Equivariant Spaces

Labix

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Abstract

Contents

1 Topological Groups and G-Spaces

The above four chapters has established deep connections between three properties of a space. Namely, the fundamental group, the fibers of the covering space and the group of homeomorphisms of the covering space. Such a deep connection between algebra and topology is not unique to covering space theory, nor to fundamental groups. In this section we will take a step back and look at the big picture.

1.1 Topological Groups and the Coset Space

Definition 1.1.1: Topological Groups

Let G be a group. We say that G is a topological group if G is also a topological space and that the following are true.

- The multiplication map $\cdot : G \times G \to G$ defined by $(g, h) \mapsto gh$ is continuous.
- The inverse map $(-)^{-1} : G \to G$ defined by $g \mapsto g^{-1}$ is continuous.

Notice that every group can be given the discrete topology, and so every group is trivially a topological group. But of course there is no guarantee that anything interesting theorems will occur in this case. We call these topological groups discrete.

Definition 1.1.2: Discrete Group

Let G be a topological group. We say that G is a discrete group if it has the discrete topology.

Proposition 1.1.3

Let G be a topological group. Let H be a subgroup of G. Then H and $\overline{H}$ are both topological groups. Moreover, if H is normal, then $\overline{H}$ is normal.

Proposition 1.1.4

Let G be a topological group. Let H be a subgroup of G. Then the normalizer $N_G(H)$ and the centralizer $C_G(H)$ are closed subgroups of G.

Definition 1.1.5: The Coset Space

Let G be a topological group and H a closed subgroup of G. Define the coset space of H in G to be the quotient space

$$G/H = \{gH \mid g \in G\}$$

together with the (topological) quotient map $p : G \to G/H$ such that $U \subseteq G/H$ is open if and only if $p^{-1}(U)$ is open.

Theorem 1.1.6

Let G be a topological group. Let H be a subgroup of G. Then the (topological) quotient map

$$p : G \to G/H$$

is an open map. Moreover, the following are true regarding the quotient.

- G/H is Hausdorff if and only if H is closed in G
- G/H is discrete if and only if H is open in G.
- If H is normal and closed in G, then G/H is a topological group.

1.2 Morphisms Between Topological Groups

Definition 1.2.1: Continuous Homomorphisms

Let G and H be topological groups. A function $f : G \to H$ is said to be a continuous homomorphism if it is continuous and a group homomorphism.

Proposition 1.2.2

Let G, H be topological groups. Let $\varphi : G \to H$ be a surjective continuous homomorphism. Then $\ker(\varphi)$ is a closed subgroup of G and φ is a continuous bijection.

When the topological group G is compact, the first isomorphism theorem in fact gives a homeomorphism.

Proposition 1.2.3

Let G, H be topological groups. Let $\varphi : G \to H$ be a surjective continuous homomorphism. If G is compact, then

$$\overline{\varphi} : \frac{G}{\ker(\varphi)} \to H$$

is a homeomomorphism.

Proposition 1.2.4

Let G be a compact topological group. Let $g \in G$. Then

$$A = \overline{\{g^n \mid n \in \mathbb{N}\}}$$

is a subgroup of G.

1.3 Some Real Topological Groups

Proposition 1.3.1: The Group of Matrices as a Topological Group

The group of real $n \times n$ matrices

$$M_n(\mathbb{R}) = \{(a_{ij})_{n\times n} \mid a_{ij} \in \mathbb{R}\}$$

with group structure given by matrix addition can be given the structure of a topological group whose topology is given by the topology of $\mathbb{R}^{n^2}$ and a choice of bijection of sets $\mathbb{R}^{n^2} \cong M_n(\mathbb{R})$.

Proposition 1.3.2

Let $n \in \mathbb{N} \setminus \{0\}$. Then $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R})$.

Beware that $\mathrm{GL}(n, \mathbb{R}), O(n, \mathbb{R}), SL(n, \mathbb{R}), SO(n, \mathbb{R})$ are not topological groups as subsets of $M_n(\mathbb{R})$ because they are not subgroups of $M_n(\mathbb{R})$ in the first place.

Proposition 1.3.3: The General Linear Group as a Topological Group

The general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \{M \in M_n(\mathbb{R}) \mid \det(M) \neq 0\}$$

with group structure given by matrix multiplication can be given the structure of a topological group whose topology is given by the subspace topology as a subset of $M_n(\mathbb{R})$.

Proposition 1.3.4

Let $n \in \mathbb{N} \setminus \{0\}$. Then the following are all topological groups with group structure and topology inherited from $\text{GL}(n, \mathbb{R})$.

- The orthogonal group $O(n, \mathbb{R})$.
- The special linear group $SL(n, \mathbb{R})$.
- The special orthogonal group $SO(n, \mathbb{R})$.

Moreover, they are closed subsets of $\text{GL}(n, \mathbb{R})$.

1.4 Some Complex Topological Groups

$\text{GL}(n, \mathbb{C}), U(n), SU(n)$

2 Equivariant Spaces

2.1 G-Spaces and G-Equivariant Maps

In algebraic topology, we have the results of considering groups acting on spaces. We can in fact consider topological groups acting on spaces.

Definition 2.1.1: Continuous Group Actions

Let G be a topological group and X a space. We say that G is a continuous group action if G is a group acting on X such that the group action map

$$\cdot : G \times X \to X$$

is continuous. In this case we say that X is a G-space.

Frequently a continuous group action is also called a (topological) transformation group, for example in Milnor's Topology of Fiber Bundles or Introduction to Compact Topological Groups.

Proposition 2.1.2

Let G be a continuous group action of X. Then for each $g \in G$, the left action map $x \mapsto g \cdot x$ is a homeomorphism of X.

Proof. Every element of g has an inverse g^{-1} which are both continuous and are bijections on X. □

Proposition 2.1.3

Let G be a topological group and $(X, \mathcal{T})$ a topological space. Then G is a continuous group action on X if and only if G acts on $\mathcal{T}$.

Proof. Suppose that G is a continuous group action on X. Then for each $g \in G$, $g \cdot U = \{g \cdot x \mid x \in U\}$ for $U \in \mathcal{T}$ is open since A_g as above is a homeomorphism. Now suppose that G acts on $\mathcal{T}$. Then for each open set U of X, $g^{-1} \cdot U$ is open. Thus G is a continuous group action. □

In particular, some authors would assume one knows this fact, so it is always nice to see it spelled out. It is also standard to denote this action just by the element g instead of A_g.

Definition 2.1.4: Group of Homeomorphisms

Let X be a space. Define the group of homeomorphisms of X to be

$$\text{Homeo}(X) = \{f : X \to X \mid f \text{ is a homeomorphism}\}$$

together with composition of functions. We say that a group A is a subgroup of homeomorphisms of X if A is isomorphic to a subgroup of Homeo(X).

Lemma 2.1.5

Let G be a topological group. Let X be a G-space. Then there is a group homomorphism $\varphi : G \to \text{Homeo}(X)$ defined by

$$g \mapsto (x \mapsto g \cdot x)$$

Proof. We have already seen that for any $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism.

Thus the above mapping is well defined. Now we have that

$$\begin{aligned}\varphi(gh)(x) &= gh \cdot x \\ &= g \cdot (h \cdot x) \\ &= (\varphi(g) \circ \varphi(h))(x)\end{aligned}$$

and so φ is a group homomorphism. $\square$

Notice that if the above group homomorphism is injective, then the structure group G is a subgroup of homeomorphisms of G.

Definition 2.1.6: G-Equivariant Maps

Let G be a topological group and let X, Y be G-spaces. A G-equivariant map is a continuous map $f : X \to Y$ such that f is equivariant. In other words, we require that

$$f(g \cdot x) = g \cdot f(x)$$

for all $x \in X$ and all $g \in G$.

Definition 2.1.7: Isomorphic G-Spaces

Let G be a topological group and let X, Y be G-space. We say that G and H are isomorphic G-spaces if there exists a G-equivariant map such that f is a homeomorphism.

Theorem 2.1.8

Let G be a topological group and let X be a G-space. Then the map

$$p : \frac{G}{\text{Stab}_G(x_0)} \to Gx_0 \subseteq X$$

induced by the map $g \mapsto g \cdot x_0$ is well defined. Moreover, it is isomorphic to the left G-space Gx_0.

Proof. To show that it s well defined, we want to show that if $g \in \text{Stab}_G(x_0)$, then $g \cdot x_0 = x_0$. But this is true by definition of the stabilizer. By definition of the induced map, it is continuous. Also, the orbit of x_0 is precisely Gx_0 and hence p is a bijection. It remains to show that p is an open map.

To show isomorphism, we also need to show that p is a G-equivariant map. We have that

$$\begin{aligned}p(g \cdot (h\text{Stab}_G(x_0))) &= p(gh\text{Stab}_G(x_0)) \\ &= (gh) \cdot x_0 \\ &= g \cdot (h \cdot x_0) \\ &= g \cdot p(h\text{Stab}_G(x_0))\end{aligned}$$

so that p is G-equivariant. $\square$

Definition 2.1.9: The Category of G-Spaces

Let G be a topological space. Define the category of G-spaces

$${}_G\mathbf{Top}$$

to consist of the following data.

- The objects are the G-spaces
- The morphisms are the G-equivariant spaces
- Composition is given by the composition of functions.

There is an obvious forgetful functor ${}_G\mathbf{Top} \to \mathbf{Top}$. One of its adjoint should assign the space to a trivial G-action.

Definition 2.1.10: The Trivial G-Space Functor

Let G be a topological group. Define the trivial G-space functor

$$\text{Triv} : \mathbf{Top} \to {}_G\mathbf{Top}$$

by the following.
- For each space X, define a group action on X by $g \cdot x = x$ for all $g \in G$ and $x \in X$.
- For each map $f : X \to Y$, $\text{Triv}(f) = f$ because f is trivially equivariant.

2.2 Induced and Restricted G-Spaces

Definition 2.2.1: Induced G-Spaces

Let G be a topological group. Let $H \leq G$ be a subgroup. Let X be an H-space. Define the induced G-space of X to be the space

$$\text{Ind}_H^G X = G \times_H X = \frac{G \times X}{\sim}$$

where the relation is generated by $(g \cdot h, x) \sim (g, h \cdot x)$ for $g \in G$, $h \in H$ and $x \in X$.

Pushout?

Lemma 2.2.2

Let G be a topological group. Let $H \leq G$ be a subgroup. Let X be an H-space. Then the following are true.
- If $X = *$, then $G \times_H X \cong \frac{G}{H}$.
- If $h \cdot x = x$ for all $h \in H$ (the action of H is trivial), then $G \times_H X \cong \frac{G}{H} \times X$.

Definition 2.2.3: Restricted G-Spaces

Let G be a topological group. Let $H \leq G$ be a subgroup. Let X be a G-space. Define the restriction

$$\text{Res}_H^G X$$

of X to H to be the space X considered as an H-space by the group action of G.

Proposition 2.2.4

Let G be a topological group. Let $H \leq G$ be a subgroup. Then there is an adjunction

$$\text{Ind}_H^G : {}_H\mathbf{Top} \rightleftarrows {}_G\mathbf{Top} : \text{Res}_H^G$$

This means that there is an isomorphism

$$\text{Hom}_{{}_G\mathbf{Top}}(\text{Ind}_H^G X, Y) \cong \text{Hom}_{{}_H\mathbf{Top}}(X, \text{Res}_H^G Y)$$

that is natural in X and Y.

2.3 Fixed Points and Orbit Spaces

Definition 2.3.1: The Fixed Points Functor

Let G be a topological group. Define the fixed points functor

$$(-)^G : {}_G\textbf{Top} \to \textbf{Top}$$

by the following.

- For each G-space, X, $X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$ is the subset of fixed points of G equipped with the subspace topology.
- For each G-equivariant map $f : X :\to Y$, $(f)^G : X^G \to Y^G$ is the restriction of f to X^G.

Check: it is well defined.

Proposition 2.3.2

Let G be a topological group. There is an adjunction

$$\text{Triv} : \textbf{Top} \rightleftarrows {}_G\textbf{Top} : (-)^G$$

This means that there is an isomorphism

$$\text{Hom}_{{}_G\textbf{Top}}(\text{Triv}X, Y) \cong \text{Hom}_{\textbf{Top}}(X, Y^G)$$

that is natural in X and Y.

Definition 2.3.3: The Orbit Space

Let X be a space and G be a group acting on X. Define the orbit space of X and G to be

$$\frac{X}{G} = \{\text{Orb}_G(x) \mid x \in X\}$$

the set of all orbits of G on X, inherited with the quotient topology of the equivalence relation of orbits.

This has the quotient topology because recall from groups and rings that $\text{Orb}_G(x)$ defines an equivalence relation on X.

2.4 The Pointed Analogue

Definition 2.4.1: Pointed G-Spaces

Let G be a topological group. A pointed G-space is a space X together with a G-equivariant map $* \to X$. We denote it by (X, x_0) where x_0 is the image of the map $* \to X$.

Lemma 2.4.2

Let G be a topological group. Let X be a G-space. Let (X, x_0) be a pointed space. Then (X, x_0) is a pointed G-space if and only if x_0 is a G-fixed point of X.

Definition 2.4.3: Induced Pointed G-Spaces

Let G be a topological group. Let $H \leq G$ be a subgroup. Let (X, x_0) be a pointed H-space. Define the induced pointed G-space of (X, x_0) to be the space

$$\text{Ind}_H^G X = G_+ \wedge_H X = \frac{G_+ \wedge X}{\sim}$$

where the relation is generated by $(g \cdot h, x) \sim (g, h \cdot x)$ for all $g \in G$, $h \in H$ and $x \in X$.

Pushout?

Proposition 2.4.4

Let G be a topological group. Let $H \leq G$ be a subgroup. Then there is an adjunction

$$\text{Ind}_H^G : {}_H\mathbf{Top}_* \rightleftarrows {}_G\mathbf{Top}_* : \text{Res}_H^G$$

This means that there is an isomorphism

$$\text{Hom}_{{}_G\mathbf{Top}_*}(\text{Ind}_H^G X, Y) \cong \text{Hom}_{{}_H\mathbf{Top}_*}(X, \text{Res}_H^G Y)$$

that is natural in X and Y.

3 Types of Actions on G-Spaces

3.1 Homogenous G-Spaces

Recall that a group action G on X is said to be transitive if for any $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

Definition 3.1.1: Homogenous G-Space

Let G be a topological group and let X be a G-space. We say that X is a Homogenous G-space if G acts transitively on X.

Much of the theorem we considered in covering space theory was in fact on homogenous G-spaces. For instance, when $\tilde{X}$ is path connected, prp2.4.7 says that the fibers $p^{-1}(x_0)$ of the covering space is a homogenous $\pi_1(X, x_0)$-space. The following corollary proves thm 2.4.9

Corollary 3.1.2

Let G be a topological group and let X be a homogenous G-space. Then there is an isomorphism of G-spaces

$$\frac{G}{\text{Stab}_G(x_0)} \cong X$$

induced by the map $g \mapsto g \cdot x_0$.

Proof. Since G is transitive on X, we have that $Gx_0 = X$. Hence by the above theorem, we obtain the desired isomorphism. □

Indeed, if $p : \tilde{X} \to X$ is a covering space, $p^{-1}(x_0)$ is a homogenous $\pi_1(X, x_0)$-space and it follows that

$$\frac{\pi_1(X, x_0)}{p_*(\pi_1(\tilde{X}, \tilde{x}_0))} \cong p^{-1}(x_0)$$

Notice that $\text{Stab}_{\pi_1(X,x_0)} = \text{im}(p_*)$ is a non-trivial fact that was proven in prp 2.4.7.

Corollary 3.1.3

Let G be a topological group and let X be a homogenous G-space. If G is more over a free action on X, there is an isomorphism of G-spaces

$$G \cong X$$

given by the map $g \mapsto g \cdot x_0$.

Proof. If G is free, then the stabilizer is trivial. By the above corollary, we obtain the desired isomorphism. □

Theorem 3.1.4

Let G be a topological group and let X be a homogenous G-space. Then the following are true.

- For any $\varphi \in \text{Homeo}(X)$, $x \in X$ and $\varphi(x)$ has the same stabilizers
- If $x, y \in X$ has the same stabilizers, then there exists $\varphi \in \text{Homeo}(X)$ such that $\varphi(x) = y$.

Lemma 3.1.5

Let G be a topological group and let X be a homogenous G-space. Let A be a subgroup of $\text{Homeo}(X)$. Then $A = \text{Homeo}(X)$ if and only if for any $x, y \in X$ such that $\text{Stab}_G(x) = \text{Stab}_G(y)$, there exists $\varphi \in A$ such that $\varphi(x) = y$.

Theorem 3.1.6

Let G be a topological group and let X be a homogenous G-space. Then there is a isomorphism of left G-spaces

$$\frac{N(\text{Stab}_G(x_0))}{\text{Stab}_G(x_0)} \cong \text{Homeo}(X)$$

3.2 Properly Discontinuous Group Actions

Definition 3.2.1: Proper Group Actions

Let G be a topological group acting continuously on a topological space X. The action is said to be proper if the map $G \times X \to X \times X$ defined by

$$(g, x) \mapsto (x, g \cdot x)$$

is a proper map.

Definition 3.2.2: Properly Discontinuous Group Actions

Let G be a group acting on a space X. Then we say that G is a properly discontinuous group action if for every compact set $K \subseteq X$, we have

$$(g \cdot K) \cap K \neq \emptyset$$

for finitely many $g \in G$.

Proposition 3.2.3

Every properly discontinuous group action is a wandering action.

Proposition 3.2.4

If G is a proper group action on a space X, then the action is properly discontinuous.

The converse is not true in general, unless we assume that X is locally compact.

Recall the notion of a covering space action. G is a covering space action on X if $g \cdot U \cap U \neq \emptyset$ implies $g = 1$. This is also related to properly discontinuous group actions. In fact, properly discontinuous group actions are in general stronger than covering space actions.

Proposition 3.2.5

Let G be a covering space action on X. If X is locally compact and Hausdorff, then G is a properly discontinuous group action on X.

3.3 Covering Space Actions

Definition 3.3.1: Wandering Actions

Let X be a space. Let G be a group acting on X. We say that G is a wandering action on X if for all $x \in X$, there exists a neighbourhood U of x such that

$$(g \cdot U) \cap U \neq \emptyset$$

for finitely many $g \in G$.

Algebraic topologists are primarily interested in the following type of group actions.

Definition 3.3.2: Covering Space Action

Let X be a space and G be a group acting on X. We say that G is a covering space action if for each $x \in X$, there is a neighbourhood U of x such that

$$(g_1 \cdot U) \cap (g_2 \cdot U) = \emptyset$$

for all $g_1, g_2 \in G$.

Lemma 3.3.3

Every covering space action is wandering and free.

Proof. Let G be a covering space action. Then $(g \cdot U) \cap U = \emptyset$ for all $g \in G$ implies that $g \cdot x$ cannot be equal to x. Thus G is a free action on X. It is clear that G is a wandering action on X since we require all actions of $g \in G$ to be disjoint while for wandering actions, we only require a finite amount of actions of $g \in G$ to be disjoint. □

The following proposition will show where the name of covering space actions comes from. In particular, we will see that if X is path connected, then there is one unique covering space action on X, namely via the deck group $\text{Deck}(p)$.

Proposition 3.3.4

Let X be a space and $p : \tilde{X} \to X$ be a covering of X. Then the action of $\text{Deck}(p)$ on $\tilde{X}$ is a covering space action.

Proof. Suppose that $\tilde{x} \in (\tau_1 \cdot U) \cap (\tau_2 \cdot U)$. Then this means that $\tau_1(\tilde{x}_1) = \tau_2(\tilde{x}_2)$ for some $\tilde{x}_1, \tilde{x}_2 \in U$. But we have that $p \circ \tau_1 = p \circ \tau_2$ which implies that $\tilde{x}_1$ and $\tilde{x}_2$ lie in the same fiber $p^{-1}(x)$ for some $x \in X$. By definition of covering spaces, the fiber $p^{-1}(x_0)$ intersects U at exactly one point so that $\tilde{x}_1 = \tilde{x}_2$. But this implies that $\tau_2^{-1}\tau_1$ fixes one point in $\tilde{X}$ so that $\tau_2^{-1}\tau_1 = 1$ and $tau_1 = \tau_2$. □

Lemma 3.3.5

Let X be a space and let $p : \tilde{X} \to X$ be a regular covering space of X. Then the orbit space of the deck group

$$\frac{\tilde{X}}{\text{Deck}(p)} \cong X$$

is isomorphic to the base space.

Proposition 3.3.6

Let X be a space and let G be a covering space action on X. Then the quotient map $p : X \to X/G$ defined by $p(x) = \text{Orb}_G(x)$ is a regular covering space of X/G.

Theorem 3.3.7

Let X be a path connected space. Let G be a covering space action on X. Then $G \cong \text{Deck}(p)$ where $p : X \to X/G$ is the regular covering space of X/G.

Corollary 3.3.8

Let X be a path connected and locally path connected space. Then there is a group isomorphism

$$\text{Deck}(p) \cong \frac{\pi_1\left(\frac{X}{\text{Deck}(p)}, x_0\right)}{p_*(\pi_1(X, p(x_0)))}$$

for any $x_0 \in X$.

4 Equivariant Homotopy Theory

4.1 G-Homotopy

Definition 4.1.1: G-Homotopy

Let G be a topological group and let X, Y be G-spaces. Let $f, g : X \to Y$ be G-equivariant maps. A G-homotopy from f to g is a homotopy $H : X \times I \to Y$ from f to g such that for each $t \in I$, the map

$$H(-, t) : X \to Y$$

is G-equivariant map.