Vector Bundles

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Abstract

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1 Vector Bundles

1.1 Basic Definitions

Definition 1.1.1: Vector Bundles

Let B, E be topological spaces and $p : E \to B$ a map. A K-vector bundle of rank r is a triple (B, E, p) such that p is a continuous surjection and that

- For every $b \in B$, the fibre $E_b = p^{-1}(b)$ is a K-vector space of dimension r.
- For every $b \in B$, there exists an open neighbourhood $U \subseteq B$ of p and a homeomorphism ((U, ϕ) is called local trivialization) $\phi : p^{-1}(U) \to U \times K^r$ such that the map
  - The following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times K^r \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

    where π is the projection.
  - The map

$$E_p \xrightarrow{\phi|_{E_b}} \{b\} \times K^r \xrightarrow{\pi} K^r$$

    is a vector space isomorphism.

Definition 1.1.2: Transition Functions

Let $p : E \to B$ be a K-vector bundle of rank r. Let (U_α, ϕ_α) and (U_β, ϕ_β) be local trivialization. Define the induced map from $\phi_\alpha \circ \phi_\beta^{-1}$ to be the transition function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, K)$ where

$$g_{\alpha\beta}(p) : K^r \to K^r$$

Proposition 1.1.3

Let $p : E \to B$ be a K-vector bundle of rank r. The transition functions of the vector bundle satisfies the following.

- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I_r$ on $U_\alpha \cap U_\beta \cap U_\gamma$
- $g_{\alpha\alpha} = I_r$ on U_α

Definition 1.1.4: Sections

A section of a vector bundle $p : E \to B$ is a map $s : B \to E$ assigning to each $b \in B$ a vector space $s(b)$ in the fiber $p^{-1}(b)$.

Proposition 1.1.5

Let $p : E \to B$ be a vector bundle. Let s, s_1, s_2 be sections of E. Then $s_1 + s_2$ and λs are also vector bundles for any $\lambda \in \mathbb{R}$. Moreover, the set of all sections $s(E)$ is a vector space.

Definition 1.1.6: Morphism of Vector Bundles

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be vector bundles. A morphism of these vector bundles is given by is a pair of continuous maps $f : E_1 \to E_2$ and $g : B_1 \to B_2$ such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

If $B = B_1 = B_2$ then the diagram collapses:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Definition 1.1.7: Isomorphism of Vector Bundles

A bundle homomorphism from E_1 to E_2 is an isomorphism if there exists an inverse bundle homomorphism from E_2 to E_1. In this case, we say that E_1 and E_2 are isomorphic.

1.2 Operations on Vector Bundles

Theorem 1.2.1: Whitney Sum

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two vector bundles. Define the direct sum of the vector bundles to be

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 | p_1(v_1) = p_2(v_2)\}$$

together with the projection $p : E_1 \oplus E_2 \to B$ defined by $(v_1, v_2) \mapsto p_1(v) = p_2(v)$.

The construction $E_1 \oplus E_2$ is a vector bundle over B.

Proposition 1.2.2: Tensor Product Bundle

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be vector bundles. Define the tensor product bundle of it to be

$$E_1 \otimes E_2 = \{p_1^{-1}(x) \otimes p_2^{-1}(x) | x \in B\}$$

The construction $E_1 \otimes E_2$ is a vector bundle over B.

Theorem 1.2.3: Pullback Bundle

Let $p : E \to Y$ be a vector bundle. Let $f : X \to Y$ be a continuous map. Then there exists E' and p' such that $p' : E' \to X$ is a vector bundle.

Theorem 1.2.4: Dual Bundle

Let $p : E \to B$ be a K-vector bundle. Then the dual bundle $p^* : E^* \to B$ defined by

$$E_b^* = (E_b)^* = \mathrm{Hom}(E_b, K)$$

is a vector bundle over B.

1.3 Trivial Bundles

Definition 1.3.1: The Trivial Bundle

Let B be a base space. Define the trivial rank n bundle to be vector bundle with total space $E = B \times \mathbb{R}^n$. We say that a vector bundle is trivial if it is isomorphic to $B \times \mathbb{R}^n$.

Proposition 1.3.2

Let $p : E \to B$ be a vector bundle over a paracompact base B and $E_0 \subset E$ is a vector subbundle, then there exists a vector subbundle $E_0^\perp$ such that $E_0 \otimes E_0^\perp = E$.

Proposition 1.3.3

For each vector bundle $E \to B$ over a compact Hausdorff space B, there exists a vector bundle $E' \to B$ such that $E \otimes E'$ is the trivial bundle