Algebraic Curves

Labix

March 7, 2025

Abstract

Contents

1 Algebraic Curves in Classical Algebraic Geometry

1.1 Basic Properties of Curves

Definition 1.1.1: Curves

Let k be a field. Let X be a variety over k. We say that X is a curve if $\dim(X) = 1$.

Proposition 1.1.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Let $p \in C$ be a non-singular point. Then $\mathcal{O}_{C,p}$ is a DVR. Moreover, the valuation is given by the degree of the regular function.

Proof. Since p is non-singular, by definition $\mathcal{O}_{C,p}$ is a regular local ring. Moreover, we know that $1 = \dim(C) = \dim(\mathcal{O}_{C,p})$ so that $\mathcal{O}_{C,p}$ has Krull dimension 1. By the equivalent characterization of DVR we conclude. □

We denote the valuation map by $v_p : \text{Frac}(\mathcal{O}_{C,p}) \to \mathbb{Z}$.

Example 1.1.3

Consider the projective curve $C = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{\mathbb{C}}$. Let $p = [p_0 : p_1 : p_2]$ be a point on the curve.

If $p_2 \neq 0$, then $p \in U_2$. Under the affine chart (U_2, φ_2), we find that $C_2 = \varphi_2(C \cap U_2) = \mathbb{V}(x^2 + y^2 - 1)$. The corresponding coordinate ring is given by $\frac{\mathbb{C}[x,y]}{(x^2+y^2-1)}$. The formula for the local ring in the affine case gives

$$\mathcal{O}_{C,p} \cong \left(\frac{\mathbb{C}[x,y]}{(x^2 + y^2 - 1)} \right)_{m_{(p_0/p_2, p_1/p_2)}}$$

Recall that the unique maximal ideal of the local ring is given as the $\mathcal{O}_{X,p}$-module $m_p = \{f \in \mathbb{C}[C_2] \mid f(p_0/p_2, p_1/p_2) = 0\}$, which under the nullstellensatz is the maximal ideal corresponding to the point $(p_0/p_2, p_1/p_2)$ and is given by $m_p = (x - r, y - s)$ where $r = p_0/p_1$ and $s = p_0/p_2$. By Nakayama's lemma, since $x - r, y - s$ generate m_p we know that $x - r + m_p^2, y - s + m_p^2$ span the vector space m_p/m_p^2 over $\mathcal{O}_{X,p}/m_p$. I claim that they are linearly dependent. This mean that I want to find $f + m_p^2$ and $g + m_p^2$ in $\mathcal{O}_{X,p}/m_p$ that are non-trivial, and that $(x - r)f + (y - s)g + m_p^2 = m_p^2$. This means that we want to find $f, g \in \mathcal{O}_{X,p} \setminus m_p$ such that $(x - r)f + (y - s)g \in m_p^2$. Choose $f = x + r$ and $g = y + s$ to get

$$(x - r)(x + r) + (y - s)(y + s) = x^2 - r^2 + y^2 - s^2 = 1 - 1 = 0$$

since (r, s) lie on the curve. Moreover, $x + r, y + s\mathcal{O}_{X,p} \setminus m_p$ since evaluating at (r, s) at the functions are non-zero. This verifies that $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1, hence is a DVR.

We can even find its uniformizer and valuation. Since $x - r$ and $y + s$ are linearly dependent and spans m_p/m_p^2, any one of the two is a basis for the vector space. WLOG take $x - r$ to be a basis. Nakayama's lemma implies that $x - r$ generates m_p. Being a DVR means that for all $f \in \mathcal{O}_{X,p}$, $f = u(x - r)^n$ where u is invertible. Then the valuation of f is n.

Proposition 1.1.4

Let C be an affine irreducible curve over $\mathbb{C}$. Then C is smooth if and only if C is a normal variety.

1.2 Morphisms between Curves

Proposition 1.2.1

Let k be a field. Let C be a smooth curve over k. Then for any projective variety $X \subseteq \mathbb{P}^n$ and rational map $\phi : C \to X$, there exists a regular map

$$\overline{\phi} : C \to X$$

such that $\overline{\phi}|_U = \phi|_U$ for some dense subset $U \subseteq C$.

Proposition 1.2.2

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Then ϕ is a finite morphism.

Proposition 1.2.3

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a rational map. If ϕ is birational, then ϕ is an isomorphism of varieties.

1.3 Blowing Up Curves and Normalization

Recall that by taking the integral closure of the coordinate ring $k[C]$ of an irreducible affine curve $C \subseteq \mathbb{A}^n$, we obtain a corresponding variety $\widetilde{C}$ called the normalization of C.

Proposition 1.3.1

Let k be an algebraically closed field. Let $C \subseteq \mathbb{A}_k^n$ be an irreducible affine curve over k. Then the normalization $\widetilde{C}$ is smooth.

Theorem 1.3.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Then C is birational to a unique non-singular projective irreducible curve.

1.4 Ramification Index

Definition 1.4.1: Ramification Index

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Let $p \in X$. Define the ramification index of ϕ at p to be

$$e_\phi(p) = v_p(\phi^*(\pi))$$

where π is a uniformizing parameter of $\mathcal{O}_{Y,\phi(p)}$.

Lemma 1.4.2

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Let $p \in X$. Then

$$e_\phi(p) = \dim_k \left(\frac{\mathcal{O}_{X,p}}{(\phi^*(\pi))} \right)$$

where π is a uniformizing parameter of $\mathcal{O}_{Y,\phi(p)}$.

Let $\phi : X \to Y$ be a non-constant regular map between smooth irreducible and projective curves. Since ϕ is finite, the notion of degree makes sense. Recall that the degree is defined to be

$$\deg(\phi) = \dim_{K(Y)} K(X)$$

Proposition 1.4.3

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Let $q \in Y$. Then we have

$$\sum_{p \in \phi^{-1}(q)} e_\phi(p) = \deg(\phi)$$

1.5 Differential Forms on Curves

Proposition 1.5.1

Let C be a smooth irreducible curve over $\mathbb{C}$. Then $\Omega^1_{\mathbb{C}(C)/\mathbb{C}}$ is a 1-dimensional vector space over $\mathbb{C}(C)$.

Proposition 1.5.2

Let C be a smooth irreducible curve over $\mathbb{C}$. Let $f \in \mathbb{C}(C)$ be non-constant. Then the following are true.

- $df \neq 0$ in $\Omega^1_{\mathbb{C}(C)/\mathbb{C}}$.
- df is a $\mathbb{C}(C)$-basis for $\Omega^1_{\mathbb{C}(C)/\mathbb{C}}$.

Definition 1.5.3: Valuation of Differential 1-Forms

Let C be a smooth irreducible curve over $\mathbb{C}$. Let $p \in C$. Let $\omega \in \Omega^1_{\mathbb{C}(C)/\mathbb{C}}$ be a differential 1-form of C. Define the valuation of ω at p as follows. Choose a uniformizing parameter $\pi \in \mathcal{O}_{C,p}$. Write $\omega = f d\pi$ for $f \in \mathbb{C}(C)$. Then define the valuation as

$$\text{val}_p(\omega) = \text{val}_p(f)$$

2 Classical Divisors on Curves

2.1 The Pullback Map of Divisors

Definition 2.1.1: Pullback Map of Divisors

Let k be an algebraically closed field. Let C be a smooth irreducible projective curve over k. Let Y be a smooth irreducible projective variety over k. Let $\phi : X \to Y$ be a non-constant regular map. Define the induced pullback map $\phi^* : \text{Div}(Y) \to \text{Div}(C)$ on generators as follows. For $H \subseteq Y$ a codimension one subvariety, define

$$\phi^*(H) = \sum_{p\in X} \text{val}_p(\phi^*(g)) \cdot p$$

where g is a generator of $\mathbb{I}(H)\mathcal{O}_{Y,\phi(p)}$.

When Y is a curve, we essentially have the formula:

$$\phi^*\left(\sum_{q\in Y} n_q \cdot q\right) = \sum_{q\in Y} n_q \cdot \left(\sum_{p\in\phi^{-1}(q)} e_\phi(p)\cdot p\right) = \sum_{p\in X} n_{\phi(p)} e_\phi(p) \cdot p$$

Proposition 2.1.2

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Then we have

$$\deg(\phi^*(D)) = \deg(\phi)\deg(D)$$

for any $D \in \text{Div}(Y)$.

Proposition 2.1.3

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D \in \text{Div}(X)$ be a principal divisor of X. Then $\deg(D) = 0$.

Proposition 2.1.4

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Then $\phi(\text{Prin}(Y)) \subseteq \text{Prin}(X)$.

Definition 2.1.5: Induced Map of Divisor Class Groups

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Define the induced map of divisor class groups $\phi^* : \text{Cl}(Y) \to \text{Cl}(X)$ by

$$\phi^*([D]) = [\phi^*(D)]$$

2.2 The Linear System of Divisors

Definition 2.2.1: The Linear System of Divisors

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D \in \text{Div}(X)$ be a divisor. Define the linear system of D to be

$$\mathcal{L}(D) = \{0\} \cup \{f \in K(X) \mid \deg(D + \text{div}(f)) \geq 0\} \subseteq K(X)$$

Lemma 2.2.2

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D \in \text{Div}(X)$ be a divisor. Then $\mathcal{L}(D)$ is a vector space over k.

Proposition 2.2.3

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D, D' \in \text{Div}(X)$ be divisors. If $D \sim D'$ are linearly equivalent, then we have

$$\dim_k(\mathcal{L}(D)) = \dim_k(\mathcal{L}(D'))$$

Proposition 2.2.4

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D \in \text{Div}(X)$ be a divisor. Then the following are true.

- If $\deg(D) < 0$, then we have
$$\dim_k(\mathcal{L}(D)) = 0$$
- If $\deg(D) = 0$, then we have
$$\dim_k(\mathcal{L}(D)) = \begin{cases} 0 & \text{if } D \not\sim 0 \\ 1 & \text{if } D \sim 0 \end{cases}$$

Proposition 2.2.5

Let k be an algebraically closed field. Let X be a smooth irreducible projective curve over k. Let $D \in \text{Div}(X)$ be a divisor. Then we have

$$\dim_k(\mathcal{L}(D)) \leq \deg(D) + 1$$

2.3 The Canonical Divisor for Curves

Definition 2.3.1: Divisors of Differential Forms

Let C be a smooth irreducible curve over $\mathbb{C}$. Let $p \in C$. Let $\omega \in \Omega^1_{\mathbb{C}(C)/\mathbb{C}}$ be a differential 1-form of C. Define the divisor of ω by

$$\text{div}(\omega) = \sum_{p \in C} \text{val}_p(\omega) \cdot p \in \text{Div}(C)$$

Proposition 2.3.2

Let C be a smooth irreducible curve over $\mathbb{C}$. Let $p \in C$. Let $\omega, \tau \in \Omega^1_C$ be non-zero. Then $\text{div}(\omega)$ and $\text{div}(\tau)$ are linearly equivalent.

Definition 2.3.3: The Canonical Divisor

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Let $p \in C$. Define the canonical divisor of C to be

$$K_C = [\omega] \in \text{Pic}(C)$$

in the divisor class group for any non-zero $\omega \in \Omega^1_C$.

Lemma 2.3.4

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Then

$$\dim_{\mathbb{C}}(\mathcal{L}(K_C)) = \dim_{\mathbb{C}}(\Omega^1_C)$$

2.4 The Riemann-Roch Theorem

Theorem 2.4.1: Riemann-Roch Theorem

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor on C. Then

$$\dim_{\mathbb{C}}(\mathcal{L}(D)) + \dim_{\mathbb{C}}(\mathcal{L}(K_C - D)) = \deg(D) + 1 - p_g(C)$$

Proposition 2.4.2

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor on C. Then

$$\deg(D) + 1 - p_g(C) \leq \dim_{\mathbb{C}}(\mathcal{L}(D)) \leq \deg(D) + 1$$

Proposition 2.4.3

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Then we have

$$\deg(K_C) = 2p_g(C) - 2$$

Proposition 2.4.4

Let C be a smooth irreducible projective curve over $\mathbb{C}$. Then the following are equivalent.

- C is isomorphic to $\mathbb{P}^1$.
- The geometric genus $p_g(C) = 0$ is zero.
- For all $p, q \in C$, $p \sim q$ are linearly equivalent.
- There exists distinct $p, q \in C$, such that $p \sim q$ are linearly equivalent.
- The degree map $\deg : \text{Pic}(C) \to \mathbb{Z}$ is an isomorphism.
- For all $D \in \text{Div}(C)$ with $\deg(D) > 0$, we have $l(D) = \deg(D) + 1$.
- There exists $D \in \text{Div}(C)$ with $\deg(D) > 0$ such that $l(D) = \deg(D) + 1$.

2.5 Base Point Free Divisors

Definition 2.5.1: Base Point Free Divisor

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor given by $D = \sum_{p \in C} n_p \cdot p$. We say that D is base point free if for all $p \in C$ and all $f \in \mathcal{L}(D)$, we have

$$\text{val}_p(f\pi^{n_p}) \neq 0$$

where π is a uniformizer of $\mathcal{O}_{C,p}$.

Definition 2.5.2: Associated Map to Divisors

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor. Define the associated rational map $F_D : C \to \mathbb{P}(\mathcal{L}(D)^*)$ by

$$p \mapsto \left(\substack{\phi_p : \mathcal{L}(D) \to \mathbb{C} \\ f \mapsto (f \cdot \pi^{n_p})(p)} \right)$$

where π is the uniformizer of $\mathcal{O}_{C,p}$.

Lemma 2.5.3

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor. Then the associated map $F_D : C \to \mathbb{P}(\mathcal{L}(D)^*)$ is a regular map.

Proposition 2.5.4

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor. Then D is base point free if and only if

$$\dim_{\mathbb{C}}(\mathcal{L}(D-p)) = \dim_{\mathbb{C}}(\mathcal{L}(D)) - 1$$

for all $p \in C$.

Corollary 2.5.5

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a divisor. If $\deg(D) \geq 2g$ then D is base point free.

Proposition 2.5.6

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$ be a base point free divisor. Then there is a one-to-one correspondence

$$\{H \subseteq \mathbb{P}(\mathcal{L}(D)^*) \mid H \text{ is a hyperplane }\} \overset{1:1}{\longleftrightarrow} \{E \in \text{Div}(C) \mid E \text{ is effective and } E \sim D\}$$

The map is given by $H \mapsto F_D^*(H)$.

2.6 Very Ample Divisors

Definition 2.6.1: Very Ample Divisor

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$. We say that D is very ample if D is base point free and the associated map $F_D : C \to \mathbb{P}(\mathcal{L}(D)^*)$ is an embedding.

Proposition 2.6.2

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$. Then D is very ample if and only if for all $p, q \in C$, we have

$$\dim_{\mathbb{C}}(\mathcal{L}(D-p-q)) = \dim_{\mathbb{C}}(\mathcal{L}(D)) - 2$$

Corollary 2.6.3

Let C be a smooth projective irreducible curve over $\mathbb{C}$. Let $D \in \text{Div}(C)$. If $\deg(D) \geq 2g+1$ then D is very ample.

3 Algebraic Curves in the Context of Schemes

Definition 3.0.1: Algebraic Curves

Let k be an algebraically closed field. A curve over k is an integral separated scheme X of finite type over k that has dimension 1.

Proposition 3.0.2

Let X be an algebraic curve. Then the arithmetic and geometric genus coincide. In particular,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

We will simply call the genus of a curve g from now on since the arithmetic genus is the same as the geometric genus.

3.1 Riemann-Roch Theorem

Definition 3.1.1: Canonical Divisor

Let X be an algebraic curve. The canonical divisor K of X is a divisor in the linear equivalence class of

$$\Omega^1_{X/k} = \omega_X$$

Theorem 3.1.2: Riemann-Roch Theorem

Let X be an algebraic curve. Let D be a divisor on X and let K be the canonical divisor of X. Let $\mathcal{L}(D)$ be the associated sheaf of the divisor D. Then

$$\dim_k(H^0(X, \mathcal{L}(D))) + \dim_k(H^0(X, \mathcal{L}(K - D))) = \deg(D) + 1 - p_g(X)$$

3.2 Classification of Curves in $\mathbb{P}^3$