Complex Manifolds

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Abstract

Contents

1 Holomorphic Functions of Several Variables

1.1 Holomorphicity in Several Variables

We have seen that analyticity and holomorphicity essentially mean the same thing. We begin the section be showing that holomorphicity is dependent on individual variables for functions of several variables.

Definition 1.1.1: Holomorphic Functions of Several Variables

Let U be an open subset of $\mathbb{C}^n$. Let $f : U \to \mathbb{C}$ be a complex valued function. We say that f is holomorphic if for each $w \in U$, there exists some open neighbourhood V of w such that

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}$$

Theorem 1.1.2: Osgood's Lemma

Let $U \subseteq \mathbb{C}^n$ be open. Let $f : U \to \mathbb{C}$ be a complex valued function. Then f is holomorphic on U if and only if f is holomorphic on each variable on U and f is continuous on $\mathbb{C}$.

Proposition 1.1.3

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a function where $F = (f_1, \dots, f_n)$ and $f_i(z) = u_i(z) + v_i(z)$ are the decomposition into its real and complex part. Then F is holomorphic if and only if $u_1, v_1, \dots, u_n, v_n$ are C^∞ functions that satisfy the Cauchy Riemann equations:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}$$

$$\frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}$$

Proposition 1.1.4

Let $f : U \to \mathbb{C}$ be a holomorphic function define on some $U \subseteq \mathbb{C}^n$ open. Let $z = (z_1, \dots, z_n) \in U$. Choose $\epsilon_1, \dots, \epsilon_n > 0$ such that $D_{\epsilon_1,\dots,\epsilon_n}(z) = D_{\epsilon_1}(z_1) \times \cdots \times D_{\epsilon_n}(z_n)$ is a subset of U. Then for each $w = (w_1, \dots, w_n) \in D_{\epsilon_1,\dots,\epsilon_n}(z)$ we have

$$f(w_1', \dots, w_n') = \frac{1}{(2\pi i)^n} \int_{\partial D_{\epsilon_1}(z_1)} \cdots \int_{\partial D_{\epsilon_1}(z_n)} \frac{f(w)}{(w_1 - w_1') \cdots (w_n - w_n')} \, dw_1 \cdots dw_n$$

1.2 The Inverse and Implicit Function Theorem

Theorem 1.2.1: The Inverse Function Theorem

Let $U \subseteq \mathbb{C}^n$ be open. Let $f = (f_1, \dots, f_n) : U \to \mathbb{C}^n$ be a holomorphic function. Assume that $w \in U$ is a point such that the Jacobian

$$J(w) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial z_1}\right|_w & \cdots & \left.\frac{\partial f_1}{\partial z_n}\right|_w \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial z_1}\right|_w & \cdots & \left.\frac{\partial f_n}{\partial z_n}\right|_w \end{pmatrix}$$

is invertible at $z = w$. Then there exists an open neighbourhood V of w and W of $f(w)$, as well as a holomorphic function $g : W \to V$ such that g is the inverse of f.

Theorem 1.2.2: The Implicit Function Theorem

Let $U \subseteq \mathbb{C}^n$ be open. Let $f = (f_1, \dots, f_k) : U \to \mathbb{C}^k$ be a holomorphic function. If $f(0) = 0$ and submatrix has the property that

$$\begin{vmatrix} \left.\frac{\partial f_1}{\partial z_1}\right|_0 & \cdots & \left.\frac{\partial f_1}{\partial z_k}\right|_0 \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_k}{\partial z_1}\right|_0 & \cdots & \left.\frac{\partial f_k}{\partial z_k}\right|_0 \end{vmatrix} \neq 0$$

Then there exists a holomorphic function $g = (g_1, \dots, g_k) : V \to \mathbb{C}^n$ such that for some neighbourhood W of 0 in U, we have $f(z) = 0$ is equivalent to $z_i = g_i(z_{k+1}, \dots, z_n)$ for $0 \leq i \leq k$ and $z \in W$.

1.3 Singularities

Theorem 1.3.1

Let $U \subseteq \mathbb{C}^n$ be open. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then any isolated singularities of f are removable.

2 Complex Manifolds

2.1 Basic Definitions

Definition 2.1.1: Atlas (Complex Structure)

Let M be a topological space. An atlas of M is a family of pairs $\{(U_i, \phi_i) | i \in I\}$, called charts, such that

- Each U_i is an open subset of M and $M = \bigcup_{i \in I} U_i$
- Each ϕ_i is a homeomorphism of U_i onto an open set $V \subseteq \mathbb{C}^n$
- Compatibility: Whenever $U_i \cap U_j$ is nonempty, the mapping $\phi_j \circ \phi_i^{-1}$ of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$ is holomorphic

Definition 2.1.2: Complete Atlas

A complete atlas on a topological space M is an atlas of M which is not contained in any other atlas of M.

Lemma 2.1.3

Every atlas of M is contained in a unique complete atlas.

By collecting atlas and using the complete atlas of a topological space, we can talk about all possible transitions between the open covers.

Definition 2.1.4: Complex Manifolds

A complex manifold is a Hausdorff topological space M that is second countable, together with a fixed complete atlas.

Proposition 2.1.5

Every complex manifold is a smooth real manifold.

Proof. Let $(U, \phi = z_1, \dots, z_n))$ be a chart of a complex manifold M. Write $z_k = x_k + iy_k$ where x_k and y_k are smooth real valued functions on U. Then $(x_1, \dots, x_n, y_1, \dots, y_n)$ gives a homeomorphism between U and an open subset of $\mathbb{R}^{2n}$. $\square$

2.2 Holomorphic Maps between Manifolds

Definition 2.2.1: Holomorphic Maps to $\mathbb{C}^k$

A continuous function $f : X \to \mathbb{C}^k$ is called holomorphic if for every chart (U, ϕ), we have that $f \circ \phi^{-1} : \mathbb{C}^n \to \mathbb{C}^k$ is holomorphic.

Definition 2.2.2: Holomorphic Maps between Manifolds

A continuous map $f : M \to N$ between two complex manifolds is called holomorphic if $\psi \circ f \circ \phi^{-1}$ is holomorphic for every pair of charts (U, ϕ) of M and (V, ψ) of N such that $f(U) \subset V$.

2.3 Complex Submanifolds

Definition 2.3.1: Complex Submanifolds

Let M be an n dimensional manifold. An m dimensional submanifold is a subset Y of M such that for every $y \in Y$, there exists a chart (U, ϕ) of M such that

$$\phi(Y \cap U) = \phi(U) \cap \{0\}^{n-m} \times \mathbb{C}^m = \{(z_1, \dots, z_n) \in \phi(U) | z_{n-m+1} = \dots = z_n = 0\}$$

3 Sheaves of Rings on Complex Manifolds

3.1 The Weierstrass Theorem

Theorem 3.1.1: The Weierstrass Preparation Theorem

Proposition 3.1.2

Let $p \in \mathbb{C}^n$. Then $\mathcal{O}_{\mathbb{C}^n,p}$ is Noetherian and is s UFD

3.2 Sheaves on Complex Manifolds

Definition 3.2.1: Ring of Continuous Complex Functions

Let $U \subseteq \mathbb{C}^n$ be open. Define the ring of continuous complex functions to be the set

$$\mathcal{C}^0(U) = \{f : U \to \mathbb{C} \mid f \in C^0\}$$

together with pointwise addition and multiplication.

TBA: Topology on $\mathcal{C}^0(U)$

Definition 3.2.2: Ring of Holomorphic Functions

Let $U \subseteq \mathbb{C}^n$ be open. Define the ring of holomorphic functions to be the subring

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic }\} \subseteq \mathcal{C}^0(U)$$

Definition 3.2.3: Local Ring of Differentiable Functions

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. For $0 \leq r \leq \infty$, define the local ring of r-differentiable functions at p by

$$\mathcal{C}^r_{U,p} = \{(V, f : V \to \mathbb{C}) \mid V \text{ is open and } f \text{ is } r\text{-differentiable }\}/\sim$$

where we say that $(V, f) \sim (W, g)$ if there exists $X \subseteq V \cap W$ open such that $f|_X = g|_X$. An element of the equivalence class is called a germ.

Definition 3.2.4: The Local Ring of Holomorphic Functions

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. Define the local ring of holomorphic functions at p by

$$\mathcal{O}_{U,p} = \{(V, f : V \to \mathbb{C}) \mid V \text{ is open and } f \text{ is holomorphic }\}/\sim$$

where we say that $(V, f) \sim (W, g)$ if there exists $X \subseteq V \cap W$ open such that $f|_X = g|_X$. An element of the equivalence class is called a germ.

Proposition 3.2.5

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. Then $\mathcal{O}_{U,p}$ is a local ring and an integral domain.

Proposition 3.2.6

Let $p \in \mathbb{C}^n$. Then $\mathcal{O}_{\mathbb{C}^n,p}$ is isomorphic to the ring of convergent power series centered at p.

Definition 3.2.7: Field of Meromorphic Germs

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. Let m_p be the unique maximal ideal of $\mathcal{O}_{U,p}$. Define the field of meromorphic germs on p by

$$\mathcal{M}_{U,p} = \frac{\mathcal{O}_{U,p}}{m_p}$$

4 More on Vector Bundles

4.1 Structure on Vector Spaces

Definition 4.1.1: Linear Complex Structure

Let V be a real vector space. A linear complex structure on V is a map $T : V \to V$ such that $T^2 = -\text{id}$.

Proposition 4.1.2

Let V be a real vector space admitting a linear complex structure T. Then V can be seen as a complex vector space.

Proposition 4.1.3

Let V be a real vector space admitting a linear complex structure T. Then $W_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ admits the decomposition

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

where $W^{1,0}$ is the $\mathbb{C}$-linear forms and $W^{0,1}$ the $\mathbb{C}$-antilinear forms.

4.2 Structure on Vector Bundles

Definition 4.2.1: Almost Complex Structure

Let $p : E \to B$ be a vector bundle. An almost complex structure J on E is a linear complex structure on each fibre varying smoothly on E. In other words, each J_x for $x \in E$ is a linear complex structure.

Definition 4.2.2: Almost Complex Manifolds

An almost complex manifold is a complex manifold M such that its tangent space TM has an almost complex structure.

Proposition 4.2.3

Every complex manifold is an almost complex manifold.

Definition 4.2.4: Integrable Complex Structure

An almost complex structure on a manifold M is said to be integrable if there exists a complex structure on M which induces I.

Definition 4.2.5: Hermitian Structure

Let $p : E \to B$ be a vector bundle. A Hermitian structure H on E is a Hermitian product on each fibre varying smoothly on E. This means that for $x \in M$, $H : E_x \times E_x \to \mathbb{C}$ satisfies the following.

- $H(u, v)$ is $\mathbb{C}$-linear for every $v \in E_x$
- $H(u, v) = \overline{H(v, u)}$
- $H(u, u) > 0$ for all $u \neq 0$
- $H(u, v)$ is a smooth function on M for every smooth sections u, v of E

Definition 4.2.6: Holomorphic Vector Bundle

Let $p : E \to M$ be a vector bundle over a complex manifold M. We say the vector bundle is holomorphic (equipped with a holomorphic structure) if the trivializations

$$\tau_i : p^{-1}(U_i) \stackrel{\cong}{\to} U_i \times \mathbb{C}^n$$

has transition matrices $\tau_{ij} = \tau_j \circ \tau_i$ that have holomorphic coefficients.

5 Tangent Spaces

5.1 The Complex Tangent Space

Since every complex manifold is a smooth real manifold, there is no need to redefine everything. We begin this section with a note that for a complex manifold M of dimension n, M has the real tangent space structure on $p \in M$ with basis

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\}$$

This is now denoted $T_{\mathbb{R}}M$.

The following is an analogue to the tangent bundle of a smooth manifold. We shall see later that by identifying a complex manifold as also a smooth manifold of double the dimension, we can decompose this tangent bundle.

Definition 5.1.1: Holomorphic Tangent Bundles

Let M be a complex manifold. Let $\{(U_i, \phi_i = (z_1, \dots, z_n)) | i \in I\}$ be an atlas. Denote $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ and the

$$\phi_{ij*} = \begin{pmatrix} \frac{\partial(\phi_{ij})_1}{\partial z_1} & \cdots & \frac{\partial(\phi_{ij})_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial(\phi_{ij})_n}{\partial z_1} & \cdots & \frac{\partial(\phi_{ij})_n}{\partial z_n} \end{pmatrix}$$

Define the tangent bundle as the union of $U_i \times \mathbb{C}^n$, glued by identifying $U_i \cap U_j \times \mathbb{C}^n \subset U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n$ by the map $(u, v) \mapsto (u, \phi_{ij*}(v))$. Denote the holormorphic tangent bundle as TM. Each fibre of TM is denoted T_pM.

Definition 5.1.2: Complexified Tangent Space

Let M be a complex manifold of dimension n. Define the complexified tangent space of M as

$$T_{\mathbb{C}}M = T_{\mathbb{R}}M \otimes \mathbb{C}$$

Locally, each fibre of $T_{\mathbb{C}}M$ for $p \in M$ has the decomposition

$$T_{\mathbb{C},p}M = T_{\mathbb{R},p}M \otimes \mathbb{C}$$

Proposition 5.1.3

Let M be a manifold. The complexified tangent space $T_{\mathbb{C}}M$ decomposes into

$$T_{\mathbb{C}}M \cong T^{1,0}M \oplus T^{0,1}M$$

where $T^{1,0}M$ is the bundle of eigenvectors for the eigenvalue i and $T^{0,1}M$ is the bundle of eigenvectors for the eigenvalue $-i$ of the almost complex structure I.

Proposition 5.1.4

Let M be a complex manifold. Then M admits an almost complex structure, and the we have an isomorphism $T^{1,0}M \cong TM$

Proposition 5.1.5

Let M be a complex manifold of dimension n. Then $T_{\mathbb{C}}M$ has basis

$$\left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_n}\right\}$$

or equivalently, with basis

$$\left\{\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n},\frac{\partial}{\partial \bar{z}_1},\dots,\frac{\partial}{\partial \bar{z}_n}\right\}$$

Proposition 5.1.6

Let M be a complex manifold. Then $T^{0,1}M=\overline{T^{1,0}M}$

Lemma 5.1.7

Let M be a complex manifold. Then the map

$$TM \hookrightarrow T_{\mathbb{C}}M \twoheadrightarrow T^{1,0}M$$

is an isomorphism.

6 Complex Differential Forms

6.1 Differential 1-Forms

Once again, we already have the notion of differential forms for a complex manifold since we already have it for smooth real manifolds. However, we can once again use the decomposition $dz = dx + idy$ and $d\bar{z} = dx - idy$. This is why we can write any k-form on a complex manifold as

$$\omega = \sum_{|I|+|J|=k} \phi_{I,J} dz_I \wedge d\bar{z}_J$$

Definition 6.1.1: Differential 1-Forms

Let M be a complex manifold. A differential 1-form on M is a function $\omega : M \to T^*M$ such that $\omega(p) \in T_p(M)^*$. In local coordinates, we have that

$$\omega = \sum_{k=1}^{n} (f_k dz_k + g_k d\bar{z}_k)$$

Denote $\Omega^{1,0}(M)$ the space of complex differential forms only containing dz_k and $\Omega^{0,1}(M)$ the space of complex differential forms only containing $d\bar{z}_k$.

Recall that for a smooth manifold M, we define the vector bundle of differential 1-forms as $\Omega^1(M) = T^*M$.

Proposition 6.1.2

The spaces $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ for a complex manifold M defines a vector bundle over M. Moreover, the complexification $T_{\mathbb{C}}M$ induces a dual decomposition

$$\Omega^1_{\mathbb{C}}(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

6.2 Differential k-Forms

Definition 6.2.1: Differential k-Forms

Let M be a complex manifold. We say that a differential k-form is of type (p,q) if it can be locally written in the form

$$\omega = \sum_{|I|=p} \sum_{|J|=q} \phi_{I,J} dz_I \wedge d\bar{z}_J$$

where $p + q = k$, and $\phi_{I,J}$ is a smooth function on U.

Denote the space of all (p,q)-forms by $\Omega^{p,q}(M)$.

Proposition 6.2.2

Let M be a complex manifold. Then

$$\Omega^{p,q}(M) = \bigwedge_{i=1}^{p} \Omega^{1,0}(M) \otimes \bigwedge_{j=1}^{q} \Omega^{0,1}(M)$$

Lemma 6.2.3

Let M be a complex manifold. We have the decomposition

$$\Omega_{\mathbb{C}}^k(M) = \bigwedge_{i=1}^k \Omega_{\mathbb{C}}^1(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

Proposition 6.2.4

Let M be a complex manifold. Let $d : \Omega^k(M) \to \Omega^{k+1}(M)$ be the de Rham differential treating M as a smooth manifold. The extended differential

$$d_{\mathbb{C}} : \Omega_{\mathbb{C}}^k(M) \to \Omega_{\mathbb{C}}^{k+1}(M)$$

is a boundary operator for the sequence $\Omega_{\mathbb{C}}^\bullet(M)$.

In other words, $\Omega_{\mathbb{C}}^\bullet$ is a cochain complex.

6.3 Dolbeault Cohomology

Definition 6.3.1: Complex de Rham Cohomology

Let M be a smooth real manifold. Define the complex de Rham cohomology groups to be

$$H_{\text{dR}}^k(M;\mathbb{C}) = \frac{\ker(d : \Omega_{\mathbb{C}}^k(M) \to \Omega_{\mathbb{C}}^{k+1}(M))}{\text{im}(d : \Omega_{\mathbb{C}}^{k-1}(M) \to \Omega_{\mathbb{C}}^k(M))} = H_{dR}^k(\Omega_{\mathbb{C}}^\bullet(M))$$

Recall that complex manifolds are in particular smooth real manifolds, hence complex de Rham cohomology also makes sense on complex manifolds.

Proposition 6.3.2

Let M be a complex manifold. The real and complex de Rham cohomology groups of M satisfy

$$H_{\text{dR}}^k(M;\mathbb{C}) = H_{\text{dR}}^k(M;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$$

We can also decompose the exterior derivative into its holomorphic and antiholomorphic part.

Lemma 6.3.3

Let M be a complex manifold. Denote d the exterior derivative of smooth manifolds. Then we can decompose

$$d_{\mathbb{C}} = \partial + \bar{\partial}$$

where $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. In local coordinates,

$$\bar{\partial}\left(\sum \phi_{I,J} dz_I \wedge dz_J\right) = \sum \frac{\partial \phi_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

Proposition 6.3.4

Let M be a complex manifold. Then $\Omega^{p,\bullet}$ together with $\bar{\partial}$ defines a chain complex.

Definition 6.3.5: Dolbeault Complex

Let M be a complex manifold. Define the Dolbeault complex of M to be the cochain complex

$$\cdots \longrightarrow \Omega^{p,q-1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\overline{\partial}} \cdots$$

denoted $\Omega^{p,\bullet}(M)$.

Definition 6.3.6: Dolbeault Cohomology

Define the dobeault cohomology of a complex manifold m to be

$$H^{p,q}(M) = \frac{\ker(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)}{\operatorname{im}(\bar{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))} = H^q(\Omega^{p,\bullet})$$

7 Hermitian Manifolds

7.1 Hermitian Manifold and its Metric

Definition 7.1.1: Hermitian Manifold

A complex manifold M is said to be Hermitian if the holomorphic tangent bundle has a Hermitian structure.

Definition 7.1.2: Hermitian Metric

A Hermitian metric on a complex vector space V is a map $h : V \times V \to \mathbb{C}$ such that

- $h(v, w) = \overline{h(w, v)}$ for all $v, w \in V$
- $h(v, v) > 0$ for all $v \in V$

A Hermitian metric on a vector bundle $p : E \to B$ is a smoothly varying Hermitian metric on each fibre E_x of E for $x \in E$.

Proposition 7.1.3

Let M be an almost complex manifold. Every Hermitian metric on M induces a Hermitian structure on M. Every Hermitian structure on M induces a Hermitian metric on M.

Proof. Let h be a Hermitian metric on M. Then $H(X, Y) = h(X, Y) - ih(JX, Y)$ defines a Hermitian structure on M. Conversely, let H be a Hermitian structure on the tangent space $T_{\mathbb{C}}M$ defines a Hermitian metric by $h(X, Y) = \text{Re}(X, Y)$. □

This shows that Hermitian metrics and Hermitian structure essentially mean the same thing, just in different presentations.

Proposition 7.1.4

Every almost complex manifold admits a Hermitian metric.

Proof. Choose any arbitrary Riemannian metric g. Then define $h(X, Y) = g(X, Y) + g(JX, JY)$. This is a Hermitian metric. □

7.2 The Riemannian Metric, The Hermitian Metric and the Associated Form

Proposition 7.2.1

Every hermitian metric h on a complex manifold M defines a Riemannian metric

$$g(u, v) = \frac{1}{2}(h + \overline{h})$$

In local coordinates, g is expressed as

$$g(u, v) = \frac{1}{2} \sum h_{\alpha\overline{\beta}}(dz_\alpha \otimes d\overline{z}_\beta + d\overline{z}_\beta \otimes dz_\alpha)$$

Lemma 7.2.2

Let M be a Hermitian manifold. Denote h the Hermitian metric of M. Then

$$\omega(x,y)=\frac{i}{2}(h-\overline{h})$$

is a $(1,1)$ form

In local coordinates, ω is expressed as

$$\omega=\frac{i}{2}\sum_{\alpha,\beta=1}^{n}h_{\alpha\overline{\beta}}dz_\alpha\wedge d\overline{z}_\beta$$

if M is a complex manifold of complex dimension n.

Definition 7.2.3: Associated Form of a Hermitian Metric

Let M be a Hermitian manifold. Let h be the Hermitian metric. Define the associated form of h to be the $(1,1)$-form

$$\omega(u,v)=\frac{i}{2}(h-\overline{h})$$

Proposition 7.2.4

Let M be a Hermitian manifold. Then the following are true in terms of the metrics.

- $\omega(u,v)=g(Ju,v)$
- $g(u,v)=\omega(u,Jv)$
- $h=g-i\omega$

Theorem 7.2.5

Let M be a Hermitian manifold. Denote h the Hermitian metric. Then h,g,ω preserve the almost complex structure J of M. This means that the following are true.

- $h(Ju,Jv)=h(u,v)$
- $g(Ju,Jv)=g(u,v)$
- $\omega(Ju,Jv)=\omega(u,v)$

Lemma 7.2.6

Let M be an almost complex manifold. Let g be a Riemannian metric on M such that $g(Ju,Jv)=g(u,v)$. Then g induces a Hermitian metric.

Lemma 7.2.7

Let M be an almost complex manifold. Let ω be a non-degenerate $(1,1)$-form such that $\omega(Ju,Jv)=\omega(u,v)$ and that $\omega(u,Ju)>0$ for all tangent vectors u. Then ω induces a Hermitian metric.