Advanced Ring Theory

Labix

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Abstract

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1 Artinian Rings and Modules

1.1 Artinian Modules

Definition 1.1.1: Artinian Modules

A left R-module M is said to be Artinian if for every descending chain of submodules

$$N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n \supseteq \cdots$$

there exists $m \in \mathbb{N}$ such that $N_n = N_m$ for all $n > m$.

We have seen that if M is semisimple then $\text{rad}(M) = 0$. The converse holds if M is Artinian.

Theorem 1.1.2

Let M be an Artinian left R-module. Then M is semisimple if and only if $\text{rad}(M) = 0$.

Proof. Lemma 2.5.4 proves one direction. So suppose that $\text{rad}(M) = 0$. Then we obtain a descending chain using intersections of cosimple submodules

$$N_1 \supseteq N_1 \cap N_2 \supseteq \cdots \text{rad}(M) = 0$$

Since M is Artinian, the chain stops after finitely many steps. Then this gives us finitely many cosimple modules N_i such that

$$N_1 \cap \cdots \cap N_k = 0$$

Consider the following homomorphism of R-modules $\psi : M \to \prod_{i=1}^{k} \frac{M}{N_i}$ defined by the individual projection homomorphism. It is injective since its kernel if $N_1 \cap \cdots \cap N_k = 0$. Since there are only finitely many submodules, together with surjectivity we have that

$$M \cong \psi(M) \cong \bigoplus_{i=1}^{k} \frac{M}{N_i}$$

Thus M is semisimple. □

1.2 Artinian Rings

Definition 1.2.1: Artinian Rings

Let R be a ring. We say that R is artinian if R as a left R-module is an Artinian module.

The following is a result of theorem 1.1.2.

Corollary 1.2.2

Let R be a ring. Then R is semisimple if and only if R is left artinian and $J(R) = 0$.

Proof. □

Theorem 1.2.3

If R is a left artinian ring, then $J(R)$ is nilpotent.

Recall that an ideal I of a ring is nilpotent if $I^n = 0$ for some $n \in \mathbb{N}$.

Corollary 1.2.4

Let R be a left artinian ring. Then the following are equivalent.

- $J(R)$ is the largest nilpotent two sided ideal of R
- $J(R)$ is the largest nilpotent left ideal of R
- $J(R)$ is the largest nilpotent right ideal of R.