Representation Theory

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Abstract

Contents

1 Group Representations

1.1 Matrix and Linear Representations

Recall the result stating that for $G = \langle S|R\rangle$ where S is finite, if H is a group with elements $h_1, \dots, h_n \in H$. Then there exists a homomorphism $\phi : G \to H$ satisfying $\phi(s_i) = h_i$ if and only if every relation $r \in R$ is also satisfied by the h_i. In this case ϕ is unique.

Definition 1.1.1: Matrix Representations

Let G be a group and F a field. A matrix representation is a homomorphism

$$\rho : G \to \mathrm{GL}(n, F)$$

for some n. The degree of ρ is the integer n.

In some sense we are enabling a geometric picture of a group by visualizing them through a subgroups consisting of matrices. And since matrices act on the plane $\mathbb{R}^n$, we can visualize what the group is doing through this.

Lemma 1.1.2

Let $\rho : G \to GL(n, F)$ be a matrix representation. Let $A \in \mathrm{GL}(n, F)$. Then the homomorphism $\rho' : G \to \mathrm{GL}(n, F)$ defined by

$$\rho'(g) = A\rho(g)A^{-1}$$

is a matrix representation.

Proof. We just have to show that ρ' is a group homomorphism. We have that

$$\begin{aligned} \rho'(gh) &= A\rho(gh)A^{-1} \\ &= A\rho(g)\rho(h)A^{-1} \\ &= A\rho(g)A^{-1}A\rho(h)A^{-1} \\ &= \rho'(g)\rho'(h) \end{aligned}$$

Thus we are done. □

Definition 1.1.3: Equivalent Representations

Let $\rho_1 : G \to \mathrm{GL}(n, F)$ and $\rho_2 : G \to \mathrm{GL}(n, F)$ be two representations. We say that ρ_1 and ρ_2 are equivalent if $n = m$ and there exists a matrix $P \in \mathrm{GL}(n, F)$ such that $\rho_2(g) = P\rho_1(g)P^{-1}$ for all $g \in G$.

Lemma 1.1.4

The equivalence of representations is an equivalence relation.

Lemma 1.1.5

Degree 1 representations $\rho_1, \rho_2 : G \to \mathrm{GL}(1, F) = F^*$ are equivalent if and only if they are equal.

Proof. Suppose that ρ_1, ρ_2 are equivalent. Then we have that $\rho_1(g) = u\rho_2(g)u^{-1}$ for some $u \in F^*$. But F is commutative so $\rho_1(g) = \rho_2(g)$.

If ρ_1 and ρ_2 are equal then they are clearly equivalent, □

Definition 1.1.6: Faithful Representations

A representation $\rho : G \to \mathrm{GL}(n, F)$ is said to be faithful if it is injective.

Definition 1.1.7: Linear Representations

Let G be a group. A linear representation of G is a pair (V, ρ) where V is a vector space and ρ is a homomorphism $\rho : G \to \mathrm{GL}(V)$. The dimension of V is called the degree of the representation.

Recalling that by choosing a basis, we can show that $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$ if $\dim(V) = n$. Linear representations are often used for when we do not want to choose a basis and leave it arbitrary. In practical calculations matrix representations may be useful but in the abstract theory itself, using an arbitrary vector space is more useful.

1.2 KG-Modules

Definition 1.2.1: Group Ring

Let G be a group and R a ring. The group ring RG is the ring whose elements are the R-linear combinations $\sum_{g\in G} \lambda_g g$ for finitely many non-zero $\lambda_g \in R$, where operations are defined as follows:

- Addition: $\left(\sum_{g\in G} \lambda_g \cdot g\right) + \left(\sum_{g\in G} \mu_g \cdot g\right) = \sum_{g\in G}(\lambda_g + \mu_g) \cdot g$
- Multiplication: $\left(\sum_{g\in G} \lambda_g g\right) \cdot \left(\sum_{h\in G} \mu_h h\right) = \sum_{g,h\in G}(\lambda_g \mu_h)h$

Lemma 1.2.2

Let G be a group and K a field. Then the group ring KG is a K-vector space with basis G. Moreover, KG is a K-algebra.

There is a very rich structure in KG-modules. In ring and modules we know that algebras over a field can be seen as a vector space. Vector spaces can also be seen as a module over a field.

Definition 1.2.3: Linear Action

Let G be a group and V a vector space. A linear action of G on V is a map $\gamma : G \times V \to V$ such that the following holds:

- Identity: $\gamma(1_G, v) = v$ for all $v \in V$
- Associativity: $\gamma(hg, v) = \gamma(h, \gamma(g, v))$ for all $g, h \in G$, $v \in V$
- Linearity on V: $\gamma(g, u + v) = \gamma(g, u) + \gamma(g, v)$ for all $g \in G$, $u, v \in V$
- Linearity on V: $\gamma(g, av) = a\gamma(g, v)$ for all $g \in G$ and $v \in V$ and $a \in K$

This means that G acts on V and that $\rho(g) : V \to V$ defined by $v \mapsto \gamma(g, v)$ is a linear map.

Proposition 1.2.4

Let G be a group. If V is a KG-module then the action of G on V is a linear action. Conversely, if V is a K-vector space with a linear action G then V is a KG-module.

There is also a 1-1 correspondence between linear representations and KG-modules.

Theorem 1.2.5

Let V be a vector space. Linear representations over V and KG-modules over V are the same in the following sense.

- If $\rho : G \to \mathrm{GL}(V)$ is a linear representation, ρ gives rise to a KG-module structure on V, where the composition law $KG \times V \to V$ is defined by
$$\left(\sum_{g\in G}\lambda_g g, v\right) \mapsto \left(\sum_{g\in G}\lambda_g g\right)\cdot v = \sum_{g\in G}\lambda_g \rho(g)(v)$$
- Conversely, given a KG-module V, the map $\rho_V : G \to \mathrm{GL}(V)$ defined by
$$g \mapsto \rho_V(g) : V \to V$$
where $\rho_V(g)$ is defined by $\rho_V(g)(v) = g \cdot v$ is in fact a linear representation.

One can think of the KG-module action on V as an extension of the K-action on V.

Lemma 1.2.6

Two representations $\rho_1 : G \to \mathrm{GL}(V_2)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ are equivalent if and only if $V_1 \cong V_2$ as KG-modules.

Essentially, one can think of KG-modules being a vector space (module) over K together with a group action. Thus later when we encounter KG-submodules and morphisms we can simply regard them as vector subspaces (submodules) and linear transformations that respect the group action.

1.3 KG-Submodules

Definition 1.3.1: KG-Submodule

Let G be a group, K a field and V a KG-module. We say that W is a KG-submodule if the following are true.

- W is a K-subspace of V
- $g \cdot w \in W$ for all $w \in W$ and $g \in G$

We know that any R-submodule N of M is also an R-module. This property is inherited and thus KG-submodules are also KG-modules in its own right.

Definition 1.3.2: Morphism of KG-modules

Let V, W be KG-modules. A map $\pi : V \to W$ is called a morphism if the following are true.

- π is a linear transformation (K-module homomorphism): $\pi(au + bv) = a\pi(u) + b\pi(v)$ for all $u, v \in V$ and $a, b \in K$
- π respects the group action: $\pi(g \cdot v) = g \cdot \pi(v)$ for $v \in V$ and $g \in G$.

An isomorphism of KG-modules is a bijective morphism.

Lemma 1.3.3

Let $\pi : V \to W$ be a morphism of KG-modules. Then $\ker(\pi)$ and $\mathrm{im}(\pi)$ are KG-submodules of V and W respectively.

Recall the notion of an irreducible module.

Definition 1.3.4: Irreducible Representations

Let V be a KG-module. We say that V is irreducible if V is a simple KG-module. Equivalently, a representation $\rho : G \to GL(V)$ is irreducible if there are no proper, non-trivial subspace of V that is invariant under the action of G.

Proposition 1.3.5

Let V be a KG-module. V is irreducible if and only if V has no proper, non-trivial subspace of V that is invariant under the action of G.

Theorem 1.3.6: Schur's Lemma III

Let G be a group. Let V be an irreducible $\mathbb{C}G$-module of finite degree. Let $\pi : V \to V$ be a morphism. Then $\pi = \lambda I_V$ for some $\lambda \in \mathbb{C}$.

1.4 Maschke's Theorem

Recall the notion of semisimple modules: An R-module is semisimple if it is the direct sum of simple submodules.

Lemma 1.4.1: The Averaging Trick

Let G be a finite group and K a field. Suppose that $|G| \cdot 1_K L \neq 0$. Let V, U be KG-modules and let $\pi : V \to U$ be a linear transformation. Define $\pi' : V \to U$ by

$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$$

Then π' is a morphism of KG-modules.

Theorem 1.4.2: Maschke's Theorem

Let G be a finite group and K a field. Suppose that $|G| \cdot 1_K \neq 0$. Let V be a KG-module of finite degree. Then V is semisimple.

Corollary 1.4.3

Let $V \neq 0$ be a KG-module of finite degree, where G is a finite group and $|G| \cdot 1_K \neq 0$. Then there exists irreducible submodules $U_1, \dots U_k$ such that

$$V = U_1 \oplus \cdots \oplus U_k$$

Character theory will then be to show that this decomposition of KG-submodules is essentially unique assuming that $K = \mathbb{C}$.

2 Character Theory

2.1 Trace of a Matrix

Definition 2.1.1: Trace of a Matrix

Let $A \in M_{n\times n}(K)$ for $K = \mathbb{R}$ or $\mathbb{C}$ where we write

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

Define the trace of A to be

$$\text{tr}(A) = \sum_{i=1}^{n} a_{ii}$$

which is the sum of the diagonal entries of A.

Proposition 2.1.2

Let $A \in M_{n\times n}(K)$ for $K = \mathbb{R}$ or $\mathbb{C}$. Then the trace of A is the coefficient of x^{n-1} in the characteristic polynomial $c_A(x)$ and the determinant is the

Lemma 2.1.3

Let A, B be similar $d \times d$ matrices. Then A and B have the same trace.

Proof. Since similar matrices have the same characteristic polynomial and that the trace of a matrix is the coefficient of the characteristic polynomial at the x^{d-1} term, we have that A and B have the same trace. □

Lemma 2.1.4

Let $A \in \text{GL}(d, \mathbb{C})$ such that $A^n = I$ for some $n \in \mathbb{N} \setminus \{0\}$. Then the following are true regarding the trace of A.

- $|\text{tr}(A)| \leq d$
- $|\text{tr}(A)| = d$ if and only if $A = \theta I_d$ where θ is some nth root of unity.
- $\text{tr}(A) = d$ if and only if $A = I$
- $\text{tr}(A^{-1}) = \overline{\text{tr}(A)}$

Proof.

- By lemma 1.2.2, there is some matrix Q and nth roots of unity $\theta_1, \dots, \theta_d$ such that $Q^{-1}AQ = \text{diag}(\theta_1, \dots, \theta_d)$. It follows that $\text{tr}(A) = \text{tr}(Q^{-1}AQ) = \sum_{i=1}^{d} \theta_i$ and that

$$|\text{tr}(A)| \leq \sum_{i=1}^{d} |\theta_i|$$

- Suppose that $|\text{tr}(A)| = d$ Then this means that $|\text{tr}(A)| = \sum_{i=1}^{d} |\theta_i|$. This happens precisely when each θ_i have the same angle, which means they are positive multiples of each other. Since $|\theta_1| = 1$, we have $\theta_1 = \cdots = \theta_d$. Thus $A = \theta I_d$ for some θ an nth root of 1.

  Conversely, If $A = \theta I_d$ then $\text{tr}(A) = d \cdot \theta$ and thus we are done.

- It follows immediately from the second item
- We have that
$$Q^{-1}A^{-1}Q = (Q^{-1}AQ)^{-1} = \text{diag}(\theta_1^{-1}, \dots, \theta_d^{-1})$$
This means that $\text{tr}(A^{-1}) = \sum_{i=1}^d \theta_i^{-1}$. But since θ_i is a root of unity, we have that $\overline{\theta_i} = \theta_i^{-1}$. Thus we are done.

$\square$

2.2 Characters of a Representation

Definition 2.2.1: Character of a Representation

Let $\rho : G \to \text{GL}(d, \mathbb{C})$ be a degree d complex matrix representation. Define the character of ρ as the function $\chi_\rho : G \to \mathbb{C}$ defined by
$$\chi(g) = \text{tr}(\rho(g))$$

Lemma 2.2.2

Equivalent matrix representations have the same character.

Proof. Suppose $\rho_1, \rho_2 : G \to \text{GL}(d, \mathbb{C})$ are equivalent matrix representations. Then ρ_1, ρ_2 are similar for each g and so they have the same trace. Thus they have the same characteristic. $\square$

In fact the inverse of this lemma is also true, which we will see later in the notes. This makes characteristics a powerful invariant for representations.

Proposition 2.2.3

Let G be a finite group. Let $\rho : G \to \text{GL}(d, \mathbb{C})$ be a complex matrix representation. Then the following are true regarding the character χ of the representation.

- $|\chi(g)| \leq d$ for all g
- $\chi(g) = d$ if and only if $\rho(g) = I_d$
- $\chi(g^{-1}) = \overline{\chi(g)}$ for all $g \in G$.
- $\chi(hgh^{-1}) = \chi(g)$ for all $g, h \in G$

In particular, χ is invariant under conjugacy classes. This means that we can think of χ as class functions instead. Class functions are functions that are constant on classes so that we can think of their input are conjugacy classes.

Lemma 2.2.4

Let V be a $\mathbb{C}G$-module of finite degree. Suppose $V = U \oplus W$ where U and W are submodules. Then
$$\chi_V = \chi_U + \chi_W$$

Definition 2.2.5: Irreducible Character

Let G be a finite group. A character is said to be irreducible if it is the character of an irreducible $\mathbb{C}G$-module.

Lemma 2.2.6

Let $V = U_1 \oplus \cdots \oplus U_k$ be a decomposition of a $\mathbb{C}G$-module into irreducible $\mathbb{C}G$-submodules. Then

$$\chi_V = \sum_{i=1}^{k} \chi_{U_i}$$

2.3 Orthogonality Relations of Characters

Definition 2.3.1: Set of Functions from Group to $\mathbb{C}$

Let G be a finite group. Denote

$$\mathbb{C}[G] = \{\phi : G \to \mathbb{C} | \phi \text{ is a map of sets }\}$$

the set of all functions from G to $\mathbb{C}$.

Lemma 2.3.2

Let V be a finite dimensional irreducible $\mathbb{C}G$-module. Let $f : V \to V$ be a linear map. Define

$$\tilde{f}(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot (f(g^{-1} \cdot v))$$

Then $\tilde{f} = \frac{\text{tr}(f)}{\dim(V)} I_V$

Proposition 2.3.3

Let G be a finite group. Then $\mathbb{C}[G]$ is an inner product space over $\mathbb{C}$ where the Hermitian product $\langle \, , \, \rangle : \mathbb{C}[G] \times \mathbb{C}[G] \to \mathbb{C}$ is defined by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}$$

Moreover, $\dim_{\mathbb{C}}(\mathbb{C}[G]) = |G|$.

Theorem 2.3.4

Let U, V be finite dimensional $\mathbb{C}G$-modules. Then

$$\langle \chi_U, \chi_V \rangle = \begin{cases} 1 & \text{if } U \cong V \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $U \cong V$ if and only if $\chi_U = \chi_V$.

Lemma 2.3.5

Let U be a finite dimensional $\mathbb{C}G$-module. Then U is irreducible if and only if $\langle \chi_U, \chi_U \rangle = 1$.

2.4 The Wedderburn Isomorphism

Definition 2.4.1: Multiplicity

Let U, W be a finite dimensional $\mathbb{C}G$-module such that U is irreducible. Define the multiplicity of U in W as

$$\text{mult}_U(W) = \langle \chi_U, \chi_W \rangle$$

Lemma 2.4.2

Let U, W be a finite dimensional $\mathbb{C}G$-module such that U is irreducible. Suppose that $W = \bigoplus_{i=1}^r U_i$ is any decomposition into irreducible $\mathbb{C}G$-submodules. Then we have

$$\text{mult}_U(W) = |\{U_i | \, U \cong U_i\}|$$

Lemma 2.4.3

Let V be a finite dimensional $\mathbb{C}G$-module. Let $W_1, \dots, W_k$ be the complete list of pairwise non-isomorphic irreducible $\mathbb{C}G$-submodules of V. Then

$$\sum_{i=1}^k (\dim(W_i))^2 = |G|$$

Theorem 2.4.4: Wedderburn's Theorem

Let V be a finite dimensional $\mathbb{C}G$-module. Let $W_1, \dots, W_k$ be the complete list of pairwise non-isomorphic irreducible $\mathbb{C}G$-submodules of V. Let

$$f : \mathbb{C}G \to \text{End}(W_1) \times \dots \times \text{End}(W_k)$$

be defined by $f(g) = (\rho_{W_1}(g), \dots, \rho_{W_k}(g))$ and extended linearly. Then f is a $\mathbb{C}$-algebra isomomorphism.

Proposition 2.4.5

Let G be a group. Denote Cl_G the set of conjugacy classes in G. Then

$$\dim(Z(\mathbb{C}G)) = |\text{Cl}_G|$$

Corollary 2.4.6

The number of pairwise non-isomorphic irreducible representations of G equals the number $|\text{Cl}_G|$ of conjugacy classes of G.

Corollary 2.4.7

The characters of the irreducible representations form a basis of the vector space $\mathbb{C}[\text{Cl}_G]$.

2.5 Character Tables

Definition 2.5.1: Character Tables

Let G be a finite group. The character table of G is a table

G	$\text{Cl}_G(g_1)$	$\text{Cl}_G(g_2)$	$\cdots$
Trivial			
χ_1			
χ_2			
$\vdots$			

where the rows are the irreducible characters and the columns are the conjugacy classes of G.

Corollary 2.5.2

Let G be a finite group and write the character table of G into a matrix A. Multiply each column of A by $\sqrt{\frac{\text{Cl}_G(g)}{|G|}}$. Then the new matrix A' is orthonormal.

Corollary 2.5.3

Let G be a finite group and $g \in G$. Then we have

$$\sum_{\chi \text{ is irr.}} \chi(g)\overline{\chi(g)} = \frac{|G|}{|\text{Cl}_G(g)|}$$

where the sum is over all irreducible characters.

Corollary 2.5.4

Let G be a finite group and $g_1, g_2 \in G$. If g_1 and g_2 are not in the same conjugacy classes then

$$\sum_{\chi \text{ is irr.}} \chi(g_1)\overline{\chi(g_2)} = 0$$

where the sum is over all irreducible characters.

2.6 The Isotypic Decomposition

Theorem 2.6.1

Let $W_1, \dots, W_k$ be a complete list of pairwise nonisomorphic irreducible representations of G. For $1 \leq i \leq k$, let

$$a_i = \frac{\dim(W)}{|G|} \sum_{g \in G} \overline{\chi_{W_i}(g)} g \in \mathbb{C}G$$

Let V be a finite dimensional $\mathbb{C}G$-module. Consider the decomposition into irreducibles:

$$V = \bigoplus_{l=1}^{k} \bigoplus_{j=1}^{\text{mult}_{W_l}(V)} U_{l,j}$$

with each $U_{l,j} \cong W_l$. Then $\rho_V(a_i) \in \text{End}(V)$ is the projection onto V_i. In particular, the space V_i is independent of the finer decomposition of V into the direct sum of the $U_{l,j}$.

Notice that the theorem gives a decomposition of the vector space.

Definition 2.6.2: Isotypic Components

Let V be a finite dimensional $\mathbb{C}G$-module. Let W_l be an irreducible representation of G. We call the spaces

$$V_l = \bigoplus_{j=1}^{\text{mult}_{W_l}(V)} U_{l,j}$$

given above where $U_{l,j} \cong W_l$ the isotypic components of V.

A representation is said to be isotypic if it contains only one non-zero isotypic component.

While the decomposition of V into irreducible subrepresentations is not unique, the isotypic decomopsition is unique up to reordering the summands.

2.7 Induced Representations

Definition 2.7.1: Subgroups

Let $H \leq G$ be a subgroup. Let V be a finite dimensional $\mathbb{C}G$-module. Then H acts on V and we denote the corresponding $\mathbb{C}H$-module by $V \downarrow_H^G$. We write the restriction of the characters as $\chi_V \downarrow_H^G = \chi_{V\downarrow_H^G}$.

Note that if V is an irreducible $\mathbb{C}G$-module, $V \downarrow_H^G$ may not be irreducible.

Definition 2.7.2: The Coset Module

Let $\mathcal{H} = \{t_1 H, \dots, t_k H\}$ be the set of all cosetrs of G. Then G acts on $\mathcal{H}$. Let $\mathbb{C}\mathcal{H}$ denote the corresponding permutation representation. The representation $\mathbb{C}\mathcal{H}$ that is a finite dimensional $\mathbb{C}G$-module is called the coset module.

Definition 2.7.3: Induced Representation

Let H be a subgroup of G. Let $\rho : H \to GL(n, \mathbb{C})$ be a representation. Define the induced representation of ρ to be $\rho \uparrow_H^G : G \to \text{End}(\mathbb{C}^{nl})$ via

$$\rho \uparrow_H^G (g) = \begin{pmatrix} \rho(t_1^{-1} g t_1) & \cdots & \rho(t_1^{-1} g t_l) \\ \vdots & \ddots & \vdots \\ \rho(t_l^{-1} g t_1) & \cdots & \rho(t_l^{-1} g t_l) \end{pmatrix}$$

where $\rho(g) = 0$ if $g \notin H$.

Theorem 2.7.4

Let H be a subgroup of G and $\rho : H \to GL(n, \mathbb{C})$ a representation of H. Then $\rho \uparrow_H^G : G \to GL(n, \mathbb{C})$ is a matrix representation.

Theorem 2.7.5

Suppose that $\mathcal{H} = \{t_1 H, \dots, t_l H\}$ are $\mathcal{H}' = \{s_1 H, \dots, s_l H\}$ are two representations of the set of cosets of H in G. Then the two representations constructed from $\mathcal{H}$ and $\mathcal{H}'$ are isomorphic.

Lemma 2.7.6

Let ρ be a finite dimensional representation of H with character χ. Then for all $g \in G$, we have that

$$\chi \uparrow_H^G (g) = \frac{1}{|H|} \sum_{x \in G} \chi(x^{-1} g x)$$

where $\chi(g) = 0$ if $g \notin H$.

Theorem 2.7.7: Frobenius Reciprocity

Let $H \leq G$ and let ψ and χ be characters of H and G respectively. Then

$$\langle \psi \uparrow_H^G, \chi \rangle = \langle \psi, \chi \downarrow_H^G \rangle$$

2.8 Decomposition of Regular Representations

Definition 2.8.1: Regular Representation

Let $\rho : G \to GL(V)$ be a representation such that V has basis $\{v_g | g \in G\}$. We say that ρ is a regular representation if $\rho(h) : V \to V$ has the property that

$$\rho(h)(v_g) = v_{hg}$$

for every $h \in H$.

3 Computations of Representations

3.1 Representations of the Cyclic Group

Proposition 3.1.1

Denote $C_n = \langle x \rangle$ the cyclic group. The set of all degree 1 complex representations (up to equivalence) of C_n are precisely

$$\{\phi_k | k = 0, \dots, n-1\}$$

where $\phi_k(x) = e^{2\pi i k/n}$

Lemma 3.1.2

Let $A \in GL(d, \mathbb{C})$ be a matrix such that $A^n = I$ for some $n \in \mathbb{N}$. Then there is a matrix $Q \in GL(d, \mathbb{C})$ such that

$$Q^{-1}AQ = \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_d \end{pmatrix}$$

where $\theta_1, \dots, \theta_d$ are nth roots of unity and the matrix is everywhere else 0.

Proof. Let $f(X) = X^n - 1$. Then Clearly $f(A) = 0$. This means that the minimal polynomial $\mu_A(X)$ of A divides $f(X) = X^n - 1$. The roots of f are the n-roots of 1, namely $1, \zeta, \dots, \zeta^{n-1}$ where $\zeta = e^{2\pi i/n}$. Since $\mu_A(X)$ divides $f(X)$, the roots of μ_A are the n-roots of 1. Moreover, $f(X) = X^n - 1$ has distinct roots, and so μ_A has distinct roots. Hence we know that A is diagonalizable with entries the n-roots of unity. □

Theorem 3.1.3

Denote $C_n = \langle x \rangle$ the cyclic group. Let $\rho : C_n \to GL(d, \mathbb{C})$ be a representation. Then there exists $\theta_1, \dots, \theta_d$ which are nth rots of unity such that the representation $\rho' : C_n \to GL(d, \mathbb{C})$ defined by

$$\rho'(x^k) = \begin{pmatrix} \theta_1^k & & \\ & \ddots & \\ & & \theta_d^k \end{pmatrix}$$

is equivalent to ρ.

Proof. Let $A = \rho(x)$. Since $x^n = 1$ and ρ is a homomorphism we have $A^n = \rho(x^n) = \rho(1) = 1$. By the above lemma there exists $Q \in GL(d, \mathbb{C})$ and $\theta_1, \dots, \theta_d$ nth roots of unity such that $Q^{-1}\rho(x)Q = \text{diag}(\theta_1, \dots, \theta_d)$. Define $\rho' : C_n \to GL(d, \mathbb{C})$ by

$$\rho'(x^k) = Q^{-1}\rho(x)Q$$

This is a representation equivalent to ρ by lemma 1.1.2. Finally we have that

$$\begin{aligned} \rho'(x^k) &= Q^{-1}\rho(x^k)Q \\ &= Q^{-1}A^kQ \\ &= (Q^{-1}AQ)^k \\ &= \begin{pmatrix} \theta_1^k & & \\ & \ddots & \\ & & \theta_d^k \end{pmatrix} \end{aligned}$$

Thus we are done. □

Definition 3.1.4: Regular Representation

Let G be a finite group. Let $n = |G|$. Let V be a $\mathbb{C}$-vector space for dimension n with basis $\{v_g|g \in G\}$. For $h \in G$, define the regular representation $\text{reg}_h : V \to V$ be the linear transformation such that $\text{reg}_h(v_g) = v_{hg}$.

Lemma 3.1.5

Let G be a finite group. Let $h \in G$. Then $\text{reg}_h \in GL(V)$.

Lemma 3.1.6

Let G be a finite group. Let $h \in G$. Then reg_h is a linear representation.

3.2 Representations of the Symmetric Group