Homotopy Theory

Labix

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Abstract

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1 Homotopy Theory

1.1 The nth Homotopy Groups

Definition 1.1.1: Pairs of Space

Let X be a topological space. A pair of space is a pair (X, A) where $A \subseteq X$ is a subspace of X. A map of pairs $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

Definition 1.1.2: Homotopy between Maps of Pairs

Let $f, g : (X, A) \to (Y, B)$ be maps of pairs. A homotopy between f and g is a homotopy $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subseteq B$.

Definition 1.1.3: The nth Homotopy Groups

Let (X, x_0) be a pointed space. Define the nth homotopy group $\pi_n(X, x_0)$ to be

$$\pi_n(X, x_0) = \frac{\left\{f : (I^n, \partial I^n) \to (X, \{x_0\}) \;\middle|\; f \text{ is continuous}\right\}}{\simeq}$$

where we say that $f \simeq g$ if there exists a homotopy between f and g.

Definition 1.1.4: Concatenation

For $n \geq 1$, define a composition law on $\pi_n(X, x_0)$ for a pointed space (X, x_0) by the formula

$$(f \cdot g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $f, g \in \pi_n(X, x_0)$.

Theorem 1.1.5

Let (X, x_0) be a pointed space and $n \geq 1$. The operation $\cdot$ on $\pi_n(X, x_0)$ is well defined and endows it with the structure of a group.

Proposition 1.1.6

Let (X, x_0) be a pointed space. Then $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

1.2 Properties of Homotopy

Definition 1.2.1: Category of Pointed Spaces

The Category of Pointed spaces Top_* is defined where

- The objects are pointed topological spaces (X, x_0) for $x_0 \in X$.
- The morphisms are continuous maps $f : X \to Y$ such that $f(x_0) = y_0$ for two pointed spaces (X, x_0) and (Y, y_0).
- Composition is defined as the composition of continuous maps that preserve the base point.

Proposition 1.2.2: Functoriality

For each $n \geq 1$, $\pi_n(-) : \text{Top}_* \to \text{Grp}$ is a functor where

- On objects, it sends (X, x_0) to the nth homotopy group $\pi_n(X, x_0)$
- On morphisms, it sends $f : (X, x_0) \to (Y, y_0)$ to the induced map

  $$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

  defined as $[\varphi] \mapsto [f \circ \varphi]$

Proposition 1.2.3

Let $(X, x_0), (Y, y_0)$ be pointed spaces and $f, g : (X, x_0) \to (Y, y_0)$ be pointed maps. If f and g are homotopic, then the induced maps

$$\pi_n(f) = \pi_n(g) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then $\pi_n(f)$ is an isomorphism.

Theorem 1.2.4

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. If $u : I \to X$ is a path from x_0 to x_1, then u induces a map

$$u_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

satisfying the following functorial properties:

- $u_\#$ is a group homomorphism
- If $v : I \to X$ is a path from x_1 to x_2 and $u \cdot v$ is the concatenation of these paths, then

  $$(u \cdot v)_\# = u_\# \circ v_\#$$

- If c_{x_0} is the constant path from x_0 to x_0 then $(c_{x_0})_\#$ is the identity

Proposition 1.2.5

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. Let $u, v : I \to X$ be paths from x_0 to x_1. If u and v are homotopic relative to end points then the induced maps

$$u_\# = v_\# : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

are equal.

Corollary 1.2.6

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. If x_0 and x_1 are path connected, then

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

where the isomorphism depends on the choice of path from x_0 to x_1.

Proposition 1.2.7

Let (X, x_0) be a pointed space and $f \in \pi_n(X, x_0)$. Let $u : I \to X$ be a loop on x_0. Then u

induces a left action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ by the map

$$(u, f) \mapsto u_\#(f) = u \cdot f \cdot u^{-1}$$

In particular, for $n \geq 2$, $\pi_n(X, x_0)$ is a $\mathbb{Z}\pi_1(X, x_0)$-module.

1.3 Relative Homotopy Groups

Definition 1.3.1: Triplets of Spaces

Let X be a topological space. A pointed pair of space is a triple (X, A_1, A_2) where $A_2 \subseteq A_1 \subseteq X$ are subspaces of X. A map between triplets of spaces $f : (X, A_1, A_2) \to (Y, B_1, B_2)$ is a map $f : X \to Y$ such that $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$.

If $A_2 = \{x_0\}$ is a single point we say that (X, A, x_0) is a pointed pair of spaces.

Definition 1.3.2: Homotopy between Maps of Triplets

Let $f, g : (X, A_1, A_2) \to (Y, B_1, B_2)$ be maps triplets of spaces. A homotopy between f and g is a homotopy between $f : X \to Y$ and $g : X \to Y$, namely $H : X \times [0, 1] \to Y$ such that $H(A_1 \times [0, 1]) \subseteq B_1$ and $H(A_2 \times [0, 1]) \subseteq B_2$.

For a pointed pair of spaces (X, A, x_0), the inclusion $\iota : (A, x_0) \to (X, x_0)$ induces a map on homotopy

$$\pi_n(\iota) = \pi_n(A, x_0) \to \pi_n(X, x_0)$$

which is in general not injective. For $[\alpha] \in \pi_n(A, x_0)$ to lie in the kernel, it must satisfy that for any map $f : (I, \partial I^n) \to (A, x_0)$ representing $[\alpha]$, $\iota \circ f$ is homotopic to the constant map c_{x_0} on x_0. Such a homotopy is a map $H : I^n \times I \to X$ satisfying the following conditions:

- $H(-, 1) = f$
- $H(-, 0) = c_{x_0}$
- $H|_{\partial I^n \times I} = c_{x_0}$

Thus if we denote

$$J^n = I^n \times \{0\} \cup \partial I^n \times I$$

which is a subspace of the boundary ∂I^{n+1}, such a homotopy H is a map of triplets of spaces

$$H : (I^{n+1}, \partial I^n, J^n) \to (X, A, x_0)$$

Definition 1.3.3: The nth Relative Homotopy Groups

Let (X, A, x_0) be a pointed pair of space. Define the relative homotopy groups of the triple by

$$\pi_n(X, A, x_0) = \frac{\left\{ f : \left(I^n, \partial I^n, J^{n-1}\right) \to (X, A, \{x_0\}) \;\middle|\; f \text{ is continuous} \right\}}{\simeq}$$

for $n \geq 2$, where $J^n = I^n \times \{0\} \cup \partial I^n \times I$ and we say that $f \simeq g$ if there exists a homotopy between f and g.

Theorem 1.3.4

Let (X, A, x_0) be a pointed pair of space. The composition law on definition 1.1.4 defines a group structure on $\pi_n(X, A, x_0)$ for $n \geq 2$.

Corollary 1.3.5

Let (X, A, x_0) be a pointed pair of space. For $n \geq 3$, $\pi_n(X, A, x_0)$ is abelian.

1.4 Induced Maps of Relative Homotopy Groups

Theorem 1.4.1

Let (X, A, x_0) and (Y, B, y_0) be pointed pairs of spaces and $f : (X, A, x_0) \to (Y, B, y_0)$ a map. Then f induces a map on the relative homotopy groups

$$f_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

for $n \geq 2$ satisfying the following functorial properties:

- f_* is a group homomorphism
- If $g : (Y, B, y_0) \to (Z, C, z_0)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*$$

- If $\text{id}_{(X,A,x_0)}$ is the identity map on (X, A, x_0), then

$$(\text{id}_{(X,A,x_0)})_* = \text{id}_{\pi_n(X,A,x_0)}$$

Proposition 1.4.2

Let $(X, A, x_0), (Y, B, y_0)$ be pointed pairs of spaces and $f, g : (X, A, x_0) \to (Y, B, y_0)$ be pointed maps. If f and g are homotopic, then the induced maps

$$f_* = g_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then f_* is an isomorphism.

Proposition 1.4.3

The relative homotopy groups of (X, A, x_0) fit into a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial_{n+1}} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, x_0) \longrightarrow 0$$

where ∂_n is defined by $[f] \mapsto [f|_{I^{n-1}}]$ and i_* and j_* are induced by inclusions.

Note that even though at the end of the sequence group structures are not defined, exactness still makes sense: kernels in this case consists of elements that map to the homotopy class of the constant map.

Theorem 1.4.4: The Hurewicz Homomorphism

Let (X, A, x_0) be a pointed pair of space. Let u_n be a generator of $H_n(S^n) \cong \mathbb{Z}$. Then the map

$$h : \pi_n(X, A, x_0) \to H_n(X, A)$$

defined by $[f] \mapsto f_*(u_n)$ is a group homomorphism.

1.5 n-Connectedness

Definition 1.5.1: n-Connected Space

We say that the pair (X, A) is n-connected if $\pi_i(X, A) = 0$ for $i \leq n$ and X is n-connected if $\pi_i(X) = 0$ for $i \leq n$.

1.6 Weakly Contractible Space

Definition 1.6.1: Weakly Contractible

Let X be a space. We say that X is weakly contractible if

$$\pi_n(X) = 0$$

for all $n \geq 0$.

2 Homotopy and CW-Complexes

2.1 The Homotopy Extension Property and Compression Lemma

Definition 2.1.1: Homotopy Extension Property

Let (X, A) be a pair of space. Let $F_0 : X \to Y$ a map and a homotopy $H : A \times I \to Y$ such that $H(-, 0) = F_0|_A$. We say that (X, A) satisfies the homotopy extension property (HEP) if there is a homotopy $F : X \times I \to Y$ extending H and F_0.

Proposition 2.1.2

Any CW pair has the homotopy extension property.

2.2 Whitehead's Theorem

Definition 2.2.1: Weak Homotopy Equivalence

We say that a map $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups π_n on any choice of base point.

Theorem 2.2.2: Whitehead's Theorem

If X and Y are CW-complexes and $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

Corollary 2.2.3

If X and Y are CW-complexes with $\pi_1(X) = \pi_1(Y) = 0$ and $f : X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.

2.3 Cellular Approximations

Definition 2.3.1: Cellular Maps

Let X and Y be CW-complexes. A map $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n$ for all n, where X_n is the n-skeleton of X.

Definition 2.3.2: Cellular Approximations

Let X and Y be CW-complexes. We say that $f : X \to Y$ has a cellular approximations if f is homotopic to a cellular map $f' : X \to Y$.

Theorem 2.3.3: Cellular Approximation Theorem

Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subseteq X$, then we can take $f'|_A = f|_A$.

Theorem 2.3.4: Relative Cellular Approximation

Any map $f : (X, A) \to (Y, B)$ between pairs of CW-complexes has a cellular approximation.

Corollary 2.3.5

Let $A \subset X$ be CW-complexes and suppose that all cells $X \setminus A$ have dimension larger than n. Then $\pi_i(X, A) = 0$ for all $i \leq n$.

Corollary 2.3.6

If X is a CW-complex, then $\pi_i(X, X_n) = 0$ for all $i \leq n$.

Corollary 2.3.7

Let X be a CW-complex. Then

$$\pi(X) \cong \pi(X_n)$$

for $i < n$.

2.4 CW Approximations

Definition 2.4.1: CW Approximation

A CW approximation of X is a weak homotopy equivalence $f : Z \to X$ where Z is a CW approximation.

Definition 2.4.2: CW Model

Let (X, A) be a non-empty pair of CW-complexes. An n-connected CW model of (X, A) is an n-connected CW pair (Z, A) together with a map $f : Z \to X$ with $f|_A = \mathrm{id}_A$ such that

$$f_* : \pi_i(Z) \to \pi_i(X)$$

is an isomorphism for $i > n$ and an injection for $i = n$ for any choice of base point.

Theorem 2.4.3

For any non-empty pair (X, A) of CW-complexes, there exists an n-connected model (Z, A). Moreover, Z can be built from A by attaching cells of dimension greater than n.

Corollary 2.4.4

Every pair of CW-complex (X, A) has a CW approximation (Z, B).

Thus we have shown existence of CW approximations, it remains to show uniqueness.

Corollary 2.4.5

CW-approximations are unique up to homotopy equivalence.

3 Applications of Approximations to Homotopy

3.1 Excision for Homotopy Groups

Theorem 3.1.1

Let X be a CW-complex decomposed as the union of subcomplexes A and B with non-empty connected intersection $C = A \cap B$. If (A, C) is m-connected and (B, C) is n-connected for $m, n \geq 0$, then the map

$$\iota_* \pi_i(A, C) \to (X, B)$$

induced by the inclusion $\iota : (A, C) \to (X, B)$ is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.

Corollary 3.1.2: Freudenthal Suspension Theorem

Let X be a $n - 1$-connected CW-complex. The suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n - 1$ and a surjection for $i = 2n - 1$.

Corollary 3.1.3

We have that

$$\pi_n(S^n) \cong \mathbb{Z}$$

for all $n \geq 1$. Moreover, it is generated by the identity map.

3.2 Hurewicz's Theorem

Theorem 3.2.1: Hurewicz's Theorem

Let X be a $(n - 1)$-connected space and $n \geq 2$. Then $\widetilde{H}_i(X) = 0$ for all $i < n$ and $\pi_n(X) \cong H_n(X)$.

Moreover, if a pair (X, A) is $(n - 1)$-connected with $n \geq 2$ and $\pi_1(A) = 0$, then $H_i(X, A) = 0$ for all $i < n$ and $\pi_n(X, A) \cong H_n(X, A)$

4 Spectral Sequences

4.1 Spectral Sequences

Definition 4.1.1: Bigraded Abelian Groups

A bigraded abelian group $A_{\bullet,\bullet}$ is an abelian group A together with a decomposition

$$A = \bigoplus_{p,q\in\mathbb{Z}} A_{p,q}$$

A degree (a,b) map $f : A_{\bullet,\bullet} \to B_{\bullet,\bullet}$ of bigraded abelian groups is a homomorphism $f : A \to B$ such that

$$f(A_{p,q}) \subseteq B_{p+a,q+b}$$

Definition 4.1.2: Spectral Sequences

A homological spectral sequence is a sequence

$$E^1_{\bullet,\bullet}, E^2_{\bullet,\bullet}, E^3_{\bullet,\bullet}, \dots$$

of bigraded abelian groups, each called pages, together with maps

$$d^r : E^r_{\bullet,\bullet} \to E^r_{\bullet,\bullet}$$

of degree $(-r, r-1)$ such that $d^r \circ d^r = 0$ and $E^{r+1}_{\bullet,\bullet} = H_\bullet(E^r_{\bullet,\bullet}, d^r)$. This means that

$$E^{r+1}_{p,q} = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E_{p+r,q-r+1} \to E^r_{p,q})}$$

Definition 4.1.3: Exact Couple

An exact couple of type r consists of bigraded abelian groups $E_{\bullet,\bullet}$ and $A_{\bullet,\bullet}$ and maps $i : A_{\bullet,\bullet} \to A_{\bullet,\bullet}$ of degree $(1,-1)$, $j : A_{\bullet,\bullet} \to E_{\bullet,\bullet}$ of degree $(-r,r)$ and $k : E_{\bullet,\bullet} \to A_{\bullet,\bullet}$ of degree $(-1,0)$ such that the triangle

$$\begin{tikzcd}
A_{\bullet,\bullet} \arrow[rr, "i"] & & A_{\bullet,\bullet} \arrow[dl, "j"] \\
& E_{\bullet,\bullet} \arrow[ul, "k"] &
\end{tikzcd}$$

is exact at each vertex ($\operatorname{im}(i) = \ker(j)$ and so on).

4.2 Serre Spectral Sequences