Differential Equations

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Abstract

These notes will act as a an introductory text with a collection of theorems and definitions for differential equations.

Contents

1 Introduction to Differential Equations

1.1 Classification

Definition 1.1.1: Dependency

We say that a variable x is dependent on y if $y = f(x)$ for some function f, otherwise it is independent.

Definition 1.1.2: Differential Equations

A differential equation is a function of variables and their derivatives.

Definition 1.1.3: Ordinary and Partial

A differential equation is said to be ordinary if there is only one independent variable. It is partial otherwise. In particular,

- The equation
$$\left(y\frac{dy}{dx} + x\right) = \frac{d^2y}{dx^2}$$
is ordinary
- The equation
$$\frac{du}{dx}\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dt^2}\right) = u$$
is partial

Definition 1.1.4: Autonomous and Non-autonomous

A differential equation is said to be autonomous if the independent variable does not appear explicitly. It is non-autonomous otherwise. In particular,

- The equation
$$y'' + 2y' = y'''y + 4$$
is autonomous
- The equation
$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = x$$
is non-autonomous

Definition 1.1.5: Linear and non-linear

An ordinary differential equation with y dependent on x is said to be linear if the coefficient of every derivative is a function of x or a constant. It is non-linear otherwise. In particular,

- The equation
$$x^2\frac{dy}{dx} = 6\frac{d^2y}{dx^2}$$
is linear
- The equation
$$y\frac{dy}{dx} = \left(\frac{d^2y}{dx^2}\right)^2$$
is non-linear

Definition 1.1.6: Homogenous and non-homogenous

An ordinary differential equation is said to be homogenous if the term without a derivative is 0. In particular,

- The equation
$$y'' + 2y' + y = 0$$
is homogenous
- The equation
$$y''' + yy' = xy + \sin(x)$$
is not homogenous

2 Ordinary Differential Equations: First Order

2.1 Existence and Uniqueness of Solutions of First Order Differential Equations

Theorem 2.1.1

[Globally Lipschitz Picard's Theorem] Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and that there exists $L > 0$ such that

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$. If $\tau L < 1$, then for every $x_0 \in \mathbb{R}^n$, there exists a unique differentiable function $x : [t_0, t_0 + \tau] \to \mathbb{R}^n$ that satisfies

$$\frac{dx}{dt} = f(x,t)$$

with $x(t_0) = x_0$ for all $t \in [t_0, t_0 + \tau]$.

Proposition 2.1.2

Suppose that a solution is guaranteed by the above theorem on $[t_0, t_1]$ given by $x_1(t)$ and another solution is guaranteed on $[t_1, t_2]$ given by $x_2(t)$. Then

$$x(t) = \begin{cases} x_1(t) & t_0 \leq t \leq t_1 \\ x_2(t) & t_1 \leq t \leq t_2 \end{cases}$$

is a solution for the differential equation $x'(t) = f(x,t)$ on $[t_0, t_2]$

Theorem 2.1.3

Suppose we have a system of linear differential equations $\mathbf{x}'(t) = A\mathbf{x}(t)$ with initial condition $x(0) = x_0$ where A is an $n \times n$ matrix. Then the solution of this is

$$x(t) = e^{tA}x_0 = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} x_0$$

for $|t| \leq \tau$ where $\tau < \|A\|^{-1}$

Theorem 2.1.4

[Locally Lipschitz Picard's Theorem] Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and that there exists $L > 0$ such that

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

for all $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$. If $\tau L < 1$, then for every $x_0 \in \mathbb{R}^n$, there exists a unique differentiable function $x : [t_0, t_0 + \tau] \to \mathbb{R}^n$ that satisfies

$$\frac{dx}{dt} = f(x,t)$$

with $x(t_0) = x_0$ for all $t \in [t_0, t_0 + \tau]$.

Theorem 2.1.5

Suppose that $U \subset \mathbb{R}^n$ is open and that $f : U \to \mathbb{R}^n$ is continuous and locally Lipschitz, meaning for every compact subset K of U there exists L_K such that

$$|f(u) - f(v)| \leq L_k|u - v|$$

for $u, v \in K$, then for some $\tau = \tau(x_0) > 0$ there exists a unique $x : [t_0 - \tau, t_0 + \tau] \to U$ that solves

$$\frac{dx}{dt} = f(x)$$

with initial condition $x(t_0) = x_0$ for all $t \in [t_0 - \tau, t_0 + \tau]$.

2.2 Separable Equations

Definition 2.2.1: Separable Equations

If $\frac{dx}{dt} = f(x)g(t)$. Then it is called separable.

Lemma 2.2.2

A separable differential equation $\frac{dx}{dt} = f(x)g(t)$ can be solved by solving

$$\int \frac{1}{f(x)}\,dx = \int g(t)\,dt$$

2.3 The Linear Case

Theorem 2.3.1

[First Order Linear Equation] Let $R(t) = \int r(t)\,dt$. The differential equation

$$\frac{dx}{dt} + r(t)x = g(t)$$

has the solution

$$x(t) = Ae^{-R(t)} + e^{-R(t)} \int e^{R(t)} g(t)\,dt$$

Proof. We first multiply both sides of the equation by $e^{R(t)}$. We have

$$\begin{aligned}
\frac{dx}{dt} + r(t)x &= g(t) \\
e^{R(t)}\frac{dx}{dt} + r(t)e^{R(t)}x &= g(t)e^{R(t)} && \text{(Multiply by } e^{R(t)}\text{)} \\
\frac{d}{dt}\left(e^{R(t)}x\right) &= g(t)e^{R(t)} \\
e^{R(t)}x + C &= \int g(t)e^{R(t)}\,dt && \text{(By the FTC)} \\
x &= e^{-R(t)} \int g(t)e^{R(t)}\,dt + Ae^{-R(t)} && \text{(Relabel } A = -C\text{)}
\end{aligned}$$

□

Corollary 2.3.2

If $r(t) = p$ is a constant function and $g(t) = 0$, then the differential equation

$$\frac{dx}{dt} + r(t)x = g(t)$$

has the solution

$$x(t) = Ae^{-\int r(t)\,dt}$$

Corollary 2.3.3

If $r(t) = p$ is a constant function, then the differential equation

$$\frac{dx}{dt} + r(t)x = g(t)$$

has the solution

$$x(t) = Ae^{-pt}$$

2.4 Substitution Method

Theorem 2.4.1: Type 1 Substitution

Consider differential equations of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Then let $u = \frac{y}{x}$ to obtain

$$x\frac{du}{dx} = F(u) - u$$

Theorem 2.4.2: Type 2 Substitution

Consider differential equations of the form

$$\frac{dy}{dx} + p(x)y = q(x)y^n$$

Then let $u = y^{1-n}$ to obtain

$$\frac{du}{dx} + (1-n)p(x)u = (1-n)q(x)$$

Proposition 2.4.3

For a differential equation in the form $f(x, y', y'') = 0$, the equation can be reduced to a first order differential equation by substitution.

Proposition 2.4.4

For a differential equation in the form $f(y, y', y'') = 0$, the equation can be reduced to a first order differential equation by substitution

2.5 Systems of First Order Equations

Definition 2.5.1

[System of First Order Equations] A system of first order equations is given by

$$\mathbf{x}'(t) = A\mathbf{x}$$

where $\mathbf{x}'(t) = \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Theorem 2.5.2

Consider a system of first order equation with constant coefficients

$$\begin{cases} \frac{dx}{dt} = px + qy \\ \frac{dy}{dt} = rx + sy \end{cases}$$

with matrix equation

$$\mathbf{x}'(t) = A\mathbf{x}$$

Suppose A has eigenvalues $\lambda_{1,2}$ and eigenvectors $\mathbf{v}_{1,2}$.

- If $\lambda_{1,2}$ are distinct, then
$$\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_2 t}\mathbf{v}_2$$
- If $\lambda_{1,2} \in \mathbb{C}$, let $\mathbf{v} = \mathbf{v}_1 + i\mathbf{v}_2$. Then
$$\mathbf{x}(t) = \mathbf{c}\mathbf{v}e^{\lambda t} + \overline{\mathbf{c}\mathbf{v}}e^{\overline{\lambda}t} = 2\,\mathrm{Re}(\mathbf{c}\mathbf{v}e^{\lambda t})$$
where c is such that $\mathbf{x}(t)$ is real. Or
$$x(t) = e^{pt}[a\cos(qt) + b\sin(qt)\mathbf{v}_1 + (b\cos(qt) - a\sin(qt))\mathbf{v}_2]$$
- If the eigenvalues are repeated, then $\mathbf{x}(t) = \mathbf{a}e^{\lambda t} + \mathbf{b}te^{\lambda t}$

Theorem 2.5.3

[Uncoupling Solutions] Let $\dot{\mathbf{x}} = A\mathbf{x}$ be a system of differential equations. Let it has distinct eigenvalues. Then there exists $P = (\mathbf{v}_1|\mathbf{v}_2)$ such that $P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ Then let $\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}}$. Then $\dot{\mathbf{y}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\mathbf{y}$

2.6 Direction Fields

Definition 2.6.1

[Vector Fields] A planar vector field is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Definition 2.6.2

[Direction Fields] Let $\frac{dx}{dt} = f(x,t)$ be a differential equation. A direction field is vector field $v : \mathbb{R}^2 \to \mathbb{R}^2$ such that each vector $(x,y) \in \mathbb{R}^2$ is assigned the vector with slope $f(x,t)$.

Definition 2.6.3

[Fixed Points] We say that $(x_0, f(x_0))$ is a fixed point if $\frac{dx}{dt} = f(x) = 0$. A fixed point is stable if $f'(x_0) < 0$. It is unstable if $f'(x_0) > 0$.

3 Ordinary Differential Equations: Second Order

3.1 Second Order Linear Ordinal Differential Equations

Definition 3.1.1

[Linearly Independent Solutions] Two functions $x_1(t), x_2(t)$ defined on an interval I are linearly independent if $\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0 \implies \alpha_1 = \alpha_2 = 0$.

Definition 3.1.2

The complete solution to a inhomogeneous solution is given by $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ is the solution to the homogeneous solution and $x_p(t)$ is the particular solution to the equation.

Definition 3.1.3

[Auxiliary Equations] Consider

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

The auxiliary equation of the differential is $ak^2 + bk + c = 0$.

Theorem 3.1.4

Consider

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

and $k_{1,2}$ roots of $ak^2 + bk + c = 0$

- If $b^2 - 4ac > 0$, then $x(t) = Ae^{k_1 t} + Be^{k_2 t}$
- If $b^2 - 4ac = 0$, then $x(t) = (A + Bt)e^{kt}$
- If $b^2 - 4ac < 0$, then $x(t) = e^{pt}(C\cos(qt) + D\sin(qt))$, where $k_{1,2} = p \pm iq$ or $x(t) = Ee^{pt}\cos(qt - \phi)$ where $E^2 = C^2 + D^2$ and $\tan(\phi) = \frac{D}{C}$

Theorem 3.1.5

Let

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$$

To find the particular solution, plug the possibility into the differential to find identities for coefficients.

- If $f(t)$ is a polynomial, then $x_p(t)$ is a general polynomial of degree n.
- If $f(t) = ae^{kt}$ where k is not a root of the auxiliary equation, then $x_p(t) = Ae^{kt}$
- If $f(t) = ae^{kt}$ where k is a root of the auxiliary equation, then $x_p(t) = Ate^{kt}$ or At^2e^{kt}
- If $f(t) = a\sin(\omega t)$ or $f(t) = a\cos(\omega t)$ then $x_p(t) = A\sin(\omega t) + B\cos(\omega t)$
- If $f(t) = at^ne^{kt}$ then $x_p(t) = P(t)e^{kt}$, where $P(t)$ is a polynomial.
- If $f(t) = t^n(a\sin(\omega t) + b\cos(\omega t))$ then $x_p(t) = P_1(t)a\sin(\omega t) + P_2(t)b\cos(\omega t)$
- If $f(t) = e^{kt}(a\sin(\omega t) + b\cos(\omega t))$ then $x_p(t) = e^{kt}(A\sin(\omega t) + B\cos(\omega t))$

4 Difference Equations

4.1 First Order Homogeneous Linear Difference Equations

Definition 4.1.1: Order of a Differential Equation

The order of a difference equation is the difference between the highest index of x and lowest.

Definition 4.1.2: First Order Difference Equations

First order difference equations are of the form

$$x_{n+1} = f(x, n)$$

Theorem 4.1.3

The solution to a first order homogeneous linear difference equation

$$x_{n+1} = ax_n$$

is given by $x_n = a^n x_0$.

Definition 4.1.4: Fixed Point

A fixed point of the difference equation $x_{n+1} = f(x_n)$ is a point x_0 such that $f(x_0) = x_0$

Proposition 4.1.5: Stability of Fixed Points

If $|a| < 1$, then for any x_0, x_0 is stable. If $|a| > 1$, then x_0 is unstable.

4.2 Second Order Linear Difference Equations

Theorem 4.2.1

Consider an equation of the form

$$x_{n+2} + ax_{n+1} + bx_n = 0$$

Let $k_{1,2}$ be roots of the equation $k^2 + ak + b = 0$

- If $k_{1,2}$ are distinct, then the general solution is $x_n = Ak_1^n + Bk_2^n$
- If $k_1 = k_2$, then the general solution is $Ak^n + Bnk^n$
- If $k_1, k_2 \in \mathbb{C}$, then let $k_{1,2} = k_\pm = re^{\pm i\theta}$. Then the general solution is $x_n = r^n(A\cos(n\theta) + B\sin(n\theta))$

Theorem 4.2.2

Consider an equation of the form

$$x_{n+2} + ax_{n+1} + bx_n = f(n)$$

Let $k_{1,2}$ be roots of the equation $k^2 + ak + b = 0$.

- If $f(n)$ is a polynomial, try a polynomial.
- If $f(n) = a^n$. Try $x_n = Ca^n$ or Cna^n or Cn^2a^n

5 Partial Differential Equations: Notations

5.1 Terminology

Definition 5.1.1: Partial Differential Equations

A partial differential equations is an equation of the form

$$F(x_1, \dots, x_n, u, \partial_{x_1} u, \dots, \partial_{x_n} u, \partial_{x_1 x_1} u, \dots, \partial_{x_n x_n} u, \dots) = 0$$

We some times use the differential operator $\mathcal{L}$ to replace the differentials of the equation. The equation thus becomes

$$\mathcal{L}(u) = r(x_1, \dots, x_n)$$

where $r(x_1, \dots, x_n)$ is a function only of the independent variables.

Definition 5.1.2: Classification of Partial Differential Equations

We say that a partial differential equation $\mathcal{L}(u) = r(x)$ is

- linear if the differential operator is linear, meaning $\mathcal{L}(au + bv) = a\mathcal{L}(u) + b\mathcal{L}(v)$ for $a, b \in \mathbb{R}$ and u, v functions
- homogenous if $r(x) = 0$, meaning $\mathcal{L}(u) = 0$
- of order n if the highest order of possibly mixed partial derivative of u is n

5.2 Well-Posedness

Definition 5.2.1: Hadmard's Well-posedness

A partial differential equation problem is well-posed if it has a unique solution that continuously depends on the data. In other words, we want existence, uniqueness and stability in the solution.

5.3 Classification of Linear Second Order PDEs

Definition 5.3.1: Classification of Linear Second Order PDEs

Let $\sum_{i=1}^d \sum_{j=1}^d a_{ij}(x) \partial_{x_i x_j} u(x) + b(x) = 0$ where $b(x)$ consists of lower order terms involving derivatives up to order 1. Define the coefficient matrix A by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

for $x \in U \subseteq \mathbb{R}^d$. We say that

- A is elliptic for $x \in U$ if A has d eigenvalues of the same sign
- A is hyperbolic for $x \in U$ if A has $d - 1$ eigenvalues of the same sign and one eigenvalue of the opposite sign
- A is parabolic for $x \in U$ if one eigenvalue is 0 and the other $d - 1$ have the same sign

6 Partial Differential Equations: Transport Equation

6.1 Homogenous Transport Equation

Definition 6.1.1: Transport Equation

The transport equation is the equation

$$\partial_t u(x,t) + v(x,t)\partial_x u(x,t) = 0$$

or

$$\frac{\partial u}{\partial t}(x,t) + v(x,t)\frac{\partial u}{\partial x}(x,t) = 0$$

Definition 6.1.2: Characteristics

The characteristic is the fucntion $\xi(t)$ satisfying the ODE

$$\xi'(t) = v(\xi(t),t)$$

where $\xi(0) = x_0 \in \mathbb{R}$.

Proposition 6.1.3

The solution to the transport equation is constant along the characteristics curve.

Proof. We have that letting the position x depends on t, meaning $x = \xi(t)$,

$$\begin{aligned}\frac{d}{dt}u(\xi(t),t) &= \partial_x u(\xi(t),t)\xi'(t) + \partial_t u(x,t)\\ &= \partial_x u(\xi(t),t)v(x,t) + \partial_u(x,t)\\ &= 0\end{aligned}$$

Thus u is constant along the characteristics. □

6.2 Inhomogenous Transport Equation

Definition 6.2.1: Transport Equation with Source

The transport equation with source term is the equation

$$\partial_t u(x,t) + v(x,t)\partial_x u(x,t) = s(u)$$

or

$$\frac{\partial u}{\partial t}(x,t) + v(x,t)\frac{\partial u}{\partial x}(x,t) = s(u)$$

Proposition 6.2.2

Given the characteristic function $\xi(t)$, the transport equation with source amounts to solving the differential equation

$$\frac{d}{dt}u(\xi(t),t) = s(\xi(t),t,u(\xi(t),t))$$

with given initial conditions.

Proof. If $u(x,t) = u(\xi(t),t)$, then

$$\begin{aligned}\frac{d}{dt}u(\xi(t),t) &= \xi'(t)\partial_x u(\xi(t),t) + \partial_t u(\xi(t),t)\\ &= v(\xi(t),t)\partial_x u(\xi(t),t) + \partial_t u(\xi(t),t)\\ &= s(\xi(t),t,u(\xi(t),t))\end{aligned}$$

thus we are done. □

6.3 Non-Linear Transpot Equation

Definition 6.3.1: Non-Linear Transport Equation

The non-linear transport equation is the equation

$$\partial_t u(x,t) + v(u)\partial_x u(x,t) = 0$$

or

$$\frac{\partial u}{\partial t}(x,t) + v(u)\frac{\partial u}{\partial x}(x,t) = 0$$

The difference being that v is now a function of u.

7 Partial Differential Equations: Wave Equation

7.1 Wave Equation

Definition 7.1.1: Wave Equation in One Spatial Dimension

The wave equation in one dimension is defined to be

$$\partial_{tt}u = c^2\partial_{xx}u$$

where c is the speed of the sound waves.

Proposition 7.1.2

The wave operator corresponds to twice applying the transport operator with opposite velocity signs.

Proof. Note that

$$\begin{aligned}(\partial_t - c\partial_x)(\partial_t + c\partial_x)u &= (\partial_t - c\partial_x)(\partial_t u + c\partial_x u)\\ &= \partial_{tt}u - c\partial_{xt}u + c\partial_{tx}u - c^2\partial_{xx}u\\ &= \partial_t tu - c^2\partial_{xx}u\end{aligned}$$

By expanding, we also have that

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = \partial_t tu - c^2\partial_{xx}u$$

mainly since the partial derivatives are well defined and thus commute. □

Theorem 7.1.3

The general solution of the wave equation is given by

$$u(x,t) = f(x+ct) + g(x-ct)$$

where $f, g : \mathbb{R} \to \mathbb{R}$, using the coordinate method.

Proof. By finding suitable coordinates $\xi = x + ct$ and $\nu = x - ct$, we can linearly transform the equation in to one the only involves $\partial_{\xi\nu}$. In particular, letting $v(\xi,\nu) = u(x,t)$, we have

$$\partial_x u = \partial_\xi v + \partial_\nu v$$

and

$$\partial_t u = c\partial_\xi - v\partial_\nu v$$

so that

$$\partial_{xx}u = \partial_{\xi\xi}v + \partial_{\nu\xi}v + \partial_{\nu\xi}v + \partial_{\nu\nu}v$$

and

$$\partial_{tt}u = c^2\partial_{\xi\xi}v - c^2\partial_{\xi\nu}v - c^2\partial_{\nu\xi}v + c^2\partial_{\nu\nu}v$$

If u, v are smooth enough, we have that

$$\partial_{tt}u - c^2\partial_{xx}u = c^2(-4\partial_{\xi\nu}v)$$

and our question becomes

$$\partial_{\xi\nu}v = 0$$

Integrating the mixed partial derivative by each of its variables once results in the form

$$u(x,t) = f(x+ct) + g(x-ct)$$

□

7.2 Initial Value Problem on Wave Equations

Definition 7.2.1: Cauchy's Initial Value Problem

Let $\Psi : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ be two arbitrary functions. The initial value problem for the wave equation in one dimension consists of finding a function $u : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_{tt} u(x,t) = c^2 \partial_{xx} u(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in \mathbb{R} \\ \partial_t u(x,0) = V(x) & x \in \mathbb{R} \end{cases}$$

Proposition 7.2.2: d'Alembert's Formula

The solution to Cauchy's Initial Value Problem is given by

$$u(x,t) = \frac{1}{2}\left(\Psi(x+ct) + \Psi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(r)\,dr$$

provided that $\Psi \in C^2(\mathbb{R})$ and $V \in C^1(\mathbb{R})$.

Proof. Using the change of coordinates method, we know that $u(x,t) = f(x+ct) + g(x-ct)$ thus

$$\Psi(x) = u(x,0) = f(x) + g(x)$$

and

$$\Psi'(x) = f'(x) + g'(x)$$

and

$$V(x) = \partial_t u(x,0) = cf'(x) - cg'(x)$$

and

$$\frac{1}{c}V(x) = f'(x) - g'(x)$$

Adding and subtracting the identities give

$$f'(x) = \frac{1}{2}(\Psi'(x) + \frac{1}{c}V(x))$$

and

$$g'(x) = \frac{1}{2}(\Psi'(x) - \frac{1}{c}V(x))$$

Integrating the two expressions with respect to x gives

$$f'(x) = \frac{1}{2}\Psi(x) + \frac{1}{2c}\int_0^x V(r)\,dr + C_f$$

and

$$g(x) = \frac{1}{2}\Psi(x) - \frac{1}{2c}\int_0^x V(r)\,dr + C_g$$

Since $f(x) + g(x) = \Psi(x)$, we see that $C_f + C_g = 0$. Thus

$$u(x,t) = \frac{1}{2}\left(\Psi(x+ct) + \Psi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} V(r)\,dr$$

□

Proposition 7.2.3: Preservation in Energy

Let $u \in C^2(\mathbb{R} \times (0, \infty))$ be a solution to the initial value problem for the wave equation. Let

$$E_k(t) = \int_{\mathbb{R}} \frac{1}{2}(\partial_t u)^2(x,t)\,dt$$

denote kinetic energy and

$$E_p(t) = \int_{\mathbb{R}} \frac{1}{2}c^2(\partial_x u)^2(x,t)\,dt$$

denote potential energy. If

- $E_{WE}(t) = E_k(t) + E_p(t)$ is continuously differentiable with respect to t
- $u(x,t)$ and its partial derivatives satisfy $u(x,t) \to 0$ as $x \to \pm\infty$ for all t
- All mixed partial derivatives of u of up to order 2 are integrable with respect to x over $\mathbb{R}$ for all t

Then the total energy $E_{WE}(t)$ is constant in time.

Proof. We want to show that $E'_{WE}(t) = 0$. Thanks to the assumptions, the following integrals and its calculations are allowed. We have

$$\begin{aligned}
E'_{WE}(t) &= \frac{d}{dt}\left(\int_{\mathbb{R}} \frac{1}{2}\left((\partial_t u)^2 + c^2(\partial_x u)^2\right)\,dx\right) \\
&= \int_{\mathbb{R}} (\partial_t u \partial_{tt} u + c^2 \partial_x u \partial_{xt} u)\,dx \\
&= \int_{\mathbb{R}} \partial_t u \partial_{tt} u\,dx + \int_{\mathbb{R}} c^2 \partial_x u \partial_x(\partial_t u)\,dx \\
&= \int_{\mathbb{R}} \partial_t u(\partial_{tt} u - c^2 \partial_{xx} u)\,dx + c^2\left(\lim_{x\to\infty}(\partial_x u(x,t)\partial_t u(x,t)) - \lim_{x\to-\infty}(\partial_x u(x,t)\partial_t u(x,t))\right) \\
&\qquad\qquad \text{(By parts)} \\
&= \int_{\mathbb{R}} (\partial_t u \cdot 0)\,dx \\
&= 0
\end{aligned}$$

thus we are done. ☐

Theorem 7.2.4: Uniqueness and Existence

Let $\Psi \in C^2(\mathbb{R})$ and $V \in C^1(\mathbb{R})$. Then there is a unique solution $u \in C^2(\mathbb{R} \times (0, \infty))$ to the initial value problem for the one dimensional wave equation , given by d'Alembert's formula.

Proof. Let u_1, u_2 be two solutions and $w = u_1 - u_2$. Then w is a solution of the initial value problem where

$$\partial_{tt} w(x,t) = c^2 \partial_{xx} w(x,t)$$

where $(x,t) \in (0,\infty) \times \mathbb{R}$, $w(x,0) = 0$ for $x \in \mathbb{R}$ and $\partial_t w(x,0) = 0$ for $x \in \mathbb{R}$. Thus the corresponding energy

$$E_{WE}(t) = \frac{1}{2}\int_{\mathbb{R}} \left((\partial_t w)^2(x,t) + c^2(\partial_x w)^2(x,t)\right)\,dx$$

is constant. Note that

$$E_{WE}(0) = \frac{1}{2}\int_{\mathbb{R}} \left((\partial_t w)^2(x,0) + c^2(\partial_x w)^2(x,0)\right)\,dx = 0$$

by the initial values. Thus we have that $E_{WE}(t) = E_{WE}(0) = 0$. Thus we have that

$$\partial_t w(x,t) = 0$$

and

$$\partial_x w(x,t) = 0$$

for all (x,t). This means that $w(x,t)$ is constant. But $w(x,0)$ thus $0 = u_1 - u_2$ and $u_1 = u_2$ and we are done. □

Definition 7.2.5: Cauchy's Initial Value Problem on Finite Spatial Domain

Let $\Psi : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ be two arbitrary functions. The initial value problem for the wave equation in one dimension consists of finding a function $u : (0,L) \times (0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_{tt} u(x,t) = c^2 \partial_{xx} u(x,t) & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in (0,L) \\ \partial_t u(x,0) = V(x) & x \in (0,L) \end{cases}$$

Proposition 7.2.6: Change of Variables

Let $\Psi : \mathbb{R} \to \mathbb{R}$ and $V : \mathbb{R} \to \mathbb{R}$ be two arbitrary functions. The initial value problem for the wave equation in one dimension on finite spatial domain is equivalent to solving

$$\begin{cases} \partial_{tt} u(x,t) = \partial_{xx} u(x,t) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \hat{\Psi}(x) = \Psi\left(\frac{L}{\pi}x\right) & x \in (0,\pi) \\ \partial_t u(x,0) = \hat{V}(x) = \frac{L}{\pi c} V\left(\frac{L}{\pi}x\right) & x \in (0,\pi) \end{cases}$$

7.3 Boundary Conditions on Wave Equations

Typical examples: motion of a string with fixed ends.

Definition 7.3.1: Homogenous Dirichlet Boundary Condition

The initial boundary value problem for the wave equation in 1D with homogenous Dirichlet boundary condition consists of finding a function $u : (0,\pi) \times (0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_{tt} u(x,t) = \partial_{xx} u(x,t) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in (0,\pi) \\ \partial_t u(x,0) = V(x) & x \in (0,\pi) \\ u(0,t) = 0 & t \in (0,\infty) \\ u(\pi,t) = 0 & t \in (0,\infty) \end{cases}$$

Definition 7.3.2: Homogenous Neumann Boundary Condition

The initial boundary value problem for the wave equation in 1D with homoegenous Neumann boundary condition consists of finding a function $u : (0,\pi) \times (0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_{tt} u(x,t) = \partial_{xx} u(x,t) & (x,t) \in (0,\pi) \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in (0,\pi) \\ \partial_t u(x,0) = V(x) & x \in (0,\pi) \\ \partial_x u(0,t) = 0 & t \in [0,\infty) \\ \partial_x u(\pi,t) = 0 & t \in [0,\infty) \end{cases}$$

Proposition 7.3.3

Suppose that u_1, u_2 both solve either Neumann or Dirichlet's boundary conditions but with different initial data. Then $w = u_1 - u_2$ is bounded.

Lemma 7.3.4

Suppose that u_1, u_2 both solve either Neumann or Dirichlet's boundary conditions with the same initial data. Then $u_1 = u_2$.

Proposition 7.3.5

If $\Psi \in C^4(\mathbb{R})$ and $V \in C^3(\mathbb{R})$ and $\Psi(x) = \sum_{k=1}^{\infty} A_k \sin(kx)$ and $V(x) = \sum_{k=1}^{\infty} B_k k \sin(kx)$, then the solution to the Dirichlet Boundary Condition is given by

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos(kt) + B_k \sin(kt)) \sin(kx)$$

Proof. Asuume that $u(x,t) = X(x)T(t)$ is separable where $X : (0.\pi) \to \mathbb{R}$ and $T : (0,\infty) \to \mathbb{R}$. Then the wave equation changes to

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

This ratio can only be constant since it cannot depend on t nor x. Let the ratio be λ. Then we now have two ODEs

$$X''(x) - \lambda X(x) = 0$$

and

$$T''(t) - \lambda T(t) = 0$$

I claim that $\lambda \leq 0$. Indeed since integrating any of the above two expression with respect to x from 0 to π gives

$$\int_0^\pi (-X''(x)X(x) + \lambda |X(x)|^2)\, dx = 0$$

$$\int_0^\pi \lambda |X(x)|^2\, dx - [X'(x)X(x)]_0^\pi + \int_0^\pi |X'(x)|^2\, dx$$

$$\int_0^\pi |X'(x)|^2\, dx + \lambda \int_0^\pi |X(x)|^2\, dx \qquad (X(\pi) = X(0) = 0)$$

If $\lambda > 0$ then $X(x) = 0$ for all $x \in (0,\pi)$ which is trivial. If $\lambda = 0$ then $X'(x) = 0$ and $X(x)$ is a constant function. But $X(0) = 0$ thus $X(x)$ is also identicaly zero.

Thus we assume that $\lambda = -\beta^2$ for some $\beta \in \mathbb{R}^+$. The solution to this ODE is precisely

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

and

$$T(t) = A\cos(\beta t) + B\sin(\beta t)$$

From $X(0) = 0$ we have that $C = 0$ and $D \neq 0$. $X(\pi) = 0$ implies that $\beta\pi = k\pi$ for $k \in \mathbb{N} \setminus \{0\}$. Taking into account $T(t)$, we now have that

$$u_j(x,t) = (A_j \cos(kt) + B_j \sin(kt)) \sin(kx)$$

for $A_k, B_k \in \mathbb{R}$. Since any $k \in \mathbb{N}$ could solve the wave equation, we sum all the frequencies to get

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos(kt) + B_k \sin(kt)) \sin(kx)$$

□

Proposition 7.3.6

If $\Psi \in C^4(\mathbb{R})$ and $V \in C^3(\mathbb{R})$ and $\Psi(x) = \sum_{k=1}^{\infty} A_k \cos(kx)$ and $V(x) = \sum_{k=1}^{\infty} B_k k \cos(kx)$, then the solution to the Dirichlet Boundary Condition is given by

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos(kt) + B_k \sin(kt)) \sin(kx)$$

Proof. We argue similarly by assuming separation of variables and take the ratio constant to be $\lambda = -\beta^2$ for some $\beta \in \mathbb{R}^+$. The solution to this ODE is precisely

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

and

$$T(t) = A\cos(\beta t) + B\sin(\beta t)$$

From $X'(0) = 0$ we have that $D = 0$ and $C \neq 0$. $X'(\pi) = 0$ implies that $\beta\pi = k\pi$ for $k \in \mathbb{N}$. Taking into account $T(t)$, we now have that

$$u_j(x,t) = (A_j \cos(kt) + B_j \sin(kt)) \cos(kx)$$

for $A_k, B_k \in \mathbb{R}$. Since any $k \in \mathbb{N}$ could solve the wave equation, we sum all the frequencies to get

$$u(x,t) = \sum_{k=0}^{\infty} (A_k \cos(kt) + B_k \sin(kt)) \cos(kx)$$

$\square$

8 Partial Differential Equations: Heat Equation

8.1 Heat Equation

Definition 8.1.1: One Dimensional Heat Equation

The one dimensional heat equation is the equation

$$\partial_t u = k\partial_{xx}u$$

for a function $u(x,t)$ where $k>0$.

8.2 Heat Equation: Dirichlet Boundary Conditions

Definition 8.2.1: Dirichlet Boundary Conditions

The initial boundary value problem for the heat equation in one dimension with inhomogenous Dirichlet boundary condition consists of finding a fuinction $u:(0,L)\times(0,\infty)\to\mathbb{R}$ such that

$$\begin{cases} \partial_t u(x,t) = k\partial_{xx}u(x,t) & (x,t)\in(0,L)\times(0,\infty) \\ u(x,0)=\Psi(x) & x\in[0,L] \\ u(0,t)=g_0(t) & t\in[0,\infty) \\ u(L,t)=g_L(t) & t\in[0,\infty) \end{cases}$$

Definition 8.2.2: Space-Time Boundaries

We denote the space time rectangle by

$$V_{L,T}=(0,L)\times(0,T]$$

and the parabolic boundary by

$$\Gamma_{L,T}=\{(x,t)\in\overline{V_{L,T}}|x\in\{0,L\}\text{ or }t=0\}$$

Theorem 8.2.3: Maximum Priniciple

Assume that $u(x,t)\in C^2(\overline{V_{L,T}})$ solves the heat equation. Then the maximum and minimum of u are attained on the parabolic boundary $\Gamma_{L,T}$

Theorem 8.2.4: Uniqueness and Stability

Let $u_1,u_2\in C^2([0,L]\times[0,\infty])$ denote two solutions to the initial boundary value problem with respective initial date Ψ_1,Ψ_2 and boundary data $g_{1,0},g_{1,L},g_{2,0},g_{2,L}$. Then

$$\max_{(x,t)\in V_{L,T}}|u_1(x,t)-u_2(x,t)|\leq\max\{A,B,C\}$$

where

- $A=\max_{x\in[0,L]}|\Psi_1(x)-\Psi_2(x)|$
- $B=\max_{t\in[0,T]}|g_{1,0}-g_{2,0}|$
- $C=\max_{t\in[0,T]}|g_{1,L}-g_{2,L}|$

In particular, if $\Psi_1=\Psi_2$, $g_{1,0}=g_{2,0}$ and $g_{1,L}=g_{2,L}$ then $u_1=u_2$.

8.3 Heat Equation: Neumann Boundary Condition

Definition 8.3.1: Inhomogenous Neumann Boundary Condition

The initial boundary value problem for the heat equation in one dimension with inhomogenous Neumann boundary condition consists of finding a function $u : (0, L) \times (0, \infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u(x,t) = k\partial_{xx} u(x,t) & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in [0,L] \\ \partial_x u(0,t) = h_0(t) & t \in [0,\infty) \\ \partial_x u(L,t) = h_L(t) & t \in [0,\infty) \end{cases}$$

where $h_0(0) = \Psi'(0)$ and $h_L(0) = \Psi'(L)$

Proposition 8.3.2

The solution to the heat equation in Neumann boundary condition is unique ad is stable in the mean square sense.

8.4 Duhamel's Principle

Theorem 8.4.1: Partial Solution to Heat Equation with Dirichlet Boundary Conditions

The heat equation with Dirichlet boundary conditions has solution of the form $u(x,t) = u_B(x,t) + u_I(x,t) + w(x,t)$ where

$$\begin{cases} h : [0,L] \to \mathbb{R} & \text{is smooth such that } h(0) = 0 \text{ and } h(L) = 1 \\ u_B(x,t) = g_0(t) + (g_L(t) - g_0(t))h(x) & (x,t) \in [0,L] \times [0,\infty) \\ u_I(x,t) = \Psi(x) - u_B(x,0) & (x,t) \in [0,L] \times [0,\infty) \\ w(x,t) & (x,t) \in [0,L] \times [0,\infty) \end{cases}$$

where $w(x,t)$ satisfies the following homogenous boundary and initial values:

$$\begin{cases} \partial_t w(x,t) - k\partial_{xx} w(x,t) = f(x,t) & (x,t) \in [0,L] \times [0,\infty) \\ w(x,0) = 0 & x \in [0,L] \\ w(0,t) = 0 & t \in [0,\infty) \\ w(L,t) = 0 & t \in [0,\infty) \end{cases}$$

where

$$f(x,t) = -(g_0'(t) + (g_L'(t) - g_0'(t))h(x)) + k(g_L(t) - g_0(t))h''(x) + k(\Psi''(x) - (g_L(0) - g_0(0)))h''(x)$$

Theorem 8.4.2: Duhamel Principle

Let f be any function. Consider the following homogenous boundary and initial values differential equation:

$$\begin{cases} \partial_t w(x,t) - k\partial_{xx} w(x,t) = f(x,t) & (x,t) \in [0,L] \times [0,\infty) \\ w(x,0) = 0 & x \in [0,L] \\ w(0,t) = 0 & t \in [0,\infty) \\ w(L,t) = 0 & t \in [0,\infty) \end{cases}$$

Let $v(x,t,\tau)$ denote a smooth solution to the following conditions with parameter τ:

$$\begin{cases} \partial_t v(x,t) = k\partial_{xx}v(x,t) & (x,t) \in (0,L) \times (\tau,\infty) \\ v(x,\tau) = f(x,\tau) & x \in [0,L] \\ v(0,t) = 0 & t \in [\tau,\infty) \\ v(L,t) = 0 & t \in [\tau,\infty) \end{cases}$$

Then

$$w(x,t) = \int_0^t v(x,t,\tau)\,d\tau$$

solves above homogenous boundary condition and initial values of the heat equation.

It now remains to solve the function v in Duhamel's principle, which is simply the heat equation and homogenous Dirichlet boundary conditions.

Proposition 8.4.3

Suppose that f has fourier series expansion as $f(x,0) = \sum_{k=1}^n E_k \sin\left(\frac{k\pi}{L}x\right)$. The solution to the following initial value problem

$$\begin{cases} \partial_t v(x,t) = k\partial_{xx}v(x,t) & (x,t) \in (0,L) \times (\tau,\infty) \\ v(x,\tau) = f(x,\tau) & x \in [0,L] \\ v(0,t) = 0 & t \in [\tau,\infty) \\ v(L,t) = 0 & t \in [\tau,\infty) \end{cases}$$

is given by

$$v(x,t) = \sum_{i=0}^{\infty} e^{-k\beta_i^2 t} D_i \sin(\beta_i x)$$

where $\beta_i = \frac{i\pi}{L}$ and $D_i =$

Proof. WLOG take $\tau = 0$. We perform separation of variables again and assume that $v(x,t) = X(x)T(t)$. Subsituting into the heat equation gives

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

This ratio can only be constant since it cannot depend on x nor t. Similar to before we notice that $\lambda < 0$ thus let $\lambda = -\beta^2$. The solution to the ODE with X is given by

$$X(x) = A_k \cos(\beta x) + B_k \sin(\beta x)$$

Using the fact that $X(0) = 0$ and $X(\pi) = L$, we obtain

$$X_k(x) = D_k \sin(\beta_k x)$$

for $\beta_k = \frac{k\pi}{L}$ for $k \in \mathbb{N} \setminus \{0\}$. Now solving the ODE for T, we get

$$T_k(t) = C_k e^{-k\beta_k^2 t}$$

Now absorbing the coefficients C_k into D_k, we obtain the general solution

$$v(x,t) = \sum_{k=1}^{\infty} D_k e^{-k\beta_k^2 t} \sin\left(\frac{k\pi}{L}x\right)$$

Now when $t = 0$,

$$v(x,0) = \sum_{k=1}^{\infty} D_k \sin\left(\frac{k\pi}{L}x\right) = f(x,0)$$

Thus we can directly compare coefficients to get $D_k = E_k$. □

Recalling all the steps we used, to solve an inhomogenous boundary conditions (the solution to Neumann is similar), we have the following procedure:

Step 1: Find u_B and u_I to get solutions with respect to the boundary conditions and initial value conditions.
Step 2: Caluculate $f(x,t)$ from $w = u - u_B - u_I$ and the heat equation.
Step 3: Solve the homogenous boundary conditions problem to obtain v Step 4: Use Duhamel's principle to find $w(x,t) = \int_0^t v(x,t,\tau)\,d\tau$

8.5 Cauchy Problem and the Fundamental Solution

Definition 8.5.1: Cauchy Problem of the Heat Equation

The Cauchy problem for the heat equation in one dimension consists of finding a fuinction $u : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u(x,t) = k\partial_{xx} u(x,t) & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \Psi(x) & x \in \mathbb{R} \end{cases}$$

Theorem 8.5.2: Fundamental Solution of the Heat Equation

Let Ψ be continuous and bounded. Then

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}}\,\Psi(y)\,dy$$

is smooth and solves the Cauchy problem of the heat equation.

9 Partial Differential Equation: Laplace's Equation

9.1 Laplace's Equation

Definition 9.1.1: Laplacian Operator

Let $u : \mathbb{R}^2 \to \mathbb{R}$. Define the laplacian operaetor on u by

$$\Delta u(x, y) = \partial_{xx} u + \partial_{yy} u = \nabla \cdot \nabla u$$

Proposition 9.1.2: Change of Variables

Using the subtitution $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we can change the laplacian operator into the form

$$\Delta u(x_1, x_2) = \Delta v(r, \theta) = \partial_{rr} u + \frac{1}{r^2}\partial_{\theta\theta} u + \frac{1}{r}\partial_r u$$

Theorem 9.1.3

The solution to the laplace's equation where u is a function of r and θ with the following conditions

$$\begin{cases} \partial_{rr} u + \frac{1}{r^2}\partial_{\theta\theta} u + \frac{1}{r}\partial_r u = 0 & u : (0, k) \times (0, \alpha) \\ u(k, \theta) = g(\theta) & \theta \in (0, \alpha) \\ u(r, 0) = u(r, \alpha) = 0 & r \in (0, k) \end{cases}$$

is given by

$$u(r, \theta) = \sum_{k=1}^{\infty} r^{\frac{k\pi}{\alpha}} A_k \sin\left(\frac{k\pi}{\alpha}\theta\right)$$

9.2 Fundamental Solution and Green's Function

Definition 9.2.1: Poisson Problem

The poisson problem consists of finding a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$-\Delta u = f$$

Theorem 9.2.2: Fundamental Solution for the Laplacian

Let $f \in C^2(\mathbb{R}^n)$ be a function such that $\{x | f(x) \neq 0\}$ is bounded. Then the solution to the Poisson problem is given by

$$u(x_1, \dots, x_n) = \int_{\mathbb{R}^n} F(x - y) f(y)\, dy$$

where its second order partial derivatives are continuous and

$$F(x) = \begin{cases} -\frac{|x|}{2} & n = 1 \\ -\frac{1}{2\pi}\ln(|x|) & n = 2 \\ \frac{1}{n(n-2)V_n(B_1(0))} \cdot \frac{1}{|x|^{n-2}} & n \geq 3 \end{cases}$$

where $V_n(B_1(0))$ is the volume of the unit ball in n dimensions.

Definition 9.2.3: Poisson Problem with Boundary Conditions

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded domain. Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$. The poisson problem with boundary conditions consists of finding a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases}$$

Theorem 9.2.4

Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is harmonic. Then

$$\max_{x \in \Omega} u(x) = \max_{x \in \partial\Omega}\{u(x)\}$$

and

$$\min_{x \in \Omega} u(x) = \min_{x \in \partial\Omega}\{u(x)\}$$

The following principle gives the uniqueness and stability of our solution for the laplacian operator.

Theorem 9.2.5: Dirichlet Principle

A function $u \in \mathcal{A} = \{v \in C^2(\partial\Omega) | v = g \text{ in } \partial\Omega\}$ solves $-\Delta u = f$ in Ω and $u = g$ in $\partial\Omega$ if and only if it minimizes I, meaning

$$I(u) = \min_{w \in \mathcal{A}}\{I(w)\}$$

where

$$I(w) = \int_\Omega \frac{1}{2}|\nabla w|^2 - fw\,dx$$