Topological K Theory

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Abstract

Contents

1 The K Group of a Space

1.1 Grothendieck Completions

Definition 1.1.1: Grothendieck Completion

Let A be an Abelian monoid. We say that a group $\mathcal{G}(A)$ together with a monoid homomorphism $i : A \to \mathcal{G}(A)$ is a Grothendieck completion of A if the following universal property is satisfied. If $j : A \to H$ is another monoid homomorphism where H is an abelian group, then there exists a unique group homomorphism $k : \mathcal{G}(A) \to H$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "j"'] & \mathcal{G}(A) \arrow[d, dashed, "\exists! k"] \\ & H \end{tikzcd}$$

is commutative.

Proposition 1.1.2

Let X be a space. Then $\text{Vect}^{\mathbb{R}}(X)$ and $\text{Vect}^{\mathbb{C}}(X)$ are both abelian monoids with the Whitney sum operator. They are moreover a commutative semiring under then tensor product operator.

1.2 The K-Group of a Space

Theorem 1.2.1

Let X be a compact Hausdorff space. Then for any vector bundle $E \to X$ over $F = \mathbb{R}$ or $\mathbb{C}$, there exists a vector bundle $\widetilde{E} \to X$ over F such that there is an isomorphism

$$E \oplus \widetilde{E} \cong X \times F^n$$

to the trivial bundle.

Definition 1.2.2: The K-Group of a Space

Let X be a space. Define the real and complex K group of X respectively to be the Grothendieck completion

$$KO(X) = \mathcal{G}(\text{Vect}^{\mathbb{R}}(X)) \quad \text{and} \quad KU(X) = \mathcal{G}(\text{Vect}^{\mathbb{C}}(X))$$

Lemma 1.2.3

Let X be a space. Then $KO(X)$ and $KU(X)$ are both commutative rings with identity.

Definition 1.2.4: The K Functor

Define the K functors

$$KO, KU : \textbf{Top} \to \textbf{CRing}$$

as follows.

- For each space X, $KO(X)$ is the real K-group of X and $KU(X)$ is the complex K-group of X
- For each map $f : X \to Y$, $KO(f) : KO(Y) \to KO(X)$ is the ring homomorphism that sends each isomorphism class of vector bundle $[E] \in KO(Y)$ to the pullback bundle $[f^*(E)] \in KO(X)$. This is similar for $KU(f) : KU(Y) \to KU(X)$.

We use $K : \textbf{Top} \to \textbf{CRing}$ to mean either the real K groups or the complex K groups when no distinction is needed.

1.3 Reduced K-Theory

Definition 1.3.1: Reduced K-Theory

Let X be a space. Define the reduced K-theory of X to be the kernel

$$\widetilde{KO}(X) = \ker(KO(X) \to KO(*)) \quad \text{and} \quad \widetilde{KU}(X) = \ker(KU(X) \to KU(*))$$

Similarly, we use $\widetilde{K} : \textbf{Top} \to \textbf{Ab}$ to mean either the reduced real K groups or the reduced complex K groups when no distinction is needed.

The universal property of the kernel turns reduced K-theory into a functorial construction.

Definition 1.3.2: Reduced K Functor

Define the K functors

$$\widetilde{KO}, \widetilde{KU} : \textbf{Top} \to \textbf{Ab}$$

as follows.

- For each space X, $\widetilde{KO}(X)$ is the reduced real K-group of X and $\widetilde{KU}(X)$ is the reduced complex K-group of X
- For each map $f : X \to Y$, $\widetilde{KO}(f) : \widetilde{KO}(Y) \to \widetilde{KO}(X)$ is the ring homomorphism that sends each isomorphism class of vector bundle $[E] \in \widetilde{KO}(Y)$ to the pullback bundle $[f^*(E)] \in \widetilde{KO}(X)$. This is similar for $\widetilde{KU}(f) : \widetilde{KU}(Y) \to \widetilde{KU}(X)$.

Theorem 1.3.3

Let X be a compact Hausdorff space. Then the natural homomorphism $K(X) \to \widetilde{K}(X)$ is surjective with kernel $\mathbb{Z}$. In particular, this gives an isomorphism

$$K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$$

A similar

Definition 1.3.4: Reduced Equivalence

Let X be a space. Let $V \to X$ and $W \to X$ be two vector bundles over X. We say that $V \sim_{\text{red}} W$ if $V \oplus T^m \cong W \oplus T^n$ for some $m, n \in \mathbb{N}$ and T the trivial bundle.

Proposition 1.3.5

Let X be a space. Then $\text{Vect}^{\mathbb{R}} / \sim_{\text{red}}$ and $\text{Vect}^{\mathbb{C}} / \sim_{\text{red}}$ form abelian groups respectively with the Whitney sum operation.

Theorem 1.3.6

Let X be a space. Then there is are natural isomorphisms

$$KO(X) \cong \frac{\text{Vect}^{\mathbb{R}}}{\sim_{\text{red}}} \quad \text{and} \quad KU(X) \cong \frac{\text{Vect}^{\mathbb{C}}}{\sim_{\text{red}}}$$

Proposition 1.3.7

Let X be a compact Hausdorff space and let $A \subseteq X$ be a closed subspace. Then the inclusion map $i : A \to X$ and the projection map $q : X \to X/A$ induces an exact sequence

$$\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

1.4 Relative K-Theory

Definition 1.4.1

Let X be a space and let $A \subseteq X$ be a closed subspace. Define the relative K theory of the pair (X, A) by

$$K(X, A) = \widetilde{K}(X/A)$$

1.5 Representability of the Reduced K Functor

They key point of being compact Hausdorff is displayed as follows.

Theorem 1.5.1: Stabilization Theorem

Let X be a compact space. Then there is a natural isomorphism

$$\lim_{\mathbb{N}} \text{Vect}_n^{\mathbb{R}}(X) \cong \widetilde{KO}(X)$$

induced by the direct limit of the maps $\text{Vect}_n^{\mathbb{R}}(X)$. Similarly, there is a natural isomorphism

$$\lim_{\mathbb{N}} \text{Vect}_n^{\mathbb{C}}(X) \cong \widetilde{KU}(X)$$

Now $\text{Vect}_n^{\mathbb{R}}(X)$ is representable by $O(n)$. Commuting the direct limit gives the following theorem.

Theorem 1.5.2

The real and complex $K : \textbf{CH} \to \textbf{CRing}$ functors defined on the full subcategory $\textbf{CH}$ of compact Hausdorff spaces are representable:

$$KO(-) \cong [-, BO \times \mathbb{Z}] \quad \text{and} \quad KU(-) \cong [-, BU \times \mathbb{Z}]$$

Similarly, the real and complex reduced $\widetilde{K} : \textbf{CH} \to \textbf{CRing}$ functors are representable:

$$\widetilde{KO}(-) \cong [-, BO] \quad \text{and} \quad \widetilde{KU}(-) \cong [-, BU]$$

1.6 Functorial Properties of the K Functors

Theorem 1.6.1: Homotopy Invariance

If X and Y are paracompact space such that $f : X \to Y$ is a homotopy equivalence, then there is an isomorphism

$$K(f) : K(Y) \xrightarrow{\cong} K(X)$$

given by the induced map.

Theorem 1.6.2: Long Exact Sequence

Let X be a compact Hausdorff space. Let $A \subseteq X$ be a closed subspace. Then there is a long exact sequence in reduced K-theory:

$$\cdots \longrightarrow \widetilde{KU}(\Sigma(X/A)) \longrightarrow \widetilde{KU}(\Sigma X) \longrightarrow \widetilde{KU}(\Sigma A) \longrightarrow \widetilde{KU}(X/A) \longrightarrow \widetilde{KU}(X) \longrightarrow \widetilde{KU}(A) \longrightarrow \cdots$$

1.7 Fundamental Product Theorem

2 Bott Periodicity

2.1 External Product

2.2 Bott Periodicity Theorem