Measure Theory

Labix

May 6, 2025

Abstract

Contents

1 Measure Theory

1.1 σ-Algebra

Definition 1.1.1: σ-algebra

Let X be a set. Let $\mathcal{F} \subseteq \mathcal{P}(X)$. We say that $\mathcal{F}$ is a σ-algebra if the following are true.

- $S \in \mathcal{F}$.
- If $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$.
- If $A_k \in \mathcal{F}$ for $k \in \mathbb{N} \setminus \{0\}$, then $\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}$

We say that $A \in \mathcal{F}$ is a measurable set.

Lemma 1.1.2

Let X be a set. Let $\mathcal{F}$ a σ-algebra. Then the following are true.

- $\emptyset \in \mathcal{F}$.
- If $A_k \in \mathcal{F}$ for $k \in \mathbb{N}$, then $\bigcap_{k=0}^{\infty} A_k \in \mathcal{F}$.

Definition 1.1.3: Smallest σ-algebra Containing a Set

Let X be a set. Let $P \subseteq P(X)$. Define the smallest σ-algebra containing P by $\sigma(P)$.

Lemma 1.1.4

Let X be a set. Let $P \subseteq P(X)$ be a subset. Then we have

$$\sigma(P) = \bigcap_{\substack{\mathcal{F} \supseteq P \\ \mathcal{F} \text{ is measurable}}} \mathcal{F}$$

1.2 Measures

Definition 1.2.1: Measure

Let X be a set. Let $\mathcal{F}$ be a σ-algebra of X. Let $\mu : \mathcal{F} \to [0, \infty)$ be a function. We say that μ is a measure if the following are true.

- $\mu(\emptyset) = 0$
- If $A_1, \dots, A_k, \dots$ are pairwise disjoint in $\mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$$

Proposition 1.2.2

Let X be a set. Let $\mathcal{F}$ be a σ-algebra of X. Let $\mu : \mathcal{F} \to [0, \infty)$ be a measure on X.

- If $A_1, A_2 \in \mathcal{F}$ and $A_1 \subseteq A_2$, then
$$\mu(A_1) \leq \mu(A_2)$$
- If $A_k \in \mathcal{F}$ for $k \in \mathbb{N} \setminus \{0\}$, then we have
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$$
- For any $A_1, A_2 \in \mathcal{F}$, we have
$$\mu(A_1) + \mu(A_2) = \mu(A_1 \cup A_2) - \mu(A_1 \cap A_2)$$

Proposition 1.2.3

Let X be a set. Let $\mathcal{F}$ be a σ-algebra of X. Let $\mu : \mathcal{F} \to [0, \infty)$ be a measure on X. The the following are true.

- If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k \subseteq \cdots$ are measurable subsets, then we have
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\in\mathbb{N}} \mu(A_k)$$
- If $B_1 \supseteq B_2 \supseteq \cdots \supseteq B_k \supseteq \cdots$ are measurable subsets, then we have
$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\in\mathbb{N}} \mu(B_k)$$

Definition 1.2.4: Outer Measures

Let X be a set. Let $\nu : P(X) \to \emptyset$. We say that ν is an outer measure.

- $\nu(\emptyset) = 0$.
- If $A_1 \subseteq A_2$, then
$$\nu(A_1) \leq \nu(A_2)$$
- If $A_1, \dots, A_k, \dots$ are subsets, then
$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \nu(A_k)$$

Lemma 1.2.5

Let X be a set. Let $\mathcal{F}$ be a σ-algebra. Let μ be a measure. Then μ is an outer measure.

1.3 Borel Measures

Definition 1.3.1: Borel σ-algebra

Let $(X, \mathcal{T})$ be a topological space. Define the Borel σ-algebra of X to be
$$\mathcal{B}(X) = \sigma(\mathcal{T})$$

Definition 1.3.2: Borel Measure

Let X be a topological space. A Borel measure is a measure $\mu : \mathcal{B}(X) \to [0, \infty)$ on X.

Definition 1.3.3: Radon Measure

Let X be a topological space. Let μ be a Borel measure. We say that X is Radon if for any compact subset $K \in \mathcal{B}(X)$, we have
$$\mu(K) < \infty$$

2 Measure Spaces

2.1 Measure Spaces

Definition 2.1.1: Measurable Space

Let X be a set. We say that X is measurable if there exists a σ-algebra $\mathcal{F}$ and a measure $\mu : \mathcal{F} \to [0, \infty)$ on X.

Definition 2.1.2: Measure Space

A measure space $(X, \mathcal{F}, \mu)$ consists of a set X, a σ-algebra $\mathcal{F}$ and a measure μ on X.

Definition 2.1.3: Finiteness of Measure Spaces

Let $(X, \mathcal{F}, \mu)$ be a measure space.
- We say that X is finite if $\mu(X) < \infty$.
- We say that X is σ-finite if there exists a collection $\{U_k \in \mathcal{F} \mid k \in \mathbb{N} \setminus \{0\}\}$ such that $X = \bigcup_{k=1}^{\infty} U_k$ and $\mu(U_k) < \infty$.

Lemma 2.1.4

Let $(X, \mathcal{F}, \mu)$ be a measure space. If X is finite, then X is σ-finite.

2.2 Measurable Functions

Definition 2.2.1: Measurable Functions

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Let $f : E \to F$. We say that f is measurable if for all $A \in \mathcal{F}$, $f^{-1}(A) \in \mathcal{E}$.

Lemma 2.2.2

Let $(E, \mathcal{E})$, $(F, \mathcal{F})$ and $(G, \mathcal{G})$ be measurable spaces. Then the following are true.
- If $f : E \to F$ and $g : F \to G$ are measurable functions, then $g \circ f$ is measurable.
- $\text{id}_E : E \to E$ is measurable.

Definition 2.2.3: Pushforward Measure

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Let $f : E \to F$ be a measurable function. Let $\mu : \mathcal{E} \to [0, \infty)$ be a measure. Define the push forward measure $\mu_* : \mathcal{F} \to [0, \infty)$ by

$$\mu_*(A) = \mu(f^{-1}(A))$$

2.3 Convergence

Definition 2.3.1: Convergence Almost Everywhere

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Let $\mu : \mathcal{E} \to [0, \infty)$ be a measure. Let $(f_n : E \to F)_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of measurable functions. We say that $(f_n)_{n \in \mathbb{N} \setminus \{0\}}$ converges almost everywhere to a measurable function $f : E \to F$ if

$$\mu(\{x \in E \mid (f_n(x))_{n \in \mathbb{N} \setminus \{0\}} \text{ does not converge to } f(x)\}) = 0$$

Definition 2.3.2: Convergence in Measure

Let $(E, \mathcal{E})$ be a measurable space. Let $\mu : \mathcal{E} \to [0, \infty)$ be a measure. Let $(f_n : E \to \mathbb{R})_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of measurable functions. We say that $(f_n)_{n \in \mathbb{N} \setminus \{0\}}$ converges in measure to a measurable function $f : E \to F$ if

$$\mu(\{x \in E \mid |f(x) - f_n(x)| > \varepsilon\}) \to 0$$

as $n \to \infty$.

3 Integration Theory

3.1 Integration of Measurable Functions

Definition 3.1.1: Simple Functions

Let $(E, \mathcal{E}, \mu)$ be a measure space. A simple function is a function of the form

$$f(x) = \sum_{k=1}^{n} a_k 1_{A_k}(x)$$

for $A_1, \dots, A_n$ disjoint measurable sets and $a_k \in [0, \infty)$.

Definition 3.1.2: Lebesgue Integral for Simple Functions

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f(x) = \sum_{k=1}^{n} a_k 1_{A_k}(x)$ be a simple function. Define the Lebesgue integral of f to be

$$\int f \, d\mu = \sum_{k=1}^{n} a_k \mu(A_k)$$

Lemma 3.1.3

Definition 3.1.4: Lebesgue Integral for Positive Functions

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f : E \to \mathbb{R}$ be a positive measurable function. Define the Lebesgue integral of f to be

$$\int f \, d\mu = \sup \left\{ \int g \, d\mu \mid g \text{ is a simple function and } g \le f \right\}$$

Definition 3.1.5: Lebesgue Integral for General Functions

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f : E \to \mathbb{R}$ be a measurable function. Let f_+ be the positive part of f and let f_- be the negative part of f. Define the Lebesgue integral of f to be

$$\int f \, d\mu = \int f_+ \, d\mu + \int -f_- \, d\mu$$

3.2 Properties of the Lebesgue Integral

Theorem 3.2.1: Monotone Convergence Theorem

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f : E \to [0, \infty)$ be a non-negative measurable function. Let $(f_n : E \to [0, \infty))_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of non-negative measurable functions. If $(f_n) \uparrow f$, then

$$\int f_n \, d\mu \uparrow \int f \, d\mu$$

Proposition 3.2.2: Beppo-Levi

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $(f_n : E \to \mathbb{R})_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of measurable functions. Then we have

$$\int \sum_n f_n \, d\mu = \sum_n \int f_n \, d\mu$$

Theorem 3.2.3: Fatou's Lemma

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $(f_n : E \to [0, \infty))_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of non-negative measurable functions. Then we have

$$\int \left(\liminf_n f_n \right) \, d\mu \leq \liminf_n \int f_n \, d\mu$$

Theorem 3.2.4: Dominated Convergence Theorem

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $(f_n : E \to \mathbb{R})_{n \in \mathbb{N} \setminus \{0\}}$ be a sequence of measurable functions. Let $f : E \to \mathbb{R}$ be a measurable function such that f_n converges to f almost everywhere. Suppose that there exists a positive function $g : E \to \mathbb{R}$ such that $|f| \leq g$ and $|f_n| \leq g$ for all n and $\int g < \infty$. Then we have

$$\lim_n \int f_n \, d\mu = \int f \, d\mu$$

3.3 Comparison to Riemann Integrability

3.4 The Space of Measurable Functions

Definition 3.4.1: The L^p Space of a Measure Space

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $p \geq 1$. Define the associated L^p space of E to be the set of measurable functions

$$L^p(E) = \{ f : E \to \mathbb{R} \mid f \text{ is measurable } \}$$

together with the norm function $\| \cdot \|_p : L^p(E) \to \mathbb{R}$ defined by

$$\|f\|_p = \left(\int |f|^p \, d\mu \right)^{1/p}$$

Lemma 3.4.2

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $p \geq 1$. Then $L^p(E)$ is a normed space.

Proposition 3.4.3: Holder's Inequality

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $p, q \geq 1$ such that $1/p + 1/q = 1$. Let $f, g : E \to \mathbb{R}$ be measurable functions. Then we have

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

Proposition 3.4.4: Minkowski's Inequality

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $p \geq 1$. Let $f, g : E \to \mathbb{R}$ be measurable functions. Then we have

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Proposition 3.4.5: Markov's Inequality

Let $(E, \mathcal{E}, \mu)$ be a measure space. Let $f : E \to [0, \infty)$ be a non-negative measurable function. Let $\lambda > 0$. Then we have
$$\mu(\{x \in E \mid f(x) > \lambda\}) \leq \frac{1}{\lambda} \int f \, d\mu$$

4 Differentiation

4.1 Existence of Anti-Derivatives