Selected Topics

Labix

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Abstract

References:

Contents

1 Smooth Algebras

1.1 Smooth Algebras

Definition 1.1.1: Smooth Algebra

Let A be a commutative algebra over a field k. We say that A is a smooth algebra over k if $\Omega^1_{A/k}$ is a projective A-module.

Definition 1.1.2: Formal Smoothness

Let A be a commutative algebra over a field k. We say that A is a formally smooth if for every k-algebra C together with a k-algebra homomorphism $u : A \to C/N$ where $N^2 = 0$, there exists a k-algebra map $v : A \to C$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, dashed, "\exists v"] \arrow[dr, "u"'] & C \arrow[d, "p"] \\ & \frac{C}{N} \end{tikzcd}$$

We say that A is etale over k if such a map $v : A \to C$ is unique.

2 Geometric Properties of Ring Homomorphisms

2.1 Finite and Quasi-Finite Morphisms

Definition 2.1.1: Finite Morphisms

Let $f : R \to S$ be a commutative ring homomorphism. We say that f is a finite morphism if S is a finitely generated R-module.

Note: An algebra A over a ring R is finite over R if A is finitely generated as an R-module.

Note: For R a ring and p a prime ideal, the residue field is defined as

$$k(p) = \frac{R_p}{pR_p}$$

Definition 2.1.2: Quasi-Finite Morphisms

Let $f : R \to S$ be a commutative ring homomorphism. We say that f is a quasi-finite morphism if for all prime ideals $q \subset S$ and $p = f^{-1}(q)$, the algebra

$$\frac{S_q}{pS_q}$$

is finitely generated as an $k(p) = \frac{R_p}{pR_p}$-module.

2.2 Unramified Morphisms

Recall that a local homomorphism consists of a homomorphism of local rings $f : R \to S$ such that $m_R S \subseteq m_S$

Definition 2.2.1

Let $(R, m_R, k(R))$ and $(S, m_S, k(S))$ be local rings. Let $f : R \to S$ be a local homomorphism of local rings. We say that f is unramified if the following are true.

- $m_R S = m_S$
- The field extension $k(R) \to k(S)$ is finite and separable

Lemma 2.2.2

Every unramified morphism is quasi-finite.

Note: Zariski's main theorem

Proposition 2.2.3

Let $f : A \to B$ be a unramified local ring homomorphism such that B is the localization of a finitely generated A-algebra. Then B is the localization of an A-algebra B_0 that is a finitely generated A-module.

2.3 Etale Morphisms

Definition 2.3.1

Let $f : R \to S$ be a homomorphism of commutative rings. We say that f is etale if the following are true.

- S is a finitely generated R-algebra
- S is a flat R-module
- f is unramified.

In this case, we say that S is etale over R, or we say that S is an etale R-algebra.

Corollary 2.3.2

Let $f : R \to S$ be an etale homomorphism. Then B is the localization of an A-algebra B_0 that is a finitely generated A-module.

Proof. We have already proved this for every every unramified ring homomorphism and every etale homomorphism is also unramified. □

Proposition 2.3.3

Let A be a k-algebra for a field k. Then $k \to A$ is an etale morphism if and only if A is isomorphic to the product of finitely many finite separable field extensions of k.

Proposition 2.3.4

Let $f : A \to B$ be a finitely generated flat algebra. Then f is etale if and only if for every prime ideal $p \subset A$, the algebra

$$B \otimes_A k(p)$$

is etale over $k(p)$.

Note: We say that a local homomorphism $f : A \to B$ is finitely generated if B is the localization of a finitely generated A-algebra.

3 Derived Categories in Algebraic Geometry

3.1 Derived Categories of Schemes

Definition 3.1.1: Derived Category of a Scheme

Let X be a scheme. Define the derived category to be

$$D^*(X) = D^*(\mathbf{Coh}(X))$$

where $*$ can be $b, +, -$ or null.

Definition 3.1.2: Derived Equivalences

Let X, Y be a schemes over a field k. We say that X and Y are derived equivalent if there exists a k-linear exact equivalence

$$D^b(X) \cong D^n(Y)$$

Let X be a scheme. Then there is a natural inclusion of categories

$$\mathbf{Coh}(X) \subset \mathbf{QCoh}(X) \subset \mathbf{Mod}_{O_X}$$

Proposition 3.1.3

Let X be a noetherian scheme. Suppose that $* = b, +, -$. Then there are natural equivalences

$$D^*(\mathbf{QCoh}(X)) \cong D^*_{\mathbf{QCoh}(X)}(\mathbf{Mod}_{O_X})$$

Proposition 3.1.4

Let X be a noetherian scheme. Then the inclusion functor induces

$$D^b(X) \to D^b(\mathbf{QCoh}(X))$$

which defines an equivalence of categories between $D^b(X)$ and $D^b_{\mathbf{Coh}(X)}(\mathbf{QCoh}(X))$.

4 Intersection Theory

Scheme = scheme + morphism of finite type to Spec(k) Variety = Irreducible variety subvariety = closed subscheme of a variety which is a variety point = closed point

4.1 The Order of Zeroes and Poles

Recall that when we defined the notion of Weil divisors, we restricted ourselves to the case where X is a Noetherian separated scheme that is regular in codimension 1. This is because we wanted to make use of the fact that $\mathcal{O}_{X,\eta}$ is a discrete valuation ring for η a generic point of any irreducible subscheme of codimension 1 of X. In general when X is Noetherian, $\mathcal{O}_{X,\eta}$ is only a local Noetherian ring. In order to define a notion of order for every element in $K(X) = \mathcal{O}_{X,\eta}$, we need a new definition.

Definition 4.1.1: Order of Functions in the Function Field

Let X be an irreducible variety. Let Y be a subvariety of codimension 1 and let η be its generic point. For every $r \in \mathcal{O}_{X,\eta}$, define

$$\text{ord}_Y(r) = \text{length}_{\mathcal{O}_{X,\eta}}\left(\frac{\mathcal{O}_{X,\eta}}{(r)}\right)$$

For every $r = ab^{-1} \in \text{Frac}(\mathcal{O}_{X,\eta}) = K(X)$, define

$$\text{ord}_Y(r) = \text{ord}_Y(a) - \text{ord}_Y(b)$$

Beware that $\mathcal{O}_{X,\eta}$ is not the function field of X. This is because η is the generic point of Y, not the generic point of X. And generic points do not coincide for subvarieties and varieties.

Lemma 4.1.2

Let X be an irreducible variety. Let Y be a subvariety of codimension 1 and let η be its generic point. Then the function $\text{ord}_Y : K(X)^* \to \mathbb{Z}$ is a group homomorphism.

Definition 4.1.3: k-Cycles on a Variety

Let X be an irreducible variety. Define the group of k-cycles to be free abelian group

$$Z_k(X) = \mathbb{Z}\langle V \mid V \text{ is a } k\text{-dimensional subvariety of } X\rangle$$

generated by k-dimensional subvarieties of X.

Definition 4.1.4: Divisors of a Function

Let X be an irreducible variety. Let W be a $(k+1)$-dimensional subvariety of X. For any $r \in K(X)$, define

$$\text{div}(r) = \sum_{\text{codim}_W(V)=1} \text{ord}_V(r) \cdot [V]$$

where $\text{ord}_V : K(W)^* \to \mathbb{Z}$.

Definition 4.1.5: Rational Equivalence

Let X be an irreducible variety. Let α and β be k-cycles. We say that α and β are rationally equivalent, denoted by $\alpha \sim_{\text{rat}} \beta$ if there exists a finite number of $(k+1)$-dimensional subvarieties W_i of X and $r_i \in K(W_i)^*$ such that

$$\alpha - \beta = \sum_i [\text{div}(r_i)] \cdot W_i$$

Theorem 4.1.6

Let X be an irreducible variety. Let α and β be k-cycles. Then α and β are rationally equivalent if and only if there are $(k+1)$-dimensional subvarieties $V_1, \dots, V_t$ of the Cartesian product $X \times \mathbb{P}^1$ such that the projections $f_i : V_i \to \mathbb{P}^1$ are dominant and

$$\alpha - \beta = \sum_{k=1}^{t} [V_i(0)] - [V_i(\infty)]$$

in $Z_k(X)$. Here, $V_i(P)$ is the subscheme of X where the projection $X \times \{P\} \to X$ maps the subscheme $f^{-1}(P)$ isomorphically to, for P a point in $\mathbb{P}^1$.

Definition 4.1.7: The Chow Group

Let X be an irreducible variety. Define

$$\text{CH}_k(X) = \frac{Z_k(X)}{\sim_{\text{rat}}}$$

for each $k \in \mathbb{N}$. Also define

$$Z_*(X) = \bigoplus_{k=0}^{\dim(X)} Z_k(X) \quad \text{and} \quad \text{CH}_*(X) = \bigoplus_{k=0}^{\dim(X)} A_k(X)$$

4.2 The Induced Map

Definition 4.2.1: Degree of Subvariety

Let X, Y be irreducible varieties. Let $f : X \to Y$ be a proper morphism. Let V be a subvariety of X and let $W = f(V)$ be the corresponding subvariety of Y. Define the degree of V over W by

$$\deg(V/W) = \begin{cases} [k(V) : k(W)] & \text{if } \dim(W) = \dim(V) \\ 0 & \text{if } \dim(W) < \dim(V) \end{cases}$$

Notice that this definition makes sense. Since $f : X \to Y$ is proper, $W = f(V)$ becomes a subvariety of Y. Such a map induces a map of fields $k(W) \to k(V)$ which is necessarily injective. From field theory we know that such a map is injective and in particular $k(V)$ is a vector space over $k(W)$.

Definition 4.2.2: The Pushfoward Map

Let X, Y be irreducible varieties. Let $f : X \to Y$ be a proper morphism. Define the pushforward of f by $f_* : Z_k(X) \to Z_k(Y)$ where

$$f_*([V]) = \deg(V/W)[W]$$

for V a closed subvariety of X and $W = f(V)$.

Lemma 4.2.3: Functorial Properties of the Pushforward Map

Let X, Y, Z be irreducible varieties. Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms. Then the following are true.

- $(g \circ f)_* = g_* \circ f_*$
- $(\text{id}_X)_* = \text{id}_{Z_k(X)}$

TBA: $Z_* : \text{IrrVar}_k \to \mathbf{GrAb}$ is a covariant functor.

Theorem 4.2.4

Let X, Y be irreducible varieties. Let $f : X \to Y$ be a proper morphism. Let α be a k-cycle on X that is rationally equivalent to 0. Then $f_*(\alpha)$ is also rationally equivalent to 0.

TBA: $A_* : \text{IrrVar}_k \to \mathbf{GrAb}$ is a covariant functor.

5 Symmetric Polynomials

5.1 Symmetric Polynomials

The theory of symmetric functions are important in combinatorics, representation theory, Galois theory and the theory of λ-rings.

Requirements: Groups and Rings
Books: Donald Yau: Lambda Rings

Definition 5.1.1: Symmetric Group Action on Polynomial Rings

Let R be a ring. Define a group action of S_n on $R[x_1, \dots, x_n]$ by

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

It is easy to check that this defines a group action.

Definition 5.1.2: Symmetric Polynomials

Let R be a ring. We say that a polynomial $f \in R[x_1, \dots, x_n]$ is symmetric if

$$\sigma \cdot f = f$$

for all $\sigma \in S_n$.

Definition 5.1.3: The Ring of Symmetric Polynomials

Let R be a ring. Define the ring of symmetric polynomials in n variables over R to be the set

$$\Sigma = \{ f \in R[x_1, \dots, x_n] \mid \sigma \text{ is a symmetric polynomial } \}$$

Definition 5.1.4: Elementary Symmetric Polynomials

Let R be a ring. Define the elementary symmetric polynomials to be the elements $s_1, \dots, s_n \in R[x_1, \dots, x_n]$ given by the formula

$$s_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

Theorem 5.1.5: The Fundamental Theorem of Symmetric Polynomials

Let R be a ring. Then $s_1, \dots, s_n$ are algebraically independent over R. Moreover,

$$\Sigma = R[s_1, \dots, s_n]$$

6 λ-Rings

6.1 λ-Rings

Complex representation of a group is a λ-ring. Topological K theory is a λ-ring.

Requirements: Category Theory, Groups and Rings, Symmetric Functions
Books: Donald Yau: Lambda Rings

We need the theory of symmetric polynomials before defining λ-structures.

Definition 6.1.1: λ-Structures

Let R be a commutative ring. A λ-structure on R consists of a sequence of maps $\lambda^n : R \to R$ for $n \geq 0$ such that the following are true.

- $\lambda^0(r) = 1$ for all $r \in R$
- $\lambda^1 = \text{id}_R$
- $\lambda^n(1) = 0$ for all $n \geq 2$
- $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$ for all $r, s \in R$
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \dots, \lambda^{mn}(r))$ for all $r \in R$

Here P_n and $P_{m,n}$ are defined as follows.

- The coefficient of t^n in the polynomial
$$h(t) = \prod_{i,j=1}^n (1 + x_i y_j t)$$
is a symmetric polynomial in x_i and y_j with coefficients in $\mathbb{Z}$. P_n is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of x_i and y_j respectively.
- The coefficient of t^n in the polynomial
$$g(t) = \prod_{1 \leq i_1 \leq \dots \leq i_m \leq nm} (1 + x_{i_1} \cdots x_{i_m} t)$$
is a symmetric polynomial in x_i with coefficients in $\mathbb{Z}$. $P_{m,n}$ is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ of x_i.

In this case, we call R a λ-ring.

Note that we do not require that the λ^n are ring homomorphisms.

Definition 6.1.2: Associated Formal Power Series

Let R be a λ-ring. Define the associated formal power series to be the function $\lambda_t : R \to R[[t]]$ given by
$$\lambda_t(r) = \sum_{k=0}^\infty \lambda^k(r) t^k$$
for all $r \in R$

Proposition 6.1.3

Let R be a λ-ring. Then the following are true regarding $\lambda_t(r)$.

- $\lambda_t(1) = 1 + t$
- $\lambda_t(0) = 1$
- $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$
- $\lambda_t(-r) = \lambda(r)^{-1}$

Proposition 6.1.4

The ring $\mathbb{Z}$ has a unique λ-structure given by

$$\lambda_t(n) = (1+t)^n$$

Proposition 6.1.5

Let R be a λ-ring. Then R has characteristic 0.

Definition 6.1.6: Dimension of an Element

Let R be a λ-ring and let $r \in R$. We say that r has dimension n if $\deg(\lambda_t(r)) = n$. In this case, we write $\dim(r) = n$.

Proposition 6.1.7

Let R be a λ-ring. Then the following are true regarding the dimension of n.
- $\dim(r+s) \leq \dim(r) + \dim(s)$ for all $r, s \in R$
- If r and s both has dimension 1, then so is rs.

6.2 λ-Ring Homomorphisms and Ideals

Definition 6.2.1: λ-Ring Homomorphisms

Let R and S be λ-rings. A λ-ring homomorphism from R to S is a ring homomorphism $f : R \to S$ such that

$$\lambda^n \circ f = f \circ \lambda^n$$

for all $n \in \mathbb{N}$.

Definition 6.2.2: λ-Ideals

Let R be a λ-ring. A λ-ideal of R is an ideal I of R such that

$$\lambda^n(i) \in I$$

for all $i \in I$ and $n \geq 1$.

TBA:λ-ideal and subring. Ker, Im, Quotient Product, Tensor, Inverse Limit are λ-rings

Proposition 6.2.3

Let R be a λ-ring. Let $I = \langle z_i \mid i \in I \rangle$ be an ideal in R. Then I is a λ-ideal if and only if $\lambda^n(z_i) \in I$ for all $n \geq 1$ and $i \in I$.

Proposition 6.2.4

Every λ-ring R contains a λ-subring isomorphic to $\mathbb{Z}$.

6.3 Augmented λ-Rings

Definition 6.3.1: Augmented λ-Rings

Let R be a λ-ring. We say that R is an augmented λ-ring if it comes with a λ-homomorphism
$$\varepsilon : R \to \mathbb{Z}$$
called the augmentation map.

TBA: tensor of augmented is augmented

Proposition 6.3.2

Let R a λ-ring. Then R is augmented if and only if there exists a λ-ideal I such that
$$R = \mathbb{Z} \oplus I$$
as abelian groups.

6.4 Extending λ-Structures

Proposition 6.4.1

Let R be a λ-ring. Then there exists a unique λ-structure on $R[x]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[x]$ and $\varepsilon(r) = 0$ or 1.

Proposition 6.4.2

Let R be a λ-ring. Then there exists a unique λ-structure on $R[[x]]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[[x]]$ and $\varepsilon(r) = 0$ or 1.

6.5 Free λ-Rings

6.6 The Universal λ-Ring

6.7 Adams Operations

7 Witt Vectors

7.1 Fundamentals of the Ring of Big Witt Vectors

Prelim: Symm Functions, Lambda Rings, Category theory, Frobenius endomorphism (Galois), Rings and Modules, Kaehler differentials (commutative algebra 2)
Leads to: K theory
Books: Donald Yau: Lambda Rings

Definition 7.1.1: Truncation Sets

Let $S \subseteq \mathbb{N}$. We say that S is a truncation set if for all $n \in S$ and $d|n$, then $d \in S$. For $n \in \mathbb{N}$ and S a truncation set, define
$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

For instance, $\mathbb{N} \setminus \{0\}$ is a truncation set. We will also use $\{1, \dots, n\}$.

Theorem 7.1.2: Dwork's Theorem

Let R be a ring and let S be a truncation set. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i \in S} \in \prod_{i \in S} R$ has the form
$$(b_i)_{i \in S} = (w_i(a))_{i \in S}$$
for some $a \in R$
- For all primes p and all $n \in S$ such that $p|n$, we have
$$b_n \equiv \sigma_p(b_{n/p}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on all the b_k for $1 \le k \le n$ and $k \in S$.

We wish to equip $\prod_{i \in S} R$ with a non-standard addition and multiplication to make it into a ring.

Proposition 7.1.3

Consider the ring $R = \mathbb{Z}[x_i, y_i \mid i \in S]$. There exists unique polynomials
$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$
for $n \in S$ such that
- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i \in S}) + w_n((y_i)_{i \in S})$
- $w_n(\pi_1, \dots, \pi_n) = w_n((x_i)_{i \in S}) \cdot w_n((y_i)_{i \in S})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i \in S})$

for all $n \in S$.

Note that the polynomials ξ_n, π_n have variables x_k and y_k for $k \le n$ and $k \in S$. This is similar for the variables of ι. From now on, this will be the convention: For S a truncation set, the sequence $a_1, \dots, a_n$ actually refers to the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \le d_1 \le \cdots \le d_k \le n$ and $d_1, \dots, d_k$ are all divisors of n. The result of this is that sequences in $\mathbb{N}$ are now restricted to S.

Definition 7.1.4: The Ring of Truncated Witt Vector

Let R be a ring. Let S be a truncation set. Define the ring of big Witt vectors $W_S(R)$ of R to consist of the following.
- The underlying set $\prod_{i \in S} R$
- Addition defined by $(a_n)_{n \in S} + (b_n)_{n \in S} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$
- Multiplication defined by $(a_n)_{n \in S} \times (b_n)_{n \in S} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$

Theorem 7.1.5

Let R be a ring. Let S be a truncation set. Then the ring of big Witt vectors $W_S(R)$ of R is a ring with additive identity $(0,0,\dots)$ and multiplicative identity $(1,0,0,\dots)$. Moreover, for $(a_n)_{n\in S}\in W(R)$, its additive inverse is given by $(\iota_n(a_1,\dots,a_n))_{n\in\mathbb{N}}$.

Proposition 7.1.6

Let $\phi:R\to R'$ be a ring homomorphism. Then the induced map $W_S(\phi):W_S(R)\to W_S(R')$ defined by
$$W(\phi)((a_n)_{n\in S})=(\phi(a_n))_{n\in S}$$
is a ring homomorphism.

Definition 7.1.7: The Witt Functor

Define the Witt functor $W_S:\textbf{Ring}\to\textbf{Ring}$ to consist of the following data.

- For each ring R, $W_S(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi:R\to R'$, $W_S(\phi):W_S(R)\to W_S(R')$ is the induced ring homomorphism defined by
$$W_S(\phi)((a_n)_{n\in S})=(\phi(a_n))_{n\in S}$$

Proposition 7.1.8

Let S be a truncation set. The Witt functor is indeed a functor.

Definition 7.1.9: The Ghost Map

Let R be a ring. Let S be a truncation set. Define the ghost map to be the map
$$w:W_S(R)\to\prod_{k\in S}R$$
by the formula
$$w((a_n)_{n\in S})=(w_n(a_1,\dots,a_n))_{n\in S}$$

Remember, by the sequence $a_1,\dots,a_n$ we mean the sequence $a_1,a_{d_1},\dots,a_{d_k},a_n$ where $1\leq d_1\leq\cdots\leq d_k\leq n$ and $d_1,\dots,d_k$ the complete collection of divisors of n.

Proposition 7.1.10

Let S be a truncation set. Then the following are true.

- For each $n\in S$, the collection of maps $w_n:W_S(R)\to R$ for a ring R defines a natural transformation $w_n:W_S\to\text{id}$.
- The collection of ghost maps $w_R:W_S(R)\to\prod_{k\in S}R$ for R a ring defines a natural transformation $w:W_S\to(-)^S$.

Proposition 7.1.11

Let S be a truncation set. The truncated Witt functor $W_S:\textbf{Ring}\to\textbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W_S(R)$ is given by $\prod_{k\in S}R$
- For a ring homomorphism $\phi:R\to S$, $W(\phi):W(R)\to W(S)$ is the induced ring homomorphism defined by
$$W(\phi)((a_n)_{n\in\mathbb{N}})=(\phi(a_n))_{n\in\mathbb{N}}$$

- For each $n \in S$, $w_n : W_S(R) \to R$ defines a natural transformation $w_n : W \to \text{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k \in S} R$ is unique under the above conditions.

7.2 Important Maps of Witt Vectors

Definition 7.2.1: The Forgetful Map

Let R be a ring. Let $T \subseteq S$ be truncation sets. Define the forgetful map $R^S_T : W_S(R) \to W_T(R)$ to be the ring homomorphism given by forgetting all elements $s \in S$ but $s \notin T$.

Definition 7.2.2: The nth Verschiebung Map

Let R be a ring. Let S be a truncation set. For $n \in \mathbb{N}$, define the nth Verschiebung map $V_n : W_{S/n}(R) \to W_S(R)$ by

$$V_n((a_d)_{d \in S/n})_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise} \end{cases}$$

Note that this is not a ring homomorphism. However, it is additive.

Lemma 7.2.3

Let R be a ring. Let S be a truncation set. Then for all $a, b \in W_{S/n}(R)$, we have that

$$V_n(a + b) = V_n(a) + V_n(b)$$

Definition 7.2.4: Frobenius Map

Let S be a truncation set. Let R be a ring. Define the Frobenius map to be a natural ring homomorphism $F_n : W_S(R) \to W_{S/n}(R)$ such that the following diagram commutes:

$$\begin{tikzcd} W_S(R) \arrow[r, "w"] \arrow[d, "F_n"'] & \prod_{k \in S} R \arrow[d, "F_n^w"] \\ W_{S/n}(R) \arrow[r, "w"'] & \prod_{k \in S/n} R \end{tikzcd}$$

if it exists.

Lemma 7.2.5

Let S be a truncation set. Let R be a ring. Then the Frobenius map exists and is unique.

The following lemma relates this notion of Frobenius map to that in ring theory.

Lemma 7.2.6

Let A be an F_p algebra. Let S be a truncation set. Let $\varphi_p : A \to A$ denote the Frobenius homomorphism given by $a \mapsto a^p$. Then

$$F_p = R^S_{S/p} \circ W_S(\varphi) : W_S(A) \to W_{S/p}(A)$$

Definition 7.2.7: The Teichmuller Representative

Let R be a ring. Let S be a truncation set. Define the Teichmuller representative to be the map $[-]_S : R \to W_S(R)$ defined by

$$([a]_S)_n = \begin{cases} a & \text{if } n = 1 \\ 0 & b \text{ otherwise} \end{cases}$$

The Teichmuller representative is in general not a ring homomorphism, but it is still multiplicative.

Lemma 7.2.8

Let R be a ring. Let S be a truncation set. The for all $a, b \in R$, we have that

$$[ab]_S = [a]_S \cdot [b]_S$$

The three maps introduced are related as follows.

Proposition 7.2.9

Let R be a ring. Let S be a truncated set. Then the following are true.

- $r = \sum_{n \in S} V_n([r_n]_{S/n})$ for all $r \in W_S(R)$
- $F_n(V_n(a)) = na$ for all $a \in W_{S/n}(R)$
- $r \cdot V_n(a) = V_n(F_n(r) \cdot a)$ for all $r \in W_S(R)$ and all $a \in W_{S/n}(R)$
- $F_m \circ V_n = V_n \circ F_m$ if $\gcd(m, n) = 1$

The remaining section is dedicated to the example of $R = \mathbb{Z}$.

Proposition 7.2.10

Let S be a truncation set. Then the ring of big Witt vectors of $\mathbb{Z}$ is given by

$$W_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = \gcd(m, n) \cdot V_d([1]_{S/d})$$

and $d = \operatorname{lcm}(m, n)$.

7.3 The Ring of p-Typical Witt Vectors

For the ring of p-typical Witt vectors, we consider the truncation set $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$ for a prime p.

Definition 7.3.1: The Ring of p-Typical Witt Vectors

Let R be a ring. Let p be a prime. Let $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$. Define the ring of p-typical Witt vectors to be

$$W_p(R) = W_P(R)$$

Define the ring of p-typical Witt vectors of length n to be

$$W_n(R) = W_{\{1, p, \dots, p^{n-1}\}}(R)$$

when the prime p is understood.

Theorem 7.3.2

Let R be a ring. Let p be a prime number. Let S be a truncation set. Write $I(S) = \{k \in S \mid k \text{ does not divide } p\}$. Suppose that all $k \in I(S)$ are invertible in R. Then there is a decomposition

$$W_S(R) = \prod_{k \in I(S)} W_S(R) \cdot e_k$$

where

$$e_k = \prod_{t \in I(S) \setminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kt} V - kt([1]_{S/kt}) \right)$$

Moreover, the composite map given by

$$W_S(R) \cdot e_k \hookrightarrow W_S(R) \xrightarrow{F_k} W_{S/k}R \xrightarrow{R^{S/k}_{S/k \cap P}} W_{S/k \cap P}(R)$$

is an isomorphism.

7.4 The λ-structure on W(R)

Lemma 7.4.1

Let R be a ring. Then every $f \in \Lambda(R)$ can be written uniquely as

$$f = \prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n)$$

Theorem 7.4.2: The Artin-Hasse Exponential

There is a natural isomorphism $E : \Lambda \to W$ given as follows. For a ring R, $E_R : \Lambda(R) \to W(R)$ is defined by

$$E_R \left(\prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n) \right) = (a_n)_{n \in \mathbb{N}}$$

Corollary 7.4.3

Let R be a ring. Then $W(R)$ has a canonical λ-structure inherited from $\Lambda(R)$.

TBA: The forgetful functor $U : \Lambda\mathbf{Ring} \to \mathbf{CRing}$ has a left adjoint Symm and has a right adjoint W.

8 Formal Group Laws

Definition 8.0.1: Formal Group Laws

Let R be a ring. A formal group law over R is a power series

$$f(x,y)\in R[[x,y]]$$

such that the following are true.

- $f(x,0)=f(0,x)=x$
- $f(x,y)=f(y,x)$
- $f(x,f(y,z))=f(f(x,y),z)$

Definition 8.0.2: The Formal Group Law Functor

Define the formal group law functor

$$FGL:\textbf{Ring}\to\textbf{Set}$$

by the following data.

- For each ring R, $FGL(R)$ is the set of all formal group laws over R
- For each ring homomorphism $f:R\to S$, $FGL(f)$ sends each formal group law $\sum_{i,j=0}^\infty c_{i,j}x^iy^j$ over R to the formal group law $\sum_{i,j=0}^\infty f(c_{i,j})x^iy^j$ over S.

Definition 8.0.3: The Lazard Ring of a Formal Group Law

Define the lazard ring by

$$L=\frac{\mathbb{Z}[c_{i,j}]}{Q}$$

where Q is the ideal generated as follows. Write $f=\sum_{i,j=0}^\infty c_{i,j}x^iy^j$. Then Q is generated by the constraints on $c_{i,j}$ for which f becomes a formal group law.

Lemma 8.0.4

The Lazard ring $L=\mathbb{Z}[c_{i,j}]/Q$ has the structure of a graded ring where $c_{i,j}$ has degree $2(i+j-1)$.

Theorem 8.0.5

The formal group law functor $FGL:\textbf{Ring}\to\textbf{Set}$ is representable

$$FGL(R)\cong\text{Hom}_{\textbf{Ring}}(L,R)$$

There exists a universal element $f\in L$ such that the map $\text{Hom}_{\textbf{Ring}}(L,R)\to FGL(R)$ given by evaluation on f is a bijection for any ring R.

Theorem 8.0.6

There is an isomorphism of the Lazard ring

$$L\cong\mathbb{Z}[t_1,t_2,\dots]$$

where each t_k has degree $2k$.

9 Calculus of Functors

9.1 Excisive Functors

Definition 9.1.1: Homotopy Functors

Let $\mathcal{C}, \mathcal{D}$ be categories with a notion of weak equivalence. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy functor if F preserves weak equivalences.

Definition 9.1.2: n-Excisive Functors

Let F be a homotopy functor. We say that F is n-excisive if it takes strongly homotopy cocartesian $(n+1)$-cubes to homotopy cartesian $(n+1)$-cubes.

9.2 The Taylor Tower

Definition 9.2.1: Fiberwise Join

Let X, Y, U be spaces. Let $f : X \to Y$ be a map. Define the fiberwise join of X and U along f to be the space

$$X *_Y U = \text{hocolim}(X \longleftarrow X \times U \longrightarrow Y \times U)$$

Lemma 9.2.2

Let X, Y, U, V be spaces. Let $f : X \to Y$ be a map. Then there is a natural isomorphism

$$(X *_Y U) *_Y V \cong X *_Y (U * V)$$

Proposition 9.2.3

Let $\mathcal{P}(n)$ denote the category of posets. Let X be a space over Y. Then the assignment

$$U \mapsto X *_Y U$$

defines an n-dimensional cubical diagram in **Top**. Moreover, it is strongly cocartesian.

Definition 9.2.4

Let Y be a space. Let $F : \textbf{Top}_Y \to \textbf{Top}$ be a homotopy functor. Define the functor

$$T_n F : \textbf{Top}_Y \to \textbf{Top}$$

to consist of the following data.

- For each $X \in \textbf{Top}$, consider the functor $\mathcal{X} : \mathcal{P}(n+1) \to \textbf{Top}$ given by $U \mapsto F(X *_Y U)$. Define
$$T_n F(X) = \text{holim}(\mathcal{X}) = \underset{U \in \mathcal{P}(n+1)}{\text{holim}} (F(X *_Y U))$$
- For each $f : X \to Z$ a morphism of spaces over Y, define a map $T_n F(X) \to T_n F(Y)$ to be the map
$$F(f *_Y \text{id}) \circ \mathcal{X}$$

Lemma 9.2.5

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then $T_n F$ is a homotopy functor.

Proposition 9.2.6

Let F be a homotopy functor. Then there exists a natural map $t_nF : F \Rightarrow T_nF$ given by the canonical map of homotopy limits. Moreover, t_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "t_nF"'] & G \arrow[d, "t_nG"] \\ T_nF \arrow[r, "T_n\lambda"'] & T_nG \end{tikzcd}$$

Definition 9.2.7

Let Y be a space. Let $F : \mathbf{Top}_Y \to \mathbf{Top}$ be a homotopy functor. Define the functor

$$P_nF : \mathbf{Top}_Y \to \mathbf{Top}$$

to consist of the following data.

- For each space X over Y, define $P_nF(X)$ to be the homotopy limit
$$P_nF(X) = \text{holim}(F(X) \to T_nF(X) \to (T_n(T_nF))(X) \to \dots)$$
- For each morphism $f : X \to Z$ of spaces over Y, define $P_nF(f) : P_nF(X) \to P_nF(Z)$ to be the map ????

Lemma 9.2.8

Let Y be a space. Let X be a space over Y. Let F be a homotopy functor. Then P_nF is a homotopy functor.

Proposition 9.2.9

Let F be a homotopy functor. Then there exists a natural map $p_nF : F \Rightarrow P_nF$ given by the canonical map of homotopy limits. Moreover, p_nF is natural in the following sense. If G is another homotopy functor and $\lambda : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation, then the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\lambda"] \arrow[d, "p_nF"'] & G \arrow[d, "p_nG"] \\ P_nF \arrow[r, "P_n\lambda"'] & P_nG \end{tikzcd}$$

Definition 9.2.10: n-Reduced Functors

Let F be a homotopy functor. We say that F is n-reduced if $P_{n-1}F \simeq *$.

Definition 9.2.11: n-Homogenous Functor

Let F be a homotopy functor. We say that F is n-homogenous if F is n-excisive and n-reduced.

9.3 Linear Functors

Definition 9.3.1: Linear Functors

Let F be a homotopy functor. We say that F is linear if F is 1-homogenous. Explicitly, this means that

- F sends homotopy pushouts to homotopy pullbacks
- $F(X)$ is homotopy equivalent to $*$

Let us consider the case $n = 1$ and $Y = *$. Now $\mathcal{P}_0(2)$ is the small category given in a diagram as follows:

$$\begin{tikzcd} & \{1\} \arrow[d] \\ \{0\} \arrow[r] & \{0,1\} \end{tikzcd}$$

Now $T_1 F$ sends every space X to the homotopy limit of the following diagram:

$$\begin{tikzcd} & F(X * \{1\}) \arrow[d] \\ F(X * \{0\}) \arrow[r] & F(X * \{0,1\}) \end{tikzcd}$$

But we know that $X * \{0\}$ is the cone CX and $X * \{0, 1\}$ is the reduced suspension. This means that we can simplify the above diagram into

$$\begin{tikzcd} & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

Now $CX \simeq *$ and F is a reduced functor. Thus we can further simplify the diagram into

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

We recognize this as the homotopy pullback, and so $T_1 F(X) \simeq \Omega F(\Sigma X)$. Now recall that

$$P_1 F(X) = \text{hocolim}(F(X) \overset{t_1 F(X)}{\longrightarrow} T_1 F(X) \overset{t_1(T_1 F)}{\longrightarrow} (T_1(T_1 F))(X) \longrightarrow)$$

Again because we know that $T_1 F(X) \simeq \Omega F(\Sigma X)$ and we care about everything only up to homotopy, we can write $P_1 F$ as

$$P_1 F(X) = \text{hocolim}(F(X) \overset{t_1 F(X)}{\longrightarrow} \Omega F(\Sigma X) \overset{t_1(T_1 F)}{\longrightarrow} \Omega(T_1 F)(\Sigma X) \longrightarrow)$$

which further simplifies to

$$P_1 F(X) = \text{hocolim}(F(X) \to \Omega F(\Sigma X) \to \Omega^2 F(\Sigma^2 X) \longrightarrow)$$

Thus in general,

$$P_1 F(X) = \underset{n \to \infty}{\text{hocolim}}(\Omega^n F(\Sigma^n X))$$

We are considering the case $n = 2$. Now $\mathcal{P}_0(3)$ is the small category given in a diagram as follows:

$$\begin{tikzcd}
 & \{2\} \arrow[ddl] \arrow[drr] & & \\
 & & & \{1,2\} \arrow[ddl] \\
\{0,2\} \arrow[drr] & & & \{1\} \arrow[u] \arrow[ddl] \\
\{0\} \arrow[u] \arrow[drr] & & \{0,1,2\} & \\
 & & \{0,1\} \arrow[u] &
\end{tikzcd}$$

If we plug it into the definition of T_nF and choose $Y = *$, we obtain a functor

$$T_2F : \mathbf{Top}_* \to \mathbf{Top}$$

that consists of the following data. For each $X \in \mathbf{Top}$, $T_2F(X)$ is precisely the homotopy limit of the diagram

$$\begin{tikzcd}
 & F(X * \{2\}) \arrow[ddl] \arrow[drr] & & \\
 & & & F(X * \{1,2\}) \arrow[ddl] \\
F(X * \{0,2\}) \arrow[drr] & & & F(X * \{1\}) \arrow[u] \arrow[ddl] \\
F(X * \{0\}) \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
 & & F(X * \{0,1\}) \arrow[u] &
\end{tikzcd}$$

which simplifies to the diagram:

$$\begin{tikzcd}
 & F(CX) \arrow[ddl] \arrow[drr] & & \\
 & & & F(\Sigma X) \arrow[ddl] \\
F(\Sigma X) \arrow[drr] & & & F(CX) \arrow[u] \arrow[ddl] \\
F(CX) \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
 & & F(\Sigma X) \arrow[u] &
\end{tikzcd}$$

Now since F is reduced and $CX \simeq *$, we can further simplify it into

$$\begin{tikzcd}
 & * \arrow[drr] \arrow[ddl] & & \\
 & & & F(\Sigma X) \arrow[ddl] \\
F(\Sigma X) \arrow[dr] & & & * \arrow[u] \arrow[ddl] \\
* \arrow[u] \arrow[drr] & & F(X * \{0,1,2\}) & \\
 & & F(\Sigma X) \arrow[u] &
\end{tikzcd}$$

(what does the maps look like?)

Definition 9.3.2: The Category of Linear Functors

Define the category

$$\mathcal{H}_1(\mathcal{C}, \mathcal{D})$$

of linear functors to be the full subcategory of $\mathcal{D}^{\mathcal{C}}$ consisting of linear functors.

Theorem 9.3.3

There is an equivalence of categories

$$\mathcal{H}_1(\mathbf{CGWH}_*, \mathbf{CGWH}_*) \cong \Omega\mathrm{Sp}^{\mathbb{N}}(\mathbf{CGWH}_*)$$

given as follows. For a linear functor F, we associate to it the sequence of spaces $\{F(S^n) \mid n \in \mathbb{N}\}$, and this defines a spectra.

9.4 Catalogue of Construction Needed

Example 9.4.1

$\mathcal{P}_0(n+1)$ for small values of n is given as follows:

- When $n = 0$, $\mathcal{P}_0(1)$ consists of only one object

$$\{1\}$$

- When $n = 1$, $\mathcal{P}_0(2)$ is given by the following diagram:

$$\begin{tikzcd}
 & \{1\} \arrow[d] \\
\{0\} \arrow[r] & \{0,1\}
\end{tikzcd}$$

Example 9.4.2: Joins

We consider the join of a space and some finite space with discrete topology.

- When $n = 1$, the join of X and $\{1\}$ is given by

$$CX = X * \{1\}$$

- When $n = 2$, the join of X and $\{0, 1\}$ is given by

$$\Sigma X = X * \{0, 1\}$$

Example 9.4.3

Let F be a homotopy functor. We consider the intermediate functors T_nF for a homotopy functor F.

- When $n = 0$, $T_0F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$T_0F(X) = F(CX) \simeq F(*)$$
because F is a homotopy functor. If F is reduced then $T_0F(X) \simeq *$.
- When $n = 1$, $T_1F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$T_1F(X) \simeq \Omega F(\Sigma X)$$
because F is a homotopy functor.

Example 9.4.4

Let F be a homotopy functor. We consider the intermediate functors P_nF for a homotopy functor F.

- When $n = 0$, $P_0F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$P_0F(X) = \operatorname{hocolim}\,(F(X) \to (T_0F)(X) \to (T_0(T_0F))(X) \to \dots) \simeq P_0F(*)$$
because F is a homotopy functor. If F is reduced then $P_0F(X) \simeq *$.
- When $n = 1$, $P_1F : \mathbf{Top}_* \to \mathbf{Top}_*$ is a functor defined by
$$P_1F(X) \simeq \operatorname*{hocolim}_{n\to\infty} \Omega^n F(\Sigma^n X)$$
because F is a homotopy functor.

9.5 Important Theorems

Denote Sp by the category of spectra. Define a map $\mathcal{L}(\mathbf{Top}_*, \mathrm{Sp}) \to \mathrm{Sp}$ that sends $F : \mathbf{Top}_* \to \mathrm{Sp}$ to the spectra $F(S^0)$. Conversely, define a map $\mathrm{Sp} \to \mathcal{L}(\mathbf{Top}_*, \mathrm{Sp})$ by sending each spectra X to the functor $X \wedge -$.

Now define a map $\mathcal{L}(\mathbf{Top}_*) \to \mathrm{Sp}$ as follows. For each $F : \mathbf{Top}_* \to \mathbf{Top}_*$, $F(S^n)$ is a collection of spaces indexed by $\mathbb{N}$. As for the bonding maps $F(S^n) \wedge S^1 \to F(S^{n+1})$, this is defined as follows:

1. Consider the identity map $\mathrm{id} : X \wedge Y \to X \wedge Y$.
2. By the smash-hom adjunction, this corresponds to a map $Y \to \mathrm{Map}(X, X \wedge Y)$.
3. Now composing with F gives a map
$$Y \to \mathrm{Map}(X, X \wedge Y) \to \mathrm{Map}(F(X), F(X \wedge Y))$$
(Why is the latter map continuous?)
4. By the smash-hom adjunction, this corresponds to a map $F(X) \wedge Y \to F(X \wedge Y)$
5. Taking $X = S^n$ and $Y = S^1$ gives the desired results.

At the same time, we can do the following:

1. We begin by noticing that
$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

is a homotopy pushout.

2. Applying F sends the homotopy pushout to a homotopy pullback:

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(*) \arrow[d] \\ F(*) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

3. Since F is reduced, the diagram can be simplified into

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd}$$

4. Now recall that $\Omega(F(\Sigma X))$ is the homotopy pullback of $* \to F(\Sigma X) \leftarrow *$.
5. We obtain maps $F(X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ and $\Omega F(\Sigma X) \to \text{holim}(* \to F(\Sigma X) \leftarrow *)$ which are both weak

Now take the first map constructed $f : F(X) \wedge Y \to F(X \wedge Y)$ and substitute X and Y with our wanted values to get a map $f : F(S^n) \wedge S^1 \to F(S^{n+1})$. Adjunct it to the map $f : F(S^n) \to \Omega(F(S^{n+1}))$. Using the weak equivalences we obtained, we conclude that there is a diagram

$$\begin{tikzcd} F(S^n) \arrow[rr, "f"] \arrow[dr, "\simeq"'] & & \Omega F(S^{n+1}) \arrow[dl, "\simeq"] \\ & \text{Holim} & \end{tikzcd}$$

which we can prove to be commutative. By the two out of three property we easily conclude that f is a weak equivalences. This is exactly where the bonding maps come from.

We now have maps $\mathcal{L}(\textbf{Top}_*) \rightleftarrows \text{Sp}$. This actually gives an equivalence of categories. In fact, one can find out that it is a two step process:

$$\mathcal{L}(\textbf{Top}_*) \rightleftarrows \mathcal{L}(\textbf{Top}_*, \text{Sp}) \rightleftarrows \text{Sp}$$

Theorem 9.5.1

There is an equivalence of categories

$$\mathcal{L}(\textbf{Top}_*, \textbf{Sp}) \to \textbf{Sp}$$

given by $F \mapsto F(S^0)$.

Proof. Firstly, note that the above assignment defines a functor. Let $\lambda : F \Rightarrow G$ be a morphism in $\mathcal{L}(\textbf{Top}_*, \textbf{Sp})$. This means that for any $X \in \textbf{Top}_*$, we have a map of spectra $\lambda_X : F(X) \to G(X)$. Applying $X = S^0$ gives our map of spectra $F(S^0) \to G(S^0)$. Composition is preserved in this construction, and if $F = G$ then the identity natural transformation $\lambda : F \Rightarrow F$ gives the identity map $F(S^0) \to F(S^0)$ of spectra.

Now define a functor $\textbf{Sp} \to \mathcal{L}(\textbf{Top}_*, \textbf{Sp})$ by sending each spectra X to the functor $X \wedge -$. We want to show that $X \wedge -$ sends homotopy pushouts to homotopy pullbacks. Cubical 10.1.9. So let

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

be a homotopy pushout in $\mathbf{Top}_*$. □

10 Stable Infinity Categories

10.1 Stable Infinity Categories

Definition 10.1.1: Zero Objects

Let $\mathcal{C}$ be an infinity category. A zero object of $\mathcal{C}$ is an object 0 of $\mathcal{C}$ such that 0 is both initial and final. We say that $\mathcal{C}$ is pointed if it contains a zero object.

Lemma 10.1.2

Let $\mathcal{C}$ be an infinity category. Then $\mathcal{C}$ is pointed if and only if the following are true.
- $\mathcal{C}$ has an initial object $\emptyset$
- $\mathcal{C}$ has a final object $*$
- There exists a morphism $* \to \emptyset$ in $\mathcal{C}$

Definition 10.1.3: Triangles

Let $\mathcal{C}$ be a pointed infinity category. A triangle in $\mathcal{C}$ consists of a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\exists!"'] & Y \arrow[d, "g"] \\ 0 \arrow[r, "\exists!"'] & Z \end{tikzcd}$$

where X, Y, Z are objects and f, g are morphisms.

Definition 10.1.4: Fiber and Cofiber Sequences

Let $\mathcal{C}$ be a pointed infinity category.
- A triangle in $\mathcal{C}$ is called a fiber sequence if it is a pullback square
- A triangle in $\mathcal{C}$ is called a cofiber sequence if it is a pushout square.

Definition 10.1.5: Stable Infinity Categories

Let $\mathcal{C}$ be an infinity category. We say that $\mathcal{C}$ is stable if the following are true.
- $\mathcal{C}$ has a zero object 0
- Every morphism in $\mathcal{C}$ admits a fiber and a cofiber
- A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence

10.2

Recall that Lurie defined the infinity category of spaces as $\mathcal{S} = N_\bullet^{\text{hc}}(\textbf{Kan})$.

10.3

11 Algebras and Coalgebras

11.1 Coalgebras

There is a need to revisit the definition of an algebra (over a field)

Proposition 11.1.1

A vector space V over a field k is an algebra if and only if there is a following collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map

such that the following two diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \otimes V \arrow[r, "\mathrm{id}\otimes m"] \arrow[d, "m\otimes\mathrm{id}"'] & V \otimes V \arrow[d, "m"] \\
V \otimes V \arrow[r, "m"'] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V \arrow[r, "u\otimes\mathrm{id}"] \arrow[d, "\cong"'] & V \otimes V \arrow[dl, "m"'] \\
V & V \otimes k \arrow[l, "\cong"] \arrow[u, "\mathrm{id}\otimes u"']
\end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Evidently, the map μ gives a multiplicative structure for V and Δ gives the unitary structure of an algebra. The diagram on the left then represent associativity of multiplication. Notice that such additional structure on V formally lives in the category $\mathbf{Vect}_k$ of vector spaces over a fixed field k.

Therefore we can formally dualize all arrows to obtain a new object.

Definition 11.1.2: Coalgebra

Let V be a vector space over a field k. We say that V is a coalgebra over k if there is a collection of data:

- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\Delta"'] \\
V \otimes V \arrow[u, "\Delta\otimes\varepsilon"] & V \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
k \otimes V & V \otimes V \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\varepsilon"] \\
V \arrow[u, "\cong"] \arrow[ur, "\Delta"] \arrow[r, "\cong"'] & V \otimes k
\end{tikzcd}$$

where the unnamed maps is the canonical isomorphisms.

Lemma 11.1.3

Every vector space V over a field k can be given the structure of a coalgebra where

- $\Delta : V \to V \otimes V$ is defined by $\Delta(v) = v \otimes v$
- $\varepsilon : V \to k$ is defined by $\varepsilon(v) = 1_k$

We would like to formally invert the definitions of algebra homomorphisms in order to define coalgebra homomorphisms.

11.2 Bialgebras

Definition 11.2.1: Bialgebras

Let V be a vector space over a field k. We say that V is a bialgebra if there is a collection of data:

- A k-linear map $m : V \otimes V \to V$ called the multiplication map
- An k-linear map $u : k \to V$ called the unital map
- A k-linear map $\Delta : V \to V \otimes V$ called the comultiplication map
- An k-linear map $\varepsilon : V \to k$ called the counital map

such that (V, m, u) is an algebra over k and (V, Δ, ε) is a coalgebra over k and that the following diagrams are commutative:

$$\begin{tikzcd}
V \otimes V \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] & V \arrow[r, "\Delta"] & V \otimes V \\
V \otimes V \otimes V \otimes V \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & V \otimes V \otimes V \otimes V \arrow[u, "m\otimes m"]
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & k \\
& V \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
V \otimes V \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & V \arrow[dl, "\varepsilon"] \\
& k \otimes k \cong k &
\end{tikzcd}
\qquad
\begin{tikzcd}
& k \otimes k \cong k \arrow[dl, "u\otimes u"'] \arrow[dr, "u"] & \\
V \otimes V & & V \arrow[ll, "\Delta"]
\end{tikzcd}$$

where $\tau : V \otimes V \to V \otimes V$ is the commutativity map defined by $\tau(x \otimes y) = y \otimes x$.

Theorem 11.2.2

Let V be a vector space over k. Suppose that (V, m, u) is an algebra and (V, Δ, ε) is a coalgebra. Then the following conditions are equivalent.

- $(V, m, u, \Delta, \varepsilon)$ is a bialgebra
- $m : V \otimes V \to V$ and $u : k \to V$ are coalgebra homomorphisms
- $\Delta : V \to V \otimes V$ and $\varepsilon : V \to k$ are algebra homomorphisms

12 Hopf Algebras

12.1 Hopf Algebras

Definition 12.1.1: Hopf Algebra

Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. We say that H is a Hopf algebra if there is a k-linear map $S : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & k \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

13 Differential Graded Algebra

13.1 Basic Definitions

Similar to how chain complexes and cochain complexes are two names of the same object, we can define differential graded algebra using either the chain complex notation or cochain complex notation. For our purposes, we will use the cochain version. This means that differentials will go up in index.

A differential graded algebra equips a graded algebra with a differential so that the algebra in the grading form a cochain complex.

Definition 13.1.1: Differential Graded Algebra

A differential graded algebra is a graded algebra $A_\bullet$ together with a map $d : A \to A$ that has degree 1 such that the following are true.

- $d \circ d = 0$
- For $a \in A_n$ and $b \in A_m$, we have $d(ab) = (da)b + (-1)^n a(db)$

Lemma 13.1.2

Let (A, d) be a differential graded algebra. Then (A, d) is also a cochain complex.

Recall that a graded commutative algebra A is a collection of algebra over some ring A_0, graded in $\mathbb{N}$ together with a multiplication $A_n \times A_m \to A_{m+n}$ such that

$$a \cdot b = (-1)^{nm} b \cdot a$$

Such a multiplication rule is said to be graded commutative.

Definition 13.1.3: Commutative Differential Graded Algebra

A differential graded algebra A is said to be a commutative differential graded algebra (CDGA) if A is also graded commutative.

We will often be concerned of differential graded algebra over a field $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. In particular this means that the algebra has the structure of a vector space.

14 Group Structures on Maps of Spaces

Req: AT3

H-spaces is a natural generalization of topological groups in the direction of homotopy theory.

Definition 14.0.1: H-Spaces

Let (X,x_0) be a pointed space. Let $\mu:(X,x_0)\times(X,x_0)\to(X,x_0)$ be a map. Let $e:(X,x_0)\to(X,x_0)$ be the constant map $x\mapsto x_0$. We say that (X,x_0,μ) is an H-space if the following diagram:

$$\begin{tikzcd} X \arrow[r, "{(e,\mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X,e)}"'] \arrow[dr, "\mathrm{id}_X"] & X\times X \arrow[d, "\mu"] \\ X\times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map μ is called H-multiplication.

Definition 14.0.2: H-Associative Spaces

Let (X,x_0,μ) be an H-space. We say that (X,x_0,μ) is an H-associative space if the following diagram:

$$\begin{tikzcd} X\times X\times X \arrow[r, "\mu\times\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X\times\mu"'] & X\times X \arrow[d, "\mu"] \\ X\times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 14.0.3: H-Group

Let (X,x_0,μ) be an H-space. Let $j:(X,x_0)\to(X,x_0)$ be a map. We say that (X,x_0,μ,j) is an H-group if the following diagram:

$$\begin{tikzcd} X \arrow[r, "{(j,\mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_X,j)}"'] \arrow[dr, "e"] & X\times X \arrow[d, "\mu"] \\ X\times X \arrow[r, "\mu"'] & X \end{tikzcd}$$

is commutative up to homotopy. The map j is called H-inverse.

Example 14.0.4

Let X be a pointed space. Then the loopspace ΩX is an H-group.

Definition 14.0.5: H-Abelian

Let (X,x_0,μ,j) be an H-group. Let $T:(X,x_0)\times(X,x_0)\to(X,x_0)$ be the map $T(x,y)=T(y,x)$. We say that (X,x_0,μ,j) is an H-abelian if the following diagram:

$$\begin{tikzcd} X\times X \arrow[r, "T"] \arrow[dr, "\mu"'] & X\times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

is commutative up to homotopy.

Definition 14.0.6: Natural Group Structure

Let (X, x_0) be pointed spaces. We say that $[Z, X]_*$ has a natural group structure for all spaces (Z, z_0) if the following are true.

- $[Z, X]_*$ has a group structure such that the constant map $[e]$ is the identity of the group.
- For every map $f : A \to B$, the induced function

$$f^* : [B, X]_* \to [A, X]_*$$

  is a group homomorphism.

15 R Project

15.1 Homotopy Axioms

Definition 15.1.1: Axioms for Homotopy Theory

A homotopy theory consists of a sequence of functors

$$\pi_n : \mathbf{CW}_*^2 \to \mathbf{Set}$$

for $n \geq 0$, together with a sequence of natural transformations

$$\partial_n : \pi_n \Rightarrow \pi_{n-1} \circ T$$

where $T(X, A, x_0) = (A, x_0, x_0) = (A, x_0)$ subject to the following enrichment:

- For $n \geq 2$, the functor lands in abelian groups:

$$\pi_n : \mathbf{Top}_*^2 \to \mathbf{Ab}$$

- For $n = 1$, if $A = x_0$ then the functor lands in groups

$$\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$$

- For $n \geq 2$, the natural transformation ∂_n is a collection of group homomorphisms
- For $n = 1$, if $A = x_0$ then the natural transformation ∂_1 is a collection of group homomorphisms

that satisfies the following axioms:

- Homotopy: If $f, g : (X, A, x_0) \to (Y, B, y_0)$ are homotopic maps, then the induced map

$$\pi_n(f) = \pi_n(g) : \pi_n(X, A, x_0) \to \pi_n(Y, B, x_0)$$

  are equal.
- Exactness: If $(X, A, x_0) \in \mathbf{Top}_*^2$ is a pointed pair of spaces, then the inclusions $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ induces a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(X, A, x_0) \xrightarrow{\partial_{n+1}} \pi_n(A, x_0) \xrightarrow{\pi_n(i)} \pi_n(X, x_0) \xrightarrow{\pi_n(j)} \pi_n(X, A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, A, x_0)$$

  where exactness of morphisms of sets is defined as follows. If $f : (P, p_0) \to (Q, q_0)$ and $g : (Q, q_0) \to (R, r_0)$ are functions of pointed sets, then it is exact at Q if $g^{-1}(r_0) = \operatorname{im}(f)$.
- Excision???? Let $p : E \to B$ be a Serre fibration. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. Then for $n \geq 1$, p induces a group isomorphism

$$\pi_n(p) : \pi_n(E, x_0) \xrightarrow{\cong} \pi_n(B, b_0)$$

- Dimension: $\pi_n(*, *) = 0$ for all $n \in \mathbb{N}$ where $*$ is the one-point space.

Axioms implies: There is a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial_{n+1}} \pi_n(F, x_0) \xrightarrow{\pi_n(\iota)} \pi_n(E, x_0) \xrightarrow{\pi_n(\text{incl.})} \pi_n(E, F, x_0) \xrightarrow{\pi_n(p), \cong} \pi_n(B, b_0) \longrightarrow \cdots \longrightarrow \pi_0(B, b_0)$$

Reason: Recall if $i : A \to X$ is a closed Hurewicz fibration, (closed meaning $i(A) \subseteq X$ is closed), then $X/i(A)$ is homotopy equivalent to Cone(f). https://ncatlab.org/nlab/show/topological+cofiber+sequence#HurewiczCofibration

In reduced homology theory, there is the following axiom:

- Exactness: If $i : A \to X$ is a cofibration, then
$$\widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A)$$
is exact.

This is why we consider fibrations.

Heuristic no.2: Recall that we only care about CW complexes (at least for now) in homology theories. Nowe every CW complex can be expressed as a direct limit

$$X = \text{colim}\left(X^{(0)} \longrightarrow X^{(1)} \longrightarrow \cdots \longrightarrow X^{(n)} \longrightarrow \cdots\right)$$

where the map $X^{(n)} \to X^{(n+1)}$ is defined by the inclusion and then the projection:

$$X^{(n)} \hookrightarrow X^{(n)} \amalg \coprod_{\alpha\in I_n} D^n_\alpha \to \frac{X^{(n)} \amalg \coprod_{\alpha\in I_n} D^n_\alpha}{\coprod_{\alpha\in I_n} S^{n-1}_\alpha} = X^{(n+1)}$$

In particular, $X^{(n)} \to X^{(n+1)} \to \coprod_{\alpha\in I_n} S^{n-1}_\alpha$ is a cofiber sequence.

On the other hand, every CW complex is weakly equivalent to a direct limit of a Postnikov tower:

$$X \simeq \lim\left(\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0\right)$$

Recall that
$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0$$
is a Postnikov tower of the space $Y = \lim$ of the direct limit if the following are true:

- $Y \to X_n$ induces isomorphisms $\pi_i(Y) \cong \pi_i(X_n)$ for all $i \leq n$.
- $\pi_i(X_n) = 0$ for all $i \geq n$.
- Each map $X_{n+1} \to X_n$ is a fibration with fiber $K(\pi_n(X), n)$.

Effectively, we want the n space in the tower to exactly capture homotopy information up to level n.

15.2 Blakers-Massey Theorem

15.3 Linear Functors and Spectra

Functor 1: $F : \mathcal{L}(\textbf{Top}_*) \to \Omega\text{Sp}$ is defined as follows. For $X : \textbf{Top}_* \to \textbf{Top}_*$ a linear functor, $F(X) = \{X(S^n) \mid n \in \mathbb{N}\}$ together with bonding maps $X(S^n) \to \Omega X(S^{n+1})$ given as follows. We notice that

$$\Omega X(S^{n+1}) = \text{Holim}(* \to X(S^{n+1}) \leftarrow *)$$

where both maps $* \to X(S^{n+1})$ are the same inclusion to the base point. At the same time,

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma S^n \end{tikzcd}$$

is a homotopy pushout. Since X is linear, the following is a homotopy pullback:

$$\begin{tikzcd} X(S^n) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X(\Sigma S^n) \end{tikzcd}$$

so there is a weak equivalence $X(S^n) \to \Omega X(S^{n+1})$.

A natural transformation $\lambda : X \Rightarrow Y$ of linear functors also give a morphism of $\Omega\mathrm{Sp}$ in the following way. The naturality condition implies that the following diagram commutes: ??????

Functor 2: $G : \Omega\mathrm{Sp} \to \mathcal{L}(\mathbf{Top}_*)$ is defined as follows. For $\{T_n, \sigma_n \mid n \in \mathbb{N}\}$ an omega spectra, define a functor

$$A \mapsto \Omega^\infty\{T_n \wedge A, \sigma_n \wedge \mathrm{id}_A \mid n \in \mathbb{N}\} = \mathrm{Hocolim}_k \Omega^k(T_k \wedge A)$$

(Recall Ω^∞ sends an omega spectra to its level 0 space, if its not an omega spectra then one needs to take fibrant replacement). This is a functor because it is a composition of three functors:

$$A \xrightarrow{\{T_n \wedge -\}} \{T_n \wedge A\} \xrightarrow{\text{fib. replace}} R_\infty\{T_n \wedge A\} \xrightarrow{\mathrm{Ev}_0^{\mathbb{N}}} \mathrm{Hocolim}_k \Omega^k(T_k \wedge A)$$

We need to check that this functor is linear.

- It is a homotopy functor because:
- It is reduced because $\Omega^\infty\{T_n \wedge *\} = \Omega^\infty\{*\} = *$.
- It is finitary because:
- If we have a homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

then the smash of the square

$$\begin{tikzcd} T_k \wedge X \arrow[r] \arrow[d] & T_k \wedge Y \arrow[d] \\ T_k \wedge Z \arrow[r] & T_k \wedge W \end{tikzcd}$$

is also a homotopy pushout. Now T_0 is (-1)-connected hence $T_k \overset{\text{weak}}{\simeq} \Omega^k T_0$ is $(k-1)$-connected. Hence $T_k \wedge X$, $T_k \wedge Y$, $T_k \wedge Z$ and $T_k \wedge W$ are all $(k-1)$-connected. Maps of n connected spaces must be n connected, in particular $T_k \wedge X \to T_k \wedge Y$ and $T_k \wedge X \to T_k \wedge Z$ are $(k-1)$-connected. By Blakers-Massey theorem, the very same square is $(2k-3)$-cartesian. This means that the map

$$T_k \wedge X \to \mathrm{Holim}(T_k \wedge (Z \to W \leftarrow X))$$

is $(2k-3)$-connected. Now recall that $\pi_n(X) = \pi_{n-1}(\Omega X)$ for a connected space X. In particular, if X is k-connected then ΩX is $(k-1)$-connected. Hence the map

$$\Omega^k\left(T_k \wedge X\right) \to \Omega^k \mathrm{Holim}(T_k \wedge (Z \to W \leftarrow Y)) \simeq \mathrm{Holim}\left(\Omega^k(T_k \wedge (Z \to W \leftarrow Y))\right)$$

is $(k-3)$-connected (Ω commutes with homotopy pullbacks). (Final Step??????)

The main theorem is as follows.

Theorem 15.3.1

$F : \mathcal{L}(\mathbf{Top}_*) \leftrightarrows \Omega\mathrm{Sp} : G$ is a weak equivalence in the following sense:
- $F(X)$ is weakly equivalent to $\mathrm{Hocolim}_k(\Omega^k(F(S^k) \wedge X))$ for all spaces X.
- $\{T_n\}$ is weakly equivalent to $\{\mathrm{Hocolim}_k(\Omega^k T_k \wedge S^n)\}$ for all Ω-spectrum $\{T_n\}$.

15.4

Recall that $\mathcal{S} = N_\bullet^{\mathrm{hc}}(\mathbf{Top}_*)$ is the infinity category of spaces.

Proposition 15.4.1

Let $\mathcal{C}$ be a pointed infinity category that admits all finite colimits. Then $\text{Exc}_*(\mathcal{C},\mathcal{S})$ is stable.

Proof. Let $F:\mathcal{C}\to\mathcal{S}$ be excisive and reduced. Then $\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F)=F\circ\Sigma_{\mathcal{C}}$. By definition of the suspension functor,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma_{\mathcal{C}}(X) \end{tikzcd}$$

is a pushout in $\mathcal{C}$. Since F is excisive,

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & (F\circ\Sigma_{\mathcal{C}})(X) \end{tikzcd}$$

is a pullback in $\mathcal{S}$. On the other hand, $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F)=\Omega_{\mathcal{S}}\circ F$. By definition of the loop functor,

$$\begin{tikzcd} (\Omega_{\mathcal{S}}\circ F\circ\Sigma_{\mathcal{C}})(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & (F\circ\Sigma_{\mathcal{C}})(X) \end{tikzcd}$$

is a pullback in $\mathcal{S}$ for any $X\in\mathcal{C}$. Therefore $F(X)$ and $(\Omega_{\mathcal{S}}\circ F\circ\Sigma_{\mathcal{C}})(X)$ are equivalent. Hence F and $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F))$ are equivalent. □

Theorem 15.4.2

There is an equivalence of infinity categories

$$\text{Sp}(\mathcal{S})\simeq\lim(\cdots\to\mathcal{S}\overset{\Omega}{\to}\mathcal{S}\overset{\Omega}{\to}\mathcal{S})=:\overline{\mathcal{S}}$$

induced by the evaluation map $\text{ev}_{S^0}:\overline{\mathcal{S}}\to\mathcal{S}$.

Proof.
Since $\mathcal{S}$ is presentable and the infinity category of presentable infinity categories admit all small limits, $\overline{\mathcal{S}}$ is also presentable. Every presentable infinity category admits all small limits and colimits. Since $\mathcal{S}$ is pointed, $\overline{\mathcal{S}}$ is also pointed. Since all limits are computed term-wise, we have that in particular $\Omega_{\overline{\mathcal{S}}}$ is computed term wise. given $\{X_n\mid n\in\mathbb{N}\}$ an object of $\overline{\mathcal{S}}$, $\{\Omega X_n\mid n\in\mathbb{N}\}$ is equivalent to $\{X_n\mid n\in\mathbb{N}\}$ because we have that ΩX_{n+1} is equivalent to X_n for all n. By a prp we conclude that $\overline{\mathcal{S}}$ is stable.

Consider the canonical functor $G:\overline{\mathcal{S}}\to\mathcal{S}$ defined by recovering the first factor: $(X_0,X_1,\dots)\mapsto X_0$. It is clear that it commutes with finite limits since limits are computed term-wise.

Let $\mathcal{C}$ be an arbitrary stable infinity category. Any functor $\mathcal{C}\to\mathcal{S}$ is left exact if and only if it is exact so that $\text{Exc}_*(\mathcal{C},\mathcal{S})=\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$. 1.4.2.16 implies that $\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$ is a stable infinity category. Thus $\Omega_{\mathcal{S}}\circ -$ is an equivalence.

On the other hand, since Ω are computed term-wise (like all limits) and since $\text{Func}(\mathcal{C},\overline{\mathcal{S}})$ is right adjoint to products we know that Func commutes with finite limits . Thus we have that

$$\text{Exc}_*^{\text{L}}(\mathcal{C},\overline{\mathcal{S}})=\lim(\cdots\to\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})\overset{\Omega\circ-}{\to}\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})\overset{\Omega\circ-}{\to}\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S}))$$

Since each $\Omega_{\overline{\mathcal{S}}} \circ -$ is an equivalence of infinity categories, we conclude that $\text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) \simeq \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$. Thus evaluation on the first factor $G \circ - : \text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) \to \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$ is an equivalence of infinity categories.

By a previous corollary, there is an equivalence of infinity categories given by

$$\Omega^\infty \circ - : \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}}, \text{Sp}(\mathcal{S})) \to \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}}, \mathcal{S})$$

The fact that G is left exact means that there is a factorization

$$\begin{tikzcd} \overline{\mathcal{S}} \arrow[rr, "G"] \arrow[dr, "G' \circ -"'] & & \mathcal{S} \\ & \text{Sp}(\mathcal{S}) \arrow[ur, "\Omega^\infty"'] & \end{tikzcd}$$

By functoriality we obtain a similar factorization:

$$\begin{tikzcd} \text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) \arrow[rr, "G \circ -"] \arrow[dr, "G' \circ -"'] & & \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S}) \\ & \text{Exc}_*^{\text{L}}(\mathcal{C}, \text{Sp}(\mathcal{S})) \arrow[ur, "\Omega^\infty \circ -"'] & \end{tikzcd}$$

Since $G \circ -$ and $\Omega^\infty \circ -$ are both equivalence of infinity categories, we conclude that $G' \circ -$ is an equivalence of infinity categories.

Since this is true for all stable infinity categories, the fact that

$$\text{Exc}_*(\mathcal{C}, \overline{\mathcal{S}}) = \text{Exc}_*^L(\mathcal{C}, \overline{\mathcal{S}}) \simeq \text{Exc}_*^L(\mathcal{C}, \text{Sp}(\mathcal{S})) = \text{Exc}_*(\mathcal{C}, \text{Sp}(\mathcal{S}))$$

is an equivalence for all stable $\mathcal{C}$ together with the Yoneda embedding implies that $\overline{\mathcal{S}}$ and $\text{Sp}(\mathcal{S})$ is an equivalence of infinity categories. $\square$

Beware that in the proof we also showed that $G \circ -$ is an equivalence of infinity categories for any stable infinity category $\mathcal{C}$. But this does not imply that $\overline{\mathcal{S}}$ and $\mathcal{S}$ are equivalent because we are applying the Yoneda embedding on the category of stable infinity categories, and a priori $\mathcal{S}$ is not stable.