Algebraic Topology 4

Labix

October 9, 2024

Abstract

References:

- Notes on Algebraic Topology by Oscar Randal-Williams:
  The first chapter gives a complete treatment of the first three sections of these notes, as well as providing the importance of fibrations on the higher homotopy groups. These notes are highly recommended to understanding the first three sections.

- Algebraic Topology by Allen Hatcher:
  A more or less complete dictionary on all topics of these notes. However it is prone to the same problem in the sense that Hatcher's book is rather terse and definitions and parts of some theorems are scattered throughout the paragraphs rather than having a complete statement for reference. Nevertheless it is still the standard reference of the notes, albeit organized in a slightly different way.

- A non-visual proof that higher homotopy groups are abelian by Shintaro Fushida-Hardy:
  This short piece of article proves that the higher homotopy groups are abelian in a purely algebraic way. Most geometric visualization of such a proof has the same underlying idea as the algebraic method.

Contents

1   The Fundamental Groupoid and Covering Space Theory

1.1   The Fundamental Groupoid

Definition 1.1.1: The Fundamental Groupoid

Let X be a space. Define the fundamental groupoid $\Pi_1 X$ of X to be the category with the following data.
- The objects are the points of X.
- Let $x, y \in X$. The morphisms of $\Pi_1 X$ are given by

$$\text{Hom}_{\Pi_1 X}(x, y) = \{\gamma : I \to X \mid \gamma(0) = x \text{ and } \gamma(1) = y \text{ is a path }\}/ \sim$$

where we say that two paths are equivalent if they are homotopic.
- Composition is defined by the concatenation of paths.

We have seen in Algebraic Topology 1 that composition of homotopy classes of paths are well defined.

Lemma 1.1.2

Let X be a space. Then $\Pi_1 X$ is a groupoid.

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Proof. Every path in X has an inverse that lies in $\Pi_1 X$ given by reversing traversal of the path. $\qquad\square$

Lemma 1.1.3

Let X be a space and $x_0 \in X$. Then there is a group isomorphism

$$\text{Hom}_{\Pi_1 X}(x_0, x_0) \cong \pi_1(X, x_0)$$

Proposition 1.1.4

Let $f : X \to Y$ be a continuous map. Then f induces a functor $\Pi_1 f : \Pi_1 X \to \Pi_1 Y$ defined by

$$\Pi_1 f([\alpha]) = [f \circ \alpha]$$

on morphisms.

- -

Proof. Direct from Algebraic Topology 1 due to the above group isomorphism. We have also seen that it is functorial in Algebraic Topology 1. $\qquad\square$

Theorem 1.1.5

The fundamental groupoid defines a functor $\Pi_1 : \mathbf{Top} \to \mathbf{Grps}$ from the category of spaces to the category of groupoids with the following data.
- Π_1 sends each space X to $\Pi_1 X$
- Π_1 sends each continuous map $f : X \to Y$ to the functor $\Pi_1 f$

1.2   The Seifert-Van Kampen Theorem on Fundamental Groupoids

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Definition 1.2.1: The Fundamental Groupoid of Subspaces

Let X be a space and $A \subseteq X$ a subspace. Define $\Pi_1 X[A]$ to be the full subcategory of $\Pi_1 X$ where the objects are A. Explicitly, $\Pi_1 X[A]$ consists of the following data.
- The objects of $\Pi_1 X[A]$ are the points of A.
- The morphisms are given by

$$\mathrm{Hom}_{\Pi_1 X[A]}(x, y) = \mathrm{Hom}_{\Pi_1 X}(x, y)$$

  for any $x, y \in X$.
- Composition is inherited from $\Pi_1 X$.

Lemma 1.2.2

Let X be a space and $A \subseteq X$ a subspace of X such that every path component of X contains a point of A. Then the inclusion

$$\Pi_1 X[A] \to \Pi_1 X$$

of groupoids is an equivalence of categories.

- -

Proof. The inclusion is already fully faithful since $\Pi_1 X[A]$ is a full subcategory. Now let $x \in X$. Let $a \in A$ lie in the same path component as x. Let $\alpha : I \to X$ be a path from x to a. Then the morphism $[\alpha] : x \to a$ of $\Pi_1 X$ is an isomorphism since $\Pi_1 X$ is a groupoid. Thus we conclude. $\square$

Corollary 1.2.3

Let X be a space. Then there is an equivalence of categories

$$\coprod_{[x_0] \in \pi_0(X)} B\pi_0(X, x_0) \cong \Pi_1 X$$

- -

Proof. This is done by choosing A to contain exactly one point of each path component, and then by applying the isomorphism

$$\Pi_1 X[x_0] = B\mathrm{Aut}_{\Pi_1 X}(x_0) = B\pi_1(X, x_0)$$

and the above lemma. $\square$

If X is path connected, then this shows that any choice of base point $x_0 \in X$ gives an equivalence of categories

$$B\pi_0(X, x_0) \cong \Pi_1 X$$

This translates roughly to the standard fact in Algebraic Topology that the fundamental group of a path connected space for any two base points are isomorphic. Indeed in the equivalence of categories exhibited, the former depends on the base point while the latter does not.

We need a lemma.

Lemma 1.2.4

Let $\mathcal{J}$ and $\mathcal{C}$ be categories and let $\mathcal{J}$ be the following category

---

$$
\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\downarrow & & \downarrow{\scriptstyle i} \\
2 & \xrightarrow{\ j\ } & 3
\end{array}
$$

such that $Y : \mathcal{J} \to \mathcal{C}$ is a pushout diagram. If $p : Y \Rightarrow X$ is a natural transformations such that p is a retraction, then $X : \mathcal{J} \to \mathcal{C}$ is also a pushout diagram.

- -

Proof. Consider the following diagram:

This diagram is commutative by the following reasons.
- The front and back face of the square commutes since X and Y are functors and functors preserve commutative diagrams.
- The rest of the faces of the square commutes by the natural transformations p and s.

Let $Z \in \mathcal{C}$ such that there are maps $\lambda_1 : X_1 \to Z$ and $\lambda_2 : X_2 \to Z$ for which the maps

$$
X_0 \to X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad X_0 \to X_2 \xrightarrow{\lambda_2} Z
$$

are equal. Then in particular the two maps

$$
Y_0 \to X_0 \to X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad Y_0 \to X_0 \to X_2 \xrightarrow{\lambda_2} Z
$$

are equal. By commutativity of the cube, the two maps

$$
Y_0 \to Y_1 \xrightarrow{p_1} X_1 \xrightarrow{\lambda_1} Z \quad \text{and} \quad Y_0 \to Y_2 \xrightarrow{p_2} X_2 \xrightarrow{\lambda_2} Z
$$

are equal. By the universal property of Y_3 as a pushout diagram, there exists a unique map $Y_3 \to Z$. such that the following diagram commutes:

Since the retraction of a map is unique, s is unique. Also the map $Y_3 \to Z$ is unique by definition of pushout diagram. Hence there is a unique map $X_3 \to Y_3 \to Z$ so that X is a pushout diagram. $\qquad\square$

Theorem 1.2.5: The Seifert-Van Kampen Theorem on Fundamental Groupoids

Let X be a space and $U, V \subseteq X$ an open cover of X. Let $A \subseteq X$ be a subspace such that every path connected component of U, V, X contains a point in A. Then the inclusions

$$\Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 U[U \cap A] \quad \text{and} \quad \Pi_1(U \cap V)[U \cap V \cap A] \to \Pi_1 V[V \cap A]$$

give a pushout diagram to $\Pi_1 X[A]$. This means that the following diagram is a pushout:

$$
\begin{array}{ccc}
\Pi_1(U \cap V)[U \cap V \cap A] & \longrightarrow & \Pi_1 U[U \cap A] \\
\downarrow & & \downarrow \\
\Pi_1 V[V \cap A] & \longrightarrow & \Pi_1 X[A]
\end{array}
$$

where each arrow is an inclusions.

Proof. First assume that $X = A$. We want to show that for any groupoid $\mathcal{G} \in \mathbf{Grp}$ with maps $\Pi_1 U, \Pi_1 V \to \mathcal{G}$, there exists a unique map $\Pi_1 X \to \mathcal{G}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\Pi_1(U \cap V) & \longrightarrow & \Pi_1 U \\
\downarrow & & \downarrow \\
\Pi_1 V & \longrightarrow & \Pi_1 X
\end{array}
$$

with f, g, and $\exists! u$ mapping to $\mathcal{G}$.

Define the functor $u : \Pi_1 X \to \mathcal{G}$ as follows. For each $x \in \Pi_1 X$, define

$$
u(x) = \begin{cases} f(x) & \text{if } x \in U \\ g(x) & \text{if } x \in V \end{cases}
$$

This is well defined on $U \cap V$ since the outer square of the above diagram commutes. Depending on the path in X, there will be different constructions. Let $[\alpha]$ be a morphism in $\Pi_1 X$. If $\alpha : I \to X$ has image in U, then define $u([\alpha]) = f([\alpha])$. Similarly, define $u([\alpha]) = g([\alpha])$ if α has image in V.

Otherwise, by the Lebesgue covering theorem, there is a finite sequence $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\alpha([a_i, a_{i+1}]) \subseteq U$ or V. Define $\alpha_i = \alpha \mid_{a_i, a_{i+1}}$. It is easy to see that

$$
\begin{aligned}
[\alpha] &= [\alpha \mid_{0, a_1}] \cdot [\alpha \mid_{a_1, a_2}] \cdots [\alpha \mid_{a_{n-1}, 1}] & \text{(Viewed as paths)} \\
&= [\alpha_{n-1}] \circ \cdots [\alpha_1] \circ [\alpha_0] & \text{(Viewed as morphisms in } \Pi_1 X\text{)}
\end{aligned}
$$

Then we can define $u(\alpha)$ as

$$
u([\alpha]) = u([\alpha_{n-1}]) \circ u(\cdots [\alpha_1]) \circ u([\alpha_0])
$$

where we have that

$$
u([\alpha_i]) = \begin{cases} f([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq U \\ g([\alpha_i]) & \text{if } \operatorname{im}(\alpha_i) \subseteq V \end{cases}
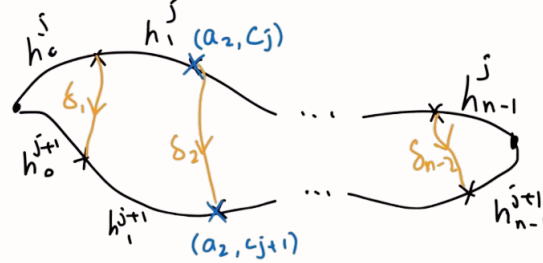$$

If u exists, then u must take the above form. Thus we have shown uniqueness.

For existence, we have to show that above construction of u is well defined. Let α, β be paths in X from x to y that are homotopic via the map $H : I \times I \to X$. We want to show that $u([\alpha]) = u([\beta])$. By the Lebesgue covering theorem, there is a grid in $I \times I$ where the x-axis is subdivided into $0 = a_0 < a_1 < \cdots < a_n = 1$ and the y-axis is subdivided into

$0 = c_0 < c_1 < \cdots < c_k = 1$. such that H sends each rectangle with vertices $\{a_i, a_{i+1}, c_j, c_{j+1}\}$ to either U or V. Let $h^j = H(-, c_j) : I \to X$ so that $h^0 = \alpha$ and $h^k = \beta$. Define

$$\delta_i = H(\alpha_i, -) \mid_{[c_j, c_{j+1}]} : I \to X$$

which are paths from (a_i, c_j) to (a_i, c_{j+1}) in $I \times I$. Also define $h^j_i = h^j \mid_{[\alpha_i, \alpha_{i+1}]}$. Now we have the following which lies entirely in X:

Now we have that

$$
\begin{aligned}
u([h^j]) &= u([h^j_{n-1}]) \circ \cdots \circ u([h^j_0]) \\
&= u([h^{j+1}_{n-1} \circ \delta_{n-2}]) \circ u([\overline{\delta_{n-2}} \circ h^{j+1}_{n-2} \circ \delta_{n-3}]) \circ \cdots \circ u([\overline{\delta_1} \circ h^{j+1}_0]) \\
&= u([h^{j+1}_{n-1}]) \circ \cdots \circ u([h^{j+1}_0]) \\
&= u([h^{j+1}])
\end{aligned}
$$

By induction, we conclude that

$$u([\alpha]) = u([h^0]) = u([h^1]) = \cdots = u([h^k]) = u([\beta])$$

Now suppose that $A \subseteq X$. By the above lemma, it is sufficient to show that the square for A is a retract of the square for X. Let $x \in U \cap V$ and $a_x \in A \cap U \cap V$ lying in the same path component as x. Choose a path $\alpha_x : I \to X$ from a_x to x with α_x being constant if $x \in A$. Do a similar choice for $x \in U \setminus (U \cap V)$ and $x \in V \setminus (U \cap V)$. Define $p_{U \cap V} : \Pi_1(U \cap V) \to \Pi_1(U \cap V)[U \cap V \cap A]$ defined by $x \mapsto a_x$ on objects and

$$[x \xrightarrow{\alpha} y] \mapsto \left(a_x \xrightarrow{[\alpha_x]} x \xrightarrow{[\alpha]} y \xrightarrow{[\alpha_y]} a_y \right)$$

and similarly for p_U and p_V. This defines the natural transformation p in lemma 5.3.4. We conclude by lemma 5.3.4. □

Take $A = \{x_0\}$ be a single point in $U \cap V$. Then this theorem shows that there is a pushout diagram

$$
\begin{array}{ccc}
\pi_1(U \cap V, x_0) & \longrightarrow & \pi_1(U, x_0) \\
\downarrow & & \downarrow \\
\pi_1(V, x_0) & \longrightarrow & \pi_1(X, x_0)
\end{array}
$$

in **Grp**, provided that A contains every path connected component of U, V, X. But A is just one point so the condition becomes that U, V, X and $U \cap V$ being path connected. Hence we recover the usual Seifert-Van Kampen theorem in Algebraic Topology 1.

1.3 Categorical Covering Space Theory

We end the section with a categorical approach of the Galois correspondence between covering spaces and the fundamental group.

Definition 1.3.1: Category of Covering Spaces of a Space

Let X be a space. Define the category $\mathrm{Cov}(X)$ of covering spaces of X by the following.
- The objects are the covering spaces $p : \tilde{X} \to X$ of X
- For two covering spaces $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ of X, a morphism is a map $q : \tilde{X}_1 \to \tilde{X}_2$ such that following diagram commutes:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\ \ q\ \ } & \tilde{X}_2 \\ & {}_{p_1}\searrow \quad \swarrow_{p_2} & \\ & X & \end{array}$$

- Composition is given by the composition of functions.

Recall the category G-Set of G-sets for a group G to consist of the following data.

- The objects are sets which have a group action G.

- For two G-sets X and Y, a morphism is a G-equivariant function $f : X \to Y$. This means that

$$f(g \cdot x) = g \cdot f(x)$$

for all $g \in G$ and $x \in X$.

- Composition is given by the composition of functions.

Theorem 1.3.2

Let X be a connected and locally simply connected space. Let $x_0 \in X$. The the functor $F : \mathrm{Cov}(X) \to \pi_1(X, x_0)$-Set defined by $\left(p : \tilde{X} \to X \right) \mapsto p^{-1}(x_0)$ and $\left(q : \tilde{X}_1 \to \tilde{X}_2 \right) \mapsto q|_{p^{-1}(x_0)}$ gives an equivalence of categories

$$\mathrm{Cov}(X) \cong \pi_1(X, x_0)\text{-Set}$$

2 Cohomology Operations

3   Spectral Sequences in Algebraic Topology

3.1   Spectral Sequences in Topology

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Theorem 3.1.1

Let X be a space. Let the following be a sequence

$$\emptyset \subset X_0 \subset X_1 \subset \cdots \subset X$$

of subspaces. Let G be an abelian group. Then the following data
- $A_{p,q} = H_{p+q}(X_p; G)$
- $E_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$
- $i : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_{p+1}; G) = A_{p+1,q-1}$ (degree $(1, -1)$)
- $j : H_{p+q}(X_p; G) = A_{p,q} \to H_{p+q}(X_p, X_{p-1}; G) = E_{p,q}$ (degree $(0, 0)$)
- $k : H_{p+q}(X_p, X_{p-1}; G) = A_{p,q} \to H_{p+q-1}(X_{p-1}; G) = A_{p-1,q}$ (degree $(-1, 0)$)

defines an exact couple and hence a spectral sequence with E^1 page given by

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1}; G)$$

where the differential $d : E^1_{p,q} \to E^1_{p-1,q}$ is given by the composition

$$H_{p+q}(X_p, X_{p-1}; G) \xrightarrow{\ k\ } H_{p+q-1}(X_{p-1}; G) \xrightarrow{\ j\ } H_{p+q-1}(X_{p-1}, X_{p-2}; G)$$

</div>

The E_1 page of such a spectral sequence is given by

$$\cdots \longleftarrow \qquad \cdots \longleftarrow \qquad \cdots \longleftarrow \cdots$$

$$H_3(X_1, X_0; G) \longleftarrow H_4(X_2, X_1; G) \longleftarrow H_4(X_3, X_2; G) \longleftarrow \cdots$$

$$H_2(X_1, X_0; G) \longleftarrow H_3(X_2, X_1; G) \longleftarrow H_4(X_3, X_2; G) \longleftarrow \cdots$$

$$H_1(X_1, X_0; G) \longleftarrow H_2(X_2, X_1; G) \longleftarrow H_3(X_3, X_2; G) \longleftarrow \cdots$$

Things get interesting when we choose X to be a CW complex and we choose the filtration of X by the skeleton of X. Recall that we have the formula

$$H_{p+q}(X_p, X_{p-1}; G) \cong \begin{cases} C_p^{\mathrm{CW}}(X; G) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the E^1 page is only left with a chain complex at $q = 0$.

Let us also compute the derived couple of this exact couple or in other words, the E^2 page of the spectral sequence. This is more intuitive then the one thinks about on the definition of the derived couple. The $E^2_{p,q}$ slot is simply the homology of the chain complex at the (p, q)th slot. The direction of the maps of the E^2 page depends not on the choice of spectral sequence at all (In fact, the direction only depends on the page). Now in our case, the homology can be given by a known construct:

$$E^2_{p,q} = \frac{\ker(d : H_{p+q}(X_p, X_{p-1}; G) \to H_{p+q-1}(X_{p-1}, X_{p-2}; G))}{\mathrm{im}(d : H_{p+q+1}(X_{p+1}, X_p; G) \to H_{p+q}(X_p, X_{p-1}; G))} = H_{p+q}^{\mathrm{CW}}(X; G)$$

Since the direction of the maps are now diagonal and when $q \neq 0$ we have $E^2_{p,q} = 0$, all maps in E^2 are 0 and we are left with

$$H_0^{\mathrm{CW}}(X; G) \qquad H_1^{\mathrm{CW}}(X; G) \qquad H_2^{\mathrm{CW}}(X; G) \qquad H_3^{\mathrm{CW}}(X; G) \qquad \cdots$$

> **Theorem 3.1.2: Leray-Serre Spectral Sequence**
>
> Let $p : E \to B$ be a Serre fibration with fibre F and path connected B. Suppose that the action of $\pi_1(B)$ on $H_*(F; G)$ is trivial. Then there is a first quadrant homological spectral sequence starting with E^2 and weakly converging to $H_\bullet(E; \mathbb{Z})$. Explicitly, there is a convergence
> $$E^2_{p,q} = H_p(B, H_1(F)) \Rightarrow H_{p+q}(E; \mathbb{Z})$$

3.2   Spectral Kunneth Theorem

4 Localizations of Spaces

4.1 T-Local Groups

Definition 4.1.1: T-Local Groups

Let G be a group and let T be a set of prime. We say that G is T-local if for all prime p not in T, The pth power map $G \to G$ is a bijection.

Lemma 4.1.2

Let G be a group. Then for any prime p, the pth power map $G \to G$ is a group homomorphism.

Definition 4.1.3: Localization on a Set of Prime

Let T be a set of primes. Define the localization of $\mathbb{Z}$ at the set T by

$$\mathbb{Z}_T = (\mathbb{Z} \setminus T)^{-1}\mathbb{Z}$$

Proposition 4.1.4

Let G be abelian. Let T be a set of primes. Then the following are equivalent.
- G is T-local
- G admits a unique structure of a $\mathbb{Z}_T$-module
- For all prime p not in T, $G \otimes \mathbb{Z}/p\mathbb{Z} = 0$.

Recall that tensoring an abelian group $\mathbb{Z}/p\mathbb{Z}$ detects whether G has torsion $\mathbb{Z}/p\mathbb{Z}$. If it is 0 it means that G contains no torsion groups of p^n for all $n \in \mathbb{N}$.

Lemma 4.1.5

Let A, B, C be abelian groups such that there is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

If two out of A, B, C are T-local, then the third one is also T-local.

TBA: Localization is an exact functor Ab to Mod $\mathbb{Z}_T$.

4.2 T-Local Spaces

Definition 4.2.1: T-Equivalent Maps

Let $f : X \to Y$ be a map and let T be a set of primes. We say that f is T-equivalent if

$$f_* : H_n(A; \mathbb{Z}_T) \to H_n(B; \mathbb{Z}_T)$$

is an isomorphism for all $n \in \mathbb{N}$.

Definition 4.2.2: T-Local Spaces

Let X be space and let T be a set of primes. We say that X is a T-local space if for all T-equivalent maps $f : A \to B$, the induced map

$$f^* : [B, X] \to [A, X]$$

is a bijection.

Theorem 4.2.3

Let X be a space and let T be a set of primes. Then the following conditions are equivalent for X.

- X is T-local
- Each homotopy group $\pi_n(X)$ is T-local
- Each homology group $H_n(X)$ is T-local

Proposition 4.2.4

Let A be an abelian group and let T be a set of prime. Then A is T-local if and only if $K(A, 1)$ is T-local.

Definition 4.2.5: Localization of a Space

Let X be a space and let T be a set of primes. A localization of X at T is a T-local space X_T together with a T-equivalent map $f : X \to X_T$ such that the following universal property is satisfied. For any map $g : X \to Z$ where Z is a T-local space, there exists a map $h : X_T \to Z$ unique up to homotopy such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & Z \\
{\scriptstyle f}\downarrow & \nearrow_{\exists h} & \\
X_T &
\end{array}
$$

To be added: functoriality of localization

Theorem 4.2.6

Every nilpotent space X admits a localization at any set of primes T.

5 Completion of Spaces