Stable Homotopy Theory

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Abstract

Contents

1 The Cobordism Groups

1.1 Unoriented Cobordism

Definition 1.1.1: Unoriented Cobordism

Let M, N be two smooth compact n-manifolds. We say that M and N are cobordant if there exists a smooth compact manifold X with boundary ∂X such that ∂X is diffeomorphic to $M \coprod -N$.

Lemma 1.1.2

Unoriented cobordism is an equivalence relation on the class of n-dimensional smooth manifolds for any $n \in \mathbb{N}$.

Definition 1.1.3: The Unoriented Cobordism Group

For $n \in \mathbb{N}$, denote

$$\mathcal{N}_n = \{[M] \mid [M] \text{ is a cobordism class of } n\text{-manifolds } \}$$

the set of cobordism classes of n-manifolds. Define

$$\mathcal{N}_* = \bigoplus_{k=0}^{\infty} \mathcal{N}_k$$

Lemma 1.1.4

For any $n \in \mathbb{N}$, $\mathcal{N}_n$ is an abelian group together with the disjoint union as the operation.

Proposition 1.1.5

The direct sum of abelian groups

$$\mathcal{N}_* = \bigoplus_{k=0}^{\infty} \mathcal{N}_k$$

forms a ring with the cartesian product. Moreover, it is graded commutative.

1.2 The Cobordism Group of a Pair of Spaces

Definition 1.2.1: Singular Manifold

Let (X, A) be a fixed pair of space. We say that a smooth compact n-manifold B is singular to (X, A) if there is a map $f : (B, \partial B) \to (X, A)$.

Definition 1.2.2: Cobordism of Oriented Singular Manifolds

Let (X, A) be a fixed pair of space. Let (B_1, f_1) and (B_2, f_2) be singular manifolds of (X, A). We say that they are cobordant if there exists a oriented smooth $(n + 1)$-manifold C together with a map $F : C \to X$ such that the following are true.

- $(B_1 \coprod -B_2, f_1 \coprod f_2)$ is contained in ∂C as a regular submanifold.
- $F|_{B_1 \coprod -B_2} = f_1 \coprod f_2$ and $f_1 \coprod f_2(\partial C \setminus B_1 \coprod -B_2) \subseteq A$

Definition 1.2.3: The Cobordism Group of a Pair of Spaces

Let (X, A) be a pair of space. Define
$$\mathcal{N}_n(X, A)$$
the n-dimensional cobordism group of (X, A) to consist of the equivalence class of cobordant n-dimensional compact smooth $(n+1)$-manifolds that are singular to (X, A). Define

$$\mathcal{N}_*(X, A) = \bigoplus_{k=0}^{\infty} \mathcal{N}_k(X, A)$$

TBA: Abelian group structure

Proposition 1.2.4

Let (X, A) be a pair of space. Then $\mathcal{N}_*(X, A)$ is a graded module over $\mathcal{N}_*$.

Theorem 1.2.5

The assignment $\mathcal{N}_n : \mathbf{Top}^2 \to \mathbf{Ab}$ is functorial in the following sense.

- For each $(X, A) \in \mathbf{Top}^2$, $\mathcal{N}_n(X, A)$ is the oriented cobordism group.
- For each map $\varphi : (X, A) \to (Y, B)$, there is an induced map
$$\varphi_* : \mathcal{N}_n(X, A) \to \mathcal{N}_n(Y, B)$$
given by $[B, f] \mapsto [B, \varphi \circ f]$

Theorem 1.2.6

Let (X, A) be a pair of spaces. There is a natural transformation

$$\partial : \mathcal{N}_n(X, A) \to \mathcal{N}_{n-1}(A, \emptyset)$$

defined by $\partial[B, f] = [\partial B, f|_{\partial B}]$.

Theorem 1.2.7

The collection of functor $\mathcal{N}_n : \mathbf{Top}^2 \to \mathbf{Ab}$ together with the natural transformation $\partial : \mathcal{N}_n(X, A) \to \mathcal{N}_{n-1}(A, \emptyset)$ defines a generalized homology theory.

1.3 Oriented Cobordism

Definition 1.3.1: Oriented Cobordism

Let M, N be two smooth compact oriented n-manifolds. We say that M and N are oriented cobordant if there exists a smooth compact oriented manifold X with boundary ∂X such that ∂X with its induced orientation is diffeomorphic to $M \coprod -N$ under an oriented diffeomorphism.

Lemma 1.3.2

Cobordism is an equivalence relation.

Definition 1.3.3: The Oriented Cobordism Group

For $n \in \mathbb{N}$, denote

$$\Omega_n = \{[M] \mid [M] \text{ is an oriented cobordism class of } n\text{-manifolds } \}$$

the set of oriented cobordism classes of n-manifolds. Define

$$\Omega_* = \bigoplus_{k=0}^{\infty} \Omega_k$$

Lemma 1.3.4

For any $n \in \mathbb{N}$, Ω_n is an abelian group together with the disjoint union as the operation.

Proposition 1.3.5

The direct sum of abelian groups

$$\Omega_* = \bigoplus_{i=0}^{\infty} \Omega_i$$

forms a ring with the cartesian product. Moreover, it is graded commutative.

1.4 The Oriented Cobordism Group of a Pair of Spaces

Definition 1.4.1: Oriented Singular Manifold

Let (X, A) be a fixed pair of space. We say that an oriented smooth compact n-manifold B is singular to (X, A) if there is a map $f : (B, \partial B) \to (X, A)$.

Definition 1.4.2: Oriented Cobordism of Oriented Singular Manifolds

Let (X, A) be a fixed pair of space. Let (B_1, f_1) and (B_2, f_2) be oriented singular manifolds of (X, A). We say that they are cobordant if there exists a compact oriented smooth $(n + 1)$-manifold C together with a map $F : C \to X$ such that the following are true.

- $(B_1 \coprod -B_2, f_1 \coprod f_2)$ is contained in ∂C as a regular submanifold with induced orientations.
- $F|_{B_1 \coprod -B_2} = f_1 \coprod f_2$ and $f_1 \coprod f_2(\partial C \setminus B_1 \coprod -B_2) \subseteq A$

Definition 1.4.3: The Oriented Cobordism Group of a Pair of Spaces

Let (X, A) be a pair of space. Define

$$\Omega_n(X, A)$$

the n-dimensional oriented cobordism group of (X, A) to consist of the equivalence class of cobordant n-dimensional compact oriented smooth $(n + 1)$-manifolds that are oriented singular to (X, A). Define

$$\Omega_*(X, A) = \bigoplus_{k=0}^{\infty} \Omega_k(X, A)$$

TBA: Abelian group structure

Proposition 1.4.4

Let (X, A) be a pair of space. Then $\Omega_*(X, A)$ is a graded module over Ω_*.

Theorem 1.4.5

The assignment $\Omega_n : \mathbf{Top}^2 \to \mathbf{Ab}$ is functorial in the following sense.

- For each $(X, A) \in \mathbf{Top}^2$, $\Omega_n(X, A)$ is the oriented cobordism group.
- For each map $\varphi : (X, A) \to (Y, B)$, there is an induced map
$$\varphi_* : \Omega_n(X, A) \to \Omega_n(Y, B)$$
given by $[B, f] \mapsto [B, \varphi \circ f]$

Theorem 1.4.6

Let (X, A) be a pair of spaces. There is a natural transformation
$$\partial : \Omega_n(X, A) \to \Omega_{n-1}(A, \emptyset)$$
defined by $\partial[B, f] = [\partial B, f|_{\partial B}]$.

Theorem 1.4.7

The collection of functor $\Omega_n : \mathbf{Top}^2 \to \mathbf{Ab}$ together with the natural transformation $\partial : \Omega_n(X, A) \to \Omega_{n-1}(A, \emptyset)$ defines a generalized homology theory.