# Measure Theory

Labix

June 28, 2022

**Abstract**

# Contents

# 1 Measure Theory

## 1.1 Sigma Fields

**Definition 1.1.1** (Sigma Fields)**.** A $\sigma$-field on a non-empty set $S$ is a collection $\mathcal{F}$ of subsets of $S$ such that

- $S, \emptyset \in \mathcal{F}$
- $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$
- $A_k \in \mathcal{F}$, $k \in \mathbb{N}$ implies $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

**Definition 1.1.2** (Measure)**.** A measure on a $\sigma$-field $\mathcal{F}$ of subsets of $S$ is a function $\mu : \mathcal{F} \to [0, +\infty]$ such that

- $\mu(\emptyset) =$
- $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu\left(A_k\right)$ where $A_k \in \mathcal{F}$ are pairwise disjoint.

**Proposition 1.1.3.** Let $\mu$ be a measure on a $\sigma$-field $\mathcal{F}$ and $A_1, A_2, \dots \in \mathcal{F}$

- If $A_1 \subseteq A_2$, then $\mu(A_1) \leq \mu(A_2)$
- $\mu(\bigcup_{k=1}^{\infty} A_k) \leq \sum_{k=1}^{\infty} \mu(A_k)$
- $\mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) + \mu(A_1 \cap A_2)$

# 2 Lebesgue Measure

## 2.1 Elementary Measure

**Definition 2.1.1** (Intervals)**.** An interval $I$ is a subset of $\mathbb{R}$ of the form

- $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
- $(a,b] = \{x \in \mathbb{R} | a < x \leq b\}$
- $[a,b) = \{x \in \mathbb{R} | a \leq x < b\}$
- $(a,b) = \{x \in \mathbb{R} | a < x < b\}$

Define the measure of an interval to be its length, $m(I) = b - a$

**Definition 2.1.2** (Boxes)**.** A box in $\mathbb{R}^n$ is a cartesian product

$$B = I_1 \times I_2 \times \cdots \times I_n$$

of $n$ intervals. Define the measure of a box to be its "volume",

$$m(B) = |B| = |I_1| \cdots |I_n|$$

**Definition 2.1.3** (Elementary Sets)**.** An elementary set is a finite union of boxes. Denote the set of all Elementary sets to be $\mathcal{M}_E$.

**Proposition 2.1.4.** Let $E, F \in \mathcal{M}_E$. $E \cap F$, $E \cup F$, $E/F \in \mathcal{M}_E$.

**Proposition 2.1.5.** Let $E$ be an elementary set of $\mathbb{R}^n$.

- $E$ can be partitioned into finite union of disjoint boxes.
- The measure of $E$ is independent of the partition.

**Proposition 2.1.6.** Let $E, F$ be an elementary set. Elementary measures have the following properties.

- $m(E) \geq 0$
- $m(E \cup F) = m(E) + m(F)$ if $E \cap F = \emptyset$
- $m(\emptyset) = 0$
- $E \subset F \implies m(E) \leq m(F)$
- $m(E \cup F) \leq m(E) + m(F)$

**Theorem 2.1.7.** The elementary measure function is unqiue up to a constant multiple.

## 2.2 Jordan Measure

**Definition 2.2.1** (Jordan Inner Measure)**.** Let $E \subset \mathbb{R}^n$ be a bounded set. Define the Jordan Inner Measure as

$$m_{J*}(E) = \sup_{A \subset E, A \in \mathcal{M}_E} m(A)$$

**Definition 2.2.2** (Jordan Outer Measure)**.** Let $E \subset \mathbb{R}^n$ be a bounded set. Define the Jordan Outer Measure as

$$m^{J*}(E) = \inf_{E \subset A, A \in \mathcal{M}_E} m(A)$$

**Definition 2.2.3** (Jordan Measurable)**.** We say that $E$ is Jordan mesurable if

$$m_{J*}(E) = m^{J*}(E)$$

Denote the set of all Jordan measurable sets to be $\mathcal{M}_J$.

**Lemma 2.2.4.** $\mathcal{M}_E \subseteq \mathcal{M}_J$ and $m(E)$ is consistent if $E \in \mathcal{M}_E$

**Proposition 2.2.5.** Let $E, F \in \mathcal{M}_J$.

- $E \cap F \in \mathcal{M}_J$
- $E \cup F \in \mathcal{M}_J$
- $E/F \in \mathcal{M}_J$

**Proposition 2.2.6.** A set $E \subset \mathbb{R}^n$ is Jordna measurable if and only if for every $\epsilon > 0$ there exists $A \subset E \subset B$ such that $m(B/A) < \epsilon$

**Proposition 2.2.7.** Let $E, F \in \mathcal{M}_J$. Elementary measures have the following properties.

- $m(E) \geq 0$
- $m(E \cup F) = m(E) + m(F)$ if $M \cap F = \emptyset$
- $m(\emptyset) = 0$
- $E \subset F \implies m(E) \leq m(F)$
- $m(E \cup F) \leq m(E) + m(F)$

**Theorem 2.2.8.** The Jordan measure function is unqiue up to a constant multiple.

**Proposition 2.2.9.** Let $E \subset \mathbb{R}^n$ and $F \subset \mathbb{R}^m$ be Jordan measurable. $E \times F$ is also Jordan Measurable and $m(E \times F) = m(E) \times m(F)$.

**Proposition 2.2.10.** $B_r(x) \subset \mathbb{R}^n$ and $\overline{B}_r(x) \subset \mathbb{R}^n$ are Jordan Measurable with Jordan Measure $c_n r^n$ for some $c_n > 0$ depending on $n$.

**Proposition 2.2.11.** Let $E \in \mathcal{M}_J$.

- $m^{J*}(E) = m^{J*}(\overline{E})$
- $m_{J*}(E) = m^{J*}(E^\circ)$
- $E \in \mathcal{M}_J$ if and only if $m^{J*}(\partial E) = 0$

**Proposition 2.2.12** (Jordan Measurable implies Riemann Integrable)**.** Let $E \in \mathcal{M}_J$ and $E \subset [a,b]$. Let

$$1_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

Then $1_E$ is Riemann Integrable and $\int_a^b 1_E(x)\,dx = m(E)$

**Proposition 2.2.13.** Let $f : [a,b] \to \mathbb{R}$ be bounded. $f$ is Riemann integrable if and only if $E_* = \{(x,t) | x \in [a,b], f(x) \leq t \leq 0\}$ and $E^* = \{(x,t) | x \in [a,b], 0 \leq t \leq f(x)\}$ are Jordan Measurable. In this case,

$$\int_a^b f(x)\,dx = m(E^*) - m(E_*)$$

**Lemma 2.2.14.** Let $E_n \in \mathcal{M}_J$ for all $n$. $\bigcup_{n=1}^{\infty} E_n$ and $\bigcap_{n=1}^{\infty} E_n$ may not be Jordan Measurable.

## 2.3 Lebesgue Outer Measure

**Definition 2.3.1** (Lebesgue Outer Measure)**.** Let $E \subset \mathbb{R}^n$ be a bounded set. Define the Lebesgue Outer Measure as

$$m^*(E) = \inf_{E \subset \bigcup_{n=1}^{\infty} A_n, A_n \in \mathcal{M}_E} \sum_{n=1}^{\infty} |A_n|$$

**Proposition 2.3.2.** $m^*(E) \leq m^{J*}(E)$

**Proposition 2.3.3.** Let $E, F \subset \mathbb{R}^n$.

- $m^*(\emptyset) = 0$
- $E \subset F$ implies $m^*(E) \leq m^*(F)$
- Let $E_n \subset \mathbb{R}^n$ for all $n$. Then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m^*(E_n)$$

**Lemma 2.3.4.** Let $E, F \subset \mathbb{R}^n$ such that $\text{dist}(E,F) > 0$. Then

$$m^*(E \cup F) = m^*(E) + m^*(F)$$

**Definition 2.3.5** (Almost Disjoint Sets)**.** Two boxes $E, F$ are almost disjoint if

$$E^\circ \cap F^\circ = \emptyset$$

**Lemma 2.3.6.** Let $E = \bigcup_{n=1}^{\infty} B_n$ be a countable union of almost disjoint boxes. Then

$$m^*(E) = \sum_{n=1}^{\infty} |B_n|$$

**Lemma 2.3.7.** If $E \subset \mathbb{R}^n$ is expressible as a countable union of almost disjoint boxes, then $m^*(E) = m_{J*}(E)$

**Lemma 2.3.8.** Open sets can be expressed as a countable union of almost disjoint boxes.

**Lemma 2.3.9.** Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then

$$m^*(E) = \inf_{E \subset U, U \text{ open}} m^*(U)$$

## 2.4 Lebesque Measure

**Definition 2.4.1** (Lebesgue Measurability)**.** A set $E \subset \mathbb{R}^n$ is Lebesgue mesurable if for every $\epsilon > 0$, there exists an open set $U \subset \mathbb{R}^n$ with $E \subset U$ such that $m^*(U/E) < \epsilon$. Denote the set of all Lebesgue measurable sets to be $\mathcal{M}_L$.

**Proposition 2.4.2.** There exists Lebesgue mesurable sets.

- Every open set is Lebesgue measurable
- Every closed set is Lebesgue measurable
- Every set of Lebesgue outer measure 0 is measurable
- $\emptyset \in \mathcal{M}_L$
- If $E \subset \mathbb{R}^n \in \mathcal{M}_L$ implies $\mathbb{R}^n/E \in \mathcal{M}_L$
- Let $E_n \in \mathcal{M}_L$ for all $n$ and $E_n \subset \mathbb{R}^n$. Then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{M}_L$ and $\bigcap_{n=1}^{\infty} E_n \in \mathcal{M}_L$

**Theorem 2.4.3** (Criteria for Measurability)**.** Let $E \subset \mathbb{R}^n$. The following are equivalent

- $E \in \mathcal{M}_L$
- For every $\epsilon > 0$ there exists an open set $U$ such that $E \subset U$ with $m^*(U/E) < \epsilon$
- For every $\epsilon > 0$ there exists an open set $U$ such that $m^*(U \bigtriangleup E) < \epsilon$
- For every $\epsilon > 0$, there exists a closed set $F$ such that $F \subset E$ with $m^*(E/F) < \epsilon$
- For every $\epsilon > 0$ there exists a closed set $F$ such that $m^*(F \bigtriangleup E) < \epsilon$
- For every $\epsilon > 0$ there exists $L \in \mathcal{M}_L$ such that $m^*(L \bigtriangleup E) < \epsilon$.

**Lemma 2.4.4.** $\mathcal{M}_J \subset \mathcal{M}_L$ and $m(E)$ is consistent if $E \in \mathcal{M}_J$.

**Proposition 2.4.5** (Measure Axioms)**.** Let $E, F \subset \mathbb{R}^n$.

- $m(\emptyset) = 0$
- Let $E_n \subset \mathbb{R}^n$ for all $n$. Then
$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)$$

**Theorem 2.4.6** (Monotone Convergence Theorem)**.** Let $E_n$ be measurable for all $n$.

- Suppose $E_1 \subset E_2 \subset \cdots \subset \mathbb{R}^d$. Then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$
- Suppose $\mathbb{R}^d \supset E_1 \supset E_2 \supset \dots$. Then $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$

**Definition 2.4.7** (Set Convergence)**.** We say that $\{E_n\}$ converges to $E$ if the indicator functionsd $1_{E_n}$ converges pointwise to $E$.

**Theorem 2.4.8.** Suppose that $\{E_n\} \subset \mathcal{M}_L$ and $\{E_n \to E\}$ pointwise. Then $E \in \mathcal{M}_L$.

**Theorem 2.4.9** (Dominated Convergence Theorem)**.** Suppose that $E_n$ are all contained in another Lebesgue measurable set $F$ of finite measure. Then

$$\lim_{n\to\infty} m(E_n) = m(E)$$

## 2.5 Lebesque Integral

**Definition 2.5.1** (Simple Functions)**.** A simple function $f : \mathbb{R}^d \to \mathbb{C}$ is a finite linear combination

$$f = c_1 1_{E_1} + \cdots + c_k 1_{E_k}$$

of indicator functions $1_{E_i}$ that are from Lesbegue measurable sets $E_i \subset \mathbb{R}^d$ for $i \in \{1, \ldots, k\}$, where $k \in \mathbb{N}$ and $c_1, \ldots, c_k \in \mathbb{C}$.

**Definition 2.5.2** (Unsigned Simple Functions)**.** A simple function is unsigned if $f : \mathbb{R}^d \to [0, +\infty]$ and $c_i$ is a function mapping to the positive reals.

**Definition 2.5.3.** Let $f$ be an unsigned simple function. Then define

$$S\left(\int_{R^d} f(x)\, dx\right) = \sum_{k=1}^{n} c_k m(E_k)$$