Simplicial Methods in Topology

Labix

November 18, 2024

Abstract

References:

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1 The Category of Simplicial Sets

1.1 The Simplex Category

Recall the simplex category in Category Theory 1.

Definition 1.1.1: Simplex Category

The simplex category Δ consists of the following data.
- The objects are $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}$.
- The morphisms are the non-strictly order preserving functions. This means that a morphism $f : [n] \to [m]$ must satisfy $f(i) \leq f(j)$ for all $i \leq j$.
- Composition is the usual composition of functions.

Definition 1.1.2: Maps in the Simplex Category

Consider the simplex category Δ. Define the face maps and the degeneracy maps as follows.
- A face map in Δ is the unique morphism $d^i : [n-1] \to [n]$ that is injective and whose image does not contain i. Explicitly, we have

$$d^i(k) = \begin{cases} k & \text{if } 0 \leq k < i \\ k+1 & \text{if } i \leq k \leq n-1 \end{cases}$$

- A degeneracy map in Δ is the unique morphism $s^i : [n+1] \to [n]$ that is surjective and hits i twice. Explicitly, we have

$$s^i(k) = \begin{cases} k & \text{if } 0 \leq k \leq i \\ k-1 & \text{if } i+1 \leq k \leq n+1 \end{cases}$$

Proposition 1.1.3

The face maps and the degeneracy maps in the simplex category Δ satisfy the following simplicial identities:
- $d^j \circ d^i = d^i \circ d^{j-1}$ if $i < j$
- $s^j \circ s^i = s^i \circ s^{j+1}$ if $i \leq j$
- $s^j \circ d^i = \begin{cases} \text{id} & \text{if } i = j \text{ or } j+1 \\ d^i \circ s^{j-1} & \text{if } i < j \\ d^{i-1} \circ s^j & \text{if } i > j+1 \end{cases}$

Proof.
- Consider the object $[n-1]$. On the left, we have that

$$\begin{aligned} d^j(d^i[n-1]) &= d^j([0, \dots, i-1, i+1, \dots, n]) \\ &= [0, \dots, i-1, i+1, \dots, j-1, j+1, \dots, n+1] \end{aligned}$$

On the right, we have that

$$\begin{aligned} d^i(d^{j-1}[n-1]) &= d^i([0, \dots, j-2, j, \dots, n]) \\ &= [0, \dots, i-1, i+1, \dots, j-1, j+1, \dots, n+1] \end{aligned}$$

and so the relation is indeed true.

□

Proposition 1.1.4

Every morphism in the simplex category Δ is a composition of the face maps and the degeneracy maps.

1.2 Simplicial Sets

Definition 1.2.1: Simplicial Sets

A simplicial set is a presheaf
$$S:\Delta^{\text{op}}\to\textbf{Set}$$
Define the n-simplicies of S to be the set
$$S_n=S([n])$$

Example 1.2.2

Let X be a topological space. The set of singular simplices of X is given by the presheaf
$$S:\Delta\to\textbf{Set}$$
defined by $[n]\mapsto\text{Hom}_{\textbf{Top}}(|\Delta^n|,X)$. In other words, an n-simplex of X is simply a continuous function $\sigma:\Delta^n\to X$. This is exactly the same as how we defined singular n-simplexes in Algebraic Topology 2.

Definition 1.2.3: Category of Simplicial Sets

The category of simplicial sets sSet is defined as follows.

- The objects are simplicial sets $S:\Delta\to\text{Sets}$
- The morphisms are just morphisms of presheaves. This means that if $S,T:\Delta\to\text{Sets}$ are simplicial sets, then a morphism $\lambda:S\to T$ consists of morphisms $\lambda_n:S([n])\to T([n])$ for $n\in\mathbb{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
S([n]) \arrow[r, "S(f)"] \arrow[d, "\lambda_n"'] & S([m]) \arrow[d, "\lambda_m"] \\
T([n]) \arrow[r, "T(f)"'] & T([m])
\end{tikzcd}$$

- Composition is defined as the usual composition of functors.

The Yoneda lemma in this context implies that there is a bijection
$$\text{Hom}_{\text{sSet}}(\text{Hom}_\Delta(-,[n]),S)\cong S([n])=S_n$$
that is natural in the variable $[n]$.

Definition 1.2.4: Standard n-Simplicies

Let $n\in\mathbb{N}$. Define the standard n-simplex to be the simplicial set
$$\Delta^n=\text{Hom}_\Delta(-,[n]):\Delta\to\textbf{Set}$$

Notice that Δ^n is a simplicial set
$$\Delta^n:\Delta\to\text{Set}$$
defined by $[m]\mapsto\text{Hom}_\Delta([m],[n])$. Notice that if $m>n$, then it is impossible to have an order preserving function $[m]\to[n]$. Hence when $m>n$, $\text{Hom}_\Delta([m],[n])$ is empty.

All such simplicial sets Δ^n are useful in determining the contents of an arbitrary simplicial set. As for any presheaf, instead of focusing between the passage of data from Δ to Set, we should instead think of what kind of structure the presheaf brings to Set. Let C be a simplicial set. Then this means the following. For each n, there is a set $C_n = \text{Hom}_{\text{sSet}}(\Delta^n, C)$. For each morphism in Δ, there is a corresponding morphism in Set, which we shall discuss now.

Proposition 1.2.5

Let $S : \Delta \to \text{Set}$ be a simplicial set. Then every morphism in $S(\Delta)$ is the composite of two kinds of maps:

- The face maps: $d_i : S_n \to S_{n-1}$ for $0 \leq i \leq n$ defined by
$$d_i = S(d^i : [n-1] \to [n])$$
- The degeneracy maps: $s_i : S_n \to S_{n+1}$ for $0 \leq i \leq n$ defined by
$$s_i = S(s^i : [n+1] \to [n])$$

Moreover, these maps satisfy the following simplicial identities:

- $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$
- $s_i \circ s_j = s_{j+1} \circ s_i$ if $i \leq j$
- $d_i \circ s_j = \begin{cases} \text{id} & \text{if } i = j \text{ or } j+1 \\ s_{j-1} \circ d_i & \text{if } i < j \\ s_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$

Proof. Results are immediate using prp 1.1.3 and the fact that S is contravariant. □

We can now explicitly determine a simplicial set using the above proposition. We can alternatively define a simplicial set to be the following data.

- For each $n \in \mathbb{N}$, a set S_n.
- For $n \in \mathbb{N}$ and $0 \leq i \leq n$, a face map
$$d_i^n : S_n \to S_{n-1}$$
determining the ith face of the n-simplex
- For $n \in \mathbb{N}$ and $0 \leq i \leq n$, a degeneracy map
$$s_i^n : S_n \to S_{n+1}$$

The above data is sufficient to determine a unique simplicial set.

Proposition 1.2.6

The category **sSet** is a symmetric monoidal category with level-wise cartesian product.

Recall the notion of a Δ-set from Algebraic Topology 2 and one might realize they look suspiciously similar to that of a simplicial set. Let us recall. A Δ-set is a collection of sets S_n for $n \in \mathbb{N}$ together with maps $d_i^n : S_n \to S_{n-1}$ for $0 \leq i \leq n$ such that
$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$$
for $i < j$. One can easily convince themselves that every simplicial set is a Δ-set. Indeed, a simplicial set satisfies five more relations than a Δ-set. Therefore we have that
$$\textbf{sSet} \subset \Delta \text{ Complexes}$$

Proposition 1.2.7

Every simplicial set is a Δ-set.

Combining with the previously learnt combinatorial objects in algebraic topology, we now have the following tower:

$$\text{Simplicial Complexes} \subset \textbf{sSet} \subset \Delta \text{ Complexes} \subset \textbf{CW}$$

Definition 1.2.8: Faces of a Simplex

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n.

- Denote $\partial_i \Delta^n \subset \Delta^n$ the simplicial subset generated by the ith face

$$d_i(\text{id} : [n] \to [n]) = d^i : [n-1] \to [n]$$

- Denote $\partial \Delta^n$ the simplicial subset generated by the faces $\partial_i \Delta^n$ for $0 \leq i \leq n$. Define $\partial \Delta^0 = \emptyset$.

Definition 1.2.9: Degenerate Simplices

Let $S : \Delta \to \textbf{Set}$ be a simplicial set. Let $x \in S_n$ be an n-simplex. We say that x is degenerate if if $x \in \text{im}(s_k)$ for some degenerate map s_k.

Intuitively, degenerate simplices refer to simplicies that consecutively have the same face. Indeed, if $x = s_k(y)$ for some k, then the kth and $k+1$th position of the simplex x is given by the same element. Collapsing the two vertices should return the simplex y.

1.3 Geometric Realization of Simplicial Sets

Definition 1.3.1: Geometric Realization of Standard n-Simplexes

Let $n \in \mathbb{N}$. Consider the standard n-simplex Δ^n. Define the geometric realization of Δ^n to be

$$|\Delta^n| = \left\{ \sum_{k=0}^n t_k v_k \,\middle|\, \sum_{k=0}^n t_k = 1 \text{ and } t_k \geq 0 \text{ for all } k = 0, \dots, n \right\}$$

This definition is exactly the same as the definition of an n-simplex in Algebraic Topology 2. Now we proceed to the general case.

Definition 1.3.2: Geometric Realization of Simplicial Sets

Let C be a simplicial set. Define the geometric realization of C to be

$$|C| = \left(\coprod_{n \geq 0} C_n \times |\Delta^n| \right) / \sim$$

where the equivalence relation is generated by the following.

- The ith face of $\{x\} \times |\Delta^n|$ is identified with $\{d_i x\} \times \left|\Delta^{n-1}\right|$ by the linear homeomorphism preserving the order of the vertices.
- $\{s_i x\} \times |\Delta^n|$ is collapsed onto $\{x\} \times \left|\Delta^{n-1}\right|$ via the linear projection parallel to the line connecting the ith and the $(i+1)$st vertiex.

This construction of geometric realization is moreover functorial. Once again, we first define a map of geometric realization of simplicial sets.

Definition 1.3.3: Induced Map of Geometric Realization of Standard Simplicial Sets

Let $f : \Delta^n \to \Delta^m$ be a map of standard simplexes. Define $f_* : |\Delta^n| \to |\Delta^m|$ by

$$(t_0, \dots, t_n) \mapsto (s_0, \dots, s_m)$$

where

$$s_i = \begin{cases} 0 & \text{if } f^{-1}(i) = 0 \\ \sum_{j \in f^{-1}(i)} t_j & \text{otherwise} \end{cases}$$

Theorem 1.3.4

The geometric realization of a simplicial set is functorial $|\cdot| : \text{sSet} \to \text{Top}$ in the following way.

- On objects, it sends any simplicial set C to its geometric realization $|C|$.
- On morphisms, it sends any morphism $C \to D$ of simplicial sets to a continuous map defined by

We thus have that

$$\substack{\text{Geometric Relizations} \\ \text{of simplicial sets}} \subset \substack{\text{Geometric Relizations} \\ \text{of } \Delta\text{-sets}} \subset \text{CW-Complexes}$$

1.4 Simplicial and Semisimplicial Objects

Definition 1.4.1: Simplicial Objects

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $S : \Delta^{\text{op}} \to \mathcal{C}$.

Hence a simplicial object in **Set** is just simplical sets.

Definition 1.4.2: Category of Simplicial Objects

Let $\mathcal{C}$ be a category. Define the category of simplicial objects $s\mathcal{C}$ of $\mathcal{C}$ as follows.

- The objects are simplicial objects $S : \Delta^{\text{op}} \to \mathcal{C}$ of $\mathcal{C}$ which are presheaves
- The morphism of simplcial objects are just morphisms of presheaves, which are natural transformations
- Composition is given by composition of natural transformations

Definition 1.4.3: The Semisimplex Category

The Semisimplex category **SS** is the subcategory of Δ consisting of strict order preserving functions.

Definition 1.4.4: Semisimplicial Objects

Let $\mathcal{C}$ be a category. A semisimplicial object in $\mathcal{C}$ is a presheaf

$$\mathbf{SS} \to \mathcal{C}$$

Lemma 1.4.5

Let S be a set. Then S is a semisimplicial set if and only if S is a Δ-set (in the sense of Algebraic Topology 2).

1.5 Completeness and Cocompleteness

Definition 1.5.1: Product of Simplicial Sets

Let S, T be simplicial sets. Define the product of S and T to be the simplicial set

$$S \times T : \Delta \to \textbf{Set}$$

by the formula

$$[n] \mapsto S_n \times T_n$$

Proposition 1.5.2

Let S, T be simplicial sets. Then the categorical product of S and T is precisely $S \times T$.

Lemma 1.5.3

Let S, T be simplicial sets. Then there is a canonical homeomorphism

$$|S \times T| \cong |S| \times |T|$$

given by geometric realization.

1.6 The Closed Monoidal Structure of sSet

Definition 1.6.1: Internal Hom of Simplicial Sets

Let S, T be simplicial sets. Define the simplicial set

$$[S, T] : \Delta \to \textbf{Set}$$

by the formula

$$[n] \mapsto \text{Hom}_{\textbf{sSet}}(\Delta^n \times S \to T)$$

By the Yoneda embedding this is well defined.

We can easily identify morphisms of simplicial sets as the vertices of the internal hom.

Definition 1.6.2: Evaluation Morphism

Let S, T be simplicial sets. Define the evaluation morphism

$$\text{Ev} : [S, T] \times S \to T$$

as follows. For $f \in [S, T]$ and n-simplex and $\sigma \in S$ an n-simplex, define an n-simplex in T by

$$\Delta^n \xrightarrow{\delta} \Delta^n \times \Delta^n \xrightarrow{\text{id} \times \sigma} \Delta^n \times S \xrightarrow{f} T$$

Theorem 1.6.3: The Closed Monoidal Adjunction

Let S, T, U be simpplicial sets. Then there is a natural isomorphism

$$\text{Hom}_{\textbf{sSet}}(S \times T, U) \cong \text{Hom}_{\textbf{sSet}}(S, [T, U])$$

induced by the post composition of the evaluation momorphism (in the opposite way). Moreover **sSet** is a closed monoidal category.

2 Simplicial Homological Algebra

2.1 Chain Complexes of Simplicial Objects

Definition 2.1.1: Associated Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a (semi)-simplicial object in $\mathcal{A}$. Define the associated chain complex of A to be

$$\cdots \longrightarrow C_{n+1}(A) \xrightarrow{\partial_{n+1}} C_n(A) \xrightarrow{\partial_n} C_{n-1}(A) \longrightarrow \cdots \longrightarrow C_0(A)$$

where $C_n(A) = A_n$ and the boundary operator given by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i^n : A_n \to A_{n-1}$$

TBA: Functoriality of associated chain complex

Definition 2.1.2: Simplicial Homology

Let R be a ring. Let X be a (semi)-simplicial set. Define the simplicial homology of X with coefficients in R to be the homology groups

$$H_n^{\Delta}(X; R) = H_n(C_\bullet(R[X]))$$

Notice that this definition coincides with that in Algebraic Topology 2. Recall that in AT2 we defined the simplicial homology of a Δ-set, but in $\mathbb{Z}$ coefficients.

2.2 The Singular Functor

Definition 2.2.1: Singular Functor

The singular functor $S : \text{Top} \to \text{sSet}$ is defined as follows.

- On objects, it sends a space X to the simplicial set $S(X) : \Delta \to \text{Set}$ called the singular set, defined by
$$S(X)[n] = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$$
- On morphisms, it sends a continuous map $f : X \to Y$ to the morphism of simplicial sets $\lambda : S(X) \to S(Y)$ defined as follows. For each $n \in \mathbb{N}$, $\lambda_n : S(X)[n] \to S(Y)[n]$ is defined by
$$(h : |\Delta^n| \to X) \mapsto (f \circ h : |\Delta^n| \to Y)$$
such that the following diagram commutes:
$$\begin{tikzcd} S(X)[n] \arrow[r, "S(X)(f)"] \arrow[d, "\lambda_n"'] & S(X)[m] \arrow[d, "\lambda_m"] \\ S(Y)[n] \arrow[r, "S(Y)(f)"'] & S(Y)[m] \end{tikzcd}$$

Notice that this is reminiscent of the definitions in Algebraic Topology 2. Indeed $S(X)[n]$ for each $n \in \mathbb{N}$ is in fact the basis of the abelian group $C_n(X)$. It represents all the possible ways that an n-simplex could fit into X. Then the passage

$$\textbf{Top} \xrightarrow{S} \textbf{sSet} \xrightarrow{H_\bullet^{\Delta}(-;R)} {}_R\textbf{Mod}$$

recovers the singular homology of a space X with coefficients in a ring R. This is formulated slightly differently in Algebraic Topology 2.

Theorem 2.2.2

The singular functor $S : \text{Top} \to \text{sSet}$ is right adjoint to the geometric realization functor $|\cdot| : \text{sSet} \to \text{Top}$. This means that there is a natural bijection

$$\text{Hom}_{\text{Top}}(|X|, Y) \cong \text{Hom}_{\text{sSet}}(X, S(Y))$$

for any space Y and any simplicial set X.

2.3 Normalized Chain Complexes

Definition 2.3.1: Normalized Chain Complexes

Let $\mathcal{A}$ be an abelian category or the category **Grp**. Let A be a simplicial object in $\mathcal{A}$. Define the normalized chain complex of A to be the chain complex:

$$\cdots \longrightarrow N_{k+1}(A) \xrightarrow{\partial_{k+1}} N_k(A) \xrightarrow{\partial_k} N_{k-1}(A) \longrightarrow \cdots \longrightarrow N_1(A)$$

where

$$N_k(A) = \bigcap_{i=1}^{k} \ker(d_i^k : A_k \to A_{k-1})$$

and the differential given by $\partial_k = d_0^K|_{N_k(A)}$. We denote the normalized chain complex by $(N_\bullet(G), \partial_\bullet)$

nLab: We may think of the elements of the complex in degree k as being k-dimensional disks in G all of whose boundary is captured by a single face.

Lemma 2.3.2

Let G be a simplicial group. Consider the normalized chain complex $(N_\bullet(G), \partial_\bullet)$. Then $\partial_n N_n(G)$ is a normal subgroup of $N - n - 1(G)$.

Because of this lemma, it now makes sense to take the homology group of the normalized chain complex even if we take a simplicial object in **Grp**.

Definition 2.3.3: Normalized Chain Complex Functor

Let $\mathcal{A}$ be an abelian category. Define the normalized chain complex functor N

Definition 2.3.4: Degenerate Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Define the degenerate chain complex $D_\bullet(A)$ of A to be the subcomplex of the associated chain complex $C_\bullet(A)$ defined by

$$D_n(A) = \langle s_i^n : A_n \to A_{n+1} \mid s_i \text{ is the degenerate maps}\rangle$$

Proposition 2.3.5

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Then there is a splitting

$$C_\bullet(A) \cong N_\bullet(A) \oplus D_\bullet(A)$$

in the abelian category of chain complexes of $\mathcal{A}$.

Theorem 2.3.6: Eilenberg-Maclane

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Then the inclusion

$$N_\bullet(A) \hookrightarrow C_\bullet(A)$$

is a natural chain homotopy equivalence. In other words, $D_\bullet(A)$ is null homotopic.

Theorem 2.3.7: The Dold-Kan Correspondence

Consider the abelian category $\mathbf{Ab}$ of abelian groups. The normalized chain complex functor

$$N : \text{s}\mathbf{Ab} \xrightarrow{\cong} \text{Ch}_{\geq 0}(\mathbf{Ab})$$

gives an equivalence of categories, with inverse as the simplicialization functor

$$\Gamma : \text{Ch}_{\geq 0}(\mathbf{Ab}) \to \text{s}\mathbf{Ab}$$

2.4 Bar Resolutions

Definition 2.4.1: Bar Construction

Let A be an algebra over a ring R. Let M be an A-algebra. Define the maps $d_i^n : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ by the following formulas:

- If $i = 0$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
- If $0 < i < n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n$$
- If $i = n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}$$

Highly related: Cotriple homology / cotriple constructions

Proposition 2.4.2

Let A be an algebra over a ring R. Let M be an A-algebra. Then the collection

$$\{M \otimes A^{\otimes n}, d_i^n \mid n \in \mathbb{N}\}$$

is a simplicial object.

Definition 2.4.3: Bar Resolutions

Let A be an algebra over a ring R. Let M be an A-algebra. Define the bar resolution of M to be the associated chain complex of the simplicial object

$$\{M \otimes A^{\otimes n}, d_i^n \mid n \in \mathbb{N}\}$$

Explicitly, it is the chain complex

$$\cdots \longrightarrow A^{\otimes n+1} \otimes M \longrightarrow A^{\otimes n} \otimes M \longrightarrow A^{\otimes n-1} \otimes M \longrightarrow \cdots \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0$$

with the boundary map $\partial : A^{\otimes n} \otimes M \to A^{\otimes n-1} \otimes M$ given by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i^n$$

3 Kan Complexes

3.1 Horns, Fillers and Kan Complexes

Definition 3.1.1: Inner and Outer Horns

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Define the ith horn Λ_i^n of Δ^n to be the the simplicial subset generated by all the faces $\partial_k \Delta^n$ except the ith one. It is called inner if $0 < i < n$. It is called outer otherwise.

Definition 3.1.2: Fillers for a Horn

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Let Λ_i^n be a horn. We say that Λ admits a filler if for all maps $F : \Lambda_i^n \to C$ there exists a map $U : \Delta^n \to C$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "F"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists U"'] & \end{tikzcd}$$

Definition 3.1.3: Kan Complexes

An infinity category is a simplicial set C such that each horn admits a filler. In other words, for all $0 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

3.2 Kan Fibrations

Definition 3.2.1: Kan Fibrations

Let $f : X \to Y$ be a morphism of simplicial sets. We say that f is a Kan fibration if the following condition is satisfied: For every commutative diagram:

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

where $n \geq 1$ and $0 \leq k \leq n$, there exists a lift $\Delta^n \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Lemma 3.2.2

Let X be a simplicial set. Then X is a Kan complex if and only if the unique map $X \to *$ is a Kan fibration.

Closed under retracts, pullbacks, filtered colimits, products with standard simplicies, composition

3.3 Anodyne Morphisms

They are the acyclic cofibrations of sSet with standard model structure.

3.4 Ex^∞ and the Subdivision Functor

3.5 Weak Factorization

Proposition 3.5.1

Let X, Y be simplicial sets. Let $f : X \to Y$ be a morphism. Then there exists a simplicial set $Q(f)$ and maps $f' : X \to Q(f)$ and $f'' : Q(f) \to Y$ such that f' is anodyne, f'' is a Kan fibration and the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "\exists f'"'] & & Y \\ & Q(f) \arrow[ur, dashed, "\exists f''"'] & \end{tikzcd}$$

Proposition 3.5.2

Let X, Y be simplicial sets. Let $f : X \to Y$ be a morphism. Then there exists a choice of $Q(f)$ such that Q defines a functor

$$Q : \text{Hom}_{\mathbf{Cat}}(\Delta^1, \mathbf{sSet}) \to \mathbf{sSet}$$

that commutes with filtered colimits.

3.6 (Co)Fibrant Replacements

Definition 3.6.1: Fibrant Replacements

Let X be a simplicial set. A fibrant replacement of X is a Kan complex QX such that the unique map $X \to *$ factorizes into $X \to QX \to *$ where $X \to QX$ is anodyne.

Secret: these are just the fibrant replacements in model category.

Lemma 3.6.2

Let X be a simplicial set. Then the following are all fibrant replacements of X.

- $Q_X = Q(X \to *)$
- $\text{Ex}^\infty(X)$

4 Simplicial Homotopy Theory

Algebraic topology is a new subject which received a name change from combinatorial topology. At that time, combinatorics was highly involved in deriving topological invariants because the combinatorial structure of any invariants or spaces of study makes computation easy. In particular, a great deal of work has been put into the homotopy theory of simplicial spaces.

Nowadays, the study of combinatorial objects in topology is less prominent, but the category of simplicial sets still play a distinguished role in algebraic topology. Indeed the category of simplicial sets is the prototypical example of a model category (we have not seen it), as well as exhibiting a Quillen equivalence between the model of a simplicial set and some model category structure of **Top**. We will not explore the concept of model categories but will instead display foundational knowledge of it in order to develop the category of simplicial sets as a workable category when studying Model Categories. This means that we will develop the notion of homotopy, fibrations and cofibrations in this category.

4.1 Homotopy of Simplicial Sets

Definition 4.1.1: Homotopy between Two Morphisms

Let X, Y be simplicial sets. Let $f, g : X \to Y$ be two morphisms. A simplicial homotopy from f to g is a morphism $\eta : X \times \Delta^1 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \simeq X \times \Delta[0] \arrow[r, "\mathrm{id}_X \times \delta_1"] \arrow[dr, "f"'] & X \times \Delta[1] \arrow[d, dashed, "\exists \eta"] & X \simeq X \times \Delta[0] \arrow[l, "\mathrm{id}_X \times \delta_1"'] \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

We say that f and g are homotopic either if there is a homotopy from f to g, or there is a homotopy from g to f.

Definition 4.1.2: Homotopy Relative to Subsets

Let X, Y be simplicial sets. Let $K \subseteq X$ be a simplicial subset. Let $\iota : K \hookrightarrow X$ denote the inclusion. Let $f, g : X \to Y$ be two morphisms. Let $\eta : X \times \Delta^1 \to Y$ be a homotopy from f to g. We say that ι is a homotopy relative to K if the following diagram commutes:

$$\begin{tikzcd}
K \times \Delta^1 \arrow[r, "\mathrm{proj}_K"] \arrow[d, hook, "\iota \times \mathrm{id}_{\Delta^1}"'] & K \arrow[d, "\alpha"] \\
X \times \Delta^1 \arrow[r, "\eta"'] & Y
\end{tikzcd}$$

where $\alpha = f|_K = g|_K$.

Proposition 4.1.3

Let X, Y be simplicial sets. Let $f, g : X \to Y$ be morphisms. Then there exists a homotopy from g to f if and only if there exists a family of morphisms $h_i^n : X_n \to Y_{n+1}$ such that the following are true.

- $d_0^n \circ h_0 = f_n$
- $d_0^{n+1} \circ h_1 = g_n$
- The composition

$$d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i^{n+1} & \text{if } 0 \leq i < j \leq n \\ d_i \circ h_{i-1} & \text{if } i = j \neq 0 \\ h_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$

- The composition
$$s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & \text{if } i \leq j \\ h_j \circ s_{i-1} & \text{if } i > j \end{cases}$$

The homotopy relation is in general not an equivalence relation. However, when Y is a Kan complex, this is true.

Proposition 4.1.4

Let X be a simplcial set. Let Y be a Kan complex. Then the relation of homotopic morphisms from X to Y is an equivalence relation.

Definition 4.1.5: Homotopy Inverses

Let X, Y be simplicial sets. Let $f : X \to Y$ be a morphism. A homotopy inverse of f is a morphism $g : Y \to X$ such that there exist a homotopy from $g \circ f$ to id_X and a homotopy from $f \circ g$ to id_Y.

Definition 4.1.6: Homotopy Equivalences

Let X, Y be simplicial sets. Let $f : X \to Y$ be a morphism. We say that f is a homotopy equivalence between X and Y if f admits a homotopy inverse.

Lemma 4.1.7

Let X, Y be simplicial sets. Let $f : X \to Y$ be a homotopy equivalence. Then the homotopy inverse of f is unique up to homotopy.

4.2 Homotopy between Vertices

We can also talk about homotopies between vertices of a simplicial set.

Definition 4.2.1: Homotopy between Vertices

Let X be a simplicial set. Let v_0, v_1 be vertices of X. A homotopy from v_0 to v_1 is a homotopy from the inclusion $\Delta^0 \cong \{v_0\} \hookrightarrow X$ to the inclusion $\Delta^0 \cong \{v_1\} \hookrightarrow X$.

Lemma 4.2.2

Let X be a simplicial set. Let v_0, v_1 be vertices of X. Then there exists a homotoyp from v_0 to v_1 if and only there exists and edge of X from v_0 to v_1.

We can generalize this to n-simplicies.

4.3 The Simplicial Homotopy Groups

Because homotopies from X to Y is an equivalence relation only when Y is a Kan complex, we can only define simplicial homotopy groups for Kan complexes. This notion can easily be extended to arbitrary simplicial sets by replacing the simplicial set with a fibrant replacement (and hence becomes a Kan complex).

Definition 4.3.1: Simplicial Homotopy Groups

Let X be a Kan complex. Let $v \in X$ be a vertex of X. Define the simplicial homotopy groups of (X, v) as follows:

- For $n \geq 1$, define the nth simplicial homotopy group $\pi_n^{\text{sSet}}(X, v)$ of X at v to be the set of homotopy classes of maps $[\alpha : \Delta^n \to X]$ relative to boundary $\partial\Delta^n$. In other words, the following diagram commutes:

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "!"] \arrow[d, hook] & \Delta^0 \arrow[d, "v"] \\ \Delta^n \arrow[r, "\alpha"'] & X \end{tikzcd}$$

- Define the 0th simplicial homotopy group $\pi_0^{\text{sSet}}(X)$ of X to be the set of homotopy classes of vertices of X.

If X is a general simplicial set, define the simplicial homotopy group of X to be

$$\pi_n^{\text{sSet}}(X, v) = \pi_n^{\text{sSet}}(PX, Pv)$$

where PX is a fibrant replacement of X.

Theorem 4.3.2

Let X be a Kan complex. Let $f, g : \Delta^n \to X$ be two representatives of two elements in $\pi_n(X, v)$ for $n \geq 1$. Then the following data

$$v_i = \begin{cases} s_0^n(v) & \text{if } 0 \leq i \leq n-2 \\ f & \text{if } i = n-1 \\ g & \text{if } i = n+1 \end{cases}$$

defines a horn $\Lambda_i^{n+1} \to X$. Such a map extends to a map $\theta : \Delta^{n+1} \to X$. Define

$$[f] \cdot [g] = d_n\theta$$

Then such an operation is well defined on the equivalence class. Moreover, it defines a group operation on $\pi_n^{\text{sSet}}(X, v)$.

Theorem 4.3.3

Let X be a Kan complex. Then for $n \geq 2$, the above group structure on $\pi_n^{\text{sSet}}(X, v)$ is abelian.

Theorem 4.3.4

Let X be a simplicial set. Let $v \in X_0$. Then there is an isomorphism

$$\pi_n^{\text{sSet}}(X, v) \cong \pi_n(|X|, v)$$

for all $n \in \mathbb{N}$.

5 Homotopy Limits and Colimits

5.1 The Standard Homotopy Pullback and Pushout

Definition 5.1.1: The Standard Homotopy Pullbacks

Let the following be a diagram of simplicial sets and morphisms.

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

Define the standard homotopy pullback to be the simplicial set

$$\text{Holim}_Q(X \to Z \leftarrow Y) = X \times_Z^h Y = X \times_{\text{Hom}_{\textbf{sSet}}(\{0\},Z)} \text{Hom}_{\textbf{sSet}}(\Delta^1, Z) \times_{\text{Hom}_{\textbf{sSet}}(\{1\},Z)} Y$$

Here we identify the vertices of Δ^1 as 0 and 1, and we use the isomorphism $\mathcal{C} \cong \text{Hom}_{\textbf{sSet}}(\Delta^0, Z)$.

Kerodon 3.4.0.3

Lemma 5.1.2

Let the following be a diagram of simplicial sets and morphisms, where Z is a Kan complex.

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

Then the following diagram commutes up to homotopy:

$$\begin{tikzcd} \text{Holim}_Q(X \to Z \leftarrow Y) \arrow[r, "\text{proj}"] \arrow[d, "\text{proj}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

5.2 Homotopy Pullback and Pushout Squares

5.3 Recognizing Homotopy Pullbacks and Pushouts

Proposition 5.3.1: Kerodon 3.4.0.7

Let the following be a diagram in simplicial sets.

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

If Z is a Kan complex and one of f or g is a Kan fibration, then the following are true.

- The following diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\text{pr}"] \arrow[d, "\text{pr}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

  is a homotopy pullback square.
- The inclusion map

$$X \times_Z Y \hookrightarrow X \times_Z^h Y$$

  is a weak homotopy equivalence.
- $X \times_Z Y$ is a homotopy pullback of the diagram.

Secret: if the diagram is fibrant in the diagram category, then the homotopy pullback is the same as the standard pullback up to weak equivalence.

6 Constructing a Category from Simplicial Sets

6.1 The Nerve of a Category

Definition 6.1.1: Nerve of a Category

Let $\mathcal{C}$ be a category. Define the nerve of the category $N(C) \in \mathbf{sSet}$ as follows.

- For $n \in \mathbb{N}$, $N(C)_n$ consists of paths of morphisms with n compositions:

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$

- The face map $d_i : C_n \to C_{n-1}$ sends the above element to

$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_i \xrightarrow{\mathrm{id}_{c_i}} c_i \longrightarrow \cdots \longrightarrow c_n$$

- The degeneracy map $s^i : C_n \to C_{n+1}$ sends the above element to

Definition 6.1.2: Nerve Functor

The nerve functor N : Cat $\to$ sSet is defined as follows.

- Each $\mathcal{C} \in$ Cat is sent to the nerve $N(C)$
- Every functor $\mathcal{C} \to \mathcal{D}$ in Cat is sent to the morphism of presheaves $\lambda : N(C) \to N(D)$ defined by $\lambda_n : N(C)([n]) \to N(D)([n])$, of which is defined as the map

$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$

$$F(c_0) \xrightarrow{F(f_1)} F(c_1) \xrightarrow{F(f_2)} F(c_2) \longrightarrow \cdots \longrightarrow F(c_n)$$

from the upper path in $\mathcal{C}$ to the lower path in $\mathcal{D}$, such that the following diagram commutes:

$$\begin{tikzcd} N(C)[n] \arrow[r, "N(C)(f)"] \arrow[d, "\lambda_n"'] & N(C)[m] \arrow[d, "\lambda_m"] \\ N(D)[n] \arrow[r, "N(D)(f)"'] & N(D)[m] \end{tikzcd}$$

where $f : [m] \to [n]$ is a morphism in Δ.

Theorem 6.1.3

The nerve functor N : Cat $\to$ sSet is fully faithful. Moreover, the nerve of a category is a complete invariant for categories.

6.2 The Homotopy Category of a Simplicial Set

Definition 6.2.1: Homotopy Category of Simplicial Sets

Let $X \in \mathbf{sSet}$. Define the homotopy category $h(X)$ and the universal map $u : X \to N(h(X))$ by the following universal property. For any category $\mathcal{D}$ and morphism of simplicial sets $f : X \to N(\mathcal{D})$, there exists a unique functor $G : h(X) \to \mathcal{D}$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[dr, "F"'] & N(h(X)) \arrow[d, dashed, "\exists ! N(G)"] \\ & N(\mathcal{D}) \end{tikzcd}$$

Note that the nerve functor N is fully faithful. In other words there is a bijection

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D}) \cong \mathrm{Hom}_{\mathbf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$$

This means that giving a morphism $h(X) \to \mathcal{D}$ is exactly the same as giving a morphism $N(h(X)) \to N(\mathcal{D})$.

Lemma 6.2.2

Let $X \in \mathbf{sSet}$. Then the homotopy category $h(X)$ exists and is unique up to unique isomorphism.

Definition 6.2.3: Induced Functors on Homotopy Category

Let $X, Y \in \mathbf{sSet}$. Let $F : X \to Y$ be a map of simplicial sets. Define the induced functor

$$h(F) : h(X) \to h(Y)$$

of F to be the unique morphism corresponding to the functor $u \circ F : X \to Y \to N(h(Y))$ under the universal property of $h(X)$.

Definition 6.2.4: The Homotopy Functor

Define the homotopy functor $h :$ sSet $\to$ Cat as follows.

- On objects, h sends a simplicial set $S : \Delta \to$ Set to $h(S)$.
- For $f : S \to T$ a morphism of simplicial sets, $h(f) : h(S) \to h(T)$ is the induced functor on the homotopy category defined above.

Theorem 6.2.5

The homotopy functor h is left adjoint

$$h : \mathbf{sSet} \rightleftarrows \mathbf{Cat} : N$$

to the nerve functor N. This means that there is a natural bijection

$$\mathrm{Hom}_{\mathrm{Cat}}(h(C), D) \cong \mathrm{Hom}_{\mathrm{sSet}}(C, N(D))$$

We have now constructed two pairs of adjoint functors:

$$\begin{tikzcd} & \mathbf{Cat} \arrow[dl, shift left, "N_\bullet"] \\ \mathbf{sSet} \arrow[ur, shift left, "h"] \arrow[dr, shift left, "|\cdot|"] & \\ & \mathbf{Top} \arrow[ul, shift left, "S_\bullet"] \end{tikzcd}$$