Solutions to Hatcher

Labix

May 10, 2024

Abstract

Solutions to the book Algebraic Topology authored by Allen Hatcher

Contents

1 The Fundamental Group

1.1 Basic Constructions

Exercise 1.1.1

Show that the composition of paths satisfy the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

Proof. From the relation $g_0 \simeq g_1$ we have that $g_1 \cdot \overline{g}_0 \simeq e$. It follows that

$$\begin{aligned} f_0 \cdot g_0 &\simeq f_1 \cdot g_1 \\ f_0 \cdot g_0 \cdot \overline{g}_0 &\simeq f_1 \cdot g_1 \cdot \overline{g}_0 \\ &= f_0 \simeq f_1 \end{aligned}$$

and so we conclude. □

Exercise 1.1.2

Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. Recall that the isomorphism is defined by $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ sending $[\alpha] \in \pi_1(X, x_1)$ to $[h \cdot \alpha \cdot \overline{h}]$. We have that $h \overset{\partial}{\simeq} h'$ implies $h \cdot \alpha \cdot \overline{h} \simeq h' \cdot \alpha \cdot \overline{h'}$ so that $\beta_h([\alpha]) = \beta_{h'}([\alpha])$. □

Exercise 1.1.3

For a path connected space X, show that $\pi_1(X)$ is abelian if and only if all base point change homomorphisms β_h depend only on the endpoints of the path h.

Proof. Suppose that $\pi_1(X)$ is abelian. We want to show that $\beta_h = \beta_{h'}$ if $h(1) = h'(1)$. We have that

$$\begin{aligned} \beta_h([\alpha]) &= [h \cdot \alpha \overline{h}] \\ \beta_{h'}([\alpha]) &= [h' \cdot \alpha \cdot \overline{h'}] \end{aligned}$$

Since $\pi_1(X)$ is abelian, we have that

$$\begin{aligned} \beta_h([\alpha]) \cdot \overline{\beta_{h'}([\alpha])} &= [h \cdot \alpha \cdot \overline{h} \cdot h' \cdot \alpha \cdot \overline{h'}] \\ &= [h \cdot (\overline{h} \cdot h') \cdot \alpha \cdot \overline{\alpha} \cdot \overline{h'}] && (\overline{h} \cdot h' \text{ is a loop on } x_1) \\ &= [h' \cdot \overline{h'}] \\ &= [e_{x_0}] \end{aligned}$$

This implies that $[h \cdot \alpha \cdot \overline{\alpha}] = [h' \cdot \alpha \cdot \overline{h'}]$ which is what is required.

Now suppose that $\pi_1(X)$ is not abelian. Then there exists $[a], [b] \in \pi_1(X)$ such that $[a] \cdot [b] \neq [b] \cdot [a]$. In other words, $[\overline{b}] \cdot [a] \cdot [b] \neq [a]$. But clearly for the constant loop e, we have that $[\overline{e}] \cdot [a] \cdot [e] = [a]$ which implies that

$$\begin{aligned} [\overline{b}] \cdot [a] \cdot [b] &\neq [\overline{e}] \cdot [a] \cdot [e] \\ \beta_b([a]) &\neq \beta_e([a]) \end{aligned}$$

even though b and e have the same end points. □

Exercise 1.1.4

Show that if a subspace $X \subset \mathbb{R}^n$ is locally star-shaped, then every path in X is homotopic in X to a piecewise linear path. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

Proof. Let γ be a path in X. Consider the open cover of $\gamma([0,1])$ by the star-shaped neighbourhood of each $x \in \gamma([0,1])$. Since $[0,1]$ is compact, $\gamma([0,1])$ is compact so the open cover has a finite subcover $U_1, \dots, U_m$ which are neighbourhoods of $\gamma(t_1) = x_1, \dots, \gamma(t_m) = x_m$ for $t_1 < \cdots < t_m$. For any $U_i \cap U_{i+1}$ (nonempty since open cover), choose $\gamma(s_i) = y_i$ and $t_1 < s_1 < t_2 < s_2 < \cdots < s_{m-1} < t_m$. Since each U_i is star-shaped at x_i, there are straight paths from x_i to y_{i-1} and y_i, say $a_{i-1} : I \to X$ and $b_i : I \to X$. Since U_i is star-shaped at x_i, any point between the paths a_{i_1} and $\gamma_{[s_{i-1}, t_i]}$ (likewise $\gamma_{[t_i, s_i]}$ and b_i) is reachable via a straight line, so that $\gamma_{[s_{i-1}, t_i]}$ is homotopic to the straight path a_{i-1} and likewise $\gamma_{[t_i, s_i]}$ is homotopic to the straight path b_i and so we are done.

If X is a union of finitely many closed convex sets, then notice that each convex set is star-shaped. Each $x \in X$ must be contained in one of the convex sets and so X is locally star-shaped. □

Exercise 1.1.5

Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^11 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic (Without regards to basepoint).

Proof.

- $(a) \implies (b)$: Suppose that $f \simeq e_{x_0}$. This means that there exists a homotopy $H : S^1 \times I \to X$ from f to e_{x_0}. Now by the universal property of quotient spaces, we have a factorization

$$\begin{tikzcd} S^1 \times I \arrow[r, "p"] \arrow[dr, "H"'] & \frac{S^1 \times I}{S^1 \times \{1\}} \arrow[d, "\tilde{H}"] \\ & X \end{tikzcd}$$

  where p is the quotient map. This is possible because $H(S^1 \times I) = \{x_0\}$. Since $\frac{S^1 \times I}{S^1 \times \{1\}} \cong D^2$, we obtain an extension.

- $(a) \implies (c)$: $S^1 \cong \frac{I}{0 \sim 1}$ so that every map $f : S^1 \to X$ is just a loop in X. Since all loops in X is homotopic to the constant map, we must have $\pi_1(X, x) = 0$.

- $(b) \implies (a)$: Suppose that $f : S^1 \to X$ is a map. By assumption, f can be extended to a map $\tilde{f} : D^2 \to X$. Since D^2 is contractible, we have $\mathrm{id}_{D^2} \simeq e$, which implies that

$$\tilde{f} \simeq f \circ e_{x_0} = e_{f(x_0)}$$

  In particular, f is also homotopic to a constant map by the same homotopy.

- $(c) \implies (a)$: Suppose that $\pi_1(X, x_0) = 0$. Then any loop $f : I \to X$ is such that $f \simeq e_{x_0}$. In particular, a loop with domain I is just a map from S^1 because $S^1 \cong \frac{I}{0 \sim 1}$.

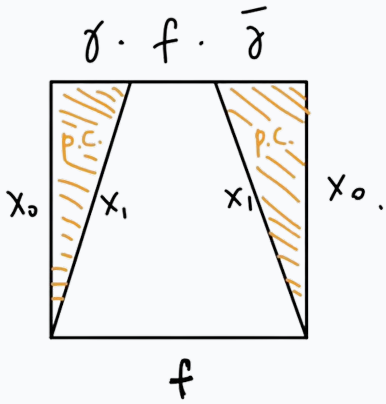
For the remainder of the question, suppose that X is simply connected. This means that $\pi_1(X, x_0) = 0$. This means that any loop $S^1 \to X$ is homotopic to a constant map. Since simply connectedness implies path connectedness, any constant paths are homotopic. This means that any $S^1 \to X$ are homotopic.

Now suppose that any $S^1 \to X$ are homotopic. Then in particular, they are all homotopic to the constant path. Thus $\pi_1(X, x_0) = 0$ for any $x_0 \in X$. □

Exercise 1.1.6

There is a natural map $\Psi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Ψ is onto if X is path-connected, and that $\Psi([f]) = \Psi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Proof. Suppose that X is path connected, and let $[f] \in [S^1, X]$. Let x_1 be the end point of the loop f. Since X is path connected, there exists $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. This means that $\gamma \cdot f \cdot \overline{\gamma}$ is a loop starting at x_0. But $[\gamma \cdot f \cdot \overline{\gamma}] = [f]$ via the homotopy



Thus $\Psi([\gamma \cdot f \cdot \overline{\gamma}) = [f]$.

Now suppose that $\Psi([f]) = \Psi([g])$. Then this implies that $f \simeq g$ are free homotopic where f and g have basepoint x_0. Let $H : I \times I \to X$ be the homotopy. Let $h : I \to X$ be defined as $h(t) = H(0, t)$. Then we have

$$h(0) = H(0,0) = f(0) = x_0$$
$$h(1) = H(0,1) = g(0) = x_0$$

so that h is a loop. By lemma 1.19, we have that $(H_0)_* = \beta_h \circ (H_1)_*$ if and only if $f_* = \beta_h \circ g_*$. Plugging in the generator ω_1 of $\pi_1(S^1, 1) \cong \mathbb{Z}$, we have that

$$f_*(\omega_1) = (\beta_h \circ g_*)(\omega_1)$$
$$f \circ \omega_1 = \beta_h(g \circ \omega_1)$$

But $f \simeq f \circ \omega_1$ and $\overline{h} \cdot g \cdot h \simeq \beta_h(g \circ \omega_1)$ and so we have that $[f] = [\overline{h}] \cdot [g] \cdot [h]$.

Suppose that $[f] = [\overline{\gamma} \cdot g \cdot \gamma]$ for some $\gamma : I \to X$ a loop so that f, g, γ are loops based at x_0. Applying Ψ, we have that $\Psi([f]) = \Psi([\overline{\gamma} \cdot g \cdot \gamma])$. Consider $\Psi([g]) \in [S^1, X]$. It is clear that $g \in \Psi([g])$. Moreover, we must have $g \simeq \overline{\gamma} \cdot g \cdot \gamma$ by the same homotopy given above (replace f with g). Thus we have that f is free homotopic to g. □

Exercise 1.1.7

Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$ so f restricts to the identity on the two boundary circles $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. Define $H : (S^1 \times I) \times I \to S^1 \times I$ by

$$(\theta, s, t) \mapsto (\theta + 2\pi s(1-t), s)$$

Clearly H is continuous. Moreover,

$$H(\theta, s, 0) = f(\theta, s)$$
$$H(\theta, s, 1) = \mathrm{id}(\theta, s)$$
$$H(\theta, 0, t) = (\theta, 0)$$

Thus we have that

$$f \overset{S^1 \times \{0\}}{\simeq} \mathrm{id}$$

Now suppose that H is a homotopy from id to f that fixes $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Let $\gamma : I \to S^1 \times I$ be a path defined as $\gamma(s) = \theta_0 + s$ for some fixed θ_0. Then the conditions on H implies that

$$H(\gamma(s), 0) = \gamma(s)$$
$$H(\gamma(s), 1) = (f \circ \gamma)(s)$$

so that we have a homotopy $\gamma \simeq f \circ \gamma$.

Consider the projection $p : S^1 \times I \to S^1$. Then we have that

$$\gamma \simeq f \circ \gamma$$
$$p \circ \gamma \simeq p \circ f \circ \gamma$$
$$e \simeq \omega_1$$

But ω_1 is a generator of $\pi_1(S^1)$ hence this is a contradiction. □

Exercise 1.1.8

Does the Borsak-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$, must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Proof. The Borsak-Ulam theorem fails on the torus. Consider the map $f : S^1 \times S^1 \subset \mathbb{R}^3 \to \mathbb{R}^2$ that forgers the z coordinate of the torus. It is clear that for two points to have the same image under f, it must have the same y value in $S^1 \times S^1$ (Think of the first circle in $S^1 \times S^1$ having the y-axis passing through and the second circle having the z-axis passing through). Assume that the theorem holds. Then $f(x, y) = f(-x, -y)$ together with $y = -y$ implies that $y = 0$ in $\mathbb{R}^3$ coordinates. But not point in the torus has $\mathbb{R}^3$ coordinate $y = 0$, which is a contradiction. □

Exercise 1.1.9

Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsa-Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Proof. Consider $S^2 \subset \mathbb{R}^3$. Let v be a vector in S^2 and consider its span which I also denote by v. For any scalar p, there is a normal plane of v that passes through pv. In particular, there is a continuous collection of planes that slices through A_i for $i = 1, 2, 3$. Define a measure of volume in $\mathbb{R}^3$. Such a measure must be continuous so that the intermediate value theorem implies that there exists one such p_i for which the normal plane at $p_i v$ slices A_i in half by volume.

This is because as p increases in $\mathbb{R}$,

$$\text{Vol}\left(A \cap \text{lower of half of } \mathbb{R}^3 \text{ bounded by the normal plane}\right)$$

increases and eventually attains full volume (full volume is finite since A_i is compact) so that we can apply IVT.

Doing this for every vector v in $\mathbb{R}^3$, we obtain a function $f : S^2 \to \mathbb{R}^2$ defined by $f(v) = (p_1 - p_3, p_2 - p_3)$. By Borsak-Ulam theorem, there exists $v \in S^2$ such that $f(v) = f(-v)$ (Showing continuity of f is hard!). In other words, we have that $(p_1 - p_3, p_2 - p_3) = (p_3 - p_1, p_3 - p_2)$. This implies that $p_1 = p_2 = p_3$. But this means that the hyperplane at $p_1 v = p_2 v = p_3 v$ cuts through all A_1, A_2, A_3 and thus we conclude. $\square$

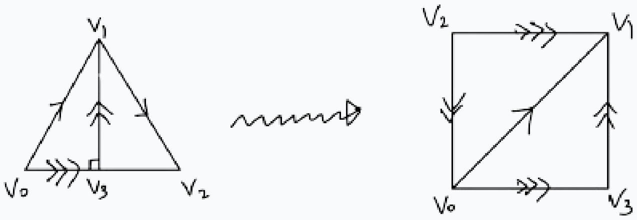
2 Homology

2.1 Simplicial and Singular Homology

Exercise 2.1.1

What familiar space is the quotient Δ-complex of a 2-simplex $[v_0, v_1, v_2]$ obtained by identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$, preserving the order of vertices?

Proof. By cutting through a straight line from v_1 down to the face $[v_0, v_2]$ perpendicularly, we can glue it back together according to the identification $[v_0, v_1] \sim [v_1, v_2]$ to obtain the following.

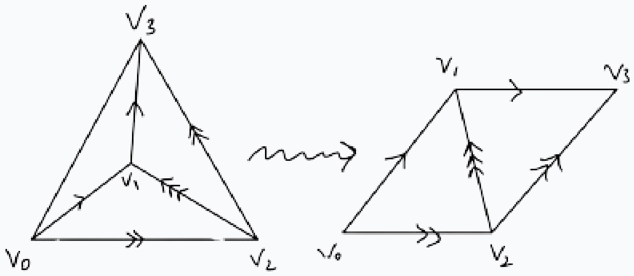


It is then clear that this is a Möbius strip. □

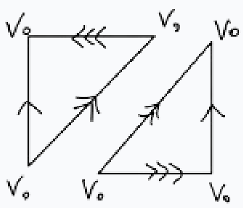
Exercise 2.1.2

Show that the Δ-complex obtained from Δ^3 by performing the edge identifications $[v_0, v_1] \sim [v_1, v_3]$ and $[v_0, v_2] \sim [v_2, v_3]$ deformation retracts onto a Klein bottle. Find other pairs of identifications of edges that produce Δ-complexes deformation retracting onto a torus, a 2-sphere and $\mathbb{RP}^2$.

Proof. Notice that the face $[v_0, v_1, v_3]$ can be deformation retracted onto the union of $[v_0, v_1]$ and $[v_1, v_3]$ so that Δ^3 is now a square with two faces $[v_0, v_2, v_1]$ and $[v_2, v_1, v_3]$ as follows



Then by cutting along the 1-simplex $[v_1, v_2]$ an gluing it back up along the identified edges $[v_0, v_2]$ and $[v_2, v_3]$, we obtain the following



It is then clear that this is precisely the Δ-complex structure of the Klein bottle.

For the torus, the identification is $[v_0, v_1] \sim [v_2, v_3]$ and $[v_0, v_2] \sim [v_1, v_3]$ and then deformation retracting the same face $[v_0, v_1, v_3]$ onto the union of $[v_0, v_1]$ and $[v_1, v_3]$. No edges should be identified to form the 2-sphere since Δ^3 is already homeomorphic to the sphere. Finally, $\mathbb{RP}^2$ is obtained by identifying $[v_0, v_2] \sim [v_3, v_1]$ and $[v_0, v_1] \sim [v_3, v_3]$ in the order of the vertices mentioned. $\square$

Exercise 2.1.3

Construct a Δ-complex structure on $\mathbb{RP}^n$ as a quotient of a Δ-complex structure on S^n having vertices the two vectors of length 1 along each coordinate axis in $\mathbb{R}^{n+1}$

Proof. Write the unit vectors of $\mathbb{R}^{n+1}$ by $e_0, \dots, e_n$ and its negatives by $-e_0, \dots, -e_n$. A Δ-complex structure of S^n can be obtained as follows. The n-simplexes for $0 \leq k \leq n$ are the simplexes of the form $[\pm e_0, \dots, \pm \hat{e}_i, \dots, \pm e_n]$ where the hat means that $\hat{e}_i$ is omitted so that it indeed denotes an n-simplex. The k-simplexes are then precisely the boundaries of the $(k+1)$-simplexes for $0 \leq k \leq n-1$ and the 0-simplexes are then precisely the points $\pm e_0, \dots, \pm e_n$. For $[(-1)^{a_0} e_0, \dots, (-1)^{a_n} e_n]$ an n-simplex, identify it with

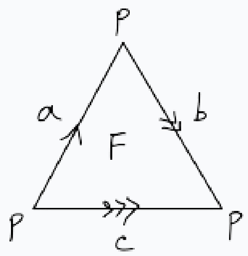
$$[(-1)^{a_0+1} e_0, \dots, (-1)^{a_n+1} e_n]$$

This identification on S^n gives precisely $\mathbb{RP}^2$ because for any point v on the n-simplex, it is identified with $-v$. $\square$

Exercise 2.1.4

Compute the simplicial homology groups of the triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.

Proof. The triangular parachute has a Δ complex structure with one 0-simplex p, three 1-simplexes a, b, c with boundary the only simplex, and 2-simplex with boundary $b - c + a$ given by the following picture:



This means that we have the following chain complex:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

The matrix of d_1 and d_2 are given by

$$d_1 = 0 \quad \text{and} \quad d_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

respectively. Then the Smith normal form of d_2 is just $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and thus the homology groups are given as

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^2 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

Exercise 2.1.5

Compute the simplicial homology groups of the Klein bottle using the Δ-complex structure described at the beginning of this section.

Proof. The Δ-complex structure of the Klein bottle described consists of one 0-simplex v, three 1-simplexes a, b, c and two 2-simplexes U and L. This means that this gives its simplicial chain complex as

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0$$

Since there is only one 0-simplex, every 1-simplex has boundary $v - v = 0$ and so d_1 is the zero map. From the Δ-complex structure, the boundary of U can be seen to be oriented by a, b and c. So we have that $d_2(U) = b - c + a$. Similarly, we have that $d_2(L) = a - b + c$. Thus the matrix of the map $d_2 : \mathbb{Z}^2 \to \mathbb{Z}^3$ can be written as

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

where SNF mean the Smith normal form. It is easy to see that $\ker(d_2) = 0$ and $\operatorname{im}(d_2) \cong \mathbb{Z} \oplus 2\mathbb{Z}$. Thus we have that

$$H_n(K) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

□

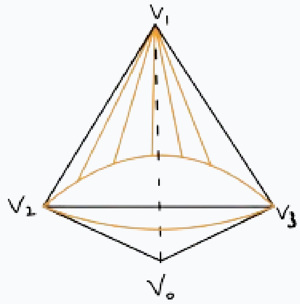
Exercise 2.1.6

Compute the simplicial homology groups of the Δ-complex obtained from $n + 1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.

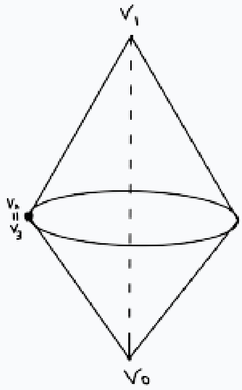
Exercise 2.1.7

Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology groups of this Δ-complex.

Proof. Identify the faces of Δ^3 via $[v_0, v_1, v_2] \sim [v_0, v_1, v_3]$ and $[v_0, v_2, v_3] \sim [v_1, v_2, v_3]$. The first identification can be visualized as follows: hold the two points v_2 and v_3 of Δ^3 and move it along a circular arc, thinking of $[v_2, v_3]$ as the diameter.



Gluing the faces, together, we obtain the following:



This is homeomorphic to D^3 by thinking of v_1 and v_0 as the north and south poles respectively, and then inflating it. Now it is known that

$$S^3 \cong \frac{D^3}{\sim}$$

where $v \sim -v$ for $v \in \partial D$. Then the identification of $[v_0, v_2, v_3] \sim [v_1, v_2, v_3]$ in Δ^3 is precisely taking D^3 and then defining the same equivalence relation on $\partial D^3 \cong S^2$. Thus we know have Δ^3 being a Δ-complex structure on S^3.
To compute the homology groups, notice that there are now two 0-simplexes $[v_0] \sim [v_1]$ and $[v_2] \sim [v_3]$. There are three 1-simplexes $[v_2, v_2 = v_3]$, $[v_1, v_2] \sim [v_0, v_2] \sim [v_1, v_3] \sim [v_0, v_3]$ and $[v_0, v_0 = v_1]$. There are two 2-simplexes $[v_0, v_2, v_3] \sim [v_1, v_2, v_3]$ and $[v_0, v_1, v_2] \sim [v_0, v_1, v_3]$ and one 3-simplex $[v_0, v_1, v_2, v_3]$. The simplicial chain complex is thus

$$0 \longrightarrow \mathbb{Z}[v_0, v_1, v_2, v_3] \longrightarrow \mathbb{Z}[v_0, v_1, v_2] \oplus \mathbb{Z}[v_0, v_2, v_3] \longrightarrow \mathbb{Z}[v_0, v_1] \oplus \mathbb{Z}[v_1, v_2] \oplus \mathbb{Z}[v_2, v_3] \longrightarrow \mathbb{Z}[v_0] \oplus \mathbb{Z}[v_2] \longrightarrow 0$$

The boundary maps can be understood as follows. The first boundary map can be computed as

$$\begin{aligned} d_1([v_0, v_1]) &= [v_1] - [v_0] = 0 \\ d_1([v_1, v_2]) &= [v_2] - [v_1] = [v_2] - [v_0] \\ d_1([v_2, v_3]) &= [v_3] - [v_2] = 0 \end{aligned}$$

Then the second boundary map:

$$\begin{aligned} d_2([v_0, v_1, v_2]) &= [v_1, v_2] - [v_0, v_2] + [v_0, v_1] = [v_1, v_2] - [v_1, v_2] + [v_0, v_1] = [v_0, v_1] \\ d_2([v_0, v_2, v_3]) &= [v_2, v_3] - [v_0, v_3] + [v_0, v_2] = [v_2, v_3] \end{aligned}$$

And the last one:

$$d_3([v_0, v_1, v_2, v_3]) = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2] = 0$$

We can write them in to a matrix so that

$$d_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\text{SNF}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then the homology groups can be read off as

$$H_n(S^3) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 3 \\ 0 & \text{otherwise} \end{cases}$$

which one can check is precisely the singular homology of S^3. □

Exercise 2.1.8

Compute the homology groups of the Δ-complex X obtained from Δ^n by identifying all faces of the same dimension. Thus X has a single k-simplex for each $k \leq n$.

Proof. It is clear that the simplicial chain complex of X is just one copy of $\mathbb{Z}$ on dimensions $0, 1, \dots, n$. We have to understand the boundary maps. For a k-simplex $[v_0, \dots, v_k]$, we have the formula

$$d_k([v_0, \dots, v_k]) = \sum_{i=0}^{k} (-1)^i \sigma[v_0, \dots, \hat{v}_i, \dots, v_k]$$

In X, there is only one $(k-1)$-simplex so the formula becomes

$$d_k([v_0, \dots, v_k]) = \begin{cases} 0 & \text{if } 0 \leq k \leq n \text{ is odd} \\ [v_0, \dots, \hat{v}_i, \dots, v_k] & \text{if } 0 \leq k \leq n \text{ is even} \end{cases}$$

This means that d_k is either the identity or the zero map. When $0 < k < n$ is even, we have that

$$H_k(X) = \frac{\ker(d_k)}{\operatorname{im}(d_{k+1})} = \frac{\ker(\mathrm{id})}{\operatorname{im}(0)} \cong 0$$

When $0 < k < n$ is odd, we have that

$$H_k(X) = \frac{\ker(d_k)}{\operatorname{im}(d_{k+1})} = \frac{\ker(0)}{\operatorname{im}(\mathrm{id})} \cong 0$$

When $k = 0$, we have $H_0(X) = \mathbb{Z}$. When $k = n$ and n is odd, we have that $H_n(X) = \mathbb{Z}$. When $k = n$ and n is even, we have that $H_n(X) = 0$. □