

# Homological Algebra

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**Abstract**

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# 1 Abelian Categories and its Properties

## 1.1 Category of Modules

### Definition 1.1.1: Category of $R$ -Modules

Define the category of  $R$ -modules to be  ${}_R\text{Mod}$  where objects are exactly modules and morphisms are morphisms between modules. Define  $\text{Hom}_R(A, B)$  to be the set of  $R$ -modules homomorphisms between  $R$ -modules  $A$  and  $B$ .

### Proposition 1.1.2

For any  $R$ -modules  $A$  and  $B$ ,  $\text{Hom}_R(A, B)$  is an  $R$ -module.

*Proof.* Trivially  $\text{Hom}_R(A, B)$  is an abelian group by defining

$$(f + g)(x) = f(x) + g(x)$$

for  $f, g \in \text{Hom}_R(A, B)$ . For  $r \in R$ , define

$$(rf)(x) = rf(x)$$

Then clearly  $\text{Hom}_R(A, B)$  is an  $R$ -module. □

## 1.2 Additive Categories

### Definition 1.2.1: Pre-Additive Categories

A category  $\mathcal{C}$  is pre-additive if it is a category that satisfies the fact that each  $\text{Hom}_{\mathcal{C}}(X, Y)$  is given the structure of an abelian group where

$$\text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$

are bilinear. This means that if  $f : X \rightarrow Y$  and  $g, h : Y \rightarrow Z$ , then  $g + h = h + g : Y \rightarrow Z$  and  $f \circ (g + h) = (f \circ g) + (f \circ h)$  and the same distributive property for the first element.

### Definition 1.2.2: Additive Categories

A category  $\mathcal{A}$  is additive if in addition to being pre-additive, it also satisfies the following:

- $\mathcal{A}$  has a zero object, denoted  $0$
- $\mathcal{A}$  has finite products

### Lemma 1.2.3

Let  $\mathcal{A}$  be an additive category. Then coproducts and products coincide, meaning that

$$X \times Y \cong X \sqcup Y$$

for any  $X, Y \in \text{Obj}(\mathcal{A})$ .

### 1.3 Abelian Categories

#### Definition 1.3.1: Abelian Categories

An additive category  $\mathcal{A}$  is said to be abelian if it satisfies the following:

- Every morphism in  $\mathcal{A}$  has a kernel and a cokernel
- Every monic morphism is the kernel of its cokernel
- Every epic morphism is the cokernel of its kernel

#### Proposition 1.3.2

The category of  $R$ -modules is an abelian category.

#### Theorem 1.3.3

Let  $\mathcal{A}$  be an abelian category whose objects form a set. Then there exists a ring  $R$  and an exact functor  $F : \mathcal{A} \rightarrow R\text{-mod}$  which is an embedding on objects and an isomorphism on Hom sets.

#### Definition 1.3.4: Injectivity and Surjectivity

Let  $f : X \rightarrow Y$  be a morphism in an abelian category.

- We say that  $f$  is injective if  $\ker(f) = 0$
- We say that  $f$  is surjective if  $\operatorname{coker}(f) = 0$

In particular, these notions coincide that of epics and monics in an abelian category.

#### Proposition 1.3.5

Let  $f : X \rightarrow Y$  be a morphism in an abelian category. Then the following are true.

- $f$  is injective if and only if  $f$  is a monomorphism
- $f$  is surjective if and only if  $f$  is epimorphism

#### Theorem 1.3.6

The category  $R\text{-mod}$  of  $R$ -modules is an abelian category.

## 2 Chain Complexes in an Abelian Category

### 2.1 Chain Complexes

#### Definition 2.1.1: Chain Complex

Let  $\mathcal{A}$  be an abelian category. A chain complex  $(C_\bullet, \partial_\bullet)$  in  $\mathcal{A}$  is a family of objects  $C_n \in \mathcal{A}$  for  $n \in \mathbb{Z}$  and morphisms  $\partial_n : C_n \rightarrow C_{n-1}$  in  $\mathcal{A}$  such that  $\partial_n \circ \partial_{n+1} = 0$  for all  $n$ .

In other words, we have the diagram:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

for which we require that

$$\text{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$$

for each  $n$ .

#### Definition 2.1.2: Homology Group

Let  $(C_\bullet, \partial_\bullet)$  be a chain complex in an abelian category  $\mathcal{A}$ . Define  $Z_n(C_\bullet) = \ker(\partial_n)$  and  $B_n(C_\bullet) = \text{im}(\partial_{n+1})$ . Define the  $n$ th homology of  $(C_\bullet, \partial_\bullet)$  to be

$$H_n(C_\bullet) = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)} = \frac{\ker(\partial_n)}{\text{im}(\partial_{n+1})}$$

Elements of  $Z_n(C_\bullet) = \ker(\partial_n)$  are called  $n$ -cycles and elements of  $B_n(C_\bullet) = \text{im}(\partial_{n+1})$  are called  $n$ -boundaries.

#### Definition 2.1.3: Chain Map

Let  $(C_\bullet, \partial_\bullet)$  and  $(C'_\bullet, \partial'_\bullet)$  be two chain complexes in an abelian category  $\mathcal{A}$ . A chain map  $f_\bullet : C_\bullet \rightarrow C'_\bullet$  is a family of maps

$$f_n : C_n \rightarrow C'_n$$

in  $\mathcal{A}$  such that  $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$  for all  $n$ .

In other words, we have the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \longrightarrow \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \longrightarrow \cdots \end{array}$$

#### Proposition 2.1.4

Let  $f_\bullet : C_\bullet \rightarrow D_\bullet$  and  $g_\bullet : D_\bullet \rightarrow E_\bullet$  be two chain maps. Then  $g_\bullet \circ f_\bullet$  is also a chain map.

#### Definition 2.1.5: Category of Chain Complexes

Let  $\mathcal{A}$  be an abelian category. Define  $\text{Ch}(\mathcal{A})$  to be the category of chain complexes where

- The objects are chain complexes of objects in  $\mathcal{A}$ .
- The morphisms are chain maps.
- Composition is given by composition of functions.

**Theorem 2.1.6**

Let  $\mathcal{A}$  be an abelian category. Then  $\text{Ch}(\mathcal{A})$  is also an abelian category.

**2.2 Exact Sequences****Definition 2.2.1: Exact Sequence**

A chain complex  $(C_\bullet, \partial_\bullet)$  is said to be exact if  $\text{im}(\partial_{n+1}) = \ker(\partial_n)$  for all  $n$ .

**Definition 2.2.2: Short Exact Sequence**

Let  $\mathcal{A}$  be an abelian category. Let  $A, B, C \in \mathcal{A}$ . A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are morphisms in  $\mathcal{A}$ .

**Proposition 2.2.3**

Let  $\mathcal{A}$  be an abelian category. Let  $A, B, C \in \mathcal{A}$  and  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be morphisms in  $\mathcal{A}$ . A chain complex

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact if and only if  $f$  is a monomorphism,  $g$  is epimorphism and  $\ker(g) = \text{im}(f)$ .

**Definition 2.2.4: Split Exact Sequence**

Let  $\mathcal{A}$  be an abelian category. Let  $A, B, C \in \mathcal{A}$  such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence. We say that it is split exact if  $B \cong A \oplus C$ .

The following is an important equivalent characterization of split exact sequence.

**Theorem 2.2.5: The Splitting Lemma**

Let  $\mathcal{A}$  be an abelian category. Let  $A, B, C \in \mathcal{A}$ . Then the following are equivalent for a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

- The short exact sequence is split exact sequence
- There exists a morphism  $p : B \rightarrow A$  such that  $p \circ f = \text{id}_A$
- There exists a morphism  $s : C \rightarrow B$  such that  $g \circ s = \text{id}_C$

**Lemma 2.2.6: Five Lemma**

Consider the commutative diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\
\downarrow l & & \downarrow m & & \downarrow n & & \downarrow p & & \downarrow q \\
A' & \xrightarrow{r} & B' & \xrightarrow{s} & C' & \xrightarrow{t} & D' & \xrightarrow{u} & E'
\end{array}$$

where all the objects lie in an abelian group  $\mathcal{A}$ . If the two rows are exact,  $m : B \rightarrow B'$ ,  $p : D \rightarrow D'$  are isomorphisms,  $l : A \rightarrow A'$  is an epimorphism and  $q : E \rightarrow E'$  is a monomorphism, then  $n$  is an isomorphism.

### Lemma 2.2.7: Snake Lemma

Consider the commutative diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
\downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array}$$

where all the objects lie in an abelian group  $\mathcal{A}$ . If the two rows are exact, then there is an exact sequence relating the kernels and cokernels of  $a, b, c$

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \xrightarrow{d} \operatorname{coker}(a) \longrightarrow \operatorname{coker}(b) \longrightarrow \operatorname{coker}(c)$$

where  $d$  is called the connecting homomorphism.

## 2.3 Chain Homotopy

### Definition 2.3.1: Chain Homotopy

Let  $\mathcal{A}$  be an abelian category. Let  $a_\bullet, b_\bullet : C_\bullet \rightarrow C'_\bullet$  be two chain maps in  $\operatorname{Ch}(\mathcal{A})$ . Then a chain homotopy from  $a$  to  $b$  is a collection of morphisms

$$\eta_n : C_n \rightarrow C'_{n+1}$$

in  $\mathcal{A}$  such that

$$b_n - a_n = \partial'_{n+1}\eta_n + \eta_{n-1}\partial_n$$

for all  $n \in \mathbb{Z}$ . In this case,  $a$  and  $b$  are said to be chain homotopic.

In other words, we have the diagram:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow \cdots \\
& & \downarrow & \swarrow \eta_{n+1} & \downarrow & \swarrow \eta_n & \downarrow & \swarrow \eta_{n-1} \\
& & b_{n+1} - a_{n+1} & & b_n - a_n & & b_{n-1} - a_{n-1} & \\
& & \downarrow & \swarrow \eta_{n+1} & \downarrow & \swarrow \eta_n & \downarrow & \swarrow \eta_{n-1} \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow \cdots
\end{array}$$

In this case we write  $f \simeq g$ .

### Lemma 2.3.2

Let  $a, b$  be chain homotopic. Then their induced maps in homology are equal. Meaning

$$a_n = b_n : H_n(X) \rightarrow H_n(Y)$$

*Proof.* Let  $c \in \ker(\partial_n)$  be an  $n$ -cycle. Using the equation for chain homotopy, we have that

$$\begin{aligned} b(c) - a(c) &= \partial'_{n+1}(\eta_n(c)) + \eta_{n-1}(\partial(c)) \\ &= \partial'_{n+1}(\eta(c)) \end{aligned}$$

is a boundary in  $\text{im}(\partial'_{n+1}) \subseteq C'_n$ . Thus  $b_n(c)$  and  $a_n(c)$  are of the same coset in  $H_n(X)$ .  $\square$

### Proposition 2.3.3

Let  $\mathcal{A}$  be an abelian group. Let  $f_1, g_1 : C_\bullet \rightarrow D_\bullet$  and  $f_2, g_2 : D_\bullet \rightarrow E_\bullet$  be chain maps in  $\text{Ch}(\mathcal{A})$ . If  $f_1$  and  $g_1$  are chain homotopic and  $f_2$  and  $g_2$  are chain homotopic, then  $f_2 \circ f_1$  is chain homotopic to  $g_2 \circ g_1$ .

*Proof.* The chain homotopies between  $f_1$  and  $g_1$  imposes the identity

$$\partial\eta + \eta\partial = g_1 - f_1$$

for  $\eta : C_\bullet \rightarrow D_\bullet$  the given chain homotopy. Similarly, for  $\nu : D_\bullet \rightarrow E_\bullet$  we have the identity

$$\partial\nu + \nu\partial = g_2 - f_2$$

Then we have that

$$\begin{aligned} g_2 \circ g_1 - f_2 \circ f_1 &= g_2 \circ g_1 - g_2 \circ f_1 + g_2 \circ f_1 - f_2 \circ f_1 \\ &= g_2(g_1 - f_1) + (g_2 - f_2) \circ f_1 \\ &= g_2(\partial\eta + \eta\partial) + (\partial\nu + \nu\partial) \circ f_1 \\ &= \partial g_2\eta + g_2\eta\partial + \partial\nu f_1 + \nu f_1\partial \\ &= \partial(g_2\eta + \nu f_1) + (g_2\eta + \nu f_1)\partial \end{aligned}$$

Thus  $g_2\eta + \nu f_1 : C_n \rightarrow E_{n+1}$  would be a valid chain homotopy from  $g_2 \circ g_1$  to  $f_2 \circ f_1$ .  $\square$

### Lemma 2.3.4

Let  $\mathcal{A}$  be an abelian category. Let  $C_\bullet$  and  $D_\bullet$  be two chain complexes in  $\text{Ch}(\mathcal{A})$ . Then the relation  $\simeq$  on the chain maps from  $C_\bullet$  to  $D_\bullet$  is an equivalence relation.

### Definition 2.3.5: Chain Homotopy Equivalence

Let  $\mathcal{A}$  be an abelian category. Let  $C_\bullet$  and  $D_\bullet$  be two chain complexes in  $\text{Ch}(\mathcal{A})$ . We say that they are chain homotopy equivalence if there exists chain maps  $a_\bullet : C_\bullet \rightarrow D_\bullet$  and  $b_\bullet : D_\bullet \rightarrow C_\bullet$  such that there are chain homotopies

$$b_\bullet \circ a_\bullet \simeq \text{id}_{C_\bullet} \quad \text{and} \quad a_\bullet \circ b_\bullet \simeq \text{id}_{D_\bullet}.$$

### Lemma 2.3.6

Let  $\mathcal{A}$  be an abelian category. Let  $C_\bullet$  and  $D_\bullet$  be chain homotopy equivalent in  $\text{Ch}(\mathcal{A})$ . Then the chain maps induces an isomorphism

$$H_n(C_\bullet) \cong H_n(D_\bullet)$$

in all degrees  $n \in \mathbb{N}$ .



*Proof.* We know that  $b_\bullet \circ a_\bullet \simeq \text{id}_{C_\bullet}$ , which means that they induce the same map:

$$b_* \circ a_* = \text{id}_{H_n(C_\bullet)}$$

Similarly the chain homotopies  $a_\bullet \circ b_\bullet \simeq \text{id}_{D_\bullet}$  induce the same map

$$a_* \circ b_* : \text{id}_{H_n(D_\bullet)}$$

as the identity. Then these two identities mean that  $a_*$  is both injective and surjective.  $\square$

### Proposition 2.3.7

Let  $\mathcal{A}$  be an abelian category. Then chain homotopy equivalence defines an equivalence relation on all chain complexes in  $\text{Ch}(\mathcal{A})$ .

*Proof.* Clearly any chain complex is chain homotopy equivalent to itself by the identity map. If  $C_\bullet$  and  $D_\bullet$  are chain homotopy equivalent by the chain maps  $a_\bullet : C_\bullet \rightarrow D_\bullet$  and  $b_\bullet : D_\bullet \rightarrow C_\bullet$ , then we have the identities  $b_\bullet \circ a_\bullet = \text{id}_{C_\bullet}$  and  $a_\bullet \circ b_\bullet = \text{id}_{D_\bullet}$ . We can then read them in reverse so that  $D_\bullet$  and  $C_\bullet$  are chain homotopy equivalence by the maps  $b_\bullet$  and  $a_\bullet$ .

Suppose further that  $D_\bullet$  and  $E_\bullet$  are chain homotopy equivalent via the maps  $u_\bullet : D_\bullet \rightarrow E_\bullet$  and  $v_\bullet : E_\bullet \rightarrow D_\bullet$ . Then the maps  $u_\bullet \circ a_\bullet$  and  $b_\bullet \circ v_\bullet$  give a chain homotopy equivalence between  $C_\bullet$  and  $E_\bullet$ . Indeed, upon composition, we have that they are chain homotopic to the identity maps.  $\square$

## 2.4 Sequences of Chain Complexes

One can even define short exact sequences of chain complexes themselves.

### Definition 2.4.1: Short Exact Sequence of Chain Complexes

Let  $A_\bullet, B_\bullet, C_\bullet$  be chain complexes in an abelian category  $\mathcal{A}$ . Let  $i : A_\bullet \rightarrow B_\bullet$  and  $j : B_\bullet \rightarrow C_\bullet$  be chain maps in  $\text{Ch}(\mathcal{A})$ . A short exact sequence of chain complexes is a diagram of the form

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & A_{n+1} & \xrightarrow{d_A} & A_n & \xrightarrow{d_A} & A_{n-1} \longrightarrow \cdots \\
 & & \downarrow i & & \downarrow i & & \downarrow i \\
 \cdots & \longrightarrow & B_{n+1} & \xrightarrow{d_B} & B_n & \xrightarrow{d_B} & B_{n-1} \longrightarrow \cdots \\
 & & \downarrow j & & \downarrow j & & \downarrow j \\
 \cdots & \longrightarrow & C_{n+1} & \xrightarrow{d_C} & C_n & \xrightarrow{d_C} & C_{n-1} \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

such that for each  $n$  (vertically in the diagram), the sequence

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0$$

is a short exact sequence. We write this as

$$0 \longrightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{j} C_{\bullet} \longrightarrow 0$$

**Theorem 2.4.2**

Let  $\mathcal{A}$  be an abelian category. Let  $A_{\bullet}, B_{\bullet}, C_{\bullet}$  be a chain complexes in  $\text{Ch}(\mathcal{A})$  such that

$$0 \longrightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{j} C_{\bullet} \longrightarrow 0$$

is a short exact sequence of chain complexes. Then there exists a connecting homomorphism  $\partial : H_n(C_{\bullet}) \rightarrow H_{n-1}(A_{\bullet})$  such that the following sequence of homology

$$\cdots \longrightarrow H_{n+1}(C_{\bullet}) \xrightarrow{\partial} H_n(A_{\bullet}) \xrightarrow{i_*} H_n(B_{\bullet}) \xrightarrow{j_*} H_n(C_{\bullet}) \xrightarrow{\partial} H_{n-1}(A_{\bullet}) \longrightarrow \cdots$$

is an exact sequence.

**Theorem 2.4.3**

Let  $A_{\bullet}, B_{\bullet}, C_{\bullet}, A'_{\bullet}, B'_{\bullet}, C'_{\bullet}$  be six chain complexes in an abelian category  $\mathcal{A}$  and let the following

$$0 \longrightarrow A_{\bullet} \xrightarrow{i} B_{\bullet} \xrightarrow{j} C_{\bullet} \longrightarrow 0$$

$$0 \longrightarrow A'_{\bullet} \xrightarrow{i'} B'_{\bullet} \xrightarrow{j'} C'_{\bullet} \longrightarrow 0$$

be two short exact sequence of chain complexes. Let the following diagram be a morphism of the two short exact sequence of chain complexes in  $\text{Ch}(\mathcal{A})$ .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longrightarrow & A'_{n+1} & \xrightarrow{\partial} & A'_n & \xrightarrow{\partial} & A'_{n-1} \longrightarrow \cdots \\
 & \nearrow \alpha & \downarrow & \nearrow \alpha & \downarrow & \nearrow \alpha & \downarrow \\
 \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} \longrightarrow \cdots \\
 & \downarrow i & \downarrow i' & \downarrow i & \downarrow i' & \downarrow i & \downarrow i' \\
 \cdots & \longrightarrow & B'_{n+1} & \xrightarrow{\partial} & B'_n & \xrightarrow{\partial} & B'_{n-1} \longrightarrow \cdots \\
 & \nearrow \beta & \downarrow & \nearrow \beta & \downarrow & \nearrow \beta & \downarrow \\
 \cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} \longrightarrow \cdots \\
 & \downarrow j & \downarrow j' & \downarrow j & \downarrow j' & \downarrow j & \downarrow j' \\
 \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial} & C'_n & \xrightarrow{\partial} & C'_{n-1} \longrightarrow \cdots \\
 & \nearrow \gamma & \downarrow & \nearrow \gamma & \downarrow & \nearrow \gamma & \downarrow \\
 \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Then the induced diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{j_*} & H_n(C) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots \\
 & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\
 \cdots & \longrightarrow & H_n(A') & \xrightarrow{i'_*} & H_n(B') & \xrightarrow{j'_*} & H_n(C') \xrightarrow{\partial} H_{n-1}(A') \longrightarrow \cdots
 \end{array}$$

is a commutative diagram.

## 2.5 Cochain Complexes

### 3 Derived Functors

#### 3.1 Exact Functors

##### Definition 3.1.1: Additive Functors

Let  $\mathcal{A}, \mathcal{B}$  be abelian categories. We say that a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is additive if for every  $X, Y \in \mathcal{A}$ , the map

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

##### Definition 3.1.2: Exact Functors

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor of abelian categories. Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence in  $\mathcal{A}$ .

- We say that  $F$  is exact if the sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is exact.

- We say that  $F$  is right exact if the sequence

$$F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$$

is exact.

- We say that  $F$  is left exact if the sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$$

is exact.

##### Proposition 3.1.3

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor. Then  $F$  preserves split exact sequences.

##### Theorem 3.1.4: Freyd-Mitchell Embedding Theorem

Let  $\mathcal{A}$  be a small abelian category. Then there exists a ring  $R$  and an exact, fully faithful functor  $F : \mathcal{A} \rightarrow R\text{-mod}$ .

This means that

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(F(M), F(N))$$

##### Lemma 3.1.5

The Freyd-Mitchell embedding preserves kernels and cokernels. Moreover, it maps the zero object to the zero object.

##### Theorem 3.1.6

Let  $\mathcal{A}$  be an abelian category. Let  $M \in \mathcal{A}$ . Then the following are true.

- The covariant functor  $\mathrm{Hom}(M, -) : \mathcal{A} \rightarrow \mathbf{Ab}$  is left exact.
- The contravariant functor  $\mathrm{Hom}(-, M) : \mathcal{A} \rightarrow \mathbf{Ab}$  is right exact.

### 3.2 Injective and Projective Objects

Injectivity and Projectivity objects are created just for the sake of allowing the Hom functor to be exact. Therefore its definition is also direct.

#### Definition 3.2.1: Projective and Injective Objects

Let  $\mathcal{A}$  be an abelian category.

- We say that an object  $Y$  of  $\mathcal{A}$  is injective if the functor  $\text{Hom}(-, Y) : \mathcal{A} \rightarrow \mathbf{Ab}$  is exact.
- We say that an object  $Y$  of  $\mathcal{A}$  is projective if the functor  $\text{Hom}(Y, -) : \mathcal{A} \rightarrow \mathbf{Ab}$  is exact.

#### Definition 3.2.2: Enough Injectives and Enough Projectives

Let  $\mathcal{A}$  be an abelian category.  $\mathcal{A}$  is said to have enough injectives if every object is the subobject of an injective object.  $\mathcal{A}$  is said to have enough projectives if every object is the quotient of a projective object.

There are however equivalent definitions from the categorical point of view.

### 3.3 Resolutions of Objects

There are in general, four types of resolutions. Namely injective resolutions, projective resolutions, free resolutions and acyclic resolutions. We will study all four of them and their relations in this section.

#### Definition 3.3.1: Injective Resolution

Let  $\mathcal{A}$  be an abelian category. An injective resolution of an object  $A \in \mathcal{A}$  is an exact sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

where each  $I^k$  is injective.

#### Theorem 3.3.2

Let  $\mathcal{A}$  be an abelian category. Then  $\mathcal{A}$  has enough injectives if and only if every object of  $\mathcal{A}$  has an injective resolution.

#### Proposition 3.3.3

Let  $\phi : A \rightarrow A'$  be a morphism in an abelian category  $\mathcal{A}$ . Suppose that there are injective resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow \dots \\ & & \phi \downarrow & & & & \\ 0 & \longrightarrow & A' & \longrightarrow & J^0 & \longrightarrow & J^1 \longrightarrow \dots \end{array}$$

for  $A$  and  $A'$  respectively. Then there exists a chain map extending  $\phi$  such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 \longrightarrow \dots \\ & & \phi \downarrow & & \phi^0 \downarrow & & \phi^1 \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & J^0 & \longrightarrow & J^1 \longrightarrow \dots \end{array}$$

Moreover, any two such chain maps are homotopic.

**Lemma 3.3.4**

Let  $\mathcal{A}$  be an abelian category. Then any two injective resolutions of an object  $A$  are homotopically equivalent.

**Definition 3.3.5: Projective Resolution**

Let  $\mathcal{A}$  be an abelian category. An projective resolution of an object  $A$  is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{d} A \longrightarrow 0$$

where each  $P_k$  is projective.

**Theorem 3.3.6**

Let  $\mathcal{A}$  be an abelian category. Then  $\mathcal{A}$  has enough projectives if and only if every object of  $\mathcal{A}$  has a projective resolution.

**Proposition 3.3.7**

Let  $\phi : A \rightarrow A'$  be a morphism in an abelian category  $\mathcal{A}$ . Suppose that there are projective resolutions

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & A \longrightarrow 0 \\ & & & & & & \downarrow \phi \\ \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & A' \longrightarrow 0 \end{array}$$

for  $A$  and  $A'$  respectively. Then there exists a chain map extending  $\phi$  such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi \\ \cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & A' \longrightarrow 0 \end{array}$$

Moreover, any two such chain maps are homotopic.

**Lemma 3.3.8**

Let  $\mathcal{A}$  be an abelian category. Then any two projective resolutions of an object  $A$  are homotopically equivalent.

**3.4 Derived Functors****Definition 3.4.1: Right Derived Functors**

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor. Suppose that  $\mathcal{A}$  has enough injectives. Define the right derived functors  $R^i F : \mathcal{A} \rightarrow \mathcal{B}$  for  $i \geq 0$  as follows.

- On objects,  $R^i F(A) = H^i(F(I^\bullet))$  where  $d : A \rightarrow I^\bullet$  is an injective resolution of  $A$
- On Morphisms,  $R^i F(\phi : A \rightarrow B) = H^i(F(\phi^\bullet : I^\bullet \rightarrow (I')^\bullet))$  where  $\phi^\bullet : I^\bullet \rightarrow (I')^\bullet$  is an extension of  $\phi$  to resolutions.

**Theorem 3.4.2**

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor. The  $n$ th right derived functor  $R^n F$  is an additive functor from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Lemma 3.4.3**

Let  $A$  be an injective object, then  $R^n F(A) = 0$  for  $n \neq 0$ .

**Corollary 3.4.4**

If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a left exact functor, then  $R^0 F = F$ .

**Theorem 3.4.5**

Let  $\mathcal{A}, \mathcal{B}$  be abelian categories with enough injectives. Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor. For any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is a canonical long exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow R^1(A) \longrightarrow R^1(B) \longrightarrow R^1(C) \longrightarrow R^2(A) \longrightarrow \dots$$

**Definition 3.4.6: Left Derived Functors**

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor. Suppose that  $\mathcal{A}$  has enough projectives. Define the left derived functors  $L_i F : \mathcal{A} \rightarrow \mathcal{B}$  for  $i \geq 0$  as follows.

- On objects,  $L_i F(A) = H_i(F(P_\bullet))$  where  $d : P_\bullet \rightarrow A$  is a projective resolution of  $A$
- On Morphisms,  $L_i F(\phi : A \rightarrow B) = L_i(F(\phi_\bullet : P_\bullet \rightarrow (P')_\bullet))$  where  $\phi_\bullet : P_\bullet \rightarrow (P')_\bullet$  is an extension of  $\phi$  to resolutions.

**Theorem 3.4.7**

Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor. The  $n$ th left derived functor  $L_n F$  is an additive functor from  $\mathcal{A}$  to  $\mathcal{B}$ .

**Lemma 3.4.8**

Let  $A$  be a projective object, then  $L_n F(A) = 0$  for  $n \neq 0$ .

**Corollary 3.4.9**

If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is a right exact functor, then  $L_0 F = F$ .

**Theorem 3.4.10**

Let  $\mathcal{A}, \mathcal{B}$  be abelian categories with enough projectives. Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a right exact functor. For any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is a canonical long exact sequence

$$\cdots \longrightarrow L_2(C) \longrightarrow L_1(A) \longrightarrow L_1(B) \longrightarrow L_1(C) \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

### 3.5 $\delta$ -Functors

#### Definition 3.5.1: $\delta$ -Functors

Let  $\mathcal{A}$  and  $\mathcal{B}$  be abelian categories. A homological  $\delta$ -functor is a collection  $\{T_n : \mathcal{A} \rightarrow \mathcal{B} \mid n \in \mathbb{N}\}$  of additive functors such that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there are morphisms  $\delta_n : T_n(C) \rightarrow T_n(A)$  for  $n \in \mathbb{N}$  such that the following are true.

- There is a long exact sequence

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \longrightarrow \cdots$$

- If there is a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

the following diagram commutes:

$$\begin{array}{ccc} T_n(C) & \xrightarrow{\delta_n} & T_{n-1}(A) \\ \downarrow & & \downarrow \\ T_n(C') & \xrightarrow{\delta'_n} & T_{n-1}(A') \end{array}$$



## 4 A Second Course on Modules

### 4.1 Projective and Injective Modules

#### Definition 4.1.1: Projective Modules

An  $R$ -module  $M$  is said to be projective if for every surjective homomorphism  $f : N \twoheadrightarrow M$  and every module homomorphism  $g : P \rightarrow M$ , there exists a module homomorphism  $h : P \rightarrow N$  such that  $f \circ h = g$ . In other words, the following diagram commutes:

$$\begin{array}{ccc} & N & \\ \nearrow \exists h & \downarrow f & \\ P & \xrightarrow{g} & M \end{array}$$

#### Lemma 4.1.2

Every free module is projective.

*Proof.* Let  $R$  be a ring and let  $F$  be a free  $R$ -module. Suppose that  $F$  has basis  $B$ . Let  $M, N$  be  $R$ -modules. Suppose that  $f : N \rightarrow M$  is surjective and there exists an  $R$ -module homomorphism  $g : F \rightarrow M$ . Since  $f$  is surjective, for each  $b \in B$ , we can choose a pre-image for  $g(b)$  in  $N$  for all  $b \in B$ . Call it  $n_b$ . Now define  $h : F \rightarrow N$  by  $b \mapsto n_b$  and then extend it  $R$ -linearly. Now if  $\sum_{b \in B} k_b b \in F$ , we have that

$$\begin{aligned} (f \circ h) \left( \sum_{b \in B} k_b b \right) &= f \left( \sum_{b \in B} k_b h(b) \right) \\ &= f \left( \sum_{b \in B} k_b n_b \right) &= f \left( \sum_{b \in B} k_b n_b \right) = \sum_{b \in B} k_b f(n_b) \\ &= \sum_{b \in B} k_b g(b) \\ &= g \left( \sum_{b \in B} k_b b \right) \end{aligned}$$

so that  $f \circ h = g$ . Thus  $F$  is projective.  $\square$

#### Theorem 4.1.3

Let  $P$  be an  $R$ -module. Then the following are equivalent.

- $P$  is projective
- For every short exact sequence  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  we have that

$$0 \rightarrow \operatorname{Hom}(P, A) \xrightarrow{f} \operatorname{Hom}(P, B) \xrightarrow{g} \operatorname{Hom}(P, C) \rightarrow 0$$

is exact.

- $P \oplus Q$  is a free  $R$ -module for some  $R$ -module  $Q$ .

*Proof.*

- (3)  $\implies$  (1): Suppose that there exists a module  $Q$  such that  $P \oplus Q$  is free. Let  $f : N \rightarrow M$  be a surjective  $R$ -module homomorphism and let  $g : P \rightarrow M$  be an  $R$ -module homomorphism.

□

**Proposition 4.1.4**

A direct sum  $\bigoplus_{i \in I} P_i$  is projective if and only if each  $P_i$  is.

**Proposition 4.1.5**

Let  $P$  be a module. Then  $P$  is projective if and only if every exact sequence of the following form splits:

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$$

**Definition 4.1.6: Injective Modules**

An  $R$ -module  $M$  is said to be projective if for every injective homomorphism  $f : N \hookrightarrow M$  and every module homomorphism  $g : N \rightarrow I$ , there exists a module homomorphism  $h : M \rightarrow I$  such that  $f \circ h = g$ . In other words, the following diagram commutes:

$$\begin{array}{ccc} N & & \\ f \downarrow & \searrow g & \\ M & \xrightarrow{\exists h} & I \end{array}$$

**Theorem 4.1.7**

An  $R$ -module  $I$  is injective if and only if for every short exact sequence  $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$  we have that

$$0 \rightarrow \operatorname{Hom}(A, I) \xrightarrow{f} \operatorname{Hom}(B, I) \xrightarrow{g} \operatorname{Hom}(C, I) \rightarrow 0$$

is exact.

**Proposition 4.1.8**

Let  $E$  be a module. Then  $E$  is injective if and only if every exact sequence of the following form splits:

$$0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$$

**4.2 Flat Modules****Definition 4.2.1: Flat Modules**

Let  $R$  be a ring. An  $R$ -module  $M$  is said to be flat if for every injective linear map  $\phi : K \rightarrow L$  of  $R$ -modules, the map

$$\phi \otimes \operatorname{id}_M : K \otimes_R M \rightarrow L \otimes_R M$$

is also injective.

**Theorem 4.2.2**

Let  $R$  be a ring and  $M$  an  $R$ -module. Let  $0 \rightarrow K \rightarrow L \rightarrow J \rightarrow 0$  be an exact sequence, then the sequence

$$K \otimes_R M \rightarrow L \otimes_R M \rightarrow J \otimes_R M \rightarrow 0$$

is also exact.

**Theorem 4.2.3**

Let  $R$  be a ring and  $M$  an  $R$ -module. Then  $M$  is a flat module if and only if for every short exact sequence  $0 \rightarrow K \rightarrow L \rightarrow J \rightarrow 0$ , the sequence

$$0 \rightarrow K \otimes_R M \rightarrow L \otimes_R M \rightarrow J \otimes_R M \rightarrow 0$$

is also exact.

**Theorem 4.2.4**

Let  $R$  be a ring. Then the following are true.

- Product: If  $A$  and  $B$  are flat over  $R$  then  $A \otimes_R B$  is flat over  $R$
- Base Change: Let  $S$  be an  $R$ -algebra ( $R \rightarrow S$  a ring hom). Then  $M \otimes_R S$  is flat over  $S$  for any flat  $R$ -module  $M$
- Transitivity: Let  $S$  be an  $R$ -algebra such that  $S$  is flat over  $R$ . If  $C$  is flat over  $S$  then  $C$  is flat over  $R$ .

We have the following inclusion of modules

$$\text{Free Modules} \subset \text{Projective Modules} \subset \text{Flat Modules} \subset \text{Torsion Free Modules}$$

**4.3 Extensions and Torsions****Definition 4.3.1: Extensions**

Let  $R$  be a ring. Let  $M, N$  be  $R$ -modules. An extension of  $M$  by  $N$  is a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

of  $R$ -modules.

**Definition 4.3.2: Equivalent Extensions**

Let  $R$  be a ring. Let  $M, N$  be  $R$ -modules. We say that two extensions

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

are equivalent if there exists an  $R$ -module homomorphism  $\phi : E \rightarrow F$  such that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \text{id}_N & & \downarrow \phi & & \downarrow \text{id}_M \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M \longrightarrow 0 \end{array}$$

**Proposition 4.3.3**

Let  $R$  be a ring. Let  $M, N$  be  $R$ -modules. Suppose that the following two extensions are equivalent:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & M \longrightarrow 0 \\ & & \downarrow \text{id}_N & & \downarrow \phi & & \downarrow \text{id}_M \\ 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M \longrightarrow 0 \end{array}$$

Then  $\phi$  is an isomorphism. Moreover, equivalent extensions is an equivalence relation.

**Definition 4.3.4: Split Extensions**

Let  $R$  be a ring. Let  $M, N$  be  $R$ -modules. We say that an extension splits if it is equivalent to the following extension

$$0 \longrightarrow N \longrightarrow N \oplus M \longrightarrow M \longrightarrow 0$$

of  $R$ -modules. The above extension is called a trivial extension.

**Theorem 4.3.5**

Let  $R$  be a ring. Let  $M, N$  be  $R$ -modules. There is a bijection

$$\frac{\{\text{Extensions of } M \text{ by } N\}}{\cong} \xleftarrow{1:1} \text{Ext}_R^1(M, N)$$

where  $\cong$  means equivalence of extensions. Moreover, the trivial extension corresponds to the zero element of  $\text{Ext}_R^1(M, N)$ .

**4.4 Derived Functors in the Category of  $R$ -Modules****Definition 4.4.1: The Ext Functor**

Denote  ${}_R\mathbf{Mod}$  the category of  $R$ -modules. Let  $A$  be an  $R$ -module. Define the right derived functor of the functor  $\text{Hom}(A, -) : {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$  to be

$$\text{Ext}_R^i(A, -) : {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$$

Explicitly, for

$$0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

an injective resolution, form the cochain complex

$$0 \rightarrow \text{Hom}_R(A, I^0) \rightarrow \text{Hom}_R(A, I^1) \rightarrow \dots$$

and define  $\text{Ext}$  to be the cohomology group

$$\text{Ext}_R^i(A, B) = \frac{\ker(\text{Hom}_R(A, I^i) \rightarrow \text{Hom}_R(A, I^{i+1}))}{\text{im}(\text{Hom}_R(A, I^{i-1}) \rightarrow \text{Hom}_R(A, I^i))}$$

**Theorem 4.4.2**

Let  $A, B$  be  $R$ -modules. Then the following are true regarding the  $\text{Ext}$  group.

- $\text{Ext}_R^0(A, B) \cong \text{Hom}_R(A, B)$
- $\text{Ext}_R^i(A, B) = 0$  for all  $i > 0$  if  $A$  is projective or  $B$  is injective

- $\text{Ext}_R^i(A, B) = 0$  for all  $i \geq 2$  if  $A, B$  are  $\mathbb{Z}$ -modules.

#### Definition 4.4.3: The Tor Functor

Denote  ${}_R\mathbf{Mod}$  the category of  $R$ -modules. Let  $B$  be an  $R$ -module. Define the right derived functor of the functor  $-\otimes_R B : {}_R\mathbf{Mod} \rightarrow {}_R\mathbf{Mod}$  to be

$$\text{Tor}_i^R(-, B) : {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$$

Explicitly, for

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow B \rightarrow 0$$

an injective resolution, form the chain complex

$$\cdots \rightarrow P_1 \otimes_R B \rightarrow P_0 \otimes_R B \rightarrow 0$$

and define Tor to be the homology group

$$\text{Tor}_i^R(A, B) = \frac{\ker(P_i \otimes_R B \rightarrow P_{i-1} \otimes_R B)}{\text{im}(P_{i+1} \otimes_R B \rightarrow P_i \otimes_R B)}$$

## 5 Triangulated Categories

### 5.1 Axioms of a Triangulated Category

#### Definition 5.1.1: Triangles

Let  $\mathcal{C}$  be a category and  $T : \mathcal{C} \rightarrow \mathcal{C}$  an automorphism functor. Let  $A, B, C \in \mathcal{C}$ . A triangle on  $(A, B, C)$  is a triple  $(u, v, w)$  of morphisms in  $\mathcal{C}$  where  $u : A \rightarrow B$ ,  $v : B \rightarrow C$ ,  $w : C \rightarrow T(A)$ .

#### Definition 5.1.2: Morphisms of Triangles

Let  $\mathcal{C}$  be a category and  $T : \mathcal{C} \rightarrow \mathcal{C}$  an automorphism functor. Let  $(u, v, w)$  and  $(u', v', w')$  be triangles in  $\mathcal{C}$ . A morphism of triangles is a triple  $(f, g, h)$  such that the following diagram commutes:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ f \downarrow & & g \downarrow & & h \downarrow & & \downarrow T(f) \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A') \end{array}$$

#### Definition 5.1.3: Triangulated Categories

Let  $\mathcal{C}$  be an additive category. We say that  $\mathcal{C}$  is a triangulated category if there is a functor  $T : \mathcal{C} \rightarrow \mathcal{C}$  and a family  $\{(u, v, w) \mid u, v, w \in \text{Mor}(\mathcal{C})\}$  of triangles called exact triangles such that the following hold.

- For any morphism  $u : A \rightarrow B$ , there exists an exact triangle  $(u, v, w)$ :

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{-\exists w} T(A)$$

If  $(u, v, w)$  is a triangle on  $(A, B, C)$  isomorphic to an exact triangle  $(u', v', w')$  on  $(A', B', C')$ , then  $(u, v, w)$  is also exact:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\ \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \downarrow \cong \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A') \end{array}$$

Finally,  $(\text{id}_A, 0, 0)$  is exact:

$$A \xrightarrow{\text{id}_A} A \longrightarrow 0 \longrightarrow T(A)$$

- Rotations: If  $(u, v, w)$  is an exact triangle on  $(A, B, C)$ , then both rotations  $(v, w, -T(u))$  and  $(-T^{-1}(w), u, v)$  are exact triangles on  $(B, C, T(A))$  and  $(T^{-1}(C), A, B)$  respectively.
- Morphisms: Let the following be exact triangles:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

$$A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} T(A')$$

Suppose that there exists morphisms  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$  such that  $g \circ u = u' \circ f$ . Then there exists  $h : C \rightarrow C'$  such that  $(f, g, h)$  is a morphism of triangles:

$$\begin{array}{ccccccc}
A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & T(A) \\
f \downarrow & & g \downarrow & & \exists h \downarrow & & \downarrow T(f) \\
A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & T(A')
\end{array}$$

- The Octahedral Axiom: Let the following be exact triangles:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} T(A)$$

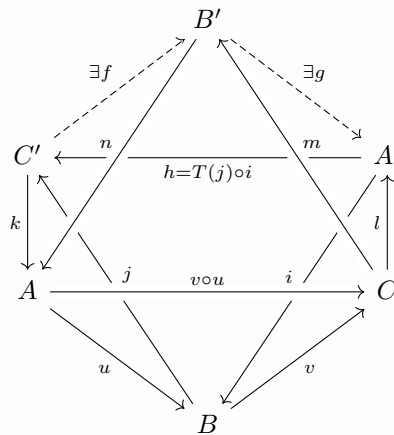
$$B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} T(B)$$

$$A \xrightarrow{v \circ u} C \xrightarrow{m} B' \xrightarrow{n} T(A)$$

Then there exists an exact triangle:

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} T(C')$$

such that  $l = g \circ m$ ,  $k = n \circ f$ ,  $h = T(j) \circ i$ ,  $i \circ g = T(u) \circ n$  and  $f \circ j = m \circ v$ . In other words, the following diagram commutes:



Where we abused notation by drawing  $k : C' \rightarrow T(A)$  as a morphism  $C' \rightarrow A$  etc so that the drawing becomes compact.

#### Lemma 5.1.4

Let  $(\mathcal{C}, T)$  be a triangulated category. Let  $(u, v, w)$  be an exact triangle. Then  $v \circ u$ ,  $w \circ v$  and  $T(u) \circ w$  are 0 in  $\mathcal{C}$ .

#### Lemma 5.1.5

Let  $(\mathcal{C}, T)$  be a triangulated category. Let  $(f, g, h)$  be a morphism of exact triangles. If both  $f$  and  $g$  are isomorphisms, then  $h$  is an isomorphism.

## 6 Derived Categories

### 6.1 The Homotopy Category of Cochain Complexes

#### Definition 6.1.1: Homotopy Category of Cochain Complexes

Let  $\mathcal{A}$  be an abelian category. Let  $\mathbf{CCh}(\mathcal{A})$  be the category of cochain complexes of  $\mathcal{A}$ . Define the homotopy category of chain complexes  $K(\mathcal{A})$  to be the category defined as follows.

- The objects are the objects of  $\mathbf{CCh}(\mathcal{A})$
- The morphisms are homotopy classes of chain maps
- Composition is given by composition of chain maps

#### Lemma 6.1.2

Let  $\mathcal{A}$  be an abelian category. Then the cohomology functors  $H^\bullet : \mathbf{CCh}(\mathcal{A}) \rightarrow \mathcal{A}$  induces a well defined functor from  $K(\mathcal{A})$  to  $\mathcal{A}$ .

#### Proposition 6.1.3

Let  $\mathcal{A}$  be an abelian category. The homotopy category of cochain complexes satisfy the following universal property.

If  $F : \mathbf{CCh}(\mathcal{A}) \rightarrow \mathcal{D}$  is a functor that sends chain homotopy equivalences to isomorphisms, then  $F$  factors uniquely through  $K(\mathcal{A})$ :

$$\begin{array}{ccc} \mathbf{CCh}(\mathcal{A}) & \longrightarrow & K(\mathcal{A}) \\ & \searrow F & \downarrow \exists! \\ & & \mathcal{D} \end{array}$$

#### Lemma 6.1.4

Let  $\mathcal{A}$  be an abelian category. Then  $K(\mathcal{A})$  is a triangulated category.

### 6.2 Localization of Categories

#### Definition 6.2.1: Localization of a Category

Let  $\mathcal{C}$  be a category and let  $S$  be a collection of morphisms in  $\mathcal{C}$ . A localization of  $\mathcal{C}$  with respect to  $S$  is a category  $S^{-1}\mathcal{C}$  together with a functor  $q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$  such that the following are true.

- For all  $s \in S$ ,  $q(s)$  is an isomorphism in  $S^{-1}\mathcal{C}$
- If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor such that  $F(s)$  is an isomorphism for all  $s \in S$ , then there exists a unique functor  $G : S^{-1}\mathcal{C} \rightarrow \mathcal{D}$  such that the following diagram commute:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{q} & S^{-1}\mathcal{C} \\ & \searrow F & \downarrow \exists! G \\ & & \mathcal{D} \end{array}$$



**Lemma 6.2.2**

Let  $\mathcal{A}$  be an abelian category. Then  $K(\mathcal{A})$  is a localization of  $\mathcal{A}$  with respect to all homotopy equivalences.

Not all localizations are well defined by set-theoretic issues. Morphisms that one wants to invert may not form a set or even a collection. We will give a way of explicitly constructing the localization of some specific categories below.

**Definition 6.2.3: Multiplicative System****Definition 6.2.4: Locally Small Multiplicative System****Theorem 6.2.5: Gabriel-Zisman Theorem****Corollary 6.2.6**

Let  $\mathcal{C}$  be a category containing the zero object  $0$  and let  $q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$  be a localization of  $\mathcal{C}$ . Then  $q(X) \cong 0$  if and only if the  $S$  contains the  $0$  map  $0 : X \rightarrow X$ .

**Corollary 6.2.7**

Let  $\mathcal{C}$  be a category and let  $q : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$  be a localization of  $\mathcal{C}$ . If  $\mathcal{C}$  is additive, then  $S^{-1}\mathcal{C}$  and  $q$  are both additive.

**6.3 Derived Categories****Definition 6.3.1: Derived Categories**

Let  $\mathcal{A}$  be an abelian category and let  $K(\mathcal{A})$  be its category of chain complexes. Define the derived category  $D(\mathcal{A})$  of  $\mathcal{A}$  to be the localization of  $K(\mathcal{A})$  with respect to all quasi-isomorphisms.

## 7 Spectral Sequences

### 7.1 Spectral Sequences

#### Definition 7.1.1: Homological Spectral Sequences

Let  $\mathcal{A}$  be an abelian category. A homological spectral sequence consists of the following data.

- A collection of objects  $E_{\bullet,\bullet}^r = \{E_{p,q}^r \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$  called pages for each  $r \in \mathbb{N}$ . So that there is a sequence

$$E_{\bullet,\bullet}^1, E_{\bullet,\bullet}^2, E_{\bullet,\bullet}^3, \dots$$

of family of objects

- A degree  $(p, q)$  map

$$d_{p,q}^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$$

for each  $p, q \in \mathbb{Z}$  and  $r \in \mathbb{N}$  such that  $d^r \circ d^r = 0$

- Isomorphisms of the form  $E_{\bullet,\bullet}^{r+1} = H_{\bullet}(E_{\bullet,\bullet}^r, d^r)$ . This means that

$$E_{p,q}^{r+1} = \frac{\ker(d^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r)}{\operatorname{im}(d^r : E_{p+r,q-r+1}^r \rightarrow E_{p,q}^r)}$$

We say that the total degree of  $E_{p,q}^r$  is  $n = p + q$ .

#### Definition 7.1.2: Cohomological Spectral Sequences

Let  $\mathcal{A}$  be an abelian category. A cohomological spectral sequence consists of the following data.

- A collection of objects  $E_r^{\bullet,\bullet} = \{E_r^{p,q} \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$  called pages for each  $r \in \mathbb{N}$ . So that there is a sequence

$$E_1^{\bullet,\bullet}, E_2^{\bullet,\bullet}, E_3^{\bullet,\bullet}, \dots$$

of family of objects

- A degree  $(p, q)$  map

$$d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p-r,q+r-1}$$

for each  $p, q \in \mathbb{Z}$  and  $r \in \mathbb{N}$  such that  $d_r \circ d_r = 0$

- Isomorphisms of the form  $E_{r+1}^{\bullet,\bullet} = H^{\bullet}(E_r^{\bullet,\bullet}, d_r)$ . This means that

$$E_{r+1}^{p,q} = \frac{\ker(d_r : E_r^{p,q} \rightarrow E_r^{p-r,q+r-1})}{\operatorname{im}(d_r : E_r^{p+r,q-r+1} \rightarrow E_r^{p,q})}$$

Notice that cohomological spectral sequences are really the same thing as homological spectral sequences, just reindex the objects by  $E_r^{p,q} = E_{-p,-q}^r$ .

#### Definition 7.1.3: Bounded Spectral Sequences

Let  $(E_{\bullet,\bullet}^r, d^r)$  be a homological spectral sequence. We say that it is bounded if for each  $n \in \mathbb{N}$ , there are only finitely many non-zero terms of total degree  $n$  in  $E_{\bullet,\bullet}^r$  for each  $r \in \mathbb{N}$ .

We say that it is bounded below if there exists  $s_n \in \mathbb{Z}$  for each  $n \in \mathbb{N}$  such that terms  $E_{\bullet,\bullet}^r$  of total degree  $n$  are 0 for all  $p < s_n$ .

**Lemma 7.1.4**

Let  $(E_{\bullet,\bullet}^r, d^r)$  be a bounded homological spectral sequence. Then for each  $(p, q) \in \mathbb{Z}^2$ , there exists  $r_0 \in \mathbb{N}$  such that  $E_{p,q}^{r+1} \cong E_{p,q}^r$  for all  $r \geq r_0$ .

**Definition 7.1.5: Stable Values**

Let  $(E_{\bullet,\bullet}^r, d^r)$  be a bounded homological spectral sequence. Let  $(p, q) \in \mathbb{Z}^2$  and  $r_0 \in \mathbb{N}$  such that  $E_{p,q}^{r+1} = E_{p,q}^r$  for all  $r \geq r_0$ . Define the stable values of the sequence to be

$$E_{p,q}^\infty = E_{p,q}^{r_0}$$

**7.2 Filtrations****Definition 7.2.1: Bigraded Abelian Groups**

A bigraded abelian group  $A_{\bullet,\bullet}$  is an abelian group  $A$  together with a decomposition

$$A = \bigoplus_{p,q \in \mathbb{Z}} A_{p,q}$$

**7.3 Exact Couples****Definition 7.3.1: Exact Couple**

An exact couple of type  $r$  consists of bigraded abelian groups  $E_{\bullet,\bullet}$  and  $A_{\bullet,\bullet}$  and maps  $i : A_{\bullet,\bullet} \rightarrow A_{\bullet,\bullet}$  of degree  $(1, -1)$ ,  $j : A_{\bullet,\bullet} \rightarrow E_{\bullet,\bullet}$  of degree  $(-r, r)$  and  $k : E_{\bullet,\bullet} \rightarrow A_{\bullet,\bullet}$  of degree  $(-1, 0)$  such that the triangle

$$\begin{array}{ccc} A_{\bullet,\bullet} & \xrightarrow{i} & A_{\bullet,\bullet} \\ & \nwarrow k & \nearrow j \\ & E_{\bullet,\bullet} & \end{array}$$

is exact at each vertex ( $\text{im}(i) = \ker(j)$  and so on).