Functional Analysis

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Abstract

Contents

1 Vector Spaces with a Topological Structure

1.1 Topological Vector Spaces

Definition 1.1.1: Topological Field

Let k be a field equipped with a topology. We say that k is a topological field if the following are true.

- The addition map $+ : k \times k \to k$ is continuous.
- The multiplication map $\cdot : k \times k \to k$ is continuous.
- The inverse map $(\cdot)^{-1} : k \to k$ is continuous.

Definition 1.1.2: Topological Vector Space

Let k be a topological field. Let V be a vector space over a field k that is also a topological space. We say that V is a topological vector space if the following are true.

- The addition map $+ : V \times V \to V$ is continuous.
- The scalar multiplication map $\cdot : k \times V \to V$ is continuous.

1.2 Normed Spaces

Let $\mathbb{F}$ be a field. Let V be a vector space over $\mathbb{F}$. Recall that a norm on V is a function $\|\cdot\| : V \to \mathbb{F}$ satisfying the following rules:

- $\|x\| \geq 0$ with equality if and only if $x = 0$.
- $\|\lambda x\| = |\lambda|\|x\|$ for any $\lambda \in \mathbb{F}$ and $x \in V$.
- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Lemma 1.2.1

Let $(V, \|\cdot\|)$ be a normed space. Then the induced topology given by the induced metric space from the norm gives V the structure of a topological space.

Proposition 1.2.2

Let V be a finite dimensional normed space. Let $U \subseteq X$ be a subset of V. Then U is compact if and only if V is closed and bounded.

Lemma 1.2.3: Riesz's Lemma

Let $(X, \|\cdot\|)$ be a normed space and Y a non-empty closed subspace of X not equal to X. Then there exists $x \in X$ with $\|x\| = 1$ such that $\|x - y\| \geq \frac{1}{2}$ for every $y \in Y$.

Proposition 1.2.4

Let V be a normed space. Let $U \subseteq V$ be a finite dimensional vector subspace of V. Then U is closed.

Proposition 1.2.5

Let V be a normed space. Then V is finite dimensional if and only if $B_1(0)$ is compact.

1.3 Isomorphisms that Preserve Distance

Definition 1.3.1: Isometrically Isomorphic Normed Spaces

Two normed spaces V and W are isometrically isomorphic if there is a surjective linear isometry $L : V \to W$. In this case, we write $V \cong W$.

Proposition 1.3.2

Let $(X, \|\cdot\|_X)$ be a normed space. Let V be a vector space over the same field and $L : V \to X$ a linear isomorphism. Then the pullback norm

$$\|v\|_V = \|L(v)\|_X$$

defines a norm on V. In particular, $L : (V, \|\cdot\|_V) \to (X, \|\cdot\|_X)$ is a linear isometry.

Theorem 1.3.3

If V is a finite dimensional vector space then all norms on V are equivalent.

Recall that every normed space is a vector space by defining $d(x, y) = \|x - y\|$ for x, y in a normed space X. We thus have the notion of convergence in normed spaces.

2 Banach Spaces

2.1 Banach Spaces

Definition 2.1.1: Banach Spaces

Let $(V, \|\cdot\|)$ be a normed space. We say that V is a Banach space if V is complete as a metric space.

Proposition 2.1.2

Let V be a finite dimensional normed space. Then V is a Banach space.

Lemma 2.1.3

Suppose that $(X, \|\cdot\|_X) \cong (Y, \|\cdot\|_Y)$ are isometrically isomorphic. Then X is a Banach space if and only if Y is a Banach space.

Lemma 2.1.4

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space X then $(X, \|\cdot\|_1)$ is complete if and only if $(X, \|\cdot\|_2)$ is complete.

Proposition 2.1.5

Both $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete.

2.2 Separability of Banach Spaces

Recall the notion of separability: A space is separable if it has a countably dense subset.

Lemma 2.2.1

Let X be a normed space. Then the following are equivalent.

- X is separable
- The set $\{x \in X | \|x\| = 1\}$ is separable
- X contains a sequence $(x_n)_{n\in\mathbb{N}}$ whose linear span is dense.

2.3 The Completion of a Normed Space

3 The Space of Bounded Linear Maps

3.1 Basic Definitions

Definition 3.1.1: Bounded Linear Maps

Let V, W be normed space over a field k. Let $T : V \to W$ be a linear map. We say that T is bounded if there exists $M > 0$ such that

$$\|T(x)\|_W \leq M\|x\|_V$$

for all $x \in V$.

Lemma 3.1.2

Let V, W be normed spaces over a field k. Let $T : V \to W$ be a linear map. Then T is bounded if and only if T is continuous.

Definition 3.1.3: The Space of Bounded Linear Maps

Let V, W be normed spaces over a field k. Define the space of bounded linear maps to be the vector subspace

$$B(V, W) = \{T \in \text{Hom}_k(V, W) \mid T \text{ is bounded }\}$$

together with the operator norm $\| \cdot \|_{\text{op}} : B(V, W) \to k$ defined by

$$\|T\|_{\text{op}} = \sup\{\|T(v)\|_W \mid v \in V \text{ such that } \|v\|_V = 1\}$$

Lemma 3.1.4

Let V, W be normed spaces. If V is finite dimensional, then we have

$$B(V, W) = \text{Hom}_k(V, W)$$

Lemma 3.1.5

Let V, W be normed spaces over a field k. Then $B(V, W)$ is a Banach space.

3.2 Invertibility

Corollary 3.2.1

Let $T \in B(X, Y)$ for X, Y vector spaces. Then $\ker(T)$ is a closed linear subspace of X.

Definition 3.2.2: Bounded Invertible

A linear map $T \in B(X, Y)$ is bounded invertible if there exists $S \in B(X, Y)$ such that $S \circ T$ and $T \circ S$ are the identity.

Lemma 3.2.3

Suppose that X and Y are normed spaces. Then for any $T \in B(X, Y)$ the following are equivalent.

- T is bounded invertible
- T is a bijection and $T^{-1} \in B(X, Y)$

- T is surjective and for some $c > 0$, $\|T(x)\|_Y \geq c\|x\|_X$ for every $x \in X$.

Corollary 3.2.4

If X is finite dimensional then a linear operator $T : X \to X$ is invertible if and only if $\ker(T) = \{0\}$.

3.3 The Hahn Banach Theorem

Definition 3.3.1: Continuous Dual Space

Let X be a normed space. Denote the continuous dual space of X to be the the subspace

$$X' = B(X, K)$$

of the dual space X^*.

Notice that since continuity is the same as boundedness, this notation of $B(X, K)$ coincides with the set of all bounded linear operators.

Lemma 3.3.2

Let X be a Banach space. Then X' is also a Banach space.

Proof. Since we know that $B(X, Y)$ is a Banach space for any normed space X and Banach space Y, choose $Y = \mathbb{R}$ and we are done. □

Theorem 3.3.3: The Real Hahn-Banach Theorem

Let X be a real vector space. Let p be a convex function on X. Let X_0 be a linear subspace of X and let f be a linear functional on Y satisfying

$$f(x) \leq p(x)$$

for all $x \in Y$. Then f can be extended to a linear functional F on all of X satisfying the condition

$$F(x) \leq p(x)$$

for all $x \in X$ and $F|_Y = f$.

4 Hilbert Spaces

4.1 Hilbert Spaces

Definition 4.1.1: Hilbert Spaces

Let V be an inner product space. We say that V is a Hilbert space if V is a Banach space.

4.2 Orthogonality in Hilbert Spaces

Definition 4.2.1: Orthonormal Set

A subset E of a Hilbert space is orthonormal if $\|e\| =$ for all $e \in E$ and $\langle e_1, e_2 \rangle = 0$ for all $e_1, e_2 \in E$ with $e_1 \neq e_2$.

Lemma 4.2.2: Bessel's Inequality

Let V be an inner product space and $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence. Then for any $x \in V$ we have

$$\sum_{k=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$$

Lemma 4.2.3

Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ an orthonormal sequence in H. Then the series $\sum_{k=1}^{\infty} a_n e_n$ converges if and only if

$$\sum_{k=1}^{\infty} |a_n|^2 < \infty$$

and

$$\left\| \sum_{k=1}^{\infty} a_n e_n \right\|^2 = \sum_{k=1}^{\infty} |a_n|^2$$

Corollary 4.2.4

Let H be a Hilbert space and $(e_n)_{n\in\mathbb{N}}$ an orthonormal sequence in H. Then for any x the sequence

$$\sum_{k=1}^{\infty} \langle x, e_n \rangle e_n$$

converges.

Proposition 4.2.5

Let $E = (e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in a Hilbert space H. Then the following are equivalent.

- E is a basis for H
- For any $x \in H$ we have $x = \sum_{k=1}^{\infty} \langle x, e_n \rangle e_n$
- $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_n \rangle|^2$ for all $x \in H$
- $\langle x, e_n \rangle = 0$ for all n implies $x = 0$
- $\overline{\text{span}(E)} = H$

4.3 Separability of Hilbert Spaces

Proposition 4.3.1

An infinite-dimensional Hilbert space is separable if and only if it has a countable orthonormal basis.

Theorem 4.3.2

Any infinite dimensional separable Hilbert space H is isometric to $l^2(K)$ for some field K.

4.4 The Adjoint of Maps of Hilbert Spaces

Theorem 4.4.1: Riesz Representation Theorem

Theorem 4.4.2

Let H, K be Hilbert spaces. Let $A : H \to K$ be a bounded linear map. Then there exists a unique bounded linear map $A^* : K \to H$ such that

$$\langle A(x), y \rangle = \langle x, A^*(y) \rangle$$

for all $x \in H$ and $y \in K$.

5 Spectral Theory

5.1 Spectrum

Definition 5.1.1: Resolvent and Spectrum

Let X be a complex Banach space and $T \in B(X)$. Define the resolvent set of T to be

$$\rho(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is invertible } \}$$

Define the spectrum of T to be

$$\sigma(T) = \mathbb{C} \setminus \rho(T) = \{\lambda \in \mathbb{C} | T - \lambda I \text{ is not invertible } \}$$

Notice that in the case that X is finite dimensional, the resolvent set of any linear operator is just its set of eigenvalues.

Lemma 5.1.2

Suppose that $T \in B(X)$ and that $(\lambda_n)_{n\in\mathbb{N}}$ are distinct eigenvalues of T. Then any set $(e_n)_{n\in\mathbb{N}}$ of corresponding eigenvectors is linearly independent.

Theorem 5.1.3

Let X be a Banach space and $T \in B(X)$ such that $T^{-1} \in B(X)$. Then for any $U \in B(X)$, with $\|U\| < \|T^{-1}\|^{-1}$, we have

$$\|(T+U)^{-1}\| \leq \frac{\|T^{-1}\|}{1 - \|U\|\|T^{-1}\|}$$

Lemma 5.1.4

Let X be a Banach space. Let $T \in B(X)$. Then $\sigma(T)$ is a closed subset of $\{\lambda \in \mathbb{C} | |\lambda| \leq \|T\|\}$.