Group Cohomology

Labix

August 15, 2024

Abstract

Contents

1 Introduction to Group Homology and Cohomology

1.1 G-Modules

Definition 1.1.1: G-Modules

Let G be a group. A G-module is an abelian group A together with a group action of G on A.

Definition 1.1.2: Morphisms of G-Modules

Let G be a group. Let M and N be G-modules. A function $f : M \to N$ is said to be a G-module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all $m \in M$ and $g \in G$.

1.2 Invariants and Coinvariants

Definition 1.2.1: The Group of Invariants

Let G be a group and let M be a G-module. Define the group of invariants of G in M to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of M for which G acts trivially.

Definition 1.2.2: Functor of Invariants

Let G be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$$

as follows.

- For each G-module M, M^G is the group of invariants
- For each morphism $f : M \to N$ of G-modules, $f^G : M^G \to N^G$ is the restriction of f to M^G.

Theorem 1.2.3

Let G be a group. The functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

Definition 1.2.4: The Group of Coinvariants

Let G be a group and let M be a G-module. Define the group of coinvariants of G in M to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M \rangle}$$

This is the largest quotient of M for which G acts trivially.

1.3 Group Cohomology and its Equivalent Forms

Definition 1.3.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H_n(G;M) = (L_n(-)_G)(M)$$

the nth left derived functor of $(-)_G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 1.3.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H^n(G;M) \cong \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

Recall that there are two descriptions of Ext by considering it as a functor of the first or second variable. Since the above theorem exhibits an isomorphism that is natural in the second variable, let us consider Ext as the right derived functor of the functor $\text{Hom}_{\mathbb{Z}[G]}(-, M)$ applied to $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

Proposition 1.3.3

Let G be a group and let M be a G-module. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules. Then there is an isomorphism

$$H^n(G;M) \cong H^n(\text{Hom}_{\mathbb{Z}[G]}(P_\bullet, M))$$

that is natural in M.

For any group G, there is always the trivial choice of projective resolution. In the following lemma, we use the notation $(g_0, \dots, \hat{g}_i, \dots, g_n)$ as a shorthand for writing the element in G^n but with the ith term omitted.

Lemma 1.3.4

Let G be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $f_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of $\mathbb{Z}$ lying in ${}_{\mathbb{Z}[G]}\mathbf{Mod}$.

Let A be an R-algebra and let M be an A-module. Recall that the bar resolution is defined to be the chain complex consisting of $M \otimes A^{\otimes n}$ for each $n \in \mathbb{N}$ together with the boundary maps defined by multiplying the $i$the element to the $i+1$th element. Now let G be a group. By considering $\mathbb{Z}[G]$ as a $\mathbb{Z}$-algebra and that and ring is a module over itself, it makes sense to talk about the bar resolution of $\mathbb{Z}[G]$.

Theorem 1.3.5

Let G be a group. Consider the bar resolution

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \longrightarrow \mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}[G]$. Then it is a free resolution, and hence a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules.

Thus, given a group G and a G-module M, the group cohomology of G with coefficients in M can be thought of in the following way:

- It is the right derived functor of the functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$
- It is the extension group $\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ (which is computable by the obvious projective resolution $\mathbb{Z}[G^\bullet]$, or the bar resolution)

1.4 Group Homology and its Equivalent Forms

Definition 1.4.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be
$$H^n(G; M) = (R^n(-)^G)(M)$$
the nth right derived functor of $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 1.4.2

Let G be a group and let M be a G-module. Then there is an isomorphism
$$H_n(G; M) \cong \text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$
that is natural in M.

2 Group Cohomology

2.1 G-Modules

2.2 The Group of Invariants

Theorem 2.2.1

Let G be a group and let M be a G-module. Then there are canonical isomorphisms

$$M^G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

2.3 The Different Forms of Cohomology of Groups

These are purely algebraic descriptions of group cohomology. There is a more informative description of group cohomology topologically.

2.4 Low Degree Interpretations

Theorem 2.4.1

Let G be a group and let M be a G-module. Then there are natural isomorphisms

$$H^0(G, M) = M^G \quad \text{and} \quad H_0(G; M) = M_G$$

Theorem 2.4.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_1(G, M) \cong \frac{G}{[G, G]} = G_{\text{ab}}$$

Theorem 2.4.3

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) = \frac{(\{f : G \to M \mid f(ab) = f(a) + af(b)\}, +)}{\langle f : G \to M \mid f(g) = gm - m \text{ for some fixed } m\rangle}$$

Corollary 2.4.4

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) \cong \text{Hom}_{\mathbf{Grp}}(G, M)$$

3 Group Homology