Sheaf Theory

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Abstract

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1 Sheaves

1.1 Basic Definition of Sheaves

As with how we equipped to each variety V its coordinate ring $k[V]$ which are functions on V, we want to equip to each spectrum some ring which are functions on them. It will not make sense that we can define functions on spectrums immediately.

Definition 1.1.1: Presheaves

Let $(X, \mathcal{T})$ be a topological space. A presheaf on X is consists of

- A function
$$\mathcal{F} : \mathcal{T} \to \text{Sets}$$
This means each open set U of X gets associated with a set, potentially with additional structures (groups / rings). Each individual element of $\mathcal{F}(U)$ is called a section. Each element of $\mathcal{F}(X)$ is instead called a global section
- For each inclusion of open sets $V \subseteq U$, there exists a restriction map $\text{res}_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ satisfying
  - $\text{res}_{U,U} : \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity
  - Whenever $W \subseteq V \subseteq U$, then $\text{res}_{W,V} \circ \text{res}_{V,U} = \text{res}_{W,U}$

In other words, let $\mathbf{Top}(X)$ be the category whose objects are the open subsets of X and morphisms are the inclusion maps. A presheaf of X is a contravariant functor from $\mathbf{Top}(X)$ to a set.

The reason that the map $\mathcal{F}(U) \to \mathcal{F}(V)$ is called a restriction is that we will soon see that elements of $\mathcal{F}(X)$ are actually functions over some ring or field.

Notation: we often use $\Gamma(U, \mathcal{F})$ to denote the set $\mathcal{F}(U)$ and $s|_V$ to denote $\text{res}_{UV}(s)$ for $s \in \mathcal{F}(U)$.

Definition 1.1.2: Sheaves

A sheaf is a presheaf satisfying two additional properties

- Identity: If $\{U_i | i \in I\}$ is an open cover of U and $\phi_1, \phi_2 \in \mathcal{F}(U)$ and $\phi_1|_{U_i} = \phi_2|_{U_i}$ for all i, then $\phi_1 = \phi_2$
- Gluing: If $\{U_i | i \in I\}$ is an open cover of U and $\phi_i \in \mathcal{F}(U_i)$ for all $i \in I$ such that $\phi_i|_{U_i \cap U_j}$ for all $i, j \in I$, then there is exists some $\phi \in \mathcal{F}(U)$ such that $\phi|_{U_i} = \phi_i$ for all $i \in I$.

We can define the category of sheaves on a topological space X where objects are all the sheaves on X and morphisms are all the morphisms between the sheaves. This will be seen formally later.

Definition 1.1.3: Stalks and Germs

Let $\mathcal{F}$ be a presheaf on a topological space $(X, \mathcal{T})$. Let $p \in X$. Define the stalk of $\mathcal{F}$ at p to be
$$\mathcal{F}_{X,p} = \{(U, s) | x \in U \subset X \text{ open}, s \in \mathcal{F}(U)\} / \sim$$
where we say that $(U_1, s_1) \sim (U_2, s_2)$ if there exists some $V \subseteq U_1 \cap U_2$ open such that $\text{res}_{V,U_1}(s_1) = \text{res}_{V,U_2}(s_2)$.

Equivalently, $\mathcal{F}_{X,p}$ is the colimit of the groups $\mathcal{F}(U)$ for all open sets U containing p.

Think of the definition of stalks as follows: Treat f and g to be sections in $\mathcal{F}(U_1)$ and $\mathcal{F}(U_2)$ where $V \subseteq U_1 \cap U_2$ is open and contains x. Then we treat f and g to be the same function in the stalk as long as they agree on some open set that contains x. Indeed, since we do not care about the entirety of the domain of f and g, and only care about what happens locally near x, it makes sense for us to treat them as a function when they appear to be the same locally.

Definition 1.1.4: Morphism of Presheaves

Let $\mathcal{F}, \mathcal{G}$ be presheaves on X. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of sets (groups, rings, etc) $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set U such that if $V \subset U$ is an inclusion, the following digram commutes, where ρ, ρ' are restriction maps.

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\phi(U)"] \arrow[d, "\mathrm{res}_{V,U}"] & \mathcal{G}(U) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

An isomorphism is just a morphism with an inverse.

In other words, morphism of presheaves is just a natural transformation between two contravariant functors $\mathcal{F}$ and $\mathcal{G}$.

Notice that the natural transformation ϕ here takes every open set U and maps it to a group homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$.

Proposition 1.1.5

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. Then ϕ is an isomorphism if and only if the induced map on the stalk $\phi_p : \mathcal{F}_{X,p} \to \mathcal{G}_{X,p}$ is an isomorphism for all $p \in X$.

Theorem 1.1.6

For every presheaf $\mathcal{F}$, there is a sheaf $\mathcal{F}^+$ and a morphism of sheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf G, and any morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, there is a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ and that $\phi = \psi \circ \theta$. Furthermore, the pair $(\mathcal{F}^+, \theta)$ is unique up to isomorphism. In other words, the following diagram commutes.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[dr, "\phi"'] & \mathcal{F}^+ \arrow[d, "\psi"] \\ & \mathcal{G} \end{tikzcd}$$

Definition 1.1.7: Sheafification

The above sheaf $\mathcal{F}^+$ defined by morphisms is called the sheafification of the presheaf $\mathcal{F}^+$.

1.2 Subsheaves of a Sheaf

Definition 1.2.1: Subsheaf

A subsheaf of a sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and that the restriction maps of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$.

It follows directly from the definition that for any point P, the stalk $\mathcal{F}'_P$ is a subgroup of $\mathcal{F}_P$.

Definition 1.2.2: Kernel of a Presheaves

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Define the presheaf kernel of ϕ to be the presheaf given by

$$U \to \ker(\phi(U))$$

Notice that the definitions here make sense because essentially $\phi(U)$ is a group (ring) homomorphism if the presheaf we are working with is a presheaf of groups or rings.

Proposition 1.2.3

The presheaf kernel of a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a subsheaf of $\mathcal{F}$.

1.3 Sheaves from a Basis

Theorem 1.3.1

Let X be a topological space. Let $\mathcal{B}$ be the basis of X. Suppose that $\mathcal{F}_0$ is a sheaf defined on the basis $\mathcal{B}$ of X. Then the natural extension to open sets U by

$$\mathcal{F}(U) = \left\{ (s_i)_i \in \prod_i \mathcal{F}_0(B_i) \middle| B_i \in \mathcal{B}, B_i \subseteq U, s_i|_{B_i \cap B_j} = s_j|_{B_i \cap B_j} \right\} = \varprojlim_{\substack{B \in \mathcal{B} \\ B \subseteq U}} \mathcal{F}(B)$$

defines a sheaf for X.

Proof. $\square$

This means that sheaves are uniquely determined by their values in the basis of X. We can simply define the sheaf on the basis elements and by this natural extension, a sheaf will be defined for all of X.

1.4 Image Sheaves

Definition 1.4.1: Direct Image Sheaf

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf on X. Define the direct image sheaf on Y as follows. For every open set $V \subseteq Y$, define

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

This means that $f_*\mathcal{F}$ is defined as follows:

$$X \xrightarrow{f} Y$$

$$\begin{tikzcd}
f^{-1}(V) \arrow[d, "\mathcal{F}"'] & V \arrow[l, "f^{-1}"'] \arrow[dl, "f_*\mathcal{F}"] \\
\mathcal{F}(f^{-1}(V))
\end{tikzcd}$$

Proposition 1.4.2

The direct image sheaf on Y is indeed a sheaf on Y.

Proof. The proof is direct since $\mathcal{F}$ is already a sheaf itself and we are only taking sparser open sets than open sets in X. $\square$

Definition 1.4.3: Inverse Image Sheaf

Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{G}$ be a presheaf on Y. Define the inverse image sheaf on X as follows. For every open set $U \subseteq X$, define

$$f^+\mathcal{G}(U) = \lim_{\substack{V \supset f(U) \\ V \subseteq Y \text{ open}}} \mathcal{G}(V)$$

The sheaffification of $f^+\mathcal{G}$, $f^{-1}\mathcal{G}$ is called the inverse image sheaf of $\mathcal{G}$ under f.

Note: The direct image sheaf and inverse image sheaf are adjoint functors. Goertz Wedhorn P.55.

1.5 Ringed Spaces

Definition 1.5.1: Ringed Space

A ringed space is a topological space X together with a sheaf of rings on X.

A locally ringed space is a ringed space X where all stalks are local rings.

Definition 1.5.2: Morphisms of Ringed Spaces

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of continuous map $f : X \to Y$ and a map $f^\# : \mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on Y.

If X and Y are locally ringed spaces, then a morphism of locally ringed spaces is a morphism of ringed spaces such that for each $p \in X$, the induced map of local rings

$$f_p^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$$

is a local homomorphism of local rings.

Definition 1.5.3: Open Embedding

Let $U \to Y$ be an isomorphism of U and an open subset of Y, together with an isomorphism ringed spaces $(U, \mathcal{O}|_U)$ and $(V, \mathcal{O}_Y|_V)$. Then this map of ringed spaces is called an open embedding or an open immersion of ringed spaces.

2 Coherent Sheaves

2.1 The Category of $\mathcal{O}_X$-Modules

Definition 2.1.1: Sheaf of $\mathcal{O}_X$-modules

Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on X such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

This means that the following diagram should commute:

$$\begin{tikzcd}
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] \arrow[d, "\text{res}_{U,V} \times \text{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\text{res}_{U,V}"] \\
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] & \mathcal{F}(V)
\end{tikzcd}$$

Denote the category of $\mathcal{O}_X$-modules by $\text{Mod}(\mathcal{O}_X)$.

Proposition 2.1.2

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Let $\varphi, \psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves. Then the map $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ defined by

$$(\varphi + \psi)(\mathcal{F}(U)(x) = \varphi(\mathcal{F}(U)(x) + \psi(\mathcal{F}(U))(x)$$

for $x \in \mathcal{F}(U)$ and each U is a bilinear map of sheaves.

Moreover, under this operation, the category $\text{Mod}(\mathcal{O}_X)$ is a pre-additive category.

Proposition 2.1.3

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Then the direct sum

$$\mathcal{F} \oplus \mathcal{G} = \mathcal{F} \times \mathcal{G}$$

is also a sheaf of $\mathcal{O}_X$-modules.

Moreover, under this operation, the category $\text{Mod}(\mathcal{O}_X)$ is an additive category.

Proposition 2.1.4

Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\text{Mod}(\mathcal{O}_X)$ is an abelian category.

Proposition 2.1.5

Denote i the trivial functor taking a sheaf to its presheaf. Then the functor i and the sheaffication functor $^+$ are adjoints. In other words,

$$\text{Hom}(i(\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, \mathcal{G}^+)$$

for a presheaf $\mathcal{G}$ and a sheaf $\mathcal{F}$.

Proposition 2.1.6

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Then the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ defined by

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = (\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U))^+$$

is also a sheaf of $\mathcal{O}_X$-modules.

2.2 Invertible Sheaves

Definition 2.2.1: Free Sheaf

An $\mathcal{O}_X$-module $\mathcal{F}$ is free if $\mathcal{F} \cong \mathcal{O}_X^{\oplus n}$.

It is locally free if X can be covered by open sets U for which $\mathcal{F}|_U \cong \mathcal{O}_X|_U^{\oplus n}$-module. In this case we say that the rank of $\mathcal{F}$ is n.

Lemma 2.2.2

If X is connected then the rank of a locally free sheaf on X is constant.

Definition 2.2.3: Invertible Sheaf

A locally free sheaf of rank 1 is called an invertible sheaf.

Theorem 2.2.4

Let $(X, \mathcal{O}_X)$ be a scheme. Then the following are equivalent characterization of a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$

- $\mathcal{F}$ is invertible
- There exists a sheaf G such that $F \otimes_{\mathcal{O}_X} G \cong \mathcal{O}_X$
- $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$

Theorem 2.2.5

The category of locally free sheaves on a space X is equivalent to the category of vector bundles over X.

2.3 Quasicoherent Sheaves

Definition 2.3.1: Quasicoherent Sheaves

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$ modules $\mathcal{F}$ is quasicoherent if X can be covered by open affine subsets $U_i = \operatorname{Spec}(A_i)$ such that for each i, there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Definition 2.3.2: Coherent Sheaves

We say that $\mathcal{F}$ is a coherent sheaf if $\mathcal{F}$ is a quasicoherent sheaf and each M_i is a finitely generated A_i-module.

In some sense, the category of quasicoherent sheaves is the smallest abelian category for which it encompasses the category of locally free sheaves. In the case that A is locally Noetherian, the category of finite rank locally free sheaves sit inside the category of coherent sheaves, which is also an abelian category.

Proposition 2.3.3

Let A be a ring and let $X = \operatorname{Spec}(A)$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules. Its inverse is the functor $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

If A is noetherian, the same functor gives an equivalence of categories between the category of finitely generated A-modules and the category of coherent $\mathcal{O}_X$-modules.

3 Sheaf Cohomology

3.1 Category of Sheaves

Definition 3.1.1: The Category of Sheaves of Abelian Groups

Let X be a topological space. The category of sheaves of abelian groups is the category $\mathbf{Ab}(X)$ where

- $Obj(\mathbf{Ab}X) = \{\text{Sheaves of Abelian Groups on } X\}$
- $Mor(\mathbf{Ab}) = \text{Morphisms of Sheaves}$

Proposition 3.1.2

The category of sheaves of abelian groups on a topological space is an abelian category.

Proposition 3.1.3

Let $\phi : F \to G$ be a morphism of sheaves. Then the categorical kernel and cokernel of ϕ is canonically isomorphic to the sheaves $\ker(\phi)$ and $\text{coker}(\phi)$.

Proposition 3.1.4

Let X be a topological space. The cochain complex

$$\cdots \longrightarrow F^{i-1} \longrightarrow F^i \longrightarrow F^{i+1} \longrightarrow \cdots$$

is exact in $\mathbf{Ab}(X)$ if and only if for every $x \in X$ the corresponding sequence of stalks

$$\cdots \longrightarrow F_x^{i-1} \longrightarrow F_x^i \longrightarrow F_x^{i+1} \longrightarrow \cdots$$

is exact.

Proposition 3.1.5

The functor f^{-1} is left adjoint to the functor f_*.

This immediately implies the following:

Proposition 3.1.6

The functor f_* is left exact, and the functor f^{-1} is right exact.

Proposition 3.1.7

Let X be a topological space. Then the category $\mathbf{Ab}(X)$ has enough injectives.

3.2 Cohomology of Sheaves

Definition 3.2.1: Global Section Functor

Let $\mathcal{F}$ be a sheaf on a space X. Define the global section functor to be the functor $\Gamma : \mathbf{Ab}(X) \to \mathbf{Ab}(X)$ defined by

$$\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$$

Lemma 3.2.2

The global section functor Γ is a left exact functor.

Definition 3.2.3: Flasque Sheaves

A sheaf $\mathcal{F}$ on a space X is said to be flasque if for every pair of open sets $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

Proposition 3.2.4

Flasque sheaves are acyclic for the functor Γ.

3.3 Čech Cohomology

Definition 3.3.1: Čech Complex

Let X be a topological space and $\mathcal{U} = \{U_i | i \in I\}$ an open cover of X where I is an indexing set. For any $(i_0, \dots, i_k) \in I^{k+1}$, denote

$$U_{i_0,\dots,i_k} = U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_k}$$

Define for each k,

$$C^k(X, \mathcal{U}, \mathcal{F}) = \bigcap_{(i_0,\dots,i_k) \in I^{k+1}} \mathcal{F}(U_{i_0,\dots,i_k})$$

Furthermore, define a boundary map $d : C^k(X, \mathcal{U}, \mathcal{F}) \to C^{k+1}(X, \mathcal{U}, \mathcal{F})$ by

$$c_{i_0 \dots, i_k} \overset{d}{\mapsto} \sum_{s=0}^{k+1} (-1)^s \operatorname{res}(c_{i_0,\dots,\hat{i}_s,\dots,i_{k+1}})$$

Define the Čech complex to be $(C^\bullet(X, \mathcal{U}, \mathcal{F}), d)$.

Lemma 3.3.2

For any space X and any open cover $\mathcal{U}$ of X, $(C^\bullet(X, \mathcal{U}, \mathcal{F}), d)$ is indeed a chain complex.

Definition 3.3.3: Čech Cohomology

Let $(C^\bullet(X, \mathcal{U}, \mathcal{F}), d)$ be a Čech complex. Define the kth cohomology group of it to be

$$\check{\mathrm{H}}^k(X, \mathcal{U}, \mathcal{F}) = \frac{\ker(C^k(X, \mathcal{U}, \mathcal{F}) \to C^{k+1}(X, \mathcal{U}, \mathcal{F}))}{\operatorname{im}(C^{k-1}(X, \mathcal{U}, \mathcal{F}) \to C^k(X, \mathcal{U}, \mathcal{F}))} = H(C^\bullet(X, \mathcal{U}, \mathcal{F}), d)$$

Lemma 3.3.4

For any Čech complex, we have that $\check{\mathrm{H}}^0(X, \mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

Theorem 3.3.5

Let X be a topological space and $\mathcal{U}$ an open cover of X. If the open sets $U_{i_0,\dots,i_k}$ satisfy that $H^k(U_{i_0,\dots,i_k}, \mathcal{F}) = 0$ for all $k > 0$, then

$$H^k(X, \mathcal{F}) = \check{\mathrm{H}}^k(X, \mathcal{U}, \mathcal{F})$$