Functional Analysis

Labix

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Abstract

Contents

1 Banach Spaces

1.1 Normed Spaces

Recall from linear algebra that a normed space is a vector space equipped with a norm.

Definition 1.1.1: Isometrically Isomorphic Normed Spaces

Two normed spaces V and W are isometrically isomorphic if there is a surjective linear isometry $L : V \to W$. IN this case, we write $V \cong W$.

Proposition 1.1.2

Let $(X, \|\cdot\|_X)$ be a normed space. Let V be a vector space over the same field and $L : V \to X$ a linear isomorphism. Then the pullback norm

$$\|v\|_V = \|L(v)\|_X$$

defines a norm on V. In particular, $L : (V, \|\cdot\|_V) \to (X, \|\cdot\|_X)$ is a linear isometry.

Theorem 1.1.3

If V is a finite dimensional vector space then all norms on V are equivalent.

Recall that every normed space is a vector space by defining $d(x, y) = \|x - y\|$ for x, y in a normed space X. We thus have the notion of convergence in normed spaces.

1.2 Condition for Finite-Dimensional

Theorem 1.2.1

A compact set in a normed space is closed and bounded.

Recall in topology that compactness is preserved by continuity. This allows us to construct an argument proving the following.

Theorem 1.2.2

Let X be a finite dimensional normed space. Then a subset U of X is compact if and only if it is closed and bounded.

Lemma 1.2.3: Riesz's Lemma

Let $(X, \|\cdot\|)$ be a normed space and Y a non-empty closed subspace of X not equal to X. Then there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geq \frac{1}{2}$ for every $y \in Y$.

Proposition 1.2.4

Every finite dimensional subspace of a normed space is closed.

Theorem 1.2.5

A normed space X is finite dimensional if and only if its closed unit ball is compact.

1.3 Banach Spaces

Definition 1.3.1: Complete Spaces

A normed space $(X, \|\cdot\|)$ is complete if any Cauchy sequence in X converges to some $x \in X$.

Definition 1.3.2: Banach Spaces

A complete normed space is called a Banach space.

Lemma 1.3.3

Suppose that $(X, \|\cdot\|_X) \cong (Y, \|\cdot\|_Y)$ are isometrically isomorphic. Then X is complete if and only if Y is complete.

Lemma 1.3.4

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space X then $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

Proposition 1.3.5

Both $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete.

Corollary 1.3.6

Every finite dimensional normed space $(V, \|\cdot\|)$ is complete.

1.4 Separability of Banach Spaces

Recall the notion of separability: A space is separable if it has a countably dense subset.

Lemma 1.4.1

Let X be a normed space. Then the following are equivalent.

- X is separable
- The set $\{x \in X | \|x\| = 1\}$ is separable
- X contains a sequence $(x_n)_{n\in\mathbb{N}}$ whose linear span is dense.

2 Linear Maps Between Banach Spaces

2.1 Boundedness and Continuity

Definition 2.1.1: Bounded Linear Maps

A linear map A from a normed space $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ is bounded if there exists a constant M such that

$$\|Ax\|_Y \leq M\|x\|_X$$

for all $x \in X$.

Denote the space of all bounded linear operators by $B(X, Y)$.

Lemma 2.1.2

If X is a finite dimensional space then any linear map $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is bounded.

Lemma 2.1.3

A linear map $T : X \to Y$ is continuous if and only if it is bounded.

Theorem 2.1.4

Let X be a normed space and Y a Banach space. Then $B(X, Y)$ is Banach space.

2.2 Invertibility

Corollary 2.2.1

Let $T \in B(X, Y)$ for X, Y vector spaces. Then $\ker(T)$ is a closed linear subspace of X.

Definition 2.2.2: Bounded Invertible

A linear map $T \in B(X, Y)$ is bounded invertible if there exists $S \in B(X, Y)$ such that $S \circ T$ and $T \circ S$ are the identity.

Lemma 2.2.3

Suppose that X and Y are normed spaces. Then for any $T \in B(X, Y)$ the following are equivalent.

- T is bounded invertible
- T is a bijection and $T^{-1} \in B(X, Y)$
- T is surjective and for some $c > 0$, $\|T(x)\|_Y \geq c\|x\|_X$ for every $x \in X$.

Corollary 2.2.4

If X is finite dimensional then a linear operator $T : X \to X$ is invertible if and only if $\ker(T) = \{0\}$.

2.3 The Hahn Banach Theorem

Definition 2.3.1: Continuous Dual Space

Let X be a normed space. Denote the continuous dual space of X to be the the subspace

$$X' = B(X, K)$$

of the dual space X^*.

Notice that since continuity is the same as boundedness, this notation of $B(X, K)$ coincides with the set of all bounded linear operators.

Lemma 2.3.2

Let X be a Banach space. Then X' is also a Banach space.

Proof. Since we know that $B(X, Y)$ is a Banach space for any normed space X and Banach space Y, choose $Y = \mathbb{R}$ and we are done. $\square$

Theorem 2.3.3: The Real Hahn-Banach Theorem

Let X be a real vector space. Let p be a convex function on X. Let X_0 be a linear subspace of X and let f be a linear functional on Y satisfying

$$f(x) \leq p(x)$$

for all $x \in Y$. Then f can be extended to a linear functional F on all of X satisfying the condition

$$F(x) \leq p(x)$$

for all $x \in X$ and $F|_Y = f$.

3 Hilbert Spaces

3.1 Hilbert Spaces

Definition 3.1.1: Hilbert Spaces

A Hilbert space is a complete inner product space, where complete means complete using the norm induced by the inner product.

3.2 Orthogonality in Hilbert Spaces

Definition 3.2.1: Orthonormal Set

A subset E of a Hilbert space is orthonormal if $\|e\| =$ for all $e \in E$ and $\langle e_1, e_2\rangle = 0$ for all $e_1, e_2 \in E$ with $e_1 \neq e_2$.

Lemma 3.2.2: Bessel's Inequality

Let V be an inner product space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence. Then for any $x \in V$ we have

$$\sum_{k=1}^{\infty} |\langle x, e_n\rangle|^2 \leq \|x\|^2$$

Lemma 3.2.3

Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ an orthonormal sequence in H. Then the series $\sum_{k=1}^{\infty} a_n e_n$ converges if and only if

$$\sum_{k=1}^{\infty} |a_n|^2 < \infty$$

and

$$\left\|\sum_{k=1}^{\infty} a_n e_n\right\|^2 = \sum_{k=1}^{\infty} |a_n|^2$$

Corollary 3.2.4

Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ an orthonormal sequence in H. Then for any x the sequence

$$\sum_{k=1}^{\infty} \langle x, e_n\rangle e_n$$

converges.

Proposition 3.2.5

Let $E = (e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. Then the following are equivalent.

- E is a basis for H
- For any $x \in H$ we have $x = \sum_{k=1}^{\infty} \langle x, e_n\rangle e_n$
- $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_n\rangle|^2$ for all $x \in H$
- $\langle x, e_n\rangle = 0$ for all n implies $x = 0$
- $\overline{\operatorname{span}(E)} = H$

3.3 Separability of Hilbert Spaces

Proposition 3.3.1

An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.

Theorem 3.3.2

Any infinite dimensional separable Hilbert space H is isometric to $l^2(K)$ for some field K.

3.4 The Adjoint of Maps of Hilbert Spaces

Theorem 3.4.1: Riesz Representation Theorem

Theorem 3.4.2

Let H, K be Hilbert spaces. Let $A : H \to K$ be a bounded linear map. Then there exists a unique bounded linear map $A^* : K \to H$ such that

$$\langle A(x), y\rangle = \langle x, A^*(y)\rangle$$

for all $x \in H$ and $y \in K$.

4 Spectral Theory

4.1 Spectrum

Definition 4.1.1: Resolvent and Spectrum

Let X be a complex Banach space and $T \in B(X)$. Define the resolvent set of T to be

$$\rho(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is invertible } \}$$

Define the spectrum of T to be

$$\sigma(T) = \mathbb{C} \setminus \rho(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible } \}$$

Notice that in the case that X is finite dimensional, the resolvent set of any linear operator is just its set of eigenvalues.

Lemma 4.1.2

Suppose that $T \in B(X)$ and that $(\lambda_n)_{n\in\mathbb{N}}$ are distinct eigenvalues of T. Then any set $(e_n)_{n\in\mathbb{N}}$ of corresponding eigenvectors is linearly independent.

Theorem 4.1.3

Let X be a Banach space and $T \in B(X)$ such that $T^{-1} \in B(X)$. Then for any $U \in B(X)$, with $\|U\| < \|T^{-1}\|^{-1}$, we have

$$\|(T+U)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|U\|\|T^{-1}\|}$$

Lemma 4.1.4

Let X be a Banach space. Let $T \in B(X)$. Then $\sigma(T)$ is a closed subset of $\{\lambda \in \mathbb{C} | |\lambda| \leq \|T\|\}$.