Stable Homotopy Theory

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Abstract

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1 The Stable Phenomena

1.1 The Stable Homotopy Groups

In Algebraic Topology, we refer to a property or invariant being stable if applying the suspension functor to the space does not change the property (up to possibly a shift in index). The first such example is one we have already encountered.

Definition 1.1.1: Stable Homotopy Groups

Let X be a space. Let $n \in \mathbb{N}$. Define the nth stable homotopy groups of X to be

$$\pi_n^s(X) = \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X)$$

This is well defined because of the Freudenthal suspension theorem, which states that the groups in the direct limit eventually stabilize. Indeed, the same theorem shows the following.

Proposition 1.1.2

Let X be a space. Then there is an isomorphism

$$\pi_n^s(X) \cong \pi_{n+1}^s(\Sigma X)$$

induced by the suspension functor between stable homotopy groups for any $n \in \mathbb{N}$.

Proof. This can be seen by simply unwinding the definitions. We have that

$$\begin{aligned}\pi_{n+1}^s(\Sigma X) &= \operatorname*{colim}_{k\to\infty} \pi_{n+k+1}(\Sigma^{k+1} X)\\ &= \operatorname*{colim}_{k\to\infty} \pi_{n+k}(\Sigma^k X)\\ &= \pi_n^s(X)\end{aligned}$$

and so we conclude. □

New: Graded ring structure on π_*^s.

Recall that for $n \geq 2$, the set $[\Sigma^2 X, Z]$ for any spaces X and Z are abelian. Moreover, the suspension map $\Sigma : [\Sigma^2 X, Z] \to [\Sigma^3 X, \Sigma Z]$ is a group homomorphism. In particular, we can choose $Z = \Sigma^2 Y$. Now since the category **Ab** of abelian groups is cocomplete, the following inverse system

$$[\Sigma^2 X, \Sigma^2 Y] \xrightarrow{\Sigma} [\Sigma^3 X, \Sigma^3 Y] \xrightarrow{\Sigma} [\Sigma^4 X, \Sigma^4 Y] \longrightarrow \cdots$$

has an inverse limit. This leads to the following definition.

Definition 1.1.3: Set of Stable Homotopy Classes of Maps

Let X and Y be space. The set of stable homotopy classes of maps from X to Y is defined to be the abelian group

$$[X, Y]^s = \operatorname*{colim}_{n\in\mathbb{N}\setminus\{0,1\}} [\Sigma^n X, \Sigma^n Y]$$

The following observation is crucial. When X and Y are pointed CW complexes, the abelian groups $[\Sigma^n X, \Sigma^n Y]$ also become stable under suspensions.

Theorem 1.1.4

Let X and Y be pointed CW complexes. Then the sequence of abelian groups,

$$[\Sigma^n X, \Sigma^n Y] \xrightarrow{\Sigma} [\Sigma^{n+1} X, \Sigma^{n+1} Y] \longrightarrow \cdots$$

are isomorphic under the suspension functor for $n > \dim(X)$ and $n \geq 2$.

New: When X is compact, $[X, QY]_* = [X, Y]^s_*$.

Theorem 1.1.5

The stable homotopy groups define a collection of functors $\pi_n^s : \mathbf{CW}_* \to \mathbf{Ab}$ as follows.

- For X a pointed CW complex, $\pi_n^s(X)$ is the nth stable homotopy group of X
- For $f : X \to Y$ a map,
$$\pi_n^s(f) : \pi_n^s(X) \to \pi_n^s(Y)$$
is the image of f under the canonical map $[X, Y] \to [X, Y]^s$

Theorem 1.1.6

The stable homotopy functors $\pi_n^s : \mathbf{CW}_* \to \mathbf{Ab}$ for each $n \in \mathbb{N}$ defines a reduced homology theory.

This is untrue for the unstable homotopy groups. In particular, the fundamental group is not necessarily abelian.

1.2 Spectra and Functions of Spectra

The stable homotopy groups inputs a space and outputs a colimit of homotopy groups which stabilizes by the Freudenthal suspension theorem. Conversely, we can extract from this result the following. If X is a space, we have a sequence of spaces

$$X, \Sigma X, \Sigma^2 X, \dots$$

For each n, the sequence

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to \cdots$$

eventually stabilizes by the Freudenthal suspension theorem. This is the guiding result for the definition of a spectrum.

Definition 1.2.1: (Sequential) Spectra

A spectrum E is a collection $\{(E_n, *) \mid n \in \mathbb{Z}\}$ of pointed spaces in $\mathbf{CG}$ together with continuous maps $e_n : \Sigma E_n \to E_{n+1}$ or equivalently, continuous maps $e_n : E_n \to \Omega E_{n+1}$.

We relate the definition with the above as follows. A spectrum consists of a sequence of spaces (let us start index it with $\mathbb{N}$)

$$E_0, E_1, E_2, \dots$$

For each n, we would like to have a sequence of maps

$$\pi_n(E_0) \to \pi_{n+1}(E_1) \to \pi_{n+2}(E_2) \to \dots$$

similar to the initial digression. These maps are in fact in our hands. For each $k \in \mathbb{N}$, one has the maps

$$\pi_{n+k}(E_k) \xrightarrow{\Sigma_*} \pi_{n+k+1}(\Sigma E_k) \xrightarrow{(e_k)_*} \pi_{n+k+1}(E_{k+1})$$

Notice that we have restricted our spaces to that in **CG**, the category of compactly generated spaces. Most of the theorems only work under such an assumption, and there is little loss forgetting the rest of the spaces.

Definition 1.2.2: Functions of Spectra

Let E and F be spectra. A function from E to F is a collection of maps $\varphi_n : E_n \to F_n$ such that the following diagrams (which are equivalent by adjunction) are commutative:

$$\begin{tikzcd} \Sigma E_n \arrow[r, "e_n"] \arrow[d, "\Sigma\varphi_n"'] & E_{n+1} \arrow[d, "\varphi_{n+1}"] \\ \Sigma F_n \arrow[r, "f_n"'] & F_{n+1} \end{tikzcd} \qquad \begin{tikzcd} E_n \arrow[r, "e_n'"] \arrow[d, "\varphi_n"'] & \Omega E_{n+1} \arrow[d, "\Omega\varphi_{n+1}"] \\ F_n \arrow[r, "f_n'"'] & \Omega F_{n+1} \end{tikzcd}$$

Definition 1.2.3: The Category of Sequential Spectra

Let $\mathcal{U}$ be a full subcategory of $\mathbf{Top}_*$. Define the category

$$\text{Sp}^{\mathbb{N}}(\mathcal{U})$$

of sequential spectra of $\mathcal{U}$ to consist of the following data.
- The objects are the sequential spectra $\{E_n \in \mathcal{U} \mid n \in \mathbb{N}\}$ of spaces in $\mathcal{U}$.
- The morphisms are the functions of spectra
- Composition is given by component-wise composition.

Definition 1.2.4: CW Spectra

A CW spectrum E is a collection $\{E_n \mid n \in \mathbb{Z}\}$ of CW-complexes with a chosen basepoint together with maps $e_n : \Sigma E_n \to E_{n+1}$ so that ΣE_n is recognized as a subcomplex of E_{n+1}. The category of CW spectra is denoted by

$$\text{Sp}^{\mathbb{N}}(\mathbf{CW}_*)$$

Definition 1.2.5: The Suspension Spectrum

Let $X \in \mathbf{CG}_*$ be a space. Define the suspension spectrum $\Sigma^\infty X$ of X to consist of the following data.
- The collection $\{\Sigma^n X \mid n \in \mathbb{N}\}$ of spaces.
- The collection $\sigma_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ of maps which is a homeomorphism.

Definition 1.2.6: Eilenberg-MacLane Spectrum

Let G be an abelian group. Define the Elienberg-Maclane spectrum to consist of the following data.
- The collection $\{K(G, n) \mid n \in \mathbb{N}\}$ of spaces.
- The collection $K(G, n) \to \Omega K(G, n+1)$ of maps which are homeomorphisms.

1.3 Brown's Representability Theorem

Specific types of spectra are related to (co)homology theories. We will introduce the names of such spectra below.

Definition 1.3.1: Ω-Spectra

Let $\{E_n \mid n \in \mathbb{Z}\}$ and $e_n : E_n \to \Omega E_{n+1}$ be a spectra. We say that it is an Ω-spectra if the induced map $(e_n)_*$ is a weak homotopy equivalence.

Lemma 1.3.2

Let X be an infinite loopspace. Define a sequence of spaces and maps inductively:

- Let $X_0 = X$
- Suppose X_n is a chosen space. Choose X_{n+1} to be a space such that
$$X_n \simeq \Omega X_{n+1}$$
Also let the bonding map σ_n be the weak equivalence $\sigma : X_n \to \Omega X_{n+1}$.

The above data defines an Ω-spectrum.

Recall that if Z is a group-like H-space (ref Concise J.P. May), then $[X, Z]$ has a group structure.

Theorem 1.3.3

Let $\{T_n \mid n \in \mathbb{Z}\}$ be a Ω-spectrum consisting of CW complexes. For any space X, define a functor $\widetilde{E}^k : \mathbf{CW}_* \to \mathbf{Ab}$ as follows.

- On objects, a space X is sent to $\widetilde{E}^k(X) = [X, T_k]$ for $k \in \mathbb{Z}$.
- For $f : X \to Y$ a morphism, $\widetilde{E}^k(f) : [Y, T_k] \to [X, T_k]$ is defined by pre composition.

Then the collection of functors $\widetilde{E}^k$ for all k defines a reduced cohomology theory on CW complexes with base point.

Theorem 1.3.4: Brown's Representability Theorem

Let $\widetilde{h}^n : \mathbf{hCW}_* \to \mathbf{Ab}$ be a reduced cohomology theory with chosen base points on the CW complexes. Then there exists a CW spectrum $\mathbb{K} = \{K_n \mid n \in \mathbb{N}\}$ and natural isomorphisms
$$\widetilde{h}^n(X) \cong [X, K_n]_*$$
for all CW-complexes X.

It is related to representability in category theory in the following sense: Since $\widetilde{h}^n$ are functors that are homotopy equivalent, we can instead consider $\widetilde{h}^n$ as a functor from the homotopy category $\mathbf{hCW}_*$ of pointed CW-complexes to $\mathbf{Ab}$. Then Hom sets in $\mathbf{hCW}$ are precisely $[X, Y]_*$ which are the base point preserving homotopic maps from X to Y. Then Brown's representability states that the functor $\widetilde{h}^n : \mathbf{hCW}_* \to \mathbf{Ab}$ is representable via $[X, K_n]_*$ and more over the K_n assemble into a spectrum.

Unfortunately, the two assignments does not give an equivalence of categories. The obstruction is called phantom maps.

Theorem 1.3.5

Every reduced cohomology theory determines and is determined by an Ω CW-spectrum.

We note here that every reduced cohomology theory induces a generalized cohomology theory, hence the above theorem hence a version for generalized cohomology theories and also pointed cohomology theories.

Note: non homotopy equivalent spectra can represent the same cohomology theory.

Theorem 1.3.6

The reduced singular cohomology theory $\widetilde{H}^k : \mathbf{CW} \to \mathbf{Ab}$ with coefficients in an abelian group G is determined by the Eilenberg-maclane spectrum $\{K(G, n) \mid n \in \mathbb{N}\}$.

Definition 1.3.7: Homology with Coefficients in a Spectrum

Let $\mathbb{K} = \{T_n \mid n \in \mathbb{Z}\}$ be a spectrum. Define a functor $H_n(-;\mathbb{K}) : \mathbf{CW}^2 \to \mathbf{Ab}$ by

$$H_n(X, A; \mathbb{K}) = \operatorname*{colim}_{k\to\infty} \pi_{n+k}\left(\frac{X_+}{A_+} \wedge T_k\right)$$

where X_+ is the space X together with a chosen base point.

Theorem 1.3.8

Let (h_n, δ_n) be a generalized homology theory. Then there exists a spectrum $\mathbb{K}$ and a natural isomorphism

$$h_n(X, A) \cong H_n(X, A; \mathbb{K})$$

for all CW pairs (X, A).

Theorem 1.3.9

Let $\{T_n \mid n \in \mathbb{Z}\}$ be a CW spectrum such that T_n is $(n-1)$-connected. Define

$$\widetilde{E}_k(X) = \operatorname*{colim}_{n\to\infty} \pi_{k+n}(X \wedge T_n)$$

Then the functors $\widetilde{E}_k$ for all k defines a reduced homology theory on CW complexes with base point. (Concise J.P. May)

Theorem 1.3.10

Any reduced homology theory determines and is determined by a CW spectrum.

1.4 The Spanier-Whitehead Category

We would like to define a category so that we can investigate the stable phenomena. Recall from Model Category Theory that the classical suspension functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ does not give an equivalence of categories. Thus **Top** is not a good category to investigate stable phenomena. Indeed we would like such a category to be stable (in the sense of equivalence of categories) with respect to the suspension and loopspace functor. More generally, we want such a category **SHC** to have the following properties:

- There is an adjunction
$$\Sigma^\infty : \mathbf{CG}_* \rightleftarrows \mathbf{SHC} : \Omega^\infty$$
- If A is a CW complex with finitely many cells, and B is a CW complex, then there is an isomorphism
$$[\Sigma^\infty A, \Sigma^\infty B] \cong [A, B]^s$$
- ?

Definition 1.4.1: The Spanier-Whitehead Category

Define the Spanier-Whitehead category **SW** as follows.
- The objects consists of a pair (X, n) where X is a pointed CW complex and $n \in \mathbb{N}$.
- For (X, n) and (Y, m) to objects,
$$\mathrm{Hom}_{\mathbf{SW}}((X, n), (Y, m)) = \operatorname*{colim}_{r\to\infty}[\Sigma^{n+r} X, \Sigma^{m+r} Y]_*$$
- Composition is given by the composition of maps.

Proposition 1.4.2

The category **SW** is additive and is a triangulated category.

$[-, X]^s_*$ almost defines a reduced cohomology theory. It fails at the wedge axiom.
Suspension gives equivalence of categories.

2 The Stable Model Structure on Sequential Spectra

2.1 The Category of Sequential Spectra

Definition 2.1.1: Smash Tensoring

Let $E \in \mathrm{Sp}^{\mathbb{N}}(\mathbf{CGWH}_*)$ be a spectrum and let $X \in \mathbf{CG}_*$ be a pointed space. Define the smash tensoring of E and X to be the spectrum

$$E \wedge X$$

given as follows.

- For each $n \in \mathbb{N}$, $(E \wedge X)_n = E_n \wedge X$
- For each map $e_n : \Sigma E \to E_{n+1}$, the structure map is given by

$$e_n \wedge \mathrm{id}_X : \Sigma(E \wedge X)_n \to (E \wedge X)_{n+1}$$

  by wed

Definition 2.1.2: Powering

Let $E \in \mathrm{Sp}^{\mathbb{N}}(\mathbf{CGWH}_*)$ be a spectrum and let $X \in \mathbf{CG}_*$ be a pointed space. Define the powering of E and X to be the spectrum

$$\mathrm{Map}(X, E)_*$$

given as follows.

- For each $n \in \mathbb{N}$, $(\mathrm{Map}(X, E)_*)_n = \mathrm{Map}(X, E_n)_*$
- For each map $e_n : \Sigma E \to E_{n+1}$, the structure map is given by

$$\Sigma(\mathrm{Map}(X, E)_*)_n \overset{(\mathrm{const}, \mathrm{id})}{\longrightarrow} \mathrm{Map}(X, \Sigma E_n)_* \overset{\mathrm{Map}(X, e_n)_*}{\longrightarrow} (\mathrm{Map}(X, E)_*)_{n+1}$$

  by wed

Next: Smash tensoring and powering are functorial (nLab), adjunction between smash tensoring and powering.

Definition 2.1.3: Shifted Spectrum

Let $X \in \mathbf{CG}$ be a space. Define the k-fold shifted spectrum

$$F_k^{\mathbb{N}} X$$

of X to consists of the following data.

- For each n, the space is given by

$$(F_k^{\mathbb{N}} X)_n = \begin{cases} S^{n-k} \wedge X & \text{if } n \geq k \\ * & \text{if } n < k \end{cases}$$

- The structure maps are given by the canonical maps and the unique map from $*$.

2.2 The Level-wise Model Structure on the Category of Spectra

Note: $X_+ = (X \coprod \{*\}, *) \in \mathbf{Top}_*$.

Theorem 2.2.1

The category $\mathcal{S}^{\mathbb{N}}$ of spectra has a pointed model structure with the following data.
- The weak equivalences are the level-wise weak homotopy equivalences of spaces.
- The fibrations are the level-wise Serre fibrations
- The cofibrations are the level-wise q-cofibrations.

This model structure is cofibrantly generated with the generating sets given by

$$I_{\text{level}} = \{F_d^{\mathbb{N}}(S_+^{a-1} \to D_+^a) \mid a, d \in \mathbb{N}\} \quad \text{and} \quad J_{\text{level}} = \{F_d^{\mathbb{N}}(D_+^a \to (D^a \times I)_+) \mid a, d \in \mathbb{N}\}$$

Definition 2.2.2: Level-wise Model Structure

The level wise model structure on the category $\mathcal{S}^{\mathbb{N}}$ of spectra is the model structure generated by
- The cofibrations $I_{\text{level}} = \{F_d^{\mathbb{N}}(S_+^{a-1} \to D_+^a) \mid a, d \in \mathbb{N}\}$
- The ayclic cofibrations $J_{\text{level}} = \{F_d^{\mathbb{N}}(D_+^a \to (D^a \times I)_+) \mid a, d \in \mathbb{N}\}$

Unfortunately, such a direct translation of model category structure from **Top** to $\mathcal{S}^{\mathbb{N}}$ does not give the appropriate stable homotopy category. Therefore we need a new model structure. For this, we turn to the homotopy groups.

2.3 Homotopy Groups of a Spectrum

Recall from the digression after def1.2.1 that we have a series of maps of the form

$$\pi_{n+k}(X_k) \xrightarrow{\Sigma_*} \pi_{n+k+1}(\Sigma X_k n) \xrightarrow{(\sigma_n)_*} \pi_{n+k+1}(X_{k+1})$$

for $n + k > 1$. We will use this to define the homotopy groups of a spectrum.

Definition 2.3.1: Homotopy Groups of a Spectrum

Let X be a spectrum. Define the nth (stable) homotopy group of X to be the colimit of the inverse system

$$\pi_{n+k}(X_k) \xrightarrow{(\sigma_n)_* \circ \Sigma_*} \pi_{n+k+1}(X_{k+1}) \xrightarrow{(\sigma_{n+1})_* \circ \Sigma_*} \pi_{n+k+2}(X_{k+2}) \longrightarrow \cdots$$

for $n + k > 1$. We write the nth stable homotopy group as

$$\pi_n(X) = \operatorname*{colim}_{k \to \infty} \pi_{n+k}(X_k)$$

Notice that this is a generalization of the stable homotopy groups in Algebraic Topology 3. Indeed if one considers the suspension spectrum of space, then the homotopy groups of the given suspension spectrum are the stable homotopy groups. This is made rigorous with the following functor.

Definition 2.3.2: The Suspension Functor

Define the suspension functor

$$\Sigma^\infty : \mathbf{CG}_* \to \mathcal{S}^{\mathbb{N}}$$

to consist of the following.
- For $X \in \mathbf{CG}_*$ a space, $\Sigma^\infty X$ is the suspension spectrum of X
- For $f : X \to Y$ a map,

  $$\Sigma^\infty f : \Sigma^\infty X \to \Sigma^\infty Y$$

  is the map induced by the nth suspension of f for all $n \in \mathbb{N}$.

Proposition 2.3.3

Let $X \in \mathbf{CG}_*$ be a space. There is an isomorphism

$$\mathbb{S} \wedge X \cong \Sigma^\infty X$$

Next: $\pi_k(-)$ is functorial.

Proposition 2.3.4

Let X be an Ω-spectrum. Then the homotopy groups of X are given as follows.

$$\pi_k(X) = \begin{cases} \pi_{k+n}(X_n) & \text{if } k+n \geq 0 \\ \pi_k(X_0) & \text{if } k \geq 0 \\ \pi_0(X_{|k|}) & \text{if } k < 0 \end{cases}$$

Unwinding the proposition, we have that

- $\pi_0(X) = \pi_0(X_0) = \pi_1(X_1) = \dots$
- $\pi_1(X) = \pi_1(X_0) = \pi_2(X_1) = \dots$

and so on. Indeed this is the effect of imposing weak equivalences on the structure maps of X.

Definition 2.3.5: π_*-Equivalence

Let $f : X \to Y$ be a map of spectra. We say that f is a π_*-equivalence if the induced map

$$\pi_n(f) : \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all n. In this case, we say that X and Y are π_*-isomorphic.

2.4 The Stable Model Structure on the Category of Spectra

Definition 2.4.1: Generating Sets of the Stable Model Structure

Define the generating sets of the stable model structure by
- $I_{\text{stable}} = I_{\text{level}}$
- $J_{\text{stable}} = J_{\text{level}} \cup \{???\}$

Definition 2.4.2: Stable Fibrations

Let $f : X \to Y$ be a map of spectra. We say that f is a stable fibration if it has the right lifting property with respect to J_{stable}.

Proposition 2.4.3

Let $f : X \to Y$ be a map of spectra. Then f is a stable fibration if and only if f is a level-wise fibration of spaces and for each $n \in \mathbb{N}$, the map

$$X_n \to Y_n \times_{\Omega Y_{n+1}} \Omega X_{n+1}$$

induced by $\widetilde{\sigma}_n^X$ and f is a weak homotopy equivalence.

Theorem 2.4.4

The above generating sets cofibrantly generates a model structure on $\mathcal{S}^{\mathbb{N}}$ with the following data.

- The weak equivalence are precisely the π_*-isomorphisms
- The cofibrations are the q-fibrations
- The fibrations are precisely the stable fibrations

Moreover, the fibrant objects are precisely the Ω-spectra.

Definition 2.4.5: The Stable Model Structure

The stable model structure on $\mathcal{S}^{\mathbb{N}}$ is the model structure described above. Explicitly,

- The weak equivalence are precisely the π_*-isomorphisms
- The cofibrations are the q-fibrations
- The fibrations are precisely the stable fibrations

3 The First Stable Homotopy Category

3.1 The Stable Homotopy Category

Definition 3.1.1: The Stable Homotopy Category

Define the stable homotopy category to be the homotopy category

$$\mathcal{SHC} = \text{Ho}(\mathcal{S}^{\mathbb{N}})$$

of the category of spectra.

Recall that a pointed model category implicitly has the notion of a suspension and loopspace functor. In our case it will prove to be useful to be able to construct it explicitly. In particular, one can see that such functors are reminiscent of the usual suspension and loopspace functors in classical algebraic topology.

Definition 3.1.2: Alternative Suspension

Define the alternative suspension functor $\Sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ by the following data.

- For a spectrum $X \in \mathcal{S}^{\mathbb{N}}$, define ΣX to be the spectrum where $(\Sigma X)_n = S^1 \wedge X_n$ and
$$\sigma_n^{\Sigma X} = S^1 \wedge (\sigma_n^X) : S^1 \wedge \Sigma X_n \to S^1 \wedge X_{n+1}$$
- For a map $f : X \to Y$ of spectra, define $\Sigma f : \Sigma X \to \Sigma Y$ level-wise by $(\Sigma f)_n = S^1 \wedge (\sigma_n(f)) : \Sigma X_n \to \Sigma Y_n$

Definition 3.1.3: Alternative Looping

Define the alternative looping functor $\Sigma : \mathcal{S}^{\mathbb{N}} \to \mathcal{S}^{\mathbb{N}}$ by the following data.

- For a spectrum $X \in \mathcal{S}^{\mathbb{N}}$, define ΩX to be the spectrum where $(\Omega X)_n = \text{Map}_*(S^1, X_n)$ and
$$\sigma_n^{\Omega X} = \text{Map}_*(S^1, \sigma_n^X) : \text{Map}_*(S^1, \Sigma X_n) \to \text{Map}_*(S^1, X_{n+1})$$
- For a map $f : X \to Y$ of spectra, define $\Omega f : \Omega X \to \Omega Y$ level-wise by $(\Omega f)_n = \text{Map}_*(S^1, \sigma_n(f)) : \Omega X_n \to \Omega Y_n$

Theorem 3.1.4

The suspension and looping functor of the stable model structure on $\mathcal{S}^{\mathbb{N}}$ is precisely given by the alternative suspension and alternative looping. In particular, there is an adjunction

$$\Sigma : \text{Ho}(\mathcal{S}^{\mathbb{N}}) \rightleftarrows \text{Ho}(\mathcal{S}^{\mathbb{N}}) : \Omega$$

Theorem 3.1.5

The category $\mathcal{S}^{\mathbb{N}}$ is a stable model category. Explicitly, this means that both $\Sigma, \Omega : \text{Ho}(\mathcal{S}^{\mathbb{N}}) \to \text{Ho}(\mathcal{S}^{\mathbb{N}})$ define equivalence of categories.

3.2 Properties of the Stable Homotopy Category

We have seen the suspension functor for spectra when defining homotopy groups for spectra. We also define the loopspace functor for spectra.

Definition 3.2.1: The Loopspace Functor

Define the loopspace functor $\Omega^\infty : \mathcal{S}^{\mathbb{N}} \to \mathbf{CG}_*$ as follows.

- For $X = \{X_n \mid n \in \mathbb{N}\}$ a spectrum, $\Omega^\infty X = X_0$ returns the first space in the sequence.
- For $f : X \to Y$ a morphism, $\Omega^\infty f : \Omega^\infty X \to \Omega^\infty Y$ is the map on the 0th level in the function of spectra.

Theorem 3.2.2

There is an adjunction given by

$$\Sigma^\infty : \mathbf{CG}_* \rightleftarrows \mathcal{S}^{\mathbb{N}} : \Omega^\infty$$

Explicitly, this means that there are isomorphisms

$$\mathrm{Hom}_{\mathcal{S}^{\mathbb{N}}}(\Sigma^\infty X, Y) \cong \mathrm{Hom}_{\mathbf{CG}_*}(X, \Omega^\infty Y)$$

that are natural in $X \in \mathbf{CG}_*$ and $Y \in \mathcal{S}^{\mathbb{N}}$.

Theorem 3.2.3

The category of Ω spectra is equivalent to the category of generalized cohomology theories on $\mathbf{Top}_*$.

4 Highly Structured Spectra

As quoted from nLab, there are a few ways to improve this stable homotopy category, only after we have seen some alternative categories of spectra. For instance, the smash product of spectra is not functorial nor is it commutative. This is important for discussion of higher algebra.

4.1 The Category of Orthogonal Spectra

Definition 4.1.1: Orthogonal Spectra

An orthogonal spectrum X consists of the following data.

- For each $n \in \mathbb{N}$, a pointed space X_n and a continuous group action of $O(n)$ that fixes the base point.
- For each $n \in \mathbb{N}$, there are maps of pointed spaces $\sigma_n : S^1 \wedge X_n \to X_{n+1}$
- For each $n, k \in \mathbb{N}$, the composite map

$$S^k \wedge X_n \xrightarrow{\mathrm{id}_{S^{k-1}} \wedge \sigma_n} S^{k-1} \wedge X_{n+1} \xrightarrow{\mathrm{id}_{S^{k-2}} \wedge \sigma_{n+1}} S^{k-2} \wedge X_{n+2} \longrightarrow \cdots \longrightarrow X_{n+k}$$

  is $O(k) \times O(n)$ equivariant, where we think of $O(k) \times O(n) \leq O(n+k)$ by $O(k)$ acting on the first k coordinate and $O(n)$ acting on the last n coordinates.

TBA: Morphism
Examples: Trivial, Sphere, Suspension, Shifted Suspension
TBA: Stable model structure
TBA: Quillen adjunction with $S^{\mathbb{N}}$, Quillen adjunction with $\mathbf{Top}_*$

4.2 The Category of Symmetric Spectra

4.3 The Symmetric Monoidal Structure

Theorem 4.3.1

Every CW spectrum is equivalent to some Ω CW spectrum.

5 More Category of Spectra

5.1 Spectra of Simplicial Sets

TBA: Def of spectra of simplicial sets
TBA: Quillen adjunction with $S^{\mathbb{N}}$ induced by geometric realization and nerve functor.

5.2 Diagram Spectra

5.3 The Category of L-Spectra

In previously defined highly structured spectra, we see that the smash product behaves better only when we pass it to the homotopy category. However, we will present here one category of spectra in which the smash product behaves decently well.

Definition 5.3.1: (Coordinate Free) Spectra

Let U be a infinite dimensional inner product space isomorphic to $\mathbb{R}^\infty$. A coordinate free spectra modelled on U consists of the following data.

- For each finite dimensional vector subspace $V \subset U$, a pointed topological space E_V.
- For each inclusion of vector subspaces $V \hookrightarrow W$, a homeomorphism of pointed spaces $\sigma_{V,W} : E_V \overset{\cong}{\to} \Omega^{W-V} E_W = \text{Map}_*(S^{W-V}, E_W)$. Here S^{W-V} means the one point compactification of the space $W - V$.