Advanced Linear Algebra

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Abstract

Linear algebra is at the heart of mathematics. Almost all areas of mathematics make some use of the notion of vector spaces and its properties. It began with the study of systems of linear equations.

Nowadays as we understand vector spaces independently from systems of linear equations, we will also treat the material differently. The first three chapter begins with the basis definitions: vector spaces and the maps between them called linear maps. Eigenvalues and eigenspaces will be an important invariant for vector spaces. Together with the aid of matrices, we will have a good grasp of how to write a given vector space in a simpler form. Chapter 4 will then improve on the further simplifying a given matrix so that we can read information from each easily.

The rest of the chapters will focus on particular properties of vector spaces and linear maps. They each correspond to an important class of matrices. For examples, quadratic form corresponds to matrices equivalent up to congruency while orthogonality of basis vectors give orthogonal matrices.

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1 The Rational Canonical Form

2 Operators on Vector Spaces

2.1 The Jordan-Chevalley Decompositions

Definition 2.1.1: Jordan-Chevalley Decompositions

Let k be a field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. A Jordan-Chevalley decomposition of V consists of $D, S \in \text{End}(V)$ such that the following are true.

- $T = D + S$
- D is diagonalizable
- S is nilpotent
- $SD = DS$

We note here that if we consider vector spaces as a k-module, then saying V semisimple is the same as saying V is diagonalizable.

Proposition 2.1.2

Let k be a field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. Then T admits a unique Jordan-Chevalley decomposition.

Proposition 2.1.3

Let k be an algebraically closed field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. Let $D, S \in \text{End}(V)$ be the Jordan-Chevalley decomposition of T. Then there exists $p, q \in k[x]$ such that $p(T) = D$ and $q(T) = S$.

2.2 The Trace of a Linear Map

Definition 2.2.1: The Trace of a Linear Map

Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$. Let $T : V \to V$ be a linear map. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis. Define the trace of T to be

$$\text{tr}(T) = \sum_{i=1}^{\dim(V)} \langle T(e_i), e_i \rangle$$

TBA: Invariance under choice of basis.

Definition 2.2.2: The Trace of a Matrix

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $M = (m_{i,j}) \in M_{n\times n}(k)$ be a matrix. Define the trace of M to be

$$\text{tr}(M) = \sum_{i=1}^{n} m_{i,i}$$

Proposition 2.2.3

Let V be an inner product space over $k = \mathbb{R}$ or $\mathbb{C}$. Let $T : V \to V$ be a linear map. Suppose that $M \in M_{n\times n}(k)$ represent the linear map T. Then

$$\text{tr}(T) = \text{tr}(M)$$

Lemma 2.2.4

Let $k = \mathbb{R}$ or $\mathbb{C}$. Let $A, B = (m_{i,j}) \in M_{n\times n}(k)$ be two square matrices. Then the following are true.

- tr : $M_{n\times n}(k) \to k$ is a linear map
- tr$(AB) =$ tr(BA)