Fourier Analysis

Labix

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Abstract

Contents

1 The Fourier Series

1.1 Introduction to the Fourier Series

Definition 1.1.1: The nth Fourier Coefficient

Let f be an integrable function on an interval $[a, b]$ with $b - a = L$, then the nth Fourier coefficient of f is defined by

$$f_n = \frac{1}{L}\int_a^b f(x)e^{-\frac{2\pi in}{L}x}\,dx$$

where $n \in \mathbb{Z}$

Definition 1.1.2: Fourier Series

Define the Fourier series of f is given by

$$f_F(x) = \sum_{n=-\infty}^{\infty} f_n e^{\frac{2\pi in}{L}x}$$

Definition 1.1.3: Nth partial sum

The Nth partial sum of the Fourier series of f, for $N \in \mathbb{N}$ is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} f_n e^{\frac{2\pi in}{L}x}$$

Theorem 1.1.4

Suppose that f is an integrable function on the circle with $f_n = 0$ for all $n \in \mathbb{N}$, then $f(\theta_0) = 0$ whenever f is continuous at the point θ_0.

Corollary 1.1.5

Let f be a continuous function on the circle and that the Fourier series of f is absolutely convergent, then the Fourier series converges uniformly to f. That is

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$$

Corollary 1.1.6

Let f be twice differentiable defined on the circle. Then

$$f_n = O\left(\frac{1}{|n|^2}\right)$$

as $|n| \to \infty$.

Definition 1.1.7: Convolution

Let f, g be 2π-periodic integrable functions on $\mathbb{R}$. Define their convolution $f * g$ on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy$$

Proposition 1.1.8

Let f, g, h be 2π-periodic integrable functions. Then

- $f * (g + h) = (f * g) + (f * h)$
- $(cf) * g = c(f * g) = f * (cg)$ for any $c \in \mathbb{C}$
- $f * g = g * f$
- $(f * g) * h = f * (g * h)$
- $f * g$ is continuous
- $(f * g)_n = f_n g_n$

Definition 1.1.9: Family of Kernels

A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of good kernels if

- For all $n \in \mathbb{N}$, $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1$
- There exists $M > 0$ such that for all $n \in \mathbb{N}$, $\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$
- For every $\delta > 0$. $\int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx \to 0$