Complex Manifolds

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Abstract

Contents

1 Holomorphic Functions of Several Variables

1.1 Holomorphicity in Several Variables

We have seen that analyticity and holomorphicity essentially mean the same thing. We begin the section be showing that holomorphicity is dependent on individual variables for functions of several variables.

Definition 1.1.1: Holomorphic Functions of Several Variables

Let U be an open subset of $\mathbb{C}^n$. Let $f : U \to \mathbb{C}$ be a complex valued function. We say that f is holomorphic if for each $w \in U$, there exists some open neighbourhood V of w such that

$$f(z) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1,\dots,k_n}(z_1 - w_1)^{k_1} \cdots (z_n - w_n)^{k_n}$$

Theorem 1.1.2: Osgood's Lemma

Let $U \subseteq \mathbb{C}^n$ be open. Let $f : U \to \mathbb{C}$ be a complex valued function. Then f is holomorphic on U if and only if f is holomorphic on each variable on U and f is continuous on $\mathbb{C}$.

Proposition 1.1.3

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a function where $F = (f_1, \dots, f_n)$ and $f_i(z) = u_i(z) + v_i(z)$ are the decomposition into its real and complex part. Then F is holomorphic if and only if $u_1, v_1, \dots, u_n, v_n$ are C^∞ functions that satisfy the Cauchy Riemann equations:

$$\frac{\partial u_k}{\partial x_j} = \frac{\partial v_k}{\partial y_j}$$

$$\frac{\partial u_k}{\partial y_j} = -\frac{\partial v_k}{\partial x_j}$$

Proposition 1.1.4

Let $f : U \to \mathbb{C}$ be a holomorphic function define on some $U \subseteq \mathbb{C}^n$ open. Let $z = (z_1, \dots, z_n) \in U$. Choose $\epsilon_1, \dots, \epsilon_n > 0$ such that $D_{\epsilon_1,\dots,\epsilon_n}(z) = D_{\epsilon_1}(z_1) \times \cdots \times D_{\epsilon_n}(z_n)$ is a subset of U. Then for each $w = (w_1, \dots, w_n) \in D_{\epsilon_1,\dots,\epsilon_n}(z)$ we have

$$f(w_1', \dots, w_n') = \frac{1}{(2\pi i)^n} \int_{\partial D_{\epsilon_1}(z_1)} \cdots \int_{\partial D_{\epsilon_1}(z_n)} \frac{f(w)}{(w_1 - w_1') \cdots (w_n - w_n')} \, dw_1 \cdots dw_n$$

1.2 The Inverse and Implicit Function Theorem

Theorem 1.2.1: The Inverse Function Theorem

Let $U \subseteq \mathbb{C}^n$ be open. Let $f = (f_1, \dots, f_n) : U \to \mathbb{C}^n$ be a holomorphic function. Assume that $w \in U$ is a point such that the Jacobian

$$J(w) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial z_1}\right|_w & \cdots & \left.\frac{\partial f_1}{\partial z_n}\right|_w \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_n}{\partial z_1}\right|_w & \cdots & \left.\frac{\partial f_n}{\partial z_n}\right|_w \end{pmatrix}$$

is invertible at $z = w$. Then there exists an open neighbourhood V of w and W of $f(w)$, as well as a holomorphic function $g : W \to V$ such that g is the inverse of f.

Theorem 1.2.2: The Implicit Function Theorem

Let $U \subseteq \mathbb{C}^n$ be open. Let $f = (f_1, \dots, f_k) : U \to \mathbb{C}^k$ be a holomorphic function. If $f(0) = 0$ and submatrix has the property that

$$\begin{vmatrix} \left.\frac{\partial f_1}{\partial z_1}\right|_0 & \cdots & \left.\frac{\partial f_1}{\partial z_k}\right|_0 \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f_k}{\partial z_1}\right|_0 & \cdots & \left.\frac{\partial f_k}{\partial z_k}\right|_0 \end{vmatrix} \neq 0$$

Then there exists a holomorphic function $g = (g_1, \dots, g_k) : V \to \mathbb{C}^n$ such that for some neighbourhood W of 0 in U, we have $f(z) = 0$ is equivalent to $z_i = g_i(z_{k+1}, \dots, z_n)$ for $0 \leq i \leq k$ and $z \in W$.

1.3 Singularities

Theorem 1.3.1

Let $U \subseteq \mathbb{C}^n$ be open. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then any isolated singularities of f are removable.

2 Complex Manifolds

2.1 Basic Definitions

Definition 2.1.1: Atlas (Complex Structure)

Let M be a topological space. An atlas of M is a family of pairs $\{(U_i, \phi_i)|i \in I\}$, called charts, such that

- Each U_i is an open subset of M and $M = \bigcup_{i\in I} U_i$
- Each ϕ_i is a homeomorphism of U_i onto an open set $V \subseteq \mathbb{C}^n$
- Compatibility: Whenever $U_i \cap U_j$ is nonempty, the mapping $\phi_j \circ \phi_i^{-1}$ of $\phi_i(U_i \cap U_j)$ onto $\phi_j(U_i \cap U_j)$ is holomorphic

Definition 2.1.2: Complete Atlas

A complete atlas on a topological space M is an atlas of M which is not contained in any other atlas of M.

Lemma 2.1.3

Every atlas of M is contained in a unique complete atlas.

By collecting atlas and using the complete atlas of a topological space, we can talk about all possible transitions between the open covers.

Definition 2.1.4: Complex Manifolds

A complex manifold is a Hausdorff topological space M that is second countable, together with a fixed complete atlas.

Proposition 2.1.5

Every complex manifold is a smooth real manifold.

Proof. Let $(U, \phi = z_1, \dots, z_n))$ be a chart of a complex manifold M. Write $z_k = x_k + iy_k$ where x_k and y_k are smooth real valued functions on U. Then $(x_1, \dots, x_n, y_1, \dots, y_n)$ gives a homeomorphism between U and an open subset of $\mathbb{R}^{2n}$. □

2.2 Holomorphic Maps between Manifolds

Definition 2.2.1: Holomorphic Maps to $\mathbb{C}^k$

A continuous function $f : X \to \mathbb{C}^k$ is called holomorphic if for every chart (U, ϕ), we have that $f \circ \phi^{-1} : \mathbb{C}^n \to \mathbb{C}^k$ is holomorphic.

Definition 2.2.2: Holomorphic Maps between Manifolds

A continuous map $f : M \to N$ between two complex manifolds is called holomorphic if $\psi \circ f \circ \phi^{-1}$ is holomorphic for every pair of charts (U, ϕ) of M and (V, ψ) of N such that $f(U) \subset V$.

2.3 Complex Submanifolds

Definition 2.3.1: Complex Submanifolds

Let M be an n dimensional manifold. An m dimensional submanifold is a subset Y of M such that for every $y \in Y$, there exists a chart (U, ϕ) of M such that

$$\phi(Y \cap U) = \phi(U) \cap \{0\}^{n-m} \times \mathbb{C}^m = \{(z_1, \dots, z_n) \in \phi(U) | z_{n-m+1} = \cdots = z_n = 0\}$$

3 Sheaves on Manifolds

3.1 The Orientation Sheaf

We can organize all the local and global orientation information into a sheaf.

Definition 3.1.1: L

t M be a topological manifold. Define the orientation sheaf

$$o_M : \mathbf{Open}(M) \to \mathbf{Ab}$$

is defined as follows.

- For each open set U, $o_M(U) = H_k(M \mid U)$
- For each inclusion $U \hookrightarrow V$, there is a map

$$H_k(M \mid V) = o_M(V) \to o_M(U) = H_k(M \mid U)$$

Lemma 3.1.2

Let M be a topological manifold. Then the orientation sheaf is locally constant, with each locally constant piece being isomorphic to $\mathbb{Z}$.

4 Sheaves of Rings on Complex Manifolds

4.1 The Weierstrass Theorem

Theorem 4.1.1: The Weierstrass Preparation Theorem

Proposition 4.1.2

Let $p \in \mathbb{C}^n$. Then $\mathcal{O}_{\mathbb{C}^n,p}$ is Noetherian and is s UFD

4.2 Sheaves on Complex Manifolds

Definition 4.2.1: Ring of Continuous Complex Functions

Let $U \subseteq \mathbb{C}^n$ be open. Define the ring of continuous complex functions to be the set

$$\mathcal{C}^0(U) = \{f : U \to \mathbb{C} \mid f \in C^0\}$$

together with pointwise addition and multiplication.

TBA: Topology on $\mathcal{C}^0(U)$

Definition 4.2.2: Ring of Holomorphic Functions

Let $U \subseteq \mathbb{C}^n$ be open. Define the ring of holomorphic functions to be the subring

$$\mathcal{O}(U) = \{f : U \to \mathbb{C} \mid f \text{ is holomorphic }\} \subseteq \mathcal{C}^0(U)$$

Proposition 4.2.3

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. Then $\mathcal{O}_{U,p}$ is a local ring and an integral domain.

Proposition 4.2.4

Let $p \in \mathbb{C}^n$. Then $\mathcal{O}_{\mathbb{C}^n,p}$ is isomorphic to the ring of convergent power series centered at p.

Definition 4.2.5: Field of Meromorphic Germs

Let $U \subseteq \mathbb{C}^n$ be open. Let $p \in U$. Let m_p be the unique maximal ideal of $\mathcal{O}_{U,p}$. Define the field of meromorphic germs on p by

$$\mathcal{M}_{U,p} = \frac{\mathcal{O}_{U,p}}{m_p}$$

5 More on Vector Bundles

5.1 Structure on Vector Spaces

Definition 5.1.1: Linear Complex Structure

Let V be a real vector space. A linear complex structure on V is a map $T : V \to V$ such that $T^2 = -\text{id}$.

Proposition 5.1.2

Let V be a real vector space admitting a linear complex structure T. Then V can be seen as a complex vector space.

Proposition 5.1.3

Let V be a real vector space admitting a linear complex structure T. Then $W_{\mathbb{C}} = \text{Hom}_{\mathbb{R}}(V, \mathbb{C})$ admits the decomposition

$$W_{\mathbb{C}} = W^{1,0} \oplus W^{0,1}$$

where $W^{1,0}$ is the $\mathbb{C}$-linear forms and $W^{0,1}$ the $\mathbb{C}$-antilinear forms.

5.2 Structure on Vector Bundles

Definition 5.2.1: Almost Complex Structure

Let $p : E \to B$ be a vector bundle. An almost complex structure J on E is a linear complex structure on each fibre varying smoothly on E. In other words, each J_x for $x \in E$ is a linear complex structure.

Definition 5.2.2: Almost Complex Manifolds

An almost complex manifold is a complex manifold M such that its tangent space TM has an almost complex structure.

Proposition 5.2.3

Every complex manifold is an almost complex manifold.

Definition 5.2.4: Integrable Complex Structure

An almost complex structure on a manifold M is said to be integrable if there exists a complex structure on M which induces I.

Definition 5.2.5: Hermitian Structure

Let $p : E \to B$ be a vector bundle. A Hermitian structure H on E is a Hermitian product on each fibre varying smoothly on E. This means that for $x \in M$, $H : E_x \times E_x \to \mathbb{C}$ satisfies the following.

- $H(u, v)$ is $\mathbb{C}$-linear for every $v \in E_x$
- $H(u, v) = \overline{H(v, u)}$
- $H(u, u) > 0$ for all $u \neq 0$
- $H(u, v)$ is a smooth function on M for every smooth sections u, v of E

Definition 5.2.6: Holomorphic Vector Bundle

Let $p : E \to M$ be a vector bundle over a complex manifold M. We say the vector bundle is holomorphic (equipped with a holomorphic structure) if the trivializations

$$\tau_i : p^{-1}(U_i) \overset{\cong}{\to} U_i \times \mathbb{C}^n$$

has transition matrices $\tau_{ij} = \tau_j \circ \tau_i$ that have holomorphic coefficients.

6 Tangent Spaces of Complex Manifolds

6.1 Holomorphic Tangent Bundles

Since every complex manifold is a smooth real manifold, there is no need to redefine everything. We begin this section with a note that for a complex manifold M of dimension n, M has the real tangent space structure on $p \in M$ with basis

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right\}$$

This is now denoted $T_{\mathbb{R}}M$.

The following is an analogue to the tangent bundle of a smooth manifold. We shall see later that by identifying a complex manifold as also a smooth manifold of double the dimension, we can decompose this tangent bundle.

Definition 6.1.1: Holomorphic Tangent Bundles

Let M be a complex manifold. Let $\{(U_i, \phi_i = (z_1, \dots, z_n)) | i \in I\}$ be an atlas. Denote $\phi_{ij} = \phi_j \circ \phi_i^{-1}$ and the

$$\phi_{ij*} = \begin{pmatrix} \frac{\partial (\phi_{ij})_1}{\partial z_1} & \cdots & \frac{\partial (\phi_{ij})_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial (\phi_{ij})_n}{\partial z_1} & \cdots & \frac{\partial (\phi_{ij})_n}{\partial z_n} \end{pmatrix}$$

Define the tangent bundle as the union of $U_i \times \mathbb{C}^n$, glued by identifying $U_i \cap U_j \times \mathbb{C}^n \subset U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n$ by the map $(u, v) \mapsto (u, \phi_{ij*}(v))$. Denote the holormorphic tangent bundle as TM. Each fibre of TM is denoted T_pM.

6.2 Complex Tangent Spaces

We mimic the definition of tangent spaces in the smooth case.

Definition 6.2.1: $\mathbb{C}$-Algebra of Germs of Holomorphic Functions

Let M be a complex manifold. Let $p \in M$. Define the $\mathbb{C}$-Algebra of Germs of Functions

$$\mathcal{O}_{M,p}^\infty = \{(f : U \to \mathbb{C}, U) \mid p \in U, U \text{ is open}, f \text{ is holomorphic } \}/ \sim = \lim_{U \subseteq M} \mathcal{O}_M(U)$$

to be the stalk of the sheaf of $\mathbb{C}$-algebras of holomorphic functions at p.

Let A be a ring. Let B be an A-algebra. Let M be a B-module. Recall that a derivation of A is a B-module homomorphism $d : B \to M$ such that the Leibniz rule

$$d(b_1 b_2) = d(b_1) b_2 + b_1 d(b_2)$$

is satisfied for all $b_1, b_2 \in B$.

Definition 6.2.2: The Complex Tangent Space

Let M be a complex manifold. Let $p \in M$. Define the tangent space of M at p to be the $\mathbb{C}$-vector space of derivations

$$T_{\mathbb{C},p}M = \text{Der}_{\mathbb{C}}(\mathcal{O}_{M,p}, \mathbb{C}) = \{d : \mathcal{O}_{M,p} \to \mathbb{C} \mid d \text{ is a derivation over } \mathbb{C}\}$$

This is the most natural way that one defines tangents on a complex manifold, just as how we did for smooth manifolds. In particular, one can write down two canonical basis for the tangent space.

Proposition 6.2.3

Let M be a complex manifold of dimension n. Then $T_{\mathbb{C}}M$ has basis

$$\left\{\frac{\partial}{\partial x_1}\bigg|_p,\dots,\frac{\partial}{\partial x_n}\bigg|_p,\frac{\partial}{\partial y_1}\bigg|_p,\dots,\frac{\partial}{\partial y_n}\bigg|_p\right\}$$

or equivalently, with basis

$$\left\{\frac{\partial}{\partial z_1}\bigg|_p,\dots,\frac{\partial}{\partial z_n}\bigg|_p,\frac{\partial}{\partial \bar{z}_1}\bigg|_p,\dots,\frac{\partial}{\partial \bar{z}_n}\bigg|_p\right\}$$

However, because complex manifolds are also smooth manifolds, we obtain another notion of tangent spaces. Namely, this tangent space consists of derivations over $\mathbb{R}$.

Definition 6.2.4: The Real Tangent Space

Let M be a complex manifold. Let $p \in M$. Consider M as a smooth manifold. Then define the real tangent space

$$T_{\mathbb{R},p}M = \text{Der}_{\mathbb{R}}(\mathcal{C}^\infty_{M,p},\mathbb{R})$$

to be the usual notion of tangent spaces in the smooth case.

Proposition 6.2.5

Let M be a complex manifold. Let $p \in M$. Then there is a $\mathbb{C}$-vector space isomorphism

$$T_{\mathbb{C},p}M \cong T_{\mathbb{R},p}M \otimes_R \mathbb{C}$$

given by the map sending $v \otimes z \in T_{\mathbb{R},p}M \otimes_R \mathbb{C}$ to the derivation $v \otimes z$ defined by $(v \otimes z)(f + ig) = z \cdot (v(f) + iv(g))$.

6.3 A Decomposition of the Complex Tangent Space

Proposition 6.3.1

Let M be a complex manifold. Then M admits an almost complex structure, and the we have an isomorphism $T^{1,0}M \cong TM$

Proposition 6.3.2

Let M be a complex manifold. Then $T^{0,1}M = \overline{T^{1,0}M}$

Lemma 6.3.3

Let M be a complex manifold. Then the map

$$TM \hookrightarrow T_{\mathbb{C}}M \twoheadrightarrow T^{1,0}M$$

is an isomorphism.

6.4 The Dual of the Complex Tangent Space

Definition 6.4.1

Let M be a complex manifold. Let $p \in M$. Define the dual of the tangent space at p to be the $\mathbb{C}$-vector space

$$T^*_{\mathbb{C},p}(M) = \mathrm{Hom}_{\mathbb{C}}(T_{\mathbb{C},p}(M), \mathbb{C})$$

dual of the complex tangent space.

6.5 The Complex Tangent Bundle

7 Complex Differential Forms

7.1 Complex Differential Forms

Definition 7.1.1: Complex Differential 1-Forms

Let M be a complex manifold. A complex differential 1-form on M is a smooth section

$$s : M \to T^*(M)$$

of the holomorphic cotangent bundle $T^*(M) \to M$.

Definition 7.1.2: The Space of Complex Differential 1-Forms

Let M be a complex manifold. Define the $\mathbb{C}$-vector space of all complex differential 1-forms to be

$$\Omega^1_{\mathbb{C}}(M) = \{s : M \to T^*M \mid s \text{ is a complex differential 1-form }\}$$

Definition 7.1.3: Differential of a Holomorphic Function

Let M be a complex manifold. Define the complex differential of $f \in \mathcal{O}_M(M)$ to be the 1-form $d_{\mathbb{C}}f : M \to T^*M$ given as follows. For each $p \in M$, $(d_{\mathbb{C}}f)_p$ is a map from $T_{\mathbb{C},p}(M)$ to $\mathbb{C}$ where

$$(d_{\mathbb{C}}f)_p(X) = X(f)$$

Let M be a complex manifold. Let $p \in M$. Inside the $\mathbb{C}$-algebra $\mathcal{O}_{M,p}$ lives the holomorphic functions

$$z^k : U \to \mathbb{C}$$

defined on some open set in M as follows. Choose a chart $(U, \varphi = (z^1, \dots, z^n))$ around p. Then $z^k(x)$ is the kth complex coordinate of x int he chart. Similarly, the holomorphic functions $\bar{z}^k : U \to \mathbb{C}$ can be expressed in a similar manner.

Proposition 7.1.4

Let M be a complex manifold. Let $p \in M$. Write $(dz^k)_p$ and $(d\bar{z}^k)_p$ the differential of the holomorphic functions z^k and $\bar{z}^k$ respectively. Then

$$\left\{(dz^1)_p, \dots, (dz^n)_p, (d\bar{z}^1)_p, \dots, (d\bar{z}^n)_p\right\}$$

is the dual basis in $T^*_{\mathbb{C},p}(M)$ of $\left\{\frac{\partial}{\partial z^1}\bigg|_p, \dots, \frac{\partial}{\partial z^n}\bigg|_p, \frac{\partial}{\partial \bar{z}^1}\bigg|_p, \dots, \frac{\partial}{\partial \bar{z}^n}\bigg|_p\right\}$.

Locally, we can express smooth sections in terms of local coordinates.

Proposition 7.1.5

Let M be a complex manifold. Let $\omega : M \to T^*M$ be a complex differential 1-form. Let $(U, \varphi = (z^1, \dots, z^n))$ be a local chart of M. Then in the chart, ω can be expressed as

$$\omega = \sum_{k=1}^n (a_k dz^k + b_k d\bar{z}^k)$$

where $a_k, b_k \in \mathcal{O}_M(M)$ are smooth functions for each k. Moreover, for each $p \in U$,

$$\omega_p = \sum_{k=1}^n \left(a_k(p)(dz^k)|_p + b_k(p)(d\bar{z}^k)|_p\right)$$

Once again, we already have the notion of differential forms for a complex manifold since we already have it for smooth real manifolds. However, we can once again use the decomposition $dz = dx + idy$ and $d\bar{z} = dx - idy$. This is why we can write any k-form on a complex manifold as

$$\omega = \sum_{|I|+|J|=k} \phi_{I,J} dz_I \wedge d\bar{z}_J$$

Definition 7.1.6: Complex Differential k-Forms

Let M be a complex manifold. A complex differential k-form on M is a smooth section $s : M \to \Lambda^k(T^*(M))$ of exterior product of the cotangent bundle $p : T^*(M) \to M$.

Definition 7.1.7: The Space of Complex Differential k-Forms

Let M be a complex manifold. Define the $\mathbb{C}$-vector space of all complex differential k-forms to be

$$\Omega^k_{\mathbb{C}}(M) = \{s : M \to \Lambda^k T^* M \mid s \text{ is a complex differential } k\text{-form }\}$$

Proposition 7.1.8

Let M be a complex manifold. Let $(U, \phi = (x^1, \dots, x^n))$ be a chart on M. Let ω be a complex differential k-form on M. Then we can write ω in terms of the chart as

$$\omega = \sum_{|I|+|J|=k} \phi_{I,J} dz_I \wedge d\bar{z}_J$$

where each $\phi_{I,J} : U \to \mathbb{C}$ are holomorphic functions in $\mathcal{O}_M(U)$.

7.2 Holomorphic de Rham Cohomology

Definition 7.2.1: Complex Differentials

Let M be a complex manifold. A complex derivative on M is a map $d_{\mathbb{C}} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ such that the following are true.

- $d(\Omega^k_{\mathbb{C}}(M)) \subseteq \Omega^{k+1}_{\mathbb{C}}(M)$ is a $\mathbb{C}$-linear map
- $d \circ d = 0$
- If $\omega \in \Omega^r_{\mathbb{C}}(M)$ and $\tau \in \Omega^s_{\mathbb{C}}(M)$ then $d_{\mathbb{C}}(\omega \wedge \tau) = d_{\mathbb{C}}\omega \wedge \tau + (-1)^r \omega \wedge d\tau$
- For any $f \in C^\infty(M)$, $d_{\mathbb{C}} f$ is the differential of f as defined above.

Proposition 7.2.2

Let M be a complex manifold. Then the complex differential exists and is unique. Moreover, if (U, ϕ) is a local chart on M and $\omega = \sum_{|I|+|J|=k} \phi_{I,J} dz_I \wedge d\bar{z}_J$ is a differential k-form, then locally on the chart

$$d\omega = \sum_{|I|+|J|=k} d(\phi_{I,J}) \wedge dz_I \wedge d\bar{z}_J = \sum_{|I|+|J|=k} \sum_{j=1}^n \left(\frac{\partial \phi_{I,J}}{\partial z^j} dz^j + \frac{\partial \phi_{I,J}}{\partial \bar{z}^j} d\bar{z}^j \right) \wedge dz_I \wedge d\bar{z}_J$$

Definition 7.2.3: The Holomorphic de Rham Chain Complex

Let M be a complex manifold. Define the holomorphic de Rham chain complex to be $(\Omega^\bullet_{\mathbb{C}}(M), d_{\mathbb{C}})$. Explicitly, it is the following chain complex:

$$\mathcal{O}_M(M) \xrightarrow{d_{\mathbb{C}}} \Omega^1(M) \xrightarrow{d_{\mathbb{C}}} \Omega^2(M) \longrightarrow \cdots$$

Definition 7.2.4: The Holomorphic de Rham Cohomology

Let M be a complex manifold. Define the complex de Rham cohomology groups to be

$$H^k_{\text{dR}}(M;\mathbb{C}) = \frac{\ker(d:\Omega^k_{\mathbb{C}}(M)\to\Omega^{k+1}_{\mathbb{C}}(M))}{\text{im}(d:\Omega^{k-1}_{\mathbb{C}}(M)\to\Omega^k_{\mathbb{C}}(M))} = H^k_{dR}(\Omega^\bullet_{\mathbb{C}}(M))$$

Relation between $H^k_{\text{dR}}(M;\mathbb{C})$ and $H^k_{\text{dR}}(M;\mathbb{R})$

Proposition 7.2.5

Let M be a complex manifold. Then there is an isomorphism

$$H^k_{\text{dR}}(M;\mathbb{C}) \cong H^k_{\text{dR}}(M;\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$$

for all $n\in\mathbb{N}$.

7.3 A Decomposition of the Space of k-Forms

In previous sections we decomposed the complex tangent space into

$$T_{\mathbb{C},p}M \cong T_p^{1,0}(M)\oplus T_p^{0,1}(M)$$

for each point p in the complex manifold M. We can also do this for the dual of the tangent pace, and hence the exterior product bundle.

Proposition 7.3.1

The spaces $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ for a complex manifold M defines a vector bundle over M. Moreover, the complexification $T_{\mathbb{C}}M$ induces a dual decomposition

$$\Omega^1_{\mathbb{C}}(M) = \Omega^{1,0}(M)\oplus\Omega^{0,1}(M)$$

Proposition 7.3.2

Let M be a complex manifold. Then

$$\Omega^{p,q}(M) = \bigwedge_{i=1}^{p}\Omega^{1,0}(M)\otimes\bigwedge_{j=1}^{q}\Omega^{0,1}(M)$$

Lemma 7.3.3

Let M be a complex manifold. We have the decomposition

$$\Omega^k_{\mathbb{C}}(M) = \bigwedge_{i=1}^{k}\Omega^1_{\mathbb{C}}(M) = \bigoplus_{p+q=k}\Omega^{p,q}(M)$$

We can also decompose the exterior derivative into its holomorphic and antiholomorphic part.

Lemma 7.3.4

Let M be a complex manifold. Consider the complex differential $d_{\mathbb{C}}:\Omega^k(M)\to\Omega^{k+1}(M)$. Then we can decompose

$$d_{\mathbb{C}} = \partial+\bar\partial$$

where $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. In local coordinates,

$$\bar{\partial}\left(\sum \phi_{I,J} dz_I \wedge dz_J\right) = \sum \frac{\partial \phi_{I,J}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I \wedge d\bar{z}_J$$

7.4 Dolbeault Cohomology

Proposition 7.4.1

Let M be a complex manifold. Then $(\Omega^{\bullet,\bullet}, \partial, \overline{\partial})$ is a double chain complex.

We can now write the bicomplex in a grid:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^{0,2}(M) \arrow[r, "\partial"] \arrow[u] & \Omega^{1,2}(M) \arrow[r, "\partial"] \arrow[u] & \Omega^{2,2}(M) \arrow[r] \arrow[u] & \cdots \\
\Omega^{0,1}(M) \arrow[r, "\partial"] \arrow[u, "\overline{\partial}"] & \Omega^{1,1}(M) \arrow[r, "\partial"] \arrow[u, "\overline{\partial}"] & \Omega^{2,1}(M) \arrow[r] \arrow[u, "\overline{\partial}"'] & \cdots \\
\mathcal{O}_M(M) \arrow[r, "\partial"] \arrow[u, "\overline{\partial}"] & \Omega^{1,0}(M) \arrow[r, "\partial"] \arrow[u, "\overline{\partial}"] & \Omega^{2,0}(M) \arrow[r] \arrow[u, "\overline{\partial}"'] & \cdots
\end{tikzcd}$$

Evidently, we can recover the complex de Rham cohomology from the bicomplex with

$$H^k_{\text{dR}}(M; \mathbb{C}) = H^k(\text{Tot}(\Omega^{\bullet,\bullet}_{\mathbb{C}}(M), \partial, \overline{\partial}))$$

Definition 7.4.2: Dolbeault Complex

Let M be a complex manifold. Let $p \in \mathbb{N}$. Define the Dolbeault complex of M to be the cochain complex

$$\cdots \longrightarrow \Omega^{p,q-1}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q}(M) \xrightarrow{\overline{\partial}} \Omega^{p,q+1}(M) \xrightarrow{\overline{\partial}} \cdots$$

denoted $\Omega^{p,\bullet}(M)$.

Definition 7.4.3: Dolbeault Cohomology

Define the dobeault cohomology of a complex manifold m to be

$$H^{p,q}(M) = \frac{\ker(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)}{\text{im}(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M))} = H^q(\Omega^{p,\bullet})$$

8 Hermitian Manifolds

8.1 Hermitian Manifold and its Metric

Definition 8.1.1: Hermitian Manifold

A complex manifold M is said to be Hermitian if the holomorphic tangent bundle has a Hermitian structure.

Definition 8.1.2: Hermitian Metric

A Hermitian metric on a complex vector space V is a map $h : V \times V \to \mathbb{C}$ such that

- $h(v, w) = \overline{h(w, v)}$ for all $v, w \in V$
- $h(v, v) > 0$ for all $v \in V$

A Hermitian metric on a vector bundle $p : E \to B$ is a smoothly varying Hermitian metric on each fibre E_x of E for $x \in E$.

Proposition 8.1.3

Let M be an almost complex manifold. Every Hermitian metric on M induces a Hermitian structure on M. Every Hermitian structure on M induces a Hermitian metric on M.

Proof. Let h be a Hermitian metric on M. Then $H(X, Y) = h(X, Y) - ih(JX, Y)$ defines a Hermitian structure on M. Conversely, let H be a Hermitian structure on the tangent space $T_{\mathbb{C}}M$ defines a Hermitian metric by $h(X, Y) = \text{Re}(X, Y)$. □

This shows that Hermitian metrics and Hermitian structure essentially mean the same thing, just in different presentations.

Proposition 8.1.4

Every almost complex manifold admits a Hermitian metric.

Proof. Choose any arbitrary Riemannian metric g. Then define $h(X, Y) = g(X, Y) + g(JX, JY)$. This is a Hermitian metric. □

8.2 The Riemannian Metric, The Hermitian Metric and the Associated Form

Proposition 8.2.1

Every hermitian metric h on a complex manifold M defines a Riemannian metric

$$g(u, v) = \frac{1}{2}(h + \overline{h})$$

In local coordinates, g is expressed as

$$g(u, v) = \frac{1}{2} \sum h_{\alpha\overline{\beta}}(dz_\alpha \otimes d\overline{z}_\beta + d\overline{z}_\beta \otimes dz_\alpha)$$

Lemma 8.2.2

Let M be a Hermitian manifold. Denote h the Hermitian metric of M. Then

$$\omega(x, y) = \frac{i}{2}(h - \overline{h})$$

is a $(1, 1)$ form

In local coordinates, ω is expressed as

$$\omega = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$$

if M is a complex manifold of complex dimension n.

Definition 8.2.3: Associated Form of a Hermitian Metric

Let M be a Hermitian manifold. Let h be the Hermitian metric. Define the associated form of h to be the $(1,1)$-form

$$\omega(u,v) = \frac{i}{2}(h - \overline{h})$$

Proposition 8.2.4

Let M be a Hermitian manifold. Then the following are true in terms of the metrics.

- $\omega(u,v) = g(Ju,v)$
- $g(u,v) = \omega(u,Jv)$
- $h = g - i\omega$

Theorem 8.2.5

Let M be a Hermitian manifold. Denote h the Hermitian metric. Then h, g, ω preserve the almost complex structure J of M. This means that the following are true.

- $h(Ju,Jv) = h(u,v)$
- $g(Ju,Jv) = g(u,v)$
- $\omega(Ju,Jv) = \omega(u,v)$

Lemma 8.2.6

Let M be an almost complex manifold. Let g be a Riemannian metric on M such that $g(Ju,Jv) = g(u,v)$. Then g induces a Hermitian metric.

Lemma 8.2.7

Let M be an almost complex manifold. Let ω be a non-degenerate $(1,1)$-form such that $\omega(Ju,Jv) = \omega(u,v)$ and that $\omega(u,Ju) > 0$ for all tangent vectors u. Then ω induces a Hermitian metric.