Topological Manifolds

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Abstract

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1 Topological Manifolds and Singular Homology

1.1 Orientability

Recall the notion of orientation in finite dimensional vector bases. We say that two bases of a vector space have the same orientation if the change of basis matrix has determinant greater than 0. Since topological manifolds locally look like finite-dimensional vector spaces, we expect that orientations can be generalized to manifolds.

The key observation in defining orientation through homology is the following proposition.

Proposition 1.1.1

Let M be a k-dimensional topological manifold and $x \in M$ a point. Then

$$H_n(M, M \setminus \{x\}) \cong H_n(\mathbb{R}^k, \mathbb{R}^k \setminus \{*\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Definition 1.1.2: Local Orientation

A local orientation of M at x is a choice of generator of $H_k(M, M \setminus \{x\})$.

Let U be a chart on a topological manifold M and that $B \subseteq M$ is such that on the chart U, B is an open / closed ball $B_r(z)$. For convention, we give a name to subsets of these type.

Definition 1.1.3: Open and Closed Ball in Manifolds

Let M be a k-dimensional topological manifold and U a chart of M. We say that B is an open / closed ball if under the homeomorphism of the chart $U \cong \mathbb{R}^k$, the image of B is a ball $B_r(x) \subseteq \mathbb{R}^k$ for some $r \in \mathbb{R}^+$ and $x \in \mathbb{R}^k$.

Notice that the inclusion $(M, M \setminus B) \hookrightarrow (M, M \setminus \{y\})$ induces a map in homology:

$$H_k(M, M \setminus B) \overset{\cong}{\to} H_k(M, M \setminus \{y\})$$

It is an isomorphism since B is homeomorphic to a ball in $\mathbb{R}^k$ which is contractible. This leads to the following definition.

Definition 1.1.4: Consistent Local Orientations

Let $(\omega_y)_{y\in B}$ be a family of local orientations. We say that it is consistent if there is a generator $\omega_B \in H_k(M, M \setminus B)$ such that $\omega_B \mapsto \omega_y$ for each $y \in B$ under the isomorphism

$$H_k(M, M \setminus B) \cong H_k(M, M \setminus \{y\})$$

With this, we can now formally define orientations in a manifold.

Definition 1.1.5: Orientation of a Manifold

Let M be a k-dimensional topological manifold. An orientation of M is a function $x \mapsto \omega_x \in H_k(M, M \setminus \{x\})$ assigning every point to a local orientation such that for every $x \in M$, there exists $x \in B$ a subset of a chart U for B homeomorphic to an open / closed ball in $\mathbb{R}^k$, for $(\omega_x)_{x\in B}$ a consistent local orientation.

Since $H_k(M, M \setminus \{x\})$ is isomorphic to $\mathbb{Z}$, this means that there are only two possible choices of distinct orientation classes for each point $x \in M$.

Definition 1.1.6: Orientation Bundle

Let M be a topological manifold. Define the orientation bundle $\widetilde{M}$ to be the set of pairs

$$\widetilde{M} = \{(x, \omega_x) | \, x \in M, \omega_x \in H_k(M, M \setminus \{x\})\}$$

together the projection map $\pi : \widetilde{M} \to M$ defined by $\pi(x, \omega_x) = x$ and with the topology defined as follows.

Let B be an open ball in M. Since there are exactly two distinct orientation classes on B, $\pi^{-1} = B_+ \amalg B_-$. Define the topology of $\widetilde{M}$ to be generated by sets of the form B_+ and B_-.

Lemma 1.1.7

For any topological manifold M, $\widetilde{M}$ is a manifold. Moreover, it is orientable with a canonical orientation.

Lemma 1.1.8

Giving an orientation of M is equivalent to giving a continuous map $s : M \to \widetilde{M}$ such that $s \circ \pi = \text{id}$ (section of the orientation bundle).

Proof. Let $s : M \to \widetilde{M}$ be continuous and that $s \circ \pi = \text{id}$. Then s assigns a orientation ω_x to each $x \in M$. The map is continuous if and only if for each open ball in M and $\pi^{-1}(B) = B_+ \amalg B_-$, the preimages $s^{-1}(B_+)$ and $s^{-1}(B_-)$ are both open in B. Since these two preimages are disjoint and jointly cover B, this condition is equivalent $s(B) = B_+$ or $s(B) = B_-$. This precisely means that the local orientations are consistent. □

Corollary 1.1.9

Let M be a connected topological manifold. Then exactly one of the following holds:

- $\widetilde{M} \to M$ is a non-trivial 2-sheeted cover and M is non-orientable
- $\widetilde{M} \cong M \amalg M$ and M admits precisely two orientations

Corollary 1.1.10

Any simply connected manifold is orientable.

1.2 Fundamental Class

Proposition 1.2.1

Let M be a connected compact smooth manifold of dimension n. If M is orientable then $H_n(M) \cong \mathbb{Z}$. Otherwise $H_n(M) = 0$.

Definition 1.2.2: Fundamental Class

Let M be a connected compact orientable smooth manifold of dimension n. A fundamental class for M is a generator for the top homology

$$H_n(M) \cong \mathbb{Z}$$

Recall that S^k and $\partial\Delta^{k+1}$ are homeomorphic.

Proposition 1.2.3

The cycle $\partial\Delta^{k+1} \in C_k(\partial\Delta^{k+1})$ represents a generator in for the top homology of S^k.

Corollary 1.2.4

Let S^k_+ and S^k_- be the northern and southern hemisphere of S^k respectively. Choose homomorphisms

$$\sigma_+ : \Delta^k \xrightarrow{\cong} S^k_+ \quad \text{and} \quad \sigma_- : \Delta^k \xrightarrow{\cong} S^k_-$$

such that both σ_+, σ_- map the boundary $\partial\Delta^k$ homeomorphically onto the equator $S^k_+ \cap S^k_-$ and the composition

$$\partial\Delta^k \xrightarrow{\sigma_+} S^k_+ \cap S^k_- \xrightarrow{(\sigma_-)^{-1}} S^k_-$$

is the identity. Then the cycle $\sigma_+ - \sigma_- \in C_k(S^k)$ represents a fundamental class for S^k.

2 The Theory of Surfaces

2.1 The Homology of Surfaces

Recall that a compact surface is a connected topological manifold of dimension 2 that is compact. Moreover, every compact surface is homeomorphic to either $\Sigma_g = \mathbb{T}\# \cdots \#\mathbb{T}$ for $g \geq 0$ or $N_h = \mathbb{RP}^2\# \cdots \#\mathbb{RP}^2$ for $h \geq 1$. We can now compute the homology groups of these surfaces and moreover, show that $\sum_g$ is orientable while N_h is not.

Proposition 2.1.1

Let $g \geq 0$. The homology of the g-holed torus Σ_g is given by

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 2.1.2

The surfaces Σ_g for $g \geq 0$ is orientable.

Proposition 2.1.3

Let $h \geq 1$. The homology of N_h is given by

$$H_n(N_h) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 2.1.4

The surfaces N_h for $h \geq 1$ is non-orientable.

2.2 The Euler Characteristic

3 Homology and Cohomology on Manifolds

3.1 de Rham Cohomology

Proposition 3.1.1

Let M be a smooth manifold. Then differential forms of M, $\Omega^0(M), \dots, \Omega^n(M), \dots$ together with the exterior derivative $d : \Omega^n(M) \to \Omega^{n+1}(M)$ form a cochain complex.

Definition 3.1.2

Let M be a smooth manifold. Define the de Rham cohomology groups of M to be the cohomology of the chain of differential forms:

$$H^n_{\text{dR}}(M; \mathbb{R}) = H^n(\Omega^\bullet(M); \mathbb{R})$$

Proposition 3.1.3

Let M be a smooth manifold of dimension n. Then the following are true for the de Rham cohomology of M.

- $H^k_{\text{dR}}(M; \mathbb{R})$ is a vector space over $\mathbb{R}$ for all $k \in \mathbb{N}$.
- For $r > n$ we have $H^r_{\text{dR}}(M; \mathbb{R}) = 0$
- If M has m connected components then $H^0_{\text{dR}}(M; \mathbb{R}) = \mathbb{R}^k$

Theorem 3.1.4

Let M be a smooth manifold of dimension n. Then the direct sum

$$H^*(M) = \bigoplus_{k=1}^{n} H^k_{\text{dR}}(M; \mathbb{R})$$

is an $\mathbb{R}$-algebra where multiplication defined by $a \wedge b \in H^{s+l}_{\text{dR}}(M; \mathbb{R})$ for $a \in H^s_{\text{dR}}(M; \mathbb{R})$ and $b \in H^l_{\text{dR}}(M; \mathbb{R})$. Moreover, this multiplication is anti-commutative, namely for $a \in H^s_{\text{dR}}(M; \mathbb{R})$ and $b \in H^l_{\text{dR}}(M; \mathbb{R})$, we have

$$a \wedge b = (-1)^{sl} b \wedge a$$

Proposition 3.1.5

Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Then f induces an $\mathbb{R}$-linear map

$$f^* : H^*(N) \to H^*(M)$$

such that $f^*(a \wedge b) = f^*(a) \wedge f^*(b)$. Moreover, it is functorial:

- If $g : N \to K$ is another smooth map of manifolds, then $(g \circ f)^* = f^* \circ g^*$
- If $\text{id} : M \to M$ is the identity map on the manifold, then $\text{id}^* : H^*(M) \to H^*(M)$ is the trivial map on $\mathbb{R}$-algebras.

Theorem 3.1.6: Homotopy Invariance of de Rham Cohomology

Let $f : M \times I \to N$ be a smooth map of manifolds varying for each $t \in I = [0, 1]$. Write $f_t(x) = f(x, t)$. Then the pull back maps $f_0^*, f_1^* : H^*(N) \to H^*(M)$ are equal:

$$f_0^* = f_1^*$$

3.2 de Rham Cohomology of Common Manifolds

Proposition 3.2.1

The real space $\mathbb{R}^n$ has the de Rham cohomology

$$H^k_{\text{dR}}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.2.2

The n-sphere S^n has the de Rham cohomology

$$H^k_{\text{dR}}(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.2.3

Let $p, q \geq 1$, the sphere S^{p+q} is not diffeomorphic to any $M \times N$ manifolds where $\dim(M) = p$ and $\dim(N) = q$.

Proposition 3.2.4

Every smooth vector fields on S^{2n} vanishes at some point of the sphere.

Proposition 3.2.5

The real projective space $\mathbb{RP}^n$ has the de Rham cohomology

$$H^k_{\text{dR}}(\mathbb{RP}^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = n \text{ where } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

4 Poincare Duality

4.1 The Cap Product

Definition 4.1.1: The Cap Product

Let $\sigma = [v_0, \dots, v_k] \in C_k(X)$ and $\phi \in C^l(X)$ where $k \geq l$ with coefficients in a ring R. Define the cap product to be

$$\sigma \frown \phi = \phi(\sigma|_{[v_0,\dots,v_l]})\sigma|_{[v_l,\dots,v_k]} \in C_{k-l}(X)$$

Lemma 4.1.2

The cap product $\frown: C_k(X) \times C^l(X) \to C_{k-l}(X)$ with coefficients in a ring R induces a cap product in homology $\frown: H_k(X) \times H^l(X, R) \to H_{k-l}(X)$ for $k \geq l$.

4.2 Cohomology with Compact Support

4.3 The Duality Theorem

Theorem 4.3.1: Poincare Duality

Let M be a compact and oriented topological n-manifold. Then the homomorphism

$$D : H^p(M) \to H_{n-p}(M)$$

is an isomorphism.