Exercise Sheet for Week 2

Question 1

Find the radius of convergence of the following power series:

- $\sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\sum_{k=0}^{\infty} \frac{x^k}{r^k}$ for some $r \in \mathbb{R}$.

Question 2

Consider the power series $\sum_{k=0}^{\infty} a_k x^k$ for $a_k \in \mathbb{R}$. Prove (it is hard) that the radius of convergence of the power series is given by

$$r = \lim_{n\to\infty} \left(\inf \left\{ |x_m|^{-1/m} : m \geq n \right\} \right)$$

Check that this formula works for all the power series you have seen.

Upshot: A hack that you are not allowed to use in assignments or exams unless they taught you this in the notes (you may use it for sanity check).

Question 3

Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series. Let $L = \lim_{n\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$. Suppose that L is finite and non-zero. Show that if $t \in \mathbb{R}$ is such that $|t| < \frac{1}{L}$, then the infinite sum $\sum_{k=0}^{\infty} a_k t^k$ converges.

Use the above to show that the power series $\sum_{k=0}^{\infty} kx^k$ has radius of convergence $R = 1$.

Question 4

Consider the equation $y^2 = x^3 + x + 10$ defined for the region $y \geq 0$. Find a parameterization of the curve. Now parameterize it for the case when $y \leq 0$. Piece the two to find a parameterization of the entire curve $y^2 = x^3 + x + 10$, making sure that it is continuous.

Upshot: Not all curves are paramterized by a single formula.

Question 5

Find two different parametrizations of the standard parabola $y = x^2$. Can you write an expression relating the two parametrizations?

Upshot: First question says that parametrization is not unique. The second question is the same as asking for a reparametrization from one curve to another.

Question 6

Find a parametrization for the intersection of the two planes whose equations are given by $x^2 + y^2 = 4$ and $z = xy$.

Question 7

7(a): Consider the following three vectors of $\mathbb{R}^3$: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Prove that no vector is a linear combination of the other two. We call these vectors linearly independent from each other.

7(b): Now consider the same set of vectors. Consider the set of all possible linear combinations of the three vectors:

$$S = \left\{ a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

Show that $S = \mathbb{R}^3$. We say that these vectors span $\mathbb{R}^3$.

7(c): Now choose your favourite vector out of the original three vectors, and take it away. Show that the remaining two vectors no longer span $\mathbb{R}^3$.