

# Algebraic Curves

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**Abstract**

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# 1 Algebraic Curves in Classical Algebraic Geometry

## 1.1 Basic Properties of Curves

### Definition 1.1.1: Curves

Let  $k$  be a field. Let  $X$  be a variety over  $k$ . We say that  $X$  is a curve if  $\dim(X) = 1$ .

### Proposition 1.1.2

Let  $k$  be an algebraically closed field. Let  $C$  be an irreducible curve over  $k$ . Let  $p \in C$  be a non-singular point. Then  $\mathcal{O}_{C,p}$  is a DVR. Moreover, the valuation is given by the degree of the regular function.

*Proof.* Since  $p$  is non-singular, by definition  $\mathcal{O}_{C,p}$  is a regular local ring. Moreover, we know that  $1 = \dim(C) = \dim(\mathcal{O}_{C,p})$  so that  $\mathcal{O}_{C,p}$  has Krull dimension 1. By the equivalent characterization of DVR we conclude.  $\square$

We denote the valuation map by  $v_p : \text{Frac}(\mathcal{O}_{C,p}) \rightarrow \mathbb{Z}$ .

### Example 1.1.3

Consider the projective curve  $C = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{P}_{\mathbb{C}}^2$ . Let  $p = [p_0 : p_1 : p_2]$  be a point on the curve.

If  $p_2 \neq 0$ , then  $p \in U_2$ . Under the affine chart  $(U_2, \varphi_2)$ , we find that  $C_2 = \varphi_2(C \cap U_2) = \mathbb{V}(x^2 + y^2 - 1)$ . The corresponding coordinate ring is given by  $\frac{\mathbb{C}[x,y]}{(x^2+y^2-1)}$ . The formula for the local ring in the affine case gives

$$\mathcal{O}_{C,p} \cong \left( \frac{\mathbb{C}[x,y]}{(x^2 + y^2 - 1)} \right)_{m_{(p_0/p_2, p_1/p_2)}}$$

Recall that the unique maximal ideal of the local ring is given as the  $\mathcal{O}_{X,p}$ -module  $m_p = \{f \in \mathbb{C}[C_2] \mid f(p_0/p_2, p_1/p_2) = 0\}$ , which under the nullstellensatz is the maximal ideal corresponding to the point  $(p_0/p_2, p_1/p_2)$  and is given by  $m_p = (x - r, y - s)$  where  $r = p_0/p_1$  and  $s = p_0/p_2$ . By Nakayama's lemma, since  $x - r, y - s$  generate  $m_p$  we know that  $x - r + m_p^2, y - s + m_p^2$  span the vector space  $m_p/m_p^2$  over  $\mathcal{O}_{X,p}/m_p$ . I claim that they are linearly dependent. This means that I want to find  $f + m_p^2$  and  $g + m_p^2$  in  $\mathcal{O}_{X,p}/m_p$  that are non-trivial, and that  $(x - r)f + (y - s)g + m_p^2 = m_p^2$ . This means that we want to find  $f, g \in \mathcal{O}_{X,p} \setminus m_p$  such that  $(x - r)f + (y - s)g \in m_p^2$ . Choose  $f = x + r$  and  $g = y + s$  to get

$$(x - r)(x + r) + (y - s)(y + s) = x^2 - r^2 + y^2 - s^2 = 1 - 1 = 0$$

since  $(r, s)$  lie on the curve. Moreover,  $x + r, y + s \in \mathcal{O}_{X,p} \setminus m_p$  since evaluating at  $(r, s)$  at the functions are non-zero. This verifies that  $\mathcal{O}_{X,p}$  is a regular local ring of dimension 1, hence is a DVR.

We can even find its uniformizer and valuation. Since  $x - r$  and  $y + s$  are linearly dependent and spans  $m_p/m_p^2$ , any one of the two is a basis for the vector space. WLOG take  $x - r$  to be a basis. Nakayama's lemma implies that  $x - r$  generates  $m_p$ . Being a DVR means that for all  $f \in \mathcal{O}_{X,p}$ ,  $f = u(x - r)^n$  where  $u$  is invertible. Then the valuation of  $f$  is  $n$ .

### Proposition 1.1.4

Let  $C$  be an affine irreducible curve over  $\mathbb{C}$ . Then  $C$  is smooth if and only if  $C$  is a normal variety.

## 1.2 Morphisms between Curves

### Proposition 1.2.1

Let  $k$  be a field. Let  $C$  be a smooth curve over  $k$ . Then for any projective variety  $X \subseteq \mathbb{P}^n$  and rational map  $\phi : C \rightarrow X$ , there exists a regular map

$$\bar{\phi} : C \rightarrow X$$

such that  $\bar{\phi}|_U = \phi|_U$  for some dense subset  $U \subseteq C$ .

### Proposition 1.2.2

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Then  $\phi$  is a finite morphism.

### Proposition 1.2.3

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a rational map. If  $\phi$  is birational, then  $\phi$  is an isomorphism of varieties.

## 1.3 Blowing Up Curves and Normalization

Recall that by taking the integral closure of the coordinate ring  $k[C]$  of an irreducible affine curve  $C \subseteq \mathbb{A}^n$ , we obtain a corresponding variety  $\tilde{C}$  called the normalization of  $C$ .

### Proposition 1.3.1

Let  $k$  be an algebraically closed field. Let  $C \subseteq \mathbb{A}_k^n$  be an irreducible affine curve over  $k$ . Then the normalization  $\tilde{C}$  is smooth.

### Theorem 1.3.2

Let  $k$  be an algebraically closed field. Let  $C$  be an irreducible curve over  $k$ . Then  $C$  is birational to a unique non-singular projective irreducible curve.

## 1.4 Ramification Index

### Definition 1.4.1: Ramification Index

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Let  $p \in X$ . Define the ramification index of  $\phi$  at  $p$  to be

$$e_\phi(p) = v_p(\phi^*(\pi))$$

where  $\pi$  is a uniformizing parameter of  $\mathcal{O}_{Y, \phi(p)}$ .

### Lemma 1.4.2

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Let  $p \in X$ . Then

$$e_\phi(p) = \dim_k \left( \frac{\mathcal{O}_{X,p}}{(\phi^*(\pi))} \right)$$

where  $\pi$  is a uniformizing parameter of  $\mathcal{O}_{Y, \phi(p)}$ .

Let  $\phi : X \rightarrow Y$  be a non-constant regular map between smooth irreducible and projective curves. Since  $\phi$  is finite, the notion of degree makes sense. Recall that the degree is defined to be

$$\deg(\phi) = \dim_{K(Y)} K(X)$$

### Proposition 1.4.3

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Let  $q \in Y$ . Then we have

$$\sum_{p \in \phi^{-1}(q)} e_\phi(p) = \deg(\phi)$$

## 1.5 Differential Forms on Curves

### Proposition 1.5.1

Let  $C$  be a smooth irreducible curve over  $\mathbb{C}$ . Then  $\Omega_{\mathbb{C}(C)/\mathbb{C}}^1$  is a 1-dimensional vector space over  $\mathbb{C}(C)$ .

### Proposition 1.5.2

Let  $C$  be a smooth irreducible curve over  $\mathbb{C}$ . Let  $f \in \mathbb{C}(C)$  be non-constant. Then the following are true.

- $df \neq 0$  in  $\Omega_{\mathbb{C}(C)/\mathbb{C}}^1$ .
- $df$  is a  $\mathbb{C}(C)$ -basis for  $\Omega_{\mathbb{C}(C)/\mathbb{C}}^1$ .

### Definition 1.5.3: Valuation of Differential 1-Forms

Let  $C$  be a smooth irreducible curve over  $\mathbb{C}$ . Let  $p \in C$ . Let  $\omega \in \Omega_{\mathbb{C}(C)/\mathbb{C}}^1$  be a differential 1-form of  $C$ . Define the valuation of  $\omega$  at  $p$  as follows. Choose a uniformizing parameter  $\pi \in \mathcal{O}_{C,p}$ . Write  $\omega = fd\pi$  for  $f \in \mathbb{C}(C)$ . Then define the valuation as

$$\text{val}_p(\omega) = \text{val}_p(f)$$

## 2 Classical Divisors on Curves

### 2.1 The Pullback Map of Divisors

#### Definition 2.1.1: Pullback Map of Divisors

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Define the induced pullback map  $\phi^* : \text{Div}(Y) \rightarrow \text{Div}(X)$  by

$$\phi^* \left( \sum_{q \in Y} n_q \cdot q \right) = \sum_{q \in Y} n_q \cdot \left( \sum_{p \in \phi^{-1}(q)} e_\phi(p) \cdot p \right) = \sum_{p \in X} n_{\phi(p)} e_\phi(p) \cdot p$$

#### Proposition 2.1.2

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Then we have

$$\deg(\phi^*(D)) = \deg(\phi) \deg(D)$$

for any  $D \in \text{Div}(Y)$ .

#### Proposition 2.1.3

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D \in \text{Div}(X)$  be a principal divisor of  $X$ . Then  $\deg(D) = 0$ .

#### Proposition 2.1.4

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Then  $\phi(\text{Prin}(Y)) \subseteq \text{Prin}(X)$ .

#### Definition 2.1.5: Induced Map of Divisor Class Groups

Let  $k$  be an algebraically closed field. Let  $X, Y$  be smooth irreducible projective curves over  $k$ . Let  $\phi : X \rightarrow Y$  be a non-constant regular map. Define the induced map of divisor class groups  $\phi^* : \text{Cl}(Y) \rightarrow \text{Cl}(X)$  by

$$\phi^*([D]) = [\phi^*(D)]$$

### 2.2 The Linear System of Divisors

#### Definition 2.2.1: The Linear System of Divisors

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D \in \text{Div}(X)$  be a divisor. Define the linear system of  $D$  to be

$$\mathcal{L}(D) = \{0\} \cup \{f \in K(X) \mid \deg(D + \text{div}(f)) \geq 0\} \subseteq K(X)$$

#### Lemma 2.2.2

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D \in \text{Div}(X)$  be a divisor. Then  $\mathcal{L}(D)$  is a vector space over  $k$ .

**Proposition 2.2.3**

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D, D' \in \text{Div}(X)$  be divisors. If  $D \sim D'$  are linearly equivalent, then we have

$$\dim_k(\mathcal{L}(D)) = \dim_k(\mathcal{L}(D'))$$

**Proposition 2.2.4**

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D \in \text{Div}(X)$  be a divisor. Then the following are true.

- If  $\deg(D) < 0$ , then we have

$$\dim_k(\mathcal{L}(D)) = 0$$

- If  $\deg(D) = 0$ , then we have

$$\dim_k(\mathcal{L}(D)) = \begin{cases} 0 & \text{if } D \not\sim 0 \\ 1 & \text{if } D \sim 0 \end{cases}$$

**Proposition 2.2.5**

Let  $k$  be an algebraically closed field. Let  $X$  be a smooth irreducible projective curve over  $k$ . Let  $D \in \text{Div}(X)$  be a divisor. Then we have

$$\dim_k(\mathcal{L}(D)) \leq \deg(D) + 1$$

**2.3 The Canonical Divisor for Curves****Definition 2.3.1: Divisors of Differential Forms**

Let  $C$  be a smooth irreducible curve over  $\mathbb{C}$ . Let  $p \in C$ . Let  $\omega \in \Omega_{\mathbb{C}(C)/\mathbb{C}}^1$  be a differential 1-form of  $C$ . Define the divisor of  $\omega$  by

$$\text{div}(\omega) = \sum_{p \in C} \text{val}_p(\omega) \cdot p \in \text{Div}(C)$$

**Proposition 2.3.2**

Let  $C$  be a smooth irreducible curve over  $\mathbb{C}$ . Let  $p \in C$ . Let  $\omega, \tau \in \Omega_C^1$  be non-zero. Then  $\text{div}(\omega)$  and  $\text{div}(\tau)$  are linearly equivalent.

**Definition 2.3.3: The Canonical Divisor**

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Let  $p \in C$ . Define the canonical divisor of  $C$  to be

$$K_C = [\omega] \in \text{Pic}(C)$$

in the divisor class group for any non-zero  $\omega \in \Omega_C^1$ .

**Lemma 2.3.4**

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Then

$$\dim_{\mathbb{C}}(\mathcal{L}(K_C)) = \dim_{\mathbb{C}}(\Omega_C^1)$$

## 2.4 The Riemann-Roch Theorem

### Theorem 2.4.1: Riemann-Roch Theorem

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Let  $D \in \text{Div}(C)$  be a divisor on  $C$ . Then

$$\dim_{\mathbb{C}}(\mathcal{L}(D)) + \dim_{\mathbb{C}}(\mathcal{L}(K_C - D)) = \deg(D) + 1 - p_g(C)$$

### Proposition 2.4.2

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Let  $D \in \text{Div}(C)$  be a divisor on  $C$ . Then

$$\deg(D) + 1 - p_g(C) \leq \dim_{\mathbb{C}}(\mathcal{L}(D)) \leq \deg(D) + 1$$

### Proposition 2.4.3

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Then we have

$$\deg(K_C) = 2p_g(C) - 2$$

### Proposition 2.4.4

Let  $C$  be a smooth irreducible projective curve over  $\mathbb{C}$ . Then the following are equivalent.

- $C$  is isomorphic to  $\mathbb{P}^1$ .
- The geometric genus  $p_g(C) = 0$  is zero.
- For all  $p, q \in C$ ,  $p \sim q$  are linearly equivalent.
- There exists distinct  $p, q \in C$ , such that  $p \sim q$  are linearly equivalent.
- The degree map  $\deg : \text{Pic}(C) \rightarrow \mathbb{Z}$  is an isomorphism.
- For all  $D \in \text{Div}(C)$  with  $\deg(D) > 0$ , we have  $l(D) = \deg(D) + 1$ .
- There exists  $D \in \text{Div}(C)$  with  $\deg(D) > 0$  such that  $l(D) = \deg(D) + 1$ .



### 3 Algebraic Curves in the Context of Schemes

#### Definition 3.0.1: Algebraic Curves

Let  $k$  be an algebraically closed field. A curve over  $k$  is an integral separated scheme  $X$  of finite type over  $k$  that has dimension 1.

#### Proposition 3.0.2

Let  $X$  be an algebraic curve. Then the arithmetic and geometric genus coincide. In particular,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

We will simply call the genus of a curve  $g$  from now on since the arithmetic genus is the same as the geometric genus.

#### 3.1 Riemann-Roch Theorem

##### Definition 3.1.1: Canonical Divisor

Let  $X$  be an algebraic curve. The canonical divisor  $K$  of  $X$  is a divisor in the linear equivalence class of

$$\Omega_{X/k}^1 = \omega_X$$

##### Theorem 3.1.2: Riemann-Roch Theorem

Let  $X$  be an algebraic curve. Let  $D$  be a divisor on  $X$  and let  $K$  be the canonical divisor of  $X$ . Let  $\mathcal{L}(D)$  be the associated sheaf of the divisor  $D$ . Then

$$\dim_k(H^0(X, \mathcal{L}(D))) + \dim_k(H^0(X, \mathcal{L}(K - D))) = \deg(D) + 1 - p_g(X)$$

#### 3.2 Classification of Curves in $\mathbb{P}^3$