# Selected Topics

Labix

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Abstract

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## 1 Excisive Functors between Spaces

#### 1.1 Homotopy Pushouts and Homotopy Pullbacks

Why we want this: pushouts dont preserve homotopies, as with any limits / colimts (therefore we have homotopy limits / colimits in model category)

Definition 1.1.1: Standard Model for Homotopy Pushouts

Definition 1.1.2: Standard Model for Homotopy Pullbacks

Definition 1.1.3: Homotopy Pushouts

Definition 1.1.4: Homotopy Pullbacks

Example 1.1.5
Suspension and loopspace.

Proposition 1.1.6

Definition 1.1.7: Excisive Functors

- 1.2 The Failure of the Identity Functor to be Excisive
- 1.3 Excisive Functors Coming From Spectra

## 2 Spectra as Reduced and Excisive Functors

#### 2.1 Stable Infinity Categories

#### **Definition 2.1.1: Infinity Pushouts**

Let  $\mathcal C$  be an infinity category. Let  $F:\Delta^1\times\Delta^1\to\mathcal C$  be a morphism of simplicial sets. Let  $X\in\mathcal C$  be an object. We say that X is a pushout in  $\mathcal C$  if there exists a natural transformation  $u:\Delta X\Rightarrow F$  such that there is a homotopy equivalence of Kan complexes:

#### **Definition 2.1.2: Infinity Pullbacks**

Why are these the correct analogue?

#### **Definition 2.1.3: Stable Infinity Categories**

Example in mind: spectra in ordinary categories: pushout=pullback.

#### **Definition 2.1.4: Excisive Functors**

#### 2.2 Suspension and Loop Functors

Own notes: Higher algebra 1.4

trivial kan fibration -> section (Kerodon 1.5.5.5)

#### 2.3 Stable Infinity Categories

Recall that  $S = N_{\bullet}^{hc}(\mathbf{Top}_{*})$  is the infinity category of spaces.

#### Proposition 2.3.1

Let  $\mathcal{C}$  be a pointed infinity category that admits all finite colimits. Then  $\operatorname{Exc}_*(\mathcal{C},\mathcal{S})$  is stable.

*Proof.* Let  $F: \mathcal{C} \to \mathcal{S}$  be excisive and reduced. Then  $\Sigma_{\operatorname{Exc}_*(\mathcal{C},\mathcal{S})}(F) = F \circ \Sigma_{\mathcal{C}}$ . By definition of the suspension functor,

$$\begin{array}{ccc}
X & \longrightarrow * \\
\downarrow & & \downarrow \\
* & \longrightarrow \Sigma_{\mathcal{C}}(X)
\end{array}$$

is a pushout in C. Since F is excisive,

$$\begin{array}{ccc}
F(X) & \longrightarrow & * \\
\downarrow & & \downarrow \\
* & \longrightarrow & (F \circ \Sigma_{\mathcal{C}})(X)
\end{array}$$

is a pullback in S. On the other hand,  $\Omega_{\operatorname{Exc}_*(\mathcal{C},S)}(F) = \Omega_S \circ F$ . By definition of the loop functor,

$$(\Omega_{\mathcal{S}} \circ F \circ \Sigma_{\mathcal{C}})(X) \xrightarrow{\qquad \qquad *} \downarrow \qquad \qquad \downarrow \downarrow \\ * \xrightarrow{\qquad \qquad } (F \circ \Sigma_{\mathcal{C}})(X)$$

is a pullback in  $\mathcal S$  for any  $X\in\mathcal C$ . Therefore F(X) and  $(\Omega_{\mathcal S}\circ F\circ \Sigma_{\mathcal C})(X)$  are equivalent. Hence F and  $\Omega_{\operatorname{Exc}_*(\mathcal C,\mathcal S)}(\Sigma_{\operatorname{Exc}_*(\mathcal C,\mathcal S)}(F))$  are equivalent.

#### Theorem 2.3.2

There is an equivalence of infinity categories

$$\operatorname{Sp}(\mathcal{S}) \simeq \lim(\cdots \to \mathcal{S} \xrightarrow{\Omega} \mathcal{S} \xrightarrow{\Omega} \mathcal{S}) =: \overline{\mathcal{S}}$$

induced by the evaluation map  $ev_{S^0}: \overline{S} \to S$ .

#### Proof.

Since  $\mathcal S$  is presentable and the infinity category of presentable infinity categories admit all small limits,  $\overline{\mathcal S}$  is also presentable. Every presentable infinity category admits all small limits and colimits. Since  $\mathcal S$  is pointed,  $\overline{\mathcal S}$  is also pointed. Since all limits are computed term-wise, we have that in particular  $\Omega_{\overline{\mathcal S}}$  is computed term wise. given  $\{X_n \mid n \in \mathbb N\}$  an object of  $\overline{\mathcal S}$ ,  $\{\Omega X_n \mid n \in \mathbb N\}$  is equivalent to  $\{X_n \mid n \in \mathbb N\}$  because we have that  $\Omega X_{n+1}$  is equivalent to  $X_n$  for all n. By a prp we conclude that  $\overline{\mathcal S}$  is stable.

Consider the canonical functor  $G : \overline{S} \to S$  defined by recovering the first factor:  $(X_0, X_1, \dots) \mapsto X_0$ . It is clear that it commutes with finite limits since limits are computed term-wise.

Let  $\mathcal C$  be an arbitrary stable infinity category. Any functor  $\mathcal C \to \mathcal S$  is left exact if and only if it is exact so that  $\operatorname{Exc}_*(\mathcal C,\mathcal S) = \operatorname{Exc}_*^L(\mathcal C,\mathcal S)$ . 1.4.2.16 implies that  $\operatorname{Exc}_*^L(\mathcal C,\mathcal S)$  is a stable infinity category. Thus  $\Omega_{\mathcal S} \circ -$  is an equivalence.

On the other hand, since  $\Omega$  are computed term-wise (like all limits) and since  $\operatorname{Func}(\mathcal{C},\overline{\mathcal{S}})$  is right adjoint to products we know that Func commutes with finite limits . Thus we have that

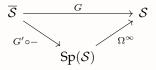
$$Exc_*^L(\mathcal{C},\overline{\mathcal{S}}) = \lim(\cdots \to Exc_*^L(\mathcal{C},\mathcal{S}) \overset{\Omega \circ -}{\to} Exc_*^L(\mathcal{C},\mathcal{S}) \overset{\Omega \circ -}{\to} Exc_*^L(\mathcal{C},\mathcal{S}))$$

Since each  $\Omega_{\overline{\mathcal{S}}} \circ -$  is an equivalence of infinity categories, we conclude that  $\operatorname{Exc}^L_*(\mathcal{C}, \overline{\mathcal{S}}) \simeq \operatorname{Exc}^L_*(\mathcal{C}, \mathcal{S})$ . Thus evaluation on the first factor  $G \circ - : \operatorname{Exc}^L_*(\mathcal{C}, \overline{\mathcal{S}}) \to \operatorname{Exc}^L_*(\mathcal{C}, \mathcal{S})$  is an equivalence of infinity categories.

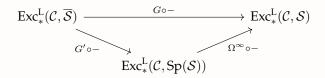
By a previous corollary, there is an equivalence of infinity categories given by

$$\Omega^{\infty} \circ - : Exc_{*}^{L}(\overline{\mathcal{S}}, Sp(\mathcal{S})) \to Exc_{*}^{L}(\overline{\mathcal{S}}, \mathcal{S})$$

The fact that G is left exact means that there is a factorization



By functoriality we obtain a similar factorization:



Since  $G \circ -$  and  $\Omega^{\infty} \circ -$  are both equivalence of infinity categories, we conclude that  $G' \circ -$  is an equivalence of infinity categories.

Since this is true for all stable infinity categories, the fact that

$$\operatorname{Exc}_*(\mathcal{C}, \overline{\mathcal{S}}) = \operatorname{Exc}_*^L(\mathcal{C}, \overline{\mathcal{S}}) \simeq \operatorname{Exc}_*^L(\mathcal{C}, \operatorname{Sp}(\mathcal{S})) = \operatorname{Exc}_*(\mathcal{C}, \operatorname{Sp}(\mathcal{S}))$$

is an equivalence for all stable  $\mathcal C$  together with the Yoneda embedding implies that  $\overline{\mathcal S}$  and  $\operatorname{Sp}(\mathcal S)$  is an equivalence of infinity categories.

Beware that in the proof we also showed that  $G \circ -$  is an equivalence of infinity categories for any stable infinity category  $\mathcal{C}$ . But this does not imply that  $\overline{\mathcal{S}}$  and  $\mathcal{S}$  are equivalent because we are applying the Yoneda embedding on the category of stable infinity categories, and a priori  $\mathcal{S}$  is not stable.

#### **3 From Functors to Excisive Functors**

#### 3.1 Goodwillie Calculus

**Definition 3.1.1** 

T1 and P1

Theorem 3.1.2

P1 is excisive.

#### 3.2 Excisive Approximations

Example 3.2.1

Id -> Infinite loop suspension

#### 3.3 Spectra and (Co)Homology Theories

Theorem 3.3.1: Brown's Representability Theorem

Definition 3.3.2: Cohomology Theory Associated to Spectra

Definition 3.3.3: Spectra Associated to Cohomology Theory

**Example 3.3.4: Singular Cohomology** 

Example 3.3.5: K theory

Example 3.3.6: Landweber-exact Spectra

Theorem 3.3.7: Landweber exact functor theorem

### 3.4 A Map From Functors to (Co)Homology Theories

Example 3.4.1

Identity Functor -> stable homotopy theory (it is a homology theory)

Example 3.4.2

Excisive functor  $F \to F(Sn) \to corresponding cohomolog theory$