Abelian Varieties

Labix

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Abstract

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1 Properties of Abelian Varieties

1.1 Group Schemes and Group Varieties

Definition 1.1.1: Group Schemes

A group scheme is a group object in the category **Sch** of schemes. A group scheme over a scheme S is a group object in the category $\mathbf{Sch}_S$ of schemes over S.

Definition 1.1.2: Group Varieties

A group variety over a field k is a group object in the category $\mathbf{Var}_k$ of varieties over k.

Definition 1.1.3: Algebraic Groups

An algebraic group over a field k is a group variety over k that is also smooth.

Proposition 1.1.4

Let k be a field with characteristic 0. Then every group scheme over k is smooth.

1.2 Basic Definitions

Let us start by recalling the definition of an abelian variety in Algebraic Geometry 3.

Definition 1.2.1: Abelian Varieties

An abelian variety over a field k is a group variety that is complete and connected.

Theorem 1.2.2: Rigidity Theorem

Corollary 1.2.3

The group law on any abelian variety is commutative, hence every abelian variety has a the structure of an abelian group.

1.3 Rational Maps into Abelian Varieties

Theorem 1.3.1

Let A be an irreducible abelian variety over k. Then for any non-singular irreducible variety V and rational map $\varphi : V \to A$, φ extends to a morphism $V \to A$.

1.4 Abelian Varieties are Projective

Theorem 1.4.1: Abelian Varieties are Projective

Every abelian variety over an algebraically closed field k is projective.

Theorem 1.4.2

Every abelian variety over $\mathbb{C}$ is a compact complex submanifold of $\mathbb{P}^n(\mathbb{C})$.