# Hopf Algebra

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Abstract

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### 1 Algebras and Coalgebras

#### 1.1 Coalgebras

There is a need to revisit the definition of an algebra (over a field)

#### **Proposition 1.1.1**

A vector space V over a field k is an algebra if and only if there is a following collection of data:

- A k-linear map  $m:V\otimes V\to V$  called the multiplication map
- An k-linear map  $u: k \to V$  called the unital map

such that the following two diagrams are commutative:

where the unnamed maps is the canonical isomorphisms.

Evidently, the map  $\mu$  gives a multiplicative structure for V and  $\Delta$  gives the unitary structure of an algebra. The diagram on the left then represent associativity of multiplication. Notice that such additional structure on V formally lives in the category  $\mathbf{Vect}_k$  of vector spaces over a fixed field k.

Therefore we can formally dualize all arrows to obtain a new object.

#### Definition 1.1.2: Coalgebra

Let V be a vector space over a field k. We say that V is a coalgebra over k if there is a collection of data:

- A k-linear map  $\Delta: V \to V \otimes V$  called the comultiplication map
- An k-linear map  $\varepsilon: V \to k$  called the counital map

such that the following diagrams are commutative:

where the unnamed maps is the canonical isomorphisms.

#### Lemma 1.1.3

Every vector space V over a field k can be given the structure of a coalgebra where

- $\Delta: V \to V \otimes V$  is defined by  $\Delta(v) = v \otimes v$
- $\bullet \ \ \varepsilon:V\to k \ \text{is defined by} \ \varepsilon(v)=1_k$

We would like to formally invert the definitions of algebra homomorphisms in order to define coalgebra homomorphisms.

#### 1.2

Every coalgebra gives rise to an algebra, but not the other way. Such an assignment is moreover functorial.

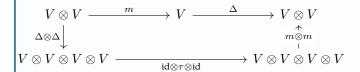
#### 1.3 Bialgebras

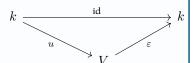
### Definition 1.3.1: Bialgebras

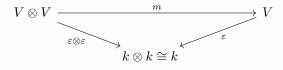
Let V be a vector space over a field k. We say that V is a bialgebra if there is a collection of data:

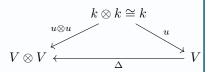
- A k-linear map  $m: V \otimes V \to V$  called the multiplication map
- ullet An k-linear map  $u:k \to V$  called the unital map
- A k-linear map  $\Delta: V \to V \otimes V$  called the comultiplication map
- An k-linear map  $\varepsilon: V \to k$  called the counital map

such that (V, m, u) is an algebra over k and  $(V, \Delta, \varepsilon)$  is a coalgebra over k and that the following diagrams are commutative:









where  $\tau: V \otimes V \to V \otimes V$  is the commutativity map defined by  $\tau(x \otimes y) = y \otimes x$ .

#### Theorem 132

Let V be a vector space over k. Suppose that (V, m, u) is an algebra and  $(V, \Delta, \varepsilon)$  is a coalgebra. Then the following conditions are equivalent.

- $(V, m, u, \Delta, \varepsilon)$  is a bialgebra
- $m: V \otimes V \to V$  and  $u: k \to V$  are coalgebra homomorphisms
- $\Delta: V \to V \otimes V$  and  $\varepsilon: V \to k$  are algebra homomorphisms

## 2 Hopf Algebras

#### 2.1

## Definition 2.1.1: Hopf Algebra

Let  $(H, m, u, \Delta, \varepsilon)$  be a bialgebra. We say that H is a Hopf algebra if there is a k-linear map  $S: H \to H$  called the antipode such that the following diagram commutes:

