Introduction to Algebraic Geometry

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Abstract

Algebraic Geometry is such a messy subject in a sense that a different books and lecture notes introduce different materials in a different orders, as well as having different prerequisites. After understanding a bit more in the subject, I believe that there is the need to give a clear distinction between traditional algebraic geometry and contemporary algebraic geometry. Although there are undoubtedly many overlappings between the two, I attempt to separate them to make clear their motivations as well as their results. Ultimately, it is simply a crucial tool in understanding different polynomials including planar curves, and surfaces and more.

This book will mainly cover traditional algebraic geometry in the sense that the construction of affine and projective varieties will be covered, as well as the Hilbert Nullstellensatz theorems, a tad bit of morphisms, and perhaps tangent maps and smoothness as well as classical constructions of morphisms. Affine schemes and sheaf theory are left for another time where they attempt to reinvent the fundamentals of algebraic geometry.

Knowledge on commutative algebra is required as a prerequisite. These set of notes make use of

- Algebraic Geometry I by I. R. Shafarevich and V. I. Danilov
- Algebraic Geometry by R. Hartshorne
- An Invitation to Algebraic Geometry by Karen. S, Pekka. K, Lauri .K, William .T

Contents

1 Introduction to Affine Varieties

1.1 Affine Varieties

Definition 1.1.1: Affine Space

For a field k, define the affine space over k to be the set

$$\mathbb{A}^n(k) = \{(a_1, \dots, a_n) | a_i \in k \text{ for } i = 1, \dots, n\}$$

In particular, there is no additional sturcture on $\mathbb{A}^n$ compared to k^n being a vector space equipped with addition and scalar multiplication.

Definition 1.1.2: Affine Algebraic Sets

Let $F = \{f_i\}$ be a collection of polynomials in $k[x_1, \dots, x_n]$. The zero locus of F is defined to be

$$V(F) = \{x \in \mathbb{A}^n | f(x) = 0 \text{ for all } f \in F\} \subseteq \mathbb{A}^n$$

Subsets of $\mathbb{A}^n$ of this form is called affine algebraic sets.

Some authors use the name algebraic sets, reserving the name algebraic variety to refer to irreducible algebraic sets.

Proposition 1.1.3

If I is the ideal of $F = \{f_i\}$ in $k[x_1, \dots, x_n]$ then $V(F) = V(I)$.

Proof. Basic properties of ideals proves the theorem. This is because every $f \in F$ is a finite sum and product of polynomials in the generating set. □

Thus from now on we need not consider ourselves with the affine variety of a countable collection of polynomials since we know that the ring of polynomials of n variables is finitely generated.

Proposition 1.1.4

Let $\{F_i \mid i \in I\}$ be a collection of subsets of $k[x_1, \dots, x_n]$. Then the following are true regarding the zero loci.

- Closed under countable intersections: $\bigcap_{i\in I} V(F_i) = V\left(\bigcup_{i\in I} F_i\right)$
- Closed under finite unions: $\bigcup_{i=1}^n V(F_i) = V\left(\bigcap_{i=1}^n F_i\right)$

Proof. □

Proposition 1.1.5: Zariski Topology

The complements of the set of all affine algebraic subsets of $X \subseteq \mathbb{A}^n$ forms a topology over X called the Zariski Topology

Definition 1.1.6: Affine Algebraic Varieties

An affine set is said to be irreducible if

$$V = V_1 \cup V_2$$

implies $V_1 = V$ or $V_2 = V$. In this case V is also said to be an affine algebraic variety.

Proposition 1.1.7

Every affine algebraic set is a finite union of affine algebraic variety. This decomposition is also unique up to reordering.

1.2 Hilbert's Nullstellensatz

Definition 1.2.1: Ideals of an Affine Variety

Let V be an affine algebraic set. Define the ideal of V to be

$$I(V) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in V\}$$

Lemma 1.2.2

Let k be a field. Let $V \subseteq \mathbb{A}_k^n$ be an affine algebraic set. Then $I(V)$ is an ideal of $k[x_1, \dots, x_n]$.

Similarly as above, $I(V)$ is finitely generated since ideals in a polynomial ring is finitely generated.

Proposition 1.2.3

Let V_1, V_2 be affine algebraic varieties. The following are true.

- If $V_1 \subseteq V_2$, then $I(V_1) \supseteq I(V_2)$
- $I(V_1 \cup V_2) = I(V_1) \cap I(V_2)$

Recall that the radical ideal is defined as

$$\sqrt{I} = \{f \in k[x_1, \dots, x_n] \mid f^r \in I \text{ for some } r > 0\}$$

Theorem 1.2.4: Hilbert's Nullstellensatz

Let k be an algebraically closed field. Let I be an ideal of $k[x_1, \dots, x_n]$. Then

$$I(V(I)) = \sqrt{I}$$

Corollary 1.2.5

Let k be an algebraically closed field. Then there is an inclusion reversing bijection

$$\left\{\begin{matrix}\text{Radical ideals of}\\ k[x_1,\dots,x_n]\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Affine algebraic}\\ \text{sets of } \mathbb{A}_k^n\end{matrix}\right\}$$

between the radical ideals of $k[x_1, \dots, x_n]$ and affine algebraic sets of $\mathbb{A}_k^n$ given by $V(-)$ and $I(-)$.

Note that this bijection is compatible with subset inclusion in the sense of proposition 1.2.3. Bijections of this form that induce a relation on subsets are called Galois connections or Galois correspondence, mimicking his work in Galois theory.

Corollary 1.2.6

Let k be an algebraically closed field. Then there is an inclusion reversing bijection

$$\left\{\begin{matrix}\text{Prime ideals of}\\ k[x_1,\dots,x_n]\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{matrix}\text{Affine algebraic}\\ \text{varieties of } \mathbb{A}_k^n\end{matrix}\right\}$$

between the prime ideals of $k[x_1, \dots, x_n]$ and affine varieties of $\mathbb{A}_k^n$ given by $V(-)$ and $I(-)$.

Corollary 1.2.7

Let k be an algebraically closed field. Then there is an inclusion reversing bijection

$$\begin{Bmatrix} \text{Maximal ideals of} \\ k[x_1,\dots,x_n] \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \text{Points in} \\ \mathbb{A}_k^n \end{Bmatrix}$$

between the maximal ideals of $k[x_1,\dots,x_n]$ and points in $\mathbb{A}_k^n$ given by $V(-)$ and $I(-)$.

Proposition 1.2.8

Every radical ideal J in $\mathbb{F}[x_1,\dots,x_n]$ is a finite intersection of prime ideals.

Proof. Given a radical ideal I, translate it over to its corresponding affine variety $V(I)$. Then the affine variety can be decomposed into a finite union of algebraic varieties $V(I) = \bigcup_{k=1}^n V_k$. These algebraic varities are able to be matched with a prime ideal by the above proposition. This bijection conjugates the union to the intersection and we are done. □

1.3 Polynomial Functions on a Variety

Definition 1.3.1: Coordinate Ring

Let k be a field and let $V \subseteq \mathbb{A}_k^n$ be an affine variety. Define the coordinate ring of V to be

$$k[V] = \frac{k[x_1,\dots,x_n]}{I(V)}$$

to be the ring of polynomial functions on V.

An example does better than its definition. Let us make an example out of $\mathbb{R}^2$. Let $f(x,y) = xy - 1$. Then $V(f) = \{(x,y) \in \mathbb{R}^2 | xy = 1\}$. Then $\mathbb{R}[V]$ can be described simply where if you see any polynomial with a factor of xy in it, treat it as 1. For example, if $g(x,y) = (x+y)^2 \in \mathbb{R}[x,y]$, then $g(x,y) = x^2 + 2xy + y^2 = x^2 + y^2 + 2 \in \mathbb{R}[V]$. This example makes the next theorem quite obvious.

Proposition 1.3.2

Let V be an affine variety over an algebraically closed field k. Then the following are equivalent.

- V is irreducible
- $I(V)$ is a prime ideal
- $k[V]$ is an integral domain.

Proof. It is clear from ring theory that $I(V)$ is a prime ideal if and only if $\mathbb{C}[V]$ is an integral domain. Suppose now that V is irreducible. Suppose for a contradiction that $\mathbb{C}[V]$ is not an integral domain. Then there exists nonzero $f_1, f_2 \in \mathbb{C}[V]$ such that $f_1 f_2 = 0$. Since they are nonzero, $V(f_1)$ and $V(f_2)$ are not V. But $V(f_1 f_2) = V$. This means that $V(f_1) \cup V(f_2) = V(f_1 f_2) = V$. Which means that V is reducible, a contradiction.

Suppose now that $\mathbb{C}[V]$ is an integral domain but V is reducible. Then there are some V_1, V_2 nonempty and closed such that $V_1 \cup V_2 = V$. By nullstellensatz, $I(V_1)$ and $I(V_2)$ are non empty since they are not the entire V. Choose nonzero $f_1 \in I(V_1)$ and $f_2 \in I(V_2)$. Then $f_1 f_2$ vanishes on V. Thus $f_1 f_2 = 0$ which contradicts the fact that $\mathbb{C}[V]$ is an integral domain. □

1.4 Morphisms of Affine Varieties

Definition 1.4.1: Regular Maps

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties. A regular map from V to W is a map

$$\phi : V \to W$$

such that for every $p \in V$, $\phi(p) = (f_1(p), \dots, f_m(p))$ for some polynomial in $f_1, \dots, f_m \in k[V]$.

The name regular maps has exactly the same meaning as morphisms of affine varieties.

Definition 1.4.2: Isomorphic Varieties

A regular map $\phi : X \to Y$ between two varieties is an isomorphism if it has an inverse that is a regular map. X and Y are said to be isomorphic in this case.

Proposition 1.4.3

Let V, W, U be affine varieties. If $f : V \to W$ and $g : W \to U$ are regular maps, then $g \circ f : V \to U$ is also a regular map.

Definition 1.4.4: Pullback of a Regular Map

Let $\phi : V \to W$ be a morphism of varieties. Then define the pull back of ϕ by

$$\phi^* : k[W] \to k[V]$$

where $\phi^*(p) = p \circ \phi$ for each $p \in k[W]$.

Lemma 1.4.5

Let V, W, U be affine varieties. If $f : V \to W$ and $g : W \to U$ are regular maps, then

$$(g \circ f)^* = f^* \circ g^*$$

Proposition 1.4.6

Let k be an algebraically closed field. Let V and W be affine varieties over k. Then there is a bijection

$$\left\{\substack{\text{Regular maps} \\ V \to W}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{\text{Algebra Homomorphisms} \\ k[W] \to k[V]}\right\}$$

between the regular maps and algebra homomorphisms.

Corollary 1.4.7

Let V and W be affine varieties over a field k. Then V and W are isomorphic if and only if $k[V] \cong k[W]$.

1.5 Rational Functions on a Variety

While polynomial rings are more than sufficient to classify varieties up to isomorphism, we would still like to enlarge the set of functions on a variety. We will do this by introducing rational functions on a variety.

Definition 1.5.1: Function Field

Let V be an affine variety over a field k. Define the function field of V to be

$$k(V) = \text{Frac}(k[V])$$

Elements of $k(V)$ are said to be rational functions on V.

Notice that functions in $k(V)$ are not well defined on all of V. Some functions may have poles on V. But by restricting to a certain open set in V (namely by removing the poles from the domain), we obtain a well defined rational function.

The remaining section constructs a sheaf of rational functions on varieties. Sheaves will not be formally in these notes.

Definition 1.5.2: Regular Functions

Let V be an affine algebraic variety over a field k. Then $f \in k(V)$ is said to be a regular function at $p \in V$ if

$$f(x) = \frac{g(x)}{h(x)}$$

for $g, h \in k[V]$ and $h(p) \neq 0$. Let U be an open subset of V. We say that $f \in k(V)$ is regular at U if f is regular at all $p \in U$.

Definition 1.5.3: Set of Regular Functions

Let V be an affine algebraic variety over a field k. Denote

$$\mathcal{O}_V(U) = \{f \in k(V) \mid f \text{ is regular at } U\}$$

the set of all regular functions on U.

Definition 1.5.4: Local Ring

Let V be an affine variety over a field k. Let $p \in V$. Define the local ring of V at p to be

$$\mathcal{O}_{V,p} = \{f \in k(V) \mid f \text{ is regular at } p\}$$

the set of all regular functions at p.

It is natural to ask that given a regular map between two varieties. How does all these regular functions and local rings transfer from one to another. In the case of polynomial functions (coordinate rings), the transferral comes from the opposite direction. The case for regular functions are similar. However we will not continue the exposition here since the language of sheaves will make definitions easier.

2 Introduction to Projective Varieties

2.1 Projective Space

Lemma 2.1.1

Let $\mathbb{A}$ be a field. The relation $\sim$ in $\mathbb{A}^{n+1}$ where $(x_0, \dots, x_n) \sim (y_0, \dots, y_n)$ if and only if $y_i = \lambda x_i$ for all $i\{1, \dots, n\}$ with $\lambda \in \mathbb{A}$ is an equivalence relation.

Definition 2.1.2: Projective Space

The equivalence relation $\sim$ on $\mathbb{A}^{n+1}$ induces the projective space with elements in it being 1 dimensional subspaces of $\mathbb{A}^n$, written as

$$\mathbb{P}^n(\mathbb{A}) = \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\sim}$$

Proposition 2.1.3

There is a bijection between $\mathbb{P}^n$ and the set of lines through the origin in $\mathbb{A}^{n+1}$.

Proposition 2.1.4

We have the following identification of projective space:

$$\mathbb{P}^n \cong \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

where the union is disjoint and

$$[x_0 : \cdots : x_n] \to \begin{cases} \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in \mathbb{A}^n & \text{for } x_0 \neq 0 \\ [x_1, \dots, x_n] \in \mathbb{P}^{n-1} & \text{for } x_0 = 0 \end{cases}$$

This is an identification of points in $\mathbb{P}^n$. They are either affine coordinates or coordinates at infinity. Recursively, we can deduce that

$$\mathbb{P}^n \cong \mathbb{A}^n \cup \cdots \cup \mathbb{A} \cup \{\infty\}$$

All of the above unions are disjoint.

Proposition 2.1.5

Let $U_k = \{[x_0 : \dots x_n] \in \mathbb{P}^n | x_k \neq 0\} = \{[x_0 : \dots x_n] \in \mathbb{P}^n | x_k = 1\}$. Then

$$\mathbb{P}^n = \bigcup_{k=0}^{n} U_k$$

Proposition 2.1.6

The projective space is an n dimensional manifold.

2.2 Homogenous Functions

Definition 2.2.1: Homogenous Functions

A polynomial f is said to be homogenous of degree d if all its terms has total degree d. Denote $\mathbb{C}_d[x_1, \dots, x_n]$ the set of all homogenous polynomials of degree d over $\mathbb{C}$.

Lemma 2.2.2

If f is homogenous of degree d then $f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n)$

Lemma 2.2.3

For all $f \in \mathbb{A}[x_0, \dots, x_n]$,

$$f = f_0 + f_1 + \cdots + f_d$$

where f_i is homogenous of degree i.

2.3 Projective Varities

Recall that an ideal is homogenous if it is generated by homogenous elements.

Definition 2.3.1: Projective Variety

Let I be an homogenous ideal. A projective variety in $\mathbb{P}^n$ is the common vanishing locus

$$V(I) = \{x \in \mathbb{P}^n | F(x) = 0 \text{ for all } F \in I\}$$

Definition 2.3.2: Ideals of Projective Varieties

Let V be a projective variety. Define

$$I(V) = \{F \in k[x_0, \dots, x_n] | F(V) = 0\}$$

It is obvious that I would be a homogenous ideal.

Proposition 2.3.3

The following are true for projective varieties.

- Projective varieties are closed under countable intersections.

$$\bigcap_{i \in I} V(F_i) = V\left(\bigcup_{i \in I} F_i\right)$$

- Projective varieties are closed under finite unions. If $F = \{f_1, \dots, f_n\}$, then

$$\bigcup_{i=1}^{n} V(F_i) = V(F)$$

Corollary 2.3.4: Zariski Topology

The complements of the set of all projective variety forms a topology over $\mathbb{P}^n$ called the Zariski Topology

2.4 The Projective Nullstellensatz

We have the projective version of the Nullstellensatz. It works exactly the same as that of the affine version.

Theorem 2.4.1: The Projective Nullstellensatz

Let $J \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an homogenous ideal. Then

$$I^H(V(J)) = \sqrt{J}$$

Corollary 2.4.2

There is an inclusion reversing bijective correspondence between projective varities in $\mathbb{P}^n$ and homogenous radical ideals in $\mathbb{C}[x_0, \dots, x_n]$.

$$\left\{\substack{\text{Homogenous Radical Ideals of} \\ k[x_1,\dots,x_n] \\ \text{such that } J \not\supseteq (x_1,\dots,x_n)}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{\text{Projective sets of} \\ \mathbb{P}^n}\right\}$$

given by $V(-)$ and $I^H(-)$.

Corollary 2.4.3

There is an inclusion reversing bijective correspondence between projective varities in $\mathbb{P}^n$ and homogenous radical ideals in $\mathbb{C}[x_0, \dots, x_n]$.

$$\left\{\substack{\text{Homogenous Prime Ideals of} \\ k[x_1,\dots,x_n] \\ \text{such that } J \not\supseteq (x_1,\dots,x_n)}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{\text{Projective varieties of} \\ \mathbb{P}^n}\right\}$$

given by $V(-)$ and $I^H(-)$.

Proposition 2.4.4

Every projective variety is a finite union of irreducible projective varieties.

2.5 The Relation Between Affine and Projective Varieties

Definition 2.5.1: The Dehomogenization Map

The map $\phi_i : \mathbb{C}_d[z_0, \dots, z_n] \to \mathbb{C}[x_1, \dots, x_n]$ defined by

$$f(z_0, \dots, z_n) \mapsto f\left(1, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right) = f(1, x_1, \dots, x_n)$$

is called dehomogenization with respect to z_i.

Lemma 2.5.2

Let $V \subseteq \mathbb{P}^n$ be a projective variety. Let U_i be a chart of $\mathbb{P}^n$ where the ith coordinate is 1. Then

$$(V \cap U_i) \subseteq U_i \cong \mathbb{A}^n$$

is an affine variety.

Theorem 2.5.3

Let (U_i, ψ_i) be the chart of $\mathbb{P}^n$ where the ith coordinate is 1. Then the map

$$\psi_i : V(f) \cap U_i \subseteq \mathbb{P}^n \to V(\phi_i(f)) \subseteq \mathbb{A}^n$$

where ϕ_i is the dehomogenization with respect to z_i.

Definition 2.5.4: Homogenization

Let $f \in k[x_1, \dots, x_n]$ where $f = g_0 + \dots + g_d$ and g_k are the terms of degree k in f. Define the homogenization of f to be the new function

$$F(x_0, \dots, x_n) = x_0^d g_0 + x_0^{d-1} g_1 + \dots + g_d$$

The new function F is homogenous of degree d, not divisble by x_0. This map from $\mathbb{C}[x_1, \dots, x_n]$ to $\mathbb{C}[x_0, \dots, x_n]$ denoted by φ is called homogenization.

Theorem 2.5.5

Let $f \in \mathbb{C}[x_1, \dots, x_n]$. Let $F \in \mathbb{C}[x_0, \dots, x_n]$ and $F = x_0^k G$ such that z_0 does not divide G. Then the following are true.

- $\phi_0(\varphi(f)) = f$
- $\varphi(\phi_0(F)) = G$

Lemma 2.5.6

Let $\{f_i | i \in I\} \subset \mathbb{C}[x_1, \dots, x_n]$. Let U_0 be the chart of $\mathbb{P}^n$ for which $x_0 = 1$. Then

$$V(\{\varphi(f_i) | i \in I\}) \cap U_0 = V(\{f_i | i \in I\})$$

Theorem 2.5.7

Suppose that U_i inherits the Zariski topology from $\mathbb{P}^n$. Then $\psi_i : U_i \to \mathbb{A}^n$ is a homeomorphism.

In general, for W an affine variety of $\mathbb{C}^n$, considering $\mathbb{C}^n$ in the open cover U_0, W may not be closed and so may not be a projective variety. However the closure certainly is.

Definition 2.5.8: Projective Closure

Let V be an affine variety of $\mathbb{C}^n$. Then define the projective closure of V to be the closure of V when considered inside $\mathbb{P}^n$.

Proposition 2.5.9

Let I be a radical ideal of $\mathbb{C}[x_1, \dots, x_n]$. Let $W = V(I) \subseteq \mathbb{A}^n$. Denote $\overline{W}$ the projective closure of W. Then

$$(\{\varphi(f) | f \in I\}) = I(\overline{W})$$

In other words, the homogenization of the radical ideal is precisely the generating set of the projective closure.

Proposition 2.5.10

Let $V \subset U_0 \cong \mathbb{A}^n$ be an affine variety. Then the closure of V in the Zariski topology of $\mathbb{A}^n$ and are exactly the projective closures of V. They coincide.

2.6 Morphisms of Projective Varieties

Definition 2.6.1: Morphisms of Projective Varieties

Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be projective varieties. Let $F : V \to W$ be a map (of sets) from V to W. We say that F is a morphism of projective varieties if for each $p \in V$, there exists homogeneous polynomials $F_0, \dots, F_m \in \mathbb{C}[x_0, \dots, x_n]$ of the same degree and an open neighbourhood U of p such that the following holds.

- $V(F_0, \dots, F_m) \cap U = \emptyset$ (They cannot all vanish at the same time)
- $F|_U : U \to W$ agrees with the map $U \to \mathbb{P}^m$ defined by

$$[z_0 : \dots : z_n] \mapsto [F_0(z_0, \dots, z_n) : \dots : F_m(z_0, \dots, z_n)]$$

This defintion ensures that when a morphism of projective varities is restricted to its open covers $\mathbb{A}^n$, it defines a morphism of affine varieties. They are locally affine varieties.

Definition 2.6.2: Isomorphism of Projective Varieties

A morphism of projective varieties $F : V \to W$ is an isomorphism if there exists a morphism $G : W \to V$ such that G is the inverse of F. In this case we say that V and W are isomorphic.

An automophism of a projective variety V is an isomorphism from V to itself.

Definition 2.6.3: Projectively Equivalent

Two projective varieties are said to be projectively equivalent if there exists a change of coordinates of $\mathbb{P}^n$ that defines an isomorphism between them.

3 Quasi-Projective Varieties

3.1 Quasi-Projective Varieties

In this section we attempt to unify the two types of varieties, affine and projective into one unified theory.

Definition 3.1.1: Locally Closed Subsets

A locally closed subset of a topological space X is a subset of the form $U \cap V$ where U is open in X and V is closed in X.

Definition 3.1.2: Quasi-Projective Varieties

A quasiprojective variety is a locally closed subset of $\mathbb{P}^n$.

Lemma 3.1.3

Affine varieties and projective varieties are both quasiprojective varieties.

Proof. Let $W \subseteq \mathbb{A}^n$ be an affine variety. Then W is closed and thus $W = \overline{W} \cap U_0$ where $\overline{W}$ is the closure of W in $\mathbb{P}^n$ by $\mathbb{A}^n \cong U_0 \subseteq \mathbb{P}^n$.

Let $V \subseteq \mathbb{P}^n$ be a projective variety. Then V being closed implies $V = V \cap \mathbb{P}^n$ trivially. □

Lemma 3.1.4

Any open subset of $\mathbb{P}^n$ or $\mathbb{A}^n$ is a quasiprojective variety.

3.2 Morphisms of Quasi-Projective Varieties

Definition 3.2.1: Morphisms of Quasi-Projective Varities

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be quasiprojective varieties. A morphism from X to Y is a map $F : X \to Y$ such that for all $p \in X$, there exists an open neighbourhood U_p together with homogenous polynomials $F_0, \dots, F_m \in k[x_0, \dots, x_n]$ of the same degree such that

- $V(F_0, \dots, F_m) \cap U = \emptyset$
- $F|_U$ agrees with $[x_0, \dots, x_n] \to [F_0(x_0, \dots, x_n), \dots, F_m(x_0, \dots, x_n)]$

Lemma 3.2.2

Every morphism of affine or projective varieties is a morphism of quasiprojective varieties.

3.3 Redefining Varieties

Definition 3.3.1: Extended Definition of Affine Varities

A quasiprojective variety is said to be affine if it is isomorphic to a closed subset of affine space.

Definition 3.3.2: Extended Definition of Projective Varities

A quasiprojective variety is said to be projective if it is isomorphic to a closed subvariety of projective space.

Definition 3.3.3: Basic Open Sets

Let V be Zariski closed on $\mathbb{A}^n$. Let $f \in k[V]$. Then

$$D(f) = V \setminus V(f)$$

is said to be a basic open set.

In other words, $D(f)$ is exactly the points of V where f is not zero. Some literature like to use V_f for notation.

Proposition 3.3.4

Let V be Zariski closed on $\mathbb{A}^n$. Let $f \in k[V]$. Then

- $D(f)$ is an affine algebraic variety
- Every open subset of V is a union of basic open sets
- The set of all basic open sets of X forms a basis for the Zariski Topology

Proposition 3.3.5

Every quasiprojective variety is locally affine.

3.4 Regular Functions

The following definition is simply a special case of regular maps, as in regular functions are simply regular maps with codomain $\mathbb{A}^1$.

Definition 3.4.1: Regular Functions on Affine Varieties

Let U be an open subset of an affine variety V. A complex valued function $f : U \to \mathbb{C}$ is regular at a point $p \in U$ if there exists functions $g, h \in \mathbb{C}[V]$ such that $h(p) \neq 0$ and that

$$f(x) = \frac{g(x)}{h(x)}$$

in some neighbourhood of p.

f is said to be regular on U if it is regular at every point of U.

The set of all regular functions is denoted by $\mathcal{O}_V(U)$.

Definition 3.4.2: Regular Functions on Quasi-Projective Varities

Let U be an open subset of a quasiprojective variety V. A complex valued function $f : U \to \mathbb{C}$ is regular at a point $p \in U$ if there some affine open set containing p on which f is regular at p. f is said to be regular on U if it is regular at every point of U.

The set of all regular functions is denoted by $\mathcal{O}_V(U)$.

Lemma 3.4.3

For any $U \subset V$ open and V a quasiprojective variety, $\mathcal{O}_V(U)$ is a $\mathbb{C}$-algebra.

Definition 3.4.4: Ring of Germs of Regular Functions

Let p be a point of a variety X. Define the local ring of p on X to be

$$\mathcal{O}_{V,p} = \{(U, f) | U \subseteq X \text{ is open}, p \in U, f \text{ is regular on } U\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if $f = g$ on $U \cap V$.

Proposition 3.4.5

Let X be a variety and $p \in X$. Then the ring of germs $\mathcal{O}_{X,p}$ is a local ring.

Definition 3.4.6: Function Field

Let X be a variety. Define the function field of X to be

$$K(X) = \{(U, f) | U \subseteq X \text{ is open}, f \text{ is regular on } U\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if $f = g$ on $U \cap V$. Elements of $K(X)$ are called rational functions on X.

Lemma 3.4.7

Let X be a variety. Then the function field $K(X)$ of X is a field.

Lemma 3.4.8

Let X be a variety. For any point p, there are natural injective maps $\mathcal{O}_X(X) \to \mathcal{O}_{X,p} \to K(X)$.

Proposition 3.4.9

Let X and Y be isomorphic varieties. Then $\mathcal{O}_X(X) \cong \mathcal{O}_Y(Y)$, $K(X) \cong K(Y)$ and if $p \in X$ maps to $q \in Y$, then $\mathcal{O}_{X,p} \cong \mathcal{O}_{Y,q}$.

Theorem 3.4.10

Let V be an affine variety. Let U be an open subset of V.

- There is a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[V] \to \mathcal{O}_V(U)$ given by restriction of functions
- The above map is injective if U is dense in V
- If V is an irreducible affine variety, then the map is surjective and an isomorphsim.

Theorem 3.4.11

Let $V \subseteq \mathbb{A}^n$ be an affine variety. Let $p \in V$ be a point. Then the following are true.

- $\mathcal{O}_V(V) \cong \mathbb{C}[V]$
- Let $m_p = \{f \in \mathbb{C}[V] | f(p) = 0\}$ be the ideal of functions that vanish at p. Then the map $p \mapsto m_p$ gives a one to one correspondence between points of V and the maximal ideals of $\mathbb{C}[V]$
- For each $p \in V$, $\mathcal{O}_{V,p} \cong \mathbb{C}[V]_{m_p}$ and $\dim(\mathcal{O}_{V,p}) = \dim(X)$
- $K(X) \cong \text{Frac}(\mathbb{C}[V])$ and $K(X)$ is a finitely generated extension field of $\mathbb{C}$.

Theorem 3.4.12

Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $p \in V$. Then the following are true.

- $\mathcal{O}_V(V) \cong \mathbb{C}$
- Let $m_p = \{f \in \mathbb{C}[V] | f$ is homogenous and $f(p) = 0\}$ be the ideal of functions that vanish at p. Then $\mathcal{O}_{V,p} = \mathbb{C}[V]_{m_p}$
- $K(X) \cong \mathbb{C}[V]_{(0)}$

4 Classical Constructions

4.1 Veronese Maps

Definition 4.1.1: Veronese Maps

The dth veronese map of $\mathbb{P}^n$ is the morphism $\nu_d : \mathbb{P}^n \to \mathbb{P}^m$ defined by

$$\nu_d([x_0 : \cdots : x_n]) = [x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d]$$

where $m = \binom{d+n}{n} - 1$.

Proposition 4.1.2

The veronese mapping ν_d defines an isomorphism of $\mathbb{P}^n$ onto its image.

4.2 Segre Maps

Definition 4.2.1: General Segre Maps

Define the general Segre map $\sum_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ by

$$\sum_{m,n}([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) = [x_0y_0 : x_0y_1 : \cdots : x_iy_j : \cdots : x_my_n]$$

4.3 Grassmannians

Definition 4.3.1: Grassmannians

Let $n \in \mathbb{N}^+$. Let $k \in \mathbb{N}$ with $0 \leq k \leq n$. Denote $G(k, n)$ the set of all k-dimensional linear subspaces of $\mathbb{C}^n$.

Similar to how $G(1, n) = \mathbb{P}^{n-1}$, $G(k, n)$ is essentially the $k-1$ dimensional projective subspaces of $\mathbb{P}^{n-1}$.

Lemma 4.3.2

$G(k, n)$ can be identified with the set

$$G = \frac{\{M \in M_{k\times n}(\mathbb{C}) | M \text{ has full rank } \}}{\text{Orbits of } GL(k)}$$

Proof. Let V be a k-dimensional vector subspace of $\mathbb{C}^n$. Choose basis vectors $(a_{j1}, \dots, a_{jn})$ where $j = 1, \dots, k$ for V. Form the row matrix of basis vectors

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

This matrix is formed by a basis thus the rows must be linearly independent, which means it achieves full rank. Two matrices span the same subspace if and only if there exists an invertible matrix of dimension k such that $(a_{ij}) = g(b_{ij})$. So we can quotient out extra elements in $M_{k\times n}(\mathbb{C})$ that represent the same vector subspace to get an identification of $G(k, n)$:

$$G = \frac{\{M \in M_{k\times n}(\mathbb{C}) | M \text{ has full rank } \}}{\text{Orbits of } GL(k)}$$

□

Definition 4.3.3: Plucker Embedding

Denote $\Delta_{i_1,\dots,i_k}$ the $k\times k$ subdeterminant of $A\in M_{k\times n}(\mathbb{C})$. formed by the columns $i_1,\dots,i_k$ in A. The Plucker embedding is the map $\phi : G \to \mathbb{P}^{\binom{n}{k}-1}$ given by

$$\phi\left(\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right) = [\Delta_{1,\dots,k} : \cdots : \Delta_{i_1,\dots,i_k} : \cdots : \Delta_{n-k+1,\dots,n}]$$

Proposition 4.3.4

The Plucker embedding is well defined and is injective.

Proof. This map is well defined since for any two matrices of rank k that span the same subspace that differ by multiplication of $G\in GL(k)$, they give the same point since multiplying G changes the subdeterminants by a factor of $\det(G)$, and in projective space they mean the same point. Moreover, since matrices in G has full rank, there must be at least one subdeterminant is nonzero. □

Theorem 4.3.5

The Grassmannians $G(k,n)$ can be embedded as a complex submanifold of $\mathbb{P}^{\binom{n}{k}-1}$.

Proof. Using the Plucker embedding, we see that $G(k,n)$ can be identified with a subset of $\mathbb{P}^{\binom{n}{k}-1}$. We now need to give an atlas to it. An open cover of $G(k,n)$ in the projective space is given by

$$U_{(i_1,\dots,i_k)} = \{V\in Gr(k,n) \mid \Delta_{i_1,\dots,i_k}\neq 0\}$$

. Since the submatrix formed by the columns $i_1,\dots,i_k$ is nonzero, we can find a representation of the subspace where each columns $i_1,\dots,i_k$ is the unit vector $e_1,\dots,e_k$. The rest of the $k(n-k)$ coordinates can be used to as an identification in the atlas. This means that we have a map $U_{i_1,\dots,i_k}\to\mathbb{C}^{k(n-k)}$.

The transition maps between two open cover is given by the rational functions $\frac{\Delta_{i_1,\dots,i_k}}{\Delta_{j_1,\dots,j_k}}$, which is clearly analytic. □

Theorem 4.3.6

The Grassmannians $G(k,n)$ is a projective algebraic variety.

5 The Tangent Space

5.1 Dimensions

Definition 5.1.1: Dimension (Topological)

Let X be a topological space. Suppose that $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ is a chain irreducible varieties of X. Define the dimension of X to be

$$\dim(X) = \sup_{\substack{Z_0,\dots,Z_n \subseteq X \\ \text{irreducible varieties}}} \{n \in \mathbb{N} | Z_0 \subset Z_1 \subset \cdots \subset Z_n\}$$

Lemma 5.1.2

If $X \subset Y$ then $\dim(X) < \dim(Y)$.

Lemma 5.1.3

$\dim(\mathbb{P}^n) = \dim(A^n) = n$.

Definition 5.1.4: Dimension (Algebraic)

Let A be a ring. Suppose that $P_0 \subset P_1 \subset \cdots \subset P_n$ is a chain of prime ideals of A. Define the dimension of A to be

$$\dim(A) = \sup_{\substack{P_0,\dots,P_n \subseteq A \\ \text{Prime ideals}}} \{n \in \mathbb{N} | P_0 \subset P_1 \subset \cdots \subset P_n\}$$

Theorem 5.1.5

Let $X \subset \mathbb{A}^n$ be an affine variety. Then

$$\dim(X) = \dim(k[X])$$

Theorem 5.1.6

Let A be a Noetherian ring. Let $f \in A$ be an element which is not a zero divisor and not a unit. Then every minimal prime ideal p containing f has height 1.

Proposition 5.1.7

A variety V in $\mathbb{A}^n$ has dimension $n-1$ if and only of it is the zero set of a single nonconstant irreducible polynomial in $k[x_1, \dots, x_n]$.

5.2 The Tangent Space of Affine Varieties

We first restrict our studies to affine varieties. There are two ways to define tangent spaces, one by the usual algebraic sense, as in the derivative being 0, the other is more geometric. The following definition comes as a result of the usual calculus we are familiar with.

Definition 5.2.1: Intersection Multiplicity

Let $L = \{ta | t \in \mathbb{C}\}$ for $a \in \mathbb{C}^n \setminus \{0\}$ be a line in $\mathbb{C}^n$. Let $X = V(f_1, \dots, f_m) \subseteq \mathbb{C}^n$ be an affine variety. We say that the intersection multiplicity of L with X is the multiplicity of $t = 0$ as a root of the polynomial $f(t) = \gcd(F_1(ta), \dots, F_m(ta))$.

Definition 5.2.2: Tangent to an Affine Variety

Let X be an affine variety. A line L is tangent to X at 0 if it has intersection multiplicity ≥ 2 with X at 0.

Definition 5.2.3: Tangent Space

Let X be an affine variety and $p \in X$. Define the tangent space of p in X to be the set of all lines tangent to X at p.

Recall that we have the notion of a differential in manifolds. Since all our functions we consider are polynomials, we can simply define derivatives by the formula for differentiating polynomials, that is without the notion of limits. This leads to the following equivalent definition of a tangent space.

Proposition 5.2.4

Let $V \subset \mathbb{A}^n$ be an affine variety. Then the tangent space of V at a point $p \in V$ is exactly equal to

$$T_pV = \left\{ q \in \mathbb{A}^n \,\middle|\, dF|_p(x-p) = \sum_{k=1}^n \frac{\partial F}{\partial x_k}\bigg|_p (q_k - p_k) = 0 \right\}$$

In particular, the tangent space is a vector space by identifying the point p as the origin and each differential $\frac{\partial F}{\partial x_1}\Big|_p, \dots, \frac{\partial F}{\partial x_n}\Big|_p$ as the standard basis.

Proposition 5.2.5

Let $V \subseteq \mathbb{C}^n$ be an affine algebraic variety. Let $x \in V$. Denote $m_x = \{f \in \mathbb{C}[V] | f(x) = 0\}$ a maximal ideal of $\mathbb{C}[V]$. Then $\dim(m_x/m_x^2) = \dim(T_p(X))$

5.3 Smooth Points of a Variety

We continue to restrict our attention to affine varieties.

Definition 5.3.1: Smooth and Singular Points

Let X be an affine variety. A point $p \in X$ is smooth if $\dim(T_p(X)) = \dim(X)$. Otherwise, $p \in X$ is singular.

Note: Some authors (IR Shafarevich) define singularity by whether the tangent space at a point has dimension higher than the minimum of all the tangent spaces. This makes sense: we can show that the set of all singularities is a closed set.

Proposition 5.3.2

A point $p \in X = V(f_1, \dots, f_m) \subseteq \mathbb{C}^n$ of an affine algebraic variety with dimension d is singular if and only the Jacobian

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_p \end{pmatrix}$$

has rank $n - d$.

Proposition 5.3.3

Let X be an affine variety. Let $p \in X$. Then X is smooth at p if and only if the local ring $\mathcal{O}_{X,p}$ is regular.

We can now motivate the definition of a smooth point using the purely algebraic characterization.

Definition 5.3.4: Smooth and Singular Points

Let X be a variety. X is smooth at a point $p \in X$ if the local ring $\mathcal{O}_{X,p}$ is a regular local ring, otherwise it is singular. X is smooth if every point of X is smooth.

Theorem 5.3.5

Let X be a variety. Then the set of singular points of X is a proper closed subset of X.

6 Birational Geometry

6.1 Birational Morphisms

Definition 6.1.1: Projective Morphism

A morphism of varieties $\pi : X \to V$ is called a projective morphism if X is a closed subvariety of a product variety, meaning that $X \subset V \times \mathbb{P}^n$ and π is the restriction of the projection onto the first coordinate.

Note that this is not the same as morphisms of projective varieties.

Definition 6.1.2: Birational Morphism

A morphism $\pi : X \to V$ of quasiprojective varieties is called a birational morphism if its restriction to some dense open set $U \subset X$ is an isomorphism onto some dense open subset $U' \subset V$.

6.2 Birational Maps

While morphisms are meant to be defined entirely for the variety, rational maps of varieties simply rely on a definition on open subsets of the variety, which makes it more versatile.

Lemma 6.2.1

Open subsets of a variety is dense.

Lemma 6.2.2

Let X, Y be varieties. Let ϕ, ψ be two morphisms from $X \to Y$. Suppose that there is a nonempty open subset $U \subseteq X$ such that $\phi|_U = \psi|_U$. Then $\phi = \psi$.

Definition 6.2.3: Rational Maps

Let X, Y he varieties. A rational map $\phi : X \to Y$ is an equivalence class of pairs $\langle U, \phi_U \rangle$, where U is a nonempty open subset of X, and $\phi|_U$ is a morphism of U to Y.

We say that $\langle U, \phi|_U \rangle$ and $\langle V, \phi|_V \rangle$ are equivalent if $\phi|_U$ and $\phi|_V$ agree on $U \cap V$.

The rational map ϕ is dominant if for some (and hence every) pair $\langle U, \phi|_U \rangle$, the image of $\phi|_U$ is dense in Y.

Definition 6.2.4: Birational Maps

A birational map $\phi : X \to Y$ is a rational map which has an inverse. In this case, we say that X and Y are birationally equivalent.

Varieties can form a category where morphisms are simply dominant rational maps. Isomorphisms in the category are birational maps.

6.3 Categorical Equivalence with Finitely Generated Field Extensions

Proposition 6.3.1

Let $\phi : X \to Y$ be a dominant rational map represented by $\langle U, \phi_U \rangle$. Let $f \in \mathbb{C}[Y]$ be a rational function represented by $\langle V, f \rangle$ where V is an open set in Y and f regular function on V. Then $f \circ \phi_U$ is a homomorphism of $\mathbb{C}$-algebras from $\mathbb{C}[Y]$ to $\mathbb{C}[X]$.

Proof. Notice that since $\phi_U(U)$ is dense in Y, $\phi_U^{-1}(V)$ is a nonempty open subset of X. Thus $f \circ \phi_U$ is a regular function on $\phi_U^{-1}(V)$. Thus $f \circ \phi_U$ is rational function on X. This means that $f \circ \phi_U \in \mathbb{C}[X]$.

In particular, the map taking f to $f \circ \phi_U$ is a $\mathbb{C}$-algebra homomorphism. □

Theorem 6.3.2

Let X and Y be two varieties. The above construction gives a bijection between the set of dominant rational maps from $X \to Y$ and the set of $\mathbb{C}$-algebra homomorphisms from $\mathbb{C}[Y]$ to $\mathbb{C}[X]$.

In other words, this correspondence is a contravariant functor from the category of varieties and the category of finitely generated field extensions of $\mathbb{C}$.

Corollary 6.3.3

Let X, Y be two varieties. The the following conditions are equivalent.

- X and Y are birationally equivalent
- There exists open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V
- $K(X)$ and $K(Y)$ are isomorphic $\mathbb{C}$-algebras

6.4 Blowing Ups

Definition 6.4.1: Blowing Up at $\mathbb{A}^n$

Define the blowing up of $\mathbb{A}^n$ at the point 0 to be the closed subset X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined by the equations $\{x_i y_j = x_j y_i | 0 \leq i, j \leq n\}$. Restricting the projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$ to the first factor gives a natural morphism $\phi : X \to \mathbb{A}^n$.

Theorem 6.4.2

The following are true with regards to blowing up at $\mathbb{A}^n$.

- X is a quasiprojective variety
- ϕ is an isomorphism for the sets $X \setminus \phi^{-1}(0)$ and $\mathbb{A}^n \setminus \{0\}$
- $\phi^{-1}(0) \cong \mathbb{P}^{n-1}$

Definition 6.4.3: Blowing Up at a Point

Let Y be a closed subvariety of $\mathbb{A}^n$ passing through 0. Define the blowing up of Y at 0 to be the the closure of $Z = \phi^{-1}(Y \setminus \{0\})$, where $\phi : X \to \mathbb{A}^n$ is obtained from the above blowing up at $\mathbb{A}^n$. Also denote $\phi : \overline{Z} \to Z$ the morphism obtained by further restricting ϕ to $\overline{Z}$.

To blow up any point other than 0, perform a linear change in coordinates sending P to 0.

Definition 6.4.4: Blowup along an Ideal

Let $F_1, \dots, F_r$ be functions in the coordinate ring $\mathbb{C}[x]$ of an affine algebraic variety X, and let I be the ideal they generate. Assume that I is a proper nonzero ideal of $\mathbb{C}[x]$. The blowup of the variety X along the ideal I is the graph B of the rational map $F : X \to \mathbb{P}^{r-1}$ defined by

$$F(x) = [F_1(x) : \cdots : F_r(x)]$$

amd the natural projection $\pi : X \times \mathbb{P}^{r-1} \to X$.

7 Theory of Divisors

7.1 Divisors of a Variety

Definition 7.1.1: Divisors of a Variety

Let X be a variety. Let $C_1, \dots, C_r$ be irreducible closed subvarieties of X of codimension 1. A divisor of X is of the form

$$D = \sum_{i=1}^{r} k_i C_i$$

for $k_i \in \mathbb{Z}$. k_i is said to be the multiplicity of C_i.

If $k_i = 0$ for all i then we write $D = 0$. If $k_i \geq 0$ for all i then D is said to be effective and we write $D > 0$. An irreducible codimension 1 subvariety C_i taken with multiplicity 1 is called a prime divisor.

Define the free group of all divisors of X by $\text{Div}(X)$.

Definition 7.1.2: Divisor of a Function

Let X be a variety such that the set of singular points of X has codimension ≥ 2. Let $f \in K(X)$. Let C be a prime divisor of X.

Definition 7.1.3: Principal Divisors

Let X be a variety. A divisor of the form $D = \text{div}(f)$ for some $f \in K(X)$ is called a principal divisor.

Define the set of all principal divisors by $P(X)$.

Proposition 7.1.4

Let X be a variety. The set of all principal divisors $P(X)$ is a group.

Definition 7.1.5: Divisor Class Group

Let X be a variety. Define the divisor class group of X to be

$$\text{Cl}(X) = \frac{\text{Div}(X)}{P(X)}$$

We say that two divisors D_1 and D_2 are linearly equivalent if they lie in the same coset of $\text{Cl}(X)$, written as $D_1 \sim D_2$. In other words, $D_1 \sim D_2$ if and only if $D_1 - D_2 = \text{div}(f)$ for some $f \in K(X)$.

Definition 7.1.6: Degree of a Divisor

Proposition 7.1.7

Let X be a variety. Then D is a principal divisor if and only if $\deg(D) = 0$.

7.2 The Linear System of a Divisor

Definition 7.2.1: Associated Vector Space of a Divisor

Let X be a nonsingular variety. Define the associated vector space of a divisor D of X to be

$$\mathcal{L}(D) = \{f \in K(X) | \text{div}(f) + D \geq 0\} \cup \{0\}$$

Lemma 7.2.2

Let X be a nonsingular variety. Then $\mathcal{L}(D)$ is a vector space over the field k.

Definition 7.2.3: Dimension of the Associated Vector Space

Let X be a nonsingular variety. Denote $\ell(D)$ the dimension of $\mathcal{L}(D)$, which is also called the dimension of D.

Theorem 7.2.4

Linearly equivalent divisors have the same dimension.

7.3 Divisors on Curves

7.4 The Riemann-Roch Theorem on Curves

Theorem 7.4.1: Riemann-Roch Theorem on Curves

Let X be a nonsingular projective algebraic curve of genus g. For any divisor D on X, we have that

$$\ell(D) - \ell(K - D) = \deg(D) - g + 1$$

where K is the canonical divisor of X.

Lemma 7.4.2

Let X be a nonsingular projective algebraic curve of genus g. Let K be the canonical divisor of X. Then

$$\deg(K) = 2g - 2$$

8 Intersection Theory