Simplicial Methods in Topology

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Abstract

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1 The Category of Simplicial Sets

1.1 The Simplex Category

Definition 1.1.1: Simplex Category

The simplex category Δ consists of the following data.

- The objects are $[n] = \{0, \dots, n\}$ for $n \in \mathbb{N}$.
- The morphisms are the non-strictly order preserving functions. This means that a morphism $f : [n] \to [m]$ must satisfy $f(i) \leq f(j)$ for all $i \leq j$.
- Composition is the usual composition of functions.

Definition 1.1.2: Maps in the Simplex Category

Consider the simplex category Δ. Define the face maps and the degeneracy maps as follows.

- A face map in Δ is the unique morphism $d^i : [n-1] \to [n]$ that is injective and whose image does not contain i. Explicitly, we have
$$d^i(k) = \begin{cases} k & \text{if } 0 \leq k < i \\ k+1 & \text{if } i \leq k \leq n-1 \end{cases}$$
- A degeneracy map in Δ is the unique morphism $s^i : [n+1] \to [n]$ that is surjective and hits i twice. Explicitly, we have
$$s^i(k) = \begin{cases} k & \text{if } 0 \leq k \leq i \\ k-1 & \text{if } i+1 \leq k \leq n+1 \end{cases}$$

Proposition 1.1.3

The face maps and the degeneracy maps in the simplex category Δ satisfy the following simplicial identities:

- $d^i \circ d^j = d^{j-1} \circ d^i$ if $i < j$
- $d^i \circ s^j = s^{j-1} \circ d^i$ if $i < j$
- $d^i \circ s^i = \text{id}$
- $d^{i+1} \circ s^i = \text{id}$
- $d^i \circ s^j = s^j \circ d^{i-1}$ if $i > j+1$
- $s^i \circ s^j = s^{j+1} \circ s^i$ if $i \leq j$

Proposition 1.1.4

Every morphism in the simplex category Δ is a composition of the face maps and the degeneracy maps.

1.2 Simplicial Sets

Definition 1.2.1: Simplicial Sets

A simplicial set is a presheaf
$$S : \Delta \to \text{Sets}$$

Definition 1.2.2: Category of Simplicial Sets

The category of simplicial sets sSet is defined as follows.

- The objects are simplicial sets $S : \Delta \to \text{Sets}$
- The morphisms are just morphisms of presheaves. This means that if $S, T : \Delta \to \text{Sets}$ are simplicial sets, then a morphism $\lambda : S \to T$ consists of morphisms $\lambda_n : S([n]) \to T([n])$ for $n \in \mathbb{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
S([n]) \arrow[r, "S(f)"] \arrow[d, "\lambda_n"'] & S([m]) \arrow[d, "\lambda_m"] \\
T([n]) \arrow[r, "T(f)"'] & T([m])
\end{tikzcd}$$

- Composition is defined as the usual composition of functors.

The Yoneda lemma in this context implies that there is a bijection
$$\text{Hom}_{\text{sSet}}(\text{Hom}_\Delta([n], -), S) \cong S([n])$$
that is natural in the variable $[n]$. We will denote
$$\Delta^n = \text{Hom}_\Delta([n], -)$$
which is the image of $[n]$ under the yoneda embedding $y : \Delta \to \text{sSet}$ defined by $[n] \mapsto \text{Hom}_\Delta([n], -)$.

Definition 1.2.3: n-Simplices

Let $S : \Delta \to \text{Set}$ be a simplicial set. For $n \in \mathbb{N}$, define the n-simplices of S to be
$$S_n = S([n]) = \text{Hom}_{\text{sSet}}(\Delta^n, S)$$

Notice that Δ^n is a simplicial set
$$\Delta^n : \Delta \to \text{Set}$$
defined by $[m] \mapsto \text{Hom}_\Delta([n], [m])$. Notice that if $n > m$, then it is impossible to have an order preserving function $[n] \to [m]$. Hence when $n > m$, $\text{Hom}_\Delta([n], [m])$ is empty. It is also clear that the m-simplices of Δ^n are precisely the order preserving maps $[m] \to [n]$.

Definition 1.2.4: Standard n-Simplex

Let $n \in \mathbb{N}$. The standard n-simplex is the simplicial set $\Delta^n : \Delta \to \text{Set}$ defined by
$$\Delta^n = \text{Hom}_\Delta([n], -)$$

All such simplicial sets Δ^n are useful in determining the contents of an arbitrary simplicial set. As for any presheaf, instead of focusing between the passage of data from Δ to Set, we should instead think of what kind of structure the presheaf brings to Set. Let C be a simplicial set. Then this means the following. For each n, there is a set $C_n = \text{Hom}_{\text{sSet}}(\Delta^n, C)$. For each morphism in Δ, there is a corresponding morphism in Set, which we shall discuss now.

Theorem 1.2.5

Let $C : \Delta \to \text{Set}$ be a simplicial set. Then every morphism in $C(\Delta)$ is the composite of two kinds of maps:

- The face maps: $d_i : C_n \to C_{n-1}$ for $0 \leq i \leq n$ defined by
$$d_i = C(d^i : [n-1] \to [n])$$
- The degeneracy maps: $s_i : C_{n+1} \to C_n$ for $0 \leq i \leq n$ defined by
$$s_i = C(s^i : [n+1] \to [n])$$

Moreover, these maps satisfy the above simplicial identities

Theorem 1.2.6

The category **sSet** is a symmetric monoidal category with level-wise cartesian product.

Recall the notion of a Δ-set from Algebraic Topology 2 and one might realize they look suspiciously similar to that of a simplicial set. Let us recall. A Δ-set is a collection of sets S_n for $n \in \mathbb{N}$ together with maps $d_i^n : S_n \to S_{n-1}$ for $0 \leq i \leq n$ such that
$$d_i^{n-1} \circ d_j^n = d_{j-1}^{n-1} \circ d_i^n$$
for $i < j$. One can easily convince themselves that every simplicial set is a Δ-set. Indeed, a simplicial set satisfies five more relations than a Δ-set. Therefore we have that
$$\textbf{sSet} \subset \Delta \text{ Complexes}$$

Theorem 1.2.7

Every simplicial set is a Δ-set.

Combining with the previously learnt combinatorial objects in algebraic topology, we now have the following tower:
$$\text{Simplicial Complexes} \subset \textbf{sSet} \subset \Delta \text{ Complexes} \subset \textbf{CW}$$

Definition 1.2.8: Faces of a Simplex

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n.

- Denote $\partial_i \Delta^n \subset \Delta^n$ the simplicial subset generated by the ith face
$$d_i(\text{id} : [n] \to [n]) = d^i : [n-1] \to [n]$$
- Denote $\partial \Delta^n$ the simplicial subset generated by the faces $\partial_i \Delta^n$ for $0 \leq i \leq n$. Define $\partial \Delta^0 = \emptyset$.

1.3 Geometric Realization of Simplicial Sets

Definition 1.3.1: Geometric Realization of Standard n-Simplexes

Let $n \in \mathbb{N}$. Consider the standard n-simplex Δ^n. Define the geometric realization of Δ^n to be
$$|\Delta^n| = \left\{ \sum_{k=0}^n t_k v_k \;\middle|\; \sum_{k=0}^n t_k = 1 \text{ and } t_k \geq 0 \text{ for all } k = 0, \dots, n \right\}$$

This definition is exactly the same as the definition of an n-simplex in Algebraic Topology 2. Now we proceed to the general case.

Definition 1.3.2: Geometric Realization of Simplicial Sets

Let C be a simplicial set. Define the geometric realization of C to be

$$|C| = \left(\coprod_{n\geq 0} C_n \times |\Delta^n|\right) / \sim$$

where the equivalence relation is generated by the following.

- The ith face of $\{x\} \times |\Delta^n|$ is identified with $\{d_i x\} \times \left|\Delta^{n-1}\right|$ by the linear homeomorphism preserving the order of the vertices.
- $\{s_i x\} \times |\Delta^n|$ is collapsed onto $\{x\} \times \left|\Delta^{n-1}\right|$ via the linear projection parallel to the line connecting the ith and the $(i+1)$st vertiex.

This construction of geometric realization is moreover functorial. Once again, we first define a map of geometric realization of simplicial sets.

Definition 1.3.3: Induced Map of Geometric Realization of Standard Simplicial Sets

Let $f : \Delta^n \to \Delta^m$ be a map of standard simplexes. Define $f_* : |\Delta^n| \to |\Delta^m|$ by

$$(t_0, \dots, t_n) \mapsto (s_0, \dots, s_m)$$

where

$$s_i = \begin{cases} 0 & \text{if } f^{-1}(i) = 0 \\ \sum_{j\in f^{-1}(i)} t_j & \text{otherwise} \end{cases}$$

Theorem 1.3.4

The geometric realization of a simplicial set is functorial $|\cdot| : \mathsf{sSet} \to \mathsf{Top}$ in the following way.

- On objects, it sends any simplicial set C to its geometric realization $|C|$.
- On morphisms, it sends any morphism $C \to D$ of simplicial sets to a continuous map defined by

We thus have that

$$\text{Geometric Relizations of simplicial sets} \subset \text{Geometric Relizations of } \Delta\text{-sets} \subset \text{CW-Complexes}$$

1.4 Simplicial and Semisimplicial Objects

Definition 1.4.1: Simplicial Objects

Let $\mathcal{C}$ be a category. A simplicial object in $\mathcal{C}$ is a presheaf $S : \Delta^{\text{op}} \to \mathcal{C}$.

Hence a simplicial object in **Set** is just simplical sets.

Definition 1.4.2: Category of Simplicial Objects

Let $\mathcal{C}$ be a category. Define the category of simplicial objects $s\mathcal{C}$ of $\mathcal{C}$ as follows.

- The objects are simplicial objects $S : \Delta^{\text{op}} \to \mathcal{C}$ of $\mathcal{C}$ which are presheaves
- The morphism of simplcial objects are just morphisms of presheaves, which are natural

transformations

- Composition is given by composition of natural transformations

Definition 1.4.3: The Semisimplex Category

The Semisimplex category $\mathbf{SS}$ is the subcategory of Δ consisting of strict order preserving functions.

Definition 1.4.4: Semisimplicial Objects

Let $\mathcal{C}$ be a category. A semisimplicial object in $\mathcal{C}$ is a presheaf

$$\mathbf{SS} \to \mathcal{C}$$

Lemma 1.4.5

Let S be a set. Then S is a semisimplicial set if and only if S is a Δ-set (in the sense of Algebraic Topology 2).

2 Simplicial Homological Algebra

2.1 Chain Complexes of Simplicial Objects

Definition 2.1.1: Associated Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a (semi)-simplicial object in $\mathcal{A}$. Define the associated chain complex of A to be

$$\cdots \longrightarrow C_{n+1}(A) \xrightarrow{\partial_{n+1}} C_n(A) \xrightarrow{\partial_n} C_{n-1}(A) \longrightarrow \cdots \longrightarrow C_1(A)$$

where $C_n(A) = A_n$ and the boundary operator given by $\partial_n = \sum_{i=0}^n (-1)^i d_i^n : A_n \to A_{n-1}$.

TBA: Functoriality of associated chain complex

Definition 2.1.2: Simplicial Homology

Let R be a ring. Let X be a (semi)-simplicial set. Define the simplicial homology of X with coefficients in R to be the homology groups

$$H_n^\Delta(X; R) = H_n(C_\bullet(R[X]))$$

Notice that this definition coincides with that in Algebraic Topology 2. Recall that in AT2 we defined the simplicial homology of a Δ-set, but in $\mathbb{Z}$ coefficients.

2.2 The Singular Functor

Definition 2.2.1: Singular Functor

The singular functor $S : \text{Top} \to \text{sSet}$ is defined as follows.

- On objects, it sends a space X to the simplicial set $S(X) : \Delta \to \text{Set}$ called the singular set, defined by
$$S(X)[n] = \text{Hom}_{\text{Top}}(|\Delta^n|, X)$$
- On morphisms, it sends a continuous map $f : X \to Y$ to the morphism of simplicial sets $\lambda : S(X) \to S(Y)$ defined as follows. For each $n \in \mathbb{N}$, $\lambda_n : S(X)[n] \to S(Y)[n]$ is defined by
$$(h : |\Delta^n| \to X) \mapsto (f \circ h : |\Delta^n| \to Y)$$
such that the following diagram commutes:
$$\begin{tikzcd} S(X)[n] \arrow[r, "S(X)(f)"] \arrow[d, "\lambda_n"'] & S(X)[m] \arrow[d, "\lambda_m"] \\ S(Y)[n] \arrow[r, "S(Y)(f)"'] & S(Y)[m] \end{tikzcd}$$

Notice that this is reminiscent of the definitions in Algebraic Topology 2. Indeed $S(X)[n]$ for each $n \in \mathbb{N}$ is in fact the basis of the abelian group $C_n(X)$. It represents all the possible ways that an n-simplex could fit into X. Then the passage

$$\textbf{Top} \xrightarrow{S} \textbf{sSet} \xrightarrow{H_\bullet^\Delta(-;R)} {}_R\textbf{Mod}$$

recovers the singular homology of a space X with coefficients in a ring R. This is formulated slightly differently in Algebraic Topology 2.

Theorem 2.2.2

The singular functor $S : \text{Top} \to \text{sSet}$ is right adjoint to the geometric realization functor $|\cdot| : \text{sSet} \to \text{Top}$. This means that there is a natural bijection

$$\text{Hom}_{\text{Top}}(|X|, Y) \cong \text{Hom}_{\text{sSet}}(X, S(Y))$$

for any space Y and any simplicial set X.

2.3 Normalized Chain Complexes

Definition 2.3.1: Normalized Chain Complexes

Let $\mathcal{A}$ be an abelian category or the category **Grp**. Let A be a simplicial object in $\mathcal{A}$. Define the normalized chain complex of A to be the chain complex:

$$\cdots \longrightarrow N_{k+1}(A) \xrightarrow{\partial_{k+1}} N_k(A) \xrightarrow{\partial_k} N_{k-1}(A) \longrightarrow \cdots \longrightarrow N_1(A)$$

where

$$N_k(A) = \bigcap_{i=1}^{k} \ker(d_i^k : A_k \to A_{k-1})$$

and the differential given by $\partial_k = d_0^K|_{N_k(A)}$. We denote the normalized chain complex by $(N_\bullet(G), \partial_\bullet)$

nLab: We may think of the elements of the complex in degree k as being k-dimensional disks in G all of whose boundary is captured by a single face.

Lemma 2.3.2

Let G be a simplicial group. Consider the normalized chain complex $(N_\bullet(G), \partial_\bullet)$. Then $\partial_n N_n(G)$ is a normal subgroup of $N - n - 1(G)$.

Because of this lemma, it now makes sense to take the homology group of the normalized chain complex even if we take a simplicial object in **Grp**.

Definition 2.3.3: Normalized Chain Complex Functor

Let $\mathcal{A}$ be an abelian category. Define the normalized chain complex functor N

Definition 2.3.4: Degenerate Chain Complex

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Define the degenerate chain complex $D_\bullet(A)$ of A to be the subcomplex of the associated chain complex $C_\bullet(A)$ defined by

$$D_n(A) = \langle s_i^n : A_n \to A_{n+1} \mid s_i \text{ is the degenerate maps} \rangle$$

Proposition 2.3.5

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Then there is a splitting

$$C_\bullet(A) \cong N_\bullet(A) \oplus D_\bullet(A)$$

in the abelian category of chain complexes of $\mathcal{A}$.

Theorem 2.3.6: Eilenberg-Maclane

Let $\mathcal{A}$ be an abelian category. Let A be a simplicial object in $\mathcal{A}$. Then the inclusion

$$N_\bullet(A) \hookrightarrow C_\bullet(A)$$

is a natural chain homotopy equivalence. In other words, $D_\bullet(A)$ is null homotopic.

Theorem 2.3.7: The Dold-Kan Correspondence

Consider the abelian category $\mathbf{Ab}$ of abelian groups. The normalized chain complex functor

$$N : \mathrm{s}\mathbf{Ab} \xrightarrow{\cong} \mathrm{Ch}_{\geq 0}(\mathbf{Ab})$$

gives an equivalence of categories, with inverse as the simplicialization functor

$$\Gamma : \mathrm{Ch}_{\geq 0}(\mathbf{Ab}) \to \mathrm{s}\mathbf{Ab}$$

2.4 Bar Resolutions

Definition 2.4.1: Bar Construction

Let A be an algebra over a ring R. Let M be an A-algebra. Define the maps $d_i^n : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ by the following formulas:

- If $i = 0$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_1 \otimes a_2 \otimes \cdots \otimes a_n$$
- If $0 < i < n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = m \otimes a_1 \otimes \cdots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n$$
- If $i = n$, then
$$d_i^n(m \otimes a_1 \otimes \cdots \otimes a_n) = ma_n \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}$$

Highly related: Cotriple homology / cotriple constructions

Proposition 2.4.2

Let A be an algebra over a ring R. Let M be an A-algebra. Then the collection

$$\{M \otimes A^{\otimes n}, d_i^n \mid n \in \mathbb{N}\}$$

is a simplicial object.

Definition 2.4.3: Bar Resolutions

Let A be an algebra over a ring R. Let M be an A-algebra. Define the bar resolution of M to be the associated chain complex of the simplicial object

$$\{M \otimes A^{\otimes n}, d_i^n \mid n \in \mathbb{N}\}$$

Explicitly, it is the chain complex

$$\cdots \longrightarrow A^{\otimes n+1} \otimes M \longrightarrow A^{\otimes n} \otimes M \longrightarrow A^{\otimes n-1} \otimes M \longrightarrow \cdots \longrightarrow A \otimes M \longrightarrow M \longrightarrow 0$$

with the boundary map $\partial : A^{\otimes n} \otimes M \to A^{\otimes n-1} \otimes M$ given by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i^n$$

3 Simplicial Homotopy Theory

Algebraic topology is a new subject which received a name change from combinatorial topology. At that time, combinatorics was highly involved in deriving topological invariants because the combinatorial structure of any invariants or spaces of study makes computation easy. In particular, a great deal of work has been put into the homotopy theory of simplicial spaces.

Nowadays, the study of combinatorial objects in topology is less prominent, but the category of simplicial sets still play a distinguished role in algebraic topology. Indeed the category of simplicial sets is the prototypical example of a model category (we have not seen it), as well as exhibiting a Quillen equivalence between the model of a simplicial set and some model category structure of **Top**. We will not explore the concept of model categories but will instead display foundational knowledge of it in order to develop the category of simplicial sets as a workable category when studying Model Categories. This means that we will develop the notion of homotopy, fibrations and cofibrations in this category.

3.1 Homotopies of Simplicial Objects

Definition 3.1.1: Homotopy between Simplicial Maps

Let $\mathcal{C}$ be a category. Let $f, g : X \to Y$ be two morphisms of simplicial objects. We say that f and g are homotopic if there is a family of morphisms $h_i^n : X_n \to Y_{n+1}$ such that the following are true.

- $d_0^n \circ h_0 = f_n$
- $d_0^{n+1} \circ h_1 = g_n$
- The composition
$$d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i^{n+1} & \text{if } 0 \leq i < j \leq n \\ d_i \circ h_{i-1} & \text{if } i = j \neq 0 \\ h_j \circ d_{i-1} & \text{if } i > j+1 \end{cases}$$
- The composition
$$s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & \text{if } i \leq j \\ h_j \circ s_{i-1} & \text{if } i > j \end{cases}$$

When $\mathcal{C}$ is either the abelian category or **Set**, we can provide an equivalence characterization that is reminiscent to the classical notion of homotopy in **Top**.

Theorem 3.1.2

Let $\mathcal{A}$ be either an abelian category or **Set**. Let $f, g : X \to Y$ be two morphisms of simplicial objects. Then f and g are homotopic if and only if there exists a morphism $\eta : X \times \Delta[1] \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \simeq X \times \Delta[0] \arrow[r, "\mathrm{id}_X \times \delta_1"] \arrow[dr, "f"'] & X \times \Delta[1] \arrow[d, dashed, "\exists \eta"] & X \simeq X \times \Delta[0] \arrow[l, "\mathrm{id}_X \times \delta_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

Here, we think of $\Delta[1]$ as playing the role of the unit interval $I = [0, 1]$ in **Top**.

Proposition 3.1.3

Let $\mathcal{A}$ be an abelian category. Let $f, g : X \to Y$ be two morphisms of simplicial objects. Then the induced map
$$f_*, g_* : N_\bullet(X) \to N_\bullet(Y)$$
between normalized chain complexes are chain homotopic $f \simeq g$.

3.2 Horns and Fillers

Definition 3.2.1: Inner and Outer Horns

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Define the ith horn Λ_i^n of Δ^n to be the the simplicial subset generated by all the faces $\partial_k \Delta^n$ except the ith one. It is called inner if $0 < i < n$. It is called outer otherwise.

Definition 3.2.2: Fillers for a Horn

Let $n \in \mathbb{N}$ and consider the standard n-simplex Δ^n. Let Λ_i^n be a horn. We say that Λ admits a filler if for all maps $F : \Lambda_i^n \to C$ there exists a map $U : \Delta^n \to C$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "F"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists U"'] & \end{tikzcd}$$

3.3 Fibrations and Cofibrations

Definition 3.3.1: Kan Fibrations

Let $f : X \to Y$ be a morphism of simplicial sets. We say that f is a Kan fibration if the following condition is satisfied: For every commutative diagram:

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y \end{tikzcd}$$

where $n \geq 1$ and $0 \leq k \leq n$, there exists a lift $\Delta^n \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} \Lambda_k^n \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Kan fibrations are the combinatorial analogue of Serre fibrations. Indeed notice that Kan fibrations satisfies a homotopy lift property similar to that of Serre fibrations.

Definition 3.3.2: Kan Complexes

Let X be a simplicial set. We say that X is a Kan complex if the unique map $X \to *$ is a Kan fibration.

Lemma 3.3.3

Let X be a space. Then $S(X)$ is a Kan complex.

Theorem 3.3.4

Let X be a simplicial set. Then X is a Kan complex if and only if every horn of X admits a filler.

Definition 3.3.5: Weak Equivalences

Let $f : X \to Y$ be a map of simplicial sets. We say that f is a weak equivalence if f induces isomorphisms

$$f_* : \pi_n(|X|, v) \xrightarrow{\cong} \pi_n(|Y|, v)$$

for all $n \in \mathbb{N}$. We say that X and Y are weakly equivalent if there exists a weak equivalence $f : X \to Y$.

Definition 3.3.6: Fibrant Replacement

Theorem 3.3.7

Every simplicial set admits a fibrant replacement.

3.4 The Simplicial Homotopy Groups

Definition 3.4.1: Simplicial Homotopy Groups

Let X be a Kan complex. Let $v \in X$ be a vertex of X. Define the simplicial homotopy groups of (X, v) as follows:

- For $n \geq 1$, define the nth simplicial homotopy group $\pi_n^{\text{sSet}}(X, v)$ of X at v to be the set of homotopy classes of maps $[\alpha : \Delta^n \to X]$ relative to boundary $\partial\Delta^n$ such that the following diagram commutes:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "!"] \arrow[d, hook] & \Delta^0 \arrow[d, "v"] \\
\Delta^n \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

- Define the 0th simplicial homotopy group $\pi_0^{\text{sSet}}(X)$ of X to be the set of homotopy classes of vertices of X.

If X is a general simplicial set, define the simplicial homotopy group of X to be

$$\pi_n^{\text{sSet}}(X, v) = \pi_n^{\text{sSet}}(PX, Pv)$$

where PX is a fibrant replacement of X.

Theorem 3.4.2

Let X be a Kan complex. Let $f, g : \Delta^n \to X$ be two representatives of two elements in

$\pi_n(X, v)$ for $n \geq 1$. Then the following data

$$v_i = \begin{cases} s_0^n(v) & \text{if } 0 \leq i \leq n-2 \\ f & \text{if } i = n-1 \\ g & \text{if } i = n+1 \end{cases}$$

defines a horn $\Lambda_i^{n+1} \to X$. Such a map extends to a map $\theta : \Delta^{n+1} \to X$. Define

$$[f] \cdot [g] = d_n \theta$$

Then such an operation is well defined on the equivalence class. Moreover, it defines a group operation on $\pi_n^{\text{sSet}}(X, v)$.

Theorem 3.4.3

Let X be a Kan complex. Then for $n \geq 2$, the above group structure on $\pi_n^{\text{sSet}}(X, v)$ is abelian.

Theorem 3.4.4

Let X be a simplicial set. Let $v \in X_0$. Then there is an isomorphism

$$\pi_n^{\text{sSet}}(X, v) \cong \pi_n(|X|, v)$$

for all $n \in \mathbb{N}$.

4 The Nerve-Homotopy Adjunction

4.1 The Nerve of a Category

Definition 4.1.1: Nerve of a Category

Let $\mathcal{C}$ be a category. Define the nerve of the category $N(C) : \Delta \to \text{sSet}$ as follows.

- For $n \in \mathbb{N}$, $N(C)_n$ consists of paths of morphisms with n compositions:
$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$
- The face map $d_i : C_n \to C_{n-1}$ sends the above element to
$$c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \longrightarrow c_i \xrightarrow{\text{id}_{c_i}} c_i \longrightarrow \cdots \longrightarrow c_n$$
- The degeneracy map $s^i : C_n \to C_{n+1}$ sends the above element to

Definition 4.1.2: Nerve Functor

The nerve functor $N : \text{Cat} \to \text{sSet}$ is defined as follows.

- Each $\mathcal{C} \in \text{Cat}$ is sent to the nerve $N(C)$
- Every functor $\mathcal{C} \to \mathcal{D}$ in Cat is sent to the morphism of presheaves $\lambda : N(C) \to N(D)$ defined by $\lambda_n : N(C)([n]) \to N(D)([n])$, of which is defined as the map
$$c_0 \xrightarrow{f_1} c_1 \xrightarrow{f_2} c_2 \longrightarrow \cdots \longrightarrow c_n$$
$$F(c_0) \xrightarrow{F(f_1)} F(c_1) \xrightarrow{F(f_2)} F(c_2) \longrightarrow \cdots \longrightarrow F(c_n)$$
from the upper path in $\mathcal{C}$ to the lower path in $\mathcal{D}$, such that the following diagram commutes:
$$\begin{tikzcd} N(C)[n] \arrow[r, "N(C)(f)"] \arrow[d, "\lambda_n"'] & N(C)[m] \arrow[d, "\lambda_m"] \\ N(D)[n] \arrow[r, "N(D)(f)"'] & N(D)[m] \end{tikzcd}$$
where $f : [m] \to [n]$ is a morphism in Δ.

Theorem 4.1.3

The nerve functor $N : \text{Cat} \to \text{sSet}$ is fully faithful. Moreover, the nerve of a category is a complete invariant for categories.

4.2 The Homotopy Category of a Simplicial Set

Definition 4.2.1: Homotopy Category of Simplicial Sets

Let $X \in \mathbf{sSet}$ be a simplicial set. Define the homotopy category $h(X)$ of X as follows:

- The objects are the zero set $h(X) = X_0$
- Every $f \in X_1 = \text{Hom}_{\mathbf{sSet}}(\Delta^1, X)$ gives a map $d_1(f) \to d_0(f)$.

- Composition is generated arbitrarily: for every f and g morphisms such that $d_0(f) = d_1(g)$, define a new morphism $h = g * f$. Then further modulo the morphisms by defining relations as follows: For every two simplex $\sigma : \Delta^2 \to X$ such that $d_0(\sigma) = g$, $d_1(\sigma) = h$ and $d_2(\sigma) = f$, define $h \sim g * f$.

Definition 4.2.2: The Homotopy Functor

Define the homotopy functor $h : \text{sSet} \to \text{Cat}$ as follows.

- On objects, h sends a simplicial set $S : \Delta \to \text{Set}$ to

Theorem 4.2.3

The homotopy functor $h : \text{sSet} \to \text{Cat}$ is left adjoint to the nerve functor $N : \text{Cat} \to \text{sSet}$. This means that there is a natural bijection

$$\text{Hom}_{\text{Cat}}(h(C), D) \cong \text{Hom}_{\text{sSet}}(C, N(D))$$

4.3 The Classifying Space

Definition 4.3.1: Classifying Space of a Category

Let $\mathcal{C}$ be a category. Define the classifying space of $\mathcal{C}$ to be

$$B\mathcal{C} = |N(\mathcal{C})|$$

the geometric realization of the nerve $N(\mathcal{C})$.

Definition 4.3.2: Classifying Space of a Group

Let G be a group. Define the classifying space of G to be

$$BG = |N(G)|$$

where G here is considered as a groupoid with one element.