Higher Category Theory

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Abstract

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1 Simplicial Objects in a Category

1.1 The Simplex Category

Definition 1.1.1: Simplex Category

The simplex category Δ consists of the following data.

- The objects are $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}$.
- The morphisms are the non-strictly order preserving functions. This means that a morphism $f : [n] \to [m]$ must satisfy $f(i) \leq f(j)$ for all $i \leq j$.
- Composition is the usual composition of functions.

Definition 1.1.2: Maps in the Simplex Category

Consider the simplex category Δ. Define the face maps and the degeneracy maps as follows.

- A face map in Δ is the unique morphism $d^i : [n-1] \to [n]$ that is injective and whose image does not contain i. Explicitly, we have

$$d^i(k) = \begin{cases} k & \text{if } 0 \leq k < i \\ k+1 & \text{if } i \leq k \leq n-1 \end{cases}$$

- A degeneracy map in Δ is the unique morphism $s^i : [n+1] \to [n]$ that is surjective and hits i twice. Explicitly, we have

$$s^i(k) = \begin{cases} k & \text{if } 0 \leq k \leq i \\ k-1 & \text{if } i+1 \leq k \leq n+1 \end{cases}$$

Proposition 1.1.3

The face maps and the degeneracy maps in the simplex category Δ satisfy the following simplicial identities:

- $d^i \circ d^j = d^{j-1} \circ d^i$ if $i < j$
- $d^i \circ s^j = s^{j-1} \circ d^i$ if $i < j$
- $d^i \circ s^i = \text{id}$
- $d^{i+1} \circ s^i = \text{id}$
- $d^i \circ s^j = s^j \circ d^{i-1}$ if $i > j+1$
- $s^i \circ s^j = s^{j+1} \circ s^i$ if $i \leq j$

Proposition 1.1.4

Every morphism in the simplex category Δ is a composition of the face maps and the degeneracy maps.

1.2 Simplicial Sets

Definition 1.2.1: Simplicial Sets

A simplicial set is a presheaf
$$S : \Delta \to \text{Sets}$$

Definition 1.2.2: Category of Simplicial Sets

The category of simplicial sets sSet is defined as follows.

- The objects are simplicial sets $S : \Delta \to \text{Sets}$
- The morphisms are just morphisms of presheaves. This means that if $S, T : \Delta \to \text{Sets}$ are simplicial sets, then a morphism $\lambda : S \to T$ consists of morphisms $\lambda_n : S([n]) \to T([n])$ for $n \in \mathbb{N}$ such that the following diagram commutes:

$$\begin{tikzcd} S([n]) \arrow[r, "S(f)"] \arrow[d, "\lambda_n"'] & S([m]) \arrow[d, "\lambda_m"] \\ T([n]) \arrow[r, "T(f)"'] & T([m]) \end{tikzcd}$$

- Composition is defined as the usual composition of functors.

The Yoneda lemma in this context implies that there is a bijection
$$\text{Hom}_{\text{sSet}}(\text{Hom}_\Delta([n], -), S) \cong S([n])$$
that is natural in the variable $[n]$. We will denote
$$\Delta^n = \text{Hom}_\Delta([n], -)$$
which is the image of $[n]$ under the yoneda embedding $y : \Delta \to \text{sSet}$ defined by $[n] \mapsto \text{Hom}_\Delta([n], -)$.

Definition 1.2.3: n-Simplices

Let $S : \Delta \to \text{Set}$ be a simplicial set. For $n \in \mathbb{N}$, define the n-simplices of S to be
$$S_n = S([n]) = \text{Hom}_{\text{sSet}}(\Delta^n, S)$$

Notice that Δ^n is a simplicial set
$$\Delta^n : \Delta \to \text{Set}$$
defined by $[m] \mapsto \text{Hom}_\Delta([n], [m])$. Notice that if $n > m$, then it is impossible to have an order preserving function $[n] \to [m]$. Hence when $n > m$, $\text{Hom}_\Delta([n], [m])$ is empty. It is also clear that the m-simplices of Δ^n are precisely the order preserving maps $[m] \to [n]$.

Definition 1.2.4: Standard n-Simplex

Let $n \in \mathbb{N}$. The standard n-simplex is the simplicial set $\Delta^n : \Delta \to \text{Set}$ defined by
$$\Delta^n = \text{Hom}_\Delta([n], -)$$

All such simplicial sets Δ^n are useful in determining the contents of an arbitrary simplicial set. As for any presheaf, instead of focusing between the passage of data from Δ to Set, we should instead think of what kind of structure the presheaf brings to Set. Let C be a simplicial set. Then this means the following. For each n, there is a set $C_n = \text{Hom}_{\text{sSet}}(\Delta^n, C)$. For each morphism in Δ, there is a corresponding morphism in Set, which we shall discuss now.

Theorem 1.2.5

Let $C : \Delta \to \text{Set}$ be a simplicial set. Then every morphism in $C(\Delta)$ is the composite of two kinds of maps:

- The face maps: $d_i : C_n \to C_{n-1}$ for $0 \leq i \leq n$ defined by
$$d_i = C(d^i : [n-1] \to [n])$$
- The degeneracy maps: $s_i : C_{n+1} \to C_n$ for $0 \leq i \leq n$ defined by
$$s_i = C(s^i : [n+1] \to [n])$$

Moreover, these maps satisfy the above simplicial identities

Theorem 1.2.6

The category **sSet** is a symmetric monoidal category with level-wise cartesian product.

Recall the notion of a Δ-set from Algebraic Topology 2 and one might realize they look suspiciously similar to that of a simplicial set. Let us recall. A Δ-set is a collection of sets S_n for $n \in \mathbb{N}$ together with maps $d_i^n : S_n \to S_{n-1}$ for $0 \leq i \leq n$ such that
$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$$
for $i < j$. One can easily convince themselves that every simplicial set is a Δ-set. Indeed, a simplicial set satisfies five more relations than a Δ-set. Therefore we have that
$$\textbf{sSet} \subset \Delta \text{ Complexes}$$

Theorem 1.2.7

Every simplicial set is a Δ-set.

Combining with the previously learnt combinatorial objects in algebraic topology, we now have the following tower:
$$\text{Simplicial Complexes} \subset \textbf{sSet} \subset \Delta \text{ Complexes} \subset \textbf{CW}$$

1.3 Geometric Realization of Simplicial Sets

Definition 1.3.1: Geometric Realization of Standard n-Simplexes

Let $n \in \mathbb{N}$. Consider the standard n-simplex Δ^n. Define the geometric realization of Δ^n to be
$$|\Delta^n| = \left\{ \sum_{k=0}^n t_k v_k \,\middle|\, \sum_{k=0}^n t_k = 1 \text{ and } t_k \geq 0 \text{ for all } k = 0, \dots, n \right\}$$

This definition is exactly the same as the definition of an n-simplex in Algebraic Topology 2. Now we proceed to the general case.

Definition 1.3.2: Geometric Realization of Simplicial Sets

Let C be a simplicial set. Define the geometric realization of C to be
$$|C| = \left(\coprod_{n \geq 0} C_n \times |\Delta^n| \right) / \sim$$
where the equivalence relation is generated by the following.

- The ith face of $\{x\} \times |\Delta^n|$ is identified with $\{d_i x\} \times |\Delta^{n-1}|$ by the linear homeomorphism preserving the order of the vertices.
- $\{s_i x\} \times |\Delta^n|$ is collapsed onto $\{x\} \times |\Delta^{n-1}|$ via the linear projection parallel to the line connecting the ith and the $(i+1)$st vertiex.

This construction of geometric realization is moreover functorial. Once again, we first define a map of geometric realization of simplicial sets.

Definition 1.3.3: Induced Map of Geometric Realization of Standard Simplicial Sets

Let $f : \Delta^n \to \Delta^m$ be a map of standard simplexes. Define $f_* : |\Delta^n| \to |\Delta^m|$ by

$$(t_0, \dots, t_n) \mapsto (s_0, \dots, s_m)$$

where

$$s_i = \begin{cases} 0 & \text{if } f^{-1}(i) = 0 \\ \sum_{j \in f^{-1}(i)} t_j & \text{otherwise} \end{cases}$$

Theorem 1.3.4

The geometric realization of a simplicial set is functorial $|\cdot| : \mathsf{sSet} \to \mathsf{Top}$ in the following way.

- On objects, it sends any simplicial set C to its geometric realization $|C|$.
- On morphisms, it sends any morphism $C \to D$ of simplicial sets to a continuous map defined by

We thus have that

$$\substack{\text{Geometric Relizations} \\ \text{of simplicial sets}} \subset \substack{\text{Geometric Relizations} \\ \text{of } \Delta\text{-sets}} \subset \text{CW-Complexes}$$

1.4 Simplicial Subsets

Definition 1.4.1: Faces of a Simplex

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n.

- Denote $\partial_i \Delta^n \subset \Delta^n$ the simplicial subset generated by the ith face

$$d_i(\text{id} : [n] \to [n]) = d^i : [n-1] \to [n]$$

- Denote $\partial \Delta^n$ the simplicial subset generated by the faces $\partial_i \Delta^n$ for $0 \leq i \leq n$. Define $\partial \Delta^0 = \emptyset$.

Definition 1.4.2: Inner and Outer Horns

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Define the ith horn Λ_i^n of Δ^n to be the the simplicial subset generated by all the faces $\partial_k \Delta^n$ except the ith one. It is called inner if $0 < i < n$. It is called outer otherwise.

Definition 1.4.3: Fillers for an Inner Horn

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Let Λ_i^n be an inner horn. We say that Λ admits a filler if for all maps $F : \Lambda_i^n \to C$ there exists a map $U : \Delta^n \to C$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "F"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists U"'] & \end{tikzcd}$$

1.5 Simplicial Objects

Definition 1.5.1: Simplicial Objects

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $S : \Delta^{\text{op}} \to \mathcal{C}$.

Hence a simplicial object in **Set** is just simplical sets.

Definition 1.5.2: Category of Simplicial Objects

Let $\mathcal{C}$ be a category. Define the category of simplicial objects $s\mathcal{C}$ of $\mathcal{C}$ as follows.

- The objects are simplicial objects $S : \Delta^{\text{op}} \to \mathcal{C}$ of $\mathcal{C}$ which are presheaves
- The morphism of simplcial objects are just morphisms of presheaves, which are natural transformations
- Composition is given by composition of natural transformations

Definition 1.5.3: Normalized Chain Complex Functor

Theorem 1.5.4: The Dold-Kan Correspondence

Consider the abelian category $\mathbf{Ab}$ of abelian groups. The normalized chain complex functor

$$N : s\mathbf{Ab} \xrightarrow{\cong} \text{Ch}_{\geq 0}(\mathbf{Ab})$$

gives an equivalence of categories, with inverse as the simplicialization functor

$$\Gamma : \text{Ch}_{\geq 0}(\mathbf{Ab}) \to s\mathbf{Ab}$$

2 Introduction to Infinity Categories

2.1 Infinity Categories and Some Examples

Definition 2.1.1: Infinity Categories

An infinity category is a simplicial set C such that each inner horn admits a filler. In other words, for all $0 < i < n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Definition 2.1.2: Nerve of a Category

Let $\mathcal{C}$ be a category. Define the nerve of the category $N(C) : \Delta \to \text{Set}$ as follows.

- For $n \in \mathbb{N}$, $N(C)_n$ consists of paths of morphisms with n compositions:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$

- The face map $d_i : C_n \to C_{n-1}$ sends the above element to

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_i \xrightarrow{\text{id}_{c_i}} c_i \longrightarrow \cdots \longrightarrow c_n$$

- The degeneracy map $s^i : C_n \to C_{n+1}$ sends the above element to

Theorem 2.1.3

Let $\mathcal{C}$ be a category. Every inner horn of $N(C)$ admits a filler and hence is an infinity category.

Definition 2.1.4: Nerve Functor

The nerve functor $N : \text{Cat} \to \text{sSet}$ is defined as follows.

- Each $\mathcal{C} \in \text{Cat}$ is sent to the nerve $N(C)$
- Every functor $\mathcal{C} \to \mathcal{D}$ in Cat is sent to the morphism of presheaves $\lambda : N(C) \to N(D)$ defined by $\lambda_n : N(C)([n]) \to N(D)([n])$, of which is defined as the map

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$

$$F(c_0) \xrightarrow{F(f_1)} F(c_1) \xrightarrow{F(f_2)} F(c_2) \longrightarrow \cdots \longrightarrow F(c_n)$$

from the upper path in $\mathcal{C}$ to the lower path in $\mathcal{D}$, such that the following diagram commutes:

$$\begin{tikzcd} N(C)[n] \arrow[r, "N(C)(f)"] \arrow[d, "\lambda_n"'] & N(C)[m] \arrow[d, "\lambda_m"] \\ N(D)[n] \arrow[r, "N(D)(f)"'] & N(D)[m] \end{tikzcd}$$

where $f : [m] \to [n]$ is a morphism in Δ.

Theorem 2.1.5

The nerve functor $N : \text{Cat} \to \text{sSet}$ is fully faithful. Moreover, the nerve of a category is a complete invariant for categories.

2.2 Homotopy Infinity Categories

Definition 2.2.1: The Homotopy Functor

Define the homotopy functor $h : \text{sSet} \to \text{Cat}$ as follows.

- On objects, h sends a simplicial set $S : \Delta \to \text{Set}$ to

Proposition 2.2.2

The homotopy functor $h : \text{sSet} \to \text{Cat}$ preserves colimits.

Theorem 2.2.3

The homotopy functor $h : \text{sSet} \to \text{Cat}$ is left adjoint to the nerve functor $N : \text{Cat} \to \text{sSet}$. This means that there is a natural bijection

$$\text{Hom}_{\text{Cat}}(h(C), D) \cong \text{Hom}_{\text{sSet}}(C, N(D))$$

Definition 2.2.4: Homotopic Morphisms

Let C be an infinity category. Two morphisms $f, g : C \to D$ are said to be homotopic if there exists a 2-simplex σ such that

- $d_0(\sigma) = \text{id}_D$
- $d_1(\sigma) = g$
- $d_2(\sigma) = f$

In this case we write $f \simeq g$.

Lemma 2.2.5

Homotopy is an equivalence relation in any infinity category.

Proposition 2.2.6

Let C be an infinity category. Let $f, f' : C \to D$ and $g, g' : D \to E$ be morphisms in C. If $f \simeq f'$ and $g \simeq g'$, then

$$g \circ f \simeq g' \circ f'$$

Definition 2.2.7: Homotopy Category

Let C be an infinity category. Define the homotopy category $h(C)$ of C to consist of the following.

- The objects are the objects of C
- The morphisms are equivalent classes of morphisms $[f]$ for f a morphism in C
- Composition is defined by

$$[g] \circ [f] = [g \circ f]$$

which is well defined by the above.

Definition 2.2.8: Isomorphisms in Infinity Categories

Let C be an infinity category. Let $f : C \to D$ be a morphism. We say that f is an isomorphism if there exists $g : D \to C$ such that $g \circ f \simeq \mathrm{id}_C$ and $f \circ g \simeq \mathrm{id}_D$.

Lemma 2.2.9

Let C be an infinity category. Let $f : C \to D$ be a morphism. Then f is an isomorphism in C if and only if $[f]$ is an isomorphism in $h(C)$.

3 Infinity Categories in Topology

3.1 The Singular Functor

The geometric realization functor actually has a right adjoint, called the singular functor.

Definition 3.1.1: Singular Functor

The singular functor $S : \text{Top} \to \text{sSet}$ is defined as follows.

- On objects, it sends a space X to the simplicial set $S(X) : \Delta \to \text{Set}$ called the singular set, defined by
$$S(X)[n] = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$$
- On morphisms, it sends a continuous map $f : X \to Y$ to the morphism of simplicial sets $\lambda : S(X) \to S(Y)$ defined as follows. For each $n \in \mathbb{N}$, $\lambda_n : S(X)[n] \to S(Y)[n]$ is defined by
$$(h : |\Delta^n| \to X) \mapsto (f \circ h : |\Delta^n| \to Y)$$
such that the following diagram commutes:
$$\begin{tikzcd} S(X)[n] \arrow[r, "S(X)(f)"] \arrow[d, "\lambda_n"'] & S(X)[m] \arrow[d, "\lambda_m"] \\ S(Y)[n] \arrow[r, "S(Y)(f)"'] & S(Y)[m] \end{tikzcd}$$

Notice that this is reminiscent of the definitions in Algebraic Topology 2. Indeed $S(X)[n]$ for each $n \in \mathbb{N}$ is in fact the basis of the abelian group $C_n(X)$. It represents all the possible ways that an n-simplex could fit into X.

Theorem 3.1.2

The singular functor $S : \text{Top} \to \text{sSet}$ is right adjoint to the geometric realization functor $| \cdot | : \text{sSet} \to \text{Top}$. This means that there is a natural bijection
$$\text{Hom}_{\text{Top}}(|X|, Y) \cong \text{Hom}_{\text{sSet}}(X, S(Y))$$
for any space Y and any simplicial set X.

We can do even better. For any X, $S(X)$ is actually an infinity category.

Lemma 3.1.3

Let X be a space. Then $S(X)$ is an infinity category.

Proposition 3.1.4

Let X be a space. Then the homotopy category of the singular set of X is equal to $h(S(X)) = \prod_1(X)$ the fundamental groupoid of X.

3.2 Kan Complexes

Definition 3.2.1: Kan Complexes

A Kan complex is a simplicial set C such that each horn (inner and outer) admits a filler. In other words, for all $0 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Since infinity catregories require only inner horns to admit a filler, we have the following inclusion relation:

$$\text{Infinity Categories} \subset \text{Kan Complexes}$$

Proposition 3.2.2

Let X be a space. Then $S(X)$ is a Kan complex.

Theorem 3.2.3

Let $\mathcal{C}$ be a small category. Then the simplicial set $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

More: Kan complexes = infinity groupoids (quillen equivalence in model category), and we should think of spaces as Kan complexes / infinity groupoids from now on.