Hochschild Homology

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Abstract

Contents

1 Differential Graded Algebras

Recall that a graded ring is a ring that can be decomposed into a direct sum

$$R = \bigoplus_{i \in \mathbb{N}} R_i$$

of abelian groups, indexed by $\mathbb{N}$, such that $R_i R_j \subseteq R_{i+j}$. A graded algebra is then an algebra that is graded as a ring. If $x \in R_i$ then we say that x has degree $\deg(x) = i$.

Definition 1.0.1: Degree of a Graded Morphism

Let M, N be graded R-modules and let $f : M \to N$ be an R-module homomorphism. We say that f has degree i if $f(M_k) \subseteq N_{k+i}$.

Definition 1.0.2: Differential

Let M be a graded R-module. We say that an R-module homomorphism $d : M \to M$ is a differential if the following are true.

- d has degree 1
- $d \circ d = 0$

It is clear the grades of M form a chain complex with a differential. Depending on whether one wants a chain or a cochain complex, we can define the differential to go upwards in degree, meaning that $d(M_i) \subseteq M_{i+1}$.

Definition 1.0.3: Derivation of Degree k

Let A be a graded algebra. We say that a degree k, A-algebra homomorphism $d : A \to A$ a derivation of degree k if

$$d(xy) = (dx)y + (-1)^{k \deg(x)} x(dy)$$

for all $x, y \in A$.

If $k = 1$ and A is not graded (so that $A = A_0$), then one recovers the notion of a derivation in chapter 1.

Definition 1.0.4: Differential Graded Algebra

A differential graded algebra is a graded algebra A together with a differential $d : A \to A$ that is a derivation.

2 Hochschild Homology

2.1 Presimplicial Modules

Definition 2.1.1: Presimplicial Modules

Let M be a graded R-module. We say that M is a presimplicial module if there are R-module homomorphisms $d_i : M_n \to M_{n-1}$ for $0 \leq i \leq n$ such that

$$d_i \circ d_j = d_{j-1} d_i$$

for $0 \leq i < j \leq n$.

Proposition 2.1.2

Let $M = \bigoplus_{n\in\mathbb{N}} M_n$ be a presimplicial module. Then the components M_n of M together with $d = \sum_{i=0}^{n} (-1)^i d_i$ forms a chain complex.

Proposition 2.1.3

Let M, N be a presimplicial R-module. A morphism of presimplicial modules is a collection of maps $f_n : M_n \to N_n$ such that $f_{n-1} \circ d_i = d_i \circ f_n$.

2.2 Hoschild Homology

Definition 2.2.1: Hochschild Complex

Let M be an R-module. Define the Hoschild complex to be the chain complex $C(R, M)$ associate to the presimplicial module $\bigoplus_{n\in\mathbb{N}} M \otimes R^{\otimes n}$. This means that

$$\cdots \longrightarrow M \otimes R^{\otimes n+1} \xrightarrow{d} M \otimes R^{\otimes n} \xrightarrow{d} M \otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M \otimes R \longrightarrow M \longrightarrow 0$$

and d is defined as follows. Define the map $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$ as follows.

- If $i = 0$, then $d_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n$
- If $i = n$, then $d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}$
- Otherwise, then $d_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n-1}$

Now define $d = \sum_{i=0}^{n} (-1)^i d_i$.

Definition 2.2.2: Hochschild Homology

Let M be an R-module. Define the Hochschild homology of M to be the homology groups of the Hochschild complex $C(R, M)$:

$$H_n(R, M) = \frac{\ker(d : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1})}{\operatorname{im}(d : M \otimes R^{\otimes n+1} \to M \otimes R^{\otimes n})} = H_n(C(R, M))$$

If $M = R$ then we simply write

$$HH_n(R) = H_n(R, R) = H_n(C(R, R))$$

TBA: Functoriality.

Proposition 2.2.3

Let A be an R-algebra. Then $HH_n(A)$ is a $Z(A)$-module.

Proposition 2.2.4

Let A be an R-algebra. Then the following are true regarding the 0th Hochschild homology.

- Let M be an A-module. Then $H_0(A,M) = \frac{M}{\{am-ma \mid a\in A, m\in M\}}$
- The 0th Hochschild homology of A is given by $HH_0(A) = \frac{A}{[A,A]}$
- If A is commutative, then the 0th Hochschild homology is given by $HH_0(A) = A$.

Theorem 2.2.5

Let A be a commutative R-algebra. Then there is a canonical isomorphism

$$HH_1(A) \cong \Omega^1_{A/R}$$

2.3 Bar Complex

Definition 2.3.1: Enveloping Algebra

Let A be an R-algebra. Define the enveloping algebra of A to be

$$A^e = A \otimes A^{\text{op}}$$

Proposition 2.3.2

Let A be an R-algebra. Then any A, A-bimodule M equal to a left (right) A^e-module.

Definition 2.3.3: Bar Complex

Proposition 2.3.4

Let A be an R-algebra. The bar complex of A is a resolution of the A viewed as an A^e-module.

Theorem 2.3.5

Let A be an R-algebra that is projective as an R-module. If M is an A-bimodule, then there is an isomorphism

$$H_n(A,M) = \text{Tor}_n^{A^e}(M,A)$$