Lie Groups and Lie Algebra

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May 19, 2024

Abstract

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1 Introduction to Lie Algebras

1.1 Lie Algebras

Definition 1.1.1: Lie Algebras

A Lie algebra is a vector space V over a field K together with a bilinear map $[-,-] : V \times V \to V$ such that for all $X, Y, Z \in V$, we have the following.

- Anti-commutativity: $[X, Y] = -[Y, X]$
- Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

The bilinear map $[-,-]$ is called a Lie bracket.

Lie Algebras are not algebras because the Lie bracket fails associativity. Therefore we have to redefine all the standard notions one has in algebra.

Definition 1.1.2: Homomorphism of Lie algebra

Let V and W be Lie algebras over K. A homomorphism from V to W is an K-linear map $F : V \to W$ such that

$$[F(a), F(b)] = [a, b]$$

for all $a, b \in V$.

Definition 1.1.3: Lie Subalgebra

Let V be a Lie algebra over K. A lie subalgebra of V is a subset $W \subseteq V$ such that

- W is a vector subspace of V
- $[w_1, w_2] \in W$ for all $w_1, w_2 \in W$

It is clear that a Lie subalgebra is also a Lie algebra in its own right. Moreover, the inclusion $W \to V$ is a Lie algebra homomorphism.

Definition 1.1.4: Ideal

Let V be a Lie algebra over K. Let I be a subset of V. Then I is an ideal of V if the following are true.

- I is a vector subspace of V
- $[v, i] \in I$ for all $v \in V$ and $i \in I$.

Proposition 1.1.5

Let V be a Lie algebra and I, J ideals of V. Then the following are also ideals of V.

- The intersection $I \cap J$
- The sum $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$
- The Lie bracket $[I, J] = \langle [i, j] \mid i \in I \text{ and } j \in J \rangle$

Proposition 1.1.6

Let V be a Lie algebra over K and U an ideal of V. Then V/U has a unique Lie algebra structure such that the quotient map $V \to V/U$ is a Lie algebra homomorphism.

Definition 1.1.7: Center

Let L be a Lie algebra. Define the center of L by

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$$

Lemma 1.1.8

Let L be a Lie algebra. Then $Z(L)$ is an ideal of L.

Definition 1.1.9: Direct Sum of Lie Algebras

Let L_1 and L_2 be Lie algebras. Define the direct sum of L_1 and L_2 by

$$L_1 \oplus L_2 = \{(a_1, a_2) \mid a_1 \in L_1, a_2 \in L_2\}$$

together with component wise addition and scalar multiplication and Lie bracket operation

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1], [a_2, b_2])$$

which is component wise application of the Lie bracket for $(a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2$.

Proposition 1.1.10

Let L_1 and L_2 be Lie algebras. Then the following are true.

- $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$
- $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$
- $\{(x, 0) \mid x \in L_1\} \cong L_1$ is an ideal of $L_1 \oplus L_2$
- $\{(0, y) \mid y \in L_2\} \cong L_2$ is an ideal of $L_1 \oplus L_2$

1.2 The Isomorphism Theorems

Theorem 1.2.1: First Isomorphism Theorem

Let $\phi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the following are true.

- $\ker(\phi)$ is an ideal of L_1
- $\text{im}(\phi)$ is a Lie subalgebra of L_2

Moreover, we have an isomorphism

$$\frac{L_1}{\ker(\phi)} \cong \text{im}(\phi)$$

Theorem 1.2.2: Second Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.

- I and J are ideals of $I + J$
- $I \cap J$ is an ideal of I and J

Moreover, we have an isomorphism

$$\frac{I + J}{J} \cong \frac{I}{I \cap J}$$

Theorem 1.2.3: Third Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L such that $I \subseteq J$. Then J/I is an ideal of L/I. Moreover, there is an isomorphism

$$\frac{L/I}{J/I} \cong \frac{L}{J}$$

Theorem 1.2.4: Correspondence Theorem

Let L be a Lie algebra with ideal I. Then there exists a bijective correspondence

$$\{J \mid J \text{ is an ideal of } L \text{ and } I \subseteq J\} \quad \overset{1:1}{\longleftrightarrow} \quad \{K \mid K \text{ is an ideal of } L/I\}$$

2 Introduction to Lie Groups

2.1 Lie Groups

Definition 2.1.1: Lie Groups

A Lie group G is a smooth manifold which is also a group such that the multiplication map $G \times G \to G$ given by $(g, h) \mapsto gh$ and the inverse map $i : G \to G$ given by $g \mapsto g^{-1}$ are smooth maps.

Proposition 2.1.2

Let G be a Lie group. A subgroup H of G has the unique structure of a Lie subgroup if H is closed in G.

2.2 Relation between Lie Groups and Lie Algebras

For a group G, denote the left multiplication map of $h \in G$ by l_h. If G is a Lie group, we have seen that l_h is a smooth map, and so it induces a differential $(l_h)_*$.

Definition 2.2.1: Left Invariant Vector Field

Let G be a Lie group and X a vector field on G. We say that X is left invariant if

$$(l_h)_*(X_g) = X_{hg}$$

for all $X_g \in T_g(G)$.

Proposition 2.2.2

Let G be a Lie group. The vector space of left invariant vector fields of G is a Lie algebra of dimension $\dim(G)$. Moreover, if $X_e \in T_e(G)$ is a tangent vector at e the identity, then there is a unique left invariant vector field X on G such that its identity is X_e.

Definition 2.2.3: Lie Algebra of a Lie Group

Let G be a Lie group. Define the Lie algebra V of G to be the vector space $T_e(G)$.

Recall that given a homomorphism of Lie groups $\phi : G \to H$, it induces a differential $\phi_* : T_g(G) \to T_{\phi(g)}(H)$.

Proposition 2.2.4

Let $\phi : G \to H$ be a homomorphism of Lie groups with Lie algebras V and W respectively. Then the induced map from the differential $\phi_* : V \to W$ is a Lie algebra homomorphism.