Higher Category Theory

Labix

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Abstract

Contents

1 Introduction to Infinity Categories

1.1 Infinity Categories as Simplicial Sets

We recall some basic facts about simplicial sets. If $S : \Delta \to \textbf{Set}$ is a simplicial set, then by Yoneda's emebdding we know that the n-simplices of S are given by

$$S([n]) = \text{Hom}_{\textbf{sSet}}(\Delta^n, S)$$

In other words, specifying an n-simplex is the same as specifying a map of simplicial sets

$$\Delta^n \to S$$

The foundations of infinity categories lay on the simpicial sets. Intuitively, any face $\partial_k\Delta$ of an n-simplex Δ captures a homotopy of the faces of $\partial_k\Delta$.

Definition 1.1.1: Infinity Categories

An infinity category is a simplicial set C such that each inner horn admits a filler. In other words, for all $0 < i < n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Definition 1.1.2: Objects and Morphisms

Let $\mathcal{C}$ be an infinity category. Define the following notions for $\mathcal{C}$.
- Define the objects of $\mathcal{C}$ to be the 0-simplices of $\mathcal{C}$.
- Define the morphisms of $\mathcal{C}$ to be the 1-simplices of $\mathcal{C}$.

Theorem 1.1.3

Let $\mathcal{C}$ be a category. Every inner horn of the nerve $N(C)$ of $\mathcal{C}$ admits a filler and hence is an infinity category.

1.2 Infinity Categories as Topological Categories

1.3 Infinity Categories as Simplicial Categories

1.4 Joins and Slices

We begin by rewriting the definition of a simplex category as follows. Instead of having distinguished names $[n]$ for the objects, we instead just think of the simplex category with objects as finite and totally ordered sets. Indeed any of these sets will be in bijection to $[n]$ for some $n \in \mathbb{N}$. This language will help us define the join.

Definition 1.4.1

Let J be a finite and totally ordered set. A cut of J consists of two subsets $I, I' \subseteq J$ such that

$$J = I \amalg I'$$

and $i < i'$ for all $i \in I$ and $i' < I'$.

Definition 1.4.2: Joins

Let X, Y be simplicial sets. Define the join of X and Y to be the simplicial set $X * Y$ as follows.

- Denote $J \neq \emptyset$ any finite and totally ordered set. Define

$$X * Y(J) = \coprod_{\substack{I \amalg I' = J \\ i < i' \text{ for } i \in I, i' \in I'}} X(I) \times Y(I') \coprod_{I, I' \text{ cuts of } J} X(I) \times Y(I')$$

  where by convention, $X(\emptyset) = Y(\emptyset) = *$.
- For two finite and totally ordered sets J and J' and a morphism $J \to J'$ preserving order, the map

$$(X * Y)[J'] \to (X * Y)[J]$$

  is defined as follows. Let K, K' be a cut of J'. Then α restricts to two well defined maps

$$\alpha|_{\alpha^{-1}(K)} : \alpha^{-1}(K) \to K \quad \text{and} \quad \alpha|_{\alpha^{-1}(K')} : \alpha^{-1}(K') \to K'$$

  In particular these are order preserving, and each are morphisms in the simplex category Δ. Thus this gives us a unique morphism

$$X(K) \times X(K') \to X(\alpha^{-1}(K)) \times X(\alpha^{-1}(K'))$$

  By taking the product of these maps, we thus obtain a morphism $(X * Y)[J'] \to (X * Y)[J]$, turning the above definition into a simplicial set.

Concrete examples:

- When $J = [0]$, we have that

$$\begin{aligned}(X * Y)[0] &= X[0] \times Y(\emptyset) \amalg X(\emptyset) \times Y[0] \\ &= X_0 \amalg Y_0\end{aligned}$$

  which means that the vertices of $X * Y$ are the vertices of X and Y combined disjointly.
- When $J = [1]$, we have that

$$\begin{aligned}(X * Y)[1] &= X[1] \times Y(\emptyset) \amalg X(\{0\}) \times Y(\{1\}) \amalg X(\emptyset) \times Y[1] \\ &= X_1 \amalg X_0 \times Y_0 \amalg Y_1\end{aligned}$$

TBA: The join of ordinary categories.

Lemma 1.4.3

Let X and Y be simplicial sets. Then $N(X * Y) \cong N(X) * N(Y)$

TBA: functoriality of join

Proposition 1.4.4

Let X, Y be simplicial sets. Then $X * Y$ is an infinity category if and only if X and Y are infinity categories.

Recall that the over category $\mathcal{C}/X$ consists of pairs $(Y, f : Y \to X)$ and morphism are given by commutative diagrams. Let us rephrase the definition as follows. The over category is the unique category such that if $\mathcal{D}$ is another category, there is a bijection

$$\mathrm{Hom}_{\mathbf{CAT}}(\mathcal{D}, \mathcal{C}/X) \cong \mathrm{Hom}_X(\mathcal{D} * [0], \mathcal{C})$$

where the right hand side indicates that we only consider morphisms $\mathcal{D} * [0] \to \mathcal{C}$ in which $[0]$ is mapped to X. This characterization is due to the fact that a morphism $[0] \to \mathcal{C}$ is essentially a choice of object in $\mathcal{C}$, in which case we choose to be X.

Definition 1.4.5: Over Category for Infinity Categories

Let K, X be simplicial sets. Let $f : K \to X$ be a map. Define the over category (which is a simplicial set)
$$f/X : \Delta \to \textbf{Set}$$
as follows.

- For each n, we have
$$(f/X)_n = \text{Hom}_{K/\textbf{sSet}}(K * \Delta^n, X)$$

TBA: Adjunction of join and slice.

1.5

For an ordinary category $\mathcal{C}$, we have the notion of Hom sets (at least for locally small categories). We would like to reproduce this notion for infinity categories.

Recall that a an n-simplex x is degenerate if any two of its consecutive vertices are given by the same element. Explicitly, this means that x lies in the image of some degeneracy map s_k.

Definition 1.5.1: The Right Mapping Space

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the right mapping space from x to y to be the simplicial set defined by
$$\text{Hom}^R_{\mathcal{C}}(x,y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(x) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = y \right\}$$
for each $n \in \mathbb{N}$.

In plain English, the hom set from x to y on the nth level consists of $n+1$-simplices h for which the face of h with the first n-vertices are given by the n simplex $[x, \dots, x]$, while the last vertex of h is given by y.

Definition 1.5.2: The Left Mapping Space

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the left mapping space from x to y to be the simplicial set defined by
$$\text{Hom}^L_{\mathcal{C}}(x,y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(y) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = x \right\}$$
for each $n \in \mathbb{N}$.

These two notions are equivalent up to homotopy (Land) Also pullbacks (Land)

Proposition 1.5.3

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$. Then both mapping spaces $\text{Hom}^R_{\mathcal{C}}(x,y)$ and $\text{Hom}^L_{\mathcal{C}}(x,y)$ are Kan complexes.

1.6 Homotopy Infinity Categories

Recall that for a simplicial set X, we defined the homotopy category $h(X)$ of X. Such an assignment is functorial. In the case of infinity categories, we can exhibit the structure of $h(X)$ more explicitly.

Definition 1.6.1: Homotopic Morphisms

Let $\mathcal{C}$ be an infinity category. Two morphisms $f, g : C \to D$ are said to be homotopic if there exists a 2-simplex σ such that

- $d_0(\sigma) = \text{id}_D$
- $d_1(\sigma) = g$
- $d_2(\sigma) = f$

In this case we write $f \simeq g$.

Lemma 1.6.2

Homotopy is an equivalence relation in any infinity category.

Proposition 1.6.3

Let C be an infinity category. Let $f, f' : C \to D$ and $g, g' : D \to E$ be morphisms in C. If $f \simeq f'$ and $g \simeq g'$, then
$$g \circ f \simeq g' \circ f'$$

Definition 1.6.4: Homotopy Category

Let C be an infinity category. Define the homotopy category $h(C)$ of C to consist of the following.

- The objects are the objects of C
- The morphisms are equivalent classes of morphisms $[f]$ for f a morphism in C
- Composition is defined by
$$[g] \circ [f] = [g \circ f]$$
which is well defined by the above.

Definition 1.6.5: Isomorphisms in Infinity Categories

Let C be an infinity category. Let $f : C \to D$ be a morphism. We say that f is an isomorphism if there exists $g : D \to C$ such that $g \circ f \simeq \text{id}_C$ and $f \circ g \simeq \text{id}_D$.

Lemma 1.6.6

Let C be an infinity category. Let $f : C \to D$ be a morphism. Then f is an isomorphism in C if and only if $[f]$ is an isomorphism in $h(C)$.

2 Relation to Model Categories

2.1 Inverting Morphisms in an Infinity Category

Definition 2.1.1

Let $\mathcal{C}$ be an infinity category. Let W be a collection of morphisms in $\mathcal{C}$. Define the category

$$\mathcal{C}[W^{-1}]$$

together with its canonical functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ by the following universal property. For every infinity category $\mathcal{D}$ together with a functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(f)$ is an equivalence for $f \in W$, there exists a unique functor $H : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists! H"] \\
& \mathcal{D}
\end{tikzcd}$$

Proposition 2.1.2

Let $\mathcal{C}$ be an infinity category. Let W be a collection of morphisms in $\mathcal{C}$. Then $\mathcal{C}[W^{-1}]$ exists and is unique up to equivalence.

2.2 Exhibiting a Model Category as an Infinity Category

Up until now, we have two ways of associating different types of categories with its homotopy category. Namely, if $\mathcal{C}$ is a model category, then we can associate to it the homotopy category $\text{Ho}(\mathcal{C})$. Similarly, if $\mathcal{D}$ is an infinity category, we can also associate to it a homotopy category $\text{Ho}(\mathcal{D})$. This constructions are highly related. In particular, there is a functor sending every model category to an infinity category such that the most important notions such as homotopy limits and colimits coincide.

Recall that for a model category $\mathcal{C}$, we denote the full subcategory spanned by cofibrant objects by $\mathcal{C}_c$.

Definition 2.2.1

Let $(\mathcal{C}, W)$ be a model category. Let $\mathcal{D}$ be an infinity category. Let $F : N(\mathcal{C}_c) \to \mathcal{D}$ be a functor. We say that F exhibits the underlying category $\mathcal{C}$ as $\mathcal{D}$ if the functor induces an equivalence of categories

$$N(\mathcal{C}_c)[W^{-1}] \simeq \mathcal{C}$$

3 Infinity Categories in Topology

Lemma 3.0.1

Let X be a space. Then applying the singular functor $S(X)$ gives an infinity category.

Proposition 3.0.2

Let X be a space. Then the homotopy category of the singular set of X is equal to $h(S(X)) = \prod_1(X)$ the fundamental groupoid of X.

3.1 Kan Complexes

Definition 3.1.1: Kan Complexes

A Kan complex is a simplicial set C such that each horn (inner and outer) admits a filler. In other words, for all $0 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Since infinity catregories require only inner horns to admit a filler, we have the following inclusion relation:

$$\substack{\text{Infinity}\\\text{Categories}} \subset \substack{\text{Kan}\\\text{Complexes}}$$

Proposition 3.1.2

Let X be a space. Then $S(X)$ is a Kan complex.

Theorem 3.1.3

Let $\mathcal{C}$ be a small category. Then the simplicial set $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

More: Kan complexes = infinity groupoids (quillen equivalence in model category), and we should think of spaces as Kan complexes / infinity groupoids from now on.

4 Limits and Colimits

4.1 Terminal and Initial Objects

Definition 4.1.1: Initial and Terminal Objects

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object.

- We say that x is initial if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\text{Hom}_{\mathcal{C}}(x, y) \simeq \Delta^0$$
- Dually, we say that x is terminal if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\text{Hom}_{\mathcal{C}}(y, x) \simeq \Delta^0$$

Proposition 4.1.2

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object. Then the following are equivalent.

- x is terminal.
- For all $n \geq 1$, every lifting problem of the form

$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r, hook] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

4.2 Limits and Colimits

Definition 4.2.1: Limits in Infinity Categories

Let K, X be infinity categories. Let $F : K \to X$ be a map. Define the limit
$$\lim_F X$$
of F over X to be the terminal object of the slice category X/F if it exists.