrow R^0F(A) \longrightarrow R^0F(A'') \longrightarrow R^1F(A') \longrightarrow R^1F(A) \longrightarrow R^1F(A'') \longrightarrow \cdots$$

3.4 The Case of Dual

Definition 3.4.1: Projective Resolution

Let $\mathcal{A}$ be an abelian category. An projective resolution of an object A is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{d} A \longrightarrow 0$$

Theorem 3.4.2

Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ has enough projectives if and only if every object of $\mathcal{A}$ has a projective resolution.

Theorem 3.4.3

The category of R-mod has enough projectives.

Definition 3.4.4: Left Derived Functors

Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Suppose that $\mathcal{A}$ has enough projectives. Define the left derived functors $L_iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ as follows.

- On objects, $L_iF(A) = H_i(F(P^\bullet))$ where $d : P_\bullet \to A$ is an projective resolution of A
- On Morphisms, $L_iF(\phi : A \to B) = L_i(F(\phi_\bullet : P_\bullet \to (P')_\bullet))$ where $\phi_\bullet : P_\bullet \to (P')_\bullet$ is an extension of ϕ to resolutions.

3.5 Applications to Module Theory

Definition 3.5.1: The Ext Functor

Denote ${}_R\mathbf{Mod}$ the category of R-modules. Let A be an R-module. Define the right derived functor of the functor $T :_R \mathbf{Mod} \to \mathbf{Ab}$ defined by $T(B) = \text{Hom}(A, B)$ to be

$$\text{Ext}_R^i(A, B) = (R^iT)(B)$$

Explicitly, for

$$0 \to B \to I^0 \to I^1 \to \cdots$$

an injective resolution, form the cochain complex

$$0 \to \text{Hom}_R(A, I^0) \to \text{Hom}_R(A, I^1) \to \cdots$$

and define Ext to be the cohomology group

$$\text{Ext}_R^i(A, B) = \frac{\ker(\text{Hom}_R(A, I^i) \to \text{Hom}_R(A, I^{i+1}))}{\text{im}(\text{Hom}_R(A, I^{i-1}) \to \text{Hom}_R(A, I^i))}$$

Theorem 3.5.2

Let A, B be R-modules. Then the following are true regarding the Ext group.

- $\text{Ext}_R^0(A, B) \cong \text{Hom}_R(A, B)$
- $\text{Ext}_R^i(A, B) = 0$ for all $i > 0$ if A is projective or B is injective
-

Definition 3.5.3: The Tor Functor

Denote ${}_R\mathbf{Mod}$ the category of R-modules. Let B be an R-module. Define the right derived functor of the functor $T :_R \mathbf{Mod} \to \mathbf{Ab}$ defined by $T(A) = A \otimes_R B$ to be

$$\text{Tor}_i^R(A, B) = (L_iT)(A)$$

Explicitly, for

$$\cdots \to P_1 \to P_0 \to A \to 0$$

an injective resolution, form the chain complex

$$\cdots \to P_1 \otimes_R B \to P_0 \otimes_R B \to 0$$

and define Tor to be the homology group

$$\text{Tor}_i^R(A, B) = \frac{\ker(P_i \otimes_R B \to P_{i-1} \otimes_R B)}{\text{im}(P_{i+1} \otimes_R B \to P_i \otimes_R B)}$$

4 Triangulated Categories