Higher Category Theory

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Abstract

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1 Introduction to Infinity Categories

1.1 Infinity Categories and Some Examples

The foundations of infinity categories lay on the simpicial sets. Intuitively, any face $\partial_k\Delta$ of an n-simplex Δ captures a homotopy of the faces of $\partial_k\Delta$.

Definition 1.1.1: Infinity Categories

An infinity category is a simplicial set C such that each inner horn admits a filler. In other words, for all $0 < i < n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Theorem 1.1.2

Let $\mathcal{C}$ be a category. Every inner horn of the nerve $N(C)$ of $\mathcal{C}$ admits a filler and hence is an infinity category.

1.2 Homotopy Infinity Categories

Recall that for a simplicial set X, we defined the homotopy category $h(X)$ of X. Such an assignment is functorial. In the case of infinity categories, we can exhibit the structure of $h(X)$ more explicitly.

Definition 1.2.1: Homotopic Morphisms

Let $\mathcal{C}$ be an infinity category. Two morphisms $f, g : C \to D$ are said to be homotopic if there exists a 2-simplex σ such that

- $d_0(\sigma) = \text{id}_D$
- $d_1(\sigma) = g$
- $d_2(\sigma) = f$

In this case we write $f \simeq g$.

Lemma 1.2.2

Homotopy is an equivalence relation in any infinity category.

Proposition 1.2.3

Let C be an infinity category. Let $f, f' : C \to D$ and $g, g' : D \to E$ be morphisms in C. If $f \simeq f'$ and $g \simeq g'$, then
$$g \circ f \simeq g' \circ f'$$

Definition 1.2.4: Homotopy Category

Let C be an infinity category. Define the homotopy category $h(C)$ of C to consist of the following.

- The objects are the objects of C
- The morphisms are equivalent classes of morphisms $[f]$ for f a morphism in C

- Composition is defined by
$$[g] \circ [f] = [g \circ f]$$
which is well defined by the above.

Definition 1.2.5: Isomorphisms in Infinity Categories

Let C be an infinity category. Let $f : C \to D$ be a morphism. We say that f is an isomorphism if there exists $g : D \to C$ such that $g \circ f \simeq \mathrm{id}_C$ and $f \circ g \simeq \mathrm{id}_D$.

Lemma 1.2.6

Let C be an infinity category. Let $f : C \to D$ be a morphism. Then f is an isomorphism in C if and only if $[f]$ is an isomorphism in $h(C)$.

2 Infinity Categories in Topology

Lemma 2.0.1

Let X be a space. Then applying the singular functor $S(X)$ gives an infinity category.

Proposition 2.0.2

Let X be a space. Then the homotopy category of the singular set of X is equal to $h(S(X)) = \prod_1(X)$ the fundamental groupoid of X.

2.1 Kan Complexes

Definition 2.1.1: Kan Complexes

A Kan complex is a simplicial set C such that each horn (inner and outer) admits a filler. In other words, for all $0 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Since infinity catregories require only inner horns to admit a filler, we have the following inclusion relation:

$$\substack{\text{Infinity}\\\text{Categories}} \subset \substack{\text{Kan}\\\text{Complexes}}$$

Proposition 2.1.2

Let X be a space. Then $S(X)$ is a Kan complex.

Theorem 2.1.3

Let $\mathcal{C}$ be a small category. Then the simplicial set $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

More: Kan complexes = infinity groupoids (quillen equivalence in model category), and we should think of spaces as Kan complexes / infinity groupoids from now on.