Algebraic K Theory

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Abstract

Contents

1 The K$_0$-Group

1.1 K$_0$ of a Symmetric Monoidal Category

Definition 1.1.1: The K$_0$-Group of a Symmetric Monoidal Category

Let $(\mathcal{C}, I, \oplus)$ be a symmetric monoidal category. Let $\mathcal{C}^{\text{iso}}$ be the category consisting of isomorphism classes of objects, which is also an abelian monoid under the operation $\oplus$. Define the K_0 group of $\mathcal{C}$ by the Grothendieck completion

$$K_0(\mathcal{C}, I, \oplus) = \left(\mathcal{C}^{\text{iso}}\right)^{-1} \mathcal{C}^{\text{iso}}$$

1.2 K$_0$ of a Ring

Definition 1.2.1: The Category of Finitely Generated Projective Modules over a Ring

Let R be a ring. Define the category $\mathbf{FGP}(R)$ of projective modules over R as follows.
- The objects are the isomorphism classes of finitely generated projective modules $[M]$ over R
- For two isomorphism classes of projective modules $[M], [N]$ over R, a morphism $[M] \to [N]$ is just an R-module homomorphism.
- Composition is given by the composition of functions.

Lemma 1.2.2

Let R be a ring. Then the category $\mathbf{FGP}(R)$ is a symmetric monoidal category with the following data.
- The binary operator $\oplus : P(R) \times P(R) \to P(R)$ is given by

$$[M] + [N] = [M \oplus N]$$

  which is the direct sum.
- The unital object is the isomorphism class $[R]$ of the R-module R.

Definition 1.2.3: The K$_0$-Group of a Ring

Let R be a ring. Define the K_0-group of R by the Grothendieck completion of the abelain monoid:

$$K_0(R) = P(R)^{-1}P(R) = K_0(P(R), R, \oplus)$$

Definition 1.2.4: The K$_0$ Functor

Define the K_0-functor

$$K_0 : \mathbf{Ring} \to \mathbf{Grp}$$

to consist of the following data.
- For each ring R, define $K_0(R)$ to be the K_0-group of R
- For each ring homomorphism $f : R \to S$, define $K_0(f) : K_0(R) \to K_0(S)$ by the formula

$$[P] \mapsto [S \otimes_R P]$$

Theorem 1.2.5: Universal Property of the K$_0$-Group

Let R be a ring.

Recall that a ring R is a principal ideal domain if every ideal of R is generated by one element. By the

structure theorem of finitely generated R-modules over PID, we can immediately conclude that there is an isomorphism
$$\mathbb{Z} \cong K_0(R)$$
given by $n \mapsto [R^n]$.

Definition 1.2.6: Reduced K$_0$-Group

Let R be a ring. Define the reduced K_0-group $\widetilde{K}_0(R)$ of R to be the quotient
$$\widetilde{K}_0(R) = \frac{K_0(R)}{\{[R^m] - [R^n] \mid n, m \in \mathbb{N}\}}$$

Lemma 1.2.7

Let R be a ring. Then the unique ring homomorphism $f : \mathbb{Z} \to R$ induces an isomorphism
$$\widetilde{K}_0(R) \cong \frac{K_0(R)}{\text{im}(K_0(f))}$$

Recall that a stably free module is an R-module M such that there exists a finitely generated free R-module T such that $M \oplus T$ is free. Now $[P] \in \widetilde{K}_0(R)$ is trivial if and only if P is stably free and finitely generated. Thus the reduced K_0 of a ring measures how far away a finitely generated R-module from also being stably free.

Theorem 1.2.8

Let R be a ring. Let $n \geq 1$. Then there is an isomorphism
$$\mu_R : K_0(R) \xrightarrow{\cong} K_0(M_n(R))$$
given by $[P] \mapsto [R^n \oplus_R P]$, where R^n here is considered as an $(M_n(R), R)$-bimodule. Moreover, the isomorphism is natural in the following sense. If $f : R \to S$ is a ring homomorphism, then the following diagram is commutative:
$$\begin{tikzcd} K_0(R) \arrow[r, "K_0(f)"] \arrow[d, "\mu_R"'] & K_0(S) \arrow[d, "\mu_S"] \\ K_0(M_n(R)) \arrow[r, "K_0(M_n(f))"'] & K_0(M_n(S)) \end{tikzcd}$$

Proposition 1.2.9

Let R, S be rings. Denote $p_1 : R \times S \to R$ and $p_2 : R \times S \to S$ the projection maps. Then the projection maps induce an isomorphism
$$K_0(p_1) \times K_0(p_2) : K_0(R \times S) \xrightarrow{\cong} K_0(R) \times K_0(S)$$

Proposition 1.2.10

Let k be a field. Let V be a vector space over k with countable basis. Then
$$K_0(\text{End}_k(V)) \cong \{1\}$$

Lemma 1.2.11

Let G be a group. Let R be a commutative integral domain with quotient field F. Then there is an isomorphism
$$K_0(R[G]) \cong \widetilde{K}_0(R[G]) \oplus \mathbb{Z}$$
given by $[P] \mapsto ([P], \dim_F(F \otimes_{R[G]} P))$

Conjecture 1.2.12: Farrell-Jones Conjecture

Let G be a torsion-free group. Let R be a regular ring. Then the map $\{1\} \hookrightarrow G$ induces an isomorphism
$$K_0(R) \cong K_0(R[G])$$

1.3 K$_0$ of an Abelian Category

1.4 K$_0$ of a Waldhaussen Category

2 The K_1-Group

2.1 K_1 of a Ring

Definition 2.1.1: The K_1-Group of a Ring

Let R be a ring. Define the K_1-group of R to be the group

$$K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]}$$

Proposition 2.1.2

Let R and S be two rings. Then there is an isomorphism

$$K_1(R \times S) \cong K_1(R) \oplus K_1(S)$$

Proposition 2.1.3

Let R be a ring. Then there is an isomorphism

$$K_1(R) \cong K_1(M_n(R))$$

for any $n \in \mathbb{N}$.

2.2 The Fundamental Theorems for K_1 and K_0

3 The Negative K-Groups

4 The K_2-Group

4.1 The Steinberg Group

Definition 4.1.1: The nth Steinberg Group

Let R be a ring. For $n \geq 3$, define the nth Steinberg group by

$$\text{St}_n(R) = \frac{\langle x_{ij}(r) \text{ for } r \in R, 1 \leq i, j \leq n \rangle}{R}$$

where R is the relation generated by

- For $r, s \in R$, $x_{ij}(r)x_{ij}(s) = x_{ij}(rs)$ for $1 \leq i, j \leq n$
- For $r, s \in R$,

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & \text{if } j \neq k \text{ and } i \neq l \\ x_{il}(rs) & \text{if } j = k \text{ and } i \neq l \\ x_{kj}(-rs) & \text{if } j \neq k \text{ and } i = l \end{cases}$$

Lemma 4.1.2

Let R be a ring. For any $n \geq 3$, the nth Steinberg group $\text{St}_n(R)$ of R includes into the $(n+1)$th Steinberg group $\text{St}_{n+1}(R)$.

Proposition 4.1.3

Let R be a ring. Let $n \geq 3$. Then the universal property of free groups with relations induce a canonical group surjection

$$\phi_n : \text{St}_n(R) \to [GL(R), GL(R)]$$

that sends $x_{ij}(r)$ to $e_{ij}(r)$.

Definition 4.1.4: The Steinberg Group of a Ring

Let R be a ring. Define the Steinberg group of R by the direct limit

$$\text{St}(R) = \varinjlim_{n \in \mathbb{N} \setminus \{0,1,2\}} \text{St}_n(R)$$

Proposition 4.1.5

Let R be a ring. The universal property of the direct limit induces a canonical group surjection

$$\phi : \text{St}(R) \to [GL(R), GL(R)]$$

4.2 K_2 of a Ring

Definition 4.2.1: The K_2-Group of a Ring

Let R be a ring. Define the K_2-group of R to be the kernel

$$K_2(R) = \ker\left(\phi : \text{St}(R) \to [GL(R), GL(R)]\right)$$

Lemma 4.2.2

Let R be a ring. Then there is an exact sequence of groups

$$0 \longrightarrow K_2(R) \longrightarrow \mathsf{St}(R) \longrightarrow [GL(R), GL(R)] \longrightarrow K_1(R) \longrightarrow 0$$

Theorem 4.2.3: (Stein)

For any ring R, the K_2-group $K_2(R)$ is an abelian group. Moreover, we have

$$Z(\mathsf{St}(R)) = K_2(R)$$

5 The K$_n$-Group

5.1 Universal Definition

Definition 5.1.1: The Plus Construction

Let R be a ring. Define $BGL(R)^+$ to be any CW complex that has a distinguished map $BGL(R) \to BGL(R)^+$ such that the following are true.

- There is an isomorphism $\pi_1(BGL(R)^+) \cong K_1(R)$ given by the induced map $\pi_1(BGL(R)) \to \pi_1(BGL(R)^+)$, which is required to be surjective with kernel $[GL(R), GL(R)]$
- For each $n \in \mathbb{N}$, there are isomorphisms

$$H_n(BGL(R); M) \cong H_n(BGL(R)^+; M)$$

  for any R-module M.

Intuitively, $BGL(R)^+$ is a modification of the classifying space of $GL(R)$ so that their homology remains the same while its fundamental group returns $K_1(R)$. The latter point is important because K_n will be defined as the nth homotopy group.

Definition 5.1.2: K$_n$ of a Ring

Let R be a ring. Define the nth K-group of R to be

$$K_n(R) = \pi_n(BGL(R)^+)$$

for $n \geq 1$.

Notice that $BGL(R)^+$ for a ring R is not defined uniquely. However, we can prove that any two such plus constructions are homotopy equivalent so that $K_n(R)$ is well defined.

In order to accommodate the 0th K-group, we make the following amendments.

Definition 5.1.3: K-Theory of a Ring

Let R be a ring. Define the K-theory of R by

$$K(R) = K_0(R) \times BGL(R)^+$$

so that $\pi_n(BGL(R)^+) = K_n(R)$ for all $n \geq 0$.

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6.1

Theorem 6.1.1: Serre-Swan Theorem I

Let M be a smooth manifold. Let E be a smooth vector bundle over M. Then the space of smooth sections $\Gamma(E)$ of E is finitely generated and projective over $C^\infty(M)$.

If M is connected, then the space of smooth section is one-to-one with the finitely generated and projective modules over $C^\infty(M)$.

Theorem 6.1.2

Let M be a smooth and connected manifold. Then the category of smooth vector bundles SVect(M) is equivalent to the category of finitely generated projective modules FinProj$_{C^\infty(M)}$Mod via the global section functor

$$\Gamma : \text{SVect}(M) \to \text{FinProj}_{C^\infty(M)}\text{Mod}$$

defined by $E \mapsto \Gamma(E)$

6.2