Algebraic Curves

Labix

February 6, 2025

Abstract

Contents

1 Algebraic Curves in Classical Algebraic Geometry

1.1 Basic Properties of Curves

Definition 1.1.1: Curves

Let k be a field. Let X be a variety over k. We say that X is a curve if $\dim(X) = 1$.

Proposition 1.1.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Let $p \in C$ be a non-singular point. Then $\mathcal{O}_{C,p}$ is a DVR. Moreover, the valuation is given by the degree of the regular function.

Proposition 1.1.3

Let k be a field. Let C be a smooth curve over k. Then for any projective variety $X \subseteq \mathbb{P}^n$ and rational map $\phi : C \to X$, there exists a regular map

$$\overline{\phi} : C \to X$$

such that $\overline{\phi}|_U = \phi|_U$ for some dense subset $U \subseteq C$.

1.2 Blowing Up Curves and Normalization

Recall that by taking the integral closure of the coordinate ring $k[C]$ of an irreducible affine curve $C \subseteq \mathbb{A}^n$, we obtain a corresponding variety $\widetilde{C}$ called the normalization of C.

Proposition 1.2.1

Let k be an algebraically closed field. Let $C \subseteq \mathbb{A}_k^n$ be an irreducible affine curve over k. Then the normalization $\widetilde{C}$ is smooth.

Theorem 1.2.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Then C is birational to a unique non-singular projective irreducible curve.

2 Algebraic Curves in the Context of Schemes

Definition 2.0.1: Algebraic Curves

Let k be an algebraically closed field. A curve over k is an integral separated scheme X of finite type over k that has dimension 1.

Proposition 2.0.2

Let X be an algebraic curve. Then the arithmetic and geometric genus coincide. In particular,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

We will simply call the genus of a curve g from now on since the arithmetic genus is the same as the geometric genus.

2.1 Riemann-Roch Theorem

Definition 2.1.1: Canonical Divisor

Let X be an algebraic curve. The canonical divisor K of X is a divisor in the linear equivalence class of

$$\Omega^1_{X/k} = \omega_X$$

Theorem 2.1.2: Riemann-Roch Theorem

Let X be an algebraic curve. Let D be a divisor on X and let K be the canonical divisor of X. Let $\mathcal{L}(D)$ be the associated sheaf of the divisor D. Then

$$\dim_k(H^0(X, \mathcal{L}(D))) + \dim_k(H^0(X, \mathcal{L}(K-D))) = \deg(D) + 1 - p_g(X)$$

2.2 Classification of Curves in $\mathbb{P}^3$