Complex Analytic Geometry

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Abstract

References: Algebraic and Analytic Geometry Neeman,
Several Complex Variables with Connections to Algebraic Geometry and Lie Groups

Contents

1 Analytic Manifolds

1.1 Real Analytic Manifolds

Definition 1.1.1: Real Analytic Manifolds

A real analytic manifold is a topological manifold with analytic transition maps.

In the real case, every analytic manifold is a smooth manifold because every analytic function is necessarily infinitely differentiable. While not every analytic function is smooth, we can still equip a smooth structure on analytic manifolds.

Theorem 1.1.2: (Grauert-Whitney)

Every smooth manifolds admits a compatible real analytic structure.

Definition 1.1.3: Analytic Functions

Let M be a real analytic manifold. Let $U \subseteq M$ be a subset. Let $f : U \to \mathbb{R}$ be a function. We say that f is analytic if for all $x \in U$, there exists a chart $(V, \varphi = x^1, \dots, x^n))$ of M such that

$$f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$$

is analytic in the usual sense.

Definition 1.1.4: Sheaf of Analytic Functions

Let M be an analytic manifold. Define the sheaf of analytic functions

$$\mathcal{A}_M : \textbf{Open} \to \textbf{CRing}$$

as follows.

- For each open set $U \subseteq M$, $\mathcal{A}_M(U)$ consists of the ring of analytic functions on U
- For each inclusion $V \subseteq U$, there is a unique morphism

  $$\text{res}^U_V : \mathcal{A}_M(U) \to \mathcal{A}_M(V)$$

  given by $f \mapsto f|_V$

1.2 Complex Analytic Manifolds

Definition 1.2.1: Complex Analytic Manifolds

A complex analytic manifold is a topological manifold with complex analytic transition maps.

Since holomorphic functions are the same as analytic functions in the complex case, this is just the usual definition of complex manifolds.

Theorem 1.2.2: Oka's (Coherence) Theorem

Let M be a complex manifold. Then the sheaf $\mathcal{O}_M$ of holomorphic functions on M is a coherent sheaf.

2 Analytic Varieties

2.1 Real Analytic Varieties

2.2 Complex Analytic Varieties

Definition 2.2.1: Complex Analytic Varieties

A complex analytic variety Z is a subset of $\mathbb{C}^n$ that is the common vanishing locus

$$Z = \mathbb{V}(f_1, \dots, f_k) = \{z \in \mathbb{C}^n \mid f_1(z) = \cdots = f_k(z) = 0\}$$

of analytic functions $f_1, \dots, f_k \in \mathcal{A}_{\mathbb{C}^n}(U)$ defined on an open set $U \supseteq Z$.

Definition 2.2.2: The Structure Sheaf

Let$U \subseteq \mathbb{C}^n$ be an open subset. Let $f_1, \dots, f_k \in \mathcal{A}_{\mathbb{C}^n}(U)$ be analytic functions on U. Let $Z = \mathbb{V}(f_1, \dots, f_k)$ be the vanishing locus of $f_1, \dots, f_k$. Define the structure sheaf

$$\mathcal{O}_Z : \textbf{Open}(Z) \to \textbf{CRing}$$

as follows.

- For each open set $U \subseteq Z$, define
$$\mathcal{O}_Z(U) = \frac{\mathcal{O}_{\mathbb{C}^n}(U)}{(f_1, \dots, f_k)}$$
- For each inclusion $U \subseteq V \subseteq Z$ of open sets, define
$$\mathcal{O}_Z(V) \to \mathcal{O}_Z(U)$$
to be the induced map of the restriction $\mathcal{O}_{\mathbb{C}^n}(V) \to \mathcal{O}_{\mathbb{C}^n}(U)$.

2.3 Complex Analytic Varieties of Manifolds

Theorem 2.3.1: Cartan's (Coherence) Theorem

Let M be a complex manifold and let A be an analytic subset of M. Then $\mathcal{I}_A$ is a coherent sheaf of $\mathcal{O}_M$-modules.

2.4 Regular and Singular Points

Definition 2.4.1: Regular and Singular Points

Let M be a complex manifold and let A be an analytic subset of M. We say that $x \in A$ is a regular point if there exists some open neighbourhood U of x such that $A \cap U$ is a complex submanifold of M. Otherwise, x is said to be singular. Denote

$$A_{\text{reg}} = \{x \in A \mid x \text{ is a regular point }\} \quad \text{and} \quad A_{\text{sing}} = \{x \in A \mid x \text{ is a singular point }\}$$

Theorem 2.4.2

Let M be a complex manifold and let A be an analytic subset of M. Then A_{sing} is an analytic subset of A.

3 Analytic Spaces

3.1 Complex Analytic Spaces

Definition 3.1.1: Complex Analytic Spaces

Let $(X, \mathcal{O}_X)$ be a locally ringed space. We say that it is a complex analytic space if for all $x \in X$, there exists an open neighbourhood U of x such that there are isomorphisms of locally ringed spaces

$$(U, \mathcal{O}_X|_U) \cong (Z = \mathbb{V}(f_1, \dots, f_k), \mathcal{O}_Z)$$

where $(Z, \mathcal{O}_Z)$ is an analytic variety.

Theorem 3.1.2

Let X be a complex space. Then X_{reg} is dense and open in X. Moreover, X_{reg} consists of a disjoin union of complex manifolds.

Theorem 3.1.3

Let X be an irreducible complex space. Then every non-constant holomorphic function $f : X \to \mathbb{C}$ is an open map.

Corollary 3.1.4

Let X be an irreducible compact complex space. Then every holomorphic function $f : X \to \mathbb{C}$ on X is constant.