Category Theory 2

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Abstract

Contents

1 Categorical Algebra

1.1 Monoid Objects

Definition 1.1.1: Monoid Objects

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. We say that $M \in \mathcal{C}$ is a monoid object if there exists
- A morphism called multiplication $\mu : M \otimes M \to M$
- A morphism called the unit $\eta : I \to M$

such that the following coherence conditions hold.
- Associativity:

$$
\begin{array}{ccc}
(M \otimes M) \otimes M & \xrightarrow{\mu \otimes \mathrm{id}_M} & M \otimes M \\
\cong \downarrow & & \downarrow \mu \\
 & & M \\
 & & \uparrow \mu \\
M \otimes (M \otimes M) & \xrightarrow{\mathrm{id}_M \otimes \mu} & M \otimes M
\end{array}
$$

- Identity:

$$
\begin{array}{ccccc}
I \otimes M & \xrightarrow{\eta \otimes \mathrm{id}_M} & M \otimes M & \xleftarrow{\mathrm{id}_M \otimes \mu} & M \otimes I \\
 & \cong \searrow & \downarrow \mu & \swarrow \cong & \\
 & & M & &
\end{array}
$$

Proposition 1.1.2

A monoid object in $(\mathbf{Set}, \times, *)$ is a monoid.

There is an interesting categorization that can be seen here from the traditional algebraic structures. Recall that a monoid is a set M together with a binary operation $+ : M \times M \to M$ such that addition is associativity, and that there exists an identity element e for which for any $m \in M$, we have $e + m = m = m + e$. Reorganizing the data, we can write this as the following: A monoid is a set together with

- A multiplication map $+ : M \times M \to M$

- An unit map $\eta : * \to M$

such that the following conditions are satisfied:

- The multiplication map is associative

- The unit map is such that multiplication with the unit returns itself.

This is starting point of formalizing monoid objects in category theory. An algebraic object in general consists of a set, some binary operations together with some coherence conditions for the binary operations. The coherence conditions are in particular provided by the associator and unitor of the underlying symmetric monoidal structure. One can in particular think of the unit map $\eta : * \to M$ as picking out a point in M to be the identity in M.

It is important to make clear what are the data of an object and what are the conditions the data are subject to. In our case, the data consists a set, a binary operation and an identity element. These data are subject to conditions called associativity and unity which is dependent on the symmetric monoidal structure.
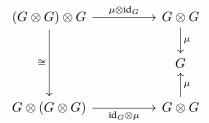
1.2 Group Objects

Definition 1.2.1: Group Objects

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. We say that $G \in \mathcal{C}$ is a group object if there exists
- A morphism called multiplication $\mu : G \otimes G \to G$
- A morphism called the unit $\eta : I \to M$
- A morphism called the inverse $\text{inv} : G \to G$

such that the following coherence conditions hold.
- Associativity:

$$
\begin{array}{ccc}
(G \otimes G) \otimes G & \xrightarrow{\mu \otimes \text{id}_G} & G \otimes G \\
\cong \downarrow & & \downarrow \mu \\
& & G \\
& & \uparrow \mu \\
G \otimes (G \otimes G) & \xrightarrow{\text{id}_G \otimes \mu} & G \otimes G
\end{array}
$$

- Identity:

$$
\begin{array}{ccccc}
I \otimes G & \xrightarrow{\eta \otimes \text{id}_G} & G \otimes G & \xleftarrow{\text{id}_G \otimes \mu} & G \otimes I \\
& \cong \searrow & \downarrow \mu & \swarrow \cong & \\
& & M & &
\end{array}
$$

- Inverse:

$$
\begin{array}{ccc}
G & \xrightarrow{(\text{id}_G, \text{inv})} & G \otimes G \\
(\text{inv}, \text{id}_G) \downarrow & \text{id}_G \searrow & \downarrow \mu \\
G \otimes G & \xrightarrow{\mu} & G
\end{array}
$$

Lemma 1.2.2

Let $\mathcal{C}$ be a symmetric monoidal category. Then every group object in $\mathcal{C}$ is also a monoid object.

Theorem 1.2.3

The following are group objects in particular categories.
- The group objects in $(\mathbf{Set}, \times, *)$ are groups
- The group objects in $(\mathbf{Grp}, \times, *)$ are abelian groups
- The group objects in $(\mathbf{Ab}, \oplus, *)$ are abelian groups
- The group objects in $(\mathbf{Top}, \times, *)$ are topological groups

1.3 Ring Objects

1.4 Algebra Objects

2   Enriched Categories

2.1   Basic Definitions

Definition 2.1.1: Enriched Categories

Let $(\mathcal{S}, \times, I, \alpha, \lambda, \rho)$ be a monoidal category. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is enriched over $\mathcal{S}$ if the following are true.
- For each $C, D \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(C, D) \in \mathcal{S}$
- For each $C \in \mathcal{C}$, there is a morphism $u_C : I \to \mathrm{Hom}_{\mathcal{C}}(C, C)$ in $\mathcal{S}$
- For each $C, D, E \in \mathcal{C}$, there is a morphism in $\mathcal{S}$ given by

$$\circ : \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(C, E)$$

such that the following diagram commutes:
- Associativity:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(E, F) \times (\mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D)) & \xrightarrow{1 \times \circ} & \mathrm{Hom}_{\mathcal{C}}(E, F) \times \mathrm{Hom}_{\mathcal{C}}(C, E) \\
\uparrow{\alpha} & & \downarrow{\circ} \\
& & \mathrm{Hom}_{\mathcal{C}}(C, F) \\
& & \uparrow{\circ} \\
(\mathrm{Hom}_{\mathcal{C}}(E, F) \times \mathrm{Hom}_{\mathcal{C}}(D, E)) \times \mathrm{Hom}_{\mathcal{C}}(C, D) & \xrightarrow{\circ \times 1} & \mathrm{Hom}_{\mathcal{C}}(D, F) \times \mathrm{Hom}_{\mathcal{C}}(C, D)
\end{array}
$$

- Identity:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(C, D) \times I & \xrightarrow{\mathrm{id} \times u} & \mathrm{Hom}_{\mathcal{C}}(C, D) \times \mathrm{Hom}_{\mathcal{C}}(C, C) \\
& {\cong}\searrow & \downarrow{\circ} \\
& & \mathrm{Hom}_{\mathcal{C}}(C, D)
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(D, D) \times \mathrm{Hom}_{\mathcal{C}}(C, D) & \xleftarrow{u \times \mathrm{id}} & I \times \mathrm{Hom}_{\mathcal{C}}(C, D) \\
{\circ}\downarrow & & \swarrow{\cong} \\
\mathrm{Hom}_{\mathcal{C}}(C, D)
\end{array}
$$

Proposition 2.1.2

Let $\mathcal{S}$ be a symmetric monoidal closed category. Then $\mathcal{S}$ is enriched over itself.

2.2   Enriched Functors and Natural Transformations

Definition 2.2.1

Let $(\mathcal{S}, \otimes, I)$ be a monoidal category. Let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched over $\mathcal{S}$. An $\mathcal{S}$-functor $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.
- $F : \mathcal{C} \to \mathcal{D}$ is a functor.
- The assignment $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$ is a morphism in $\mathcal{S}$.

Moreover, the following diagrams are commutative:
- Composition axiom:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(B, C) \otimes \mathrm{Hom}_{\mathcal{C}}(A, B) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{C}}(A, C) \\
{F \otimes F}\downarrow & & \downarrow{F} \\
\mathrm{Hom}_{\mathcal{C}}(F(B), F(C)) \otimes \mathrm{Hom}_{\mathcal{C}}(F(A), F(B)) & \xrightarrow{\circ} & \mathrm{Hom}_{\mathcal{C}}(F(A), F(C))
\end{array}
$$

- Unit axiom:

$$I \xrightarrow{\quad u \quad} \operatorname{Hom}_{\mathcal{C}}(A, A)$$

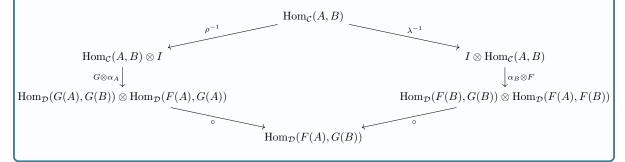with arrow u diagonal and arrow F down to $\operatorname{Hom}_{\mathcal{C}}(F(A), F(A))$

Definition 2.2.2: Enriched Natural Transformations

Let $(\mathcal{S}, \otimes, I)$ be a monoidal category. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{S}$-categories. Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be two $\mathcal{S}$-functors. An $\mathcal{S}$-natural transformation $\alpha : F \Rightarrow G$ consists of the following data. For each $C \in \mathcal{C}$, a morphism

$$\alpha_C : I \to \operatorname{Hom}_{\mathcal{D}}(F(C), G(C))$$

such that the following diagram commutes in $\mathcal{S}$:

$$\operatorname{Hom}_{\mathcal{C}}(A, B)$$

with ρ^{-1} to $\operatorname{Hom}_{\mathcal{C}}(A, B) \otimes I$ and λ^{-1} to $I \otimes \operatorname{Hom}_{\mathcal{C}}(A, B)$

$$\operatorname{Hom}_{\mathcal{C}}(A, B) \otimes I \qquad\qquad I \otimes \operatorname{Hom}_{\mathcal{C}}(A, B)$$

with $G \otimes \alpha_A$ down and $\alpha_B \otimes F$ down

$$\operatorname{Hom}_{\mathcal{D}}(G(A), G(B)) \otimes \operatorname{Hom}_{\mathcal{D}}(F(A), G(A)) \qquad \operatorname{Hom}_{\mathcal{D}}(F(B), G(B)) \otimes \operatorname{Hom}_{\mathcal{D}}(F(A), F(B))$$

with $\circ$ arrows to

$$\operatorname{Hom}_{\mathcal{D}}(F(A), G(B))$$

Definition 2.2.3: Equivalence of Enriched Categories

Let $\mathcal{S}$ be a monoidal category. Let $\mathcal{C}, \mathcal{D}$ be $\mathcal{S}$-categories. We say that $\mathcal{C}$ and $\mathcal{D}$ are $\mathcal{S}$-equivalent if there exists $\mathcal{S}$-functors $F : \mathcal{C} \to mD$ and $G : \mathcal{D} \to \mathcal{C}$ such that

$$\lambda : 1 \overset{\cong}{\Rightarrow} F \circ G \quad \text{and} \quad \eta : G \circ F \overset{\cong}{\Rightarrow} 1$$

are $\mathcal{S}$-isomorphisms.

TBA: Fully faithful essentially surjective.

Fully faithful implies isomorphism of Hom sets in $\mathcal{S}$.