Stable Homotopy Theory

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Abstract

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1 The Category of Spectra

Recall the reduced suspension ΣX of a space X.

1.1 Spectra and Ω-Spectra

Definition 1.1.1: Spectra

A spectrum E is a collection $\{(E_n, *) \mid n \in \mathbb{Z}\}$ of pointed spaces together with continuous maps $e_n : \Sigma E_n \to E_{n+1}$.

By the adjunction formula between Σ and Ω, we can reformulate the map e_n to mean $e_n : E_n \to \Omega E_{n+1}$.

Definition 1.1.2: The Suspension Spectrum

Let X be a compactly generated space. Define the suspension spectrum of X to consist of the following data.

- The collection $\{\Sigma^n X \mid n \in \mathbb{N}\}$ of spaces.
- The collection $\sigma_n : \Sigma(\Sigma^n X) \to \Sigma^{n+1} X$ of maps which is a homeomorphism.

Definition 1.1.3: The CW Spectrum

The CW spectrum E is a collection $\{E_n \mid n \in \mathbb{Z}\}$ of CW-complexes with a chosen basepoint together with maps $e_n : \Sigma E_n \to E_{n+1}$ so that ΣE_n is recognized as a subcomplex of E_{n+1}.

Lemma 1.1.4

The suspension spectrum and the CW spectrum for the n-sphere S^n

Definition 1.1.5: Ω-Spectra

Let $\{E_n \mid n \in \mathbb{Z}\}$ and $e_n : E_n \to \Omega E_{n+1}$ be a spectra. We say that it is an Ω-spectra if the induced map $(e_n)_*$ is a weak homotopy equivalence.

Proposition 1.1.6

Let G be an abelian group. Then there is an isomorphism

$$\Omega K(G, n) \cong K(G, n-1)$$

Definition 1.1.7: Eilenberg-MacLane Spectrum

Let X be a space. Define the Elienberg-Maclane spectrum

1.2 Brown's Representability Theorem

Given a spectrum, we can construct a (co)homology theory. The converse is given by Brown's representability theorem.

Theorem 1.2.1

Let $\{T_n \mid n \in \mathbb{Z}\}$ be a CW spectrum such that T_n is $(n-1)$-connected. Define

$$\widetilde{E}_k(X) = \operatorname*{colim}_{n \to \infty} \pi_{k+n}(X \wedge T_n)$$

Then the functors $\widetilde{E}_k$ for all k defines a reduced homology theory on CW complexes with base point.

Theorem 1.2.2

Let $\{T_n \mid n \in \mathbb{Z}\}$ be a Ω-spectrum consisting of CW complexes. For any space X, define

$$\widetilde{E}_k(X) = [X, T_k]$$

for $k \in \mathbb{Z}$. Then the functors $\widetilde{E}_k$ for all k defines a reduced cohomology theory on CW complexes with base point.

Definition 1.2.3: Homology with Coefficients in a Spectrum

Let $\mathbb{K} = \{T_n \mid n \in \mathbb{Z}\}$ be a spectrum. Define a functor $H_n(-;\mathbb{K}) : \mathbf{CW}^2 \to \mathbf{Ab}$ by

$$H_n(X, A; \mathbb{K}) = \lim_{k\to\infty} \pi_{n+k}\left(\frac{X_+}{A_+} \wedge T_k\right)$$

where X_+ is the space X together with a chosen base point.

Theorem 1.2.4: Brown's Representability Theorem I

Let (h_n, δ_n) be a generalized homology theory. Then there exists a spectrum $\mathbb{K}$ and a natural isomorphism

$$H_n(X, A) \cong H_n(X, A; \mathbb{K})$$

for all CW pairs (X, A).