

# Group Cohomology

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**Abstract**

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# 1 Introduction to Group Homology and Cohomology

## 1.1 G-Modules

### Definition 1.1.1: G-Modules

Let  $G$  be a group. A  $G$ -module is an abelian group  $A$  together with a group action of  $G$  on  $A$ .

### Definition 1.1.2: Morphisms of G-Modules

Let  $G$  be a group. Let  $M$  and  $N$  be  $G$ -modules. A function  $f : M \rightarrow N$  is said to be a  $G$ -module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all  $m \in M$  and  $g \in G$ .

## 1.2 Invariants and Coinvariants

### Definition 1.2.1: The Group of Invariants

Let  $G$  be a group and let  $M$  be a  $G$ -module. Define the group of invariants of  $G$  in  $M$  to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of  $M$  for which  $G$  acts trivially.

### Definition 1.2.2: Functor of Invariants

Let  $G$  be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \rightarrow \mathbf{Ab}$$

as follows.

- For each  $G$ -module  $M$ ,  $M^G$  is the group of invariants
- For each morphism  $f : M \rightarrow N$  of  $G$ -modules,  $f^G : M^G \rightarrow N^G$  is the restriction of  $f$  to  $M^G$ .

### Theorem 1.2.3

Let  $G$  be a group. The functor of invariants  $(-)^G : {}_G\mathbf{Mod} \rightarrow \mathbf{Ab}$  is left exact.

### Definition 1.2.4: The Group of Coinvariants

Let  $G$  be a group and let  $M$  be a  $G$ -module. Define the group of coinvariants of  $G$  in  $M$  to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M \rangle}$$

This is the largest quotient of  $M$  for which  $G$  acts trivially.

### 1.3 Group Cohomology and its Equivalent Forms

#### Definition 1.3.1: The $n$ th Cohomology Group

Let  $G$  be a group. Define the  $n$ th cohomology group of  $G$  with coefficients in a  $G$ -module  $M$  to be

$$H_n(G; M) = (L_n(-)_G)(M)$$

the  $n$ th left derived functor of  $(-)_G : {}_G\mathbf{Mod} \rightarrow \mathbf{Ab}$ .

#### Theorem 1.3.2

Let  $G$  be a group and let  $M$  be a  $G$ -module. Then there is an isomorphism

$$H^n(G; M) \cong \mathrm{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M)$$

that is natural in  $M$ .

Recall that there are two descriptions of  $\mathrm{Ext}$  by considering it as a functor of the first or second variable. Since the above theorem exhibits an isomorphism that is natural in the second variable, let us consider  $\mathrm{Ext}$  as the right derived functor of the functor  $\mathrm{Hom}_{\mathbb{Z}[G]}(-, M)$  applied to  $\mathbb{Z}$  as a  $\mathbb{Z}[G]$ -module.

#### Proposition 1.3.3

Let  $G$  be a group and let  $M$  be a  $G$ -module. Let  $P_\bullet \rightarrow \mathbb{Z}$  be a projective resolution of  $\mathbb{Z}$  with  $\mathbb{Z}[G]$ -modules. Then there is an isomorphism

$$H^n(G; M) \cong H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet, M))$$

that is natural in  $M$ .

For any group  $G$ , there is always the trivial choice of projective resolution. In the following lemma, we use the notation  $(g_0, \dots, \hat{g}_i, \dots, g_n)$  as a shorthand for writing the element in  $G^n$  but with the  $i$ th term omitted.

#### Lemma 1.3.4

Let  $G$  be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

where  $f_n : \mathbb{Z}[G^{n+1}] \rightarrow \mathbb{Z}[G^n]$  is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^n (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of  $\mathbb{Z}$  lying in  ${}_{\mathbb{Z}[G]}\mathbf{Mod}$ .

Let  $A$  be an  $R$ -algebra and let  $M$  be an  $A$ -module. Recall that the bar resolution is defined to be the chain complex consisting of  $M \otimes A^{\otimes n}$  for each  $n \in \mathbb{N}$  together with the boundary maps defined by multiplying the  $i$ th element to the  $i + 1$ th element. Now let  $G$  be a group. By considering  $\mathbb{Z}[G]$  as a  $\mathbb{Z}$ -algebra and that and ring is a module over itself, it makes sense to talk about the bar resolution of  $\mathbb{Z}[G]$ .

#### Theorem 1.3.5

Let  $G$  be a group. Consider the bar resolution

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \longrightarrow \mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

of  $\mathbb{Z}[G]$ . Then it is a free resolution, and hence a projective resolution of  $\mathbb{Z}$  with  $\mathbb{Z}[G]$ -modules.

Thus, given a group  $G$  and a  $G$ -module  $M$ , the group cohomology of  $G$  with coefficients in  $M$  can be thought of in the following way:

- It is the right derived functor of the functor of invariants  $(-)^G : {}_G\mathbf{Mod} \rightarrow \mathbf{Ab}$
- It is the extension group  $\mathrm{Ext}_{\mathbb{Z}[G]}^n(\mathbb{Z}, M)$  (which is computable by the obvious projective resolution  $\mathbb{Z}[G^\bullet]$ , or the bar resolution)

## 1.4 Group Homology and its Equivalent Forms

### Definition 1.4.1: The $n$ th Cohomology Group

Let  $G$  be a group. Define the  $n$ th cohomology group of  $G$  with coefficients in a  $G$ -module  $M$  to be

$$H^n(G; M) = (R^n(-)^G)(M)$$

the  $n$ th right derived functor of  $(-)^G : {}_G\mathbf{Mod} \rightarrow \mathbf{Ab}$ .

### Theorem 1.4.2

Let  $G$  be a group and let  $M$  be a  $G$ -module. Then there is an isomorphism

$$H_n(G; M) \cong \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

that is natural in  $M$ .

## 1.5 Low Degree Interpretations

### Theorem 1.5.1

Let  $G$  be a group and let  $M$  be a  $G$ -module. Then there are natural isomorphisms

$$H^0(G, M) = M^G \quad \text{and} \quad H_0(G; M) = M_G$$

### Theorem 1.5.2

Let  $G$  be a group and let  $M$  be a  $G$ -module. Then there is an isomorphism

$$H_1(G, M) \cong \frac{G}{[G, G]} = G_{\mathrm{ab}}$$

### Theorem 1.5.3

Let  $G$  be a group and let  $M$  be a trivial  $G$ -module. Then there is a natural isomorphism

$$H^1(G; M) = \frac{(\{f : G \rightarrow M \mid f(ab) = f(a) + af(b)\}, +)}{\langle f : G \rightarrow M \mid f(g) = gm - m \text{ for some fixed } m \rangle}$$

### Corollary 1.5.4

Let  $G$  be a group and let  $M$  be a trivial  $G$ -module. Then there is a natural isomorphism

$$H^1(G; M) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, M)$$