Operad Theory

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Abstract

Contents

1 The Theory of Operads

1.1 General Operads

Definition 1.1.1: (Planar) Operads

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. An operad in $\mathcal{C}$ consists of the following data

- A sequence $P = \{P(n) \mid n \in \mathbb{N}\}$ of objects in $\mathcal{C}$
- A composition function

$$\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$$

- An unit $\mu : I \to P(1)$

such that the following compatibility conditions are satisfied.

- Associativity in the first symmetric product:

$$\begin{tikzcd}
P(n) \otimes \left(\bigotimes_{k=1}^n \left(P(i_k) \otimes \bigotimes_{t=1}^{i_k} P(j_{k,t})\right)\right) \arrow[r, "\mathrm{id}_{P(n)} \otimes \gamma"'] \arrow[dd, "\cong"] & P(n) \otimes \left(\bigotimes_{k=1}^n P\left(\sum_{t=1}^{i_k} j_{k,t}\right)\right) \arrow[dr, "\gamma"] & \\
& & P\left(\sum_{k=1}^n \sum_{t=1}^{i_k} j_{k,t}\right) \\
P(n) \otimes (\bigotimes_{k=1}^n P(j_k)) \otimes \left(\bigotimes_{k=1}^n \bigotimes_{t=1}^{t_k} P(j_{k,t})\right) \arrow[r, "\gamma \otimes \mathrm{id}_{\otimes\otimes}"] & P\left(\sum_{u=1}^k i_u\right) \otimes \left(\bigotimes_{k=1}^n \bigotimes_{t=1}^{i_k} P(j_{k,t})\right) \arrow[ur, "\gamma"] &
\end{tikzcd}$$

- Unitality:

$$\begin{tikzcd}
P(n) \otimes P(1)^{\otimes n} \arrow[dr, "\gamma"'] & P(n) \otimes I^{\otimes n} \arrow[l, "\mathrm{id}_{P(n)} \otimes \mu^{\otimes n}"'] \arrow[d, "\cong"] \\
& P(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
I \otimes P(n) \arrow[r, "\mu \otimes \mathrm{id}_{P(n)}"] \arrow[d, "\cong"'] & P(1) \otimes P(n) \arrow[dl, "\gamma"] \\
P(n) &
\end{tikzcd}$$

TBA: An operad is a monoid in the monoidal category $(\mathrm{Func}(\mathbf{S}, \mathcal{C}), \circ, I)$

Definition 1.1.2: Symmetric Operads

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category. A symmetric operad is an operad (P, γ, μ) on $\mathcal{C}$ such that the following are true.

- Each $P(n)$ is an S_n-module for each $n \in \mathbb{N}$.
- $\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$ is equivariant in the following sense:

$$\begin{aligned}
\gamma(c \cdot \sigma, d_1, \dots, d_n) &= \gamma(c, d_{\sigma^{-1}(1)}, \dots, d_{\sigma^{-1}(n)}) \cdot \sigma(k_1, \dots, k_n) \\
\gamma(c, d_1 \cdot \tau_1, \dots, d_n \cdot \tau_n) &= \gamma(c, d_1, \dots, d_n) \cdot (\tau_1 \oplus \cdots \oplus \tau_n)
\end{aligned}$$

The symmetric monoidal category one usually considers are algebraic. For instance, ${}_R\mathbf{Mod}$, $\mathbf{Ch}(R)_{\geq 0}$

Definition 1.1.3: Morphisms of Operads

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. Let (P, γ, μ) and (P, γ', μ') be operads in $\mathcal{C}$. A morphism $\varphi : P \to P'$ of operads consists of the following data.

- For each $n \in \mathbb{N}$, a map $\varphi_n : P(n) \to P'(n)$ in $\mathcal{C}$
- Compatible with composition. This means that for any $n, k_1, \dots, k_n \in \mathbb{N}$, the following diagram commutes:

$$\begin{tikzcd} P(n)\otimes P(k_1)\otimes\cdots\otimes P(k_n) \arrow[r, "\varphi_n\otimes\varphi_{k_1}\otimes\cdots\otimes\varphi_{k_n}"] \arrow[d, "\gamma"'] & P'(n)\otimes P'(k_1)\otimes\cdots\otimes P'(k_n) \arrow[d, "\gamma'"] \\ P(k_1+\cdots+k_n) \arrow[r, "\varphi_{k_1+\cdots+k_n}"] & P'(k_1+\cdots+k_n) \end{tikzcd}$$

- Compatible with the identity. This means that the following diagram holds:

$$\begin{tikzcd} & I \arrow[dl, "\mu"'] \arrow[dr, "\mu'"] & \\ P(1) \arrow[rr, "\varphi_1"'] & & P'(1) \end{tikzcd}$$

1.2 Operads as a Monoid in a Monoidal Category

1.3 The Endomorphism Operad and Algebra Operads

Definition 1.3.1: The Endomorphism Operad

Let $(\mathcal{C},\otimes,I)$ be a symmetric monoidal category. Let $X\in\mathcal{C}$ be an object. Define the endomorphism operad $\mathcal{E}\text{nd}(X)$ on X to consist of the following data.

- For each $n\in\mathbb{N}$, $\mathcal{E}\text{nd}(X)(n)=\text{Hom}_{\mathcal{C}}(X^{\otimes n},X)$
- Composition is given by the following:

$$\text{Hom}_{\mathcal{C}}(X^{\otimes n},X)\otimes\bigotimes_{i=1}^n\text{Hom}_{\mathcal{C}}(X^{\otimes k_i},X)\xrightarrow{\text{id}\otimes\bigotimes}\text{Hom}_{\mathcal{C}}(X^{\otimes n},X)\otimes\text{Hom}_{\mathcal{C}}(X^{\otimes k_1+\cdots+k_n},X^{\otimes n})\xrightarrow{\text{Composition}}\text{Hom}_{\mathcal{C}}(X^{\otimes k_1+\cdots+k_n},X)$$

- The unit map $\mu:I\to\text{Hom}_{\mathcal{C}}(X,X)$ is the image of identity map id_X under the adjunction
$$\text{Hom}_{\mathcal{C}}(X,X)\cong\text{Hom}_{\mathcal{C}}(I,\text{Hom}_{\mathcal{C}}(X,X))$$

Definition 1.3.2: The (Non-Unital) Planar Associative Operad

Let $(\mathcal{C},\otimes,I)$ be a symmetric monoidal category with initial object $\emptyset$. Define the planar associative operad $\mathcal{A}\text{ss}$ of $\mathcal{C}$ to consist of the following data.

- For each $n\in\mathbb{N}\setminus\{0\}$, $\mathcal{A}\text{ss}(n)=\{I\}$ and $\mathcal{A}\text{ss}(0)=\emptyset$
- The composition function is uniquely determined by the unique map of the terminal object (in other words the identity id_I)
- The unit $\{*\}\to\mathcal{A}\text{ss}(1)=\{*\}$ is also uniquely determined by the unique object

Definition 1.3.3: The Symmetric Associative Operad

Define the planar associative operad $\mathcal{A}\text{ss}$ to consist of the following data.

- For each $n\in\mathbb{N}$, $\mathcal{A}\text{ss}(n)=S_n$ where S_n is the symmetric group
-

Definition 1.3.4: The Commutative Operad

2 Coloured Operads

2.1 Coloured Operads

Definition 2.1.1: Coloured Operads

Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A coloured operad in $\mathcal{C}$ consists of the following data.

- A set S of objects in $\mathcal{C}$ called colours
- For each $n \in \mathbb{N}$, and each $C_1, \dots, C_n, C \in S$, an object $P(C_1, \dots, C_n, C) \in \mathcal{C}$
- For each $(n+1)$-tuple $C_1, \dots, C_n, C$ and n other tuples $(D_{1,1}, \dots, D_{1,k_1}), \dots, (D_{n,1}, \dots, D_{n,k_n})$, there is a morphism
$$P(C_1, \dots, C_n, C) \otimes \bigotimes_{i=1}^{n} P(D_{i,1}, \dots, D_{i,k_i}, C_i) \to P(D_{1,1}, \dots, D_{n,k_n}, C)$$
called the composition operation
- For each $C \in S$, a morphism
$$1_C : I \to P(C, C)$$
called the identity

such that composition is associativity and unital.

2.2 The Category of Operators

Every coloured operad defines and is defined by a category called the category of operators. Recall the category of finite sets to consist of objects of the form $[n] = \{0, \dots, n\}$.

Definition 2.2.1: The Category of Pointed Finite Sets

Define the category of pointed finite sets
$$\mathbf{Fin}_*$$
to consist of the following data.

- The objects consists of $\langle n \rangle = \{*, 1, \dots, n\}$ together with the chosen point $*$
- For $\langle n \rangle$ and $\langle m \rangle$ in $\mathbf{Fin}_*$, a morphism is a function $f : \langle n \rangle \to \langle m \rangle$ such that $f(*) = *$.
- Composition is given by the composition of functions.

Definition 2.2.2: The Category of Operators

Let A be a symmetric coloured operad in $\mathbf{Set}$. Define the category of operators $\mathcal{C}_A$ as follows.

- The objects are finite sequences $C_1, \dots, C_n$ of colours in A
- For two tuples $(C_1, \dots, C_n)$ and $(D_1, \dots, D_m)$, a morphism $F : (C_1, \dots, C_n) \to (D_1, \dots, D_m)$ is given as a tuple $(\phi, f_1, \dots, f_m)$ where $\phi : \langle n \rangle \to \langle m \rangle$ is a morphism in $\mathbf{Fin}_*$ and each f_i for $1 \leq i \leq m$ an operation
$$f_i \in \text{Hom}_A((C_k)_{k \in \phi^{-1}(i)}, D_i)$$
in A
- Composition is given component wise in $\mathbf{Fin}_*$ and in A.

This is often paired with the canonical forgetful functor $p : \mathcal{C}_A \to \mathbf{Fin}_*$.

We can reconstruct the operad A from the category of operators in the following way.

3 Infinity Operads