# The Topology of Fiber Bundles

## Labix

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## Abstract

• Notes on Algebraic Topology by Oscar Randal-Williams

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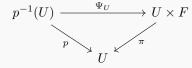
## 1 Fibrations

#### 1.1 Fiber Bundles

#### Definition 1.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and  $p: E \to B$  a trivial map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$  for all  $b \in B$
- $p: E \to B$  is surjective
- For every  $x \in B$ , there is an open neighbourhood  $U \subset B$  of x and a fiber preserving homomorphism  $\Psi_U : p^{-1}(U) \to U \times F$  that is a homeomorphism such that the following diagram commutes:



where  $\pi$  is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

#### Definition 1.1.2: Map of Fiber Bundles

Let  $(F_1, E_1, B_1)$  and  $(F_2, E_2, B_2)$  be fiber bundles. A morphism of fiber bundles is a pair of basepoint preserving continuous maps  $(\tilde{f}: E_1 \to E_2, f: B_1 \to B_2)$  such that the following diagram commutes:

$$E_1 \xrightarrow{\tilde{f}} E_2$$

$$\downarrow^{p_1} \qquad \qquad \downarrow^{p_2}$$

$$B_1 \xrightarrow{f} B_2$$

Such a map of fibrations determine a continuous of the fibers  $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$ .

A map of fibrations  $(\tilde{f}, f)$  is said to be an isomorphism if there is a map  $(\tilde{g}: E_2 \to E_1, g: B_2 \to B_1)$  such that  $\tilde{g}$  is the inverse of  $\tilde{f}$  and g is the inverse of f.

#### Definition 1.1.3: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration  $B \times F \to B$ .

#### **Definition 1.1.4: Sections**

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map  $s: B \to E$  such that  $p \circ s = \mathrm{id}_B$ . Let  $U \subset B$  be an open set. A local section of the fiber bundle on U is a map  $s: U \to B$  such that  $p \circ s = \mathrm{id}_U$ .

## 1.2 G-Bundles and the Structure Groups

Notice that for non empty intersections  $U_i \cap U_j$  for  $U_i, U_j$  open sets in B, there is a well defined homeomorphism

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

This is reminiscent of properties of an atlas on M.

## Definition 1.2.1: G-Atlas

Let (F, E, B) be a fiber bundle. Let G be topological group with a continuous faithful action on F. A G-atlas on (F, E, B) is a set of local trivalization charts  $\{(U_k, \varphi_k) \mid k \in I\}$  such that the following are true.

• For  $(U_k, \varphi_k)$  a chart, define  $\varphi_{i,x}: F \to F$  by  $f \mapsto \varphi_i(x, f)$ . Then the homeomorphism

$$\varphi_{j,x} \circ \varphi_{i,x}^{-1} : F \to F$$

for  $x \in U_i \cap U_j \neq \emptyset$  is an element of G.

• For  $i, j \in I$ , the map  $g_{ij}: U_i \cap U_j \to G$  defined by

$$g_{ij}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$$

is continuous.

If the (F, E, B) is a fiber bundle with  $F = \mathbb{R}$ , then it is often seen that  $G = GL(n, \mathbb{R})$ . Similarly, if  $F = \mathbb{C}$  then the structure group is  $G = GL(n, \mathbb{C})$ .

#### Definition 1.2.2: Equivalent G-Atlas

Two G-atlases on a fiber bundle (F, E, B) is said to be equivalent if their union is a G-atlas.

#### Definition 1.2.3: G-Bundle

Let G be a topological group. A G-bundle is a fiber bundle (F, E, B) together with an equivalence class of G-atlas. In this case, G is said to be the structure group of the fiber bundle.

The structure group and the trivialization charts completely determine the isomorphism type of the fiber bundle.

## 1.3 Morphisms of G-Bundles

#### Definition 1.3.1: Morphisms of G-Bundles

Let G be a topological group. A morphism of G-bundles is a morphism of fiber bundles  $(\tilde{h}, h)$ :  $(F, E_1, B_1) \to (F, E_2, B_2)$  where the two are G-bundles, such that the following are true.

• Let  $U_i$  be open in  $B_1$  and  $V_j$  be open in  $B_2$ . Let  $x \in U_u \cap h^{-1}(V_j)$ . Let  $h_{(E_1)_x}: (E_1)_x \to (E_2)_{f(x)}$  be the map induced by  $\tilde{h}: E_1 \to E_2$ . Then the map

$$\varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1} : F \to F$$

is an element of G.

• The map  $\widetilde{g_{ij}}: U_i \cap h^{-1}(V_j) \to G$  defined by

$$\widetilde{g_{ij}}(x) = \varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1}$$

is continuous.

It is easy to see that the mapping transformations  $\widetilde{g_{ij}}$  satisfy the following two relations:

- $\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$  for all  $x \in U_i \cap U_j \cap h^{-1}(V_k)$
- $g'_{ik}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$  for all  $x \in U_i \cap h^{-1}(V_j \cap V_k)$

 $g'_{ik}$  here refers to the transition charts in  $(F, E_2, B_2)$ .

Just as the structure groups and trivialization charts determine the isomorphism type of a fiber bundle, the  $\widetilde{g_{ij}}$  and a map of base space  $h: B_1 \to B_2$  completes determines a morphism of G-bundle.

#### Lemma 1.3.2

Let  $(F, E_1, B_1)$  and  $(F, E_2, B_2)$  be two G-bundles for a topological group G with the same fiber F. Suppose that we have the following.

- A map  $h: B_1 \to B_2$  of base space
- $\widetilde{g_{ij}}: U_i \cap h^{-1}(V_j) \to G$  a set of continuous maps such that

$$\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$$
 for all  $x \in U_i \cap U_j \cap h^{-1}(V_k)$   
 $g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$  for all  $x \in U_i \cap h^{-1}(V_j \cap V_k)$ 

Then there exists a unique G-bundle morphism having h as the map of base space and having  $\{\widetilde{g_{ij}} \mid i, j \in I\}$  as its mapping transformations.

## 1.4 Principal G-Bundles

## Definition 1.4.1: Principal Bundles

Let G be a topological group. A principal G-bundle is a G-bundle (F, E, B) together with a continuous group action G on E such that the following are true.

- The action of G preserves fibers. This means that  $g \cdot x \in E_b$  if  $x \in E_b$ . (This also means that G is a group action on each fiber)
- $\bullet$  The action of G on each fiber is free and transitive
- For each  $x \in E_b$ , the map  $G \to E_b$  defined by  $g \mapsto g \cdot x$  is homeomorphism.
- Local triviality condition: If  $\Psi_U: p^{-1}(U) \to U \times F$  are the local triviality maps, then each  $\Psi_U$  are G-equivariant maps.

Note that since G is homeomorphic to each fiber  $E_b$  of the total space, we can think of the action of G on the fiber simply becomes left multiplication.

For those who know what homogenous spaces are, principal bundles are G-bundles such that F is a principal homogenous space for the left action of G itself.

Conversely, given a continuous group action on a space, we can ask in what conditions will the space be a principal bundle over the orbit space.

#### Proposition 1.4.2

Let E be a space with a free G action. Let  $p: E \to E/G$  be the projection map to the orbit space. If for all  $x \in E/G$ , there is a neighbourhood U of x and a continuous map  $s: U \to E$  such that  $p \circ s = \mathrm{id}_U$ , then (G, E, E/G) is a principal G-bundle.

This proposition essentially means that if for each point in E/G, there is a local section, then it is sufficient for E to be a principal G bundle over E/G.

#### Theorem 1.4.3

A principal G-bundle is trivial if and only if it admits a global section.

This is entirely untrue for general bundles. For examples, the zero section of a fiber bundle is a global section.

## 1.5 Classifying Space

#### Definition 1.5.1: Universal G-Bundles

Let G be a topological group. A principal G-bundle (F, E, B) is said to be universal if for any space X, the induced pullback map

$$\psi: [X, B] \to \operatorname{Prin}_G(X)$$

defined by  $f \mapsto f^*(E)$  is a bijective correspondence.

## Theorem 1.5.2

Let (F, E, B) be a principal G-bundle. If E is contractible then (F, E, B) is a universal G-bundle.

## Theorem 1.5.3

Let  $(F, E_1, B_1)$  and  $(F, E_2, B_2)$  be universal principal G-bundles. Then there exists a bundle map

$$E_1 \xrightarrow{\tilde{f}} E_2$$

$$\downarrow^{p_1} \downarrow \qquad \qquad \downarrow^{p_2}$$

$$B_1 \xrightarrow{f} B_2$$

such that f is a homotopy equivalence.

## Definition 1.5.4: Classifying Space

Let G be a topological group. The classifying space BG of G is the homotopy type of the universal principal G-bundle. Also denote EG as the total space of the universal G-bundle.

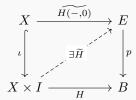
## 2 Fibrations and Cofibrations

#### 2.1 Fibrations

#### Definition 2.1.1: Fibrations

We say that a map  $p: E \to B$  is a fibration if it has the homotopy lifting property with respect to all topological spaces X.

In other words, for any space X together with a homotopy  $H: X \times I \to B$  and a lift  $H(-,0): X \to E$  of H(-,0), there exists a homotopy  $\widetilde{H}: X \times I \to E$  lifting  $\widetilde{H}$  and extending  $H(-,0): X \to E$ 



We call B the base space and E the total space. Define the fiber over  $b \in B$  to be the subspace

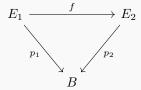
$$F_b = p^{-1}(b) \subseteq E$$

## Definition 2.1.2: Fibration Homomorphism

Let  $p_1: E_1 \to B$  and  $p_2: E_2 \to B$  be two fibrations. We say that a map  $f: E_1 \to E_2$  is a fibration homomorphism if

$$p_2 \circ f = p_1$$

In other words, the following diagram commutes:



## Definition 2.1.3: Fiber Homotopy Equivalence

We say that a fiber homomorphism  $f: E_1 \to E_2$  is a fiber homotopy equivalence if there exists a fiber homomorphism  $g: E_2 \to E_1$  such that  $f \circ g$  and  $g \circ f$  are homotopic by fibration homomorphisms to the identities  $\mathrm{id}_{E_2}$  and  $\mathrm{id}_{E_1}$  respectively.

#### Definition 2.1.4: Serre Fibration

We say that a map  $p: E \to B$  is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

It is clear that every (Hurewicz) fibration is a Serre fibration. Moreover, every fiber bundle is also a Serre fibration.

## Proposition 2.1.5

Every (Hurewicz) fibration is a Serre fibration. Every fiber bundle is a Serre fibration.