Electromagnetism

Labix

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Abstract

Contents

1 Coulumb's Law

1.1 Charge

Definition 1.1.1. The charge of an electron is negative, denoted $-e$. The charge of a proton is positive, denoted e. Like charges repel each other and opposite charges attract.

Axiom 1.1.2 (Conservation of Charge)**.** The total charge in an isolated system never changes.

Axiom 1.1.3 (Quantization of Charge)**.**

Definition 1.1.4 (Coulomb's Law)**.** The interaction between electric charges at rest is described by Coulomb's Law.

$$\mathbf{F}_{21} = \frac{q_1 q_2}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|^2}\mathbf{r}_{21}$$

where q_1 and q_2 are the value of the electric charges.

Definition 1.1.5. The force on charge q_0 exerted by $q_1, \dots, q_n$ is given by

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_0 q_k}{|\mathbf{r}_k - \mathbf{r}_0|}\mathbf{r}_{k0}$$

Theorem 1.1.6. The work required to bring two particles to have the distance of r is

$$W = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}$$

Theorem 1.1.7. The potential energy of a system of charge is

$$U = \frac{1}{2}\sum_{j=1}^{n}\sum_{k\neq j}\frac{1}{4\pi\epsilon_0}\frac{q_j q_k}{|\mathbf{r}_j - \mathbf{r}_k|^2}$$

Definition 1.1.8. The electric field $\mathbf{E}$ of a charge distribution is

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k}{|\mathbf{r}_k - \mathbf{r}_0|}\mathbf{r}_{k0}$$

such that $\mathbf{F} = q\mathbf{E}$

By developing the concept of an electric field, we can predict what force the charge receives at (x, y, z) and where it moves.

Definition 1.1.9. For a continuous charge distribution, the electric field it generates is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{3/2}}\, d^3\mathbf{r}'$$

Let there be an arbitrarily volume in 3 dimensions. Let there be a sphere enclosed by the volume. Let the normal of the sphere be $\mathbf{a}$. Then project the patch onto the volume with a cone starting from the origin. The patch projected scales with a factor of $\left(\frac{R}{r}\right)^2$. And owing to its inclination $\frac{1}{\cos(\theta)}$, where θ is the angle made between the normal of the old patch and the normal of the new patch. E_R is also scaled by a factor of E_r due to being further away from the origin. We have that the flux through the outer patch is given by $\mathbf{E}_R \cdot \mathbf{A} = E_R A\cos(\theta)$ and $vbE_r \cdot \mathbf{a} = E_r a$.

The flux of the electric field through any surface enclosing a point charge q is $\frac{q}{\epsilon_0}$

Theorem 1.1.10 (Gauss's Law)**.** The flux of the electric field is equal to $\frac{1}{\epsilon_0}$ times the total charge enclosed by the surface.

$$\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} \int \rho \, dV = \frac{1}{\epsilon_0} \sum_{k=1}^{n} q_k$$

Theorem 1.1.11 (Field of a Sphere)**.**

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

where Q is the total charge on the sphere. if it is uniformly distributed, is equal to $4\pi r_0^2 \sigma$.

Theorem 1.1.12 (Field of a Line Charge)**.**

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

Theorem 1.1.13 (Field of an Infinite Flat Sheet of Charge)**.**

$$E = \frac{\sigma}{2pi\epsilon_0}$$

where σ is the surface charge distribution

Theorem 1.1.14. The force per unit area on a layer of charfe equals the density times the average of the fields on either side

$$\frac{F}{A} = \frac{1}{2}(E_1 + E_2)\sigma$$

Definition 1.1.15 (Energy Density)**.** The energy density of an electric field is $\frac{\epsilon_0 E^2}{2}$

Theorem 1.1.16. The total energy in a system equals

$$U = \frac{\epsilon_0}{2} \int E^2 \, dV$$

2 Electric Potential

Proposition 2.0.1. The line integral

$$\int_{p_1}^{p_2} \mathbf{E} \cdot ds$$

is path independent.

Proposition 2.0.2. The line integral

$$\int \mathbf{E} \cdot ds$$

around any closed path in an electric field is 0.

Definition 2.0.3 (Electric Potential Difference)**.** Define

$$\phi_{21} = -\int_{p_1}^{p_2} \mathbf{E} \cdot ds$$

the work per unit charge done by external agency in moving a positive charge from p_1 to p_2 in the field $\mathbf{E}$. We call work per unit charge done the electric potential difference.

Note the differenecs. The potential energy of a system of charges is the total work required to assemble it. The Electric potential is the work per unit charge required to move a unit positive test charge from some reference point to the point (x, y, z) in the field.

Theorem 2.0.4. The electric field can be derived from the electric potential function

$$\mathbf{E} = -\nabla\phi$$

Theorem 2.0.5 (Superposition)**.** The potential function given from multiple sources is given by

$$\phi = \int_{\text{all sources}} \frac{\rho(x', y', z')}{4\pi\epsilon_0 r}\, dx'\, dy'\, dz' = \sum_{\text{all sources}} \frac{q_i}{4\pi\epsilon_0 r}$$

where r is the distance from the volume element or charge to the point.

Note that the above sources must be confined to some finite region of space.

Theorem 2.0.6.

$$U = \frac{1}{2}\int \rho\phi\, dV$$

Theorem 2.0.7.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Theorem 2.0.8 (Poisson's Equation)**.**

$$\nabla \cdot \mathbf{E} = -\nabla^2\phi = \frac{\rho}{\epsilon_0}$$

2.1 Laplace's Equation

Definition 2.1.1 (Laplace's Equation)**.**

$$\nabla^2\phi = 0$$

Theorem 2.1.2. If ϕ satisfies Laplace's Equation, then the average value of ϕ over the surface of any sphere is equal to the value of ϕ at the center of the sphere

Theorem 2.1.3 (Earnshaw's Theorem)**.** It is impossible to construct an electrostatic field that will hold a charged particle in stable equilibrium in empty space.

Theorem 2.1.4. Electrostatic fields must have $\nabla \times \mathbf{E} = 0$

3 Electric Fields around Conductors

Definition 3.0.1. Given a system of conductors, the following holds.

- $\mathbf{E} = 0$ inside the material of a conductor
- $\rho = 0$ inside the material of a conductor
- $\phi = \phi_k$ for all points inside the material and on the surface of the kth conductor
- At any point just outside the conductor, $\mathbf{E}$ is perpendicular to the surface and $E = \frac{\sigma}{\epsilon_0}$ where σ is the local density of surface charge
- $Q_k = \int_{S_k} \sigma \, da = \epsilon_0 \int_{S_k} \mathbf{E} \cdot d\mathbf{a}$ where Q_k is the charge on conductor S_k

Definition 3.0.2. An isolated conductor carrying a charge Q has a certain potential ϕ_0. Q would be proportional to ϕ_0, depending linearly on the size and shape of the conductor.

$$Q = C\phi_0$$

where C is the capacitance of the conductor.

Definition 3.0.3.

$$Q = C(\phi_1 - \phi_2)$$

where C isc called the capacitance of the capacitor.

Theorem 3.0.4.

$$W = \frac{Q_f^2}{2C}$$

Theorem 3.0.5.

$$U = \frac{1}{2}C\phi^2$$

Theorem 3.0.6.

$$F = \frac{Q^2}{2}\frac{d}{dx}\frac{1}{C}$$

4 Electric Currents

Definition 4.0.1.

$$A = \frac{C}{t}$$

Definition 4.0.2. Current through a frame of n particles of equal charge and direction is

$$I = nq\mathbf{a} \cdot \mathbf{u}$$

where a is the normal of the frame, u is the velocity vector of the charge. The sum of different classes of particles is

$$I = \mathbf{a} \cdot \sum_k n_k q_k \mathbf{u}_k$$

Definition 4.0.3 (Current Density)**.**

$$\mathbf{J} = \sum_k n_k q_k \mathbf{u}_k$$

Proposition 4.0.4.

$$\mathbf{J} = -eN_e \overline{\mathbf{u}_e}$$

where N_e is the total electron in a volume, $\mathbf{u}_e$ the average velocity vector.

Theorem 4.0.5.

$$I = \int_S \mathbf{J} \cdot d\mathbf{a}$$

where S is a surface.

Theorem 4.0.6.

$$\nabla \cdot \mathbf{J} = 0$$

if $\mathbf{J}$ is time independent

Theorem 4.0.7. $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$ if $\mathbf{J}$ is time dependent.

Theorem 4.0.8.

$$\mathbf{J} = \sigma \mathbf{E}$$

where σ is called the conductivity of the material.

Theorem 4.0.9. Denote V the electric potential difference $\phi_1 - \phi_2$.

$$V = IR$$

where R is the constant called the resistance of the conductor. R depends on the shape, size and conductivity of the material.

Definition 4.0.10.

$$\rho = \frac{1}{\sigma}$$

resistivity is the inverse of conductivity

Theorem 4.0.11. Resistance of a wire

$$R = \frac{\rho L}{A}$$

where L is the length and A is the cross-sectional area.

Theorem 4.0.12. Average momentum of N positive ions

$$M\overline{\mathbf{u}_+} = \frac{1}{N}\sum_j (M\mathbf{u}_j^c + e\mathbf{E}t_j)$$

$\mathbf{u}_j^c$ is the velocity of the jth ion after its last collision.

Proposition 4.0.13. Average velocity of a positive ion in $\mathbf{E}$ is

$$\overline{\mathbf{u}_+} = \frac{\mathbf{E}e\overline{t_+}}{M_+}$$

Proposition 4.0.14.

$$\sigma \approx e^2\left(\frac{N_+\tau_+}{M_+} + \frac{N_-\tau_-}{M_-}\right)$$

where τ is the mean time between collisions.

Theorem 4.0.15. Resistance in series

$$R = R_1 + R_2$$

Resistance in parallel $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

Theorem 4.0.16. Current is found via Ohm's law and Kirchhoff's rules.

- $V = IR$
- At a node of the network, a point where three or more connecting wires meet, the algebraic sum of currents into the node must be 0.
- The sum of potential differences taken in order around a loop of the network is 0.

Theorem 4.0.17. Work done

$$P = I^2R$$

Theorem 4.0.18. A battery utilizes chemical reactions to supply an electromotive foce. Since the line integral of the electric field around a complete circuit is 0, there must be locations where ions move against the electric field. This force is provided by the chemical reaction.

Theorem 4.0.19 (Thevenin's Theorem)**.** Any circuit is equivalent to a single voltage source and a single resistor.

5 Magnetic Force

Theorem 5.0.1 (Lorentz Force)**.** Total force experienced by a particle with charge q is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic field.

Theorem 5.0.2 (Moving charges)**.**

$$Q = \epsilon_0 \int_{S(t)} \mathbf{E} \cdot d\mathbf{a}$$

Gauss's law holds for moving charges.

Theorem 5.0.3. Total charge in a system is not changed by the motion of the charge carriers.

Theorem 5.0.4.

$$\mathbf{E}'_{||} = \mathbf{E}_{||}$$

and

$$\mathbf{E}'_{\perp} = \gamma \mathbf{E}_{\perp}$$

Theorem 5.0.5. The field of a point charge moving with constant velocity $v = \beta c$ is radial and has magnitude

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \frac{1-\beta^2}{(1-\beta^2 \sin^2(\theta))^{3/2}}$$

Theorem 5.0.6.

$$\mathbf{F}'_{||} = \mathbf{F}_{||}$$

and

$$\mathbf{F}'_{\perp} = \frac{1}{\gamma} \mathbf{F}_{\perp}$$

Theorem 5.0.7. If a charge is moving with respect to other changes that are also moving in the lab frame, then the charge experiences a magnetic force. This force can also be viewed as an electric force in the particle's frame.

6 Magnetic Field

Theorem 6.0.1.

$$\mathbf{B} = \mathbf{k}\frac{\mu_0 I}{2\pi r}$$

where

$$\mu_0 = \frac{1}{\epsilon_0 c^2}$$

Theorem 6.0.2. Force between two parallel wires

$$F = \frac{\mu_0 I_1 I_2 \mathbf{l}}{2\pi r}$$

and

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

Theorem 6.0.3 (Ampere's Law)**.**

$$\int \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

where I is the current enclosed by the path.

Theorem 6.0.4.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Theorem 6.0.5.

$$\nabla \cdot \mathbf{B} = 0$$

Definition 6.0.6. Define $\mathbf{A}$ to be

$$\mathbf{B} = \nabla \times \mathbf{A}$$

the vector potential.

Theorem 6.0.7.

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{r}\, dv$$

Theorem 6.0.8.

$$d\mathbf{B} = \frac{\mu_0 \mathbf{I}}{4\pi} \frac{d\mathbf{l} \times \mathbf{r}}{r^2}$$