Algebraic Topology 3

Labix

August 13, 2024

Abstract

Algebraic Topology 3 picks up from Algebraic Topology 2 and defines the final invariant for homotopy equivalence called the homotopy groups. We shall see that such homotopy groups is a complete invariant for CW-complexes up to homotopy equivalence. CW-complexes also benefit from the homotopy groups with the homotopy analogue of excision and a unique new theorem called the suspension theorem that implies stability of the homotopy groups.

The notes will then take a break from homotopy theory and redefine all the concepts (and some new ones) in the language of category theory. The point is that by looking into the picture, it is hoped that readers are able to understand how everything from Algebraic Topology 1-3 piece together into a coherent story.

Equipped with categorical constructions, we are then ready to tackle on the covering space analogue for higher homotopy groups called fibrations. They will provide a long exact sequence for computations of higher homotopy groups. However, fibrations are not just useful for understanding the higher homotopy groups. They serve as the fundamental object of study in general topology, as well as algebraic topology.

Finally, we will once again delve into a categorical setting and discuss a generalization of the fundamental group using category theory. Such a generalization also gives the full picture of how covering spaces interact with the fundamental group, as well as proving a more general version of Seifert-van Kampen theorem that now works when the intersection is not connected.

In the last chapter, we will put both homology and cohomology into a general framework, and define axioms that ensure that homology and cohomology as an invariant is unique up to having properties such as excision. It will also pave way to stable homotopy theory, of which one important theorem called Brown's representability theorem states that cohomology theories and spectra (object of study in stable homotopy theory) determines each other in a functorial way.

References:

- Notes on Algebraic Topology by Oscar Randal-Williams:
The first chapter gives a complete treatment of the first three sections of these notes, as well as providing the importance of fibrations on the higher homotopy groups. These notes are highly recommended to understanding the first three sections.

- Algebraic Topology by Allen Hatcher:
A more or less complete dictionary on all topics of these notes. However it is prone to the same problem in the sense that Hatcher's book is rather terse and definitions and parts of some theorems are scattered throughout the paragraphs rather than having a complete statement for reference. Nevertheless it is still the standard reference of the notes, albeit organized in a slightly different way.

- A non-visual proof that higher homotopy groups are abelian by Shintaro Fushida-Hardy:
This short piece of article proves that the higher homotopy groups are abelian in a purely algebraic way. Most geometric visualization of such a proof has the same underlying idea as the algebraic method.

Contents

1 The Higher Homotopy Groups

The journey of Algebraic Topology began with the fundamental group, where we assigned a group to every space functorially. The notion of fundamental group heavily involves the notion of homotopy and therefore is heavily related to the notion of homotopy. However, one realizes that even with Seifert-van Kampen theorem and the theory of covering spaces, it is not easy to compute the fundamental group of a space. This is party, but not wholly due to the fundamental group is in general not abelian. If we instead work in an abelian setting, one is able to distinguish two non-isomorphic groups simply by analysing the torsion subgroups. Therefore we refine the concept of the fundamental group and procured the notion of homology and cohomology. Both functorial invariants now produce graded abelian groups for each space, one for each dimension $n \in \mathbb{N}$. In the case of cohomology, there is a canonical ring structure on cohomology that interacts with the topology of the underlying space.

Now we turn to the final main invariant of topological spaces. The homotopy groups $\pi_n(X, x_0)$ serves as both a generalization of the fundamental group $\pi_1(X, x_0)$ in higher dimensions and a homotopic analogue to homology via the Hurewicz homomorphism

$$h : \pi_n(X) \to H_n(X)$$

It is a strong invariant that is closely related to the notion of homotopy, all the while having mostly abelian groups as its output. The trade off is that the homotopy groups are very hard to compute. Such trade off has led to the blossoming of Algebraic Topology in its fullest. For instance, stable homotopy theory stems from a crucial fact called the Freudenthal suspension theorem, which states that such a sequence

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots$$

eventually stabilizes for large enough n.

In this chapter we will closely study the nth homotopy groups such as its properties and develop tools to compute them.

1.1 The nth Homotopy Groups

We begin not with the definition of the homotopy groups, but rather a slight generalization of pointed spaces and maps between them.

Definition 1.1.1: Pairs of Space

Let X be a topological space. A pair of space is a pair (X, A) where $A \subseteq X$ is a subspace of X. A map of pairs $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

Definition 1.1.2: Homotopy between Maps of Pairs

Let $f, g : (X, A) \to (Y, B)$ be maps of pairs. A homotopy between f and g is a homotopy $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subseteq B$.

Definition 1.1.3: The nth Homotopy Groups

Let (X, x_0) be a pointed space. Define the nth homotopy group $\pi_n(X, x_0)$ to be

$$\pi_n(X, x_0) = \frac{\left\{\gamma : (I^n, \partial I^n) \to (X, \{x_0\}) \;\middle|\; \gamma \text{ is continuous}\right\}}{\simeq}$$

where we say that $f \simeq g$ if there exists a homotopy between f and g.

Notice that the definition coincides with that of the fundamental group when $n = 1$, and hence π_n is indeed a generalization.

Lemma 1.1.4

For any $n \in \mathbb{N}$, the two spaces $(I^n, \partial I^n)$ and (S^n, s_0) are homotopy equivalent.

Therefore an alternate viewpoint of the homotopy groups is instead the collection of maps from the pointed n-sphere to the space X quotient homotopy. Indeed an n-dimensional sphere has an n-dimensional hole enclosed by the sphere itself. Therefore in order to detect n-dimensional holes in a space, we are permitted to try and fit n-spheres into the space.

Spheres are also advantageous for the definition of π_n because spheres only has an n-dimensional hole and no other holes in any dimension. Therefore we are capturing the minimal amount of information on n-dimensional holes without producing excess data.

Now we have defined the set $\pi_n(X, x_0)$ for a pointed space to have the word group in its name. We will also need to procure a canonical group structure on the set $\pi_n(X, x_0)$. This will be similar with that of the fundamental group.

Definition 1.1.5: Concatenation

Let $n \geq 1$. Let (X, x_0) be a pointed space. Let $f, g : (I^n, \partial I^n) \to (X, x_0)$ be maps. Define the composition of f and g by the formula

$$(f \cdot g)(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_n) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $f, g \in \pi_n(X, x_0)$.

Notice that concatenation is really just the same concatenation between elements of the fundamental group but instead with more coordinates. The group structure on $\pi_n(X, x_0)$ uses concatenation and such a proof also uses the same homotopies as in Algebraic Topology 1, but with more coordinates.

Theorem 1.1.6

Let (X, x_0) be a pointed space and $n \geq 1$. The operation $\cdot$ on the equivalence classes in $\pi_n(X, x_0)$ is well defined and endows it with the structure of a group.

Proof. We first show that the operation is well defined on $\pi_n(X, x_0)$. Suppose that $f_1 \overset{\partial}{\simeq} g_1 : (I^n, \partial I^n) \to (X, x_0)$ via the homotopy H_1 and $f_2 \overset{\partial}{\simeq} g_2 : (I^n, \partial I^n) \to (X, x_0)$ via the homotopy H_2. Consider the map $H : I^n \times [0, 1] \to X$ defined by

$$H(x_1, \dots, x_n, t) = \begin{cases} H_1(2x_1, \dots, x_n, t) & \text{if } 0 \leq x_1 \leq \frac{1}{2} \\ H_2(2x_1 - 1, \dots, x_n, t) & \text{if } \frac{1}{2} \leq x_1 \leq 1 \end{cases}$$

Now when $t = 0$, we have that $H(x_1, \dots, x_n, 0) = f_1 \cdot f_2$. When $t = 1$, we have that $H(x_1, \dots, x_n, 1) = g_1 \cdot g_2$. Now notice that by definition of H_1 and H_2, if one of $x_1, \dots, x_n$ is equal to 0 or 1, then H_1 and H_2 is constant and maps to x_0. This means that H also has such property and hence H is a homotopy $(I, \partial I^n)$ to (X, x_0).

We now have an appropriate binary operation on $\pi_n(X, x_0)$. It is clearly associative since the composition of maps are associativity and one can re-parametrize homotopies with different traversal speeds. I claim that the constant map $e_{x_0} : (I, \partial I^n) \to (X, x_0)$ defined by $e_{x_0}(x) = x_0$ is the identity. Let $f : (I^n, \partial I^n) \to (X, x_0)$ be arbitrary. Define the homotopy from $e_{x_0} \cdot f$ to f by

$$H(x_1, \dots, x_n, t) = \begin{cases} e_{x_0}(x_1, \dots, x_n) = x_0 & \text{if } 0 \leq x_1 \leq \frac{1-t}{2} \\ f\left(\frac{2s+t-1}{t+1}\right) & \text{if } \frac{1-t}{2} \leq x_1 \leq 1 \end{cases}$$

A similar homotopy proves that $f \cdot e_{x_0} \simeq f$. For the inverse, I claim that $\overline{f} : (I^n, \partial I^n) \to (X, x_0)$ defined by $\overline{f}(1 - x_1, \dots, x_n)$ is the inverse of f. Indeed, define a homotopy from $f \cdot \overline{f}$ to e_{x_0} by

$$H(x_1, \dots, x_n, t) = \begin{cases} e_{x_0}(x_1, \dots, x_n) = x_0 & \text{if } 0 \leq x_1 \leq \frac{t}{2} \text{ or } \frac{1-t}{2} \leq x_1 \leq 1 \\ f(2x_1 - t, x_2, \dots, x_n) & \text{if } \frac{t}{2} \leq x_1 \leq \frac{1}{2} \\ \overline{f}(2s + t - 1) & \text{if } \frac{1}{2} \leq x_1 \leq \frac{1-t}{2} \end{cases}$$

□

However, what makes each $\pi_n(X, x_0)$ for $n \geq 2$ different from the fundamental group $\pi_1(X, x_0)$ is the abelian group structure on $\pi_n(X, x_0)$.

Theorem 1.1.7

Let (X, x_0) be a pointed space. Then the nth homotopy group

$$\pi_n(X, x_0)$$

together with concatenation is abelian.

Proof. Define a new operation $\star : \pi_n(X, x_0) \times \pi_n(X, x_0) \to \pi_n(X, x_0)$ by

$$[f] \star [g] = \begin{cases} f(t_1, 2t_2, \dots, t_n) & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(t_1, 2t_2 - 1, \dots, t_n) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Such an operation clearly also defines an abelian group structure on $\pi_n(X, x_0)$ using the same argument. Now I want to prove that

$$([f] * [g]) \star ([h] * [k]) = ([f] \star [h]) * ([g] \star [k])$$

This is true because

$$([f] * [g]) \star ([h] * [k]) = \begin{cases} f(2x_1, 2x_2, x_3, \dots, x_n) & \text{if } 0 \leq x_1, x_2 \leq \frac{1}{2} \\ g(2x_1, 2x_2 - 1, x_3, \dots, x_n) & \text{if } 0 \leq x_1 \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq x_2 \leq 1 \\ h(2x_1 - 1, 2x_2, x_3, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq \frac{1}{2} \\ k(2x_1, 2x_2 - 1, x_3, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1, x_2 \leq 1 \end{cases}$$

and

$$([f] \star [h]) * ([g] \star [k]) = \begin{cases} f(2x_1, 2x_2, x_3, \dots, x_n) & \text{if } 0 \leq x_1, x_2 \leq \frac{1}{2} \\ h(2x_1 - 1, 2x_2, x_3, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq \frac{1}{2} \\ g(2x_1, 2x_2 - 1, x_3, \dots, x_n) & \text{if } 0 \leq x_1 \leq \frac{1}{2} \text{ and } \frac{1}{2} \leq x_2 \leq 1 \\ k(2x_1, 2x_2 - 1, x_3, \dots, x_n) & \text{if } \frac{1}{2} \leq x_1, x_2 \leq 1 \end{cases}$$

which are entirely the same. Now I claim that $* = \star$. It is clear that both binary operations have the same identity element e_{x_0}. Now we have that

$$f * g = (f \star 1) * (1 \star g) = (f * 1) \star (1 * g) = f \star g$$

Finally, I claim that $*$ is commutative. We have that

$$f * g = (1 \star f) * (g \star 1) = (1 * g) \star (f * 1) = g \star f = g * f$$

Thus we conclude. □

The above technique is actually called the Eckmann-Hilton argument. In particular, it shows that concatenation of paths need not be defined via the first coordinate. Any choice of coordinate to perform concatenation will result in the same group structure.

Geometrically speaking,

1.2 Properties of Homotopy

The homotopy groups also satisfy functorial properties similar to the fundamental group and the (co)homology groups.

Theorem 1.2.1: Functoriality

Let (X, x_0) and (Y, y_0) be pointed spaces and let $f : (X, x_0) \to (Y, y_0)$ be a pointed map. Then the induced map
$$\pi_n(f) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$
defined by $[\gamma] \mapsto [f \circ \gamma]$ is a group homomorphism. Moreover, it satisfies the following functorial properties.

- If $g : (Y, y_0) \to (Z, z_0)$ is a pointed map then
$$\pi_n(g \circ f) = \pi_n(g) \circ \pi_n(f)$$
- If $\text{id}_{(X,x_0)} : (X, x_0) \to (X, x_0)$ is the identity map then
$$\pi_n(\text{id}_{(X,x_0)}) = \text{id}_{\pi_n(X,x_0)}$$

Proof. Firstly, let us show that it is a group homomorphism. Let $\gamma_1, \gamma_2 \in \pi_n(X, x_0)$. We have that
$$\pi_n(f)([\gamma_1] \cdot [\gamma_2]) = [f \circ (\gamma_1 \cdot \gamma_2)] = [f \circ \gamma_1 \cdot f \circ \gamma_2] = \pi_n(f)([\gamma_1]) \cdot \pi_n(f)([\gamma_2])$$
where the second equality is true because homotopies are preserved under function composition. It remains to show associativity and unitality.

- Associativity: We have that
$$\pi_n(g \circ f)([\gamma]) = [g \circ f \circ \gamma] = \pi_n(g)([f \circ \gamma]) = (\pi_n(g) \circ \pi_n(f))([\gamma])$$
- Unitality: We have that
$$\pi_n(\text{id}_{(X,x_0)})([\gamma]) = [\text{id}_{(X,x_0)} \circ \gamma] = [\gamma] = \text{id}_{\pi_n(X,x_0)}([\gamma])$$

And so we conclude. □

Similar to all other functorial properties we have seen throughout algebraic topology, a homeomorphism of spaces give an isomorphism on homotopy groups. Now that we know about category theory, we see that such a result does not depend on the definition of the homotopy groups or the (co)homology groups, but is in fact due to the functorial properties of each invariant.

Similar to (co)homology and the fundamental group, the homotopy groups are defined via a quotient with homotopy. Therefore we expect the homotopy groups to not be able to distinguish between homotopy equivalent spaces but not homeomorphic spaces.

Theorem 1.2.2: Homotopy Equivalence

Let $(X, x_0), (Y, y_0)$ be pointed spaces and $f, g : (X, x_0) \to (Y, y_0)$ be pointed maps. If f and g

are homotopic, then the induced maps

$$\pi_n(f) = \pi_n(g) : \pi_n(X, x_0) \to \pi_n(Y, y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then $\pi_n(f)$ is an isomorphism.

Proof. Let $[\gamma] \in \pi_n(X, x_0)$. Suppose that f and g are homotopic via $F : X \times I \to Y$. now define

$$H(x_1, \dots, x_n, t) = F(\gamma(x_1, \dots, x_n), t)$$

Then it is clear that $H(x_1, \dots, x_n, 0) = f \circ \gamma$ and $H(x_1, \dots, x_n, 1) = g \circ \gamma$. Thus $[f \circ \gamma] = [g \circ \gamma]$ and so we conclude that $\pi_n(f)([\gamma]) = \pi_n(g)([\gamma])$.

If f is a homotopy equivalence, then there exists $g : (Y, y_0) \to (X, x_0)$ such that $g \circ f \simeq \text{id}_{(X,x_0)}$ and $f \circ g \simeq \text{id}_{(Y,y_0)}$. By funtoriality and homotopy equivalence, we have that

$$\pi_n(g) \circ \pi_n(f) = \text{id}_{\pi_n(X,x_0)} \quad \text{and} \quad \pi_n(f) \circ \pi_n(g) = \text{id}_{\pi_n(Y,y_0)}$$

and so we conclude. $\square$

While the theory of covering spaces provided great insight for the structure of the fundamental group as well the space itself, the theory no longer works for higher homotopy groups due to the following proposition.

Proposition 1.2.3

Let (X, x_0) be a pointed space and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space. Then p induces isomorphisms

$$\pi_n(p) : \pi_n(\tilde{X}, \tilde{x}_0) \xrightarrow{\cong} \pi_n(X, x_0)$$

for all $n \geq 2$.

While covering spaces no longer prove to be useful for insights on the homotopy groups, fibrations will be the correct analogue of covering spaces to computing the higher homotopy groups. In fact, covering spaces themselves are also fibrations. We will see fibrations in later sections.

Similar to the fundamental group, changing the base point via a path induces isomorphisms on homotopy groups with the same space but different base point.

Theorem 1.2.4

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. Let $u : I \to X$ be a path from x_0 to x_1. Define the induced map

$$u_\# : \pi_n(X, x_0) \to \pi_n(X, x_1)$$

as follows. For $[\gamma] \in \pi_n(X, x_0)$ define $u_\#([\gamma])$ by first shrinking the domain of γ to a smaller concentric cube in I^n. Then inserting the path γ on each radical segment of the shell between the smaller cube and ∂I^n.

The construction of $u_\#$ is a group isomorphism. Moreover, it satisfies the following universal properties.

- If $v : I \to X$ is a path from x_1 to x_2 and $u \cdot v$ is the concatenation of these paths, then

$$(u \cdot v)_\# = u_\# \circ v_\#$$

- If c_{x_0} is the constant path from x_0 to x_0 then $(c_{x_0})_\#$ is the identity

Proposition 1.2.5

Let (X, x_0) and (X, x_1) be pointed spaces with the same base space. Let $u, v : I \to X$ be paths from x_0 to x_1. If u and v are homotopic relative to end points then the induced maps

$$u_\# = v_\# : \pi_n(X, x_0) \to \pi_n(X, x_1)$$

are equal.

This shows that if X is path connected, then $\pi_n(X, x_0)$ no longer depends on the choice of base point. Although there are no canonical isomorphisms between $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$, we still forget about the base point in this case and write the homotopy groups as $\pi_n(X)$.

Proposition 1.2.6

Let (X, x_0) be a pointed space and $f \in \pi_n(X, x_0)$. Let $u : I \to X$ be a loop on x_0. Then u induces a left action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ by the map

$$(u, \gamma) \mapsto u_\#(\gamma)$$

In particular, for $n \geq 2$, $\pi_n(X, x_0)$ is a $\mathbb{Z}\pi_1(X, x_0)$-module.

Proposition 1.2.7

Let X_i for $i \in I$ be a family of path connected spaces. Then there are isomorphisms

$$\pi_n\left(\prod_{i \in I} X_i\right) \cong \prod_{i \in I} \pi_n(X_i)$$

1.3 Relative Homotopy Groups

Definition 1.3.1: Triplets of Spaces

Let X be a topological space. A pointed pair of space is a triple (X, A_1, A_2) where $A_2 \subseteq A_1 \subseteq X$ are subspaces of X. A map between triplets of spaces $f : (X, A_1, A_2) \to (Y, B_1, B_2)$ is a map $f : X \to Y$ such that $f(A_1) \subseteq B_1$ and $f(A_2) \subseteq B_2$.

If $A_2 = \{x_0\}$ is a single point we say that (X, A, x_0) is a pointed pair of spaces.

Definition 1.3.2: Homotopy between Maps of Triplets

Let $f, g : (X, A_1, A_2) \to (Y, B_1, B_2)$ be maps triplets of spaces. A homotopy between f and g is a homotopy between $f : X \to Y$ and $g : X \to Y$, namely $H : X \times [0,1] \to Y$ such that $H(A_1 \times [0,1]) \subseteq B_1$ and $H(A_2 \times [0,1]) \subseteq B_2$.

Definition 1.3.3: The nth Relative Homotopy Groups

Let (X, A, x_0) be a pointed pair of space. Let $n \geq 2$. Regard I^{n-1} sitting inside I^n by $I^{n-1} = \{(x_1, \dots, x_n) \in I^n \mid x_n = 0\}$ and let $J^{n-1} = \partial I^n \setminus I^{n-1}$. Define the relative homotopy groups of the triple by

$$\pi_n(X, A, x_0) = \frac{\left\{\gamma : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0) \;\middle|\; \gamma \text{ is continuous}\right\}}{\simeq}$$

where we say that $f \simeq g$ if there exists a homotopy between f and g.

It is easy to see that $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$ so that homotopy groups are a special case of the relative homotopy groups.

Lemma 1.3.4

For any $n \in \mathbb{N}$, the two triplets $(I^n, \partial I^n, J^{n-1})$ and (D^n, S^{n-1}, s_0) are homotopy equivalent.

Theorem 1.3.5

Let (X, A, x_0) be a pointed pair of space. The composition law on definition 1.1.4 defines a group structure on $\pi_n(X, A, x_0)$ for $n \geq 2$. Moreover, $\pi_n(X, A, x_0)$ is abelian for $n \geq 3$.

1.4 Induced Maps of Relative Homotopy Groups

Theorem 1.4.1

Let (X, A, x_0) and (Y, B, y_0) be pointed pairs of spaces and $f : (X, A, x_0) \to (Y, B, y_0)$ a map. Then f induces a map on the relative homotopy groups

$$f_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

for $n \geq 2$ satisfying the following functorial properties:

- f_* is a group homomorphism
- If $g : (Y, B, y_0) \to (Z, C, z_0)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*$$

- If $\text{id}_{(X,A,x_0)}$ is the identity map on (X, A, x_0), then

$$(\text{id}_{(X,A,x_0)})_* = \text{id}_{\pi_n(X,A,x_0)}$$

Theorem 1.4.2

Let $(X, A, x_0), (Y, B, y_0)$ be pointed pairs of spaces and $f, g : (X, A, x_0) \to (Y, B, y_0)$ be pointed maps. If f and g are homotopic, then the induced maps

$$f_* = g_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$$

are equal. Moreover, if f is a homotopy equivalence, then f_* is an isomorphism.

TBA: change of base point isomorphisms.

Theorem 1.4.3: The Hurewicz Homomorphism

Let (X, A, x_0) be a pointed pair of space. Let u_n be a generator of $H_n(S^n) \cong \mathbb{Z}$. Then the map

$$h : \pi_n(X, A, x_0) \to H_n(X, A)$$

defined by $[f] \mapsto f_*(u_n)$ is a group homomorphism.

1.5 Long Exact Sequence in Homotopy Groups

Lemma 1.5.1: Compression Criterion

Let (X, A, x_0) be a pair of spaces with basepoint. Let $f : (D^n, S^{n-1}, *) \to (X, A, x_0)$ be a map. Then $[f] = [e_{x_0}] \in \pi_n(X, A, x_0)$ if and only if

$$(f : D^n \to X) \overset{S^{n-1}}{\simeq} (g : D^n \to X)$$

where g is any map such that $g(X) \subseteq A$.

Proof. Suppose that the second criterion is satisfied. Then it clearly shows that $[f] = [g] \in \pi_n(X, A, x_0)$. Let $r : D^n \times I \to D^n$ be a deformation retract from D^n to $* \in S^{n-1} \subset D^n$. Consider the map $g \circ r : D^n \times I \to X$. When $t = 0$, this is the map g. When $t = 1$, $g \circ r$ factors through $*$ and so becomes a map $* \to X$. In other words, it is the constant map e_{x_0}. Moreover, it $g \circ r$ has image in A and so in particular it sends S^{n-1} to A. Thus $g \circ r$ is a homotopy between e_{x_0} and g. We conclude that $[f] = [g] = [e_{x_0}]$.

Now suppose that $[f] = [e_{x_0}] \in \pi_n(X, A, x_0)$ is given by the homotopy $H : D^n \times I \to X$. This means that $H(D^n \times \{1\}) \subseteq \{x_0\} \subset A$ and $H(S^{n-1} \times I) \subset A$. Now $D^n \times I$ deformation retracts to the cup $D^n \times \{1\} \cup S^{n-1} \times I$ by radical projection from the center point of $D^n \times \{0\}$. Thus H can be converted into a map from $D^n \times \{1\} \cup S^{n-1} \times I$ to X. Then H is now a homotopy from f to a map $H(-, 1) : D^n \to X$ which has image in A, relative to S^{n-1}. Thus we conclude. ☐

Theorem 1.5.2

Let X be a space and A, B be subspaces of X such that $B \subseteq A \subseteq X$. Let $x_0 \in B$. Then there is a long exact sequence in relative homotopy groups:

$$\cdots \longrightarrow \pi_n(A, B, x_0) \xrightarrow{i_*} \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, B, x_0) \longrightarrow \cdots \longrightarrow \pi_1(X, A, x_0)$$

where $i : (A, B, x_0) \to (X, B, x_0)$ and $j : (X, B, x_0) \to (X, A, x_0)$ are the inclusions and $\partial : \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0)$ is given by $[\gamma] \mapsto [\gamma|_{I^{n-1}}]$

Proof. ☐

TBA: Naturality of the sequence.

Theorem 1.5.3

Let (X, A, x_0) be a pointed pair of spaces. The relative homotopy groups and (absolute) homotopy groups of (X, A, x_0) fit into a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial_{n+1}} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_n} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, x_0) \longrightarrow 0$$

where ∂_n is defined by $[f] \mapsto [f|_{I^{n-1}}]$ and i_* and j_* are induced by inclusions.

Note that even though at the end of the sequence group structures are not defined, exactness still makes sense: kernels in this case consists of elements that map to the homotopy class of the constant map.

1.6 n-Connectedness

Definition 1.6.1: n-Connected Space

Let X be a space. We say that it is n-connected if

$$\pi_k(X, x_0) = 0$$

for $0 \leq k \leq n$ and some $x_0 \in X$.

Note that $\pi_0(X, x_0)$ implies that X is path connected. Hence the notion of n-connectedness does not depend on the base point by the change of base point isomorphism. In particular, $\pi_k(X, x_0) = 0$ for $0 \leq k \leq n$ and some $x_0 \in X$ if and only if $\pi_k(X, x_0) = 0$ for $0 \leq k \leq n$ for all $x_0 \in X$. (Hatcher)

Definition 1.6.2: n-Connected Pair of Spaces

Let (X, A) be a pair of space. We say that it is n-connected if

$$\pi_k(X, A, x_0) = 0$$

for $0 \leq k \leq n$ and all $x_0 \in A$.

TBA: conditions in P.346 of Hatcher

Definition 1.6.3: Weakly Contractible

Let X be a space. We say that X is weakly contractible if

$$\pi_n(X) = 0$$

for all $n \geq 0$.

2 Weak Equivalences and CW-Complexes

2.1 Weak Homotopy Equivalence

Definition 2.1.1: Weak Homotopy Equivalence

We say that a map $f : X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on all homotopy groups π_n on any choice of base point.

TBA: compression lemma in Hatcher

Theorem 2.1.2

Let X, Y be spaces and let $f : X \to Y$ be a weak homotopy equivalence. Then f induces isomorphisms

$$f_* : H_n(X; G) \xrightarrow{\cong} H_n(Y; G) \quad \text{and} \quad f^* : H^n(Y; G) \xrightarrow{\cong} H^n(X; G)$$

for any group G and all $n \in \mathbb{N}$.

This theorem shows that the higher homotopy groups is not a weaker invariant than homology and cohomology. Indeed, the theorem states that if the all homotopy groups are isomorphic, then all their (co)homology groups will be isomorphic.

Proposition 2.1.3

Let X, Y be spaces and let $f : X \to Y$ be a weak homotopy equivalence. Then f induces bijections

$$[Z, X] \cong [Z, Y] \quad \text{and} \quad [Z, X]_* \cong [Z, Y]_*$$

for all CW-complexes Z.

2.2 Whitehead's Theorem

Theorem 2.2.1: Whitehead's Theorem

If X and Y are CW-complexes and $f : X \to Y$ is a weak homotopy equivalence, then f is a homotopy equivalence.

TBA: extension lemma in Hatcher.

Corollary 2.2.2

If X and Y are CW-complexes with $\pi_1(X) = \pi_1(Y) = 0$ and $f : X \to Y$ induces isomorphisms on homology groups H_n for all n, then f is a homotopy equivalence.

2.3 Cellular Approximations

Definition 2.3.1: Cellular Maps

Let X and Y be CW-complexes. A map $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n$ for all n, where X_n is the n-skeleton of X.

Definition 2.3.2: Cellular Approximations

Let X and Y be CW-complexes. We say that $f : X \to Y$ has a cellular approximations if f is homotopic to a cellular map $f' : X \to Y$.

Theorem 2.3.3: Cellular Approximation Theorem

Any map $f : X \to Y$ between CW-complexes has a cellular approximation $f' : X \to Y$. Moreover, if f is already cellular on a subcomplex $A \subseteq X$, then we can take $f'|_A = f|_A$.

Theorem 2.3.4: Relative Cellular Approximation

Any map $f : (X, A) \to (Y, B)$ between pairs of CW-complexes has a cellular approximation.

Corollary 2.3.5

Let $A \subset X$ be CW-complexes and suppose that all cells $X \setminus A$ have dimension larger than n. Then (X, A) is n-connected.

Corollary 2.3.6

Let X be a CW complex and let X^n be its n-skeleton. Then (X, X^n) is n-connected. Moreover, the inclusion $X^n \hookrightarrow X$ induces an isomorphism

$$\pi_k(X^n) \to \pi_k(X)$$

for $0 \leq k < n$ and a surjection for $k = n$.

2.4 CW Approximations

Definition 2.4.1: CW Approximation

Let X be a space. A CW approximation of X is a weak homotopy equivalence $f : Z \to X$ where Z is a CW complex.

The goal of this section is that every space has a CW approximation. The given homotopy equivalence makes this notion powerful because this means that for any space X, there exists a CW-complex such that X and Z are homotopy equivalent, and moreover, has isomorphic homotopy, homology and cohomology groups.

Definition 2.4.2: CW Model

Let (X, A) be a non-empty pair of CW-complexes. An n-connected CW model of (X, A) is an n-connected CW pair (Z, A) together with a map $f : Z \to X$ with $f|_A = \mathrm{id}_A$ such that

$$f_* : \pi_i(Z) \to \pi_i(X)$$

is an isomorphism for $i > n$ and an injection for $i = n$ for any choice of base point.

Theorem 2.4.3

For any non-empty pair (X, A) of CW-complexes, there exists an n-connected model (Z, A). Moreover, Z can be built from A by attaching cells of dimension greater than n.

Theorem 2.4.4

Every pair of spaces (X, A) has a CW approximation. Such a CW approximation is unique up to homotopy equivalence.

3 Main Results of Homotopy Theory on CW-Complexes

3.1 Excision for Homotopy Groups

Theorem 3.1.1: The Homotopy Excision Theorem

Let X be a CW-complex and A, B be sub complexes such that $X = A \cup B$ and $A \cap B \neq \emptyset$. If $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected for $m, n \geq 0$, then the map

$$\iota_* : \pi_i(A, A \cap B) \to (X, B)$$

induced by the inclusion $\iota : (A, A \cap B) \to (X, B)$ is an isomorphism for $0 \leq i < m + n$ and a surjection for $i = m + n$.

Proposition 3.1.2

Let (X, A) be a pair of r-connected CW complexes and let A be s-connected. Then the map

$$p_* : \pi_k(X, A) \to \pi_k(X/A)$$

induced by the quotient map $p : X \to X/A$ is an isomorphism for $0 \leq k \leq r + s$ and a surjection for $k = r + s + 1$.

3.2 Freudenthal Suspension Theorem

Definition 3.2.1: Reduced Suspension

Let (X, x_0) be a pointed space. Define the reduced suspension of X to be the space

$$\Sigma X = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)}$$

The reduced suspension defines a continuous map sending a space X to its reduced suspension ΣX.

Theorem 3.2.2: Freudenthal Suspension Theorem

Let X be an n-connected CW complex. Then for $0 \leq k \leq 2n$, the induced map

$$\Sigma_* : \pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism. For $k = 2n + 1$, Σ_* is a surjection.

We can keep on suspending the space and the maps. Indeed if X is n-connected then, by Freudenthal suspension theorem ΣX is $(n + 1)$-connected. We can then apply the suspension theorem again on ΣX and we see that $\Sigma^2 X$ is $(n + 2)$-connected.

Corollary 3.2.3

There is an isomorphism

$$\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1})$$

for all $n \geq k + 2$.

Proposition 3.2.4

Let X be a space. Let $k \in \mathbb{N}$. Then the the following sequence of suspensions

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \cdots$$

are eventually isomorphisms.

Proof. Let X be n-connected. There are two cases.

Let $k \leq 2n$. By Freudenthal suspension theorem, if $k \leq 2n$ then $\pi_k(X) \cong \pi_{k+1}(\Sigma X)$. Then ΣX is $(n+1)$-connected hence $\pi_{k+1}(\Sigma X) \cong \pi_{k+2}(\Sigma^2 X)$ is an isomorphism since $k+1 \leq 2n+2$. More generally, for $r \in \mathbb{N}$, $\Sigma^r X$ is $(r+n)$-connected hence

$$\pi_{k+r}(\Sigma^r X) \cong \pi_{k+r+1}(\Sigma^{r+1} X)$$

is an isomorphism since $k+r \leq 2n+2r$.

Now if $k > 2n$, then there exists $r \in \mathbb{N}$ such that $k+r \leq 2n+2r$. Such an r is given by say $k-2n$. Then by Freudenthal suspension theorem,

$$\pi_{k+r}(\Sigma^r X) \cong \pi_{k+r+1}(\Sigma^{r+1} X)$$

is an isomorphism. More generally, for $m \in \mathbb{N}$, $\Sigma^{r+m} X$ is $(r+m+n)$-connected hence

$$\pi_{k+r+m}(\Sigma^{r+m} X) \cong \pi_{k+r+m+1}(\Sigma^{r+m+1} X)$$

is an isomorphism since $k+r+m \leq 2n+2r+2m$. □

Definition 3.2.5: Stable Homotopy Groups

Let X be a space. Let $n \in \mathbb{N}$. Define the nth stable homotopy groups of X to be

$$\pi_n^s(X) = \operatorname*{colim}_{k \to \infty} \pi_{n+k}(\Sigma^k X)$$

3.3 Hurewicz's Theorem

Theorem 3.3.1: Hurewicz's Homomorphism

Let X be a path connected space. Then for any $n \in \mathbb{N}$, there is a group homomorphism

$$h_n : \pi_n(X) \to H_n(X)$$

called the Hurewicz homomorphism, defined as follows. Let $[u_n] \in H_n(S^n)$ be a canonical generator. Then $h_n([f]) = f_*(u_n)$.

Theorem 3.3.2: Hurewicz's Theorem

Let X be a space. Then the following are true regarding Hurewicz's homomorphism.

- Let $n \geq 2$. If X is $(n-1)$-connected, then $\widetilde{H}_k(X) = 0$ for all $0 \leq k < n$. Moreover, the Hurewicz homomorphism
$$h_n : \pi_n(X) \to H_n(X)$$
is an isomorphism. Moreover, h_{n+1} is a surjection.
- Let $n = 1$, then Hurewicz's homomorphism induces an isomorphism

$$\overline{h_1} : \pi_1(X)^{\text{ab}} \to H_1(X)$$

Theorem 3.3.3: Relative Hurewicz's Homomorphism

Let (X, A) be a pair of spaces. Then for any $n \geq 1$, there is a group homomorphism

$$h_n : \pi_n(X, A) \to H_n(X, A)$$

called the relative Hurewicz homomorphism, defined as follows. Let $[u_n] \in H_n(S^n, \partial S^n)$ be a canonical generator. Then $h_n([f]) = f_*(u_n)$.

Theorem 3.3.4: Relative Hurewicz's Theorem

Let (X, A) be a pair of spaces. Let $n \geq 2$. If X and A are path connected and (X, A) is $(n-1)$-connected, then $H_k(X, A) = 0$ for all $0 \leq k < n$. Moreover, the Hurewicz homomorphism

$$h_n : \pi_n(X, A, x_0) \to H_n(X, A)$$

is an isomorphism.

Theorem 3.3.5: Naturality of Hurewicz's Homomorphism

Let (X, x_0) and (Y, y_0) be pointed spaces and let $f : (X, x_0) \to (Y, y_0)$ be a map. Then the following diagram is commutative:

$$\begin{tikzcd} \pi_k(X, x_0) \arrow[r, "\pi_k(f)"] \arrow[d, "h_k"'] & \pi_k(Y, y_0) \arrow[d, "h_k"] \\ H_k(X) \arrow[r, "f_*"'] & H_k(Y) \end{tikzcd}$$

where h is the Hurewicz homomorphism. Moreover, a similar diagram is also commutative for the relative Hurewicz homomorphism.

The connection between the homotopy groups and the homology groups begs the question of whether there is a relationship between the homotopy groups and cohomology groups that is not implicit by the relation between homology and cohomology. This is answered in Stable Homotopy Theory, when we introduced Brown's representability theorem.

3.4 Eilenberg-MacLane Spaces

Definition 3.4.1: Eilenberg-MacLane Space

Let G be a group and $n \in \mathbb{N}$. We say that a space X is an Eilenberg-MacLane space of type $K(G, n)$ if

$$\pi_k(X) = \begin{cases} K(G, n) & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}$$

We often denote this space X directly by $X = K(G, n)$.

Proposition 3.4.2

Let G be a group. Then there exists a $K(G, 1)$-CW complex.

Theorem 3.4.3

Let G be an abelian group and $n \geq 2$. Then there exists a $K(G, n)$-CW complex. Moreover, it is uniquely determined by G and n.

The Eilenberg-Maclane spaces are a fundamental object of study in algebraic topology because it is a universal object. This is again part of Stable Homotopy Theory and is the same theorem that gives the connection between homotopy groups and cohomology groups.

We will not prove this here, but we will give the theorem: If G is an abelian group, then there are natural isomorphisms

$$H^n(X;G) \cong [X, K(G,n)]_*$$

that is natural in the following sense. If $f : X \to Y$ is a map, then there is a commutative diagram:

$$\begin{tikzcd} H^n(Y;G) \arrow[r, "f^*"] \arrow[d, "\cong"'] & H^n(X;G) \arrow[d, "\cong"] \\ {[Y, K(G,n)]_*} \arrow[r, "f^*"'] & {[X, K(G,n)]_*} \end{tikzcd}$$

4 The Categorical Viewpoint

Recall that the category of topological spaces **Top** is complete and cocomplete. This means that all kinds of limits and colimits exists in **Top**. We have already seen the product space and disjoint union with their universal property as a limit / colimit. There are also more constructs that can be recognized / defined in terms of the universal property.

4.1 Pullbacks and Pushouts

Definition 4.1.1: Adjunction Spaces

Let X, Y be spaces and $A \subseteq X$ a subspace. Let $f : A \to Y$ be a map. Define the adjunction space of X and Y to be

$$X \amalg_f Y = \frac{X \amalg Y}{a \sim f(a)}$$

together with the quotient topology.

Proposition 4.1.2

Let X, Y be spaces and $A \subseteq X$ a subspace of X. Let $f : A \to Y$ be a map. Then the adjunction space $X \amalg_f Y$ is a pushout of f and $i : A \to X$ in **Top**.

Proposition 4.1.3

Let X, Y be spaces with chosen base point x_0 and y_0 respectively. Then the wedge product

$$X \vee Y = X \amalg_f Y$$

is an adjunction space with $Z = \{x_0\}$ and map $f : Z \to Y$ defined by $f(x_0) = y_0$.

Definition 4.1.4: Mapping Cylinder

Let X, Y be spaces and let $f : X \to Y$ a map. Define the mapping cylinder of f to be

$$M_f = \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)} = (X \times I) \amalg_f Y$$

for $f : X \times \{1\} \cong X \to Y$ together with the quotient topology.

Lemma 4.1.5

Let X, Y be spaces and let $f : X \to Y$ be a map. Then Y is a deformation retract of M_f.

Definition 4.1.6: Mapping Cones

Let X, Y be spaces and let $f : X \to Y$ be a map. Define the mapping cone of f to be

$$C_f = \frac{(X \times I) \amalg Y}{(x, 1) \sim f(x), (x, 0) \sim (x', 0)}$$

Definition 4.1.7: The Mapping Path Space

Let X, Y be spaces and let $f : X \to Y$ be a map. Define the map $\pi : Y^I \to Y$ by $\pi(\phi) = \phi(0)$. Define the mapping path space to be

$$P_f = f^*(Y^I) = \{(x, \phi) \subseteq X \times Y^I \mid f(x) = \pi(\phi) = \phi(1)\}$$

The mapping path space satisfy the dual of the universal property of the mapping cylinder. In particular, it is a pullback in **Top**.

Proposition 4.1.8

Let X, Y be spaces and let $f : X \to Y$ be a map. Then the mapping path space P_f is the pullback of $\pi : Y^I \to Y$ and f in **Top**.

Definition 4.1.9: Mapping Fiber

Let X, Y be spaces and let $f : X \to Y$ be a map. Define the mapping fiber of f to be

$$F_f = \{(x, \phi) \in X \times Y^I \mid f(x) = \phi(1)\}$$

The mapping fiber is a natural dual of the mapping cone in **Top**.

4.2 The Category of Pointed Topological Spaces

Definition 4.2.1: The Category of Pointed Topological Spaces

Define the category of pointed topological spaces $\mathbf{Top}_*$ to consist of the following data.

- The objects are a pair (X, x_0) where X is a topological space and $x_0 \in X$ is a chosen base point.
- For (X, x_0) and (Y, y_0) two pointed spaces, the morphisms

$$\text{Hom}_{\mathbf{Top}_*}((X, x_0), (Y, y_0)) = \{f : X \to Y \mid f \text{ is continuous and } f(x_0) = y_0\}$$

  are the continuous maps from X to Y such that base points are preserved.
- Composition is defined as the composition of functions such that base point is preserved.

Proposition 4.2.2

Let (X, x_0) and (Y, y_0) be pointed spaces. Then the product and coproduct of the two spaces in $\mathbf{Top}_*$ are

$$(X \times Y, (x_0, y_0)) \quad \text{and} \quad (X \vee Y, x_0 = y_0)$$

respectively.

4.3 More Categories of Spaces

Definition 4.3.1: The Category of CW Complexes

Define the category of CW complexes **CW** to consist of the following data.

- The objects are CW complexes.

- For X and Y two CW complexes, the morphisms

$$\text{Hom}_{\mathbf{CW}}(X,Y)=\{f:X\to Y\mid f\text{ is continuous}\}$$

are the continuous maps from X to Y.

- Composition is defined as the composition of functions.

Define similarly the category $\mathbf{CW}_*$ of pointed topological spaces.

Definition 4.3.2: The Category of Pairs of Spaces

Define the category of pairs of topological spaces $\mathbf{Top}^2$ to consist of the following data.

- The objects are a pair (X,A) where X is a topological space $A\subseteq X$ is a subspace of X.
- For (X,A) and (Y,B) two pointed spaces, the morphisms

$$\text{Hom}_{\mathbf{Top}^2}((X,A),(Y,B))=\{f:X\to Y\mid f\text{ is continuous and }f(A)\subseteq B\}$$

are the continuous maps from X to Y such that subspaces are mapped to subspaces.

- Composition is defined as the composition of functions such that subspaces are mapped to subspaces.

Define similarly the category $\mathbf{CW}^2$ of pairs of CW complexes.

Definition 4.3.3: Homotopy Category of Spaces

Define the homotopy category of topological spaces $\mathbf{hTop}$ to consist of the following data.

- The objects are topological spaces.
- For X and Y two spaces, the morphisms

$$\text{Hom}_{\mathbf{CW}}(X,Y)=\{f:X\to Y\mid f\text{ is continuous}\}/\sim$$

are the homotopy classes of continuous maps from X to Y.

- Composition is defined as the composition of functions.

Define similar the homotopy category $\mathbf{hTop}_*$ of pointed topological spaces and pointed homotopy classes of maps.

4.4 Reduced Suspension and Loop Space Adjunction

Definition 4.4.1: Loop Spaces

Let X be a space with a chosen basepoint. Define the loop space of (X,x_0) to be

$$\Omega X=\text{Hom}_{\mathbf{Top}}(S^1,X)$$

together with the compact open topology. If X is pointed with $x_0\in X$ then we choose the constant loop c_{x_0} to be the base point of ΩX.

Lemma 4.4.2

Let G be an abelian group and let $n\in\mathbb{N}$. Then there is a homeomorphism

$$\Omega K(G,n)\cong K(G,n-1)$$

Theorem 4.4.3

The operations Σ and Ω define functors on **Top**, **Top**$_*$, **hTop** and **hTop**$_*$ as follows.

- Σ and Ω sends a pointed space (X, x_0) to
$$(\Sigma X, (x_0, 0)) \quad \text{and} \quad (\Omega X, c_{x_0})$$
respectively. The non-basepoint version is obtained by forgetting the base point.
- For the non homotopy categories, Σ and Ω sends a map $f : X \to Y$ to
$$\Sigma f : \Sigma X \to \Sigma Y \quad \text{and} \quad \Omega f : \Omega X \to \Omega Y$$
respectively defined by $\Sigma f([x, t]) = [f(x), t]$ and $\Omega f(\gamma) = f \circ \gamma$. It is in particular base point preserving.
- For the homotopy categories, Σ and Ω sends a homotopy class of maps $[X, Y]$ to
$$[\Sigma X, \Sigma Y] \quad \text{and} \quad [\Omega X, \Omega Y]$$
respectively given by the same formula as above. It is in particular also base point preserving.

The following theorem is also said to be the Freudenthal suspension theorem.

Theorem 4.4.4

Let Y be $(n-1)$-connected. Consider the reduced suspension functor $\Sigma : \mathbf{hTop}_* \to \mathbf{hTop}_*$. Then $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is bijective if $\dim(X) < 2n - 1$. Moreover, it is a surjection if $\dim(X) = 2n - 1$.

Theorem 4.4.5

The functor $\Sigma : \mathbf{hTop} \to \mathbf{hTop}$ is a left adjoint to the functor $\Omega : \mathbf{hTop} \to \mathbf{hTop}$. Explicitly, if X, Y are spaces, there is a bijection of sets
$$[\Sigma X, Y] \cong [X, \Omega Y]$$
that is natural in the following sense. If $f : X \to X'$ and $g : Y \to Y'$ are maps, then the following squares are commutative:
$$\begin{tikzcd} {[\Sigma X, Y]} \arrow[r, "\cong"] \arrow[d, "(\Sigma f)^*"'] & {[X, \Omega Y]} \arrow[d, "f^*"] \\ {[\Sigma X', Y]} \arrow[r, "\cong"'] & {[X', \Omega Y]} \end{tikzcd} \qquad \begin{tikzcd} {[\Sigma X, Y]} \arrow[r, "\cong"] \arrow[d, "g_*"'] & {[X, \Omega Y]} \arrow[d, "(\Omega g)_*"] \\ {[\Sigma X, Y']} \arrow[r, "\cong"'] & {[X, \Omega Y']} \end{tikzcd}$$

Theorem 4.4.6

The functor $\Sigma : \mathbf{hTop}_* \to \mathbf{hTop}_*$ is a left adjoint to the functor $\Omega : \mathbf{hTop}_* \to \mathbf{hTop}_*$. Explicitly, if X, Y are pointed spaces, there is a bijection of sets
$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*$$
that is natural in the following sense. If $f : X \to X'$ and $g : Y \to Y'$ are pointed maps, then the following squares are commutative:

$$\begin{tikzcd}
{[\Sigma X, Y]_*} \arrow[r, "\cong"] \arrow[d, "(\Sigma f)^*"'] & {[X, \Omega Y]_*} \arrow[d, "f^*"] \\
{[\Sigma X', Y]_*} \arrow[r, "\cong"'] & {[X', \Omega Y]_*}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[\Sigma X, Y]_*} \arrow[r, "\cong"] \arrow[d, "g_*"'] & {[X, \Omega Y]_*} \arrow[d, "(\Omega g)_*"] \\
{[\Sigma X, Y']_*} \arrow[r, "\cong"'] & {[X, \Omega Y']_*}
\end{tikzcd}$$

Switzer / Hatcher

Definition 4.4.7: Group Structure on Loop Spaces

Let X be a space. Define a group structure on ΩX as follows. Let $\cdot : \Omega X \times \Omega X \to \Omega X$ be defined as the concatenation: $(f, g) \mapsto f \cdot g$.

Proposition 4.4.8

Let X, Y be spaces. Then the group structure on ΩY endows $[X, \Omega Y]_*$ with a group structure defined as follows. The binary operation $+ : [X, \Omega Y]_* \times [X, \Omega Y]_* \to [X, \Omega Y]_*$ is defined by

$$([f], [g]) \mapsto [f + g]$$

where $f + g : X \to \Omega Y$ is defined by $(f + g)(x) = f(x) \cdot g(x)$.

Proposition 4.4.9

Let X, Y be spaces. Then for $n \geq 2$, the group

$$[X, \Omega^n Y]_*$$

is abelian.

By the set bijection $[\Sigma^n X, Y]_* \cong [X, \Omega^n Y]_*$, we can endow the structure of a group on $[\Sigma^n X, Y]_*$.

5 The Category of Compactly Generated Weakly Hausdorff Spaces

There is a huge inconvenience when working with **Top** and $\textbf{Top}_*$ and that is because in general, the mapping space X^Y only exists for Y when Y is imposed with extra condition. Such a space is important for a few reasons.

For that reason, it is better to work with a category in which the exponential object X^Y exists and lies inside such a category, while not restricting a wide number of classes of spaces so that the notion of homotopies still make sense and is well defined within such a category.

The category of compactly generated spaces has the following advantages:

- The smash product $X \wedge Y$ is associative. This means that there are natural isomorphisms $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
- X^Y now has a canonical topology.
- There is an adjunction between the smash product $- \wedge -$ and the mapping space $\text{Map}_*(-,-)$

While we have not encountered such notions yet, we also like to add that geometric realization of compact generated spaces preserves products.

Due to the huge advantages given to the smash product and mapping spaces, such advantages descend to the two important functors: the suspension and loopspace functors, making such a category an ideal universe for working with fibrations and cofibrations in the next section.

5.1 Compactly Generated Spaces

Definition 5.1.1: Compactly Generated Spaces

Let X be a space. We say that X is compactly generated (k-space) if for every set $A \subseteq X$, A is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Definition 5.1.2: Category of Compactly Generated Spaces

Define the category of compactly generated spaces **CG** to be the full subcategory of **Top** consisting of spaces that are compactly generated. In other words, **CG** consists of the following data:

- $\text{Obj}(\textbf{CG})$ consists of all spaces that are compactly generated.
- For $X, Y \in \text{Obj}(\textbf{CG})$, the morphisms are
$$\text{Hom}_{\textbf{CG}}(X,Y) = \text{Hom}_{\textbf{Top}}(X,Y)$$
- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces $\textbf{CG}_*$.

Definition 5.1.3: New k-space from Old

Let X be a space. Define $k(X)$ to be the set X together with the topology defined as follows: $A \subseteq X$ is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Lemma 5.1.4

Let X be a space. Then $k(X)$ is a compactly generated space.

Unfortunately $X \times Y$ may not be compactly generated even when X and Y are. But as it turns out, products do exists in $\mathcal{K}$ and are given by $k(X \times Y)$.

Proposition 5.1.5

Let X, Y be compactly generated spaces. Then the categorical product of X and Y in the category of compactly generated spaces is given by

$$k(X \times Y)$$

Proposition 5.1.6

Every CW complex is compactly generated.

5.2 Adjunctions in CG Spaces

Definition 5.2.1: The Mapping Space

Let X and Y be compactly generated. Define the mapping space of X and Y by

$$\text{Map}(X, Y) = Y^X = k(\text{Hom}_{\mathcal{K}}(X, Y))$$

Theorem 5.2.2

Let X, Y, Z be compactly generated. Then the functors $k(- \times Y) : \mathcal{K} \to \mathcal{K}$ and $\text{Map}(Y, -) : \mathcal{K} \to \mathcal{K}$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathcal{K}}(k(X \times Y), Z) \cong \text{Hom}_{\mathcal{K}}(X, \text{Map}(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}(k(X \times Y), Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

Aside from the adjunction between the product space and the mapping space, another major reason one considers compactly generated spaces is that the smash product gives another adjunction.

Definition 5.2.3: The Smash Product

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Define the smash product of the two pointed spaces to be the pointed space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

together with the point (x_0, y_0).

Proposition 5.2.4

Let X, Y, Z be compactly generated spaces with a chosen base point. Then the following are true.

- $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
- $X \wedge Y \cong Y \wedge X$

Theorem 5.2.5

The category **CG** of compactly generated spaces is a symmetric monoidal category with operator the smash product $\wedge : \mathbf{CG} \times \mathbf{CG} \to \mathbf{CG}$ and the unit S^0.

Note that this is not true if we do not restrict the spaces to the category of compactly generated spaces.

Lemma 5.2.6

Let X be a space. Then the reduced suspension and the smash product with the circle

$$\Sigma X \cong X \wedge S^1$$

are homeomorphic spaces.

Theorem 5.2.7

Let X, Y, Z be compactly generated with a chosen basepoint. Then the functors $- \wedge Y : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(Y, -) : \mathcal{K}_* \to \mathcal{K}_*$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathcal{K}_*}(X \wedge Y, Z) \cong \text{Hom}_{\mathcal{K}_*}(X, \text{Map}_*(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}_*(X \wedge Y, Z) \cong \text{Map}_*(X, \text{Map}_*(Y, Z))$$

Corollary 5.2.8

Let X be a compactly generated space with a chosen basepoint. Then there is a natural homeomorphism

$$\text{Map}_*(\Sigma X, Y) \cong \text{Map}_*(X, k(\Omega Y))$$

given by adjunction of the functors $- \wedge S^1 : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(S^1, -) : \mathcal{K}_* \to \mathcal{K}_*$.

5.3 Weakly Hausdorff Spaces

Definition 5.3.1: Weakly Hausdorff Spaces

A space X is said to be weakly Hausdorff if for any compact space M and map $f : M \to X$, the image $f(M)$ is closed in X.

Proposition 5.3.2

Every Hausdorff space is a weakly Hausdorff space.

5.4 The Working Category CGWH of Spaces

Definition 5.4.1: The Category of Compactly Generated Weakly Hausdorff Spaces

The category **CGWH** of compactly generated weakly Hausdorff spaces consists of the following data.

- The objects consists of compactly generated weakly Hausdorff spaces.
- The morphisms consists of continuous map between spaces.
- Composition is given by the composition of functions.

Define similarly the category $\mathbf{CGWH}_*$ of pointed compactly generated weakly Hausdorff spaces.

Theorem 5.4.2

The categories $\mathbf{CGWH}$ and $\mathbf{CGWH}_*$ is a closed monoidal category that is complete and cocomplete.

6 Fibrations and Cofibrations

6.1 Fibrations and The Homotopy Lifting Property

Definition 6.1.1: The Homotopy Lifting Property

Let $p : E \to B$ be a map and let X be a space. We say that p has the homotopy lifting property with respect to X if for every homotopy $H : X \times I \to B$ and a lift $\widetilde{H(-,0)} : X \to E$ of $H(-,0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \widetilde{H}"] & B
\end{tikzcd}$$

Definition 6.1.2: Fibrations

We say that a map $p : E \to B$ is a fibration if it has the homotopy lifting property with respect to all topological spaces X. We call B the base space and E the total space.

Definition 6.1.3: Pullbacks of a Fibration

Let $p : E \to B$ be a fibration and let $f : B' \to B$ be a continuous map. Define the pullback of p by f to be
$$f^*(E) = \{(b', e) \in B' \times E \mid f(b') = p(e)\}$$
together with the projection map $p_f : f^*(E) \to B'$.

Proposition 6.1.4

Let $p : E \to B$ be a fibration and let $f : B' \to B$ be continuous. Then the map $f^*(E) \to B'$ is a fibration. Moreover, the following diagram commutes:

$$\begin{tikzcd}
f^*(E) \arrow[r] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

where the top map is given by the projection to E.

Recall that we defined the mapping path space to be

$$P_f = f^*(Y^I) = \{(x, \phi) \subseteq X \times Y^I \mid f(x) = \pi(\phi) = \phi(1)\}$$

where $\pi : Y^I \to Y$ is defined as $\pi(\phi) = \phi(1)$. We can factorize any continuous map into a fibration and a homotopy equivalence through the mapping path space. Because we are working with the mapping path space here, we need to restrict our attention to compactly generated space.

Theorem 6.1.5

Let $f : X \to Y$ be a map with Y compactly generated. Then $\pi : P_f \to Y$ defined by $\pi(x, \phi) = \phi(1)$ is a fibration. Moreover, there exists a homotopy equivalence $h : X \to P_f$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "\exists h"'] & & Y \\ & P_f \arrow[ur, "\pi"'] & \end{tikzcd}$$

6.2 Cofibrations and The Homotopy Extension Property

Definition 6.2.1: The Homotopy Extension Property

Let $i : A \to X$ be a map and let Y be a space. Denote i_0 the inclusion map $A \times \{0\} \hookrightarrow A \times I$. We say that i has the homotopy extension property with respect to Y if for every homotopy $H : A \times I \to Y$ and every map $f : X \to Y$ such that

$$H \circ i_0 = f \circ i$$

there exists a homotopy $\widetilde{H} : X \times I \to Y$ such that the following diagram commute:

$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "H"'] \arrow[dd, "i \times \mathrm{id}"] \\ & Y & \\ X \times \{0\} \arrow[rr, hook] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\exists \widetilde{H}"'] \end{tikzcd}$$

The reason we had the entire digression on compactly generated spaces is because cofibrations can be redefined as a Eckmann-Hilton dual in the following form.

Lemma 6.2.2

Let X, Y be compactly generated. Let $i : A \to X$ be a map and let Y be a space. Denote $\pi_0 : Y^I \to Y$ to be the map $(\gamma : I \to Y) \mapsto \gamma(0)$ Then i has the homotopy extension property with respect to Y if and only if for all maps $f : X \to Y$ and $F : A \to Y^I$, there exists a map $\widetilde{F} : X \to Y^I$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "i"'] & Y^I \arrow[d, "\pi_0"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\widetilde{F}"] & Y \end{tikzcd}$$

Definition 6.2.3: Cofibrations

We say that a map $i : A \to X$ is a cofibration if it has the homotopy extension property for all spaces Y.

Definition 6.2.4: Pullbacks of a Cofibration

Let $i : A \to X$ be a cofibration and let $g : A \to C$ be a map. Define the pullback of i by g to be

$$f_*(X) = \frac{X \amalg C}{i(a) \sim g(a)}$$

together with the inclusion map $i_f : X \to f_*(X)$.

Proposition 6.2.5

Let $i : A \to X$ be a cofibration and let $g : A \to C$ be a map. Then the map $C \to f^*(X)$ is a cofibration. Moreover, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d] \\ X \arrow[r, "i_f"'] & f_*(X) \end{tikzcd}$$

where the map $C \to f_*(X)$ is the inclusion map.

Dual to the factorization of the mapping path space, we can factorize a map into a homotopy equivalence and a cofibration through the mapping cylinder

$$M_f = \frac{(X \times I) \amalg Y}{(x,0) \sim f(x)} = (X \times I) \amalg_f Y$$

Theorem 6.2.6

Let $f : A \to X$ be a map. Then the inclusion map $i : A \to M_f$ defined by $i(a) = [a, 0]$ is a cofibration. Moreover, there exists a homotopy equivalence $h : M_f \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & X \\ & M_f \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

6.3 Fibers and Cofibers

Definition 6.3.1: Fibers of a Fibration

Let $p : E \to B$ be a fibration. Define the fiber of p at $b \in B$ to be

$$E_b = p^{-1}(b)$$

The following definition is a supporting notion for our proof that fibers of a fibration are homotopy equivalent.

Definition 6.3.2: Induced Map of Fibers

Let $p : E \to B$. Let $\gamma : I \to B$ be a path from b_1 to b_2. Define the induced map of fibers of γ as follows: The map $H : E_{b_1} \times I \to B$ defined by $H(x,t) = \gamma(t)$ is a homotopy. Using the HLP of p, we obtain a lift:

$$\begin{tikzcd} E_{b_1} \times \{0\} \arrow[r, hook, "\widetilde{H(-,0)}"] \arrow[d, hook] & E \arrow[d, "p"] \\ E_{b_1} \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

Since $p \circ \widetilde{H}(x,t) = \gamma(t)$, we have that $\widetilde{H}(x,1) \in E_{b_2}$. The induced map of fibers is then the map

$$L_\gamma : E_{b_1} \to E_{b_2}$$

defined by $L_\gamma = \widetilde{H(-,1)}$

Lemma 6.3.3

Let $p : E \to B$ be a fibration. Let $\gamma : I \to B$ be a path from b_1 to b_2. Then the following are true regarding L_γ.

- If $\gamma \simeq \gamma'$ relative to boundary, then $L_\gamma \simeq L_{\gamma'}$.
- If $\gamma : I \to B$ and $\gamma' : I \to B$ are two composable paths, there is a homotopy equivalence $L_{\gamma\cdot\gamma'} \simeq L_{\gamma'} \circ L_\gamma$

Proof.

- Let $F : I \times I \to B$ be a homotopy equivalence from γ to γ'. Now consider the map $G : E_{b_1} \times I \times I \to B$ defined by $G(x, s, t) = F(s, t)$. Notice that $G(x, s, 0) = F(s, 0) = \gamma(s)$ and $G(x, s, 1) = F(s, 1) = \gamma'(s)$. Thus, we proceed as above by lifting $G(x, s, 0)$ and $G(x, s, 1)$ to obtain respectively $\widetilde{G(x, s, 0)}$ and $\widetilde{G(x, s, 1)}$ for which $\widetilde{G(x, 1, 0)} = L_\gamma$ and $\widetilde{G(x, 1, 1)} = L_{\gamma'}$. Now define $K : E_{b_1} \times I \times \partial I \to E$ by
$$K(x, s, t) = \begin{cases} \widetilde{G(x, s, 1)} & \text{if } t = 0 \\ G(x, s, 1) & \text{if } t = 1 \end{cases}$$
We now obtain a homotopy called $\widetilde{G} : E_{b_1} \times I \times I \to E$ by the homotopy lifting property:
$$\begin{tikzcd} X \times I \times \partial I \arrow[r, "K"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$
Now $\widetilde{G}(-, 1, -) : E_b \times I \to E$ is then a homotopy equivalence from $\widetilde{G}(x, 1, 0) = L_\gamma$ to $\widetilde{G}(x, 1, 1) = L_{\gamma'}$.
- We can repeat the above construction for γ and γ' to obtain homotopies $G : E_{b_1} \times I \to E$ and $G' : E_{b_1} \times I \to E$ such that when $t = 1$ we recover $\tilde{\gamma}$, $\tilde{\gamma'}$ and $\widetilde{\gamma \cdot \gamma'}$ respectively. Now the composition of G and G' by traversing along $t \in I$ with twice the speed gives precisely a lift of $\gamma \cdot \gamma'$ (one can check the boundary conditions). Thus $L_{\gamma\cdot\gamma'}$ obtained in this manner coincides up to homotopy equivalence to $L_{\gamma'} \circ L_\gamma$ by invoking part a).

$\square$

Theorem 6.3.4

Let $p : E \to B$ be a fibration. Let b_1 and b_2 lie in the same path component of B. Then there is a homotopy equivalence
$$E_{b_1} \simeq E_{b_2}$$
given by the lift of any path $\gamma : I \to B$ from b_1 to b_2.

Proof. Let $\gamma : I \to B$ be a path from b_1 to b_2. From the above, it follows that $L_{\overline{\gamma}} \circ L_\gamma \simeq \text{id}_{E_b}$ for any loop $\gamma : I \to B$ with basepoint b. We conclude that L_γ is a homotopy equivalence and so the fibers of $p : E \to B$ are homotopy equivalent. $\square$

Definition 6.3.5: Fiber of a Fibration

Let $p : E \to B$ be a fibration where B is path connected. Define the fiber of p to be a space F such that each fiber E_b for $b \in B$ is homotopy equivalent to.

Definition 6.3.6: Homotopy Fibers and Cofibers

Let $f:X\to Y$ be a map. Define the homotopy fiber of f to be the mapping fiber

$$F_f=\{(x,\phi)\in X\times Y^I\mid f(x)=\phi(1)\}$$

Define the homotopy cofiber of f to be the mapping cone

$$C_f=\frac{(A\times I)\amalg X}{(a,1)\sim f(a),A\setminus\{0\}}$$

Note the difference between homotopy fibers and the mapping path space. The latter is defined by considering the fibration $\pi:X^I\to X$ where $\pi(\phi)=\phi(0)$. But homotopy fibers are defined the end point $\phi(1)$. In fact, this is the main ingredient in proving that this notion is homotopy equivalent to the usual notion of fibers.

We have previously seen that the mapping fiber and the mapping cone of a map are dual notions in **Top**.

Proposition 6.3.7

Let $p:E\to B$ be a fibration. Then the homotopy fibers of p are homotopy equivalent to the fibers of p.

6.4 The Fiber and Cofiber Sequences

Definition 6.4.1: Path Spaces

Let (X,x_0) be a pointed space. Define the path space of (X,x_0) to be

$$PX=\{\phi:(I,0)\to(X,x_0)\mid\phi(0)=x_0\}=\text{Map}_*((I,0),(X,x_0))$$

together with the topology of the mapping space.

Theorem 6.4.2

Let X be a space. Then the following are true.

- The map $\pi:PX\to X$ defined by $\pi(\phi)=\phi(1)$ is a fibration with fiber ΩX
- The map $\pi:X^I\to X$ defined by $\pi(\phi)=\phi(1)$ is a fibration with fiber homeomorphic to PX.

We now write a fibration as a sequence $F\to E\to B$ for F the fiber of the fibration $p:E\to B$. This compact notation allows the following theorem to be formulated nicely.

Theorem 6.4.3

Let $f:X\to Y$ be a fibration with homotopy fiber F_f. Let $\iota:\Omega Y\to F_f$ be the inclusion map and $\pi:F_f\to X$ the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$\cdots\longrightarrow\Omega^2X\xrightarrow{\Omega^2 f}\Omega^2Y\xrightarrow{-\Omega\iota}\Omega F_f\xrightarrow{-\Omega\pi}\Omega X\xrightarrow{-\Omega f}\Omega_Y\xrightarrow{\iota}F_f\xrightarrow{\pi}X\xrightarrow{f}Y$$

where any two consecutive maps form a fibration. Moreover, $-\Omega f:\Omega X\to\Omega Y$ is defined as

$$(-\Omega f)(\zeta)(t)=(f\circ\zeta)(1-t)$$

for $\zeta\in\Omega X$.

There is then the dual notion of loop spaces and the corresponding sequence. Write a cofibration $f : A \to X$ with homotopy cofiber B as $B \to A \to X$.

Theorem 6.4.4

Let $f : X \to Y$ be a cofibration with homotopy cofiber C_f. Let $i : Y \to C_f$ be the inclusion map and $\pi : C_f \to C_f/Y \cong \Sigma X$ be the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

where any two consecutive maps form a cofibration. Moreover, $-\Sigma f : \Sigma X \to \Sigma Y$ is defined by

$$(-\Sigma f)(x \wedge t) = f(x) \wedge (1 - t)$$

Theorem 6.4.5

Let $p : E \to B$ be a fibration over a path connected space B with fiber F. Let $\iota : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{\iota_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

for $e_0 \in E$ and $b_0 = p(e_0)$. Moreover, p_* is an isomorphism.

6.5 Serre Fibrations

Definition 6.5.1: Serre Fibration

We say that a map $p : E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

Lemma 6.5.2

Every (Hurewicz) fibration is a Serre fibration.

Proof. This is true since Hurewicz fibrations satisfies the homotopy lifting property with respect to all topological spaces, including CW complexes. $\square$

Proposition 6.5.3

Let $p : E \to B$ be a fibration where B is path connected. Let F be the fiber of p. Let $b \in B$. Then the map

$$\cdot : \pi_1(B) \times E_b \to E_b$$

defined by $[\gamma] \cdot x = L_\gamma(x)$ induces an action of $\pi_1(B)$ on the homology groups $H_*(F; G)$ given by $[\gamma] \cdot [z] = (L_\gamma)_*([z])$ for any $g \in G$.

Proof. Notice first that such a map is well defined by lemma 6.3.3. Associativity follows from the second point of lemma 6.3.3. Identity follows the unique lift of the identity loop e_b that gives L_{e_b} is also the identity. $\square$

Theorem 6.5.4: Leray-Serre Spectral Sequence

Let $p : E \to B$ be a Serre fibration with fibre F and path connected B. Suppose that the action of $\pi_1(B)$ on $H_*(F; G)$ is trivial. Then there is a first quadrant homological spectral sequence starting with E^2 and weakly converging to $H_\bullet(E; \mathbb{Z})$. Explicitly, there is a convergence

$$E^2_{p,q} = H_p(B, H_1(F)) \Rightarrow H_{p+q}(E; \mathbb{Z})$$

6.6 Postnikov Towers

Definition 6.6.1: Postnikov Towers

Let X be a path connected space. A Postnikov tower is the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] & & & & & \\
\cdots \arrow[r] & X_n \arrow[r, "p_n"'] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r, "p_2"'] & X_1 \arrow[r, "p_1"'] & *
\end{tikzcd}$$

such that the following are true.

- The maps $X \to X_n$ for each $n \in \mathbb{N}$ induces isomorphisms $\pi_i(X) \cong \pi_i(X_n)$ for $i \leq n$.
- $\pi_i(X_n)$ for $i > n$.
- Each $p_n : X_n \to X_{n-1}$ for $n \in \mathbb{N}$ is a fibration with fiber $K(\pi_n(X), n)$.

Theorem 6.6.2

Suppose that there is an inverse system of spaces

$$\begin{tikzcd}
 & \lim_{n\to\infty} X_n \arrow[d] \arrow[dr] \arrow[drrr] \arrow[drrrr] & & & & & \\
\cdots \arrow[r] & X_n \arrow[r, "p_n"'] & X_{n-1} \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r, "p_2"'] & X_1 \arrow[r, "p_1"'] & *
\end{tikzcd}$$

The functor π_i for $i \in \mathbb{N}$ induces a cone in **Grp**. By definition of $\lim_\leftarrow \pi_i(X_n)$, there is a unique map

$$\lambda : \pi_i\left(\lim_{\leftarrow} X_n\right) \to \lim_{\leftarrow} \pi_i(X_n)$$

Then the following are true regarding λ.

- λ is surjective
- λ is injective if the maps $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ are surjective for sufficient large n.

Proposition 6.6.3

Let X be a connected CW complex. Then there exists a Postnikov tower for X.

Proposition 6.6.4

Let X be a connected CW complex. Choose a Postnikov tower of X. Then there is a weak homotopy equivalence

$$X \simeq \lim_{\leftarrow} X_n$$

so that X is a CW approximation of $\lim_{\leftarrow} X_n$.

7 The Fundamental Groupoid and Covering Space Theory

7.1 The Fundamental Groupoid

Definition 7.1.1: The Fundamental Groupoid

Let X be a space. Define the fundamental groupoid $\Pi_1 X$ of X to be the category with the following data.

- The objects are the points of X.
- Let $x, y \in X$. The morphisms of $\Pi_1 X$ are given by

$$\text{Hom}_{\Pi_1 X}(x, y) = \{\gamma : I \to X \mid \gamma(0) = x \text{ and } \gamma(1) = y \text{ is a path }\}/\sim$$

  where we say that two paths are equivalent if they are homotopic.
- Composition is defined by the concatenation of paths.

We have seen in Algebraic Topology 1 that composition of homotopy classes of paths are well defined.

Lemma 7.1.2

Let X be a space. Then $\Pi_1 X$ is a groupoid.

Proof. Every path in X has an inverse that lies in $\Pi_1 X$ given by reversing traversal of the path. □

Lemma 7.1.3

Let X be a space and $x_0 \in X$. Then there is a group isomorphism

$$\text{Hom}_{\Pi_1 X}(x_0, x_0) \cong \pi_1(X, x_0)$$

Proposition 7.1.4

Let $f : X \to Y$ be a continuous map. Then f induces a functor $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$ defined by

$$\Pi_1 f([\alpha]) = [f \circ \alpha]$$

on morphisms.

Proof. Direct from Algebraic Topology 1 due to the above group isomorphism. We have also seen that it is functorial in Algebraic Topology 1. □

Theorem 7.1.5

The fundamental groupoid defines a functor $\Pi_1 : \textbf{Top} \to \textbf{Grps}$ from the category of spaces to the category of groupoids with the following data.

- Π_1 sends each space X to $\Pi_1 X$
- Π_1 sends each continuous map $f : X \to Y$ to the functor $\Pi_1 f$

7.2 The Seifert-Van Kampen Theorem on Fundamental Groupoids

Definition 7.2.1: The Fundamental Groupoid of Subspaces

Let X be a space and $A \subseteq X$ a subspace. Define $\Pi_1 X[A]$ to be the full subcategory of $\Pi_1 X$ where the objects are A. Explicitly, $\Pi_1 X[A]$ consists of the following data.

- The objects of $\Pi_1 X[A]$ are the points of A.
- The morphisms are given by
$$\text{Hom}_{\Pi_1 X[A]}(x, y) = \text{Hom}_{\Pi_1 X}(x, y)$$
for any $x, y \in X$.
- Composition is inherited from $\Pi_1 X$.

Lemma 7.2.2

Let X be a space and $A \subseteq X$ a subspace of X such that every path component of X contains a point of A. Then the inclusion
$$\Pi_1 X[A] \to \Pi_1 X$$
of groupoids is an equivalence of categories.

Proof. The inclusion is already fully faithful since $\Pi_1 X[A]$ is a full subcategory. Now let $x \in X$. Let $a \in A$ lie in the same path component as x. Let $\alpha : I \to X$ be a path from x to a. Then the morphism $[\alpha] : x \to a$ of $\Pi_1 X$ is an isomorphism since $\Pi_1 X$ is a groupoid. Thus we conclude. $\square$

Corollary 7.2.3

Let X be a space. Then there is an equivalence of categories
$$\coprod_{[x_0] \in \pi_0(X)} B\pi_0(X, x_0) \cong \Pi_1 X$$

Proof. This is done by choosing A to contain exactly one point of each path component, and then by applying the isomorphism
$$\Pi_1 X[x_0] = B\text{Aut}_{\Pi_1 X}(x_0) = B\pi_1(X, x_0)$$
and the above lemma. $\square$

If X is path connected, then this shows that any choice of base point $x_0 \in X$ gives an equivalence of categories
$$B\pi_0(X, x_0) \cong \Pi_1 X$$
This translates roughly to the standard fact in Algebraic Topology that the fundamental group of a path connected space for any two base points are isomorphic. Indeed in the equivalence of categories exhibited, the former depends on the base point while the latter does not.

We need a lemma.

Lemma 7.2.4

Let $\mathcal{J}$ and $\mathcal{C}$ be categories and let $\mathcal{J}$ be the following category

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 1 \arrow[d, "i"] \\
2 \arrow[r, "j"'] & 3
\end{tikzcd}$$

such that $Y : \mathcal{J} \to \mathcal{C}$ is a pushout diagram. If $p : Y \Rightarrow X$ is a natural transformations such that p is a retraction, then $X : \mathcal{J} \to \mathcal{C}$ is also a pushout diagram.

Proof. Consider the following diagram:

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dd] \arrow[dr, shift right, "s_0"'] & & X_1 \arrow[dd, "X(i)"' near start] \arrow[dr, shift right, "s_1"'] & \\
& Y_0 \arrow[ul, shift right, "p_0"'] \arrow[rr] \arrow[dd] & & Y_1 \arrow[ul, shift right, "p_1"'] \arrow[dd, "Y(i)"'] \\
X_2 \arrow[rr, "X(j)" near start] \arrow[dr, shift right, "s_2"'] & & X_3 \arrow[dr, shift right, "s_3"'] & \\
& Y_2 \arrow[ul, shift right, "p_2"'] \arrow[rr, "Y(j)"'] & & Y_3 \arrow[ul, shift right, "p_3"']
\end{tikzcd}$$

This diagram is commutative by the following reasons.

- The front and back face of the square commutes since X and Y are functors and functors preserve commutative diagrams.
- The rest of the faces of the square commutes by the natural transformations p and s.

Let $Z \in \mathcal{C}$ such that there are maps $\lambda_1 : X_1 \to Z$ and $\lambda_2 : X_2 \to Z$ for which the maps

$$X_0 \to X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad X_0 \to X_2 \xrightarrow{\lambda_2} Z$$

are equal. Then in particular the two maps

$$Y_0 \to X_0 \to X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad Y_0 \to X_0 \to X_2 \xrightarrow{\lambda_2} Z$$

are equal. By commutativity of the cube, the two maps

$$Y_0 \to Y_1 \xrightarrow{p_1} X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad Y_0 \to Y_2 \xrightarrow{p_2} X_2 \xrightarrow{\lambda_2} Z$$

are equal. By the universal property of Y_3 as a pushout diagram, there exists a unique map $Y_3 \to Z$. such that the following diagram commutes:

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dd] \arrow[dr, shift right, "s_0"'] & & X_1 \arrow[dd, "X(i)"'] \arrow[dr, shift right, "s_1"'] \arrow[rrddd, bend left, "\lambda_1"] & & \\
& Y_0 \arrow[ul, shift right, "p_0"'] \arrow[rr] \arrow[dd] & & Y_1 \arrow[ul, shift right, "p_1"'] \arrow[dd, "Y(i)"'] & \\
X_2 \arrow[rr, "X(j)"] \arrow[dr, shift right, "s_2"'] \arrow[rrrrd, bend right, "\lambda_2"'] & & X_3 \arrow[dr, shift right, "s_3"'] & & \\
& Y_2 \arrow[ul, shift right, "p_2"'] \arrow[rr, "Y(j)"'] & & Y_3 \arrow[ul, shift right, "p_3"'] \arrow[dr, dashed, "\exists!"] & \\
& & & & Z
\end{tikzcd}$$

Since the retraction of a map is unique, s is unique. Also the map $Y_3 \to Z$ is unique by definition of pushout diagram. Hence there is a unique map $X_3 \to Y_3 \to Z$ so that X is a pushout diagram. ☐

Theorem 7.2.5: The Seifert-Van Kampen Theorem on Fundamental Groupoids

Let X be a space and $U, V \subseteq X$ an open cover of X. Let $A \subseteq X$ be a subspace such that every path connected component of U, V, X contains a point in A. Then the inclusions

$$\Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 U[U \cap A] \quad \text{and} \quad \Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 V[V \cap A]$$

give a pushout diagram to $\Pi_1 X[A]$. This means that the following diagram is a pushout:

$$\begin{tikzcd}
\Pi_1(U \cap V)[U \cap V \cap A] \arrow[r] \arrow[d] & \Pi_1 U[U \cap A] \arrow[d] \\
\Pi_1 V[V \cap A] \arrow[r] & \Pi_1 X[A]
\end{tikzcd}$$

where each arrow is an inclusions.

Proof. First assume that $X = A$. We want to show that for any groupoid $\mathcal{G} \in \mathbf{Grp}$ with maps $\Pi_1 U, \Pi_1 V \to \mathcal{G}$, there exists a unique map $\Pi_1 X \to \mathcal{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\Pi_1(U \cap V) \arrow[r] \arrow[d] & \Pi_1 U \arrow[d] \arrow[ddr, bend left, "f"] & \\
\Pi_1 V \arrow[r] \arrow[drr, bend right, "g"'] & \Pi_1 X \arrow[dr, dashed, "\exists! u"] & \\
& & \mathcal{G}
\end{tikzcd}$$

Define the functor $u : \Pi_1 X \to \mathcal{G}$ as follows. For each $x \in \Pi_1 X$, define

$$u(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}$$

This is well defined on $U \cap V$ since the outer square of the above diagram commutes. Depending on the path in X, there will be different constructions. Let $[\alpha]$ be a morphism in $\Pi_1 X$. If $\alpha : I \to X$ has image in U, then define $u([\alpha]) = f([\alpha])$. Similarly, define $u([\alpha]) = g([\alpha])$ if α has image in V.

Otherwise, by the Lebesgue covering theorem, there is a finite sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\alpha([a_i, a_{i+1}]) \subseteq U$ or V. Define $\alpha_i = \alpha \mid_{a_i, a_{i+1}}$. It is easy to see that

$$\begin{aligned} [\alpha] &= [\alpha \mid_{0,a_1}] \cdot [\alpha \mid_{a_1,a_2}] \cdots [\alpha \mid_{a_{n-1},1}] && \text{(Viewed as paths)} \\ &= [\alpha_{n-1}] \circ \cdots [\alpha_1] \circ [\alpha_0] && \text{(Viewed as morphisms in } \Pi_1 X\text{)} \end{aligned}$$

Then we can define $u(\alpha)$ as

$$u([\alpha]) = u([\alpha_{n-1}]) \circ u(\cdots [\alpha_1]) \circ u([\alpha_0])$$

where we have that

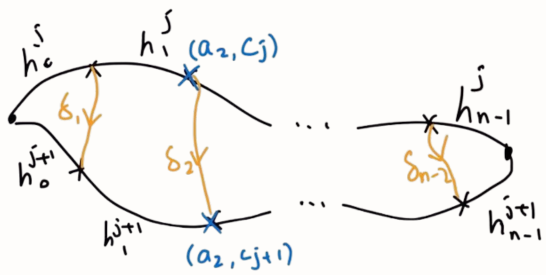
$$u([\alpha_i]) = \begin{cases} f([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq U \\ g([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq V \end{cases}$$

If u exists, then u must take the above form. Thus we have shown uniqueness.

For existence, we have to show that above construction of u is well defined. Let α, β be paths in X from x to y that are homotopic via the map $H : I \times I \to X$. We want to show that $u([\alpha]) = u([\beta])$. By the Lebesgue covering theorem, there is a grid in $I \times I$ where the x-axis is subdivided into $0 = a_0 < a_1 < \cdots < a_n = 1$ and the y-axis is subdivided into $0 = c_0 < c_1 < \cdots < c_k = 1$. such that H sends each rectangle with vertices $\{a_i, a_{i+1}, c_j, c_{j+1}\}$ to either U or V. Let $h^j = H(-, c_j) : I \to X$ so that $h^0 = \alpha$ and $h^k = \beta$. Define

$$\delta_i = H(\alpha_i, -) \mid_{[c_j, c_{j+1}]} : I \to X$$

which are paths from (a_i, c_j) to (a_i, c_{j+1}) in $I \times I$. Also define $h_i^j = h^j \mid_{[\alpha_i, \alpha_{i+1}]}$. Now we have the following which lies entirely in X:



Now we have that

$$\begin{aligned} u([h^j]) &= u([h_{n-1}^j]) \circ \cdots \circ u([h_0^j]) \\ &= u([h_{n-1}^{j+1} \circ \delta_{n-2}]) \circ u([\overline{\delta_{n-2}} \circ h_{n-2}^{j+1} \circ \delta_{n-3}]) \circ \cdots \circ u([\overline{\delta_1} \circ h_0^{j+1}]) \\ &= u([h_{n-1}^{j+1}]) \circ \cdots \circ u([h_0^{j+1}]) \\ &= u([h^{j+1}]) \end{aligned}$$

By induction, we conclude that

$$u([\alpha]) = u([h^0]) = u([h^1]) = \cdots = u([h^k]) = u([\beta])$$

Now suppose that $A \subseteq X$. By the above lemma, it is sufficient to show that the square for A is a retract of the square for X. Let $x \in U \cap V$ and $a_x \in A \cap U \cap V$ lying in the same path component as x. Choose a path $\alpha_x : I \to X$ from a_x to x with α_x being constant if $x \in A$. Do a similar choice for $x \in U \setminus (U \cap V)$ and $x \in V \setminus (U \cap V)$. Define $p_{U \cap V} : \Pi_1(U \cap V) \to \Pi_1(U \cap V)[U \cap V \cap A]$ defined by $x \mapsto a_x$ on objects and

$$[x \xrightarrow{\alpha} y] \mapsto \left(a_x \overset{[\alpha_x]}{\to} x \overset{[\alpha]}{\to} y \overset{[\alpha_y]}{\to} a_y \right)$$

and similarly for p_U and p_V. This defines the natural transformation p in lemma 5.3.4. We conclude by lemma 5.3.4. □

Take $A = \{x_0\}$ be a single point in $U \cap V$. Then this theorem shows that there is a pushout diagram

$$\begin{tikzcd} \pi_1(U \cap V, x_0) \arrow[r] \arrow[d] & \pi_1(U, x_0) \arrow[d] \\ \pi_1(V, x_0) \arrow[r] & \pi_1(X, x_0) \end{tikzcd}$$

in **Grp**, provided that A contains every path connected component of U, V, X. But A is just one point so the condition becomes that U, V, X and $U \cap V$ being path connected. Hence we recover the usual Seifert-Van Kampen theorem in Algebraic Topology 1.

7.3 Categorical Covering Space Theory

We end the section with a categorical approach of the Galois correspondence between covering spaces and the fundamental group.

Definition 7.3.1: Category of Covering Spaces of a Space

Let X be a space. Define the category $\text{Cov}(X)$ of covering spaces of X by the following.

- The objects are the covering spaces $p : \tilde{X} \to X$ of X
- For two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ of X, a morphism is a map $q : \tilde{X}_1 \to \tilde{X}_2$ such that following diagram commutes:

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "q"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

- Composition is given by the composition of functions.

Recall the category G-Set of G-sets for a group G to consist of the following data.

- The objects are sets which have a group action G.
- For two G-sets X and Y, a morphism is a G-equivariant function $f : X \to Y$. This means that
$$f(g \cdot x) = g \cdot f(x)$$
for all $g \in G$ and $x \in X$.
- Composition is given by the composition of functions.

Theorem 7.3.2

Let X be a connected and locally simply connected space. Let $x_0 \in X$. The the functor $F : \text{Cov}(X) \to \pi_1(X, x_0)$-Set defined by $\left(p : \tilde{X} \to X\right) \mapsto p^{-1}(x_0)$ and $\left(q : \tilde{X}_1 \to \tilde{X}_2\right) \mapsto q|_{p^{-1}(x_0)}$ gives an equivalence of categories

$$\text{Cov}(X) \cong \pi_1(X, x_0)\text{-Set}$$

8 Homology and Cohomology Theories

We have seen that the homotopy groups, the homology groups and the cohomology groups all satisfy a functorial property. This means that they can be considered as functors from the category of spaces to the category of some algebraic structures. It is meaningful to study all of them at once, and to compare different versions of homology and cohomology.

In general, there are different parameters of the (co)homology theories.

- Including "for CW Pairs" means that the theory is tailored for CW pairs. If we want a general theory for any topological spaces, we must add a new axiom so that weak equivalences gives isomorphism. This is true for CW complexes but not for arbitrary topological spaces.
- "Generalized" means that that dimension axiom is dropped. This enables wilder (co)homology theories such as (co)bordism to appear. "Ordinary" means that the dimension axiom is included. In the case that we restrict to CW complexes, this axiom ensures that all (co)homology theories of this type are isomorphic to each other.
- "Reduced" theories typically throw away the relative context in order to gain more concrete computations. A theorem says that determining a generalized (co)homology theory is the "same" determining a reduced theory and vice versa.

8.1 Generalized Homology Theories

The homotopy groups and the homology groups share many properties. The point of the axioms is to separate the notion of homotopy groups and homology groups. Indeed the homotopy groups together are a stronger invariant for spaces. Since homology is a functor, a natural question to ask is: what are the properties of singular / cellular / simplicial homology that make homology unique in the sense that only such a functor gives this unique identification of spaces through the invariant?

Definition 8.1.1: Generalized Homology Theory for CW Pairs

A Generalized Homology Theory is a collection of functors and natural transformations

$$h_n : \mathbf{CW}^2 \to \mathbf{Ab} \quad \text{and} \quad \delta_n : h_n \to h_n \circ F$$

where $F(X,Y) = (Y,\emptyset)$, for each $n \in \mathbb{N}$, such that the following are true.

- Homotopy Invariance: If $f \simeq g : (X,A) \to (Y,B)$ then
$$h_n(f) = h_n(g) : h_n(X,A) \to h_n(Y,B)$$
- Exactness: There exists an exact sequence
$$\cdots \longrightarrow h_{n+1}(X,A) \xrightarrow{\delta_{n+1}} h_n(A,\emptyset) \xrightarrow{h_n(i)} h_n(X,\emptyset) \xrightarrow{h_n(j)} h_n(X,A) \xrightarrow{\delta_n} h_{n-1}(A,\emptyset) \longrightarrow \cdots$$
where $i : (A,\emptyset) \to (X,\emptyset)$ and $j : (X,\emptyset) \to (X,A)$ are inclusions.
- Additivity: If $(X,A) = \coprod_{i\in I}(X_i,A_i)$, then the direct sum of the inclusion maps
$$\bigoplus_{i\in I} h_n(X_i,A_i) \cong h_n(X,A)$$
is an isomorphism
- Excision: If $\overline{E} \subseteq A^\circ \subseteq X$, then there is an isomorphism
$$h_n(X\setminus E, A\setminus E) \cong h_n(X,A)$$
induced by the inclusion map.

We mention for once and for all that the additivity axiom is required only when the CW complexes are non-finite. In particular, in order for the homology theory to be meaningful, we must restrict the underlying category of spaces to be finite CW complexes if one drops the additivity axiom.

Also, the homotopy invariance axiom means that homology descends to a functor $\mathbf{HoCW} \to \mathbf{Ab}$. Therefore some authors write the axiom implicit in the definition of the functors h_n, and then saying that a homology theory consists of functors $h_n : \mathbf{HoCW} \to \mathbf{Ab}$ instead.

Lemma 8.1.2

The excision axiom is equivalent to saying that $X = A^\circ \cup B^\circ$ with inclusion map $\iota : (B, A \cap B) \to (X, A)$ implies $h_n(\iota) : h_n(B, A \cap B) \to h_n(X, A)$ is an isomorphism.

Definition 8.1.3: Generalized Homology Theory for Spaces

A Generalized Homology Theory is a collection of functors

$$h_n : \mathbf{Top}^2 \to \mathbf{Ab} \qquad \text{and} \qquad \delta_n : h_n(X, Y) \to h_{n-1}(Y, \emptyset)$$

satisfying the firs four axioms together with the following.

- Weak Equivalence: If $f : (X, A) \to (Y, B)$ is a weak equivalence, then

  $$f_* : h_n(X, A) \to h_n(Y, B)$$

  is an isomorphism.

By adding on the axiom of weak equivalence and the fact that every space admits a weak equivalence to a CW complex, we can see that the two theories are the same.

Theorem 8.1.4

Any generalized homology theory on $\mathbf{Top}^2$ determines and is determined by a generalized homology theory on $\mathbf{CW}^2$.

However, note that in this case some of the working homology theories are not a generalized homology theory in this sense (when we encounter the dual notion, sheaf cohomology is not a generalized cohomology theory).

Definition 8.1.5: Ordinary Homology Theory

Let G be an abelian group. If a generalized homology theory (h_n, δ_n) in addition satisfies

- Dimension:

  $$h_n(*) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Then h_n is called an ordinary homology theory.

Theorem 8.1.6: Eilenberg-Steenrod Uniqueness Theorem

Let $T : (h_n, \delta_n) \to (h'_n, \delta'_n)$ be a natural transformation of generalized homology theories defined on $\mathbf{CW}^2$ such that $h_n(*) \cong h'_n(*)$, then T is a natural isomorphism

$$(h_n, \delta_n) \cong (h'_n, \delta'_n)$$

8.2 Reduced Homology Theory

In Algebraic Topology 2, we have also encountered the notion of reduced singular homology. This is derived directly from singular homology, where one simply defines reduced singular homology by also considering the augmented chain complex. Such a construction can easily be extended to arbitrary homology theories: Indeed there is no need for a topological argument in the definition of reduced singular homology.

In fact, this subsection will also prove a unification theorem for the following four homolog theories:

- Generalized homology theory for CW complexes
- Generalized homology theory for spaces
- Reduced homology theory for CW complexes
- Reduced homology theory for spaces

and they can be unified only because of the unnatural condition posed for homology theory for spaces. The axiom of weak equivalences enables the use of CW approximations.

Definition 8.2.1: Reduced Homology Theory for CW Complexes

A reduced Homology Theory is a collection of functors

$$\widetilde{h}_n : \mathbf{CW}_* \to \mathbf{Ab}$$

that satisfies the following.

- Homotopy Invariance: If $f \simeq g : (X, x_0) \to (Y, y_0)$ then
$$\widetilde{h}_n(f) = \widetilde{h}_n(g) : \widetilde{h}_n(X, x_0) \to \widetilde{h}_n(Y, y_0)$$
- Exactness: If X is a CW-complex and $A \subseteq X$ and $x_0 \in A$, then there is a short exact sequence
$$\widetilde{h}_n(A, x_0) \xrightarrow{\iota_*} \widetilde{h}_n(X, x_0) \xrightarrow{p_*} \widetilde{h}_n(X/A, *)$$
where $\iota : A \to X$ is the inclusion and $p : X \to X/A$ is the projection.
- Suspension: There is a natural isomorphism
$$\Sigma : \widetilde{h}_n(X, x_0) \overset{\cong}{\longleftrightarrow} \widetilde{h}_{n+1}(\Sigma X, *)$$
- Additivity: If $X = \coprod_{i\in I} X_i$, then the direct sum of the inclusion maps
$$\bigoplus_{i\in I} \widetilde{h}_n(X_i) \cong \widetilde{h}_n(X)$$
is an isomorphism

Lemma 8.2.2

Let $\widetilde{h}_n : \mathbf{CW}_* \to \mathbf{Ab}$ be a reduced homology theory. Then

$$\widetilde{h}_n(*) = 0$$

Theorem 8.2.3

Let $h_n : \mathbf{CW}^2 \to \mathbf{Ab}$ be a generalized homology theory for CW complexes. Define a collection of functors
$$\widetilde{h}_n : \mathbf{CW}_* \to \mathbf{Ab}$$
by $(X, x_0) \mapsto \widetilde{h}_n(X, x_0) = h_n(X, \{x_0\})$. Then $\widetilde{h}_n$ defines a reduced homology theory for CW complexes.

Theorem 8.2.4

Let $\widetilde{h}_n : \mathbf{CW}_* \to \mathbf{Ab}$ be a reduced homology theory for CW complexes. Define a collection of functors
$$\widetilde{h}_n : \mathbf{CW}^2 \to \mathbf{Ab}$$
by $(X, A) \mapsto \widetilde{h}_n(X/A, * = A)$ and a collection of natural transformations
$$\delta_n : h_n(X, Y) \to (Y, \emptyset)$$
by ??? Then h_n and δ_n defines a generalized homology theory for CW complexes.

Theorem 8.2.5

Any generalized homology theory for CW complexes determines and is determined by a reduced homology theory for CW complexes.

We have now showed that reduced homology theory and generalized homology theory for CW complexes really are the same thing:

$$\begin{array}{c}\text{Reduced Homology Theory}\\ \text{for CW Complexes}\end{array} \overset{1:1}{\longleftrightarrow} \begin{array}{c}\text{Generalized Homology Theory}\\ \text{for CW Complexes}\end{array}$$

Let us now show the same for homology theories for spaces in general and moreover, establish such a relation between spaces and CW complexes.

Definition 8.2.6: Reduced Homology Theory for Spaces

A reduced Homology Theory for spaces is a collection of functors
$$\widetilde{h}_n : \mathbf{Top}_* \to \mathbf{Ab}$$
that the above axioms (Homotopy Invariance, Exactness, Suspension, Additivity) and the following:

- Weak Equivalences: If $f : (X, x_0) \to (Y, y_0)$ is a weak equivalence then
$$f_* : \widetilde{h}_n(X, x_0) \to \widetilde{h}_n(Y, y_0)$$
is an isomorphism for all n.

Theorem 8.2.7

Any reduced homology theory for spaces determines and is determined by a reduced homology theory on spaces.

Theorem 8.2.8

Any reduced homology theory for spaces determines and is determined by a generalized homology theory on spaces.

Thus we have showed that under the conditions of all the given axioms of all such homology theories, we really just have the same thing:

$$\begin{matrix}\text{Generalized Homology Theory} \\ \text{for Spaces}\end{matrix} \xleftrightarrow{1:1} \begin{matrix}\text{Reduced Homology Theory} \\ \text{for Spaces}\end{matrix} \xleftrightarrow{1:1} \begin{matrix}\text{Reduced Homology Theory} \\ \text{for CW Complexes}\end{matrix} \xleftrightarrow{1:1} \begin{matrix}\text{Generalized Homology Theory} \\ \text{for CW Complexes}\end{matrix}$$

We once again note that such a bijection is only possible when we introduce the weak equivalence axiom for homology theories on spaces. Such an axiom in fact restricts the amount of valid homology theories, but it is only through this axiom that we can declare all such homology theories are essentially determined by one another.

Some authors indeed does not require the weak equivalence axiom to exist. In such cases the bijection between generalized homology theories and reduced homology theories can be constructed, but there is no passage between homology of CW complexes and homology of spaces in general.

8.3 Cohomology Theories

Definition 8.3.1: Generalized Cohomology Theory for CW Pairs

A Generalized cohomology theory is a collection of contravariant functors

$$h^n : \mathbf{CW}_2 \to \mathbf{Ab} \qquad \text{and} \qquad \delta^n : h^n(A, \emptyset) \to h^{n+1}(X, A)$$

satisfying the following.

- Homotopy Invariance: If $f \simeq g : (X, A) \to (Y, B)$ then
$$h^n(f) = h^n(g) : h^n(X, A) \to h^n(Y, B)$$
- Exactness: If X is a CW-complex and $A \subseteq X$, then there is a short exact sequence
$$\cdots \longrightarrow h^n(X/A) \longrightarrow h^n(X) \longrightarrow h^n(A) \xrightarrow{\partial_n} h^{n+1}(X/A) \longrightarrow h^{n+1}(X) \longrightarrow \cdots$$
- Additivity: If $(X, A) = \coprod_{i\in I}(X_i, A_i)$, then the direct sum of the inclusion maps
$$\bigoplus_{i\in I} h^n(X_i, A_i) \cong h^n(X, A)$$
is an isomorphism
- Excision: If $\overline{E} \subseteq A^\circ \subseteq X$, then
$$h^n(X \setminus E, A \setminus E) \cong h^n(X, A)$$
induced by the inclusion map

Definition 8.3.2: Generalized Cohomology Theory

A Generalized cohomology theory is a collection of contravariant functors

$$h^n : \mathbf{Top}_2 \to \mathbf{Ab} \qquad \text{and} \qquad \delta^n : h^n(A, \emptyset) \to h^{n+1}(X, A)$$

satisfying the above first four axioms and the following.

- Weak Equivalence: If $f : (X, A) \to (Y, B)$ is a weak equivalence, then
$$f_* : h^n(Y, B) \to h^n(X, A)$$
is an isomorphism.

Definition 8.3.3: Reduced Cohomology Theory for CW Pairs

A reduced cohomology theory is a collection of contravariant functors

$$\widetilde{h}^n : \mathbf{CW} \to \mathbf{Ab} \quad \text{and} \quad \delta^n : \widetilde{h}^n(A, \emptyset) \to \widetilde{h}^{n+1}(X, A)$$

satisfying the following.

- Homotopy Invariance: If $f \simeq g : X \to Y$ then
$$\widetilde{h}^n(f) = \widetilde{h}^n(g) : \widetilde{h}^n(X) \to \widetilde{h}^n(Y)$$
- Exactness: There exists a short exact sequence
$$\cdots \longrightarrow \widetilde{h}^n(X, A) \xrightarrow{\widetilde{h}_n(\pi)} \widetilde{h}^n(X) \xrightarrow{\widetilde{h}_n(\iota)} \widetilde{h}^n(A) \xrightarrow{\delta_n} \widetilde{h}^{n+1}(X, A) \xrightarrow{\widetilde{h}_{n+1}(\pi)} \widetilde{h}^{n+1}(X) \longrightarrow \cdots$$
where $\iota : A \to X$ is the inclusion and $\pi : X \to X/A$ is the projection.
- Additivity: If $X = \coprod_{i \in I} X_i$, then the direct sum of the inclusion maps
$$\bigoplus_{i \in I} \widetilde{h}^n(X_i) \cong \widetilde{h}^n(X)$$
is an isomorphism

Lemma 8.3.4

Let $\widetilde{h}_n : \mathbf{CW} \to \mathbf{Ab}$ be a reduced homology theory. Then

$$\widetilde{h}_n(*) = 0$$

Proposition 8.3.5

Let $\widetilde{h}^n : \mathbf{CW} \to \mathbf{Ab}$ be a reduced cohomology theory. Then there is a natural isomorphism

$$\widetilde{h}_{n+1}(\Sigma X) = \widetilde{h}_n(X)$$

TBA: Unreduced = reduced.

9 Bonus?

Theorem 9.0.1

Let X be a space. Let the following be a sequence

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X$$

of subspaces. Let G be an abelian group. Then the following data

- $A_{p,q} = H_{p+q}(X_p; G)$
- $E_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$
- $i : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_{p+1}; G) = A_{p+1,q-1}$ (degree $(1, -1)$)
- $j : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_p, X_{p-1}; G) = E_{p,q}$ (degree $(0, 0)$)
- $k : H_{p+q}(X_p, X_{p-1}; G) = A_{p,q} \to H_{p+q-1}(X_{p-1}; G) = A_{p-1,q}$ (degree $(-1, 0)$)

defines an exact couple and hence a spectral sequence with E^1 page given by

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$$

where the differential $d : E^1_{p,q} \to E^1_{p-1,q}$ is given by the composition

$$H_{p+q}(X_p, X_{p-1}; G) \xrightarrow{k} H_{p+q-1}(X_{p-1}; G) \xrightarrow{j} H_{p+q-1}(X_{p-1}, X_{p-2}; G)$$

The E_1 page of such a spectral sequence is given by

$$\begin{array}{ccccccc}
\cdots & \longleftarrow & \cdots & \longleftarrow & \cdots & \longleftarrow & \cdots \\
H_3(X_1, X_0; G) & \longleftarrow & H_4(X_2, X_1; G) & \longleftarrow & H_4(X_3, X_2; G) & \longleftarrow & \cdots \\
H_2(X_1, X_0; G) & \longleftarrow & H_3(X_2, X_1; G) & \longleftarrow & H_4(X_3, X_2; G) & \longleftarrow & \cdots \\
H_1(X_1, X_0; G) & \longleftarrow & H_2(X_2, X_1; G) & \longleftarrow & H_3(X_3, X_2; G) & \longleftarrow & \cdots
\end{array}$$

Things get interesting when we choose X to be a CW complex and we choose the filtration of X by the skeleton of X. Recall that we have the formula

$$H_{p+q}(X_p, X_{p-1}; G) \cong \begin{cases} C_p^{\text{CW}}(X; G) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the E^1 page is only left with a chain complex at $q = 0$.
Let us also compute the derived couple of this exact couple or in other words, the E^2 page of the spectral sequence. This is more intuitive then the one thinks about on the definition of the derived couple. The $E^2_{p,q}$ slot is simply the homology of the chain complex at the (p, q)th slot. The direction of the maps of the E^2 page depends not on the choice of spectral sequence at all (In fact, the direction only depends on the page). Now in our case, the homology can be given by a known construct:

$$E^2_{p,q} = \frac{\ker(d : H_{p+q}(X_p, X_{p-1}; G) \to H_{p+q-1}(X_{p-1}, X_{p-2}; G))}{\text{im}(d : H_{p+q+1}(X_{p+1}, X_p; G) \to H_{p+q}(X_p, X_{p-1}; G))} = H^{\text{CW}}_{p+q}(X; G)$$

Since the direction of the maps are now diagonal and when $q \neq 0$ we have $E^2_{p,q} = 0$, all maps in E^2 are 0 and we are left with

$$H_0^{\text{CW}}(X; G) \qquad H_1^{\text{CW}}(X; G) \qquad H_2^{\text{CW}}(X; G) \qquad H_3^{\text{CW}}(X; G) \qquad \cdots$$