Higher Category Theory

Labix

November 17, 2024

Abstract

Contents

1 Introduction to Infinity Categories

1.1 Infinity Categories as Simplicial Sets

We recall some basic facts about simplicial sets. If $S : \Delta \to \textbf{Set}$ is a simplicial set, then by Yoneda's emebdding we know that the n-simplices of S are given by

$$S([n]) = \text{Hom}_{\textbf{sSet}}(\Delta^n, S)$$

In other words, specifying an n-simplex is the same as specifying a map of simplicial sets

$$\Delta^n \to S$$

The foundations of infinity categories lay on the simpicial sets. Intuitively, any face $\partial_k\Delta$ of an n-simplex Δ captures a homotopy of the faces of $\partial_k\Delta$.

Definition 1.1.1: Infinity Categories

An infinity category is a simplicial set C such that each inner horn admits a filler. In other words, for all $0 < i < n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Definition 1.1.2: Objects and Morphisms

Let $\mathcal{C}$ be an infinity category. Define the following notions for $\mathcal{C}$.
- Define the objects of $\mathcal{C}$ to be the 0-simplices of $\mathcal{C}$.
- Define the morphisms of $\mathcal{C}$ to be the 1-simplices of $\mathcal{C}$.

Definition 1.1.3: Functors

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. A functor from $\mathcal{C}$ to $\mathcal{D}$ is a morphism $F : \mathcal{C} \to \mathcal{D}$ of simplicial sets.

Definition 1.1.4: The Category of Infinity Categories

Define the category $\infty-\textbf{Cat}$ of infinity categories to be the full subcategory of $\textbf{sSet}$ consisting of infinity categories.

Lemma 1.1.5

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the following are true.
- F sends an object of $\mathcal{C}$ to an object of $\mathcal{D}$.
- F sends a morphism in $\mathcal{C}$ to a morphism in $\mathcal{D}$.
- F sends the identity morphism of $X \in \mathcal{C}$ to the identity morphism of $F(X) \in \mathcal{D}$.
- If $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$

Explicitly, morphisms of infinity categories behave exactly what we want it to be like: A generalization of functors between ordinary categories. However, note that it is not enough to specify a morphism of infinity categories just from specifying it on objects. This is because we also need to tell the functor where to map the n-simplices. In other words, we need to tell the functor where to send the homotopy data.

Proposition 1.1.6

Let $\mathcal{C}$ be a category. Then the nerve $N_\bullet(\mathcal{C})$ of $\mathcal{C}$ is an infinity category.

In other words, the nerve functor $N_\bullet(-)$ actually lands in $N_\bullet$.

Proposition 1.1.7

The nerve functor $N_\bullet : \mathbf{Cat} \to \infty\!-\!\mathbf{Cat}$ is fully faithful. This means that there is a bijection

$$\{\text{Functors from } \mathcal{C} \text{ to } \mathcal{D}\} \overset{1:1}{\longleftrightarrow} \{\text{Functors from} N_\bullet(\mathcal{C}) \text{ to } N_\bullet(\mathcal{D})\}$$

given by $N_\bullet$.

Recall that **sSet** is closed monoidal. Moreover, we can identify functors from two infinity categories $\mathcal{C}$ to $\mathcal{D}$ as the zero simplicies of the simplicial set $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$ defined by

$$[n] \mapsto \mathrm{Hom}\,\mathbf{sSet}(\mathcal{C} \times \Delta^n, \mathcal{D})$$

In fact, we can show that $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$ is an infinity category.

Proposition 1.1.8

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Then

$$\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$$

is an infinity category.

1.2 Natural Transformations

Definition 1.2.1: Natural Transformations

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation from F to G is a morphism $\mathcal{C} \times \Delta^1 \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\mathrm{id}_\mathcal{C} \times \delta_0"] \arrow[dr, "f"'] & \mathcal{C} \times \Delta^1 \arrow[d, dashed, "\exists \eta"] & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C} \times \delta_1"'] \arrow[dl, "g"] \\
& \mathcal{D} &
\end{tikzcd}$$

Lemma 1.2.2

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G \in \mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$ be functors. Then the following are equivalent.

- $\eta : \mathcal{C} \times \Delta^1 \to \mathcal{D}$ is a natural transformation from F to G
- η is an 1-simplex in $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$.
- η is a homotopy from F to G.

Lemma 1.2.3

Let $\mathcal{C}, \mathcal{D}$ be categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then $\alpha : F \Rightarrow G$ is a natural transformation if and only if $N(\alpha) : N(\mathcal{C}) \to N(\mathcal{D})$ is a natural transformation of infinity categories.

Definition 1.2.4: Natural Isomorphisms

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. We say that F and G are naturally isomorphic if

1.3 The Homotopy Category of Infinite Categories

Definition 1.3.1: Homotopic Morphisms

Let $\mathcal{C}$ be an infinity category. Two morphisms $f, g : C \to D$ are said to be homotopic if there exists a 2-simplex σ in $\mathcal{C}$ such that

- $d_0(\sigma) = \mathrm{id}_D$
- $d_1(\sigma) = g$
- $d_2(\sigma) = f$

In this case we write $f \simeq g$.

Pictorially, we denote the existence of such a σ by

$$\begin{tikzcd} & D \arrow[dr, "\mathrm{id}_D"] \arrow[d, phantom, "\sigma"] & \\ C \arrow[ur, "f"] \arrow[rr, "g"'] & {} & D \end{tikzcd}$$

This diagram here does not denote commutative, but instead denotes the existence of a 2-simplex σ that has the above as vertices and edges. Rewriting the above definition, we can say that $g \circ f : C \to E$ is homotopic to $h : C \to E$ if there exists a 2-simplex of the form

$$\begin{tikzcd} & D \arrow[dr, "g"] \arrow[d, phantom, "\sigma"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"'] & {} & E \end{tikzcd}$$

By definition of an infinity category, every inner horn admits a filler. This means that for any composable morphisms f and g giving $g \circ f$, we can always find a morphism h such that $g \circ f$ is homotopic to h. However, this h may not be unique, so we cannot conclude that infinity categories have a well defined notion of composition.

Lemma 1.3.2

Let $\mathcal{C}$ be an infinity category. Then the relation of homotopic between morphisms is an equivalence relation.

QUESTION: IS THE FOLLOWING TRUE (Kerodon 1.4.3.7)

Proposition 1.3.3

Let $\mathcal{C}$ be an infinity category. Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$. Then f and g are homotopic if and only if $\Delta^1 \xrightarrow{f,g} \mathcal{C}$ are homotopic as morphisms of simplicial sets.

Proposition 1.3.4

Let C be an infinity category. Let $f, f' : C \to D$ and $g, g' : D \to E$ be morphisms in C. If $f \simeq f'$ and $g \simeq g'$, then
$$g \circ f \simeq g' \circ f'$$

We can explicitly write out the homotopy category of an infinity category as follows.

Proposition 1.3.5

Let $\mathcal{C}$ be an infinity category. Then the homotopy category $h(\mathcal{C})$ is isomorphic (as categories) to the category defined as follows.

- The objects of $h(\mathcal{C})$ are the objects of $\mathcal{C}$
- For $A, B \in \mathcal{C}$ two objects, the morphisms are the equivalence classes of morphisms $[f]$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- Composition is defined by
$$[g] \circ [f] = [g \circ f]$$
which is well defined by .2

Definition 1.3.6: Isomorphisms in Infinity Categories

Let $\mathcal{C}$ be an infinity category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. We say that f is an isomorphism if $[f]$ is an isomorphism in $h(\mathcal{C})$. Equivalently, f is an isomorphism if there exists a morphism $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Because the data of a functor between infinity categories carry 2-simplicies to 2-simplicies, we can easily deduce the following.

Lemma 1.3.7

Let $\mathcal{C}, \mathcal{D}$ be infinity category. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the following are true.

- If $f \simeq g$ are homotopic in $\mathcal{C}$, then $F(f) \simeq F(g)$ are homotopic in $\mathcal{D}$.
- If f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.

When $\mathcal{C}, \mathcal{D}$ are ordinary categories, we can talk about diagrams of shape $\mathcal{C}$ in $\mathcal{D}$. This just means that we only care about the shape of $\mathcal{C}$, and we consider this shape inside $\mathcal{D}$. This was the foundations for limits and colimits of a category. We can also do this for infinity categories, but recall that a functor between infinity categories carries much more data than just the shape of the domain infinity category: it also carries homotopy information.

1.4 Equivalence of Infinity Categories

Definition 1.4.1: Natural Isomorphisms

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Let η be a natural transformation from F to G. We say that η is a natural isomorphism if η is an isomorphism as a morphism in $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{D})$.

We say that F and G are naturally isomorphic if there exists a natural isomorphism from F to G.

Kerodon 4.4.4.1

Definition 1.4.2: Homotopy Inverses

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ an $G : \mathcal{C} \to \mathcal{D}$ be functors. We say that G is a homotopy inverse of F

- $G \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$ in $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{C})$
- $F \circ G$ is naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ in $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{D}, \mathcal{D})$

Kerodon 4.5.1.10

Definition 1.4.3: Equivalence of Infinity Categories

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is an equivalence of infinity categories if F admits a homotopy inverse. In this case, we say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Recall that two objects in an infinity category $\mathcal{C}$ is isomorphic if they are isomorphic in $h(\mathcal{C})$ in the ordinary sense. In our case, this means that we consider $G \circ F$ and $\mathrm{id}_{\mathcal{C}}$ to be objects of the infinity category $\mathrm{Hom}_{\mathbf{sSet}}(\mathcal{C}, \mathcal{C})$, and they are isomorphic if $[G \circ F] = [\mathrm{id}_{\mathcal{C}}]$. This is the same as saying that $G \circ F$ and $\mathrm{id}_{\mathcal{C}}$ are homotopic. (It is also the same as saying $\mathcal{C}$ and $\mathcal{D}$ are homotopy equivalent as simplicial sets)

Lemma 1.4.4

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. If $\mathcal{C}$ and $\mathcal{D}$ are naturally isomorphic, then $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Proposition 1.4.5

Let $\mathcal{C}, \mathcal{D}$ be ordinary categories. Let $F : \mathcal{C} \to \mathcal{D}$ be functor. Then $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories if and only if $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ induces an equivalence of categories.

Proposition 1.4.6

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is an equivalence of infinity categories, then $h(F) : h(\mathcal{C}) \to h(\mathcal{D})$ is an equivalence of ordinary categories.

Proposition 1.4.7

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of infinity categories if and only if

$$F \circ - : \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C}) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{D})$$

is an equivalence of infinity categories for all simplicial sets K.

2 Infinity Categorical Constructions

2.1 Joins and Slices

We begin by rewriting the definition of a simplex category as follows. Instead of having distinguished names $[n]$ for the objects, we instead just think of the simplex category with objects as finite and totally ordered sets. Indeed any of these sets will be in bijection to $[n]$ for some $n \in \mathbb{N}$. This language will help us define the join.

Definition 2.1.1

Let J be a finite and totally ordered set. A cut of J consists of two subsets $I, I' \subseteq J$ such that

$$J = I \amalg I'$$

and $i < i'$ for all $i \in I$ and $i' < I'$.

Definition 2.1.2: Joins

Let X, Y be simplicial sets. Define the join of X and Y to be the simplicial set $X * Y$ as follows.

- Denote $J \neq \emptyset$ any finite and totally ordered set. Define

$$X * Y(J) = \coprod_{\substack{I \amalg I' = J \\ i < i' \text{ for } i \in I, i' \in I'}} X(I) \times Y(I') \quad \coprod_{I, I' \text{ cuts of } J} X(I) \times Y(I')$$

  where by convention, $X(\emptyset) = Y(\emptyset) = *$.
- For two finite and totally ordered sets J and J' and a morphism $J \to J'$ preserving order, the map

$$(X * Y)[J'] \to (X * Y)[J]$$

  is defined as follows. Let K, K' be a cut of J'. Then α restricts to two well defined maps

$$\alpha|_{\alpha^{-1}(K)} : \alpha^{-1}(K) \to K \quad \text{and} \quad \alpha|_{\alpha^{-1}(K')} : \alpha^{-1}(K') \to K'$$

  In particular these are order preserving, and each are morphisms in the simplex category Δ. Thus this gives us a unique morphism

$$X(K) \times X(K') \to X(\alpha^{-1}(K)) \times X(\alpha^{-1}(K'))$$

  By taking the product of these maps, we thus obtain a morphism $(X * Y)[J'] \to (X * Y)[J]$, turning the above definition into a simplicial set.

Concrete examples:

- When $J = [0]$, we have that

$$\begin{aligned}(X * Y)[0] &= X[0] \times Y(\emptyset) \amalg X(\emptyset) \times Y[0] \\ &= X_0 \amalg Y_0\end{aligned}$$

  which means that the vertices of $X * Y$ are the vertices of X and Y combined disjointly.
- When $J = [1]$, we have that

$$\begin{aligned}(X * Y)[1] &= X[1] \times Y(\emptyset) \amalg X(\{0\}) \times Y(\{1\}) \amalg X(\emptyset) \times Y[1] \\ &= X_1 \amalg X_0 \times Y_0 \amalg Y_1\end{aligned}$$

TBA: The join of ordinary categories.

Lemma 2.1.3

Let X and Y be simplicial sets. Then $N(X * Y) \cong N(X) * N(Y)$

TBA: functoriality of join

Proposition 2.1.4

Let X, Y be simplicial sets. Then $X * Y$ is an infinity category if and only if X and Y are infinity categories.

Recall that the over category $\mathcal{C}/X$ consists of pairs $(Y, f : Y \to X)$ and morphism are given by commutative diagrams. Let us rephrase the definition as follows. The over category is the unique category such that if $\mathcal{D}$ is another category, there is a bijection

$$\mathrm{Hom}_{\mathbf{CAT}}(\mathcal{D}, \mathcal{C}/X) \cong \mathrm{Hom}_X(\mathcal{D} * [0], \mathcal{C})$$

where the right hand side indicates that we only consider morphisms $\mathcal{D} * [0] \to \mathcal{C}$ in which $[0]$ is mapped to X. This characterization is due to the fact that a morphism $[0] \to \mathcal{C}$ is essentially a choice of object in $\mathcal{C}$, in which case we choose to be X.

Definition 2.1.5: Over Category for Infinity Categories

Let K, X be simplicial sets. Let $f : K \to X$ be a map. Define the over category (which is a simplicial set)

$$f/X : \Delta \to \mathbf{Set}$$

as follows.

- For each n, we have
$$(f/X)_n = \mathrm{Hom}_{K/\mathbf{sSet}}(K * \Delta^n, X)$$

TBA: Adjunction of join and slice.

3 The Joyal Model Structure

3.1 Homotopy Pullbacks and Pushouts

Because the standard model structure on **sSet** and the Joyal model structure are not Quillen equivalent, there are differences also in homotopy limits and colimits.

Definition 3.1.1: Homotopy Pullbacks

Let the following be a diagram of infinity categories and functors.

$$\mathcal{C}_0 \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{C}_1$$

Let Isom$(\mathcal{C})$ denote the full subcategory of $\text{Hom}_{\textbf{sSet}}(\Delta^1, \mathcal{C})$ consisting of isomorphisms. Define the homotopy limit of the diagram with respect to the Joyal structure to be the two times pullback

$$\mathcal{C}_0 \times_{\mathcal{C}}^{hJ} \mathcal{C}_1 = \mathcal{C}_0 \times_{\text{Hom}_{\textbf{sSet}}(\{0\},\mathcal{C})} \text{Isom}(\mathcal{C}) \times_{\text{Hom}_{\textbf{sSet}}(\{1\},\mathcal{C})} \mathcal{C}_1$$

Here we identify the vertices of Δ^1 as 0 and 1, and we use the isomorphism $\mathcal{C} \cong \text{Hom}_{\textbf{sSet}}(\Delta^0, \mathcal{C})$.

Kerodon 4.5.2.1

4 Simplicial Categories as Models for Infinity Categories

4.1 Simplicial Categories

Definition 4.1.1: Simplicial Categories

A simplicial category $\mathcal{C}$ is a category enriched over **sSet**. Explicitly, it consists of the following data:

- $\mathcal{C}$ is a category (and so consisting of objects, morphisms that satisfy associativity and unitality)
- For each $X, Y \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(X, Y)$ is a simplicial set.
- For each $XZ, Y, Z \in \mathcal{C}$, the composition rule

$$\circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Z)$$

  is a morphism of simplicial sets.

These data are organized such that the following coherence diagrams are commutative:

- Associativity:

$$\begin{tikzcd}
\text{Hom}_{\mathcal{C}}(Z, W) \times \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\text{id}\times\circ"] \arrow[d, "\circ\times\text{id}"'] & \text{Hom}_{\mathcal{C}}(Z, W) \times \text{Hom}_{\mathcal{C}}(X, Z) \arrow[d, "\circ"] \\
\text{Hom}_{\mathcal{C}}(Y, W) \times \text{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\circ"'] & \text{Hom}_{\mathcal{C}}(X, W)
\end{tikzcd}$$

- Unitality:

$$\begin{tikzcd}
\{\text{id}_Y\} \times \text{Hom}_{\mathcal{C}}(X, Y) \arrow[d, hook] \arrow[dr, "\text{proj}"] & \\
\text{Hom}_{\mathcal{C}}(Y, Y) \times \text{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\circ"'] & \text{Hom}_{\mathcal{C}}(X, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Hom}_{\mathcal{C}}(X, Y) \times \{\text{id}_X\} \arrow[r, hook] \arrow[dr, "\text{proj}"'] & \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(X, X) \arrow[d, "\circ"] \\
& \text{Hom}_{\mathcal{C}}(X, Y)
\end{tikzcd}$$

Given an ordinary category, one can construct a simplicial category in which the the higher morphisms are trivial.

Definition 4.1.2: The Constant Simplicial Category

Let $\mathcal{C}$ be a category. Define a simplicial category $\underline{\mathcal{C}}$ as follows.

- The objects of $\underline{\mathcal{C}}$ are the objects of $\mathcal{C}$
- For $X, Y \in \mathcal{C}$, define $\text{Hom}_{\underline{\mathcal{C}}} X, Y)$ to be the simplicial set given by

$$\Delta^{\text{op}} \to \{\text{Hom}_{\mathcal{C}}(X, Y)\} \hookrightarrow \textbf{Set}$$

- For $X, Y, Z \in \mathcal{C}$, define the composition law

$$\circ : \text{Hom}_{\underline{\mathcal{C}}}(Y, Z) \times \text{Hom}_{\underline{\mathcal{C}}}(X, Y) \to \text{Hom}_{\underline{\mathcal{C}}}(X, Z)$$

  by the composition law $\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Z)$

Conversely, if we start with a simplicial category, we can recover a family of ordinary categories.

Definition 4.1.3: The Ordinary Categories Associated to a Simplicial Category

Let $\mathcal{C}$ be a simplicial category. Let $n \in \mathbb{N}$. Define a category $\mathcal{C}_n$ as follows.

- The objects of $\mathcal{C}_n$ are the objects of $\mathcal{C}$
- For $X, Y \in \mathcal{C}_n$, the set of morphisms are given by

$$\text{Hom}_{\mathcal{C}_n}(X, Y) = (\text{Hom}_{\mathcal{C}}(X, Y))_n$$

  are given by the n-simplicies of the simplicial set $\text{Hom}_{\mathcal{C}}(X, Y)$. In particular, the identity morphism is given by the n-simplex corresponding to the map $\Delta^n \to \Delta^0 \xrightarrow{\text{id}_X} \text{Hom}_{\mathcal{C}}(X, X)$

- For $f : X \to Y$ and $g : Y \to Z$ two morphisms in $\mathcal{C}_n$, the composition $g \circ f$ is given by the image of (g, f) in the composition law

$$\circ : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{C}}(X, Z)$$

We call $\mathcal{C}_0$ the underlying category of the simplicial category. Indeed, there is a forgetful functor that sends every simplicial category to its underlying category. (It should be adjoint to constant simplicial category?)

Definition 4.1.4: Simplicial Functors

Let $\mathcal{C}, \mathcal{D}$ be simplicial categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is a simplicial functor if the induced map

$$F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

for each $X, Y \in \mathcal{C}$ is a morphism of simplicial sets.

Definition 4.1.5: Simplicial Categories

Define the category of simplicial categories

$$\textbf{Cat}_{\textbf{sSet}}$$

by the following data.

- The objects are the simplicial categories
- The morphisms are the simplicial functors
- Composition is given by the usual composition of functors.

Proposition 4.1.6

Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a simplicial category if and only if $\mathcal{C}$ is a simplicial object in **Cat** such that the underlying simplicial set of objects is constant.

1.1.4.2 HTT

Definition 4.1.7: Weakly Equivalent Simplicial Categories

Let $\mathcal{C}, \mathcal{D}$ be simplicial categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor. We say that F is a weak equivalence if the following are true.

- For all $A, B \in \mathcal{C}$, the induced map of simplicial sets

$$F : \text{Hom}_{\mathcal{C}}(A, B) \to \text{Hom}_{\mathcal{C}}(F(A), F(B))$$

  is weakly equivalent.
- For all $D \in \mathcal{D}$, there exists some $C \in \mathcal{C}$ such that $F(C) \cong D$

Note: Markus land says this is weak equivalence, HTT says that this equivalence.

4.2 The Homotopy Coherent Nerve

Definition 4.2.1: Simplicial Path Category

Let $(P, \leq)$ be a partially ordered set. Define the simplicial path category

$$\mathbf{Path}[P]$$

to consist of the following data.

- The objects of $\mathbf{Path}[P]$ are the elements of P.
- Let $x, y \in P$ be objects. Define a partially ordered set $\mathbf{FinLin}(x, y)$ where elements are given by finite linearly ordered subsets

  $$\{x = x_0 < \cdots < x_m = y\} \subseteq P$$

  of P with start point x and end point y and ordering given by reverse inclusion. Denote by the same name its associated category. Define the set of morphisms from x to y to be

  $$\mathrm{Hom}_{\mathbf{Path}[P]}(x, y) = N_\bullet(\mathbf{FinLin}(x, y))$$

  In particular, the identity morphism is $\mathrm{id}_x \in \mathrm{Hom}_{\mathbf{Path}[P]}(x, x)$.
- Let $x, y, z \in P$ be objects. The composition law

  $$\circ \, \mathrm{Hom}_{\mathbf{Path}[P]}(y, z) \times \mathrm{Hom}_{\mathbf{Path}[P]}(x, y) \to \mathrm{Hom}_{\mathbf{Path}[P]}(x, z)$$

  is given on vertices by $S \circ T = S \cup T$.

Note: the assignment $\Delta \to \mathbf{Path}[\Delta]$ defines a cosimplicial set given by

$$[n] \mapsto \mathbf{Path}[\Delta^n] = \mathbf{Path}[n]$$

Definition 4.2.2: The Homotopy Coherent Nerve

Let $\mathcal{C}$ be a simplicial category. Define the homotopy coherent nerve

$$N_\bullet^{\mathrm{hc}}(\mathcal{C}) : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

to be the simplicial set given at the n level by

$$[n] \mapsto \mathrm{Hom}_{\mathbf{sSet}}(\mathbf{Path}[n], \mathcal{C})$$

By the Yoneda embedding, we obtain a natural isomorphism

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, N_\bullet^{\mathrm{hc}}(\mathcal{C})) \cong \mathrm{Hom}_{\mathbf{sSet}}(\mathbf{Path}[n], \mathcal{C})$$

Definition 4.2.3: The Homotopy Coherent Nerve Functor

Define the homotopy coherent nerve functor

$$N_\bullet^{\mathrm{hc}} : \mathbf{Cat}_{\mathbf{sSet}} \to \mathbf{sSet}$$

to consist of the following data.

- For each simplicial category $\mathcal{C}$, $N_\bullet^{\mathrm{hc}}(\mathcal{C})$ is the homotopy coherent nerve of $\mathcal{C}$
- For each simplicial functor $F : \mathcal{C} \to \mathcal{D}$, the morphism

  $$N_\bullet^{\mathrm{hc}}(F) : N_\bullet^{\mathrm{hc}}(\mathcal{C}) \to N_\bullet^{\mathrm{hc}}(\mathcal{D})$$

  is defined on the level of n simplicies by

  $$\mathrm{Hom}_{\mathbf{sSet}}(\mathbf{Path}[n], \mathcal{C}) \xrightarrow{F \circ -} \mathrm{Hom}_{\mathbf{sSet}}(\mathbf{Path}[n], \mathcal{D})$$

Proposition 4.2.4

Let $\mathcal{C}$ be a simplicial category. Let $\mathcal{C}_0$ denote the underlying ordinary category of $\mathcal{C}$. Let P be a partially ordered set. Let $\pi : \mathbf{Path}[P] \to \underline{P}$ be the simplicial functor given by the identity on objects. Then there is a monomorphism

$$\{F \mid F : P \to \mathcal{C}_0 \text{ is a functor }\} \hookrightarrow \{G : \mathbf{Path}[P] \to \mathcal{C}\}$$

given by sending an ordinary functor F to the simplicial functor

$$\mathbf{Path}[P] \xrightarrow{\pi} P \xrightarrow{F} \mathcal{C}_0 \hookrightarrow \mathcal{C}$$

Lemma 4.2.5

Let $\mathcal{C}$ be an ordinary category. Let $\underline{\mathcal{C}}$ denote the constant simplicial category. Then there is an isomorphism of simplicial sets

$$N_\bullet(\mathcal{C}) \cong N_\bullet^{\text{hc}}(\underline{\mathcal{C}})$$

induced by the above monomorphism.

Theorem 4.2.6: Cordier-Porter

Let $\mathcal{C}$ be a simplicial category. If $\text{Hom}_{\mathcal{C}}(X, Y)$ is a Kan complex for all $X, Y \in \mathcal{C}$, then $N_\bullet^{\text{hc}}(\mathcal{C})$ is an infinity category.

4.3 The Comparison between Simplicial Categories and Infinity Categories

Definition 4.3.1: The Generalized Path Category

Let S be a simplicial set. Define the generalized path category of S to be the limit

$$\mathbf{Path}[S]_\bullet = \lim_{\Delta} \mathbf{Path}[-]$$

where $\mathbf{Path}[\Delta^n] = \mathbf{Path}[n]$.

Kerodon 2.4.4.3

Theorem 4.3.2: The Nerve-Path Adjunction

The homotopy coherent nerve and the path category forms an adjunction

$$\mathbf{Path}[-]_\bullet : \mathbf{sSet} \rightleftarrows \mathbf{Cat}_{\mathbf{sSet}} : N_\bullet^{\text{hc}}$$

Explicitly, this means that there is an isomorphism

$$\text{Hom}_{\mathbf{Cat}_{\mathbf{sSet}}}(\mathbf{Path}[S]_\bullet, \mathcal{C}) \cong \text{Hom}_{\mathbf{sSet}}(S, N_\bullet^{\text{hc}}(\mathcal{C}))$$

Infinity categorise are fibrant objects of the Joyal model structure. In order to show that simplicial categories also give a theory of infinity categories, we need to show that all of its homotopy data and constructions are equivalent. We show this using a Quillen equivalence type theorem.

4.4 The Infinity Category of Spaces

Definition 4.4.1: The Category of Kan Complexes

Define the category of Kan complexes
$$\mathbf{Kan}$$
to be the full subcategory of $\mathbf{sSet}$ consisting of Kan complexes. Explicitly, it consists of the following data.

- The objects are the Kan complexes
- For X, Y two Kan complexes, a morphism $f : X \to Y$ is a morphism of simplicial sets
- Composition is given by the composition of morphisms of simplicial sets

Lemma 4.4.2

The category $\mathbf{Kan}$ is a simplicial category. Moreover, $\mathrm{Hom}_{\mathbf{Kan}}(X, Y)$ is a Kan complex.

Lurie def vs Dwyer-Kan def is different.

Definition 4.4.3: ∞-Category of Spaces

Define the ∞-category of spaces to be the simplicial set
$$\mathcal{S} = N_\bullet^{\text{hc}}(\mathbf{Kan})$$

Lemma 4.4.4

The ∞-category of spaces $\mathcal{S}$ is an ∞-category.

Digression: We began with two pairs of adjoint functors in Simplicial Methods in Topology:

$$\begin{tikzcd}
& \mathbf{Cat} \arrow[dl, shift left, "N_\bullet"] \\
\mathbf{sSet} \arrow[ur, shift left, "h"] \arrow[dr, shift left, "|\cdot|"] & \\
& \mathbf{Top} \arrow[ul, shift left, "S_\bullet"]
\end{tikzcd}$$

We have introduced the notion of Kan complexes and ∞-categories, which are also simplicial sets. We also saw that the two pairs of adjunction descend to these special simplicial sets:

$$\begin{tikzcd}
\mathbf{sSet} & \\
\infty-\mathbf{Cat} \arrow[u, hook, "\text{full}"] \arrow[r, shift left, "h"] & \mathbf{Cat} \arrow[l, shift left, "N_\bullet"] \\
\mathbf{Kan} \arrow[u, hook, "\text{full}"] \arrow[r, shift left, "|\cdot|"] & \mathbf{Top} \arrow[l, shift left, "S_\bullet"]
\end{tikzcd}$$

Finally in the last section we introduced another pair of adjunction from simplicial categories. Together with the forgetful functor sending a simplicial category $\mathcal{C}$ to $\mathcal{C}_0$ we obtain:

$$\begin{tikzcd}
\mathbf{sSet} \arrow[r, shift left, "\mathbf{Path}[-]_\bullet"] & \mathbf{Cat}_{\mathbf{sSet}} \arrow[l, shift left, "N_\bullet^{\text{hc}}"] \arrow[d, "\text{Forgetful}"] \\
\infty-\mathbf{Cat} \arrow[u, hook, "\text{full}"] \arrow[r, shift left, "\text{Ho}"] & \mathbf{Cat} \arrow[l, shift left, "N_\bullet"] \\
\mathbf{Kan} \arrow[u, hook, "\text{full}"] \arrow[r, shift left, "|\cdot|"] & \mathbf{Top} \arrow[l, shift left, "S_\bullet"]
\end{tikzcd}$$

5 Space of Morphisms between Vertices

5.1 The Mapping Spaces

For an ordinary category $\mathcal{C}$, we have the notion of Hom sets (at least for locally small categories). We would like to reproduce this notion for infinity categories.

Definition 5.1.1: Mapping Spaces

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the mapping space from x to y to be the homotopy pullback
$$\text{Hom}_{\mathcal{C}}(x, y) = \{x\} \times^h_{\mathcal{C}} \{y\}$$

Recall that the homotopy pullback is defined to be an iterated pullback. Explicitly, we can write the mapping space as

$$\text{Hom}_{\mathcal{C}}(x, y) = \{x\} \times_{\text{Hom}_{\textbf{sSet}}(\{0\},\mathcal{C})} \times \text{Hom}_{\textbf{sSet}}(\Delta^1, \mathcal{C}) \times_{\text{Hom}_{\textbf{sSet}}(\{1\},\mathcal{C})} \{y\}$$

Note: Land 1.3.47, Kerodon 4.6

Recall that a an n-simplex x is degenerate if any two of its consecutive vertices are given by the same element. Explicitly, this means that x lies in the image of some degeneracy map s_k.

Definition 5.1.2: The Right Mapping Space

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the right mapping space from x to y to be the simplicial set defined by

$$\text{Hom}^R_{\mathcal{C}}(x, y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(x) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = y \right\}$$

for each $n \in \mathbb{N}$.

In plain English, the hom set from x to y on the nth level consists of $n+1$-simplices h for which the face of h with the first n-vertices are given by the n simplex $[x, \dots, x]$, while the last vertex of h is given by y.

Definition 5.1.3: The Left Mapping Space

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the left mapping space from x to y to be the simplicial set defined by

$$\text{Hom}^L_{\mathcal{C}}(x, y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(y) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = x \right\}$$

for each $n \in \mathbb{N}$.

These two notions are equivalent up to homotopy (Land) Also pullbacks (Land)

Proposition 5.1.4

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$. Then both mapping spaces $\text{Hom}^R_{\mathcal{C}}(x, y)$ and $\text{Hom}^L_{\mathcal{C}}(x, y)$ are Kan complexes.

Proposition 5.1.5

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$. Then the following are true.

- The right mapping space is isomorphic to the pullback
$$\mathrm{Hom}^R_{\mathcal{C}}(x, y) \cong \{x\} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{0\},\mathcal{C})} \mathcal{C}/y$$
- The left mapping space is isomorphic to the pullback
$$\mathrm{Hom}^L_{\mathcal{C}}(x, y) \cong x/\mathcal{C} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{1\},\mathcal{C})} \{y\}$$

5.2 Composition of Morphisms in Infinity Categories

Definition 5.2.1

Let $\mathcal{C}$ be an infinity category. Let $x, y, z \in \mathcal{C}$ be objects. Define the ???
$$\mathrm{Map}_{\mathcal{C}}(x, y, z)$$
of x, y, z to be the pullback of the diagram:
$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{sSet}}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(x,y,z)"] & \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0 \times \Delta^0 \times \Delta^0, \mathcal{C}) \simeq \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{tikzcd}$$
in **sSet**, where the vertical map is given by the inclusion of each Δ^0 to a vertex of Δ^2.

Rewriting notation: $\mathrm{Map}_{\mathcal{C}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(x, y)$.

Lemma 5.2.2

Let $\mathcal{C}$ be an infinity category. Let $x, y, z \in \mathcal{C}$ be objects. Then the map
$$\mathrm{Map}_{\mathcal{C}}(x, y, z) \xrightarrow{d_0 \times d_2} \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y)$$
is a trivial Kan fibration.

Definition 5.2.3: Composition of Morphisms

Let $\mathcal{C}$ be an infinity category. Let $x, y, z \in \mathcal{C}$ be objects. Let $f : x \to y$ and $g : y \to z$ be morphisms in $\mathcal{C}$. Define the composite of f and g to be the image of (g, f) in the following:
$$\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \xrightarrow{k} \mathrm{Map}_{\mathcal{C}}(x, y, z) \xrightarrow{d_1} \mathrm{Map}_{\mathcal{C}}(x, z)$$
where k is a choice of the homotopy inverse of $d_0 \times d_2 : \mathrm{Map}_{\mathcal{C}}(x, y, z) \to \mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y)$.

Upshot: Composition of morphisms are only well defined up to an equivalence class of homotopic maps.

6 Limits and Colimits

6.1 Terminal and Initial Objects

Definition 6.1.1: Initial and Terminal Objects

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object.

- We say that x is initial if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\mathrm{Hom}_{\mathcal{C}}(x, y) \simeq \Delta^0$$
- Dually, we say that x is terminal if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\mathrm{Hom}_{\mathcal{C}}(y, x) \simeq \Delta^0$$

Proposition 6.1.2

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object. Then the following are equivalent.

- x is terminal.
- For all $n \geq 1$, every lifting problem of the form
$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r, hook] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$
has a solution.

initial / terminal carries over by equivalence

initial in i-cat imply initial in hCat

6.2 Limits and Colimits

Definition 6.2.1: Functor of Constant Diagrams

Let K be a simplicial set. Let $\mathcal{C}$ be an infinity category. Define the functor of constant diagram
$$\Delta : \mathcal{C} \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C})$$
to be the morphism of simplicial sets defined by
$$\mathcal{C}_n \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathcal{C}) \longrightarrow \mathrm{Hom}_{\mathbf{sSet}}(K \times \Delta^n, \mathcal{C})$$
induced by the projection map $K \times \Delta^n \to \Delta^n$.

In particular, notice that for $X \in \mathcal{C}$ an object, ΔX is precisely the unique morphism from K to the simplicial set $\{X\}$.

Definition 6.2.2: Limits in Infinity Categories

Let $K, \mathcal{C}$ be infinity categories. Let $F : K \to \mathcal{C}$ be a morphism. We say that X is the limit of F exhibited by $u : \Delta X \Rightarrow F$ if for all $Z \in \mathcal{C}$, the following composite:
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C})}(\Delta X, \Delta Y) \xrightarrow{[u]\circ -} \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C})}(\Delta X, F)$$
is a homotopy equivalence of simplicial sets.

TBA: If $K, \mathcal{C}$ are nerves of ordinary category, then the above definition recovers the usual notion of limits.

6.3 Equivalent Formulations of the (Co)Limit

Proposition 6.3.1

Let K be a Kan complex. Let $\mathcal{C}$ be an infinity category. Let $F : K \to \mathcal{C}$ be a functor. Let $X \in \mathcal{C}$ be an object.

- X the limit of F exhibited by $u : \Delta X \Rightarrow F$ if and only if u is the final object in

$$\mathcal{C} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{0\}, \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C}))} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C})) \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{1\}, \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C}))} \{u\}$$

7 Relation to Model Categories

7.1 Inverting Morphisms in an Infinity Category

Definition 7.1.1

Let $\mathcal{C}$ be an infinity category. Let W be a collection of morphisms in $\mathcal{C}$. Define the category

$$\mathcal{C}[W^{-1}]$$

together with its canonical functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ by the following universal property.

For every infinity category $\mathcal{D}$ together with a functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(f)$ is an equivalence for $f \in W$, there exists a unique functor $H : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists! H"] \\ & \mathcal{D} \end{tikzcd}$$

Proposition 7.1.2

Let $\mathcal{C}$ be an infinity category. Let W be a collection of morphisms in $\mathcal{C}$. Then $\mathcal{C}[W^{-1}]$ exists and is unique up to equivalence of infinity categories.

Given a category $\mathcal{C}$ with weak equivalences $\mathcal{W}$, we now have a way to systematically construct an infinity category associated to $\mathcal{C}$. Namely,

$$(\mathcal{C}, \mathcal{W}) \mapsto N(\mathcal{C})[\mathcal{W}^{-1}]$$

7.2 Exhibiting a Model Category as an Infinity Category

Up until now, we have two ways of associating different types of categories with its homotopy category. Namely, if $\mathcal{C}$ is a model category, then we can associate to it the homotopy category $\text{Ho}(\mathcal{C})$. Similarly, if $\mathcal{D}$ is an infinity category, we can also associate to it a homotopy category $\text{Ho}(\mathcal{D})$. This constructions are highly related. In particular, there is a functor sending every model category to an infinity category such that the most important notions such as homotopy limits and colimits coincide.

Recall that for a model category $\mathcal{C}$, we denote the full subcategory spanned by cofibrant objects by $\mathcal{C}_c$.

Definition 7.2.1

Let $(\mathcal{C}, W)$ be a model category. Let $\mathcal{D}$ be an infinity category. Let $F : N(\mathcal{C}_c) \to \mathcal{D}$ be a functor. We say that F exhibits the underlying category $\mathcal{C}$ as $\mathcal{D}$ if the functor induces an equivalence of categories

$$N(\mathcal{C}_c)[W^{-1}] \simeq \mathcal{D}$$

Ref:1.3.4.20 HA

Theorem 7.2.2: [Dwyer-Kan]

Let $(\mathcal{C}, \mathcal{W})$ be a model category. ??? determines a map $N(\mathcal{C}_c) \to N(\mathcal{C}_{cf})$ that induces an equivalence of infinity categories

$$N(\mathcal{C}_c)[\mathcal{W}^{-1}] \simeq N(\mathcal{C}_{cf})$$

TBA: Left Quillen equivalence implies equivalence of infinity categories.

7.3

Presentable iff $\mathcal{D} \simeq N(\mathcal{C}_c f)$ where $\mathcal{C}$ is a combinatorial simplicial model category.