Selected Topics

Labix

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Abstract

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1 Symmetric Polynomials

1.1 Symmetric Polynomials

The theory of symmetric functions are important in combinatorics, representation theory, Galois theory and the theory of λ-rings.

Requirements: Groups and Rings
Books: Donald Yau: Lambda Rings

Definition 1.1.1: Symmetric Group Action on Polynomial Rings

Let R be a ring. Define a group action of S_n on $R[x_1, \dots, x_n]$ by

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

It is easy to check that this defines a group action.

Definition 1.1.2: Symmetric Polynomials

Let R be a ring. We say that a polynomial $f \in R[x_1, \dots, x_n]$ is symmetric if

$$\sigma \cdot f = f$$

for all $\sigma \in S_n$.

Definition 1.1.3: The Ring of Symmetric Polynomials

Let R be a ring. Define the ring of symmetric polynomials in n variables over R to be the set

$$\Sigma = \{f \in R[x_1, \dots, x_n] \mid \sigma \text{ is a symmetric polynomial } \}$$

Definition 1.1.4: Elementary Symmetric Polynomials

Let R be a ring. Define the elementary symmetric polynomials to be the elements $s_1, \dots, s_n \in R[x_1, \dots, x_n]$ given by the formula

$$s_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

Theorem 1.1.5: The Fundamental Theorem of Symmetric Polynomials

Let R be a ring. Then $s_1, \dots, s_n$ are algebraically independent over R. Moreover,

$$\Sigma = R[s_1, \dots, s_n]$$

2 λ-Rings

2.1 λ-Rings

Complex representation of a group is a λ-ring. Topological K theory is a λ-ring.

Requirements: Category Theory, Groups and Rings, Symmetric Functions
Books: Donald Yau: Lambda Rings

We need the theory of symmetric polynomials before defining λ-structures.

Definition 2.1.1: λ-Structures

Let R be a commutative ring. A λ-structure on R consists of a sequence of maps $\lambda^n : R \to R$ for $n \geq 0$ such that the following are true.

- $\lambda^0(r) = 1$ for all $r \in R$
- $\lambda^1 = \text{id}_R$
- $\lambda^n(1) = 0$ for all $n \geq 2$
- $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$ for all $r, s \in R$
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \dots, \lambda^{mn}(r))$ for all $r \in R$

Here P_n and $P_{m,n}$ are defined as follows.

- The coefficient of t^n in the polynomial
$$h(t) = \prod_{i,j=1}^n (1 + x_i y_j t)$$
is a symmetric polynomial in x_i and y_j with coefficients in $\mathbb{Z}$. P_n is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of x_i and y_j respectively.
- The coefficient of t^n in the polynomial
$$g(t) = \prod_{1 \leq i_1 \leq \dots \leq i_m \leq nm} (1 + x_{i_1} \cdots x_{i_m} t)$$
is a symmetric polynomial in x_i with coefficients in $\mathbb{Z}$. $P_{m,n}$ is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ of x_i.

In this case, we call R a λ-ring.

Note that we do not require that the λ^n are ring homomorphisms.

Definition 2.1.2: Associated Formal Power Series

Let R be a λ-ring. Define the associated formal power series to be the function $\lambda_t : R \to R[[t]]$ given by
$$\lambda_t(r) = \sum_{k=0}^\infty \lambda^k(r) t^k$$
for all $r \in R$

Proposition 2.1.3

Let R be a λ-ring. Then the following are true regarding $\lambda_t(r)$.

- $\lambda_t(1) = 1 + t$
- $\lambda_t(0) = 1$
- $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$
- $\lambda_t(-r) = \lambda(r)^{-1}$

Proposition 2.1.4

The ring $\mathbb{Z}$ has a unique λ-structure given by

$$\lambda_t(n) = (1+t)^n$$

Proposition 2.1.5

Let R be a λ-ring. Then R has characteristic 0.

Definition 2.1.6: Dimension of an Element

Let R be a λ-ring and let $r \in R$. We say that r has dimension n if $\deg(\lambda_t(r)) = n$. In this case, we write $\dim(r) = n$.

Proposition 2.1.7

Let R be a λ-ring. Then the following are true regarding the dimension of n.

- $\dim(r+s) \leq \dim(r) + \dim(s)$ for all $r, s \in R$
- If r and s both has dimension 1, then so is rs.

2.2 λ-Ring Homomorphisms and Ideals

Definition 2.2.1: λ-Ring Homomorphisms

Let R and S be λ-rings. A λ-ring homomorphism from R to S is a ring homomorphism $f : R \to S$ such that

$$\lambda^n \circ f = f \circ \lambda^n$$

for all $n \in \mathbb{N}$.

Definition 2.2.2: λ-Ideals

Let R be a λ-ring. A λ-ideal of R is an ideal I of R such that

$$\lambda^n(i) \in I$$

for all $i \in I$ and $n \geq 1$.

TBA:λ-ideal and subring. Ker, Im, Quotient Product, Tensor, Inverse Limit are λ-rings

Proposition 2.2.3

Let R be a λ-ring. Let $I = \langle z_i \mid i \in I \rangle$ be an ideal in R. Then I is a λ-ideal if and only if $\lambda^n(z_i) \in I$ for all $n \geq 1$ and $i \in I$.

Proposition 2.2.4

Every λ-ring R contains a λ-subring isomorphic to $\mathbb{Z}$.

2.3 Augmented λ-Rings

Definition 2.3.1: Augmented λ-Rings

Let R be a λ-ring. We say that R is an augmented λ-ring if it comes with a λ-homomorphism

$$\varepsilon : R \to \mathbb{Z}$$

called the augmentation map.

TBA: tensor of augmented is augmented

Proposition 2.3.2

Let R a λ-ring. Then R is augmented if and only if there exists a λ-ideal I such that

$$R = \mathbb{Z} \oplus I$$

as abelian groups.

2.4 Extending λ-Structures

Proposition 2.4.1

Let R be a λ-ring. Then there exists a unique λ-structure on $R[x]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[x]$ and $\varepsilon(r) = 0$ or 1.

Proposition 2.4.2

Let R be a λ-ring. Then there exists a unique λ-structure on $R[[x]]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[[x]]$ and $\varepsilon(r) = 0$ or 1.

2.5 Free λ-Rings

2.6 The Universal λ-Ring

2.7 Adams Operations

3 Witt Vectors

3.1 Fundamentals of the Ring of Big Witt Vectors

Prelim: Symm Functions, Lambda Rings, Category theory, Frobenius endomorphism (Galois), Rings and Modules, Kaehler differentials (commutative algebra 2)
Leads to: K theory
Books: Donald Yau: Lambda Rings

Definition 3.1.1: Truncation Sets

Let $S \subseteq \mathbb{N}$. We say that S is a truncation set if for all $n \in S$ and $d|n$, then $d \in S$. For $n \in \mathbb{N}$ and S a truncation set, define

$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

For instance, $\mathbb{N} \setminus \{0\}$ is a truncation set. We will also use $\{1, \dots, n\}$.

Theorem 3.1.2: Dwork's Theorem

Let R be a ring and let S be a truncation set. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i \in S} \in \prod_{i \in S} R$ has the form

$$(b_i)_{i \in S} = (w_i(a))_{i \in S}$$

  for some $a \in R$
- For all primes p and all $n \in S$ such that $p|n$, we have

$$b_n \equiv \sigma_p(b_{n/p}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on all the b_k for $1 \leq k \leq n$ and $k \in S$.

We wish to equip $\prod_{i \in S} R$ with a non-standard addition and multiplication to make it into a ring.

Proposition 3.1.3

Consider the ring $R = \mathbb{Z}[x_i, y_i \mid i \in S]$. There exists unique polynomials

$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$

for $n \in S$ such that

- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i \in S}) + w_n((y_i)_{i \in S})$
- $w_n(\pi_1, \dots, \pi_n) = w_n((x_i)_{i \in S}) \cdot w_n((y_i)_{i \in S})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i \in S})$

for all $n \in S$.

Note that the polynomials ξ_n, π_n have variables x_k and y_k for $k \leq n$ and $k \in S$. This is similar for the variables of ι. From now on, this will be the convention: For S a truncation set, the sequence $a_1, \dots, a_n$ actually refers to the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \dots \leq d_k \leq n$ and $d_1, \dots, d_k$ are all divisors of n. The result of this is that sequences in $\mathbb{N}$ are now restricted to S.

Definition 3.1.4: The Ring of Truncated Witt Vector

Let R be a ring. Let S be a truncation set. Define the ring of big Witt vectors $W_S(R)$ of R to consist of the following.

- The underlying set $\prod_{i \in S} R$
- Addition defined by $(a_n)_{n \in S} + (b_n)_{n \in S} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$
- Multiplication defined by $(a_n)_{n \in S} \times (b_n)_{n \in S} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$

Theorem 3.1.5

Let R be a ring. Let S be a truncation set. Then the ring of big Witt vectors $W_S(R)$ of R is a ring with additive identity $(0, 0, \dots)$ and multiplicative identity $(1, 0, 0, \dots)$. Moreover, for $(a_n)_{n\in S} \in W(R)$, its additive inverse is given by $(\iota_n(a_1, \dots, a_n))_{n\in\mathbb{N}}$.

Proposition 3.1.6

Let $\phi : R \to R'$ be a ring homomorphism. Then the induced map $W_S(\phi) : W_S(R) \to W_S(R')$ defined by

$$W(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

is a ring homomorphism.

Definition 3.1.7: The Witt Functor

Define the Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ to consist of the following data.

- For each ring R, $W_S(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi : R \to R'$, $W_S(\phi) : W_S(R) \to W_S(R')$ is the induced ring homomorphism defined by

$$W_S(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

Proposition 3.1.8

Let S be a truncation set. The Witt functor is indeed a functor.

Definition 3.1.9: The Ghost Map

Let R be a ring. Let S be a truncation set. Define the ghost map to be the map

$$w : W_S(R) \to \prod_{k\in S} R$$

by the formula

$$w((a_n)_{n\in S}) = (w_n(a_1, \dots, a_n))_{n\in S}$$

Remember, by the sequence $a_1, \dots, a_n$ we mean the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \leq d_1 \leq \dots \leq d_k \leq n$ and $d_1, \dots, d_k$ the complete collection of divisors of n.

Proposition 3.1.10

Let S be a truncation set. Then the following are true.

- For each $n \in S$, the collection of maps $w_n : W_S(R) \to R$ for a ring R defines a natural transformation $w_n : W_S \to \text{id}$.
- The collection of ghost maps $w_R : W_S(R) \to \prod_{k\in S} R$ for R a ring defines a natural transformation $w : W_S \to (-)^S$.

Proposition 3.1.11

Let S be a truncation set. The truncated Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W_S(R)$ is given by $\prod_{k\in S} R$
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by

$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$

- For each $n \in S$, $w_n : W_S(R) \to R$ defines a natural transformation $w_n : W \to \text{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k \in S} R$ is unique under the above conditions.

3.2 Important Maps of Witt Vectors

Definition 3.2.1: The Forgetful Map

Let R be a ring. Let $T \subseteq S$ be truncation sets. Define the forgetful map $R_T^S : W_S(R) \to W_T(R)$ to be the ring homomorphism given by forgetting all elements $s \in S$ but $s \notin T$.

Definition 3.2.2: The nth Verschiebung Map

Let R be a ring. Let S be a truncation set. For $n \in \mathbb{N}$, define the nth Verschiebung map $V_n : W_{S/n}(R) \to W_S(R)$ by

$$V_n((a_d)_{d \in S/n})_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise} \end{cases}$$

Note that this is not a ring homomorphism. However, it is additive.

Lemma 3.2.3

Let R be a ring. Let S be a truncation set. Then for all $a, b \in W_{S/n}(R)$, we have that

$$V_n(a + b) = V_n(a) + V_n(b)$$

Definition 3.2.4: Frobenius Map

Let S be a truncation set. Let R be a ring. Define the Frobenius map to be a natural ring homomorphism $F_n : W_S(R) \to W_{S/n}(R)$ such that the following diagram commutes:

$$\begin{tikzcd} W_S(R) \arrow[r, "w"] \arrow[d, "F_n"'] & \prod_{k \in S} R \arrow[d, "F_n^w"] \\ W_{S/n}(R) \arrow[r, "w"'] & \prod_{k \in S/n} R \end{tikzcd}$$

if it exists.

Lemma 3.2.5

Let S be a truncation set. Let R be a ring. Then the Frobenius map exists and is unique.

The following lemma relates this notion of Frobenius map to that in ring theory.

Lemma 3.2.6

Let A be an F_p algebra. Let S be a truncation set. Let $\varphi_p : A \to A$ denote the Frobenius homomorphism given by $a \mapsto a^p$. Then

$$F_p = R_{S/p}^S \circ W_S(\varphi) : W_S(A) \to W_{S/p}(A)$$

Definition 3.2.7: The Teichmuller Representative

Let R be a ring. Let S be a truncation set. Define the Teichmuller representative to be the map $[-]_S : R \to W_S(R)$ defined by

$$([a]_S)_n = \begin{cases} a & \text{if } n = 1 \\ 0 & b \text{ otherwise} \end{cases}$$

The Teichmuller representative is in general not a ring homomorphism, but it is still multiplicative.

Lemma 3.2.8

Let R be a ring. Let S be a truncation set. The for all $a, b \in R$, we have that

$$[ab]_S = [a]_S \cdot [b]_S$$

The three maps introduced are related as follows.

Proposition 3.2.9

Let R be a ring. Let S be a truncated set. Then the following are true.

- $r = \sum_{n \in S} V_n([r_n]_{S/n})$ for all $r \in W_S(R)$
- $F_n(V_n(a)) = na$ for all $a \in W_{S/n}(R)$
- $r \cdot V_n(a) = V_n(F_n(r) \cdot a)$ for all $r \in W_S(R)$ and all $a \in W_{S/n}(R)$
- $F_m \circ V_n = V_n \circ F_m$ if $\gcd(m, n) = 1$

The remaining section is dedicated to the example of $R = \mathbb{Z}$.

Proposition 3.2.10

Let S be a truncation set. Then the ring of big Witt vectors of $\mathbb{Z}$ is given by

$$W_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = \gcd(m, n) \cdot V_d([1]_{S/d})$$

and $d = \operatorname{lcm}(m, n)$.

3.3 The Ring of p-Typical Witt Vectors

For the ring of p-typical Witt vectors, we consider the truncation set $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$ for a prime p.

Definition 3.3.1: The Ring of p-Typical Witt Vectors

Let R be a ring. Let p be a prime. Let $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$. Define the ring of p-typical Witt vectors to be

$$W_p(R) = W_P(R)$$

Define the ring of p-typical Witt vectors of length n to be

$$W_n(R) = W_{\{1, p, \dots, p^{n-1}\}}(R)$$

when the prime p is understood.

Theorem 3.3.2

Let R be a ring. Let p be a prime number. Let S be a truncation set. Write $I(S) = \{k \in S \mid k \text{ does not divide } p\}$. Suppose that all $k \in I(S)$ are invertible in R. Then there is a decomposition

$$W_S(R) = \prod_{k \in I(S)} W_S(R) \cdot e_k$$

where

$$e_k = \prod_{t \in I(S) \setminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kt} V - kt([1]_{S/kt}) \right)$$

Moreover, the composite map given by

$$W_S(R) \cdot e_k \hookrightarrow W_S(R) \xrightarrow{F_k} W_{S/k} R \xrightarrow{R^{S/k}_{S/k \cap P}} W_{S/k \cap P}(R)$$

is an isomorphism.

3.4 The λ-structure on $\mathbf{W(R)}$

Lemma 3.4.1

Let R be a ring. Then every $f \in \Lambda(R)$ can be written uniquely as

$$f = \prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n)$$

Theorem 3.4.2: The Artin-Hasse Exponential

There is a natural isomorphism $E : \Lambda \to W$ given as follows. For a ring R, $E_R : \Lambda(R) \to W(R)$ is defined by

$$E_R \left(\prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n) \right) = (a_n)_{n \in \mathbb{N}}$$

Corollary 3.4.3

Let R be a ring. Then $W(R)$ has a canonical λ-structure inherited from $\Lambda(R)$.

TBA: The forgetful functor $U : \Lambda\mathbf{Ring} \to \mathbf{CRing}$ has a left adjoint Symm and has a right adjoint W.

4 Formal Group Laws

Definition 4.0.1: Formal Group Laws

Let R be a ring. A formal group law over R is a power series

$$f(x, y) \in R[[x, y]]$$

such that the following are true.

- $f(x, 0) = f(0, x) = x$
- $f(x, y) = f(y, x)$
- $f(x, f(y, z)) = f(f(x, y), z)$

Definition 4.0.2: The Formal Group Law Functor

Define the formal group law functor

$$FGL : \mathbf{Ring} \to \mathbf{Set}$$

by the following data.

- For each ring R, $FGL(R)$ is the set of all formal group laws over R
- For each ring homomorphism $f : R \to S$, $FGL(f)$ sends each formal group law $\sum_{i,j=0}^{\infty} c_{i,j} x^i y^j$ over R to the formal group law $\sum_{i,j=0}^{\infty} f(c_{i,j}) x^i y^j$ over S.

Definition 4.0.3: The Lazard Ring of a Formal Group Law

Define the lazard ring by

$$L = \frac{\mathbb{Z}[c_{i,j}]}{Q}$$

where Q is the ideal generated as follows. Write $f = \sum_{i,j=0}^{\infty} c_{i,j} x^i y^j$. Then Q is generated by the constraints on $c_{i,j}$ for which f becomes a formal group law.

Lemma 4.0.4

The Lazard ring $L = \mathbb{Z}[c_{i,j}]/Q$ has the structure of a graded ring where $c_{i,j}$ has degree $2(i + j - 1)$.

Theorem 4.0.5

The formal group law functor $FGL : \mathbf{Ring} \to \mathbf{Set}$ is representable

$$FGL(R) \cong \mathrm{Hom}_{\mathbf{Ring}}(L, R)$$

There exists a universal element $f \in L$ such that the map $\mathrm{Hom}_{\mathbf{Ring}}(L, R) \to FGL(R)$ given by evaluation on f is a bijection for any ring R.

Theorem 4.0.6

There is an isomorphism of the Lazard ring

$$L \cong \mathbb{Z}[t_1, t_2, \dots]$$

where each t_k has degree $2k$.

5 Homotopy Pullbacks and Pushouts

Homotopy pullbacks and pushouts are a special case of homotopy limits and colimits. It would be fruitful for us to first consider this case also because of how it is related to maps of spaces and (co)fibrations.

5.1 Homotopy Pullbacks

Definition 5.1.1: Homotopy Pullbacks

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $\mathcal{S}$ denote the following diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

in $\mathbf{CGWH}$. Define the homotopy pullback $\text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$ of the diagram to be the subspace of $X \times \text{Map}(I, Z) \times Y$ consisting of

$$\{(x, \alpha, y) \in X \times \text{Map}(I, Z) \times Y \mid \alpha(0) = f(x), \alpha(1) = g(y)\}$$

The idea is that normally in pullbacks, we require that under f and g the elements of the pullback must arrive at the same point in Z. But here we relax the requirement by simply allowing elements of the homotopy pullback to arrive at the same path component of Z (so up to the existence of an homotopy of the two points in Z).

Definition 5.1.2: The Canonical Map of Homotopy Pullbacks

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in $\mathbf{CGWH}$. Define the canonical map of the homotopy pullback of the diagram to be the map

$$c : \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

defined by $(x, y) \mapsto (x, c_{f(x)=g(y)}, y)$.

Theorem 5.1.3: The Matching Lemma

Suppose that we have a commutative diagram of spaces

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e_X"'] & Z \arrow[d, "e_Z"'] & Y \arrow[l, "g"'] \arrow[d, "e_Y"] \\
X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"]
\end{tikzcd}$$

in $\mathbf{CGWH}$. Define the map

$$\phi_{X,Z,Y}^{X',Z',Y'} : \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X' \xrightarrow{f'} Z' \xleftarrow{g'} Y')$$

by the formula $(x, \gamma, y) \mapsto (e_X(x), e_Z \circ \gamma, e_Y(y))$. Then the following are true.

- If each e_X, e_Y, e_Z are homotopy equivalences, then ϕ is a homotopy equivalence.
- If each e_X, e_Y, e_Z are weak equivalences, then ϕ is a weak equivalence.

Proof. We first prove the case for homotopy equivalence. Consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Z \arrow[d, "e_Z"] & Y \arrow[l, "g"'] \arrow[d, "\mathrm{id}_Y"] \\
X \arrow[r, "e_Z \circ f"] \arrow[d, "e_X"'] & Z' \arrow[d, "\mathrm{id}_{Z'}"] & Y \arrow[l, "e_Z \circ g"'] \arrow[d, "e_Y"] \\
X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"]
\end{tikzcd}$$

We prove that the homotopy pullback of the first row is homotopy equivalent to that of the second, and we prove that the homotopy pullback of the second row is homotopy equivalent to that of the third.

Since e_Z is a homotopy equivalence, we can find a homotopy inverse k for e_Z and a homotopy $H : Z \times I \to Z$ such that $H(-, 0) = \mathrm{id}_Z$ and $H(-, 1) = k \circ e_Z$. Define a map

$$\rho : \mathrm{holim}(X \xrightarrow{f} Z' \xleftarrow{g} Y) \to \mathrm{holim}(X \xrightarrow{e_Z \circ f} Z \xleftarrow{e_Z \circ g} Y)$$

by the formula

$$(x, \gamma', y) \mapsto (x, H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)} : I \to Z, y)$$

where $*$ denotes concatenation of paths. The path concatenation is well defined because we have that $H(f(x), 1) = (k \circ e_Z \circ f)(x) = (k \circ \gamma')(0)$ and $k(\gamma'(1)) = k(e_Z(g(y))) = H(g(y), 1)$. This is well defined on the homotopy pullback because we have that

- $H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)}(0) = H(f(x), 0) = \mathrm{id}_Z(f(x)) = f(x)$
- $H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)}(1) = H(g(y), 0) = \mathrm{id}_Z(g(y)) = g(y)$

I claim that this map is inverse to the map $\phi = \phi^{X,Z',Y}_{X,Y,Z}$ where we take $e_X = \mathrm{id}_X$ and $e_Y = \mathrm{id}_Y$. We have that

$$\begin{aligned}
\rho(\phi(x, \gamma, y)) &= \rho(x, e_Z \circ \gamma, y) \\
&= (x, H(f(x), -) * k(e_Z(\gamma(-)) * \overline{H(g(y), -)}, y)
\end{aligned}$$

Now I claim that the middle path is homotopic to γ. For the first part, the path $H(f(x), t) : I \to Z$ can be contracted to $H(f(x), 0) = f(x) = \gamma(0)$ so you can homotope the traversal along $H(f(x), -)$ to the single point $f(x) = \gamma(0)$. For the third part, this is similar so we can homotope the traversal of $\overline{H(g(y), -)}$ to the single point $g(y) = \gamma(1)$. The middle part of the path is homotopic to γ because $k \circ e_Z$ is homotopic to id_Z. Thus we conclude. $\square$

When one of the maps f or g is a fibration, then the notion of a pullback coincides with that of homotopy pullback.

Proposition 5.1.4

Let $X, Y, Z \in$ **CGWH** be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in **CGWH**. Then the following spaces are homeomorphic.

- $\mathrm{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$
- $\lim(P_f \to Z \xleftarrow{g} Y)$
- $\lim(X \xrightarrow{f} Z \leftarrow P_g)$
- $\lim(P_f \to Z \leftarrow P_g)$

Proposition 5.1.5

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in **CGWH**. If f or g is a fibration, then the canonical map

$$\lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

is a homotopy equivalence.

5.2 Homotopy Pushouts

We now want to measure how far away is a square diagram from being a homotopy pullback and dually, how far away is a square diagram from being a homotopy pushout.

Definition 5.2.1: Homotopy Pushouts

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $\mathcal{S}$ denote the following diagram

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

in **CGWH**. Define the homotopy pushout of the diagram to be the quotient space

$$\text{hocolim}(\mathcal{S}) = \frac{X \amalg (Z \times I) \amalg Y}{\sim}$$

where $\sim$ is the equivalence relation generated by $f(z) \sim (z, 0)$ and $g(z) \sim (z, 1)$ for $z \in Z$. If (Z, z_0) is a based space, then the equivalence relation is also generated by $(x_0, t) \sim (z_0, s)$ for $s, t \in I$.

Definition 5.2.2: The Canonical Map of Homotopy Pushouts

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $\mathcal{S}$ denote the following diagram

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

in **CGWH**. Define the canonical map of the homotopy pushout of the diagram to be the map

$$s : \text{hocolim}(\mathcal{S}) \to \text{colim}(\mathcal{S})$$

given by the formula

$$u \mapsto \begin{cases} u & \text{if } u \in X \\ f(z) = g(z) & \text{if } u = (z, t) \in Z \times I \\ u & \text{if } u \in Y \end{cases}$$

Theorem 5.2.3: The Gluing Lemma

Suppose that we have a commutative diagram of spaces

$$\begin{tikzcd} X \arrow[d, "e_X"'] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "e_Z"'] & Y \arrow[d, "e_Y"] \\ X' & Z' \arrow[l, "f'"] \arrow[r, "g'"'] & Y' \end{tikzcd}$$

in **CGWH**. If each e_X, e_Y, e_Z are (homotopy) weak equivalences, then the induced map

$$\text{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y) \to \text{hocolim}(X' \xleftarrow{f'} Z' \xrightarrow{g'} Y')$$

defined by the formula

$$u \mapsto \begin{cases} e_X(u) & \text{if } u \in X \\ (e_Z(v), t) & \text{if } u = (v, t) \in Z \times I \\ e_Y(u) & \text{if } u \in Y \end{cases}$$

is a (homotopy) weak equivalence.

Proposition 5.2.4

Let $X, Y, Z \in$ **CGWH** be spaces. Let $\mathcal{S}$ denote the following diagram

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

in **CGWH**. Then the following spaces are homeomorphic.

- $\text{hocolim}(\mathcal{S})$
- $\text{colim}(M_f \leftarrow Z \rightarrow Y)$
- $\text{colim}(X \leftarrow Z \rightarrow M_g)$
- $\text{colim}(M_f \leftarrow Z \rightarrow M_g)$

5.3 Homotopy Squares

Definition 5.3.1: Homotopy Pullback and Pushout Squares

Let $W, X, Y, Z \in$ **CGWH** be spaces such that there is a (not necessarily commutative) diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

- We say that the diagram is a homotopy pullback if the map

$$\alpha : W \to \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \xrightarrow{c} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

  is a weak equivalence.
- We say that the diagram is k-cartesian if α is k-connected.
- Dually, we say that the square is a homotopy pushout square if the map

$$\beta : \text{hocolim}(X \xleftarrow{f} W \xrightarrow{g} Y) \xrightarrow{s} \text{colim}(X \xleftarrow{f} W \xrightarrow{g} Y) \to Z$$

  is a weak equivalence.
- Also dually, we say that the diagram is k-cocartesian if β is k-connected.

We can rephrase 5.1.5 in the following way: if

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is a commutative square diagram and either $X \to Z$ or $Y \to Z$ is a fibration, then the square is a homotopy pullback square.

Lemma 5.3.2

Let X be a space. Let $A, B \subseteq X$ be subspace of X such that $X = A \cup B$. Then the following commutative square

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

given by inclusions is a homotopy pushout square.

Theorem 5.3.3: Seifert-van Kampen Theorem

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that the following square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is a homotopy pullback. Suppose that W, X, Y are path connected. Then the square

$$\begin{tikzcd} \pi_1(W) \arrow[r] \arrow[d] & \pi_1(Y) \arrow[d] \\ \pi_1(X) \arrow[r] & \pi_1(Z) \end{tikzcd}$$

is a pushout in $\mathbf{Grp}$. In other words, there is a canonical isomorphism

$$\pi_1(X) *_{\pi_1(W)} \pi_1(Y) \xrightarrow{\cong} \pi_1(Z)$$

5.4 Relation to Homotopy (Co)Fibers

Recall that the mapping path space P_f of a map $f : X \to Y$ is defined to be

$$P_f = f^*(\text{Map}(I, Y)) = \{(x, \phi) \subseteq X \times \text{Map}(I, Y) \mid f(x) = \pi_0(\phi) = \phi(0)\}$$

we can now prove that P_f is a homotopy invariance.

Corollary 5.4.1

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f, g : X \to Y$ be maps. Then there is a homotopy equivalence

$$P_f \simeq P_g$$

Moreover, there is a homotopy equivalence

$$\text{hofiber}_y(f) \simeq \text{hofiber}_y(g)$$

for any $y \in Y$.

Recall that the fiber of a map $f : X \to Y$ behaves poorly because the fibers are not homeomorphic and not even homotopy equivalent. However, we can now prove that the homotopy fibers are the correct notion of a fiber to study because they are homotopy equivalent.

Corollary 5.4.2

Let $X, Y \in \mathbf{CGWH}$ be space. Let $f : X \to Y$ be a map. If y_1 and y_2 lie in the same path

component of Y then there is a homotopy equivalence

$$\text{hofiber}_{y_1}(f) = \text{hofiber}_{y_2}(f)$$

Corollary 5.4.3

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f, g : X \to Y$ be maps. Then there is a homotopy equivalence

$$M_f \simeq M_g$$

Moreover, there is a homotopy equivalence

$$\text{hocofiber}(f) \simeq \text{hocofiber}(g)$$

for any $y \in Y$.

We can we interpret homotopy pullbacks and pushouts using homotopy (co)fibers.

Proposition 5.4.4

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that following is a (not necessarily commutative) square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Then the following are true.

- The square is a homotopy pullback if and only if for all $x \in X$, the map

$$\text{hofiber}_x(W \to X) \to \text{hofiber}_{f(x)}(Y \to Z)$$

  is a weak equivalence.
- The square is k-carteisna if and only if for all $x \in X$, the map

$$\text{hofiber}_x(W \to X) \to \text{hofiber}_{f(x)}(Y \to Z)$$

  is k-connected.

5.5 Connectedness of Homotopy Squares

6 Blakers-Massey Theorem

6.1 The Blakers-Massey Theorem for Squares

The Blakers-Massey theorem is a direct generalization of the homotopy excision theorem. Its proof takes a similar form to the homotopy excision theorem. Let us recall some definitions used.

Definition 6.1.1: (Degenerative) Cubes

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Let $\delta > 0$. Let $L \subseteq \{1, \dots, n\}$. A cube in $\mathbb{R}^n$ is a set of the form

$$W = W(a, \delta, L) = \{x \in \mathbb{R}^n \mid a_i \leq x \leq a_i + \delta \text{ for } i \in L \text{ and } x_i = a_i \text{ for } i \notin L\}$$

Lemma 6.1.2

Let Y be a space. Let $B \subseteq Y$ be a subspace of Y. Let W be a cube in $\mathbb{R}^n$. Let $f : W \to Y$ be a map. Let $j = 1$ or 2. Suppose that there exists some $p \leq |L|$ such that

$$f^{-1}(B) \cap C \subset K_p^j(C) = \left\{x \in C \mid \frac{\delta(j-1)}{2} + a_i < x_i < \frac{\delta j}{2} + a_i \text{ for at least } p \text{ values of } i \in L\right\}$$

for all cubes $C \subset \partial W$. Then there exists a map $g : W \to Y$ such that $g \overset{\partial W}{\simeq} f$ and

$$g^{-1}(B) \subset K_p^j(C)$$

Proposition 6.1.3

Let X be a space. Let $X_0, X_1, X_2 \subseteq X$ be subspaces of X such that

$$X = X_1 \amalg_{X_0} X_2$$

Let $f : I^n \to X$ be a map. Suppose that $W \subseteq I^n$ is any cube given by the Lebesgue covering lemma for which $f(W) \subseteq X_i$ for one of $i = 0, 1, 2$. Assume that for each $i = 1, 2$, (X_i, X_0) is k_i-connected with $k_i \geq 0$. Then there exists a homotopy

$$H : I^n \times I \to X$$

from $f = H(-, 0)$ such that the following are true.

- If $f(W) \subset X_i$, then $H(W, t) \subset X_i$ for all $t \in I$.
- If $f(W) \subset X_0$, then $H(W, t) = f(W)$ for all $t \in I$.
- If $f(W) \subset X_i$, then $f^{-1}(X_i \setminus X_0) \cap W \subset K_{k_j+1}^j(W)$.

The reasons for the above setup is that we want to prove the following lemma.

Lemma 6.1.4

Let X be a space. Let e^{d_i} be a cell of dimension d_i for $i = 1, 2$. Then the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \cup e^{d_1} \arrow[d] \\ X \cup e^{d_2} \arrow[r] & X \cup e^{d_1} \cup e^{d_2} \end{tikzcd}$$

given by inclusion maps is $(d_1 + d_2 - 3)$-cartesian.

This lemma has a less geometric way of proving it that does not involve any of the preparative lemmas and propositions, however it does use significant material that have not been covered.

Theorem 6.1.5: Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

is a homotopy pushout. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cartesian. Explicitly, this means that

$$\alpha : X_0 \to \text{holim}(X_1 \to X_{12} \leftarrow X_2)$$

is $(k_1 + k_2 - 1)$-connected.

This theorem directly generalizes the homotopy excision theorem in the following way. For X a CW complex and A, B two subcomplexes with non-empty intersection and $X = A \cup B$, consider the following square of inclusions:

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

We have seen that such a square diagram is a homotopy pushout diagram. Now any inclusion map $W \hookrightarrow Z$ is k-connected if and only if (Z, W) is k-connected. So $(A, A \cap B)$ is k_1-connected and (X, B) is k_2-connected. Blaker's-Massey theorem implies that

$$\text{hofiber}(A \cap B \to A) \to \text{hofiber}(B, X)$$

is $(k_1 + k_2 - 1)$-connected. But by definition we have an isomorphism $\pi_k(\text{hofiber}(U \to V) \cong \pi_{k+1}(V, U)$. So we are really just saying that $\pi_k(A, A \cap B) \to \pi_k(X, B)$ given by the inclusion is $(k_1 + k_2)$-connected.

6.2 The Dual Blakers-Massey Theorem for Squares

Theorem 6.2.1: Dual Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

is a homotopy pullback. Suppose the map $X_i \to X_{12}$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cocartesian. Explicitly, this means that

$$\beta : \text{hocolim}(X_1 \leftarrow X_0 \to X_2) \to X_{12}$$

is $(k_1 + k_2 - 1)$-connected.

7 n-Cubes

In algebraic topology, we have learnt about spaces, maps of spaces and maps of maps of spaces. We can say this in a more compact way. Namely, if we think of maps of maps of space as a square (2-cube), we can think of spaces as 0-cubes and maps of spaces as 1-cube. We have studied 2-cubes extensively under the guise of homotopy pullbacks and pushouts. We can now take this further and consider general n-cubes.

Definition 7.0.1: n-Cubes of Spaces

Let $n \in \mathbb{N}$. Let $P(n)$ denote the category of posets of the set $\{1, \dots, n\}$. An n-cube of spaces is a functor

$$X : P(n) \to \mathbf{CGWH}$$

An n-cube of based spaces is a functor $X : P(n) \to \mathbf{CGWH}_*$.

Explicitly, an n-cube of spaces $X : P(n) \to \mathbf{CGWH}$ consists of the following data.

- For each $S \subseteq \{1, \dots, n\}$ a space X_S
- For each $S \subseteq T$, a map $f_{S \subseteq T} : X_S \to X_T$ such that $f_{S \subseteq S} = 1_{X_S}$ and for all $R \subseteq S \subseteq T$, we have a commutative diagram

$$\begin{tikzcd} X_R \arrow[r, "f_{R \subseteq S}"] \arrow[dr, "f_{R \subseteq T}"'] & X_S \arrow[d, "f_{S \subseteq T}"] \\ & X_T \end{tikzcd}$$

Omit drawing composite arrows and omit drawing identities.
Also: punctured cubes def

Definition 7.0.2: Cube of Cubes

An n-cube of m-cubes is a functor

$$X : P(n) \times P(m) \to \mathbf{CGWH}$$

Lemma 7.0.3

An n-cube of m-cubes X is precisely an $(n+m)$-cube.

Definition 7.0.4: Map of n-Cubes

Let $X, Y : P(n) \to \mathbf{CGWH}$ be n-cubes. A map of n-cubes is a natural transformation $F : X \to Y$ such that the assignment $Z : P(n+1) \to \mathbf{CGWH}$ given by

$$Z(S) = \begin{cases} X(S) & \text{if } S \subseteq \{1, \dots, n\} \\ Y(S \setminus \{1, \dots, n+1\}) & \text{if } \{1, \dots, n+1\} \subseteq S \end{cases}$$

defines an $(n+1)$-cube.

objectwise (co)fibration, homotopy (weak) equivalence. homeomorphism

Definition 7.0.5: Strongly Homotopy Cartesian

Let X be an n-cube of spaces. We say that X is strongly homotopy cartesian if each of its faces of dimension $n \geq 2$ is homotopy cartesian.

8 Homotopy Limits and Colimits

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Denote the constant functor of the one point space by $\Delta* : \mathbf{X} \to \mathbf{Top}$. The data of the constant functor is given as follows.

- For each $I \in \mathcal{J}$, $\Delta * (I) = *$
- For each morphism $f : I \to J$ in X, define $\Delta * (f) = \text{id}_*$.

Consider the set of all natural transformations $\Delta* \Rightarrow X$ denoted by $\text{Nat}(\Delta*, X)$. Now this set can inherit a subspace topology via the isomorphism (of sets)

$$\text{Nat}(\Delta*, X) \cong \prod_{I \in \mathcal{J}} \text{Hom}_{\mathbf{Top}}(*, X_I) \subset \prod_{I \in \mathcal{J}} X_I$$

There is in fact a canonical homeomorphism between the set of natural transformations and the limit of X.

Theorem 8.0.1

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Then there is a canonical homeomorphism

$$\lim_{\mathcal{J}} X \cong \text{Nat}(\Delta*, X)$$

Ref: Cubical diagrams

Theorem 8.0.2

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Then there is a canonical homeomorphism

$$\operatorname*{colim}_{\mathcal{J}} X \cong \frac{\coprod_{I \in \mathcal{J}} X_I}{\sim}$$

where $x \in X_I \sim y \in X_J$ if and only if there exists $f : I \to J$ such that $X(f)(x) = y$.

8.1 Homotopy Limits and Colimits

Let $\mathcal{J}$ be a small diagram. Let $I \in \mathcal{J}$ and denote $\mathcal{J}/I$ to be the overcategory with the distinguished object I. We can turn it into a topological space by constructing its classifying space

$$\mathcal{B}(\mathcal{J}/I) = |N(\mathcal{J}/I)|$$

which is the geometric realization of the nerve of $\mathcal{J}/I$. We aim to use the overcategory $\mathcal{J}/I$ to record homotopy information. Recall that the limit is canonically isomorphic to the equalizer of

$$f, g : \prod_{J \in \text{Obj}\, \mathcal{J}} X_J \to \prod_{(\alpha : J \to I) \in \mathcal{J}} X_I$$

as follows.

- Define f to be the unique map such that $\pi_I \circ f = \pi_{\text{cod}(\alpha)}$ where both π are projections.
- Define g to be the unique map such that $\pi_{\text{cod}(\alpha)} \circ g = F(\alpha) \circ \pi_{\text{dom}(\alpha)}$

We aim to replace each space X_I by the space $\text{Map}(\mathcal{B}(\mathcal{J}/I), X_I)$ (which is why we work with **CGWH**). Indeed the hom space and X_I are homotopy equivalent since $\mathcal{B}(\mathcal{J}/I)$ is contractible. Notice that this is no longer a functor in I, but rather a bifunctor. This all works with a simplicial model category.

Definition 8.1.1: Homotopy Limits

Let $\mathcal{C}$ be a simplicial model category. Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. Define two maps

$$f, g : \prod_{J \in \operatorname{Obj} \mathcal{J}} \operatorname{Map}(\mathcal{B}(\mathcal{J}/J), X_J) \to \prod_{(\alpha : J \to I) \in \mathcal{J}} \operatorname{Map}(\mathcal{B}(\mathcal{J}/J), X_I)$$

as follows.

- Define f to be the unique map such that

$$\pi_I \circ f = \operatorname{Map}(\mathcal{B}(\mathcal{J}/J), X(\alpha) : X_J \to X_I)$$

  for any $\alpha : J \to I$ a morphism in $\mathcal{J}$.
- Define g to be the unique map such that

$$\pi_I \circ g = \operatorname{Map}\left(\mathcal{B}\left(\mathcal{J}/X : \mathcal{J}/I \to \mathcal{J}/J\right), X_I\right)$$

  for any $\alpha : J \to I$ a morphism in $\mathcal{J}$.

Define the homotopy limit $\operatorname{holim} X$ of X to be the equalizer of the maps

$$\operatorname{holim} X = \operatorname{Eq}(f, g)$$

Let $\mathcal{J}$ be the small diagram consisting of two objects A, B and two non-trivial morphisms $f, g : A \to B$. Let us illustrate the homotopy limit of $\mathcal{J}$. The slice category $\mathcal{J}/A$ consists only of a single object corresponding to the identity map $\operatorname{id}_A : A \to A$. The slice category $\mathcal{J}/B$ consists of 3 objects corresponding to id_B, f and g. Non-trivial morphisms are given as follows:

$$\begin{tikzcd}
f \arrow[dr] & \\
 & \operatorname{id}_B \\
g \arrow[ur] &
\end{tikzcd}$$

Passing to the classifying space, $\mathcal{B}(\mathcal{J}/A)$ is just the one point space and $\mathcal{B}(\mathcal{J}/B)$ becomes the interval I.

Expanding things out show that

$$f, g : \operatorname{Map}(, A) \times \operatorname{Map}(, B)$$

Proposition 8.1.2

Let $X : \mathcal{J} \to \mathbf{CGWH}$ be a diagram. Then there is a natural transformation

$$\operatorname{holim}_{\mathcal{J}} X \cong \operatorname{Nat}(\mathcal{B}(\mathcal{J}/-) : \mathcal{J} \to \mathbf{CGWH}, X)$$

Definition 8.1.3: Homotopy Colimits

Let $\mathcal{C}$ be a simplicial model category. Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. Define two maps

$$f, g : \coprod_{(\alpha : J \to I) \in \mathcal{J}} X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \to \coprod_{J \in \operatorname{Obj} \mathcal{J}} X_J \times \mathcal{B}\left((J/\mathcal{J})^{\mathrm{op}}\right)$$

as follows.

- On each summand of the domain of f, define f to be the map

$$X(\alpha) \times \operatorname{id}_{\mathcal{B}((I/\mathcal{J})^{\mathrm{op}})} : X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \longrightarrow X_I \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right)$$

  and then injecting the into the coproduct.

- On each summand of the domain of g, define g to be the map

$$\mathrm{id}_{X_J} \times \mathcal{B}(\mathcal{J}/X)^{\mathrm{op}} : X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \longrightarrow X_J \times \mathcal{B}\left((J/\mathcal{J})^{\mathrm{op}}\right)$$

  and then injecting into the coproduct.

Define the homotopy colimit $\operatorname{hocolim} X$ of X to be the coequalizer of the maps

$$\operatorname{hocolim} X = \mathrm{Coeq}(f, g)$$

9 Calculus of Functors

9.1 Excisive Functors

Definition 9.1.1: Homotopy Functors

Let $\mathcal{C}, \mathcal{D}$ be categories with a notion of weak equivalence. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy functor if F preserves weak equivalences.

Definition 9.1.2: n-Excisive Functors

Let F be a homotopy functor. We say that F is n-excisive if it takes strongly homotopy cocartesian $(n+1)$-cubes to homotopy cartesian $(n+1)$-cubes.

9.2 The Taylor Tower

Definition 9.2.1: Fiberwise Join

Let X, Y, U be spaces. Let $f : X \to Y$ be a map. Define the fiberwise join of X and U along f to be the space

$$X *_Y U = \text{hocolim}(X \longleftarrow X \times U \longrightarrow Y \times U)$$

Lemma 9.2.2

Let X, Y, U, V be spaces. Let $f : X \to Y$ be a map. Then there is a natural isomorphism

$$(X *_Y U) *_Y V \cong X *_Y (U * V)$$

Proposition 9.2.3

Let $\mathcal{P}(n)$ denote the category of posets. Let X be a space over Y. Then the assignment

$$U \mapsto X *_Y U$$

defines an n-dimensional cubical diagram in **Top**. Moreover, it is strongly cocartesian.

Definition 9.2.4

Let Y be a space. Let $F : \mathbf{Top}_Y \to \mathbf{Top}$ be a homotopy functor. Define the functor

$$T_n F : \mathbf{Top}_Y \to \mathbf{Top}$$

to consist of the following data.

- For each $X \in \mathbf{Top}$, consider the functor $\mathcal{X} : \mathcal{P}(n+1) \to \mathbf{Top}$ given by $U \mapsto F(X *_Y U)$. Define
$$T_n F(X) = \text{holim}(\mathcal{X}) = \underset{U \in \mathcal{P}(n+1)}{\text{holim}} (F(X *_Y U))$$
- For each $f : X \to Z$ a morphism of spaces over Y, define a map $T_n F(X) \to T_n F(Y)$ to be the map
$$F(f *_Y \text{id}) \circ \mathcal{X}$$

Lemma 9.2.5

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then $T_n F$ is a homotopy functor.

Proposition 9.2.6

Let F be a homotopy functor. Then there exists a natural map $t_nF : F \Rightarrow T_nF$ given by the canonical map of homotopy limits. Moreover, t_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "t_nF"'] & G \arrow[d, "t_nG"] \\ T_nF \arrow[r, "T_n\lambda"'] & T_nG \end{tikzcd}$$

Definition 9.2.7

Let Y be a space. Let $F : \mathbf{Top}_Y \to \mathbf{Top}$ be a homotopy functor. Define the functor

$$P_nF : \mathbf{Top}_Y \to \mathbf{Top}$$

to consist of the following data.

- For each space X over Y, define $P_nF(X)$ to be the homotopy limit

$$P_nF(X) = \text{holim}(F(X) \to T_nF(X) \to (T_n(T_nF))(X) \to \dots)$$

- For each morphism $f : X \to Z$ of spaces over Y, define $P_nF(f) : P_nF(X) \to P_nF(Z)$ to be the map ????

Lemma 9.2.8

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then P_nF is a homotopy functor.

Proposition 9.2.9

Let F be a homotopy functor. Then there exists a natural map $p_nF : F \Rightarrow P_nF$ given by the canonical map of homotopy limits. Moreover, p_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "p_nF"'] & G \arrow[d, "p_nG"] \\ P_nF \arrow[r, "P_n\lambda"'] & P_nG \end{tikzcd}$$

Definition 9.2.10: n-Reduced Functors

Let F be a homotopy functor. We say that F is n-reduced if $P_{n-1}F \simeq *$.

Definition 9.2.11: n-Homogenous Functor

Let F be a homotopy functor. We say that F is n-homogenous if F is n-excisive and n-reduced.

9.3 Linear Functors

Definition 9.3.1: Linear Functors

Let F be a homotopy functor. We say that F is linear if F is 1-homogenous. Explicitly, this means that

- F sends homotopy pushouts to homotopy pullbacks
- $F(X)$ is homotopy equivalent to $*$

Let us consider the case $n = 1$ and $Y = *$. Now $\mathcal{P}_0(2)$ is the small category given in a diagram as follows:

$$\begin{tikzcd} & \{1\} \arrow[d] \\ \{0\} \arrow[r] & \{0,1\} \end{tikzcd}$$

Now T_1F sends every space X to the homotopy limit of the following diagram:

$$\begin{tikzcd} & F(X * \{1\}) \arrow[d] \\ F(X * \{0\}) \arrow[r] & F(X * \{0,1\}) \end{tikzcd}$$

But we know that $X * \{0\}$ is the cone CX and $X * \{0,1\}$ is the reduced suspension. This means that we can simplify the above diagram into

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

Now $CX \simeq *$ and F is a reduced functor. Thus we can further simplify the diagram into

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

We recognize this as the homotopy pullback, and so $T_1F(X) \simeq \Omega F(\Sigma X)$. Now recall that

$$P_1F(X) = \text{hocolim}(F(X) \overset{t_1F(X)}{\longrightarrow} T_1F(X) \overset{t_1(T_1F)}{\longrightarrow} (T_1(T_1F))(X) \longrightarrow)$$

Again because we know that $T_1F(X) \simeq \Omega F(\Sigma X)$ and we care about everything only up to homotopy, we can write P_1F as

$$P_1F(X) = \text{hocolim}(F(X) \overset{t_1F(X)}{\longrightarrow} \Omega F(\Sigma X) \overset{t_1(T_1F)}{\longrightarrow} \Omega(T_1F)(\Sigma X) \longrightarrow)$$

which further simplifies to

$$P_1F(X) = \text{hocolim}(F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \longrightarrow)$$

Thus in general,

$$P_1F(X) = \underset{n\to\infty}{\text{hocolim}}(\Omega^n F(\Sigma^n X))$$

We are considering the case $n = 2$. Now $\mathcal{P}_0(3)$ is the small category given in a diagram as follows:

$$\begin{tikzcd}
 & \{2\} \arrow[dl] \arrow[rr] & & \{1,2\} \arrow[dl] \\
\{0,2\} \arrow[rr] & & \{0,1,2\} & \{1\} \arrow[u] \arrow[dl] \\
\{0\} \arrow[u] \arrow[rr] & & \{0,1\} \arrow[u] &
\end{tikzcd}$$

If we plug it into the definition of T_nF and choose $Y = *$, we obtain a functor

$$T_2F : \mathbf{Top}_* \to \mathbf{Top}$$

that consists of the following data. For each $X \in \mathbf{Top}$, $T_2F(X)$ is precisely the homotopy limit of the diagram

$$\begin{tikzcd}
 & F(X * \{2\}) \arrow[dl] \arrow[rr] & & F(X * \{1,2\}) \arrow[dl] \\
F(X * \{0,2\}) \arrow[rr] & & F(X * \{0,1,2\}) & F(X * \{1\}) \arrow[u] \arrow[dl] \\
F(X * \{0\}) \arrow[u] \arrow[rr] & & F(X * \{0,1\}) \arrow[u] &
\end{tikzcd}$$

which simplifies to the diagram:

$$\begin{tikzcd}
 & F(CX) \arrow[dl] \arrow[rr] & & F(\Sigma X) \arrow[dl] \\
F(\Sigma X) \arrow[rr] & & F(X * \{0,1,2\}) & F(CX) \arrow[u] \arrow[dl] \\
F(CX) \arrow[u] \arrow[rr] & & F(\Sigma X) \arrow[u] &
\end{tikzcd}$$

Now since F is reduced and $CX \simeq *$, we can further simplify it into

$$\begin{tikzcd}
 & * \arrow[dl] \arrow[rr] & & F(\Sigma X) \arrow[dl] \\
F(\Sigma X) \arrow[rr] & & F(X * \{0,1,2\}) & * \arrow[l] \arrow[dl] \\
* \arrow[u] \arrow[rr] & & F(\Sigma X) \arrow[u] &
\end{tikzcd}$$

(what does the maps look like?)

Definition 9.3.2: The Category of Linear Functors

Define the category

$$\mathcal{H}_1(\mathcal{C}, \mathcal{D})$$

of linear functors to be the full subcategory of $\mathcal{D}^{\mathcal{C}}$ consisting of linear functors.

Theorem 9.3.3

There is an equivalence of categories

$$\mathcal{H}_1(\mathbf{CGWH}_*, \mathbf{CGWH}_*) \cong \Omega\mathrm{Sp}^{\mathbb{N}}(\mathbf{CGWH}_*)$$

given as follows. For a linear functor F, we associate to it the sequence of spaces $\{F(S^n) \mid n \in \mathbb{N}\}$, and this defines a spectra.

9.4 Important Theorems

Denote Sp by the category of spectra. Define a map $\mathcal{L}(\mathbf{Top}_*, \mathrm{Sp}) \to \mathrm{Sp}$ that sends $F : \mathbf{Top}_* \to \mathrm{Sp}$ to the spectra $F(S^0)$. Conversely, define a map $\mathrm{Sp} \to \mathcal{L}(\mathbf{Top}_*, \mathrm{Sp})$ by sending each spectra X to the functor $X \wedge -$.

Now define a map $\mathcal{L}(\mathbf{Top}_*) \to \mathrm{Sp}$ as follows. For each $F : \mathbf{Top}_* \to \mathbf{Top}_*$, $F(S^n)$ is a collection of spaces indexed by $\mathbb{N}$. As for the bonding maps $F(S^n) \wedge S^1 \to F(S^{n+1})$, this is defined as follows:

1. Consider the identity map $\mathrm{id} : X \wedge Y \to X \wedge Y$.
2. By the smash-hom adjunction, this corresponds to a map $Y \to \mathrm{Map}(X, X \wedge Y)$.
3. Now composing with F gives a map

$$Y \to \mathrm{Map}(X, X \wedge Y) \to \mathrm{Map}(F(X), F(X \wedge Y))$$

   (Why is the latter map continuous?)
4. By the smash-hom adjunction, this corresponds to a map $F(X) \wedge Y \to F(X \wedge Y)$
5. Taking $X = S^n$ and $Y = S^1$ gives the desired results.

This way we also need to show that $F(S^n) \wedge S^1 \to F(S^{n+1})$ is a weak equivalence.

Alternatively, we can construct the map $F(S^n) \to \Omega F(S^{n+1})$ as follows.

1. We begin by noticing that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

   is a homotopy pushout.
2. Applying F sends the homotopy pushout to a homotopy pullback:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(*) \arrow[d] \\ F(*) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

3. Since F is reduced, the diagram can be simplified into

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

4. Now recall that $\Omega(F(\Sigma X))$ is the homotopy pullback of $* \to F(\Sigma X) \leftarrow *$.
5. We obtain maps $F(X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ and $\Omega F(\Sigma X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ which are both weak equivalences. Hence we obtain a weak equivalence $F(X) \simeq \Omega F(\Sigma X)$.
6. When $X = S^n$ we obtain exactly what we want because $\Sigma S^n = S^{n+1}$.

In fact one can prove that the second constructed map and the first constructed map coincide by the smash-hom adjunction (how to prove?).

We now have maps $\mathcal{L}(\mathbf{Top}_*) \rightleftarrows \text{Sp}$. This actually gives an equivalence of categories. In fact, one can find out that it is a two step process:

$$\mathcal{L}(\mathbf{Top}_*) \rightleftarrows \mathcal{L}(\mathbf{Top}_*, \text{Sp}) \rightleftarrows \text{Sp}$$

10 Algebras and Coalgebras

10.1 Coalgebras

There is a need to revisit the definition of an algebra (over a field)

Proposition 10.1.1

A vector space V over a field k is an algebra if and only if there is a following collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map

such that the following two diagrams are commutative:

$$\begin{tikzcd} V \otimes V \otimes V \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & V \otimes V \arrow[d, "m"] \\ V \otimes V \arrow[r, "m"'] & V \end{tikzcd} \qquad \begin{tikzcd} k \otimes V \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\cong"'] & V \otimes V \arrow[dl, "m"'] \\ V & V \otimes k \arrow[l, "\cong"] \arrow[u, "\mathrm{id}\otimes u"'] \end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Evidently, the map μ gives a multiplicative structure for V and Δ gives the unitary structure of an algebra. The diagram on the left then represent associativity of multiplication. Notice that such additional structure on V formally lives in the category $\mathbf{Vect}_k$ of vector spaces over a fixed field k.

Therefore we can formally dualize all arrows to obtain a new object.

Definition 10.1.2: Coalgebra

Let V be a vector space over a field k. We say that V is a coalgebra over k if there is a collection of data:

- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that the following diagrams are commutative:

$$\begin{tikzcd} V \otimes V \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\Delta"'] \\ V \otimes V \arrow[u, "\Delta\otimes\varepsilon"] & V \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} k \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\ V \arrow[u, "\cong"] \arrow[ur, "\Delta"] \arrow[r, "\cong"'] & V \otimes k \end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Lemma 10.1.3

Every vector space V over a field k can be given the structure of a coalgebra where

- $\Delta : V \to V \otimes V$ is defined by $\Delta(v) = v \otimes v$
- $\varepsilon : V \to k$ is defined by $\varepsilon(v) = 1_k$

We would like to formally invert the definitions of algebra homomorphisms in order to define coalgebra homomorphisms.

10.2 Bialgebras

Definition 10.2.1: Bialgebras

Let V be a vector space over a field k. We say that V is a bialgebra if there is a collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map
- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that (V, m, u) is an algebra over k and (V, Δ, ε) is a coalgebra over k and that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V \otimes V \otimes V \otimes V \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & V \otimes V \otimes V \otimes V \arrow[u, "m\otimes m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & k \\
& V \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V \otimes V \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & V \\
& k \otimes k \cong k \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \otimes k \cong k \arrow[dl, "u\otimes u"'] \arrow[dr, "u"] & \\
V \otimes V & & V \arrow[ll, "\Delta"]
\end{tikzcd}$$

where $\tau : V \otimes V \to V \otimes V$ is the commutativity map defined by $\tau(x \otimes y) = y \otimes x$.

Theorem 10.2.2

Let V be a vector space over k. Suppose that (V, m, u) is an algebra and (V, Δ, ε) is a coalgebra. Then the following conditions are equivalent.

- $(V, m, u, \Delta, \varepsilon)$ is a bialgebra
- $m : V \otimes V \to V$ and $u : k \to V$ are coalgebra homomorphisms
- $\Delta : V \to V \otimes V$ and $\varepsilon : V \to k$ are algebra homomorphisms

11 Hopf Algebras

11.1 Hopf Algebras

Definition 11.1.1: Hopf Algebra

Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. We say that H is a Hopf algebra if there is a k-linear map $S : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

12 Differential Graded Algebra

12.1 Basic Definitions

Similar to how chain complexes and cochain complexes are two names of the same object, we can define differential graded algebra using either the chain complex notation or cochain complex notation. For our purposes, we will use the cochain version. This means that differentials will go up in index.

A differential graded algebra equips a graded algebra with a differential so that the algebra in the grading form a cochain complex.

Definition 12.1.1: Differential Graded Algebra

A differential graded algebra is a graded algebra $A_\bullet$ together with a map $d : A \to A$ that has degree 1 such that the following are true.

- $d \circ d = 0$
- For $a \in A_n$ and $b \in A_m$, we have $d(ab) = (da)b + (-1)^n a(db)$

Lemma 12.1.2

Let (A, d) be a differential graded algebra. Then (A, d) is also a cochain complex.

Recall that a graded commutative algebra A is a collection of algebra over some ring A_0, graded in $\mathbb{N}$ together with a multiplication $A_n \times A_m \to A_{m+n}$ such that

$$a \cdot b = (-1)^{nm} b \cdot a$$

Such a multiplication rule is said to be graded commutative.

Definition 12.1.3: Commutative Differential Graded Algebra

A differential graded algebra A is said to be a commutative differential graded algebra (CDGA) if A is also graded commutative.

We will often be concerned of differential graded algebra over a field $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In particular this means that the algebra has the structure of a vector space.

13 Introduction to Group Homology and Cohomology

13.1 G-Modules

Definition 13.1.1: G-Modules

Let G be a group. A G-module is an abelian group A together with a group action of G on A.

Definition 13.1.2: Morphisms of G-Modules

Let G be a group. Let M and N be G-modules. A function $f : M \to N$ is said to be a G-module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all $m \in M$ and $g \in G$.

13.2 Invariants and Coinvariants

Definition 13.2.1: The Group of Invariants

Let G be a group and let M be a G-module. Define the group of invariants of G in M to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of M for which G acts trivially.

Definition 13.2.2: Functor of Invariants

Let G be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$$

as follows.

- For each G-module M, M^G is the group of invariants
- For each morphism $f : M \to N$ of G-modules, $f^G : M^G \to N^G$ is the restriction of f to M^G.

Theorem 13.2.3

Let G be a group. The functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

Definition 13.2.4: The Group of Coinvariants

Let G be a group and let M be a G-module. Define the group of coinvariants of G in M to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M \rangle}$$

This is the largest quotient of M for which G acts trivially.

13.3 Group Cohomology and its Equivalent Forms

Definition 13.3.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H_n(G; M) = (L_n(-)_G)(M)$$

the nth left derived functor of $(-)_G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 13.3.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H^n(G; M) \cong \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

Recall that there are two descriptions of Ext by considering it as a functor of the first or second variable. Since the above theorem exhibits an isomorphism that is natural in the second variable, let us consider Ext as the right derived functor of the functor $\text{Hom}_{\mathbb{Z}[G]}(-, M)$ applied to $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module.

Proposition 13.3.3

Let G be a group and let M be a G-module. Let $P_\bullet \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules. Then there is an isomorphism

$$H^n(G; M) \cong H^n(\text{Hom}_{\mathbb{Z}[G]}(P_\bullet, M))$$

that is natural in M.

For any group G, there is always the trivial choice of projective resolution. In the following lemma, we use the notation $(g_0, \dots, \hat{g}_i, \dots, g_n)$ as a shorthand for writing the element in G^n but with the ith term omitted.

Lemma 13.3.4

Let G be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $f_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^{n} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of $\mathbb{Z}$ lying in ${}_{\mathbb{Z}[G]}\mathbf{Mod}$.

Let A be an R-algebra and let M be an A-module. Recall that the bar resolution is defined to be the chain complex consisting of $M \otimes A^{\otimes n}$ for each $n \in \mathbb{N}$ together with the boundary maps defined by multiplying the $i$the element to the $i+1$th element. Now let G be a group. By considering $\mathbb{Z}[G]$ as a $\mathbb{Z}$-algebra and that and ring is a module over itself, it makes sense to talk about the bar resolution of $\mathbb{Z}[G]$.

Theorem 13.3.5

Let G be a group. Consider the bar resolution

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \longrightarrow \mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}[G]$. Then it is a free resolution, and hence a projective resolution of $\mathbb{Z}$ with $\mathbb{Z}[G]$-modules.

Thus, given a group G and a G-module M, the group cohomology of G with coefficients in M can be thought of in the following way:

- It is the right derived functor of the functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$
- It is the extension group $\text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ (which is computable by the obvious projective resolution $\mathbb{Z}[G^\bullet]$, or the bar resolution)

13.4 Group Homology and its Equivalent Forms

Definition 13.4.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H^n(G; M) = (R^n(-)^G)(M)$$

the nth right derived functor of $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$.

Theorem 13.4.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_n(G; M) \cong \text{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in M.

13.5 Low Degree Interpretations

Theorem 13.5.1

Let G be a group and let M be a G-module. Then there are natural isomorphisms

$$H^0(G, M) = M^G \quad \text{and} \quad H_0(G; M) = M_G$$

Theorem 13.5.2

Let G be a group and let M be a G-module. Then there is an isomorphism

$$H_1(G, M) \cong \frac{G}{[G, G]} = G_{\text{ab}}$$

Theorem 13.5.3

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) = \frac{(\{f : G \to M \mid f(ab) = f(a) + af(b)\}, +)}{\langle f : G \to M \mid f(g) = gm - m \text{ for some fixed } m\rangle}$$

Corollary 13.5.4

Let G be a group and let M be a trivial G-module. Then there is a natural isomorphism

$$H^1(G; M) \cong \text{Hom}_{\mathbf{Grp}}(G, M)$$

14 Hochschild Homology

14.1 Hochschild Homology

Definition 14.1.1: Hochschild Complex

Let M be an R-module. Define the Hoschild complex to be the chain complex $C(R, M)$ given as follows.

$$\cdots \longrightarrow M \otimes R^{\otimes n+1} \xrightarrow{d} M \otimes R^{\otimes n} \xrightarrow{d} M \otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M \otimes R \longrightarrow M \longrightarrow 0$$

The map d is defined by $d = \sum_{i=0}^{n} (-1)^i d_i$ where $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$ is given by the following formula.

- If $i = 0$, then $d_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n$
- If $i = n$, then $d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}$
- Otherwise, then $d_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n-1}$

Definition 14.1.2: Hochschild Homology

Let M be an R-module. Define the Hochschild homology of M to be the homology groups of the Hochschild complex $C(R, M)$:

$$H_n(R, M) = \frac{\ker(d : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1})}{\operatorname{im}(d : M \otimes R^{\otimes n+1} \to M \otimes R^{\otimes n})} = H_n(C(R, M))$$

If $M = R$ then we simply write

$$HH_n(R) = H_n(R, R) = H_n(C(R, R))$$

TBA: Functoriality.

Proposition 14.1.3

Let A be an R-algebra. Then $HH_n(A)$ is a $Z(A)$-module.

Proposition 14.1.4

Let A be an R-algebra. Then the following are true regarding the 0th Hochschild homology.

- Let M be an A-module. Then $H_0(A, M) = \frac{M}{\{am - ma \mid a \in A, m \in M\}}$
- The 0th Hochschild homology of A is given by $HH_0(A) = \frac{A}{[A, A]}$
- If A is commutative, then the 0th Hochschild homology is given by $HH_0(A) = A$.

Theorem 14.1.5

Let A be a commutative R-algebra. Then there is a canonical isomorphism

$$HH_1(A) \cong \Omega^1_{A/R}$$

14.2 Bar Complex

Definition 14.2.1: Enveloping Algebra

Let A be an R-algebra. Define the enveloping algebra of A to be

$$A^e = A \otimes A^{\text{op}}$$

Proposition 14.2.2

Let A be an R-algebra. Then any A, A-bimodule M equal to a left (right) A^e-module.

Definition 14.2.3: Bar Complex

Proposition 14.2.4

Let A be an R-algebra. The bar complex of A is a resolution of the A viewed as an A^e-module.

Theorem 14.2.5

Let A be an R-algebra that is projective as an R-module. If M is an A-bimodule, then there is an isomorphism

$$H_n(A, M) = \text{Tor}_n^{A^e}(M, A)$$

14.3 Relative Hochschild Homology

14.4 The Trace Map

Definition 14.4.1: The Generalized Trace Map

Let R be a ring and let M be an R-module. Define the generalized trace map

$$\text{tr} : M_r(M) \otimes M_r(A)^{\oplus n} \to M \otimes A^{\otimes n}$$

by the formula

$$\text{tr}((m_{i,j}) \otimes (a_{i,j})_1 \otimes \cdots \otimes (a_{i,j})_n) = \sum_{0 \leq i_0, \dots, i_n \leq r} m_{i_0, i_1} \otimes (a_{i_1, i_2})_1 \otimes \cdots \otimes (a_{i_n, i_0})_n$$

Theorem 14.4.2

The trace map defines a morphism of chain complex

$$\text{tr} : C_\bullet(M_r(A), M_r(M)) \to C_\bullet(A, M)$$

14.5 Morita Equivalence and Morita Invariance

Definition 14.5.1

Let R and S be rings. We say that R and S are Morita equivalent if there is an equivalence of categories

$$\mathbf{Mod}_R \cong \mathbf{Mod}_S$$

Theorem 14.5.2: Morita Invariance for Matrices

15 Group Structures on Maps of Spaces

Req: AT3

H-spaces is a natural generalization of topological groups in the direction of homotopy theory.

Definition 15.0.1: H-Spaces

Let (X, x_0) be a pointed space. Let $\mu : (X, x_0) \times (X, x_0) \to (X, x_0)$ be a map. Let $e : (X, x_0) \to (X, x_0)$ be the constant map $x \mapsto x_0$. We say that (X, x_0, μ) is an H-space if the following diagram:

$$\begin{tikzcd} X \arrow[r, "{(e,\mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X,e)}"'] \arrow[dr, "\mathrm{id}_X"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map μ is called H-multiplication.

Definition 15.0.2: H-Associative Spaces

Let (X, x_0, μ) be an H-space. We say that (X, x_0, μ) is an H-associative space if the following diagram:

$$\begin{tikzcd} X \times X \times X \arrow[r, "\mu \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_X \times \mu"'] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 15.0.3: H-Group

Let (X, x_0, μ) be an H-space. Let $j : (X, x_0) \to (X, x_0)$ be a map. We say that (X, x_0, μ, j) is an H-group if the following diagram:

$$\begin{tikzcd} X \arrow[r, "{(j,\mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X,j)}"'] \arrow[dr, "e"] & X \times X \arrow[d, "\mu"] \\ X \times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map j is called H-inverse.

Example 15.0.4

Let X be a pointed space. Then the loopspace ΩX is an H-group.

Definition 15.0.5: H-Abelian

Let (X, x_0, μ, j) be an H-group. Let $T : (X, x_0) \times (X, x_0) \to (X, x_0)$ be the map $T(x, y) = T(y, x)$. We say that (X, x_0, μ, j) is an H-abelian if the following diagram:

$$\begin{tikzcd} X \times X \arrow[r, "T"] \arrow[dr, "\mu"'] & X \times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 15.0.6: Natural Group Structure

Let (X, x_0) be pointed spaces. We say that $[Z, X]_*$ has a natural group structure for all spaces (Z, z_0) if the following are true.

- $[Z, X]_*$ has a group structure such that the constant map $[e]$ is the identity of the group.
- For every map $f : A \to B$, the induced function

$$f^* : [B, X]_* \to [A, X]_*$$

  is a group homomorphism.