Fiber Bundles and Fibrations

Labix

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Abstract

- Notes on Algebraic Topology by Oscar Randal-Williams

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1 The Category of Fiber Bundles

1.1 Fiber Bundles

Definition 1.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and $p : E \to B$ a trivial map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$ for all $b \in B$
- $p : E \to B$ is surjective
- For every $x \in B$, there is an open neighbourhood $U \subset B$ of x and a fiber preserving homomorphism $\Psi_U : p^{-1}(U) \to U \times F$ that is a homeomormorphism such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Psi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  where π is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

Definition 1.1.2: Map of Fiber Bundles

Let (F_1, E_1, B_1) and (F_2, E_2, B_2) be fiber bundles. A morphism of fiber bundles is a pair of basepoint preserving continuous maps $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

Such a map of fibrations determine a continuous of the fibers $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$.

A map of fibrations $(\tilde{f}, f)$ is said to be an isomorphism if there is a map $(\tilde{g} : E_2 \to E_1, g : B_2 \to B_1)$ such that $\tilde{g}$ is the inverse of $\tilde{f}$ and g is the inverse of f.

Definition 1.1.3: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration $B \times F \to B$.

Definition 1.1.4: Sections

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map $s : B \to E$ such that $p \circ s = \text{id}_B$. Let $U \subset B$ be an open set. A local section of the fiber bundle on U is a map $s : U \to B$ such that $p \circ s = \text{id}_U$.

Definition 1.1.5: The Pullback Bundle

Let $p : E \to B$ be a fiber bundle with fiber F. Let $f : B' \to B$ be a continuous function. Define the pullback of p by f to be the space

$$f^*(E) = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$$

1.2 G-Bundles and the Structure Groups

Notice that for non empty intersections $U_i \cap U_j$ for U_i, U_j open sets in B, there is a well defined homeomorphism

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

This is reminiscent of properties of an atlas on M.

Definition 1.2.1: G-Atlas

Let (F, E, B) be a fiber bundle. Let G be topological group with a continuous faithful action on F. A G-atlas on (F, E, B) is a set of local trivalization charts $\{(U_k, \varphi_k) \mid k \in I\}$ such that the following are true.

- For (U_k, φ_k) a chart, define $\varphi_{i,x} : F \to F$ by $f \mapsto \varphi_i(x, f)$. Then the homeomorphism

$$\varphi_{j,x} \circ \varphi_{i,x}^{-1} : F \to F$$

  for $x \in U_i \cap U_j \neq \emptyset$ is an element of G.
- For $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to G$ defined by

$$g_{ij}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$$

  is continuous.

If the (F, E, B) is a fiber bundle with $F = \mathbb{R}$, then it is often seen that $G = GL(n, \mathbb{R})$. Similarly, if $F = \mathbb{C}$ then the structure group is $G = GL(n, \mathbb{C})$.

Definition 1.2.2: Equivalent G-Atlas

Two G-atlases on a fiber bundle (F, E, B) is said to be equivalent if their union is a G-atlas.

Definition 1.2.3: G-Bundle

Let G be a topological group. A G-bundle is a fiber bundle (F, E, B) together with an equivalence class of G-atlas. In this case, G is said to be the structure group of the fiber bundle.

The structure group and the trivialization charts completely determine the isomorphism type of the fiber bundle.

1.3 Morphisms of G-Bundles

Definition 1.3.1: Morphisms of G-Bundles

Let G be a topological group. A morphism of G-bundles is a morphism of fiber bundles $(\tilde{h}, h) : (F, E_1, B_1) \to (F, E_2, B_2)$ where the two are G-bundles, such that the following are true.

- Let U_i be open in B_1 and V_j be open in B_2. Let $x \in U_u \cap h^{-1}(V_j)$. Let $\widetilde{h_{(E_1)_x}} : (E_1)_x \to (E_2)_{f(x)}$ be the map induced by $\tilde{h} : E_1 \to E_2$. Then the map
$$\varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1} : F \to F$$
is an element of G.
- The map $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ defined by
$$\widetilde{g_{ij}}(x) = \varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1}$$
is continuous.

It is easy to see that the mapping transformations $\widetilde{g_{ij}}$ satisfy the following two relations:

- $\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap U_j \cap h^{-1}(V_k)$
- $g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap h^{-1}(V_j \cap V_k)$

g'_{jk} here refers to the transition charts in (F, E_2, B_2).

Just as the structure groups and trivialization charts determine the isomorphism type of a fiber bundle, the $\widetilde{g_{ij}}$ and a map of base space $h : B_1 \to B_2$ completes determines a morphism of G-bundle.

Lemma 1.3.2

Let (F, E_1, B_1) and (F, E_2, B_2) be two G-bundles for a topological group G with the same fiber F. Suppose that we have the following.

- A map $h : B_1 \to B_2$ of base space
- $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ a set of continuous maps such that
$$\begin{aligned} \widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap U_j \cap h^{-1}(V_k) \\ g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap h^{-1}(V_j \cap V_k) \end{aligned}$$

Then there exists a unique G-bundle morphism having h as the map of base space and having $\{\widetilde{g_{ij}} \mid i, j \in I\}$ as its mapping transformations.

1.4 Principal G-Bundles

Definition 1.4.1: Principal Bundles

Let G be a topological group. A principal G-bundle is a G-bundle (F, E, B) together with a continuous group action G on E such that the following are true.

- The action of G preserves fibers. This means that $g \cdot x \in E_b$ if $x \in E_b$. (This also means that G is a group action on each fiber)
- The action of G on each fiber is free and transitive
- For each $x \in E_b$, the map $G \to E_b$ defined by $g \mapsto g \cdot x$ is homeomorphism.
- Local triviality condition: If $\Psi_U : p^{-1}(U) \to U \times F$ are the local triviality maps, then each Ψ_U are G-equivariant maps.

Note that since G is homeomorphic to each fiber E_b of the total space, we can think of the action of G on the fiber simply becomes left multiplication.

For those who know what homogenous spaces are, principal bundles are G-bundles such that F is a principal homogenous space for the left action of G itself.

Conversely, given a continuous group action on a space, we can ask in what conditions will the space be a principal bundle over the orbit space.

Proposition 1.4.2

Let E be a space with a free G action. Let $p : E \to E/G$ be the projection map to the orbit space. If for all $x \in E/G$, there is a neighbourhood U of x and a continuous map $s : U \to E$ such that $p \circ s = \text{id}_U$, then $(G, E, E/G)$ is a principal G-bundle.

This proposition essentially means that if for each point in E/G, there is a local section, then it is sufficient for E to be a principal G bundle over E/G.

Theorem 1.4.3

A principal G-bundle is trivial if and only if it admits a global section.

This is entirely untrue for general bundles. For examples, the zero section of a fiber bundle is a global section.

1.5 Classifying Space

Definition 1.5.1: Universal G-Bundles

Let G be a topological group. A principal G-bundle (F, E, B) is said to be universal if for any space X, the induced pullback map

$$\psi : [X, B] \to \text{Prin}_G(X)$$

defined by $f \mapsto f^*(E)$ is a bijective correspondence.

Theorem 1.5.2

Let (F, E, B) be a principal G-bundle. If E is contractible then (F, E, B) is a universal G-bundle.

Theorem 1.5.3

Let (F, E_1, B_1) and (F, E_2, B_2) be universal principal G-bundles. Then there exists a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

such that f is a homotopy equivalence. In particular, this means that any two universal principal G-bundles are homotopy equivalent.

Definition 1.5.4: Classifying Space

Let G be a topological group. The classifying space BG of G is the homotopy type of the universal principal G-bundle. Also denote EG as the total space of the universal G-bundle.

2 The Category of Compactly Generated Spaces

2.1 Compactly Generated Spaces

Definition 2.1.1: Compactly Generated Spaces

Let X be a space. We say that X is compactly generated (k-space) if for every set $A \subseteq X$, A is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$. We denote $\mathcal{K}$ as the category of compactly generated spaces.

Definition 2.1.2: Category of Compactly Generated Spaces

Define the category of compactly generated spaces $\mathcal{K}$ to be the full subcategory of Top consisting of spaces that are compactly generated. In other words, $\mathcal{K}$ consists of the following data:

- Obj($\mathcal{K}$) consists of all spaces that are compactly generated.
- For $X, Y \in$ Obj($\mathcal{K}$), the morphisms are

$$\text{Hom}_{\mathcal{K}}(X, Y) = \text{Hom}_{\text{Top}}(X, Y)$$

- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces $\mathcal{K}_*$.

Definition 2.1.3: New k-space from Old

Let X be a space. Define $k(X)$ to be the set X together with the topology defined as follows: $A \subseteq X$ is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Lemma 2.1.4

Let X be a space. Then $k(X)$ is a compactly generated space. Moreover, k defines a functor

$$k : \mathcal{T}_2 \to \mathcal{K}$$

from the category of Hausdorff spaces to $\mathcal{K}$.

Unfortunately $X \times Y$ may not be compactly generated even when X and Y are. But as it turns out, products do exists in $\mathcal{K}$ and are given by $k(X \times Y)$.

Proposition 2.1.5

Let X, Y be compactly generated spaces. Then the product of X and Y in the category of compactly generated spaces is given by

$$k(X \times Y)$$

Definition 2.1.6: The Mapping Space

Let X and Y be compactly generated. Define the mapping space of X and Y by

$$\text{Map}(X, Y) = Y^X = k(\text{Hom}_{\mathcal{K}}(X, Y))$$

Theorem 2.1.7

Let X, Y, Z be compactly generated. Then the functors $k(- \times Y) : \mathcal{K} \to \mathcal{K}$ and $\text{Map}(Y, -) : \mathcal{K} \to \mathcal{K}$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathcal{K}}(k(X \times Y), Z) \cong \text{Hom}_{\mathcal{K}}(X, \text{Map}(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}(k(X \times Y), Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

Definition 2.1.8: Loop Spaces

Let X be a space with a chosen basepoint. Define the loop space of (X, x_0) to be

$$\Omega X = \text{Map}_*(S^1, X)$$

2.2 The Smash Product

Definition 2.2.1: The Smash Product

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Define the smash product of the two pointed spaces to be the pointed space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

together with the point (x_0, y_0).

Proposition 2.2.2

Let X, Y, Z be compactly generated spaces with a chosen base point. Then the following are true.

- $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
- $X \wedge Y \cong Y \wedge X$

Note that this is not true if we do not restrict the spaces to the category of compactly generated spaces.

Lemma 2.2.3

Let X be a space. Then the reduced suspension and the smash product with the circle

$$\Sigma X \cong X \wedge S^1$$

are homeomorphic spaces.

Theorem 2.2.4

Let X, Y, Z be compactly generated with a chosen basepoint. Then the functors $- \wedge Y : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(Y, -) : \mathcal{K}_* \to \mathcal{K}_*$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathcal{K}_*}(X \wedge Y, Z) \cong \text{Hom}_{\mathcal{K}_*}(X, \text{Map}_*(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}_*(X \wedge Y, Z) \cong \text{Map}_*(X, \text{Map}_*(Y, Z))$$

Corollary 2.2.5

Let X be a compactly generated space with a chosen basepoint. Then there is a homeomorphism
$$\text{Map}_*(\Sigma X, Y) \cong \text{Map}_*(X, \Omega Y)$$
given by adjunction of the functors $- \wedge S^1 : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(S^1, -) : \mathcal{K}_* \to \mathcal{K}_*$.

3 Fibrations and Cofibrations

From here onwards we assume that all spaces are compactly generated unless otherwise stated.

3.1 Fibrations and The Homotopy Lifting Property

Definition 3.1.1: The Homotopy Lifting Property

Let $p : E \to B$ be a map and let X be a space. We say that p has the homotopy lifting property with respect to X if for every homotopy $H : X \times I \to B$ and a lift $\widetilde{H(-,0)} : X \to E$ of $H(-,0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \widetilde{H}"] & B
\end{tikzcd}$$

Definition 3.1.2: Fibrations

We say that a map $p : E \to B$ is a fibration if it has the homotopy lifting property with respect to all topological spaces X. We call B the base space and E the total space.

Definition 3.1.3: Pullbacks of a Fibration

Let $p : E \to B$ be a fibration and let $f : B' \to B$ be a continuous map. Define the pullback of p by f to be

$$f^*(E) = \{(b', e) \in B' \times E \mid f(b') = p(e)\}$$

together with the projection map $p_f : f^*(E) \to B'$.

Proposition 3.1.4

Let $p : E \to B$ be a fibration and let $f : B' \to B$ be continuous. Then the map $f^*(E) \to B'$ is a fibration. Moreover, the following diagram commutes:

$$\begin{tikzcd}
f^*(E) \arrow[r] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\
B' \arrow[r, "f"'] & B
\end{tikzcd}$$

where the top map is given by the projection to E.

3.2 Replacing Maps by Fibrations

Definition 3.2.1: The Mapping Path Space

Let $f : X \to Y$ be a map of spaces. Denote $\pi : X^I \to X$ the fibration of the mapping space defined by $\pi(\phi) = \phi(0)$. Define the mapping path space to be

$$P_f = f^*(Y^I) = \{(x, \phi) \subseteq X \times Y^I \mid f(x) = \pi(\phi) = \phi(0)\}$$

We can factorize any continuous map into a fibration and a homotopy equivalence.

Theorem 3.2.2

Let $f : X \to Y$ be a map. Then $\pi : P_f \to Y$ defined by $\pi(x, \phi) = \phi(1)$ is a fibration. Moreover, there exists a homotopy equivalence $h : X \to P_f$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "\exists h"'] & & Y \\ & P_f \arrow[ur, "\pi"'] & \end{tikzcd}$$

3.3 Cofibrations and The Homotopy Extension Property

Definition 3.3.1: The Homotopy Extension Property

Let $i : A \to X$ be a map and let Y be a space. We say that i has the homotopy lifting property with respect to Y if for every homotopy $H : A \times I \to Y$ such that

$$H \circ i_0 = f \circ i$$

for $i_0 : A \times \{0\} \to A \times I$ the inclusion map, there exists a homotopy $\widetilde{H} : X \times I \to Y$ such that the following diagram commute:

$$\begin{tikzcd} A \times \{0\} \arrow[rr, hook, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "H"'] \arrow[dd, "i \times \mathrm{id}"] \\ & Y & \\ X \times \{0\} \arrow[ur, "f"] \arrow[rr, hook] & & X \times I \arrow[ul, dashed, "\exists \widetilde{H}"'] \end{tikzcd}$$

Definition 3.3.2: Cofibrations

We say that a map $i : A \to X$ is a fibration if it has the homotopy extension property for all spaces Y.

Definition 3.3.3: Pullbacks of a Cofibration

Let $i : A \to X$ be a cofibration and let $g : A \to C$ be a map. Define the pullback of i by g to be

$$f_*(X) = \frac{X \amalg C}{i(a) \sim g(a)}$$

together with the inclusion map $i_f : X \to f_*(X)$.

Proposition 3.3.4

Let $i : A \to X$ be a cofibration and let $g : A \to C$ be a map. Then the map $C \to f^*(X)$ is a cofibration. Moreover, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d] \\ X \arrow[r, "i_f"'] & f_*(X) \end{tikzcd}$$

where the map $C \to f_*(X)$ is the inclusion map.

3.4 Replacing Maps by Cofibrations

Definition 3.4.1: Mapping Cylinder

Let $f : A \to X$ be a map. Define the mapping cylinder to be

$$M_f = \frac{(A \times I) \amalg X}{(a, 1) \sim f(a)}$$

together with the induced topology.

Theorem 3.4.2

Let $f : A \to X$ be a map. Then the inclusion map $i : A \to M_f$ defined by $i(a) = [a, 0]$ is a cofibration. Moreover, there exists a homotopy equivalence $h : M_f \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & X \\ & M_f \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

3.5 Fibers and Cofibers

Definition 3.5.1: Fibers of a Fibration

Let $p : E \to B$ be a fibration. Define the fiber of p at $b \in B$ to be

$$E_b = p^{-1}(b)$$

Proposition 3.5.2

Let $p : E \to B$ be a fibration. Let b_1 and b_2 lie in the same path component of B. Then there is a homotopy equivalence

$$E_{b_1} \simeq E_{b_2}$$

Definition 3.5.3: Homotopy Fibers

Let $f : X \to Y$ be a map. Define the homotopy fiber of f to be

$$F_f = \{(x, \phi) \in X \times Y^I \mid f(x) = \phi(1)\}$$

where P_f is the mapping path space of f.

Note the difference between homotopy fibers and the mapping path space. The latter is defined by considering the fibration $\pi : X^I \to X$ where $\pi(\phi) = \phi(0)$. But homotopy fibers are defined the end point $\phi(1)$. In fact, this is the main ingredient in proving that this notion is homotopy equivalent to the usual notion of fibers.

Proposition 3.5.4

Let $p : E \to B$ be a fibration. Then the homotopy fibers of p are homotopy equivalent to the fibers of p.

Instead of defining cofibers and then showing homotopy equivalence cofiberwise, we will take the approach of homotopy cofibers and straight up define it without mentioning the choice of a point on the cofibration.

Definition 3.5.5: Mapping Cone

Let $f : A \to X$ be a map. Define the mapping cone to be

$$C_f = \frac{(A \times I) \amalg X}{(a,1) \sim f(a), A \setminus \{0\}}$$

Definition 3.5.6: Homotopy Cofibers

Let $f : X \to Y$. Define the homotopy cofiber of f to be the mapping cone C_f.

3.6 The Fiber and Cofiber Sequences

Definition 3.6.1: Path Spaces

Let (X, x_0) be a pointed space. Define the path space of (X, x_0) to be

$$PX = \{\phi : (I,0) \to (X,x_0) \mid \phi(0) = x_0\} = \text{Map}((I,0),(X,x_0))$$

together with the topology of the mapping space.

Theorem 3.6.2

Let X be a space. Then the following are true.

- The map $\pi : PX \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber ΩX
- The map $\pi : X^I \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber homeomorphic to PX.

We now write a fibration as a sequence $F \to E \to B$ for F the fiber of the fibration $p : E \to B$. This compact notation allows the following theorem to be formulated nicely.

Theorem 3.6.3

Let $f : X \to Y$ be a fibration with homotopy fiber F_f. Let $\iota : \Omega Y \to F_f$ be the inclusion map and $\pi : F_f \to X$ the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega F_f \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega_Y \xrightarrow{\iota} F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

where any two consecutive maps form a fibration. Moreover, $-\Omega f : \Omega X \to \Omega Y$ is defined as

$$(-\Omega f)(\zeta)(t) = (f \circ \zeta)(1-t)$$

for $\zeta \in \Omega X$.

There is then the dual notion of loop spaces and the corresponding sequence. Write a cofibration $f : A \to X$ with homotopy cofiber B as $B \to A \to X$.

Theorem 3.6.4

Let $f : X \to Y$ be a cofibration with homotopy cofiber C_f. Let $i : Y \to C_f$ be the inclusion map and $\pi : C_f \to C_f/Y \cong \Sigma X$ be the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

where any two consecutive maps form a cofibration. Moreover, $-\Sigma f : \Sigma X \to \Sigma Y$ is defined by
$$(-\Sigma f)(x \wedge t) = f(x) \wedge (1-t)$$

Theorem 3.6.5

Let $p : E \to B$ be a fibration over a connected space B with fiber F. Let $\iota : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{\iota_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

for $e_0 \in E$ and $b_0 = p(e_0)$.

3.7 Serre Fibrations

Definition 3.7.1: Serre Fibration

We say that a map $p : E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

Proposition 3.7.2

Every (Hurewicz) fibration is a Serre fibration. Every fiber bundle is a Serre fibration.

We can provide a partial converse for the fact that every fiber bundle is a Serre fibration.

Proposition 3.7.3

Let $p : E \to B$ be a fiber bundle. If B is paracompact, then p is a (Hurewicz) fibration.

4 Characteristic Classes

Definition 4.0.1: Characteristic Classes

Let G be a topological group and X a space. Denote $\text{Prin}_G(X)$ the isomorphism classes of principal G-bundles over X. Let H^* be a cohomology functor. A characteristic class for G is a natural transformation c from Prin_G to H^*.

Explicitly, if $p : E \to B$ is a principal G-bundle, then c assigns p to the collection of cohomology groups $c(p) \in H^*(X)$.

5 Obstruction Theory