Classifying Spaces

Labix

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Abstract

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Contents

1 The Category of Fiber Bundles

1.1 Fiber Bundles

Definition 1.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and $p : E \to B$ a trivial map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$ for all $b \in B$
- $p : E \to B$ is surjective
- For every $x \in B$, there is an open neighbourhood $U \subset B$ of x and a fiber preserving homomorphism $\Psi_U : p^{-1}(U) \to U \times F$ that is a homeomormorphism such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Psi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  where π is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

Lemma 1.1.2

Every vector bundle is a fiber bundle.

Proposition 1.1.3

Every fiber bundle is a Serre fibration.

We can provide a partial converse for the fact that every fiber bundle is a Serre fibration.

Proposition 1.1.4

Let $p : E \to B$ be a fiber bundle. If B is paracompact, then p is a (Hurewicz) fibration.

Definition 1.1.5: Map of Fiber Bundles

Let (F_1, E_1, B_1) and (F_2, E_2, B_2) be fiber bundles. A morphism of fiber bundles is a pair of basepoint preserving continuous maps $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ such that the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

Such a map of fibrations determine a continuous of the fibers $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$.

A map of fibrations $(\tilde{f}, f)$ is said to be an isomorphism if there is a map $(\tilde{g} : E_2 \to E_1, g : B_2 \to B_1)$ such that $\tilde{g}$ is the inverse of $\tilde{f}$ and g is the inverse of f.

Definition 1.1.6: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration $B \times F \to B$.

Definition 1.1.7: Sections

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map $s : B \to E$ such that $p \circ s = \text{id}_B$. Let $U \subset B$ be an open set. A local section of the fiber bundle on U is a map $s : U \to B$ such that $p \circ s = \text{id}_U$.

Definition 1.1.8: The Pullback Bundle

Let $p : E \to B$ be a fiber bundle with fiber F. Let $f : B' \to B$ be a continuous function. Define the pullback of p by f to be the space

$$f^*(E) = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$$

1.2 G-Bundles and the Structure Groups

Notice that for non empty intersections $U_i \cap U_j$ for U_i, U_j open sets in B, there is a well defined homeomorphism

$$\varphi_j \circ \varphi_i^{-1} : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

This is reminiscent of properties of an atlas on M.

Definition 1.2.1: G-Atlas

Let (F, E, B) be a fiber bundle. Let G be topological group with a continuous faithful action on F. A G-atlas on (F, E, B) is a set of local trivalization charts $\{(U_k, \varphi_k) \mid k \in I\}$ such that the following are true.

- For (U_k, φ_k) a chart, define $\varphi_{i,x} : F \to F$ by $f \mapsto \varphi_i(x, f)$. Then the homeomorphism
$$\varphi_{j,x} \circ \varphi_{i,x}^{-1} : F \to F$$
for $x \in U_i \cap U_j \neq \emptyset$ is an element of G.
- For $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to G$ defined by
$$g_{ij}(x) = \varphi_{j,x} \circ \varphi_{i,x}^{-1}$$
is continuous.

If (F, E, B) is a fiber bundle with $F = \mathbb{R}$, then it is often seen that $G = GL(n, \mathbb{R})$. Similarly, if $F = \mathbb{C}$ then the structure group is $G = GL(n, \mathbb{C})$.

Definition 1.2.2: Equivalent G-Atlas

Two G-atlases on a fiber bundle (F, E, B) is said to be equivalent if their union is a G-atlas.

Definition 1.2.3: G-Bundle

Let G be a topological group. A G-bundle is a fiber bundle (F, E, B) together with an equivalence class of G-atlas. In this case, G is said to be the structure group of the fiber bundle.

The structure group and the trivialization charts completely determine the isomorphism type of the fiber bundle.

Definition 1.2.4: Morphisms of G-Bundles

Let G be a topological group. A morphism of G-bundles is a morphism of fiber bundles $(\tilde{h}, h) : (F, E_1, B_1) \to (F, E_2, B_2)$ where the two are G-bundles, such that the following are true.

- Let U_i be open in B_1 and V_j be open in B_2. Let $x \in U_u \cap h^{-1}(V_j)$. Let $\widetilde{h_{(E_1)_x}} : (E_1)_x \to (E_2)_{f(x)}$ be the map induced by $\tilde{h} : E_1 \to E_2$. Then the map
$$\varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1} : F \to F$$
is an element of G.
- The map $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ defined by
$$\widetilde{g_{ij}}(x) = \varphi_{j,x} \circ \widetilde{h_{(E_1)_x}} \circ \varphi_{i,x}^{-1}$$
is continuous.

It is easy to see that the mapping transformations $\widetilde{g_{ij}}$ satisfy the following two relations:

- $\widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap U_j \cap h^{-1}(V_k)$
- $g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x)$ for all $x \in U_i \cap h^{-1}(V_j \cap V_k)$

g'_{jk} here refers to the transition charts in (F, E_2, B_2).

Just as the structure groups and trivialization charts determine the isomorphism type of a fiber bundle, the $\widetilde{g_{ij}}$ and a map of base space $h : B_1 \to B_2$ completes determines a morphism of G-bundle.

Lemma 1.2.5

Let (F, E_1, B_1) and (F, E_2, B_2) be two G-bundles for a topological group G with the same fiber F. Suppose that we have the following.

- A map $h : B_1 \to B_2$ of base space
- $\widetilde{g_{ij}} : U_i \cap h^{-1}(V_j) \to G$ a set of continuous maps such that
$$\begin{aligned} \widetilde{g_{jk}}(x) \cdot g_{ij}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap U_j \cap h^{-1}(V_k) \\ g'_{jk}(h(x)) \cdot \widetilde{g_{ij}}(x) = \widetilde{g_{ik}}(x) \quad &\text{for all} \quad x \in U_i \cap h^{-1}(V_j \cap V_k) \end{aligned}$$

Then there exists a unique G-bundle morphism having h as the map of base space and having $\{\widetilde{g_{ij}} \mid i, j \in I\}$ as its mapping transformations.

1.3 Principal G-Bundles

Definition 1.3.1: Principal Bundles

Let G be a topological group. A principal G-bundle is a G-bundle (F, E, B) together with a continuous group action G on E such that the following are true.

- The action of G preserves fibers. This means that $g \cdot x \in E_b$ if $x \in E_b$. (This also means that G is a group action on each fiber)
- The action of G on each fiber is free and transitive
- For each $x \in E_b$, the map $G \to E_b$ defined by $g \mapsto g \cdot x$ is homeomorphism.
- Local triviality condition: If $\Psi_U : p^{-1}(U) \to U \times F$ are the local triviality maps, then each Ψ_U are G-equivariant maps.

Note that since G is homeomorphic to each fiber E_b of the total space, we can think of the action of G on the fiber simply becomes left multiplication.

For those who know what homogenous spaces are, principal bundles are G-bundles such that F is a principal homogenous space for the left action of G itself.

Conversely, given a continuous group action on a space, we can ask in what conditions will the space be a principal bundle over the orbit space.

Proposition 1.3.2

Let E be a space with a free G action. Let $p : E \to E/G$ be the projection map to the orbit space. If for all $x \in E/G$, there is a neighbourhood U of x and a continuous map $s : U \to E$ such that $p \circ s = \text{id}_U$, then $(G, E, E/G)$ is a principal G-bundle.

This proposition essentially means that if for each point in E/G, there is a local section, then it is sufficient for E to be a principal G bundle over E/G.

Theorem 1.3.3

A principal G-bundle is trivial if and only if it admits a global section.

This is entirely untrue for general bundles. For examples, the zero section of a fiber bundle is a global section.

1.4 Classifying Space

Theorem 1.4.1

Let X, Y be spaces and let $f, g : X \to Y$ be homotopic maps. If $p : E \to B$ is a fiber bundle, then there is an isomorphism

$$f^*(E) \cong g^*(E)$$

This allows the principal bundles functor, defined below, to be well defined in homotopy classes of maps.

Definition 1.4.2: Principal Bundle Functor

Let G be a topological group and X a space. Define a contravariant functor $\text{Prin}_G : \textbf{hTop} \to \textbf{Set}$ as follows.

- For X a topological space, $\text{Prin}_G(X)$ is the set of isomorphism classes of principal G-bundles over X.
- If $[f : X \to Y]$ is a homotopy class of continuous maps,

$\text{Prin}_G([f]) : \text{Prin}_G(Y) \to \text{Prin}_G(X)$ is defined as follows. If $[p : E \to Y]$ is an isomorphism class of principal G-bundles over Y, then it is sent to $[f^*(E)]$ the isomorphism class of the pullback of p.

Theorem 1.4.3

Let G be ta topological group. Then the principal bundle functor is representable. Explicitly, this means that there exists a principal G-bundle $EG \to BG$ together with a natural isomorphism

$$\psi : [X, BG] \to \text{Prin}_G(X)$$

This natural isomorphism is defined by $f \mapsto [f^*(EG)]$.

Definition 1.4.4: Universal G-Bundles

Let G be a topological group. A principal G-bundle (F, E, B) is said to be universal if it represents the principal bundle functor.

Theorem 1.4.5

Let (F, E, B) be a principal G-bundle. If E is contractible then (F, E, B) is a universal G-bundle.

A surprising thing is that BG is not determined by its isomorphism type but instead by the weaker condition of its homotopy type.

Theorem 1.4.6

Let (F, E_1, B_1) and (F, E_2, B_2) be universal principal G-bundles. Then there exists a bundle map

$$\begin{tikzcd} E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"'] & B_2 \end{tikzcd}$$

such that f is a homotopy equivalence. In particular, this means that any two universal principal G-bundles are homotopy equivalent.

Definition 1.4.7: Classifying Space

Let G be a topological group. The classifying space BG of G is the homotopy type of the universal principal G-bundle. Denote the total space of BG by EG. For a principal G-bundle $f : Y \to X \in \text{Prin}_G(X)$, define the classifying map to be the associated map $X \to BG$ given in 1.5.3.

TBA: Functoriality of $B : \mathbf{Grp} \to \mathbf{Top}$.

2 Vector Bundles as Principal Bundles

2.1 The Frame Bundle

Definition 2.1.1: Frame Bundle

Theorem 2.1.2

Let X be a space. Then there is a natural bijection

$$\phi : \text{Prin}_{\text{GL}(n,\mathbb{R})}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{R}}(X)$$

given by mapping $p : E \to B$ to the frame bundle $F(E)$. Similarly, there is a natural bijection

$$\phi : \text{Prin}_{\text{GL}(n,\mathbb{C})}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{C}}(X)$$

Theorem 2.1.3

Let $n \in \mathbb{N}$, then there is an isomorphism in the classifying spaces

$$B\text{GL}(n, \mathbb{R}) \cong BO(n) \cong \text{GL}_n(\mathbb{R}^\infty)$$

Theorem 2.1.4

Let $n \in \mathbb{N}$, then there is an isomorphism in the classifying spaces

$$B\text{GL}(n, \mathbb{C}) \cong BU(n)$$

Theorem 2.1.5

Let X be a paracompact space. Then there is a natural bijection

$$\phi : \text{Prin}_{O(n)}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{R}}(X)$$

given by mapping $p : E \to B$ to the frame bundle $F(E)$. Similarly, there is a natural bijection

$$\phi : \text{Prin}_{U(n)}(X) \xrightarrow{\cong} \text{Vect}_n^{\mathbb{C}}(X)$$

2.2 The Tautological Bundle

2.3 The Thom Isomorphism

Definition 2.3.1: Unit Sphere and Unit Disc Bundle

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$. Let $\langle -, - \rangle : E \times E \to \mathbb{R}$ be a smoothly varying inner product on E. Define the disc bundle to be

$$D(E) = \{e \in E \mid \langle e, e \rangle \leq 1\}$$

together with the map $p|_{D(E)} : D(E) \to B$. Define the sphere bundle to be

$$S(E) = \{e \in E \mid \langle e, e \rangle = 1\}$$

together with the map $p|_{S(E)} : S(E) \to B$.

Definition 2.3.2: Thom Space

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$ such that B is paracompact. Define the Thom space of E to be

$$\frac{D(E)}{S(E)}$$

The base point is taken as the equivalent class $S(E)$ if needed.

Theorem 2.3.3: The Thom Isomorphism

Let $p : E \to B$ be an n-dimensional vector bundle over $\mathbb{R}$. Let E_0 denote the zero section of E. Then there exists a unique $u \in H^n(E, E \setminus E_0; \mathbb{Z}/2\mathbb{Z})$ such that

$$u|_{(F_b, F_b \setminus \{0\})} \in H^n(F_b, F_b \setminus \{0\}; \mathbb{Z}/2\mathbb{Z})$$

is non-zero for all $b \in B$. Moreover, there is an isomorphism

$$\Phi : H^k(E; \mathbb{Z}/2\mathbb{Z}) \to \widetilde{H}^{k+n}(E, E \setminus E_0; \mathbb{Z}/2\mathbb{Z})$$

given by $y \mapsto y \smile u$ for all $k \in \mathbb{Z}$.

Ref: Milnor

2.4 Orientation of a Bundle

Definition 2.4.1: Orientation of a Vector Space

Let V be a finite dimensional vector space over F. An orientation on V is an equivalence class of bases, where we say that two ordered bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are equivalent if the matrix defined by the equations

$$w_i = \sum_{k=0}^{n} a_k v_k$$

has positive determinant.

Lemma 2.4.2

Let V be a finite dimensional vector space. Then there are only two possible orientations on V.

Definition 2.4.3

Let $p : E \to B$ be a vector bundle with fiber F. An orientation on E is an assignment of an orientation to each fiber of E such that the following local compatibility condition is satisfied.

For every $b \in B$, there exists a local coordinate system (U, φ) of b and $\varphi : U \times \mathbb{R}^n \to p^{-1}(U)$ such that for all $x \in U$, the homomorphism $\varphi(b, -) : \mathbb{R}^n \to F$ is orientation preserving.

Theorem 2.4.4

Let $p : E \to B$ be a vector bundle with fiber F. An orientation on E is equivalent to the following data. To each $b \in B$ there is assignment

$$u_b \in H^n(F_b, F_b \setminus \{0\}; \mathbb{Z})$$

called the orientation class of F_b, such that for every $b \in B$, there exists a neighbourhood U of b and a cohomology class

$$u \in H^n(p^{-1}(U), p^{-1}(U) \setminus 0;)$$

where 0 is the zero section such that for every $x \in U$,

$$u|_{(F_x, F_x \setminus \{0\})} \in H^n(F_x, F_x \setminus \{0\} \mathbb{Z})$$

is equal to u_b.

Theorem 2.4.5: The Thom Isomorphism

Let $p : E \to B$ be an orientable n-dimensional vector bundle over $\mathbb{R}$. Let R be a ring. Let E_0 denote the zero section of E. Then there exists a unique $u \in H^n(E, E \setminus E_0; R)$ such that

$$u|_{(F_b, F_b \setminus \{0\})} \in H^n(F_b, F_b \setminus \{0\}; R)$$

gives precisely the orientation class on F_b for all $b \in B$. Moreover, there is an isomorphism

$$\Phi : H^k(E; R) \to \widetilde{H}^{k+n}(E, E \setminus E_0; R)$$

given by $y \mapsto y \smile u$ for all $k \in \mathbb{Z}$.

3 Characteristic Classes

3.1 Characteristic Classes as a Ring

Definition 3.1.1: Characteristic Classes

Let G be a topological group and X a space. Denote $\text{Prin}_G(X)$ the isomorphism classes of principal G-bundles over X. Let $H^*(-)$ be a cohomology functor. A characteristic class for G is a natural transformation c from $\text{Prin}_G(-)$ to $H^*(-)$.

Explicitly, if $p : E \to X$ is a principal G-bundle, then c assigns p to the collection of cohomology groups $c(p) \in H^*(X)$.

Here cohomology can be taken for example singular cohomology with coefficients in a fixed group.

Lemma 3.1.2

Let G be a topological group. Let c be a characteristic class for G. If e is the trivial G-bundle, then $c(e) = 0$.

Definition 3.1.3: Ring of Characteristic Classes

Let G be a topological group. Let R be a commutative ring. Define $\text{Char}_G(R)$ to be the set of all characteristic classes for principal G-bundles that take values in $H^*(-; R)$.

Proposition 3.1.4

Let G be a topological group. Let R be a commutative ring. Then $\text{Char}_G(R)$ is a ring with unit the constant characteristic class.

Theorem 3.1.5

Let G be a topological group and let R be a commutative ring. Then there is an isomorphism
$$\text{Char}_G(R) \cong H^*(BG; R)$$

3.2 The Stiefel-Whiteny Class

Definition 3.2.1: The Stiefel-Whitney Class

Consider the group $O(n)$. We say that a characteristic class $w : \text{Prin}_{O(n)}(-) \to H^*(-, \mathbb{Z}/2\mathbb{Z})$ for $O(n)$ is a Stiefel-Whitney Class if the following are satisfied.

1. Rank: If E is a principal $O(n)$-bundle, then $w_0(E) = 1$ and $w_i(E) = 0$ for $i > \text{rank}(E)$.
2. Naturality: Let $p : E \to X$ be a principal $O(n)$-bundle and let $f : Y \to X$ be a map. Then
$$w_i(f^*(E)) = f^*(w_i(E))$$
3. Whitney Product Formula: If E_1, E_2 are principal $O(n)$-bundles, then
$$w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \smile w_{k-i}(E_2)$$
4. Normalization: If γ is the tautological line bundle over $\mathbb{P}^1(\mathbb{R})$, then $w_1(\gamma)$ is non-zero.

TBA: Existence and uniqueness

Proposition 3.2.2

The following are true regarding the Stiefel-Whitney class.

- If $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ are isomorphic principal $O(n)$-bundles, then $w(E_1) = w(E_2)$
- If $e = B \otimes \mathbb{R}^n$ is the trivial bundle, then $w(e \oplus E) = w(E)$ for any principal $O(n)$-bundle E.
-

Theorem 3.2.3

Let $n \in \mathbb{N}$, then the ring of characteristic classes of $O(n)$ is isomorphic to

$$\text{Char}_{O(n)}(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[c_1, \dots, c_n]$$

a polynomial ring in n variables for $w_i \in H^i(BO(n), \mathbb{Z}/2\mathbb{Z})$.

3.3 The Chern Class

Theorem 3.3.1

Let E be an n-dimensional complex vector bundle over X. Then $c_1(E) = 0$ if and only if E has an $SU(n)$-structure.

TBA: First chern class is complete invariant of complex line bundles. First Stiefel-Whitney class is a complete invariant of real line bundle.

Theorem 3.3.2

Let $n \in \mathbb{N}$, then the ring of characteristic classes of $U(n)$ is isomorphic to

$$\text{Char}_{U(n)}(\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$$

a polynomial ring in n variables for $c_i \in H^{2i}(BU(n), \mathbb{Z})$.

3.4 The Euler Class

Definition 3.4.1: The Euler Class

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. Let $E_0 \subseteq E$ denote the zero section. Consider the inclusion $B \hookrightarrow E$ as E_0. Let $u \in H^n(E, E \setminus E_0; \mathbb{Z})$ be the orientation class. Define the euler class of E

$$e(E) \in H^n(B; \mathbb{Z})$$

to be the image of u under the compositions

$$H^n(E, E \setminus E_0; \mathbb{Z}) \longrightarrow H^n(E, \mathbb{Z}) \longrightarrow H^n(B; \mathbb{Z})$$

that is induced by the sequence of inclusions $(B, \emptyset) \hookrightarrow (E, \emptyset) \hookrightarrow (E, E \setminus E_0)$.

Proposition 3.4.2

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. Then the following are true regarding the Euler class.

- If $f : C \to B$ is a map, then $e(f^*(E)) = f^*(e(E))$
- If the orientation of E is reversed, then $e(E)$ changes sign.
- If F is another orientable vector bundle, then $e(E \oplus F) = e(E) \smile e(F)$.

Proposition 3.4.3: L

t $p : E \to B$ be an orientable vector bundle over $\mathbb{R}$. If the dimension of the bundle is odd, then $2e(E) = 0$.

Proposition 3.4.4

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. The natural homomorphism

$$H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}/2\mathbb{Z})$$

sends the Euler class $e(E)$ to the top Stiefel-Whitney class $w_n(E)$.

Proposition 3.4.5

Let $p : E \to B$ be an n-dimensional orientable vector bundle over $\mathbb{R}$. If E possess a nowhere 0 section, then $e(E) = 0$.

3.5 The Pontrjagin Class

4 Obstruction Theory