Fourier Analysis

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Abstract

These notes will act as a an introductory text with a collection of theorems and definitions for differential equations.

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1 Fourier Series

1.1 Fourier Series

Definition 1.1.1: Fourier Series

Let $n \in \mathbb{N}$. Fourier polynomials of degree $2n$ on $[-\pi, \pi]$ are complex polynomials of the form

$$\sum_{k=-n}^{n} c_k e^{ikx}$$

where $c_k \in \mathbb{C}$ and $x \in [-\pi, \pi]$. When $n = \infty$, we have the fourier series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

where $c_k \in \mathbb{C}$ and $x \in [-\pi, \pi]$.

Proposition 1.1.2

Every fourier polynomial $\sum_{k=-n}^{n} c_k e^{ikx}$ can be written in real variable

$$a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

where

$$c_k = \begin{cases} \frac{1}{2}(a_k - ib_k) & k > 0 \\ \frac{1}{2}(a_{-k} + ib_{-k}) & k < 0 \\ a_0 & k = 0 \end{cases}$$

Lemma 1.1.3: Orthogonality Property

The monomials of the fourier polynomial satisfy the following property.

$$\int_{-\pi}^{\pi} e^{ikx} e^{-ilx} \, dx = \begin{cases} 2\pi & \text{if } k = l \\ 0 & \text{otherwise} \end{cases}$$

where $k, l \in \mathbb{Z}$

1.2 Approximation of Functions

Proposition 1.2.1: Fourier Coefficients

Suppose that $\phi : [-\pi, \pi] \to \mathbb{R}$ can be represented by a fourier series. Then the fourier coefficients are given by

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(x) e^{-ikx} \, dx$$

for $k \in \mathbb{Z}$.

Definition 1.2.2: nth Fourier Polynomial

The nth fourier polynomial of ϕ that can be represented by a fourier series is given by

$$S_n(\phi)(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

where c_k are the fourier coefficients of ϕ.

Lemma 1.2.3

Let ϕ be able to be represnted by a fourier series. Then

$$c_{-k} = \overline{c_k}$$

Lemma 1.2.4: Riemann-Lebesgue

For any continuous function $\phi \in C^0([-\pi,\pi],\mathbb{C})$, the fourier coefficients converge to 0. Menaing

$$\lim_{k\to\pm\infty} c_k = \lim_{k\to\pm\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)e^{-ikx}\,dx = 0$$

Lemma 1.2.5: L

t $\phi : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function and suppose that its fourier series exists.

- If ϕ is odd then $\phi(x) = 2i\sum_{k=1}^{n} c_k \sin(kx)$
- If ϕ is even then $\phi(x) = c_0 + 2\sum_{k=1}^{n} c_k \cos(kx)$

1.3 Convergence of Fourier Series

Definition 1.3.1: Dirichlet Kernel

The function

$$K_n(\theta) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ik\theta} = \frac{1}{2\pi}\frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$$

Lemma 1.3.2

Let $\phi : [-\pi,\pi] \to \mathbb{R}$. Then

$$S_n(\phi)(x) = \int_{-\pi}^{\pi} K_n(x-z)\phi(z)\,dz$$

Theorem 1.3.3: Pointwise Convergence

Let $\phi \in C^1(\mathbb{R})$ be 2π periodic. Then

$$S_n(\phi)(x) \to \phi(x)$$

for all $x \in [-\pi,\pi]$.

Lemma 1.3.4: Decay of Fourier Coefficients

Assume that $\phi \in C^s(\mathbb{R})$ is 2π periodic where $s \in \mathbb{N}$. Then for $k \neq 0$,

$$\frac{|c_k|}{|k|^s} \leq \|\phi^{(s)}(x)\|_\infty$$

Theorem 1.3.5: Uniform Convergence

Let $\phi \in C^2(\mathbb{R})$ be 2π periodic. Then

$$\lim_{n\to\infty} S_n(\phi) \to \phi$$

uniformly as $n \to \infty$.