Sheaf Theory

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Abstract

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1 Sheaves

1.1 Basic Definition of Sheaves

As with how we equipped to each variety V its coordinate ring $k[V]$ which are functions on V, we want to equip to each spectrum some ring which are functions on them. It will not make sense that we can define functions on spectrums immediately.

In its full generality, sheaves are designed to encode all local information into one compact notation. It often has uses in complex geometry for its differential forms and smooth functions. In algebraic geometry it is fundamental for redefining the notion of a variety.

Definition 1.1.1: Presheaves

Let $(X, \mathcal{T})$ be a topological space. Define the category $\mathbf{Open}(X)$ to consist of the following data.

- The objects $\text{Ob}(\mathbf{Open}(X)) = \mathcal{T}$ are the open sets of X
- For $U, V \subseteq X$ two open sets of X, the morphisms

$$\text{Hom}_{\mathbf{Open}(X)}(U, V) = \begin{cases} \{\iota : U \hookrightarrow V \text{ the inclusion map}\} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

- Composition is given as the composition of functions.

A presheaf on X is then a contravariant functor

$$\mathcal{F} : \mathbf{Open}(X) \to \mathbf{Sets}$$

with potentially additional structures on the sets such as $\mathbf{Grp}, \mathbf{Rings}$ etc.

Explicitly, a presheaf consists of the following data.

- A function
$$\mathcal{F} : \mathcal{T} \to \text{Sets}$$
This means each open set U of X gets associated with a set, potentially with additional structures (groups / rings). Each individual element of $\mathcal{F}(U)$ is called a section. Each element of $\mathcal{F}(X)$ is instead called a global section
- For each inclusion of open sets $V \subseteq U$, there exists a restriction map $\text{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying
  - $\text{res}_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity
  - Whenever $W \subseteq V \subseteq U$, then $\text{res}_{W,V} \circ \text{res}_{V,U} = \text{res}_{W,U}$

The reason that the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is called a restriction is that we will soon see that elements of $\mathcal{F}(X)$ are actually functions over some ring or field.

Notation: we often use $\Gamma(U, \mathcal{F})$ to denote the set $\mathcal{F}(U)$ and $s|_V$ to denote $\text{res}_{UV}(s)$ for $s \in \mathcal{F}(U)$.

Definition 1.1.2: Sheaves

Let X be a space. A sheaf on X is a presheaf $\mathcal{F} : \mathbf{Open} \to \mathbf{Sets}$ satisfying two additional properties

- Identity: If $\{U_i | i \in I\}$ is an open cover of U and $\phi_1, \phi_2 \in \mathcal{F}(U)$ and $\phi_1|_{U_i} = \phi_2|_{U_i}$ for all i, then $\phi_1 = \phi_2$
- Gluing: If $\{U_i | i \in I\}$ is an open cover of U and $\phi_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $\phi_i|_{U_i \cap U_j}$ for all $i, j \in I$, then there is exists some $\phi \in \mathcal{F}(U)$ such that $\phi|_{U_i} = \phi_i$ for all $i \in I$.

We can define the category of sheaves on a topological space X where objects are all the sheaves on X and morphisms are all the morphisms between the sheaves. This will be seen formally later.

Given a space X and a point $p \in X$, the neighbourhoods of p form a diagram in $\mathbf{Open}(X)$. Instead of going through the categorical definition, we write the colimit of $F : \mathbf{Open}(X) \to \mathbf{Set}$ under this diagram as

$$\operatorname*{colim}_{V_p} \mathcal{F}(V_p)$$

denoting that the V_p varies by restriction.

Definition 1.1.3: Stalks and Germs

Let $\mathcal{F}$ be a presheaf on a topological space $(X, \mathcal{T})$. Let $p \in X$. Consider the full subcategory $\mathcal{J}_p$ of $\mathbf{Open}(X)$ consisting of open sets in X that contain p. Define the stalk of $\mathcal{F}$ at p to be the colimit

$$\mathcal{F}_{X,p} = \operatorname*{colim}_{V_p} \mathcal{F}(V_p)$$

Lemma 1.1.4

Let X be a space and let $F : \mathbf{Open}(X) \to \mathbf{Sets}$ be a presheaf on X. Then for any $p \in X$, the stalk at p is given by

$$\mathcal{F}_{X,p} = \{(U, s) \mid x \in U \subset X \text{ open}, s \in \mathcal{F}(U)\}/\sim$$

where we say that $(U_1, s_1) \sim (U_2, s_2)$ if there exists some $V \subseteq U_1 \cap U_2$ open such that $\text{res}_{V,U_1}(s_1) = \text{res}_{V,U_2}(s_2)$.

Think of the definition of stalks as follows: Treat f and g to be sections in $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$ where $V \subseteq U_1 \cap U_2$ is open and contains x. Then we treat f and g to be the same function in the stalk as long as they agree on some open set that contains x. Indeed, since we do not care about the entirety of the domain of f and g, and only care about what happens locally near x, it makes sense for us to treat them as a function when they appear to be the same locally.

Definition 1.1.5: Morphism of Presheaves

Let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ consists of a collection of morphism of sets (groups, rings, etc)

$$\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

for each open set U such that if $V \subseteq U$ is an inclusion, the following digram commutes, where ρ, ρ' are restriction maps.

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\text{res}_{V,U}"] & \mathcal{G}(U) \arrow[d, "\text{res}_{V,U}"] \\
\mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

An isomorphism of presheaves is just a morphism of presheaves such that each $\phi(U)$ is an isomorphism. In other words, morphism of presheaves is just a natural transformation between $\mathcal{F}$ and $\mathcal{G}$.

Notice that the natural transformation ϕ here takes every open set U and maps it to a group homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ if $\mathcal{F}$ and $\mathcal{G}$ are presheaves with values in $\mathbf{Grp}$.

Proposition 1.1.6

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on the stalk $\phi_p : \mathcal{F}_{X,p} \to \mathcal{G}_{X,p}$ is an isomorphism for all $p \in X$.

Proof. Suppose that ϕ is an isomorphism of sheaves. Then we obtain the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{F}_{X,p} & \\
\mathcal{F}(U) \arrow[ur] \arrow[d, "\phi(U)"'] & \mathcal{G}_{X,p} \arrow[u, dashed, "\exists! a"] & \mathcal{F}(V) \arrow[ul] \arrow[d, "\phi(V)"] \\
\mathcal{G}(U) \arrow[ur] \arrow[rr, "\text{res}_{V,U}"] & & \mathcal{G}(V) \arrow[ul]
\end{tikzcd}$$

which is due to the fact that $\phi(U)$ and $\phi(V)$ are isomorphisms. This unique map $a : \mathcal{G}_{X,P} \to \mathcal{F}_{X,p}$ exists by the universal property of the colimit. Similarly, we can construct a map $b : \mathcal{F}_{X,P} \to G_{X,p}$. Since $\mathcal{F}_{X,p}$ and $\mathcal{G}_{X,p}$ are colimits, they unique map from $\mathcal{F}_{X,p}$ to itself must be the identity. Hence $a \circ b$ is the identity. Similarly, $b \circ a$ is also the identity hence $\mathcal{F}_{X,p}$ and $\mathcal{G}_{X,p}$ are isomorphic.

Now suppose that ϕ_p is an isomorphism for each $p \in X$. We show that $\phi(U)$ is bijective for all $U \subseteq X$ open. Let $s \in \mathcal{F}(U)$ such that $\phi(U)(s) = 0$. Then for each $p \in U$, $\phi(U)(s)$ considered as an element in $\mathcal{G}_{X,p}$ is equal to 0. Since ϕ_p is injective, $s = 0$ in $\mathcal{F}_{X,p}$. This means that there exists an open neighbourhood $V_p \subseteq U$ of p such that $s \in \phi(V_p)$ is 0 for each $p \in U$. Since U is covered by the open neighbourhoods V_p for $p \in U$, by the identity axiom we conclude that $s = 0$ in $\phi(U)$. Thus ϕ is injective.

Now let $t \in \mathcal{G}(U)$. Let t_p be the corresponding element of t in $\mathcal{G}_{X,p}$ for each $p \in U$. Since ϕ_p is surjective, there exists $s_p \in \mathcal{F}_{X,p}$ such that $\phi_p(s_p) = t_p$ for each $p \in U$. Suppose that on a neighbourhood V_p of p, s_p is represented as r_p. Then $\phi(r_p) \in \mathcal{G}(V_p)$ and $t|_{V_p}$ are two elements whose germs in $\mathcal{G}_{X,p}$ are equal. Hence there exists $W_p \subseteq V_p$ containing p such that $\phi(r_p) = t|_{W_p}$ in $\mathcal{G}(W_p)$. Now U is covered by open sets of the form W_p for each $p \in U$. Let $p \in W_p$ and $q \in W_q$. Then both $r_p|_{W_p \cap W_q}$ and $r_q|_{W_p \cap W_q}$ are two sections in $\mathcal{F}(W_p \cap W_q)$ that are sent to $t|_{W_p \cap W_q} \in \mathcal{G}(W_p \cap W_q)$ by $\phi(W_p \cap W_q)$. By injectivity, they are equal. We conclude that

$$r_p|_{W_p \cap W_q} = r_q|_{W_p \cap W_q}$$

By the gluing axiom, we conclude that there exists a section $s \in \mathcal{F}(U)$ such that $s|_{W_p} = r_p$. Now $\phi(s)$ and t are two sections in $\mathcal{G}(U)$ such that for any $p \in U$, $\phi(s)|_{W_p} = t|_{W_p}$. By the identity axiom, we conclude that $\phi(s) = t$. □

Note that the above proposition is very untrue for presheaves because as one can see, proving both injective and surjective requires to use of the sheaf axioms instead of just the presheaf datum.

Definition 1.1.7: Sheafification

Let X be a space and let $\mathcal{F}$ be a presheaf on X. We say that a sheaf $\mathcal{F}^+$ is the sheaffication of $\mathcal{F}$ if there exists a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that the following universal property.

If G is a sheaf such that there is a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism of sheaves $\psi : \mathcal{F} \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, dashed, "\exists!\psi"] \\ & \mathcal{G} \end{tikzcd}$$

Theorem 1.1.8

Let X be a space and $\mathcal{F}$ a presheaf on X. Then $\mathcal{F}^+$ exists and is unique up to isomorphism. Explicitly, $\mathcal{F}^+$ can be defined as follows.

- For each open set $U \subseteq X$,
$$\mathcal{F}^+(U) = \left\{ s : U \to \bigcup_{p \in U} \mathcal{F}_p \;\middle|\; \begin{smallmatrix} \forall p \in U,\ s(p) \in \mathcal{F}_{X,p} \text{ and } \exists V_p \subseteq U \text{ and} \\ t \in \mathcal{F}(V_p) \text{ s.t. } \forall q \in V_p, s(q) = t_q \in \mathcal{F}_{X,q} \end{smallmatrix} \right\}$$
- For each $V \subseteq U$ an inclusion, there is a unique morphism $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$ that sends $s \in U$ to its restriction
$$s|_V : V \to \bigcup_{p \in V} \mathcal{F}_{X,p}$$

We end the section with a formula to construct a sheaf given its values on just the basis of a space.

Theorem 1.1.9

Let X be a topological space. Let $\mathcal{B}$ be the basis of X. Suppose that $\mathcal{F}_0$ is a sheaf defined on the basis $\mathcal{B}$ of X. Then the natural extension to open sets U by

$$\mathcal{F}(U) = \left\{ (s_i)_i \in \prod_i \mathcal{F}_0(B_i) \;\middle|\; B_i \in \mathcal{B}, B_i \subseteq U, s_i|_{B_i \cap B_j} = s_j|_{B_i \cap B_j} \right\} = \varprojlim_{\substack{B \in \mathcal{B} \\ B \subseteq U}} \mathcal{F}(B)$$

defines a sheaf for X.

Proof. $\square$

This means that sheaves are uniquely determined by their values in the basis of X. We can simply define the sheaf on the basis elements and by this natural extension, a sheaf will be defined for all of X.

1.2 Subsheaves of a Sheaf

Definition 1.2.1: Subsheaf

A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and that the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$.

It follows directly from the definition that for any point P, the stalk $\mathcal{F}'_P$ is a subgroup of $\mathcal{F}_P$.

Definition 1.2.2: Kernel of a Presheaves

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Define the presheaf kernel of ϕ to be the presheaf given by
$$U \to \ker(\phi(U))$$

Notice that the definitions here make sense because essentially $\phi(U)$ is a group (ring) homomorphism if the presheaf we are working with is a presheaf of groups or rings.

Proposition 1.2.3

The presheaf kernel of a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a subsheaf of $\mathcal{F}$.

Definition 1.2.4: The Image Presheaf

Let X be a space and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves. Define the image presheaf

$$\mathrm{im}(\phi) : \mathbf{Open}(X) \to \mathbf{Set}$$

of ϕ as follows.

- For $U \subseteq X$ an open set, $\mathrm{im}(\phi)(U) = \mathrm{im}(\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$
- For $V \subseteq U$ an inclusion, there is a unique morphism $\mathrm{im}(\phi)(U) \to \mathrm{im}(\phi)(V)$ given by the restriction of the morphism $\mathcal{G}(U) \to \mathcal{G}(V)$ to the set $\mathrm{im}(\phi)(U) \subseteq \mathcal{G}(U)$.

1.3 Image Sheaves

Definition 1.3.1: Direct Image Sheaf

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X. Define the direct image sheaf on Y as follows. For every open set $V \subseteq Y$, define

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

This means that $f_*\mathcal{F}$ is defined as follows:

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, "\mathcal{F}"'] & V \arrow[l, "f^{-1}"'] \arrow[dl, "f_*\mathcal{F}"] \\
\mathcal{F}(f^{-1}(V)) &
\end{tikzcd}$$

Proposition 1.3.2

The direct image sheaf on Y is indeed a sheaf on Y.

Proof. The proof is direct since $\mathcal{F}$ is already a sheaf itself and we are only taking sparser open sets than open sets in X. $\square$

Definition 1.3.3: Inverse Image Sheaf

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Define the inverse image sheaf on X as follows. For every open set $U \subseteq X$, define

$$f^+\mathcal{G}(U) = \lim_{\substack{V \supseteq f(U) \\ V \subseteq Y \text{ open}}} \mathcal{G}(V)$$

The sheaffification of $f^+\mathcal{G}$, $f^{-1}\mathcal{G}$ is called the inverse image sheaf of $\mathcal{G}$ under f.

Note: The direct image sheaf and inverse image sheaf are adjoint functors. Goertz Wedhorn P.55.

1.4 The Category of Sheaves

Definition 1.4.1: The Category of Sheaves of Abelian Groups

Let X be a topological space. The category of sheaves of abelian groups is the category $\mathbf{Ab}(X)$ where

- The objects of $\mathbf{Ab}(X)$ are the sheaves on X
- Given two sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, a morphism from $\mathcal{F}$ to $\mathcal{G}$ is a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ which is just a natural transformation
- Composition is given by the composition of natural transformations

Theorem 1.4.2

Let X be a space. Then the category $\mathbf{Ab}(X)$ of sheaves of abelian groups is an abelian category.

Since $\mathbf{Ab}(X)$ is an abelian category, all kinds of limiting objects can be formed. This includes kernels, cokernels and direct sums. Moreover, we can now talk about morphism of sheaves that are injective and surjective.

Lemma 1.4.3

Let $\phi : F \to G$ be a morphism of sheaves. Then the categorical kernel and cokernel of ϕ is canonically isomorphic to the sheaves $\ker(\phi)$ and $\text{coker}(\phi)$.

Proposition 1.4.4

Let X be a space. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then the following are true.

- $\ker(\phi)_{X,p} = \ker(\phi_p)$
- $\text{im}(\phi)_{X,p} = \text{im}(\phi_p)$

Proof. □

Proposition 1.4.5

Let X be a space. Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Denote $\phi_p : \mathcal{F}_{X,p} \to \mathcal{G}_{X,p}$ the induced map on stalks. Then then following are true.

- ϕ is injective if and only if ϕ_p is injective
- ϕ is surjective if and only if ϕ_p is surjective

Proposition 1.4.6

Let X be a topological space. Let $\mathcal{F}^i$ be a collection of sheaves on X. The cochain complex

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\phi^{i-1}} \mathcal{F}^i \xrightarrow{\phi^i} \mathcal{F}^{i+1} \longrightarrow \cdots$$

is exact in $\mathbf{Ab}(X)$ if and only if for every $p \in X$ the corresponding sequence of stalks

$$\cdots \longrightarrow \mathcal{F}^{i-1}_{X,p} \xrightarrow{\phi^{i-1}_p} \mathcal{F}^{i}_{X,p} \xrightarrow{\phi^{i}_p} \mathcal{F}^{i+1}_{X,p} \longrightarrow \cdots$$

is exact.

Proof. Suppose that we are given an exact sequence of sheaves $\phi^i : \mathcal{F}^i \to \mathcal{F}^{i+1}$. Then we have that $\mathrm{im}(\phi^{i-1}) = \ker(\phi^i)$. This passes to the stalks hence $\mathrm{im}(\phi^{i-1})_{X,p} = \ker(\phi^i)_{X,p}$. By the above proposition above, we conclude that $\mathrm{im}(\phi^{i-1}_p) = \ker(\phi^i_p)$ hence there is an exact sequence of stalks.

Now suppose that there is an exact sequence of stalks $\phi^i_p : \mathcal{F}^i_{X,p} \to \mathcal{F}^{i+1}_{X,p}$. Then similarly we have that

$$\ker(\phi^i)_{X,p} = \ker(\phi^i_p) = \mathrm{im}(\phi^{i-1}_p) = \mathrm{im}(\phi^{i-1})_{X,p}$$

We conclude that this gives $\ker(\phi^i) = \mathrm{im}(\phi^{i-1})$ by 1.1.6. $\square$

1.5 Adjunction of the Image Functors

Definition 1.5.1: Direct Image Functor

Let $f : X \to Y$ be a continuous map of spaces. Define the direct image functor

$$f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$$

as follows.

- f_* sends a sheaf $\mathcal{F}$ on X to the direct image sheaf $f_*\mathcal{F}$ on Y
- For a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X, f_* sends it to the morphism $f_*(\phi) : f_*\mathcal{F} \to f_*\mathcal{G}$ defined by the components

  $$\big(f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))\big) \overset{\phi(f^{-1}(V))}{\mapsto} \big(\mathcal{G}(f^{-1}(V)) = f_*\mathcal{G}(V)\big)$$

  for each $V \subseteq X$ open.

Definition 1.5.2: Inverse Image Functor

Let $f : X \to Y$ be a continuous map of spaces. Define the inverse image functor

$$f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$$

as follows.

- f^{-1} sends a sheaf $\mathcal{F}$ on Y to the inverse image sheaf $f^{-1}\mathcal{F}$ on X
- For a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X, $^{-1}$ sends it to the morphism $f^{-1}(\phi) : f^{-1}\mathcal{F} \to f^{-1}\mathcal{G}$ defined by ???

Theorem 1.5.3

Let $f : X \to Y$ be a continuous map. Then the inverse image functor f^{-1} is left adjoint to the direct image functor f_*. Explicitly, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}(X)}(f^{-1}(-), -) \cong \mathrm{Hom}_{\mathbf{Ab}(Y)}(-, f_*(-))$$

This immediately implies the following:

Corollary 1.5.4

Let $f : X \to Y$ be a continuous map. Then the functor f_* is left exact, and the functor f^{-1} is right exact.

Theorem 1.5.5

Let X be a topological space. Then the category $\mathbf{Ab}(X)$ has enough injectives.

1.6 Ringed Spaces

Definition 1.6.1: Ringed Space

A ringed space is a topological space X together with a sheaf of rings on X. A locally ringed space is a ringed space X where all stalks are local rings.

Definition 1.6.2: Residue Fields

Let X be a locally ringed space. Let $p \in X$. Define the residue field of p to be

$$k(p) = \frac{\mathcal{O}_{X,p}}{m}$$

where m is the unique maximal ideal of the local ring.

Definition 1.6.3: Morphisms of Ringed Spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

Suppose that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are locally ringed spaces. Given a point $p \in X$, we obtain induced maps

$$f^\# : \mathcal{O}_Y(V) \to f_*\mathcal{O}_X(f^{-1}(V))$$

for any neighbourhood V of $f(p)$. Taking the colimit, we obtain a map

$$\mathcal{O}_{Y,f(p)} = \operatorname*{colim}_V \mathcal{O}_Y(V) \to \operatorname*{colim}_V \mathcal{O}_X(f^{-1}(V))$$

As V ranges through the neighbourhood of $f(p)$, $f^{-1}(V)$ ranges over a subset of the neighbourhoods of p so that we obtain a map

$$\mathcal{O}_{Y,f(p)} = \operatorname*{colim}_V \mathcal{O}_Y(V) \to \operatorname*{colim}_V \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,p}$$

Definition 1.6.4: Morphisms of Locally Ringed Spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. Then a morphism of locally ringed spaces is a morphism of ringed spaces such that for each $p \in X$, the induced map of local rings

$$f_p^\# : \mathcal{O}_{Y,f(p)} = \operatorname*{colim}_V \mathcal{O}_Y(V) \to \operatorname*{colim}_V \mathcal{O}_X(f^{-1}(V)) \to \mathcal{O}_{X,p}$$

is a local homomorphism of local rings.

Definition 1.6.5: Open Embedding

Let $U \to Y$ be an isomorphism of U and an open subset of Y, together with an isomorphism ringed spaces $(U, \mathcal{O}|_U)$ and $(V, \mathcal{O}_Y|_V)$. Then this map of ringed spaces is called an open embedding or an open immersion of ringed spaces.

2 Coherent Sheaves

2.1 The Category of $\mathcal{O}_X$-Modules

Definition 2.1.1: Sheaf of $\mathcal{A}$-modules

Let $\mathcal{A}$ be a sheaf of rings over X. Let U be an open set of X. A sheaf of $\mathcal{A}$-modules over X is a sheaf $\mathcal{F}$ such that each $\mathcal{F}(U)$ is an $\mathcal{A}(U)$-modules. Moreover, for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] \arrow[d, "\text{res}_{U,V} \times \text{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\text{res}_{U,V}"] \\
\mathcal{A}(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] & \mathcal{F}(V)
\end{tikzcd}$$

Denote the category of $\mathcal{A}$-modules by

$$\text{Mod}_X(\mathcal{A})$$

Proposition 2.1.2

Let $\mathcal{A}$ be a sheaf of rings over X and let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{A}$-modules. Then the direct sum

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

is also a sheaf of $\mathcal{A}$-modules.

Theorem 2.1.3

Let $\mathcal{A}$ be a sheaf of rings over X. Then the category $\text{Mod}(\mathcal{A})$ of $\mathcal{A}$-modules is an abelian category.

Proposition 2.1.4

Denote i the trivial functor taking a sheaf to its presheaf. Then the functor i and the sheafification functor $^+$ are adjoints. In other words,

$$\text{Hom}(i(\mathcal{F}), \mathcal{G}) \cong \text{Hom}(\mathcal{F}, \mathcal{G}^+)$$

for a presheaf $\mathcal{G}$ and a sheaf $\mathcal{F}$.

Proposition 2.1.5

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ defined by

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = (\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U))^+$$

is also a sheaf of $\mathcal{O}_X$-modules.

2.2 Invertible Sheaves

Definition 2.2.1: Free Sheaf

Let $\mathcal{A}$ be a sheaf of rings over X and let $\mathcal{F}$ be a sheaf of $\mathcal{A}$-module. We say that $\mathcal{F}$ is free if

$$\mathcal{F} \cong \mathcal{A}^{\oplus n}$$

It is locally free if X can be covered by open sets $\{U_i \mid i \in I\}$ for which

$$\mathcal{F}|_{U_i} \cong (\mathcal{A}|_{U_i})^{\oplus n}$$

for each U. In this case we say that the rank of $\mathcal{F}$ is n. Denote the category of locally free sheaves of rank n by

$$\text{Loc}_n(X)$$

Lemma 2.2.2

If X is connected then the rank of a locally free sheaf on X is constant.

Definition 2.2.3: Invertible Sheaf

A locally free sheaf of rank 1 is called an invertible sheaf.

Theorem 2.2.4

Let $(X, \mathcal{O}_X)$ be a scheme. Then the following are equivalent characterization of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$

- $\mathcal{F}$ is invertible
- There exists a sheaf G such that $F \otimes_{\mathcal{O}_X} G \cong \mathcal{O}_X$
- $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$

Theorem 2.2.5

Let X be a space. Then there is an equivalence of categories

$$\text{Vect}_n(X) \cong \text{Loc}_n(X)$$

between vector bundles over X and sheaves of free $\mathcal{A}$-modules over X for any $n \in \mathbb{N}$.

2.3 Quasicoherent Sheaves

Definition 2.3.1: Quasicoherent Sheaves

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is quasicoherent if X can be covered by open affine subsets $U_i = \text{Spec}(A_i)$ such that for each i, there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Definition 2.3.2: Coherent Sheaves

We say that $\mathcal{F}$ is a coherent sheaf if $\mathcal{F}$ is a quasicoherent sheaf and each M_i is a finitely generated A_i-module.

In some sense, the category of quasicoherent sheaves is the smallest abelian category for which it encompasses the category of locally free sheaves. In the case that A is locally Noetherian, the category of finite rank locally free sheaves sit inside the category of coherent sheaves, which is also an abelian category.

Proposition 2.3.3

Let A be a ring and let $X = \operatorname{Spec}(A)$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

If A is noetherian, the same functor gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules.

3 Sheaf Cohomology

3.1 The Global Section Functor

Definition 3.1.1: Global Section Functor

Let X be a space. Define the global section functor to be the functor $\Gamma(X,-):\mathbf{Ab}(X)\to\mathbf{Ab}$ defined as follows.

- A sheaf $\mathcal{F}$ of groups on X is sent to $\Gamma(X,\mathcal{F})=\mathcal{F}(X)$
- For a morphism of sheaves $\phi:\mathcal{F}\to\mathcal{G}$, $\Gamma(X,-)$ sends it to the ring homomorphism $\phi(X):\mathcal{F}(X)\to\mathcal{G}(X)$

In general, we can define functor by sending every sheaf to a the local section of a chosen open set. This operation in general is left exact. This means that we have the following proposition.

Proposition 3.1.2

Let X be a space and let $\mathcal{F},\mathcal{F}',\mathcal{F}''$ be sheaves on X such that there is an exact sequence

$$0\longrightarrow\mathcal{F}\xrightarrow{\phi}\mathcal{F}'\xrightarrow{\psi}\mathcal{F}''\longrightarrow 0$$

Then there is an exact sequence

$$0\longrightarrow\mathcal{F}(U)\xrightarrow{\phi(U)}\mathcal{F}'(U)\xrightarrow{\psi(U)}\mathcal{F}''(U)$$

of local sections for any chosen open set $U\subseteq X$.

Proof. We want to show that $\ker(\phi(U))=0$. Suppose that $s\in\ker(\phi(U))$. Then $\phi(U)(s)=0$. Then for any $p\in U$, $(\phi(U),\phi(U)(s))\in\mathcal{G}_{X,p}$ is equivalent to the 0 section. By 1.4.5, ϕ_p is injective hence s is equivalent to the zero section in $\mathcal{F}_{X,p}$ for some open subset U_p of U. Since this is true for any p, we conclude that $s=0$ in $\mathcal{F}(U)$ by the gluing axiom.

It remains to show that $\operatorname{im}(\phi(U))=\ker(\psi(U))$. Suppose that $t\in\operatorname{im}(\phi(U))$. Then there exists a section $s\in\mathcal{F}(U)$ such that $\phi(U)(s)=t$. By 1.4.6, $\psi(U)(t)=(\psi\circ\phi)(U)(s)=0$ $\mathcal{F}''_{X,p}$ for every $p\in U$. Since this is true for all p, we conclude that $\psi(U)(t)=0$ by gluing axiom. Hence $\operatorname{im}(\phi(U))\subseteq\ker(\psi(U))$.

Now suppose that $t\in\ker(\psi(U))$. Then $\psi(U)(t)=0$ and hence $(\psi(U),\psi(U)(t))$ is equivalent to to the zero section in $\mathcal{F}''_{X,p}$ for every $p\in U$. $\square$

Corollary 3.1.3

Let X be a space. The global section functor $\Gamma(X,-):\mathbf{Ab}(X)\to\mathbf{Ab}$ is a left exact functor.

Proof. The global section functor is a special case of the above proposition. $\square$

Definition 3.1.4: Cohomology Functors

Let X be a space. Define the cohomology functors

$$H^i(X,-)=R^i\Gamma(X,-):\mathbf{Ab}(X)\to\mathbf{Ab}$$

of X to be the right derived functors of $\Gamma(X,-):\mathbf{Ab}(X)\to\mathbf{Ab}$. For any sheaf $\mathcal{F}$ of abelian groups on X, the groups $H^i(X,\mathcal{F})$ are called the cohomology groups of $\mathcal{F}$.

3.2 Flasque Sheaves

Definition 3.2.1: Flasque Sheaves

A sheaf $\mathcal{F}$ on a space X is said to be flasque if for every pair of open sets $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Lemma 3.2.2

If $(X, \mathcal{O}_X)$ is a ringed space, then any injective $\mathcal{O}_X$-module is flasque.

Lemma 3.2.3

Let X be a space and let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be sheaves on X such that there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of sheaves. Then the following are true.

- If $\mathcal{F}$ and $\mathcal{F}'$ are flasque, then $\mathcal{F}''$ is flasque.
- If $\mathcal{F}$ is flasque, then for any $U \subseteq X$ open, there is an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi} \mathcal{F}'(U) \xrightarrow{\psi} \mathcal{F}''(U) \longrightarrow 0$$

Proof. Suppose that $\mathcal{F}$ and $\mathcal{F}'$ are flasque. Then by naturality we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, two heads] & \mathcal{F}'(U) \arrow[r, "\psi(U)"] \arrow[d, two heads] & \mathcal{F}''(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{F}'(V) \arrow[r, "\psi(V)"] & \mathcal{F}''(V) \arrow[r] & 0
\end{tikzcd}$$

for any $V \subseteq U$ open sets. Let $s \in \mathcal{F}''(V)$. Then by surjectivity of $\psi(V)$ and $\mathcal{F}'(U) \to \mathcal{F}'(V)$, there exists $t \in \mathcal{F}'(U)$ that maps correspondingly by the same maps to s. Then using the map $\phi(U)$ we can send s to an element $r \in \mathcal{F}''(U)$ such that r is sent to s under the map $\mathcal{F}''(U) \to \mathcal{F}''(V)$. Thus this map is surjective and $\mathcal{F}''$ is flasque.

Now suppose that $\mathcal{F}$ is flasque. □

Proposition 3.2.4

Let X be a space and let $\mathcal{F}$ be a sheaf on X. If $\mathcal{F}$ is flasque then $\mathcal{F}$ is acyclic for $\Gamma(X, -)$. Explicitly, this means that

$$H^i(X, \mathcal{F}) = 0$$

for all $i > 0$.

Proof. Since $\mathbf{Ab}(X)$ has enough injective objects, suppose that $\mathcal{I}$ is an injective object of $\mathbf{Ab}(X)$ such that $\mathcal{F}$ is a subobject of $\mathcal{I}$. Let $\mathcal{G}$ be the quotient sheaf $\mathcal{F}/\mathcal{I}$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0$$

By lemma 3.2.2, $\mathcal{I}$ is flasque. By 3.2.3, $\mathcal{G}$ is flasque. By 3.2.3, there is an exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{I}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$

Since $\mathcal{I}$ is injective, we have that $H^i(X,\mathcal{I}) = 0$ for all $i > 0$. By passing to the long exact sequence in cohomology, we deduce that $H^1(X,\mathcal{F}) = 0$ and $H^i(X,\mathcal{F}) = H^{i-1}(X,\mathcal{G})$ for each $i \geq 2$. Applying the same method again gives $H^1(X,\mathcal{G}) = 0$. Thus by induction, we conclude that $H^i(X,\mathcal{F}) = 0$ for all $i > 0$. □

Proposition 3.2.5

Let $(X,\mathcal{O}_X)$ be a ringed space. Then the derived functors of

$$\Gamma(X,-) : {}_{\mathcal{O}_X}\mathbf{Mod}(X) \to \mathbf{Ab}$$

coincide with the derived functors of $\Gamma(X,-) : \mathbf{Ab}(X) \to \mathbf{Ab}$.

Thus there is no distinction in considering the cohomology groups of sheaf of mO_X-modules and when considering them as simply a sheaf of abelian groups.

Theorem 3.2.6

Let X be a Noetherian topological space of dimension n. Let $\mathcal{F}$ be a sheaf on X. Then

$$H^i(X,\mathcal{F}) = 0$$

for all $i > n$.

3.3 Čech Cohomology

Definition 3.3.1: Čech Complex

Let X be a topological space and $\mathcal{U} = \{U_i | i \in I\}$ an open cover of X where I is an indexing set. For any $(i_0,\dots,i_k) \in I^{k+1}$, denote

$$U_{i_0,\dots,i_k} = U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_k}$$

Define for each k,

$$C^k(X,\mathcal{U},\mathcal{F}) = \bigcap_{(i_0,\dots,i_k)\in I^{k+1}} \mathcal{F}(U_{i_0,\dots,i_k})$$

Furthermore, define a boundary map $d : C^k(X,\mathcal{U},\mathcal{F}) \to C^{k+1}(X,\mathcal{U},\mathcal{F})$ by

$$c_{i_0\dots,i_k} \overset{d}{\mapsto} \sum_{s=0}^{k+1}(-1)^s \text{res}(c_{i_0,\dots,\hat{i}_s,\dots,i_{k+1}})$$

Define the Čech complex to be $(C^\bullet(X,\mathcal{U},\mathcal{F}),d)$.

Lemma 3.3.2

For any space X and any open cover $\mathcal{U}$ of X, $(C^\bullet(X,\mathcal{U},\mathcal{F}),d)$ is indeed a chain complex.

Definition 3.3.3: Čech Cohomology

Let $(C^\bullet(X,\mathcal{U},\mathcal{F}),d)$ be a Čech complex. Define the kth cohomology group of it to be

$$\check{\mathrm{H}}^k(X,\mathcal{U},\mathcal{F}) = \frac{\ker(C^k(X,\mathcal{U},\mathcal{F}) \to C^{k+1}(X,\mathcal{U},\mathcal{F}))}{\mathrm{im}(C^{k-1}(X,\mathcal{U},\mathcal{F}) \to C^k(X,\mathcal{U},\mathcal{F}))} = H(C^\bullet(X,\mathcal{U},\mathcal{F}),d)$$

Lemma 3.3.4

For any Čech complex, we have that $\check{\mathrm{H}}^0(X, \mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Theorem 3.3.5

Let X be a topological space and $\mathcal{U}$ an open cover of X. If the open sets $U_{i_0,\dots,i_k}$ satisfy that $H^k(U_{i_0,\dots,i_k}, \mathcal{F}) = 0$ for all $k > 0$, then

$$H^k(X, \mathcal{F}) = \check{\mathrm{H}}^k(X, \mathcal{U}, \mathcal{F})$$