Topological Manifolds

Labix

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Abstract

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1 Point Set Topology of Topological Manifolds

1.1 Triangulation of Manifolds

Definition 1.1.1: Triangulable Manifolds

Let M be a k-manifold. We say that M is triangulable if M is a δ-complex structure consisting of a finite number of top simplicies.

1.2 Covering Spaces of Manifolds

Proposition 1.2.1

Let M be a manifold. Let $p : \tilde{M} \to M$ be a covering space. Then $\tilde{M}$ is also a manifold.

2 Orientability of a Topological Manifold

2.1 Classical Orientability

The key observation in defining orientation through homology is the following proposition, which shows that the local homology groups on a manifold are isomorphic to $\mathbb{Z}$ on the top dimension.

Proposition 2.1.1

Let M be a k-dimensional topological manifold and $x \in M$ a point. Then

$$H_n(M \mid \{x\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $x \in M$. Since M is a manifold, there exists an open neighbourhood of x such that $U \cong \mathbb{R}^k$ via some transition map $\varphi : U \to \mathbb{R}^k$. Since $M \setminus U$ is closed, we can apply excision to obtain an isomorphism

$$H_n(M \mid \{x\}) \cong H_n(U \mid \{x\})$$

By the homeomorphism $(U, U \setminus \{x\}) \cong (\mathbb{R}^k, \mathbb{R}^k \setminus \{\varphi(x)\})$ and 6.3.2 in AT2, we obtain the desired result. □

We can now consider locally what it means to have an orientation (because we excised the data to that of $\mathbb{R}^k$), and then try and glue all the choices of orientation into a coherent global orientation.

Definition 2.1.2: Local Orientation

Let M be a k-dimensional topological manifold and let $x \in M$. A local orientation of M at x is a choice of generator of

$$H_k(M \mid \{x\}) \cong \mathbb{Z}$$

Definition 2.1.3: Open and Closed Ball in Manifolds

Let M be a k-dimensional topological manifold and let (U, φ) be a chart of M. We say that $B \subset U$ is an open / closed ball if $\varphi(B) \subseteq \mathbb{R}^k$ is an open / closed ball of $\mathbb{R}^k$.

The point of the definition is that we have the following. We have homotopy equivalences

$$\begin{tikzcd} & (\mathbb{R}^k, \mathbb{R}^k \setminus B) & \\ (\mathbb{R}^k, \mathbb{R}^k \setminus \{x\}) \arrow[ur, "\simeq"] & & (\mathbb{R}^k, \mathbb{R}^k \setminus \{y\}) \arrow[ul, "\simeq"'] \end{tikzcd}$$

given by deformation retracts. If M is a k-manifold and $U \subseteq M$ is an open ball, we can use excision to obtain isomorphisms

$$\begin{tikzcd} & H_k(M \mid U) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_k(M \mid \{x\}) & & H_k(M \mid \{y\}) \end{tikzcd}$$

All of the above groups are just $\mathbb{Z}$, and a choice of local orientation is a choice of generators of the lower two homology groups. If we want the choice to be consistent, then we better have the two generators coincide to the same generator in $H_n(M \mid U)$ under the above isomorphisms. We note here that the isomorphism

$$H_n(M \mid U) \xrightarrow{\cong} H_n(M \mid \{x\})$$

for any $x \in U$ came from the inclusion map $(M, M \setminus U) \hookrightarrow (M, M \setminus \{x\})$.

Definition 2.1.4: Consistent Local Orientations

Let M be a k-manifold. Let B be an open ball in M. For each $x \in B$, let ω_x be a choice of local orientation at x. We say that the choices of local orientations at B is consistent if there exists a generator $\omega_B \in H_k(M \mid B)$ such that for any $x, y \in B$, under the isomorphisms

$$\begin{array}{ccccc} H_k(M \mid \{x\}) & \xrightarrow{\cong} & H_k(M \mid B) & \xleftarrow{\cong} & H_k(M \mid \{y\}) \\ \\ \omega_x & \longmapsto & \omega_B & \longleftarrow\!\shortmid & \omega_y \end{array}$$

the choice of local orientation maps to the same generator ω_B.

With this, we can now formally define orientations in a manifold.

Definition 2.1.5: Orientation of a Manifold

Let M be a k-dimensional topological manifold. An orientation of M is a function

$$x \mapsto \omega_x \in H_k(M, M \setminus \{x\})$$

assigning every point to a local orientation such that for every $x \in M$, there exists an open ball $x \in B$ such that $(\omega_x)_{x \in B}$ a consistent local orientation.

2.2 The Orientation Double Cover

In order to deduce orientability of a manifolds, we appeal to the theory of vector bundles.

Definition 2.2.1: Orientation Bundle

Let M be a topological manifold. Define the orientation bundle $\widetilde{M}$ to be the set of pairs

$$\widetilde{M} = \left\{ (x, \omega_x) \;\middle|\; x \in M, \omega_x \text{ is a generator of } H_k(M \mid x) \right\}$$

together the projection map $\pi : \widetilde{M} \to M$ defined by $\pi(x, \omega_x) = x$.

Definition 2.2.2: Topology on the Orientation Bundle

Let M be a topological manifold. Define the topology on the orientation bundle $\widetilde{M}$ as follows. Let B be an open ball in M. Since there are exactly two distinct orientation classes on B we have that

$$\pi^{-1}(B) = B_+ \amalg B_-$$

where B_+ and B_- are homeomorphic to B. Define the topology of $\widetilde{M}$ to be generated by sets of the form B_+ and B_-.

Lemma 2.2.3

For any topological manifold M, $\widetilde{M}$ is a manifold and is a 2-sheeted covering.

Proof. Let (x, ω_x) and (y, ω_y) in $\widetilde{M}$ be distinct. If $x = y$ then $\omega_x = -\omega_y$. We know that there are two distinct orientation classes so π^{-1} is a disjoint union consisting of those with positive orientation and those with negative. Since ω_x and ω_y are opposite, they lie in the disjoint union separately so that they are disjoint. If $x \neq y$, then since M is Hausdorff then we can choose U_1 and U_2 disjoint neighbourhoods of x and y respectively. Then this means that $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint. Thus we have shown that M is Hausdorff.

Now let $(x, \omega_x) \in \widetilde{M}$. Then since M is manifold, there is an open ball B around x so that B is homeomorphic to $\mathbb{R}^k$. $\pi^{-1}(B)$ is then a disjoint union of two copies of B, one such copy contains (x, ω_x). Then we have found a neighbourhood for (x, ω_x) that is homeomorphic to $\mathbb{R}^k$. Thus we are done.

It is clear that it is a two sheeted covering because for any open set $B \subseteq M$, $\pi^{-1}(B) = B_+ \amalg B_-$. $\square$

Lemma 2.2.4

Let M be a topological k-manifold. Then the orientation bundle $\widetilde{M}$ is orientable.

Proof. Suppose that (x, ω_x) and (y, ω_y) in $\widetilde{M}$ share an open ball $\widetilde{B}$ of $\widetilde{M}$. By definition of $\widetilde{M}$, the topology is generated by $\pi^{-1}(B) = B_+ \amalg B_-$ for any open ball B of M. Hence any open ball of $\widetilde{M}$ must be equal to some B_+ or B_-. Without loss of generality, suppose that B is an open ball of M such that $\widetilde{B}$ is one of B_+ or B_- in $\pi^{-1}(B)$. Then x and y share an open ball B. We have seen the following isomorphisms induced by inclusions:

$$\begin{tikzcd}
 & & H_k(M \mid B) & & \\
 & H_k(M \mid x) \arrow[ur, "\cong"] & H_k(\widetilde{M} \mid \widetilde{B}) & H_k(M \mid y) \arrow[ul, "\cong"'] & \\
 & H_k(\widetilde{M} \mid (x, \omega_x)) \arrow[ur, "\cong"] & & H_k(\widetilde{M} \mid (y, \omega_y)) \arrow[ul, "\cong"'] &
\end{tikzcd}$$

By excision, we can connect the above diagram with isomorphisms:

$$\begin{tikzcd}
 & & H_k(M \mid B) & & \\
 & H_k(M \mid x) \arrow[ur, "\cong"] & H_k(\widetilde{M} \mid \widetilde{B}) \arrow[u, dashed, "\cong"] & H_k(M \mid y) \arrow[ul, "\cong"'] & \\
 & H_k(\widetilde{M} \mid (x, \omega_x)) \arrow[u, dashed, "\cong"] \arrow[ur, "\cong"] & & H_k(\widetilde{M} \mid (y, \omega_y)) \arrow[u, dashed, "\cong"] \arrow[ul, "\cong"'] &
\end{tikzcd}$$

By definition of $\pi^{-1}(B) = B_+ \amalg B_-$, if (x, ω_x) and (y, ω_y) lie in the same ball, they have a consistent local orientation. Hence there exists a generator ω_B of $H_k(M \mid B)$ such that the above diagram maps elements in the following way:

$$\begin{tikzcd}
 & \omega_B & \\
\omega_x \arrow[ur, mapsto] & & \omega_y \arrow[ul, mapsto]
\end{tikzcd}$$

Under the isomorphism $H_k(M \mid x) \cong H_k(\widetilde{M} \mid (x, \omega_x))$ let ω_x be sent to the generator $\mu_{(x,\omega_x)}$. Define $\mu_{(y,\omega_y)}$ and $\mu_{\widetilde{B}}$ similarly. Then using the above diagram with 6 local homology groups, we can see that elements are sent in the following way:

$$\begin{tikzcd}
 & \omega_B \arrow[d, mapsto] & \\
\omega_x \arrow[ur, mapsto] & \mu_{\widetilde{B}} & \omega_y \arrow[ul, mapsto] \\
\mu_{(x,\omega_x)} \arrow[u, mapsto] \arrow[ur, dashed, mapsto] & & \mu_{(y,\omega_y)} \arrow[u, mapsto] \arrow[ul, dashed, mapsto]
\end{tikzcd}$$

This show that we made a choice of generators for the local homology groups of $\widetilde{M}$ for which they are consistent (they both map to the same generator $\mu_{\widetilde{B}}$). Hence $\widetilde{M}$ is orientable. □

Lemma 2.2.5

Let M be a k-manifold. Then M is orientable if and only if there exists a section $M \to \widetilde{M}$. In particular, the given section is then the assignment required in the definition of orientability.

Proof. Let M be orientable. Then there exists an assignment $x \mapsto \omega_x \in H_k(M \mid x)$ for each $x \in M$. We can rewrite the assignment into $x \mapsto (x, \omega_x)$ so that the codomain is now $\widetilde{M}$. It is clear that composing with the projection map gives the identity. It remains to show that the assignment is continuous. Since the topology of $\widetilde{M}$ is generated by open balls, it suffices to check continuity on open balls. So let $\widetilde{B}$ be an open ball of $\widetilde{M}$. It is clear that the preimage of $\widetilde{B}$ is given by $x \in M$ such that $(x, \omega_x) \in \widetilde{B}$. But this is the same set as $B = \pi(\widetilde{B})$, which by definition is an open ball. Hence s is continuous.

Now let $s : M \to \tilde{M}$ be a section. By restricting to the second factor we obtain an assignment $x \mapsto \omega_x \in H_k(M \mid x)$. I claim that defines an orientation. By continuity of s, the preimage of each open ball $\widetilde{B}$ of $\widetilde{M}$ by s is also an open ball B of M. For $x, y \in B$, ω_x and ω_y is in $\widetilde{B}$. But $\widetilde{B}$ is one of the factors of the disjoint union $\pi^{-1}(B) = B_+ \amalg B_-$, which by definition consists of consistent local orientations. Hence ω_x and ω_y are consistent. Thus we conclude. □

Theorem 2.2.6

Let M be a connected topological manifold. Then the following are true.

- M is orientable if and only if $\widetilde{M} \cong M \amalg M$. In this case, M admits exactly two possible orientations.
- M is non-orientable if and only if $\widetilde{M} \to M$ is a non-trivial two sheeted cover.

Proof. Let M first be orientable. Then there exists a section $s : M \to \widetilde{M}$ to the covering space. Assume for a contradiction that $\widetilde{M}$ is connected. Let γ be a path from (x, ω_x) to $(x, -\omega_x)$. Then notice that γ and $s \circ \pi \circ \gamma$ are distinct lifts of the path $\pi \circ \gamma$ in M. This contradicts the uniqueness of path lifting. Thus $\widetilde{M}$ is disconnected. The unique disconnected two sheeted cover of a space is precisely the disjoin union of the space. So we are done.

Now let $\widetilde{M} \cong M \amalg M$. Then it is easy to see that there exists a section $s : M \to \widetilde{M}$ simply be mapping homeomorphically to one of the disjoint components.

When M is orientable, we have that $\widetilde{M} \cong M \amalg M$. By the above lemma, each section $M \to \widetilde{M}$ corresponds to one choice of orientation. There can only be two choices of distinct sections $M \to M \amalg M$. Hence M has exactly two orientations.

The second statement is precisely the contrapositive of the equivalent characterization of orientability. □

An overview of what is happening: Let M be a manifold. Then the orientation sheaf is a locally constant sheaf with constant value $\mathbb{Z}$. Since M is locally connected, there is an equivalence between locally constant sheaves and covering spaces induced by the presheaf-bundle adjunction. The orientation sheaf then corresponds to the orientation bundle. (Why does the existence of global sections imply oreientability?)

Corollary 2.2.7

Any simply connected manifold is orientable.

Proof. By Galois theory of covering spaces, any 2-sheeted cover of a simply connected space is disconnected. □

Proposition 2.2.8

Let $k \geq 1$. Then $\mathbb{RP}^k$ is orientable if and only if k is odd.

Proof. The quotient map $q : S^k \to \mathbb{RP}^k$ is the unique connected two-sheeted cover of $\mathbb{RP}^k$ by Galois theory for covering spaces. The non-trivial deck transformation is given by the antipodal map which has degree $(-1)^{k+1}$. If k is odd then this degree is 1 so that the deck transformation is orientation preserving. Since deck transformations of the orientation bundle must be orientation reversing, we conclude that $S^k \neq \widetilde{\mathbb{RP}^k}$. This means that the orientation bundle of $\mathbb{RP}^k$ is disconnected.

Now assume that k is even. By the lifting criterion, there exists a lift of q called $\tilde{q}$ such that

$$\begin{tikzcd} & \widetilde{\mathbb{RP}^k} \arrow[d, "p"] \\ S^k \arrow[ur, "\tilde{q}"] \arrow[r, "q"'] & \mathbb{RP}^k \end{tikzcd}$$

where p is the covering map. Then $\tilde{q}$ must also be a covering space. Assume that q is not injective. This means that $\tilde{q} \circ (-\text{id}) = \tilde{q}$ since $-\text{id}$ is the only other deck transformation of S^k over $\mathbb{RP}^k$. This means that for any $x \in S^k$, we have that

$$H_k(S^k) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}) \longrightarrow H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

where the second map is given by the long exact sequence in relative homology. Denoting this entire map by α, we have that $\alpha \circ (-\text{id})_* = \alpha$ since $\tilde{q} \circ (-\text{id}) = \tilde{q}$. But α is a map from $\mathbb{Z}$ to $\mathbb{Z}$. Since $\alpha \circ (-\text{id})_* = \alpha$ this implies that $\alpha = 0$. But α also factors as

$$H_k(S^k) \xrightarrow{\cong} H_k(S^k, S^k \setminus \{x\}) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

by the long exact sequence in relative homology and naturality. But the second map is also an isomorphism since covering spaces of manifolds induces a an isomorphism in local homology groups.

Now S^k being compact and $\mathbb{RP}^k$ being Hausdorff together with $\tilde{q}$ being injective implies that $\tilde{q}$ is a homeomorphism onto an open and closed subspace of $\mathbb{RP}^k$. Assume that $\tilde{q}$ is not surjective, then we have that $\widetilde{\mathbb{RP}^k} \cong S^k \amalg X$ for some other space X. But this is impossible thus q is surjective and $\tilde{q}$ gives a homeomorphism between S^k and $\widetilde{\mathbb{RP}^k}$. Since S^k is connected, $\mathbb{RP}^k$ is thus non orientable. □

One has to be careful that homotopy equivalence does not preserve orientability. For example, the Mobius strip is homotopy equivalent to S^1 but the former is non-orientable while the latter is.

2.3 Orientability in Arbitrary Coefficient Ring

Proposition 2.3.1

Let M be a k-dimensional topological manifold and $x \in M$ a point. Let R be a ring. Then

$$H_n(M \mid \{x\}; R) \cong \begin{cases} R & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $x \in M$. Since M is a manifold, there exists an open neighbourhood of x such that $U \cong \mathbb{R}^k$ via some transition map $\varphi : U \to \mathbb{R}^k$. Since $M \setminus U$ is closed, we can apply excision to obtain an isomorphism

$$H_n(M \mid \{x\}; R) \cong H_n(U \mid \{x\}; R)$$

By the homeomorphism $(U, U \setminus \{x\}) \cong (\mathbb{R}^k, \mathbb{R}^k \setminus \{\varphi(x)\})$ and 6.3.2 in AT2, we obtain the desired result. $\square$

Definition 2.3.2: Local Orientation

Let M be a k-dimensional topological manifold and let $x \in M$. A local orientation of M at x is a choice of generator of

$$H_k(M \mid \{x\}; R) \cong R$$

Notice that being a generator of R is the same as saying that it is a unit of R.

If M is a k-manifold and $U \subseteq M$ is an open ball, a similar argument as the case of $R = \mathbb{Z}$ shows that there are isomorphisms

$$\begin{tikzcd} & H_k(M \mid U; R) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_k(M \mid \{x\}; R) & & H_k(M \mid \{y\}; R) \end{tikzcd}$$

where all of the above groups are just R. We also obtain the same definition for consistent local orientations.

Definition 2.3.3: Consistent Local Orientations

Let M be a k-manifold. Let B be an open ball in M. For each $x \in B$, let ω_x be a choice of local orientation at x. We say that the choices of local orientations at B is consistent if there exists a generator $\omega_B \in H_k(M \mid B; R)$ such that for any $x, y \in B$, under the isomorphisms

$$\begin{tikzcd} H_k(M \mid \{x\}; R) \arrow[r, "\cong"] & H_k(M \mid B; R) & H_k(M \mid \{y\}; R) \arrow[l, "\cong"'] \\ \omega_x \arrow[r, mapsto] & \omega_B & \omega_y \arrow[l, mapsto] \end{tikzcd}$$

the choice of local orientation maps to the same generator ω_B.

Definition 2.3.4: R-Orientation of a Manifold

Let M be a k-dimensional topological manifold. An orientation of M is a function

$$x \mapsto \omega_x \in H_k(M, M \setminus \{x\}; R)$$

assigning every point to a local orientation such that for every $x \in M$, there exists an open ball $x \in B$ such that $(\omega_x)_{x \in B}$ a consistent local orientation.

In order to deduce interesting results, we need to define a more general version than that of the orientation double cover.

Definition 2.3.5: Generalized Orientation Bundle

Let M be a k-manifold. Let R be a ring. Define the generalized orientation bundle M_R of M to be the set

$$M_R = \{(x, \mu_r) \mid x \in M, r \in H_k(M \mid x; R) \cong R\}$$

together with the topology generated by each B_r in

$$\pi^{-1}(B) = \coprod_{r \in R \text{ is a unit}} B_r$$

When $R = \mathbb{Z}$, we notice that $M_{\mathbb{Z}} \to M$ is infinite sheeted, and contains a copy of M as the subspace of $M_{\mathbb{Z}}$ by choosing $\mu_r = 0 \in \mathbb{Z}$. More generally, if we write

$$M_k = \{(x, \mu_x) \in M_{\mathbb{Z}} \mid \mu_x = \pm k\}$$

$M_{\mathbb{Z}}$ contains a copy of M_k for $k \in \mathbb{N} \setminus \{0\}$, and each copy M_k is homeomorphic to the orientation double cover $\widetilde{M}$.

If we instead consider an arbitrary ring R, then we can similarly define

$$M_r = \left\{(x, \mu_x) \in M_R \mid {}^{\mu_x \otimes r \in H_k(M \mid x) \otimes R \cong H_k(M \mid x; R)}_{\mu_x \text{ is a generator } H_k(M \mid x) \cong \mathbb{Z}}\right\}$$

If $2r = 0$ in the ring then M_r becomes only one copy of M. Otherwise for each $r \in R$, M_r is homemorphic to $\widetilde{M}$. Hence the covering space M_R is a disjoint union of M_r for $r \in R$, except that M_r and M_{-r} are not disjoint.

Lemma 2.3.6

Let M be a topological manifold. Let R be a ring. Then M is R-orientable if and only if there exists a section $M \to M_R$. In particular, the section is precisely the assignment required in the definition of R-orientability.

Proof. Let M be R-orientable. Then there exists an assignment $x \mapsto \omega_x \in H_k(M \mid x; R)$ for each $x \in M$. We can rewrite the assignment into $x \mapsto (x, \omega_x)$ so that the codomain is now M_R. It is clear that composing with the projection map gives the identity. It remains to show that the assignment is continuous. Since the topology of M_R is generated by open balls, it suffices to check continuity on open balls. So let $\widetilde{B}$ be an open ball of M_R. It is clear that the preimage of $\widetilde{B}$ is given by $x \in M$ such that $(x, \omega_x) \in \widetilde{B}$. But this is the same set as $B = \pi(\widetilde{B})$, which by definition is an open ball. Hence s is continuous.

Now let $s : M \to M_R$ be a section. By restricting to the second factor we obtain an assignment $x \mapsto \omega_x \in H_k(M \mid x; R)$. I claim that defines an orientation. By continuity of s, the preimage of each open ball $\widetilde{B}$ of M_R by s is also an open ball B of M. For $x, y \in B$, ω_x and ω_y is in $\widetilde{B}$. But $\widetilde{B}$ is one of the factors of the disjoint union $\pi^{-1}(B) = \coprod_{r \in R \text{ is a unit}} B_r$, which by definition consists of consistent local orientations. Hence ω_x and ω_y are consistent. Thus we conclude. $\square$

Lemma 2.3.7

Let M be a topological manifold. Let R be a ring. Then the following are true.

- If M is orientable, then M is R-orientable.
- If M is non-orientable, then M is R-orientable if and only if R contains a unit of order 2.

2.4 Implications of R-Orientability

Definition 2.4.1: Ring of Sections

Let M be a compact k-manifold. Denote $p : M_R \to M$ the projection map. Define the ring of sections of M to the orientation cover to be the set

$$\Gamma(M, M_R) = \{s : M \to M_R \mid s \circ p = \text{id}_M\}$$

together with addition / multiplication defined by addition / multiplication of ring elements in the second variable.

Proposition 2.4.2

Let M be a compact k-manifold. Then the following are true.

- If M is R-orientable, then there is an R-module isomorphism

$$\Gamma(M, M_R) \cong \text{Hom}(\pi_0(M), R)$$

  of sets. This assignment is given by sending $s : \pi_0(M) \to R$ to the map $x \mapsto (x, s([x]))$.
- If M is connected and not R-orientable, then there are no global sections.

Proposition 2.4.3

Let M be a k-manifold. Let $K \subseteq M$ be compact. Then the following are true.

- If $s : M \to M_R$ is a section sending x to (x, μ_x), there exists a unique $\mu_K \in H_k(M \mid K; R)$ such that under induced map of inclusions

$$H_k(M \mid K; R) \to H_k(M \mid x; R)$$

  μ_K is mapped to μ_x.
- The local homology groups $H_i(M \mid K; R) = 0$ for all $i > k$.

Theorem 2.4.4

Let M be a compact and connected k-dimensional manifold. Let R be a ring. Then the following are true.

- If M is R-orientable, then the map

$$H_k(M; R) \to H_k(M \mid x; R) \cong R$$

  is an isomorphism for all $x \in M$.
- If M is not R-orientable, then the map

$$H_k(M; R) \to H_k(M \mid x; R) \cong R$$

  is injective and has image $\{r \in R \mid 2r = 0\}$ for all $x \in M$

Notice this theorem is not a definitive criterion for orientability in general. However, if $R = \mathbb{Z}$, then it becomes a sufficient criterion.

Corollary 2.4.5

Let M be a compact and connected k-dimensional manifold. Then the following are true.

- M is orientable if and only if $H_k(M) \cong \mathbb{Z}$
- M is non-orientable if and only if $H_k(M) = 0$
- In any case, $H_k(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

Proof. Let M be orientable. Then by the above theorem it is clear that $H_k(M) \cong \mathbb{Z}$. Now let $H_k(M) \cong \mathbb{Z}$. Suppose for a contradiction that M is not orientable. Then the map

$$\mathbb{Z} \cong H_k(M) \to H_k(M \mid x) \cong \mathbb{Z}$$

is injective and has image $\{k \in \mathbb{Z} \mid 2k = 0\} = \{0\}$. But if $\mathbb{Z} \to \mathbb{Z}$ is injective, its image must be non-trivial. Hence we have a contradiction, so that M is orientable.

Let M be non-orientable. Then the map $H_k(M) \to \mathbb{Z}$ must be injective with trivial image. Hence $H_k(M) = 0$. Now let that $H_k(M) = 0$. Then 0 and $\mathbb{Z}$ are not isomorphic, by the contrapositive of the first statement of the above theorem, we conclude that M is not orientable.

If M is orientable then by the above theorem we are done. If M is not orientable, then the image of the map $H_k(M; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is equal to $\{k \in \mathbb{Z}/2\mathbb{Z} \mid 2k = 0\} = \mathbb{Z}/2\mathbb{Z}$. Since the map is also injective, we conclude that $H_k(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

We can summarize the situation as follows:

$$M \text{ is orientable} \implies \substack{TFAE: \\ M \text{ is } R\text{-orientable} \\ \text{there exists a section } M \to M_R} \implies \substack{\Gamma(M, M_R) \cong \operatorname{Hom}(\pi_0(M), R) \\ H_k(M;R) \cong H_k(M \mid x) \cong R}$$

$$\substack{TFAE: \\ M \text{ is not } R\text{-orientable} \\ \text{there are no sections } M \to M_R} \implies H_k(M;R) \longrightarrow H_k(M \mid x) \cong R \text{ is injective}$$

Corollary 2.4.6

Let M be a compact, connected and orientable k-dimensional manifold. Let R be a ring. Then the following are true.

- There is an isomorphism
$$H_k(M; R) \cong R$$
- The higher homology groups
$$H_i(M; R) = 0$$
for all $i > k$.

Corollary 2.4.7

Let M be a compact and connected k-dimensional manifold. Then the following are true.

- If M is orientable, then the torsion part of $H_{k-1}(M)$ is trivial.
- If M is not-orientable, then the torsion part of $H_{k-1}(M)$ is $\mathbb{Z}/2\mathbb{Z}$.

Corollary 2.4.8

Let M be a non-compact connected k-dimensional manifold. Let R be a ring. Then we have

$$H_i(M; R) = 0$$

for all $i \geq k$.

2.5 Fundamental Class

Definition 2.5.1: Fundamental Class

Let M be a compact and connected manifold of dimension n. Let R be a ring. A fundamental class for M with coefficients in R is an element $[c] \in H_n(M;R)$ such that the element $[c]$ is sent to a generator under the induced map of inclusion

$$H_n(M;R) \to H_n(M \mid x;R) \cong R$$

for any $x \in M$.

Lemma 2.5.2

Let M be a compact and connected manifold of dimension n. Let R be a ring. Then M is R-orientable if and only if M has a fundamental class for M with coefficients in R.

When M is a δ-complex, we can represent the fundamental class as a linear combination of top-simplicies satisfying some conditions.

Proposition 2.5.3

Let M be a compact and connected n-manifold. Let

$$\rho = \sum_{\sigma_i \text{ is an } n \text{ simplex}} k_i \sigma_i \in C_n(M)$$

be an n-chain. Then $[\rho]$ is a fundamental class of M if and only if each $k_i = \pm 1$ and ρ is an n-cycle. Moreover, M is orientable if and only if there exists such a ρ.

We can explicitly give a fundamental class of the sphere in integral coefficients.

Proposition 2.5.4

Let $\sigma : \Delta^{k+1} \to \Delta^{k+1}$ be the identity singular n-simplex in the space Δ^{k+1}. Then the cycle $\partial\sigma \in C_k(\partial\Delta^{k+1})$ represents a generator in for the top homology of S^k.

Proof. It is clear that it is a cycle since it is a boundary in the chain complex $C_\bullet(\Delta^{k+1})$. We proceed by induction. When $k = 0$, the statement is clear. So suppose that $k > 0$. Let U_1, U_2 be open subspaces of Δ^{k+1} as follows. U_1 is an open neighbourhood of the last face of $\partial_{k+1}\Delta^{k+1}$ which deformation retracts onto $\partial\Delta^{k+1}$. U_2 is an open neighbourhood of the remaining faces $U_2 = \bigcup_{i=0}^k \partial_i\Delta^{k+1}$ which deformation retracts onto $\bigcup_{i=0}^k \partial_i\Delta^{k+1}$. Moreover, choose them in such a way that $U_1 \cap U_2$ deformation retract onto $\partial\partial\Delta^{k+1} = \partial[v_0, \dots, v_k]$ and $U_1 \cup U_2$ deformation retracts onto $\partial[v_0, \dots, v_{k+1}]$. By induction hypothesis, we know that $\widetilde{H}_{k-1}(U_1 \cap U_2) \cong \mathbb{Z}$ is generated by

$$\partial([v_0, \dots, v_k]) = \sum_{i=0}^k (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k]$$

From Mayer-Vietoris sequence, since $U_1 \cup U_2$ deformation retracts onto $\partial\Delta^{k+1}$, the connecting homomorphism

$$\widetilde{H}_k(U_1 \cup U_2) \to \widetilde{H}_{k-1}(U_1 \cap U_2)$$

since U_1 and U_2 are contractible so we only need to show that $\partial\sigma$ is sent to the generator or its negative.

For this we will explicitly compute the connecting homomorphism. It is clear that

$$\left((-1)^{k+1}[v_0,\dots,v_k],\sum_{i=0}^{k}(-1)^i[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]\right)\in C_k(U_1)\oplus C_k(U_2)$$

is such that it is a lift of the cycle $\partial\sigma$. Its image under the connecting homomorphism is the unique $(k-1)$-cycle τ in $U_1\cap U_2$ which satisfies

$$((i_1)_*(\tau),-(i_2)_*(\tau))=\left((-1)^{k+1}\partial([v_0,\dots,v_k]),\sum_{i=0}^{k}(-1)^i\partial([v_0,\dots,\hat{v}_i,\dots,v_{k+1}])\right)$$

It is clear that $\tau=(-1)^{k+1}\partial([v_0,\dots,v_k])$ is a generator in $\widetilde{H}_k(U_1\cap U_2)$. □

Corollary 2.5.5

Let S^k_+ and S^k_- be the northern and southern hemisphere of S^k respectively. Choose homomorphisms

$$\sigma_+:\Delta^k\xrightarrow{\cong}S^k_+\quad\text{and}\quad\sigma_-:\Delta^k\xrightarrow{\cong}S^k_-$$

such that both σ_+,σ_- map the boundary $\partial\Delta^k$ homeomorphically onto the equator $S^k_+\cap S^k_-$ and the composition

$$\partial\Delta^k\xrightarrow{\sigma_+}S^k_+\cap S^k_-\xrightarrow{(\sigma_-)^{-1}}\partial\Delta^k$$

is the identity. Then the cycle $\sigma_+-\sigma_-\in C_k(S^k)$ represents a fundamental class for S^k.

Proof. For $k=1$, $\sigma_+:\Delta^1\to S^1$ is the upper half circle oriented anticlockwise and $\sigma_-:\Delta^1\to S^1$ is the lower half circle oriented clockwise. It is clear that by the isomorphism $\pi_1(S^1,1)^{\text{ab}}\cong H_1(S^1)$, $\sigma_+-\sigma_-$ is a generator. Now assume that $k>1$. It is clear from the assumptions that $\sigma_+-\sigma_-$ is a cycle. Choose open neighbourhoods U_+ and U_- of S^k_+ and S^k_- respectively which deformation retracts onto S^k_+ and S^k_- and that $U_1\cap U_2\simeq S^{k-1}$ the equator. The connecting homomorphism

$$H_k(S^k)\to H_{k-1}(U_+\cap U_-)$$

in the Mayer-Vietoris sequence is an isomorphism that sends $\sigma_+-\sigma_-$ to $\partial\sigma_+=\partial\sigma_-$. By the above proposition, $\partial\sigma_+=\partial\sigma_-$ is a generator of $H_{k-1}(\partial\Delta^k)$ and so we are done. □

2.6 Relation to Orientability of Smooth Manifolds

3 Poincare Duality

3.1 The Cap Product

Definition 3.1.1: The Cap Product

Let $\sigma = [v_0, \dots, v_k] \in C_k(X)$ and $\phi \in C^l(X)$ where $k \geq l$ with coefficients in a ring R. Define the cap product to be
$$\sigma \frown \phi = \phi(\sigma|_{[v_0,\dots,v_l]})\sigma|_{[v_l,\dots,v_k]} \in C_{k-l}(X)$$

Lemma 3.1.2

The cap product $\frown: C_k(X) \times C^l(X) \to C_{k-l}(X)$ with coefficients in a ring R induces a cap product in homology $\frown: H_k(X) \times H^l(X, R) \to H_{k-l}(X)$ for $k \geq l$.

3.2 The Duality Theorem

Theorem 3.2.1: Poincare Duality

Let M be a compact and oriented topological n-manifold. Then the map
$$D : H^p(M) \to H_{n-p}(M)$$
defined by $D(\alpha) = \alpha \frown z$ for z a fundamental class of M, is an isomorphism.

3.3 The Smooth Poincare Duality

4 The Theory of Surfaces

4.1 Connected Sums

Recall that a compact surface is a connected topological manifold of dimension 2 that is compact.

Definition 4.1.1: Connected Sum

Let S_1 and S_2 be two compact surfaces. Let $D_i \subseteq S_i$ be two small closed disks for $i = 1, 2$. Define the connected sum to be

$$S_1 \# S_2 = \frac{(S_1 \setminus D_1^\circ) \amalg (S_2 \setminus D_2^\circ)}{\partial D_1 \cong \partial D_2}$$

Lemma 4.1.2

The connected sum of two compact surfaces is again a compact surface.

Proposition 4.1.3

The connected sum is invariant under the choice of homeomorphism and the location of the small discs.

4.2 Classification of Compact Surfaces

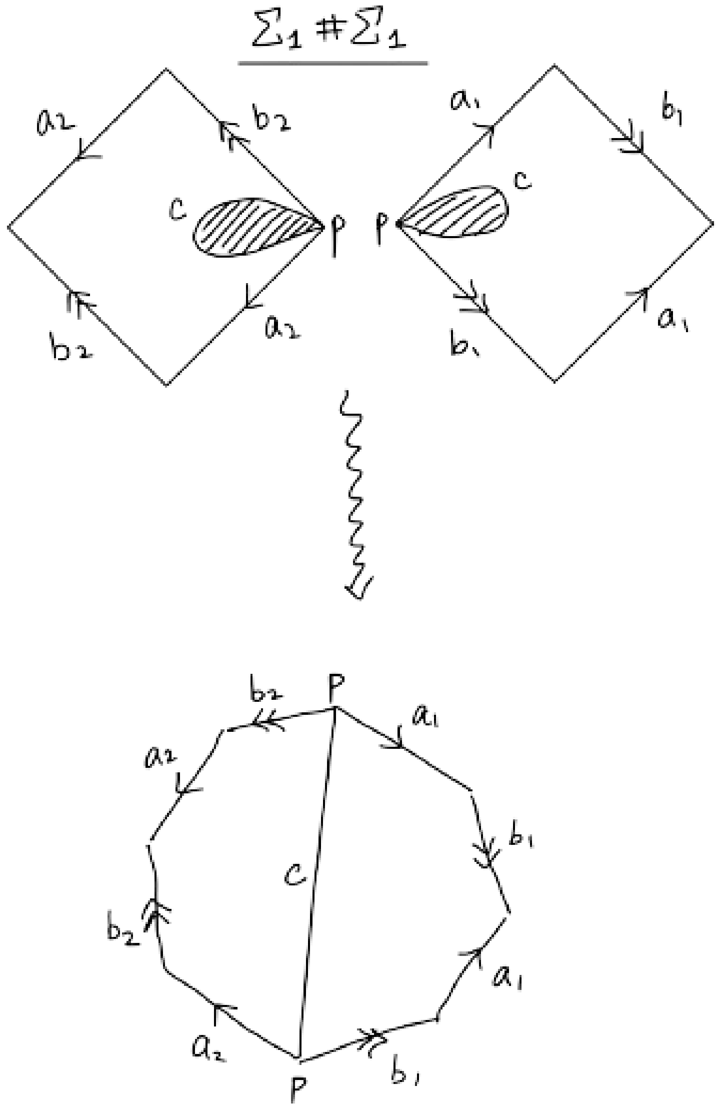
Definition 4.2.1: g-Holed Torus

For $g \geq 0$, define the g-holed torus to be

$$\Sigma_g = \mathbb{T} \# \cdots \# \mathbb{T}$$

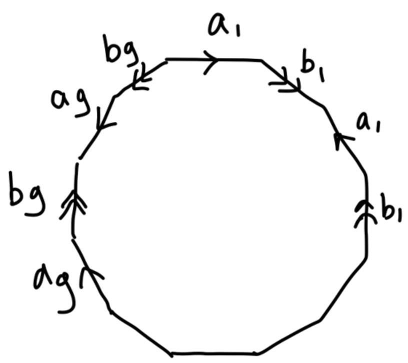
the connected sum of g toruses. By convention when $g = 0$, Σ_g is the 2-sphere.

Recall the CW complex of the torus. We can visualize the connected sum of two toruses using the CW complex.



This is done by cutting a hole at the CW complex at the point p, and the pushing the boundary c out, and then connecting them together. The cut-out hole is exactly a disc in the torus. By gluing the two toruses along the boundary c, we are effectively gluing the two toruses along the discs.

The new hectagon obtained is precisely then the CW complex of Σ_2. In general, we can perform the operation of connected sum on a $(4g-4)$-gon and a square. We then obtain the CW complex of the g-holed torus.

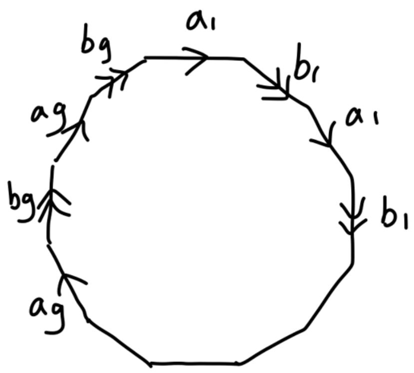


Another class of compact surfaces is the connected sum of projective spaces $\mathbb{RP}^2$.

Definition 4.2.2: Non-Orientable Surface

For $h \geq 1$, define
$$N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$
the connected sum of h projective spaces.

We can do the same process of gluing the CW complexes just like that of the torus to obtain the $4h$-gon that represents N_h:



It is also meaningful to ask what would happen if we perform connected sums through the two class of compact surfaces. We obtain the following.

Proposition 4.2.3

Let N_3 denote the connected sum of three projective spaces $\mathbb{RP}^2$. Then we have that
$$T \# \mathbb{RP}^2 = N_3$$

The above two classes of compact surfaces together with the sphere exhausts all possible cases for compact surfaces.

Theorem 4.2.4

Every compact surface is homeomorphic to exactly one of the following.
- Σ_g for $g \geq 0$
- N_h for $h \geq 1$

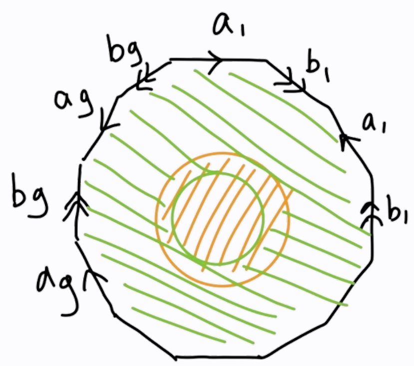
4.3 Algebraic Invariants of the Orientable Surfaces

Proposition 4.3.1

Let $g \geq 0$. The homology of the g-holed torus Σ_g is given by

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Cut an open disc along the middle of the CW complex as follows



and label it V (the orange part). Label the green part as U. It is clear that $U \cap V \simeq S^1$, U is contractible and V deformation retracts to the boundary, which is actually just a wedge sum of $2g$ circles. By the formula for the homology of wedge sums we have that

$$H_n(V) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

By the reduced Mayer-Vietoris sequence, the only non-trivial homology groups in the sequence are

$$0 \longrightarrow \widetilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \widetilde{H}_1(\Sigma_g) \longrightarrow 0$$

and the exact sequence

$$0 \longrightarrow \widetilde{H}_0(\Sigma_g) \longrightarrow 0$$

in which the latter immediately shows that $H_0(\Sigma_g) \cong \mathbb{Z}$. Now the map $\mathbb{Z} \to \mathbb{Z}^{2g}$ sends a generator of the first homology of $U \cap V \simeq S^1$ to the loop

$$a_1 + b_1 - a_1 - b_1 + \cdots + a_g + b_g - a_g - b_g$$

Since $\mathbb{Z}^{2g}$ is abelian, we conclude that this map is actually the zero map. It follows that $H_2(\Sigma_g) \cong \mathbb{Z}$ and $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$. $\square$

We can immediate deduce the orientability of Σ_g using the machinery in section 1.

Corollary 4.3.2

The surfaces Σ_g for $g \geq 0$ is orientable.

Proof. By the above, we have that $H_2(\Sigma_g) \cong \mathbb{Z}$. The long exact sequence for relative homology groups give

$$\cdots \longrightarrow H_2(\Sigma_g \setminus \{x\}) \longrightarrow H_2(\Sigma_g) \longrightarrow H_2(\Sigma_g, \Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g) \longrightarrow \cdots$$

Let U be as the proof above. Then the inclusion from U to $\Sigma \setminus \{x\}$ is a homotopy equivalence. Moreover, $\Sigma \setminus \{x\}$ is a $2g$-fold wedge of circles labelled $a_1, b_1, \dots, a_g, b_g$ and $H_2(\Sigma_g \setminus \{x\}) = 0$. Also, we have that $H_1(U) \cong H_1(\Sigma_g)$ from above and hence $H_1(\Sigma_g \setminus \{x\}) \cong H_1(\Sigma_g)$. The last map is invertible so that by exactness, the third map is the zero map. Then what remains is an isomorphism

$$H_2(\Sigma_g) \cong H_2(\Sigma_g, \Sigma_g \setminus \{x\})$$

Now since this isomorphism factors through $H_2(\Sigma_g, \Sigma_g \setminus B)$ for any ball B containing x, we thus have a consistent local orientation throughout all of Σ_g. $\square$

Proposition 4.3.3

Let $g \geq 0$. The singular cohomology of the g-holed torus Σ_g with coefficients in $\mathbb{Z}$ is given by

$$H^n(\Sigma_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Applying the universal coefficient theorem easily gives all the required cohomology groups. $\square$

We use the cohomology of Σ_g to illustrate generators of cohomology. Ref: Hatcher Ex3.7.

Proposition 4.3.4

Let $g \geq 0$. The integral cohomology ring of Σ_g is given by

$$H^*(\Sigma_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_1, \beta_1, \dots, \alpha_g, \beta_g]}{(\alpha_i^2, \beta_i^2, \alpha_i\alpha_j, \beta_i\beta_j, \alpha_i\beta_j, \beta_i\alpha_j, \alpha_i\beta_i + \beta_i\alpha_i \mid 1 \leq i \neq j \leq g)} \cong \Lambda^2(\mathbb{Z}^{2g})$$

where each α_i and β_i are of degree 1 in the graded ring.

Proof. Consider the following CW complex of Σ_g. Recall that a basis for $H_1(\Sigma_g)$ is given by $a_1, b_1, \dots, a_g, b_g$. Consider the dual basis $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$. By definition these are precisely the generators of $H^1(\Sigma_g; \mathbb{Z})$. By definition the value of α_i is 1 on a_i and 0 otherwise. This is similar for β_i. We want to find elements of $Z^1(\Sigma_g; \mathbb{Z})$ that represent α_i and β_i. Consider the following diagram: Define $\phi_i \in C^1(\Sigma_g; \mathbb{Z})$ to be the map that gives 1 for any edge that intersects with s_i and 0 otherwise. Similarly, define $\psi_i :\in C^1(\Sigma_g; \mathbb{Z})$ to be the map that gives 1 for any edge that intersects with t_i and 0 otherwise. It is easy to see that $\delta(\phi_i) = 0$ and $\delta(\psi_i) = 0$ so that ϕ and ψ are indeed cocycles. Moreover, they represent α_i and β_i respectively.

Now notice that I have indicated orientations for each 2-simplices in Σ_g depending on whether they are oriented clockwise or anti-clockwise. Define an element of $C_2(\Sigma_g)$ by the sum of the 2-simplices with ± 1 as their coefficient depending on their orientation. It is easy to see that the sum is a 2-cycle that generates $H_2(\Sigma_g)$. Let γ be its dual generator.

It is easy to check that

$$\phi_i \smile \psi_j = -\psi_j \smile \phi_i = \begin{cases} \gamma & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and $\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0$ for $1 \leq i, j \leq g$. We conclude that the cohomology ring of Σ_g is given by desired form. □

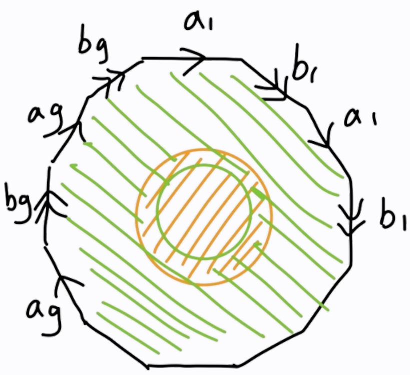
4.4 Algebraic Invariants of the Non-Orientable Surfaces

Proposition 4.4.1

Let $h \geq 1$. The homology of N_h is given by

$$H_n(N_h) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. Similar to the proof in that of Σ_g, cut an open disc along the middle of the CW complex of N_h as follows



and again label the green part U and the orange part V. Then apply Mayer-Vietoris sequence to acquire a similar exact sequence

$$0 \longrightarrow \widetilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \widetilde{H}_1(\Sigma_g) \longrightarrow 0$$

together with $\widetilde{H}_0(N_h) \cong 0$. Notice that the third non-zero term counting from the left is now $\mathbb{Z}^h$ instead of $\mathbb{Z}^{2g}$ as in the torus because the boundary circle is the wedge sum of h circles labelled $a_1 b_1, \dots, a_h b_h$. The map $\mathbb{Z}$ to $\mathbb{Z}^h$ is now given by sending the generator 1 to

$$2(a_1 + b_1 + \cdots + a_h + b_h)$$

The Smith Normal form of the matrix is an $h \times 1$ matrix with 2 at the first entry and 0 everywhere else. In particular, it means that this map is injective so that $\widetilde{H}_2(N_h) \to \mathbb{Z}$ is the 0 map so that $\widetilde{H}_2(N_h) \cong 0$. Now it remains an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \widetilde{H}_1(N_h) \longrightarrow 0$$

The image of the matrix is $2\mathbb{Z}$ and by exactness this is the kernel of the map $\mathbb{Z}^h \to \widetilde{H}_1(N_h)$. Thus we have an isomorphism
$$\widetilde{H}_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}$$
and so we conclude. □

Corollary 4.4.2

The surfaces N_h for $h \geq 1$ is non-orientable.

Proof. Notice that removing a small closed disk from $\mathbb{RP}^2$ yields a space homeomorphic to the open Mobius strip. It follows that for $h > 0$, the space N_h contains the open Mobius strip as a subspace. Since the Mobius strip is non-orientable, N_h is also non-orientable. □

Proposition 4.4.3

Let $h \geq 1$. The singular cohomology of non-orientable surface N_h with coefficients in $\mathbb{Z}$ is given by
$$H^n(N_h; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} & \text{if } n = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We use the universal coefficient theorem in all dimensions. When $n = 0$, we have that
$$\begin{aligned} H^0(N_h; \mathbb{Z}) &\cong \text{Hom}(H_0(N_h; R); \mathbb{Z}) \oplus \text{Ext}(H_{-1}(N_h; \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus 0 \\ &\cong \mathbb{Z} \end{aligned}$$
When $n = 1$, we have that
$$\begin{aligned} H^1(N_h; \mathbb{Z}) &\cong \text{Hom}(H_1(N_h), \mathbb{Z}) \oplus \text{Ext}(H_0(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}^{h-1} \oplus 0 \\ &\cong \mathbb{Z}^{h-1} \end{aligned}$$
When $n = 2$, we have that
$$\begin{aligned} H^2(N_h; \mathbb{Z}) &\cong \text{Hom}(H_2(N_h), \mathbb{Z}) \oplus \text{Ext}(H_1(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &\cong 0 \oplus \text{Ext}(\mathbb{Z}^{h-1}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &\cong 0 \oplus 0 \oplus \mathbb{Z}/2\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$
When $n \geq 3$, we have that
$$\begin{aligned} H^n(N_h; \mathbb{Z}) &\cong \text{Hom}(H_n(N_h), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) \\ &\cong 0 \end{aligned}$$
and so we conclude. □

4.5 The Euler Characteristic

Recall that if X is a CW complex such that $U \cap V = X$ and U and V are open subsets, then we have the formula
$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

Corollary 4.5.1

Let $S_1 \# S_2$ be the connected sum of two compact surfaces, then we have that
$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

Proof. Let D_i be the gluing discs for S_i for $i = 1, 2$. Using the above formula, we have that
$$\chi(S_i) = \chi(D_i) + \chi(S_i \setminus D_i^\circ) - \chi(S^1)$$
since the intersection of the disc and S_i is S^1. It follows that
$$\begin{aligned}\chi(S_1 \# S_2) &= \chi(S_1 \setminus D_1^\circ) + \chi(S_2 \setminus D_2^\circ) - \chi(S^1) \\ &= \chi(S_1) + \chi(S_2) - 2\end{aligned}$$
and so we conclude. □

Corollary 4.5.2

For $g \geq 0$ and $h > 1$, the Euler characteristic of any compact surface is given by
$$\chi(\Sigma_g) = 2 - 2g \quad \text{and} \quad \chi(N_h) = 2 - h$$

Proof. It follows directly by repeated applications of the above corollary. □

Recall that if $p : \widetilde{X} \to X$ is a d-sheeted covering and X is a finite CW complex, then we have the formula
$$\chi(\widetilde{X}) = d \cdot \chi(X)$$