Commutative Algebra 1

Labix

February 23, 2025

Abstract

Contents

1 Ideals Of a Commutative Ring

1.1 Basic Operations on Ideals

Let R be a commutative ring. Recall that an ideal of R is a subset $I \subseteq R$ such that

- If $a, b \in I$, then $a + b \in I$.
- If $r \in R$ and $a \in I$, then $ra \in I$.

Proposition 1.1.1

Let R be a commutative ring. Let $I_1, \dots, I_n$ be ideals of R. Let $P_1, \dots, P_k$ be prime ideals of R.

- Let I be an ideal of R. If $I \subseteq \bigcup_{i=1}^k P_i$, then $I \subseteq P_i$ for some i.
- Let P be an ideal of R. If $P \subseteq \bigcap_{i=1}^n I_i$, then $I_i \subseteq P$ for some i.
- Let P be an ideal of R. If $P = \bigcap_{i=1}^n I_i$, then $I_i = P$ for some i.

Proof.

- We prove the contrapositive by induction k. When $k = 1$, the case is clear. Suppose that $I \not\subseteq P_i$ for $1 \leq i \leq k-1$ implies $I \not\subseteq \bigcup_{i=1}^{k-1} P_i$. Suppose also that $I \not\subseteq P_k$.

$\square$

Proposition 1.1.2

Let R be a commutative ring. Let $S, T \subseteq R$ be subsets of R. Then

$$\langle S \cup T \rangle = \langle S \rangle + \langle T \rangle$$

Proposition 1.1.3

Let R be a commutative ring. Let I, J be ideals of R. Suppose that $I \subseteq J$. Let $\overline{J}$ denote the ideal of R/I corresponding to J under the correspondence theorem. Then there is an isomorphism

$$\frac{R/I}{\overline{J}} \cong \frac{R}{I+J}$$

given by the formula $(r + I) + \overline{J} \mapsto r + (I + J)$.

Example 1.1.4

There is an isomorphism given by

$$\frac{\mathbb{Z}[x]}{(x+1, x^2+2)} \cong \mathbb{Z}/3\mathbb{Z}$$

Proof. Using the above propositions, we have that

$$\begin{aligned} \frac{\mathbb{Z}[x]}{(x+1, x^2+2)} &= \frac{\mathbb{Z}[x]}{(x+1) + (x^2+2)} \\ &\cong \frac{\mathbb{Z}[x]/(x+1)}{(3)} \end{aligned}$$

Indeed, the ideal $(x^2 + 2)$ corresponds to the ideal (3) in $\frac{\mathbb{Z}[x]}{(x+1)}$ because the remainder of $x^2 + 2$ divided by $(x + 1)$ is (3). Now $\mathbb{Z}[x]/(x + 1) \cong \mathbb{Z}$ by the evaluation homomorphism. Thus quotieting by the ideal (3) gives the field $\mathbb{Z}/3\mathbb{Z}$. $\square$

Some more important results from Groups and Rings and Rings and Modules include:

- If I and J are coprime, then $IJ = I \cap J$
- Chinese Remainder Theorem: If I and J are coprime, then there is an isomorphism

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}$$

1.2 The Radical of an Ideal

The radical of an ideal is a very different notion from the radical of module.

Definition 1.2.1: Radical of an Ideal

Let I be an ideal of a ring R. Define the radical of I to be

$$\sqrt{I} = \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}$$

Proposition 1.2.2

Let R be a commutative ring. Let I be an ideal. Then the following are true.

- $I \subseteq \sqrt{I}$
- $\sqrt{\sqrt{I}} = \sqrt{I}$
- $\sqrt{I^m} = \sqrt{I}$ for all $m \geq 1$
- $\sqrt{I} = R$ if and only if $I = R$

Proof.

- Let $r \in I$. Then $r^1 \in I$ Thus by choosing $n = 1$ we shows that $r^n \in I$. Thus $r \in \sqrt{I}$.
- By the above, we already know that $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. So let $r \in \sqrt{\sqrt{I}}$. Then there exists some $n \in \mathbb{N}$ such that $r^n \in \sqrt{I}$. But $r^n \in \sqrt{I}$ means that there exists some $m \in \mathbb{N}$ such that $(r^n)^m \in I$. But $nm \in \mathbb{N}$ is a natural number such that $r^{nm} \in I$. Hence $r \in \sqrt{I}$ and so we conclude.

□

Proposition 1.2.3

Let R be a commutative ring. Let I, J be ideals of R. Then the following are true.

- If $I \subseteq J$ then $\sqrt{I} \subseteq \sqrt{J}$
- $\sqrt{IJ} = \sqrt{I \cap J}$
- $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$

Proof.

- Let $x \in \sqrt{IJ}$. Then $x^n \in IJ$. This means that there exists $i \in I$ and $j \in J$ such that $x^n = ij$. Since I and J are two sided ideals, we can conclude that $x^n = ij \in I, J$. Hence $x^n = ij \in I \cap J$. We conclude that $x \in \sqrt{I \cap J}$. Now let $x \in \sqrt{I \cap J}$. Then there exists $n \in \mathbb{N}$ such that $x^n \in I \cap J$. Then $x^n \in I$ and $x^n \in J$ implies that $x^{2n} = x^n \cdot x^n \in IJ$. We conclude that $x \in \sqrt{IJ}$.

□

Proposition 1.2.4

Let R be a commutative ring. Let I be an ideal. Then

$$\sqrt{I} = \bigcap_{\substack{p \text{ a prime ideal} \\ I \subseteq p \subseteq R}} p$$

Definition 1.2.5: Radical Ideals

Let R be a commutative ring. Let I be an ideal of R. We say that I is radical if

$$\sqrt{I} = I$$

In particular, by the above lemma it follows that the radical of an ideal is a radical ideal.

Lemma 1.2.6

Let R be a ring. Let P be a prime ideal of R. Then P is radical.

We conclude that there is an inclusion of types of ideal in which each inclusion is strict:

$$\substack{\text{Maximal}\\\text{ideals}} \subset \substack{\text{Prime}\\\text{ideals}} \subset \substack{\text{Radical}\\\text{ideals}}$$

1.3 The Nilradical Ideal of Commutative Rings

Let R be a ring. Recall that an element $r \in R$ is nilpotent if $r^n = 0_R$ for some $n \in \mathbb{N}$. When R is commutative, we can form an ideal out of nilpotent elements.

Definition 1.3.1: Nilradicals

Let R be a ring. Define the nilradical of R to be

$$N(R) = \{r \in R \mid r \text{ is nilpotent}\}$$

Note that this is different from nilpotent ideals, as nilpotency is a property of an ideal. However the Nilradical ideal is a nil ideal and every sub-ideal of the nilradical is a nil ideal.

Proposition 1.3.2

Let R be a ring and $N(R)$ its nilradical. Then the following are true.

- $N(R)$ is an ideal of R
- $N(R/N(R)) = 0$

Proof.

- Suppose that r, s are nilpotent, meaning that $r^n = 0$ and $s^m = 0$. Then $(r + s)^{n+m} = 0$. Moreover, if $t \in R$ then $t \cdot r$ is also nilpotent
- Let $r \notin N(R)$. Every element $r + N(R) \in R/N(R)$ has the property that $r^n \neq 0$. Consider $(r + N(R))^n = r^n + N(R)$. If $r^n \in N(R)$ then $r^n = u$ for some nilpotent u, which means that r^n is nilpotent and thus r is nilpotent, a contradiction. This means that $r + N(R) \notin N(R/N(R))$ for all $r \notin N(R)$ and thus $N(R/N(R)) = 0$

□

Proposition 1.3.3

Let R be a commutative ring. Then we have

$$N(R) = \bigcap_{\substack{P \text{ is a prime} \\ \text{ideal of } R}} P = \sqrt{(0)}$$

Proof. Let $x \in N(R)$. Let P be an arbitrary prime ideal. Since x is nilpotent, $x^n = 0$ for some $n \in \mathbb{N}$. If $x \notin P$, then $x^2 \notin P$ since P is a prime ideal. Recursively we see that $x^k \notin P$ for all $k \in N \setminus \{0\}$. But $x^n = 0 \in P$ is a contradiction. Hence $N(R) \subseteq \bigcap_{P \in \text{Spec}(R)} P$.

Now suppose that $x \in R$ is not nilpotent. Consider the set

$$\Sigma = \{I \trianglelefteq R \mid x^k \notin I \text{ for all } k \geq 1\}$$

Notice that $(0) \in \Sigma$ and hence it is non-empty. Let $I_1 \subseteq I_2 \subseteq \cdots$ be a chain in Σ. Define $I = \bigcup_{k=1}^{\infty} I_k$. I claim that $I \in \Sigma$. First of all if $a, b \in I$ and $r \in R$, then $a \in I_m$ and $b \in I_n$ for some $m, n \geq 1$. Then $a, b \in I_{\max\{m,n\}}$ so that $a + b \in I_{\max\{m,n\}} \subseteq I$. Also $ra \in I_m \subseteq I$ since I_m is an ideal. Hence I itself is an ideal of R. Suppose for a contradiction that $x^n \in I$ for some n. Then $x^n \in I_k$ for some k. This is a contradiction since $I_k \in \Sigma$. Thus we know that $I \in \Sigma$. In particular, I is an upper bound of $I_1 \subseteq I_2 \subseteq \cdots$. By Zorn's lemma, we conclude that Σ has a maximal element, say P.

Suppose for a contradiction that P is not a prime ideal. Let $ab \in P$ and $a, b \notin P$. Then $P \subset P + (a), P + (b)$. Since P is maximal in Σ, $P + (a)$ and $P + (b)$ cannot be in Σ, and there exists $x^m \in P + (a)$ and $x^n \in P + (b)$ for some m, n. Then

$$x^{m+n} = x^m \cdot x^n \in (P + (a))(P + (b)) = P + (ab)$$

Hence $P + (ab) \notin \Sigma$. But $ab \in P$ implies that $P + (ab) = P$. We have reached a contradiction. Thus P is a prime ideal that does not contain x. We show that $x \notin N(R)$ implies $x \notin P$ for some prime ideal P. The contrapositive of this statement is $x \in P$ for all prime ideals P implies $x \in N(R)$. Hence we are done. □

Example 1.3.4

Consider the ring

$$R = \frac{\mathbb{C}[x, y]}{(x^2 - y, xy)}$$

Then its nilradical is given by $N(R) = (x, y)$.

Proof. Notice that in the ring R, $x^3 = x(x^2) = xy = 0$ and $y^3 = x^6 = (x^3)^2 = 0$ and hence x and y are both nilpotent elements of R. By definition of the nilradical, we conclude that $(x, y) \subseteq N(R)$. Now (x, y) is a maximal ideal of $\mathbb{C}[x, y]$ because $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$. Also notice that $(x, y) \supseteq (x^2 - y, xy)$ because for any element $f(x)(x^2 - y) + g(x)(xy) \in (x^2 - y, xy)$, we have that

$$\begin{aligned} f(x)(x^2 - y) + g(x)(xy) \in (x^2 - y, xy) &= (xf(x))x - f(x)y + (g(x)x)y \\ &= (xf(x))x + (xg(x) - f(x))y \in (x, y) \end{aligned}$$

By the correspondence theorem, $(x, y)/(x^2 - y)$ is an maximal ideal of R. In particular, (x, y) is also a prime ideal. But the $N(R)$ is the intersection of all prime ideals and hence $N(R) \subseteq (x, y)$. We conclude that $N(R) = (x, y)$. □

Proposition 1.3.5

Let R be a commutative ring. Then the following are equivalent.
- R has exactly one prime ideal given by $N(R)$.
- Every element of R is either a unit or nilpotent.
- $N(R)$ is a maximal ideal.

Definition 1.3.6: Reduced Rings

Let R be a commutative ring. We say that R is reduced if $N(R) = 0$.

Proposition 1.3.7

Let R be a commutative ring. Let I be an ideal of R. Then R/I is reduced if and only if I is a radical ideal.

So radical, prime and maximal ideals all have characterizations using the quotient ring:

- I is maximal if and only if R/I is a field.
- I is prime if and only if R/I is an integral domain.
- I is radical if and only if R/I is reduced.

1.4 The Correspondence between Ideals and the Quotient

Definition 1.4.1: Max Spectrum of a Ring

Let A be a commutative ring. Define the max spectrum of A to be

$$\text{maxSpec}(A) = \{m \subseteq A \mid m \text{ is a maximal ideal of } A\}$$

Definition 1.4.2: Spectrum of a Ring

Let A be a commutative ring. Define the spectrum of A to be

$$\text{Spec}(A) = \{p \subseteq A \mid p \text{ is a prime ideal of } A\}$$

Example 1.4.3

Consider the following commutative rings.

- $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2 + 6\mathbb{Z}), (3 + 6\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/8\mathbb{Z}) = \{(2 + 8\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/24\mathbb{Z}) = \{(2 + 24\mathbb{Z}), (3 + 24\mathbb{Z})\}$
- $\text{Spec}(\mathbb{R}[x]) = \{(f) \mid f \text{ is irreducible }\}$

Proof.

- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$. We need to find which ones are prime ideals. Now $\mathbb{Z}/6\mathbb{Z} \setminus (2 + 6\mathbb{Z})$ consists of $1 + 6\mathbb{Z}$, $3 + 6\mathbb{Z}$ and $5 + 6\mathbb{Z}$. No multiplication of these elements give an element of $(2 + 6\mathbb{Z})$. So any two elements in $\mathbb{Z}/6\mathbb{Z}$ which multiply to an element of $(2 + 6\mathbb{Z})$ must contain one element that lie in $(2 + 6\mathbb{Z})$. Hence $(2 + 6\mathbb{Z})$ is prime. This is similar for $(3 + 6\mathbb{Z})$. Hence $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2 + 6\mathbb{Z}), (3 + 6\mathbb{Z})\}$.
- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2 + 8\mathbb{Z})$ and $(4 + 8\mathbb{Z})$. A similar argument as above shows that $(2 + 8\mathbb{Z})$ is a prime ideal. However, $6 + 8\mathbb{Z} \notin (4 + 8\mathbb{Z})$ while $(6 + 8\mathbb{Z})^2 = 4 + 8\mathbb{Z} \in (4 + 8\mathbb{Z})$ which shows that $(4 + 8\mathbb{Z})$ is not a prime ideal.
- A similar proof as above ensues.
- Recall that $\mathbb{R}[x]$ is a principal ideal domain. Let $I = (f)$ be a prime ideal of $\mathbb{R}[x]$. Then f is irreducible. Thus every prime ideal of $\mathbb{R}[x]$ is of the form (f) for f an irreducible polynomial.

□

Lemma 1.4.4

Let R, S be commutative rings. Let $f_1 : R \times S \to R$ and $f_2 : R \times S \to S$ denote the projection maps. Then the map

$$f_1^* \amalg f_2^* : \text{Spec}(R) \amalg \text{Spec}(S) \to \text{Spec}(R \times S)$$

is a bijection.

Proof. The core of the proof is the fact that P is a prime ideal of $R \times S$ if and only if $P = R \times Q$ or $P = V \times S$ for either a prime ideal Q of P or a prime ideal V of S. It is clear that if Q is a prime ideal of S and V is a prime ideal of R, then $R \times Q$ and $V \times S$ are both prime ideals of $R \times S$.

So suppose that P is a prime ideal in $R \times S$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Since $P \neq R$, at least one of e_1 or e_2 is not in P. Without loss of generality assume that $e_1 \notin P$. But $e_1 e_2 = 0 \in P$ and P being prime implies that $e_2 \in P$. Since e_2 is the identity of $\{0\} \times S \cong S$, we conclude that $\{0\} \times S \subseteq P$. By the correspondence theorem, the projection map $f_1 : R \times S \to R$ gives a bijection between prime ideals of $R \times S$ that contain $\{0\} \times S$ and prime ideals of R. So $f_1(P)$ is a prime ideal of R. Thus $P = f_1(P) \times S$ which is exactly what we wanted.

Now the bijection is clear. $f_1^* \amalg f_2^*$ sends a prime ideal P of R to $P \times S$ and it sends a prime ideal Q of S to $R \times Q$. This map is surjective by the above argument. It is injective by inspection. $\square$

Theorem 1.4.5

Let R be a commutative ring. Let I be an ideal of R. Denote φ to be the inclusion preserving one-to-one bijection

$$\left\{ \substack{\text{Ideals of } R \\ \text{containing } I} \right\} \overset{1:1}{\longleftrightarrow} \{\text{Ideals of } R/I\}$$

from the correspondence theorem for rings. In other words, $\varphi(A) = A/I$. Let $J \subseteq R$ be an ideal containing I. Then the following are true.

- J is a radical ideal if and only if $\varphi(J) = J/I$ is a radical ideal.
- J is a prime ideal if and only if $\varphi(J) = J/I$ is a prime ideal.
- J is a maximal ideal if and only if $\varphi(J) = J/I$ is a maximal ideal.

Proof.

- Let J be a radical ideal. Suppose that $r + I \in \sqrt{J/I}$. This means that $(r + I)^n = r^n + I \in J/I$ for some $n \in \mathbb{N}$. But this means that $r^n \in J$. This implies that $r \in \sqrt{J} = J$. Thus $r + I \in J/I$ and we conclude that $\sqrt{J/I} \subseteq J/I$. Since we also have $J/I \subseteq \sqrt{J/I}$, we conclude.

  Now suppose that J/I is a radical ideal. Let $r \in \sqrt{J}$. This means that $r^n \in J$ for some $n \in \mathbb{N}$. Now $r^n + I = (r + I)^n \in J/I$ implies that $r + I \in \sqrt{J/I} = J/I$. Hence $r \in J$ and so $\sqrt{J} \subseteq J$. Since we also have that $J \subseteq \sqrt{J}$, we conclude.
- Let J be a prime ideal. Then R/J is an integral domain. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also an integral domain. Hence J/I is a prime ideal. The converse is also true.
- Let J be a maximal ideal. Then R/J is a field. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also a field. Hence J/I is a maximal ideal. The converse is also true.

$\square$

Another way to write the bijections is via spectra:

$$\text{Spec}(R/I) \overset{1:1}{\longleftrightarrow} \{P \in \text{Spec}(R) \mid I \subseteq P\}$$

and

$$\text{maxSpec}(R/I) \overset{1:1}{\longleftrightarrow} \{m \in \text{maxSpec}(R) \mid I \subseteq m\}$$

1.5 Extensions and Contractions of Ideals

Definition 1.5.1: Extension of Ideals

Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I be an ideal of R. Define the extension I^e of I to S to be the ideal

$$I^e = \langle f(i) \mid i \in I \rangle$$

Proposition 1.5.2

Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I, I_1, I_2 be an ideal of R. Then the following are true regarding the extension of ideals.

- Closed under sum: $(I_1 + I_2)^e = I_1^e + I_2^e$
- $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$
- Closed under products: $(I_1 I_2)^e = I_1^e I_2^e$
- $(I_1 / I_2)^e \subseteq I_1^e / I_2^e$
- $\text{rad}(I)^e \subseteq \text{rad}(I^e)$

Definition 1.5.3: Contraction of Ideals

Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let J be an ideal of S. Define the contraction J^c of J to R to be the ideal

$$J^c = f^{-1}(J)$$

Proposition 1.5.4

Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let J, J_1, J_2 be an ideal of S. Then the following are true regarding the extension of ideals.

- $(J_1 + J_2)^e \supseteq J_1^e + J_2^e$
- Closed under intersections: $(J_1 \cap J_2)^e = J_1^e \cap J_2^e$
- $(J_1 J_2)^e \supseteq J_1^e J_2^e$
- $(J_1 / J_2)^e \subseteq J_1^e / J_2^e$
- Closed under taking radicals: $\text{rad}(J)^e = \text{rad}(J^e)$

Proposition 1.5.5

Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I be an ideal of R and let J be an ideal of S. Then the following are true.

- $I \subseteq I^{ec}$
- $J^{ce} \subseteq J$
- $I^e = I^{ece}$
- $J^c = J^{cec}$

1.6 Revisiting the Polynomial Ring

Proposition 1.6.1

Let R be a commutative ring. Then we have

$$N(R[x]) = N(R)[x]$$

Proof. Let $f = \sum_{k=0}^n a_k x^k \in N(R)[x]$. Then each a_k is nilpotent in R, and there exists $n_k \in \mathbb{N}$ such that $a_k^{n_k} = 0$. This also proves that $a_k x^k$ is nilpotent. Since the sum of nilpotents is a nilpotent, we conclude that f is nilpotent.

Now suppose that $f \in N(R[x])$. We induct on the degree of f. Let $\deg(f) = 0$. Then f is nilpotent and f lies in R. Thus $f \in N(R)[x]$. Now suppose that the claim is true for $\deg(f) \leq n-1$. Let $\deg(g) = n$ with leading coefficient b_n. Since g is nilpotent in $R[x]$, there exists $m \in \mathbb{N}$ such that $g^m = 0$. Then in particular, $b_n^m = 0$ so that b_n is nilpotent. Then $b_n x^n$ is also nilpotent. Now since $N(R[x])$ is an ideal of $R[x]$, we have that $g - b_n x^n \in N(R[x])$. By inductive hypothesis, $g - b_n x^n \in N(R)[x]$. Since $N(R)$ is an ideal of R, we have that $N(R)[x]$ is an ideal of $R[x]$. So $g = (g - b_n x^n) + b_n x^n \in N(R)[x]$. Thus we are done. $\square$

Some more important results from Groups and Rings and Rings and Modules include:

- If R is an integral domain, then $R[x]$ is an integral domain.
- R is a UFD if and only if $R[x]$ is a UFD
- If F is a field, then $F[x]$ is an Euclidean domain, a PID and a UFD
- If F is a field, then the ideal generated by p is maximal if and only if p is irreducible.

Regarding ideals of the polynomial ring, the following maybe useful:

- $I[x]$ is an ideal of R
- There is an isomorphism $\frac{R[x]}{I[x]} \cong \frac{R}{I}[x]$ given by the map

$$\left(f = \sum_{k=0}^{n} a_k x^k + I[x]\right) \mapsto \left(\sum_{k=0}^{n} (a_k + I) x^k\right)$$

- If I is a prime ideal of R, then $I[x]$ is a prime ideal of $R[x]$.

2 Basic Notions of Commutative Rings

2.1 Noetherian Commutative Rings

We recall some facts about Noetherian rings. In the following, let R be a commutative ring, although they are also true if R is non-commutative if we take all modules defined below to be left (right) R-modules.

- If we have a short exact sequence of R-modules:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

  Then M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

- If M and N are R-modules, then $M \oplus N$ is Noetherian if and only if M and N are Noetherian.
- If M is an R-module and N is an R-submodule of M, then M is Noetherian if and only if N and M/N are Noetherian.
- If R is Noetherian and I is an ideal of R, then R/I is Noetherian.
- Later when once has seen localization, we can also prove that: If R is Noetherian then $S^{-1}R$ is Noetherian for any multiplicative subset S of R.

Theorem 2.1.1: Hilbert's Basis Theorem

Let R be a commutative ring. If R is Noetherian, then

$$R[x_1, \dots, x_n]$$

is a Noetherian ring.

Proposition 2.1.2

Let R be a Noetherian commutative ring. Let I be an ideal of R. Then there exists $n \in \mathbb{N}$ such that

$$\sqrt{I}^n \subseteq I \subseteq \sqrt{I}$$

Proof. It is clear that $I \subseteq \sqrt{I}$. Since R is Noetherian, $\sqrt{I}$ is finitely generated by say $x_1, \dots, x_n$. Then $x_i^{n_i} \in I$ for some $n_i \in \mathbb{N}$. Let $m = 1 + \sum_{i=1}^n (n_i - 1)$. Then $\sqrt{I}^m$ is generated by $x_1^{r_1} \cdots x_n^{r_n}$ for $\sum_{i=1}^n r_i = m$. If $r_i < n_i$ for i then

$$m = \sum_{i=1}^n r_i \leq \sum_{i=1}^n (n_i - 1) < m$$

is a contradiction. Hence there exists some i for which $r_i \geq n_i$. Thus $x_1^{r_1} \cdots x_n^{r_n} \in I$. Thus $\sqrt{I}^m \subseteq I$. $\square$

Proposition 2.1.3

Let R be a Noetherian commutative ring. Then $N(R)$ is a nilpotent ideal.

Proof. By the above, there exists $n \in \mathbb{N}$ such that $(N(R))^n = \sqrt{(0)}^n \subseteq (0) \subseteq \sqrt{(0)}$. Hence $(N(R))^n = (0)$ for some $n \in \mathbb{N}$. $\square$

2.2 Artinian Commutative Rings

We recall some facts about Artinian modules.

- If we have a short exact sequence of R-modules:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

  Then M_2 is Artinian if and only if M_1 and M_3 are Artinian.
- If M and N are R-modules, then $M \oplus N$ is Artinian if and only if M and N are Artinian.
- If M is an R-module and N is an R-submodule of M, then M is Artinian if and only if N and M/N are Artinian.

Let R be a (not necessarily commutative ring). If R is left Artinian, then the following are true.

- If I is an ideal of R, then R/I is Artinian.
- Every prime ideal of R is maximal.
- R only has finitely many maximal ideals.
- $J(R)$ is a nilpotent ideal.
- R is Noetherian.

There are also properties of Artinian rings that only commutative rings can realize.

Proposition 2.2.1

Let R be an integral domain. Then R is Artinian if and only if R is a field.

Proof. It is clear that every field is Artinian. Conversely, let R be Artinian. Consider the following descending chain of ideals in R:

$$R \supseteq (x) \supseteq (x^2) \supseteq$$

for any $0 \neq x \in R$. Since R is Artinian, the chain terminates and $(x^n) = (x^{n+1})$ for some $n \in \mathbb{N}$. Then there exists $y \in R$ such that $x^n = yx^{n+1}$. This means that $x^n(1 - yx) = 0$. Since R is an integral domain, R has no nilpotents. Hence x^n is non-zero and $1 = xy$. Thus x has an inverse so that R is a field. $\square$

Proposition 2.2.2

Let R be a commutative ring. Let R be Artinian. Then every prime ideal in R is maximal.

Proof. Let P be a prime ideal. Since quotients of Artinian rings are Artinian, R/P is Artinian. Since R/P is also an integral domain, we conclude by the above that R/P is a field. Hence P is maximal. $\square$

Recall some properties of the Jacobson radical from Rings and Modules. For a (not necessarily commutative ring R),

- $J(R/J(R)) = 0$
- $J(R) = \bigcap_{m \in \text{maxSpec}(R)} m$

Proposition 2.2.3

Let R be a commutative ring. If R is Artinian, then

$$N(R) = J(R)$$

Proof. Since every prime ideal in R is maximal, we have that

$$N(R) = \bigcap_{P \text{ a prime ideal}} P = \bigcap_{P \text{ a maximal ideal}} P = J(R)$$

and so we conclude. □

Proposition 2.2.4

Let R be a commutative ring. If R is Artinian, then R has finitely many maximal ideals.

Proof. Consider the collection

$$\{m_1 \cap \cdots \cap m_k \mid m_1, \dots, m_k \text{ are maximal ideals of } R\}$$

of R-submodules of R. Since R is Artinian, every collection of R-submodules of R has a minimal element. Hence this collection also has a minimal element, say $m_1 \cap \cdots \cap m_k$. Let m be another maximal ideal of R. Then

$$m \cap m_1 \cap \cdots \cap m_k \subseteq m_1 \cap \cdots \cap m_k$$

Since $m_1 \cap \cdots \cap m_k$ is minimal, they are equal. By prp1.1.1, we conclude that $m \supseteq m_i$ for some i. Since they are maximal, we have $m = m_i$. Hence $m_1, \dots, m_k$ gives the full list of distinct maximal ideals of R. □

2.3 Local Rings

Definition 2.3.1: Local Rings

Let R be a commutative ring. We say that R is a local ring if it has a unique maximal ideal m. In this case, we say that R/m is the residue field of R.

Example 2.3.2

Consider the following commutative rings.
- $\mathbb{Z}/6\mathbb{Z}$ is not a local ring.
- $\mathbb{Z}/8\mathbb{Z}$ is a local ring.
- $\mathbb{Z}/24\mathbb{Z}$ is not a local ring.
- $\mathbb{R}[x]$ is not a local ring.

Proof.
- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$. They do not contain each other and so they are both maximal.
- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2 + 8\mathbb{Z})$ and $(4 + 8\mathbb{Z})$. But $(2 + 8\mathbb{Z}) \supseteq (4 + 8\mathbb{Z})$. Hence $\mathbb{Z}/8\mathbb{Z}$ has a unique maximal ideal.
- A similar proof as above ensues.
- Any irreducible polynomial $f \in \mathbb{R}[x]$ is such that (f) is a maximal ideal. Indeed the evaluation homomorphism gives an isomorphism $\frac{\mathbb{R}[x]}{(f)} \cong \mathbb{R}$.

□

Proposition 2.3.3

Let R be a ring and I an ideal of R. Then I is the unique maximal ideal of R if and only if I is the set containing all non-units of R.

Proof. Let I be the unique maximal ideal of R. Clearly I does not contain any unit else $I = R$. Now suppose that r is a non-unit. Suppose that $r \notin I$. Define $J = \{sr | s \in R\}$ Clearly J is an ideal. It must be contained in some maximal ideal. Since I is the unique maximal ideal, $J \subseteq I$. But this means that $r \in I$, a contradiction. Thus every non-unit is in I.

Suppose that I contains all non-units of R. Let $r \notin I$. Then there exists $s \notin I$ such that $rs = 1$. Then $(r + I)(s + I) = 1 + I$ in R/I. This means that every element of R/I has a multiplicative inverse which means that R/I is a field and thus I is a maximal ideal. Now let $J \neq I$ be another maximal ideal. Then J contains some unit r. This implies that $J = R$ and thus I is the unique maximal ideal. □

Example 2.3.4

Let k be a field. Then the ring of power series $k[[x]]$ is a local ring.

Proof. Let M be the set of all non-units of $k[[x]]$. I first show that $f \in M$ if and only if the constant term of f is non-zero. Let g be a power series. Then the nth coefficient of $f \cdot g$ is given by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

If the constant term of f is 0, then $c_0 = 0$ and so $f \cdot g \neq 1$. Now if the constant term of f is $a_0 \neq 0$, then set $b_0 = \frac{1}{a_0}$. Now we can use the formula $0 = c_n$ to deduce

$$b_n = -\frac{\sum_{k=1}^{n} a_k b_{n-k}}{a_0}$$

This is such that $a_n \cdot b_n = 0$. Define $g = \sum_{k=0}^{\infty} b_k x^k$. Then $f \cdot g = 1$. Thus f is a unit.

By the above proposition, we conclude that M is the unique maximal ideal of $k[[x]]$. □

Proposition 2.3.5

Let R be a Noetherian commutative ring. Then the following are equivalent.
- R is an Artinian local ring.
- R has a nilpotent maximal ideal.
- R has a unique proper radical ideal.
- R has a unique prime ideal.
- $N(R)$ is a maximal ideal of R.

Proof.
- (1) $\implies$ (2): Let R be Artinian and local. By 2.1.4 we have $N(R) = J(R) = m$ since $J(R)$ is the intersection of all maximal ideals. Since R is Artinian, R is Noetherian. Then by 2.1.3 $N(R) = m$ is nilpotent.

□

Since every Artinian ring is Noetherian, the above proposition implies the following.

Corollary 2.3.6

Let R be an Artinian commutative ring. Then the following are true.
- R is local.
- $N(R)$ is the unique maximal ideal of R.

- $N(R)$ is the unique prime ideal of R.
- $N(R)$ is the unique radical ideal of R.
- $N(R)^n = (0)$ for some $n \in \mathbb{N}$.

We will discuss more of local rings in the topic of localizations.

3 Modules over a Commutative Ring

Recall from Rings and Modules that a module consists of an abelian group M and a ring R such that there is a binary operation $\cdot : R \times M \to M$ that mimic the notion of a group action:

- For $r, s \in R$, $s \cdot (r \cdot m) = (sr) \cdot m$ for all $m \in M$.
- For $1_R \in R$ the multiplicative identity, $1_R \cdot m = m$ for all $m \in M$.

When R is a commutative ring, the first axiom is relaxed so that the resulting element of M makes no difference whether you apply r first or s first. This makes module act even more similarly than fields (although one still need the notion of a basis, which appears in free modules). Therefore the first section concerns transferring techniques in linear algebra such as the Cayley Hamilton theorem to module over a ring that mimic the notion of vector spaces.

3.1 Cayley-Hamilton Theorem

Definition 3.1.1: Characteristic Polynomial

Let R be a commutative ring. Let $A \in M_{n\times n}(R)$ be a matrix. Define the characteristic polynomial of A to be the polynomial

$$c_A(x) = \det(A - xI)$$

Theorem 3.1.2: Cayley-Hamilton Theorem for Rings

Let R be a commutative ring. Let $A \in M_{n\times n}(R)$ be a matrix. Then $c_A(A) = 0$.

Theorem 3.1.3: Cayley-Hamiliton Theorem for Modules

Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R. Let $\varphi \in \text{End}_R(M)$. If $\varphi(M) \subseteq IM$, then there exists $a_1, \dots, a_n \in I$ such that

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_{n-1}\varphi + \text{id}_M = 0 : M \to M$$

Proof. Suppose that M is generated by $x_1, \dots, x_n$. There exists a surjective map $\rho : R^n \to M$ given by $(r_1, \dots, r_n) \mapsto \sum_{k=1}^n r_k x_n$. Since $\varphi(M) \subseteq IM$, we havt that

$$\varphi(x_k) = \sum_{i=1}^n r_{ki} x_i$$

for some $r_{ki} \in I$. Write A to be the matrix $A = (a_{ki})$. We now have a commutative diagram:

In other words, we have the diagram:

$$\begin{tikzcd} R^n \arrow[r, two heads, "\rho"] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^n \arrow[r, two heads, "\rho"'] & M \end{tikzcd}$$

By Cayley-Hamilton theorem, we have that $c_A(A) = 0$ is the zero function. For all $x \in R^n$, we have that

$$\begin{aligned} c_A(A)(x) &= 0 \\ c_A(Ax) &= 0 \\ \rho(c_A(Ax)) &= \rho(0) \\ c_A(\rho(Ax)) &= 0 && (\rho \text{ is } R\text{-linear}) \\ c_A(\varphi(\rho(x))) &= 0 && (\text{Diagram is commutative}) \end{aligned}$$

Since ρ is surjective, we conclude that for any $m \in M$, the above calculation gives $c_A(\varphi(m)) = 0$ so that $c_A(\varphi)$ is the zero map. □

3.2 Nakayama's Lemma

Lemma 3.2.1: Nakayama's Lemma I

Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R. If $IM = M$, then there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I$.

Proof. Choose $\varphi = \mathrm{id}_M$. Then φ is surjective so that $M = \varphi(M) \subseteq IM$. By crl 4.1.3, there exists $r_1, \dots, r_n \in I$ such that $(1 + r_1 + \cdots + r_n)M = 0$. By choosing $r = 1 + r_1 + \cdots + r_n$, we see that $rM = 0$ and $r - 1 \in I$ so that we conclude. □

Lemma 3.2.2: Nakayama's Lemma II

Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R such that $I \subseteq J(R)$ and $IM = M$. Then $M = 0$.

Proof. By Nakayama's lemma I, there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I \subseteq J(R)$. By 2.3.8, we have that $1 - (r-1)(-1) = r \in R^\times$. This means that r is invertible. Hence $rM = 0$ implies $M = r^{-1}rM = 0$. □

Corollary 3.2.3

Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R such that $I \subseteq J(R)$. Let N be an R-submodule of M. If

$$M = IM + N$$

then $M = N$.

Proof. Since quotients of finitely generated modules are finitely generated, we know that M/N is finitely generated. Define the map

$$\phi : IM + N \to I\frac{M}{N}$$

by $\phi(im + n) = i(m + N)$. This map is clearly surjective. Now I claim that $\ker(\phi) = N$. For any $im + n \in \ker(\phi)$, we see that $i(m + N) = N$ means that $im \in N$. Hence $im + n \in N$. On the other hand, if $im + n \in N$ then $im \in N$. But this means that $im + N = N$. Hence $im + n \in \ker(\phi)$. By the first isomorphism theorem for modules, we conclude that

$$\frac{M}{N} = \frac{IM + N}{N} \cong I\frac{M}{N}$$

We can now apply Nakayama's lemma II to conclude that $M/N = 0$ so that $M = N$. □

Corollary 3.2.4

Let R be a commutative ring. Let m be a maximal ideal of R. Let M be a finitely generated R-module. Then the following are true.

- M/mM is a finite dimensional vector space over R/m.
- $a_1, \dots, a_n \in M$ generates M as an R-module if and only if $a_1 + mM, \dots, a_n + mM$

generates M/mM as a R/m vector space.

Proof. For the first part, we already know that M/mM is an R-module. We notice that for any $k \in m$ and $t + mM \in M/mM$ we have that $k(t + mM) = kt + kmM$. But $kt \in m$ means that $kt + kmM = mM$. Hence M/mM is well defined as an R/m-module. Now suppose that M is finitely generated by the elements $a_1, \dots, a_n$. Let $x + mM \in M/mM$. Then there exists $r_k \in R$ such that $x = r_1 a_1 + \dots + r_n a_n$. But this means that

$$x + mM = r_1(a_1 + mM) + \dots + r_n(a_n + mM)$$

This means that M/mM is generated by $a_1 + mM, \dots, a_n + mM$. We conclude that M/mM is finite dimensional.

Suppose that $a_1, \dots, a_n \in M$ generates M as an R-module. By the same argument as above, we can see that $a_1 + mM, \dots, a_n + mM$ is a set of generators for M/mM. For the other direction, suppose that $a_1 + mM, \dots, a_n + mM$ generates M/mM as an R/m-vector space. Define $N = Ra_1 + \dots + Ra_n \leq M$. Set $I = J(R) = m$. We want to show that $M = IM + N$. It is clear that $IM + N \leq M$. If $x \in M$, then there exists $r_k \in R$ such that $x + mM = r_1(a_1 + mM) + \dots + r_n(a_n + M)$. In particular, this means that

$$x - \sum_{k=1}^{n} r_k a_k \in mM$$

Hence $x \in IM + N$. We can now apply the above corollary to deduce that $M = N = Ra_1 + \dots + Ra_n$ so that M is generated by $a_1, \dots, a_n$. And so we are done. $\square$

Proposition 3.2.5

Let (R, m) be a local ring. Let M be a finitely generated R-module. Then $a_1, \dots, a_n \in M$ is a minimal set of generators of M as an R-module if and only if $a_1 + mM, \dots, a_n + mM$ is a basis for M/mM as a R/m vector space.

Proof. Suppose that $a_1, \dots, a_n$ generate M. The above shows that $a_1 + mM, \dots, a_n + mM$ spans M/mM. So suppose for a contradiction that $a_1, \dots, a_n$ is a minimal generating set but $a_1 + mM, \dots, a_n + mM$ is not a basis for m/m^2. This means that after relabelling, $a_1 + mM, \dots, a_{n-1} + mM$ spans M/mM. By the above, this means that $a_1, \dots, a_{n-1}$ generate M. This is a contradiction of the minimality of the generating set $a_1, \dots, a_n$. Hence $a_1 + mM, \dots, a_n + mM$ is a basis for m/m^2.

Now suppose that $a_1 + mM, \dots, a_n + mM$ is a basis for m/m^2. We have seen above that $a_1, \dots, a_n$ generate M. If this is not minimal, then there is some smaller generating set $b_1, \dots, b_k$ that still generates M where $k < n$. By the above, $b_1 + mM, \dots, b_k + mM$ spans m/m^2 hence $n = \dim_{R/m}(m/m^2) \leq k$. This is a contradiction since $k < n$. Hence we are done. $\square$

3.3 Change of Rings

Definition 3.3.1: Extension of Scalars

Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Let M be an R-module. Define the extension of M to the ring S to be the S-module

$$S \otimes_R M$$

Definition 3.3.2: Restriction of Scalars

Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Let M be an S-module. Define the restriction of M to the ring R to be the R-module M equipped with the action

$$r \cdot_R m = \varphi(r) \cdot_S m$$

for all $r \in R$.

Theorem 3.3.3

Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Then there is an isomorphism

$$\text{Hom}_S(S \otimes_R M, N) \cong \text{Hom}_R(M, N)$$

for any R-module M and S-module N given as follows.

- For $f \in \text{Hom}_S(S \otimes_R M, N)$, define the map $f^+ \in \text{Hom}_R(M, N)$ by

$$f^+(m) = f(1 \otimes m)$$

- For $g \in \text{Hom}_R(M, N)$, define the map $g^- \in \text{Hom}_S(S \otimes_R M, N)$ by

$$g^-(s \otimes m) = s \cdot g(m)$$

3.4 Properties of the Hom Set

Let R be a ring. Let M, N be R-modules. Recall that in Rings and Modules that $\text{Hom}_R(M, N)$ is a $Z(R)$-modules. When R is commutative, $Z(R) = R$ so that the Hom set becomes an R-module.

Proposition 3.4.1

Let R be a commutative ring. Let M, N be R-modules. Then

$$\text{Hom}_R(M, N)$$

is an R-module with the following binary operations.

- For $\phi, \varphi : M \to N$ two R-module homomorphisms, define $\phi + \varphi : M \to N$ by $(\phi + \varphi)(m) = \phi(m) + \varphi(m)$ for all $m \in M$
- For $\phi : M \to N$ an R-module homomorphism and rR, define $r\phi : M \to N$ by $(r\phi)(m) = r \cdot \phi(m)$ for all $m \in M$.

In particular, it is an abelian group.

Proof. We first show that the addition operation gives the structure of a group.

- Since M is associative as an additive group, associativity follows
- Clearly the zero map $0 \in \text{Hom}_R(M, N)$ acts as the additive inverse since for any $\phi \in \text{Hom}_R(M, N)$, we have that $\phi(m) + 0 = 0 + \phi(m) = \phi(m)$ since 0 is the additive identity for M
- For every $\phi \in \text{Hom}_R(M, N)$, the map taking m to $-\phi(m)$ also lies in $\text{Hom}_R(M, N)$. Since $-\phi(m)$ is the inverse of $\phi(m)$ in M for each $m \in M$, we have that $-\phi$ is the inverse of ϕ

We now show that

- Let $r, s \in R$, we have that $((sr)\phi)(m) = (sr) \cdot \phi(m) = s \cdot (r \cdot \phi(m)) = s(r(\phi))(m)$ and hence we showed associativity.
- It is clear that $1_R \in R$ acts as the identity of the operation.

Thus we are done. $\square$

Proposition 3.4.2

Let R be a ring. Let I be an indexing set. Let M_i, N be R-modules for $i \in I$. Then the following are true.

- There is an isomorphism
$$\text{Hom}\left(\bigoplus_{i\in I} M_i, N\right) \cong \bigoplus_{i\in I}\text{Hom}(M_i, N)$$
- There is an isomorphism
$$\text{Hom}\left(\prod_{i\in I} M_i, N\right) \cong \prod_{i\in I}\text{Hom}(M_i, N)$$

Definition 3.4.3: Induced Map of Hom

Let R be a commutative ring. Let M_1, M_2, N be R-modules. Let $f : M_1 \to M_2$ be an R-module homomorphism. Define the induced map

$$f^* : \text{Hom}_R(M_2, N) \to \text{Hom}(M_1, N)$$

by the formula $\varphi \mapsto \varphi \circ f$

Lemma 3.4.4

Let R be a commutative ring. Let M_1, M_2, N be R-modules. Let $f : M_1 \to M_2$ be an R-module homomorphism. Then the induced map

$$f^* : \text{Hom}(M_2, N) \to \text{Hom}(M_1, N)$$

is an R-module homomorphism.

3.5 Failure of Exactness of Hom and Tensoring

Proposition 3.5.1

Let R be a commutative ring. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Let N be an R-module. Then the following two sequences

$$0 \longrightarrow \text{Hom}_R(M_3, N) \longrightarrow \text{Hom}_R(M_2, N) \longrightarrow \text{Hom}_R(M_1, N)$$

$$\text{Hom}_R(N, M_1) \longrightarrow \text{Hom}_R(N, M_2) \longrightarrow \text{Hom}_R(N, M_3) \longrightarrow 0$$

are exact.

Proof.

- We first show that g^* is injective. Let $\phi, \rho \in \text{Hom}(C, G)$ such that $g^*(\phi) = g^*(\rho)$. This means that $\phi \circ g = \rho \circ g$. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that

$g(b) = c$. Then
$$\phi(c) = \phi(g(b)) = \rho(g(b)) = \rho(c)$$
Hence $\phi = \rho$.

Now we show that $\text{im}(g^*) \subseteq \ker(f^*)$. Let $g^*(\phi) \in \text{Hom}(B, G)$ for $\phi \in \text{Hom}(C, G)$. We want to show that $f^*(g^*(\phi)) = 0$. But we have that
$$(\phi \circ g \circ f)(a) = \phi(g(f(a)) = \phi(0) = 0$$
since $\text{im}(f) = \ker(g)$. Thus we conclude.

Finally we show that $\ker(f^*) \subseteq \text{im}(g^*)$. Let $f^*(\phi) = 0$ for $\phi \in \text{Hom}(B, G)$. This means that $\phi \circ f = 0$ or in other words, $\text{im}(f) \subseteq \ker(\phi)$. Since $\phi(k) = 0$ for all $k \in \text{im}(f)$, ϕ descends to a map $\overline{\phi} : \frac{B}{\text{im}(f)} \to G$. But $\text{im}(f) = \ker(g)$ hence this is equivalent to a map $\overline{\phi} : \frac{B}{\ker(g)} \to G$. But by the first isomorphism theorem and the fact that g is surjective, we conclude that $\overline{g} : \frac{B}{\ker(g)} \overset{g}{\cong} C$, where $b + \ker(g) \mapsto g(b)$. Thus we have constructed a map $\overline{\phi} \circ \overline{g}^{-1} : C \to G$ given by $g(b) \mapsto b + \ker(g) \mapsto \phi(b)$. But now $g^*(\overline{\phi} \circ \overline{g}^{-1})$ is the map defined by
$$b \mapsto g(b) \mapsto b + \ker(g) \mapsto \phi(b)$$
and so this map is exactly ϕ. Thus $\phi \in \text{im}(g^*)$.

$\square$

Proposition 3.5.2

Let R be a commutative ring. Let the following be an exact sequence of R-modules.
$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$
Let N be an R-module. Then the following sequence
$$M_1 \otimes N \xrightarrow{f \otimes \text{id}_N} M_2 \otimes N \xrightarrow{g \otimes \text{id}_N} M_3 \otimes N \longrightarrow 0$$
is exact.

However, one can observe that we did not imply that $M_1 \otimes N \to M_2 \otimes N$ is injective. Indeed, this is because tensoring does not preserve injections.

4 Algebra Over a Commutative Ring

4.1 Commutative Algebras

Definition 4.1.1: Commutative Algebras

Let R be a commutative ring. A commutative R-algebra is an R-algebra A that is commutative.

Proposition 4.1.2

Let R be a commutative ring. Then the following are equivalent characterizations of a commutative R-algebra.
- A is a commutative R-algebra
- A is a commutative ring together with a ring homomorphism $f : R \to A$

Proof. Suppose that A is an R-algebra. Then define a map $f : R \to A$ by $f(r) = r \cdot 1$ where $r \cdot 1$ is the module operation on A. Then clearly this is a ring homomorphism.

Suppose that A is a commutative ring together with a ring homomorphism $f : R \to A$. Define an action $\cdot : R \times A \to A$ by $r \cdot a = f(r)a$. Then this action clearly allows A to be an R-module. □

Under the correspondence of associative algebra, the above proposition gives a another correspondence between the first one.

$$\left\{(A, R) \;\middle|\; \substack{A \text{ is a commutative} \\ R\text{-algebra}}\right\} \overset{1:1}{\longleftrightarrow} \left\{\phi : R \to A \;\middle|\; \substack{\phi \text{ is a ring homomorphism} \\ \text{such that } f(R) \subseteq Z(A) = A}\right\}$$

In particular, the construction above are inverses of each other so that it gives the one-to-one correspondence.

4.2 Finitely Generated Algebra

Definition 4.2.1: Finitely Generated Algebras

Let R be a commutative ring. Let A be an R-algebra. We say that A is finitely generated if there exists $a_1, \dots, a_n \in A$ such that every element $a \in A$ can be written as a polynomial in $a_1, \dots, a_n$. This means that

$$a = \sum_{i_1, \dots, i_n} r_{i_1, \dots, i_n} a_1^{i_1} \cdots a_n^{i_n}$$

Finitely generated algebras are also called algebra of finite type.

Theorem 4.2.2

Let A be a commutative algebra over a ring R. Then the following are equivalent.
- A is a finitely generated algebra over R
- There exists elements $a_1, \dots, a_n \in A$ such that the evaluation homomorphism

  $$\phi : R[x_1, \dots, x_n] \to A$$

  given by $\phi(f) = f(a_1, \dots, a_n)$ is a surjection
- There is an isomorphism

  $$A \cong \frac{R[x_1, \dots, x_n]}{I}$$

  for some ideal I

Definition 4.2.3: Finitely Presented Algebra

Let R be a ring. Let $A = R[x_1, \dots, x_n]/I$ be a finitely generated algebra over R for some ideal I. We say that A is finitely presented if I is finitely generated.

Lemma 4.2.4

Let R be a ring, considered as an algebra over $\mathbb{Z}$. If R is finitely generated over $\mathbb{Z}$, then R is finitely presented.

Proof. Trivial since $\mathbb{Z}$ is a principal ideal domain. $\square$

4.3 Finite Algebras

Definition 4.3.1: Finite Algebras

Let R be a commutative ring. Let A be an R-algebra. We say that A is finite if A is finitely generated as an R-module.

Example 4.3.2

Let R be a commutative ring. Then $R[x]$ is a finitely generated algebra over R but is not a finite R-algebra.

4.4 Zariski's Lemma

Lemma 4.4.1

Let F be a field. Let $f \in F[x]$. Then the localization $F[x]_f$ is not a field.

Theorem 4.4.2: Zariski's Lemma

Let F be a field. Let K be a field that is also a finitely generated algebra over F. Then K is a finite algebra. In particular, K is a finite dimensional vector space over F.

Corollary 4.4.3

Let F be an algebraically closed field. Let K be a field that is also a finitely generated algebra over F. Then the inclusion homomorphism $F \hookrightarrow K$ is an F-algebra isomorphism.

Corollary 4.4.4

Let F be an algebraically closed field. Then every maximal ideal of $F[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in F$.

5 Localization

5.1 Localization of a Ring

Definition 5.1.1: Multiplicative Set

Let R be a commutative ring. $S \subseteq R$ is a multiplicative set if $1 \in S$ and S is closed under multiplication: $x, y \in S$ implies $xy \in S$

Definition 5.1.2: Localization of a Ring

Let R be a commutative ring and $S \subseteq R$ be a multiplicative set. Define the ring of fractions of R with respect to S by

$$S^{-1}R = \left\{ \frac{r}{s} \,\middle|\, r \in R, s \in S \right\} / \sim$$

where we say that $r/s \sim r'/s'$ if there exists $t \in S$ such that $t(rs' - r's) = 0$.

Lemma 5.1.3

Let R be a commutative ring. Let $f \in R$ be non-zero. Then the set $\{f^n \mid n \in \mathbb{N}\}$ is a multiplicative set.

Definition 5.1.4: Localization at an Element

Let R be a commutative ring. Let $f \in R$ be non-zero. Define the localization of R at f to be the ring

$$R_f = \{f^n \mid n \in \mathbb{N}\}^{-1}R$$

It is also denoted as $R[1/f]$.

Proposition 5.1.5

Let $S^{-1}R$ be a ring of fractions.

- $\sim$ as defined in the ring of fractions is an equivalence relation
- $(S^{-1}R, +, \times)$ is a ring
- The map $k : R \to S^{-1}R$ defined by $r \mapsto r/1$ is a ring homomorphism, called the localization map.

Proof.

- Trivial
- Define addition by $\frac{r}{s} + \frac{r'}{s'} = \frac{rs'+r's}{ss'}$ and multiplication by $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. Clearly addition is abelian, and has identity $\frac{0}{1}$ and inverse $\frac{-r}{s}$ for any $\frac{r}{s} \in S^{-1}R$. Multiplication also has identity $\frac{1}{1}$.

□

Proposition 5.1.6: Universal Property

Let R be a commutative ring. Let S be a multiplicative set. Then $S^{-1}R$ and the localization map $k : R \to S^{-1}R$ satisfies the following universal property.

For any commutative ring B and ring homomorphism $\phi : R \to B$ such that $\phi(s) \in B^\times$ for all $s \in S$, there exists a unique ring homomorphism $\phi : S^{-1}R \to B$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "k"] \arrow[dr, "\phi"'] & S^{-1}R \arrow[d, dashed, "\exists!\psi"] \\ & B \end{tikzcd}$$

Moreover, $S^{-1}R$ is the unique commutative ring (up to unique isomorphism) that has such a property.

Lemma 5.1.7

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset of R. If R is Noetherian, then $S^{-1}R$ is Noetherian.

5.2 Localization Away from Prime Ideals

Lemma 5.2.1

Let R be a ring and P a prime ideal of R. Then $R \setminus P$ is a multiplicative set.

Proof. By definition, $xy \in P$ implies $x \in P$ or $y \in P$, since $R \setminus P$ removes all these elements, we have that $x \notin P$ and $y \notin P$ implies that $xy \notin P$. □

Definition 5.2.2: Localization at Prime Ideals

Let R be a commutative ring. Let P be a prime ideal. Denote

$$R_p = (R \setminus P)^{-1}R$$

the localization of R at P.

5.3 Localization of a Module

Definition 5.3.1: Localization of a Module

Let R be a commutative ring and $S \subseteq R$ be a multiplicative set Let M be a R-module. Define the ring of fractions of M with respect to S by

$$S^{-1}M = \left\{ \frac{m}{s} | m \in M, s \in S \right\} / \sim$$

where $\sim$ is defined by

$$\frac{m}{s} \sim \frac{m'}{s'} \text{ if and only if } \exists v \in S \text{ such that } v(mu' - m'u) = 0$$

If $S = \{1, f, f^2, \dots\}$ then we write

$$S^{-1}M = M_f = M[1/f]$$

Lemma 5.3.2

Let R be a commutative ring. Let M be an R-module. Let $S \subseteq R$ be a multiplicative subset. Then $S^{-1}M$ is an $S^{-1}R$-module with operation given by

$$\left(\frac{r}{s_1}, \frac{m}{s_2} \right) \mapsto \frac{r \cdot m}{s_1 s_2}$$

Definition 5.3.3: Induced Map of Localization

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Define the induced map

$$S^{-1}\phi : S^{-1}M \to S^{-1}N$$

by the formula $\frac{m}{s} \mapsto \frac{\phi(n)}{s}$.

Lemma 5.3.4

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Then the induced map

$$S^{-1}\phi : S^{-1}M \to S^{-1}N$$

is a well defined ring homomorphism.

Proposition 5.3.5

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then the following is an exact sequence of $S^{-1}R$-modules.

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0$$

Corollary 5.3.6

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M be an R-module. Then the following are true.

- If N_1, N_2 are R-submodules of M, then
$$S^{-1}(N_1 + N_2) = S^{-1}N_1 + S^{-1}N_2$$
as $S^{-1}R$-submodules of $S^{-1}M$.
- If N_1, N_2 are R-submodules of M, then
$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2$$
as $S^{-1}R$-submodules of $S^{-1}M$.
- If N is an R-submodule of M, then
$$S^{-1}\frac{M}{N} \cong \frac{S^{-1}M}{S^{-1}N}$$
as $S^{-1}R$-modules.
- If N is an R-module, then
$$S^{-1}(M \oplus N) \cong S^{-1}M \oplus S^{-1}N$$
as $S^{-1}R$-modules.

Proposition 5.3.7

Let R be a commutative ring. Let M be an R-module. Then there is an isomorphism

$$S^{-1}M \cong S^{-1}R \otimes_R M$$

of $S^{-1}R$-modules given by $\frac{m}{s} \mapsto \frac{1}{s} \otimes m$.

Lemma 5.3.8

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Then the following are true.

- Localization commutes with kernels:

$$S^{-1} \ker(\phi) \cong \ker(S^{-1}\phi)$$

- Localization commutes with images:

$$S^{-1}(\operatorname{im} \phi) \cong \operatorname{im}(S^{-1}\phi)$$

- Localization commutes with cokernels:

$$S^{-1}\frac{N}{\operatorname{im}(\phi)} \cong \frac{S^{-1}N}{\operatorname{im}(S^{-1}\phi)}$$

5.4 Localization of Integral Domains

Lemma 5.4.1

Let R be a commutative ring. Let S be a multiplicative subset of R. If R is an integral domain, then then following are true.

- The localization map $R \to S^{-1}R$ is injective.
- If $0 \notin S$, then $S^{-1}R$ is an integral domain.

Proof. Suppose that $0 = \frac{a}{s} \cdot \frac{b}{t}$. By the equivalence relation this is the same as saying that $uab = 0$ for some $u \in S$. Since R is an integral domain and $0 \neq S$, we conclude that $u \notin S$ so that $ab = 0$. Again since R is an integral domain this implies that $a = 0$ or $b = 0$. Hence either $a/s = 0$ or $b/t = 0$ in $S^{-1}R$. Hence $S^{-1}R$ is an integral domain. □

Proposition 5.4.2

Let R be an integral domain. Then the following are true.

- $\text{Frac}(R) = R_{(0)}$
- $R = \bigcap_{m \text{ a maximal ideal}} R_m$

5.5 Ideals of a Localization

Definition 5.5.1: Ideals Closed Under Division

Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. We say that I is closed under division by s if for all $s \in S$ and $a \in R$ such that $sa \in I$, we have $a \in I$.

Lemma 5.5.2

Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Then we have
$$I^e = S^{-1}I$$

Proposition 5.5.3

Let R be a commutative ring. Let S be a multiplicative subset of R. Let P be a prime ideal of R. Then the following are true.

- $S^{-1}P$ is a prime ideal of $S^{-1}R$ if and only if $S \cap P = \emptyset$.
- $S^{-1}P = S^{-1}R$ if and only if $S \cap P \neq \emptyset$.

Proof. Recall that R/P is an integral domain if P is prime. Since S^{-1} commutes with quotients, we have that
$$\frac{S^{-1}R}{S^{-1}P} \cong S^{-1}\frac{R}{P}$$
If $S \cap P = \emptyset$, then $0 \in P$ implies that $0 \notin S$. This means that $0 \notin \phi(S)$. By 5.3.1 we conclude that $S^{-1}(R/P)$ is an integral domain. Hence $S^{-1}P$ is a prime ideal. If $S \cap P \neq \emptyset$, suppose that $x \in S \cap P$. Then ????? $\square$

Theorem 5.5.4

Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Let $\phi : R \to S^{-1}R$ denote the localization map. Then there is a one-to-one bijection
$$\{J \mid J \text{ is an ideal of } S^{-1}R\} \overset{1:1}{\longleftrightarrow} \{I \mid {}^{I \text{ is an ideal of } R \text{ and}}_{I \text{ is closed under division by } S}\}$$
whose map is given by $J \mapsto J^c = \phi^{-1}(J)$ and inverse is given by $I \mapsto I^e = S^{-1}I$.

Proof. We first show that our map of sets are well defined. Let J be an ideal of $S^{-1}R$. We first show that $\phi^{-1}(J)$ is closed under division by S. Suppose that $s \in S$ and $r \in R$ such that $sr \in \phi^{-1}(J)$. Then $sr/1 \in J$. Now since J is an ideal of $S^{-1}R$, we know that $1/s \cdot sr/1 \in J$. But $1/s \cdot sr/1 = r/1 = \phi(r)$. This means that $\phi(r) \in J$ and hence $r \in \phi^{-1}(J)$. Thus $\phi^{-1}(J)$ is an ideal closed under division by S.

Now let I be an ideal of R closed under division. I claim that $S^{-1}I$ is an ideal of $S^{-1}R$. Let $a/s, b/t \in S^{-1}I$. Then $a/s + b/t = (at + bs)/st$. Since I is an ideal, we know that $at + bs \in I$. Also since S is a multiplicative subset, $st \in S$. Hence $(at + bs)/st \in I$. Now let $a/s \in S^{-1}I$ and $r/t \in S^{-1}R$. Then $(a/s) \cdot (r/t) = ar/st$. Since I is an ideal, $ar \in I$. Thus $ar/st \in S^{-1}I$ so that I is an ideal.

It remains to show that the two maps are inverses of each other. Let J be an ideal of $S^{-1}R$. We want to show that $J = S^{-1}(\phi^{-1}(J))$. Let $a/s \in J$. Since J is an ideal, we have $\phi(a) = a/1 = 1/s \cdot a/s \in J$. Hence $a \in \phi^{-1}J$ so that $a/s \in S^{-1}\phi^{-1}(J)$. Thus $J \subseteq S^{-1}(\phi^{-1}(J))$. Now by 1.5.5 the extension of the contraction of J is a subset of J. Hence we conclude.

On the other hand, we also want to show that $I = \phi^{-1}(S^{-1}I)$. Again by 1.5.5 we know that $I \subseteq \phi^{-1}(S^{-1}I)$. Conversely, let $x \in \phi^{-1}(S^{-1}I)$. Then $\phi(x) = x/1 \in S^{-1}I$. This means that $x/1 = b/t$ for some $b \in I$ and $t \in S$. Then there exists $u \in S$ such that $uxt = ub$. Since $b \in I$, $ub \in I$ hence $uxt \in I$. Since $ut \in S$ and I is closed under division, we have $x \in I$.

This shows that $S^{-1}(-)$ and $\phi^{-1}(-)$ are mutual inverses of each others. Thus we conclude. $\square$

Using the theorem we conclude that every ideal of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R such that I is closed under division by S.

Proposition 5.5.5

Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Then the above bijection restricts to the following bijection

$$\{J \mid J \text{ is a prime ideal of } S^{-1}R\} \overset{1:1}{\longleftrightarrow} \left\{I \;\middle|\; \substack{I \text{ is a prime ideal of } R \\ \text{and } I\cap S=\emptyset}\right\}$$

Proof. Let $\phi : R \to S^{-1}R$ be the localization map. From the above we know that $Q = S^{-1}\phi^{-1}(Q)$ for any prime ideal Q of $S^{-1}R$. This implies that $S^{-1}\phi^{-1}(Q)$ is prime. By 5.4.3 this implies that $\phi^{-1}(Q) \cap S = \emptyset$. Thus the map $J \mapsto \phi^{-1}(J)$ induces a well defined map on our given sets of prime ideals.

Conversely, by 5.4.3 we know that if P is a prime ideal of R such that $S \cap P = \emptyset$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$. Hence the inverse map is also well defined on our domain and codomain. By the above theorem it is already a bijection, hence we are done. □

Proposition 5.5.6

Let R be a commutative ring. Let P be a prime ideal of R. Then the above bijection gives

$$\{J \mid J \text{ is a prime ideal of } R_P\} \overset{1:1}{\longleftrightarrow} \left\{I \;\middle|\; \substack{I \text{ is a prime ideal of } R \\ \text{and } I\subseteq P}\right\}$$

Proof. Notice that the condition that $I \cap S = \emptyset$ in the above proposition translates to $I \cap (R \setminus P) = \emptyset$, which is the same as saying $I \subseteq P$. □

Proposition 5.5.7

Let R be a commutative ring and let P be a prime ideal of R. Then R_P is a local ring with unique maximal ideal given by

$$PR_P = \left\{\frac{r}{s} \mid r \in P, s \notin P\right\}$$

Proof. We show that PR_P is the only unique maximal ideal. Suppose that I is an ideal in R_P such that I is not a subset of PR_P. Then there exists $a/s \in I$ such that $a \notin P$ and $s \notin P$. It is clear that s/a is then an element of R_P. So a/s is invertible. Hence $I = R_P$. □

Be wary that in general localizations does not result in a local ring. This happens only when we are localizing with respect to a prime ideal. The importance of prime ideals is not explicit in the above because only using prime ideals P can $R \setminus P$ be a multiplicative set which ultimately allows localization to make sense.

Proposition 5.5.8: Localization of a Localization

Let R be a commutative ring. Let S be a multiplicative subset of R. Let P be a prime ideal of R such that $S^{-1}P$ is a prime ideal of $S^{-1}R$. Then

$$(S^{-1}R)_{S^{-1}P} \cong R_P$$

Proof. Define a map $S^{-1}R \to R_P$ by the identity map. This is well defined because if $s \in S$, then we know $S^{-1}P$ is a prime ideal implies $S \cap P = \emptyset$, so $s \notin P$. Thus r/s is a well defined fraction in R_P. Since it is just the identity map, it is a well defined ring homomorphism. Now let $r/s \in S^{-1}R \setminus S^{-1}P$. Then $r \notin P$ implies that r is invertible in R_P. Hence $r/s \cdot s/r = 1$ in R_P. Thus r/s is invertible in R_P. Thus we can invoke the universal property to obtain a unique map

$$(S^{-1}R)_{S^{-1}P} \to R_P$$

Conversely, define a map $R \to (S^{-1}R)_{S^{-1}P}$ by the identity map $r \mapsto (r/1)/(1/1)$. This is well defined because $1 \notin P$ implies $1/1 \in S^{-1}R \setminus S^{-1}P$. Clearly this is a well defined ring homomorphism. For $s \in S$, notice that $(s/1)/(1/1)$ is invertible in $(S^{-1}R)_{S^{-1}P}$ via the element $(1/s)/(1/1)$. Thus we can invoke the universal property of $S^{-1}R$ to obtain a unique map

$$S^{-1}R \to (S^{-1}R)_{S^{-1}P}$$

We now have two unique maps going both directions between $S^{-1}R$ and $(S^{-1}R)_{S^{-1}P}$. This implies that they are isomorphic. □

5.6 Localization of Graded Rings

Proposition 5.6.1

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Then R_P is a graded ring in which the grading structure is given as follows: $f/g \in R_P$ has degree $\deg(f) - \deg(g)$.

Definition 5.6.2: Localization of a Graded Ring

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Define the localization of R with respect to P to be

$$(R_P)_0 = \{f \in R_P \mid f \text{ lies in the 0th graded component of } R_P\}$$

Proposition 5.6.3

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Then $(R_P)_0$ is a local ring with unique maximal ideal given by

$$(PR_P) \cap (R_P)_0$$

5.7 Local Properties

Definition 5.7.1: Local Properties of Modules

Let R be a commutative ring. A property of R-modules is local if for any R-modules M, the following are equivalent.

- M has the property
- M_P has the property for all primes ideals P
- M_m has the property for all maximal ideals m

Proposition 5.7.2: Injectivity and Surjectivity are Local Properties

Let R be a commutative ring. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Let S be a multiplicative subset of R. Then the following are equivalent.

- ϕ is injective (surjective)
- For each prime ideal P of R, the induced map $\phi_P : S^{-1}M \to S^{-1}N$ is injective (surjective)
- For each maximal ideal m of R, the induced map $\phi_m : S^{-1}M \to S^{-1}N$ is injective (surjective)

More local properties: zero, nilpotent
Non-local properties: freeness, domain

Proposition 5.7.3: Exactness is Local

Let R be a commutative ring. Let M_1, M_2, M_3 be R-modules. Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be R-module homomorphisms. Then the following conditions are equivalent.

- The following sequence is exact:
$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$
- The following sequence is exact:
$$0 \longrightarrow (M_1)_P \xrightarrow{f_P} (M_2)_P \xrightarrow{g_P} (M_3)_P \longrightarrow 0$$
for all prime ideals P of R.
- The following sequence is exact:
$$0 \longrightarrow (M_1)_m \xrightarrow{f_m} (M_2)_m \xrightarrow{g_m} (M_3)_m \longrightarrow 0$$
for all maximal ideals m of R.

Definition 5.7.4: Local Properties of Elements

A property of an element of M is local if the following is true. $m \in M$ has the property if and only if $m \in M_P$ has the property.

6 Primary Decomposition

6.1 The Annihilator and the Support of a Module

Let R be a commutative ring. Let M be an R-module. Recall that we define the annihilator of a subset $S \subseteq M$ to be the ideal

$$\text{Ann}_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$$

When R is a commutative ring, the annihilator is a two sided ideal and consequently has some nice properties.

Proposition 6.1.1

Let R be a commutative ring. Let M be an R-module. Let $\text{Ann}_R(x)$ for $x \in M$ be a maximal element in the set

$$\{\text{Ann}_R(x) \mid 0 \neq x \in M\}$$

Then $\text{Ann}_R(x)$ is a prime ideal.

Proof. Suppose that $ab \in \text{Ann}_R(x)$ and $b \notin \text{Ann}_R(x)$. Notice that if $rx = 0$ then $r(bx) = brx = 0$ so that r annihilates bx. Hence $\text{Ann}_R(x) \subseteq \text{Ann}_R(bx)$. Since x is non-zero and $b \notin I$, bx is also non-zero hence $\text{Ann}_R(bx)$ lies in the given set of annihilators. Since $\text{Ann}_R(x)$ is maximal we conclude that

$$\text{Ann}_R(x) = \text{Ann}_R(bx)$$

But ab annihilates x by definition so that a annihilates bx. Hence $a \in \text{Ann}_R(bx) = \text{Ann}_R(x)$. Hence $\text{Ann}_R(x)$ is prime. $\square$

Recall that if $S \subseteq M$ is a subset and R is not a commutative ring, then in general we only have the relation

$$\text{Ann}_R(\langle S \rangle) \subseteq \text{Ann}_R(S)$$

Proposition 6.1.2

Let R be a commutative ring. Let M be an R-module. Let $S \subseteq M$ be a subset. Then

$$\text{Ann}_R(\langle S \rangle) = \text{Ann}_R(S)$$

Definition 6.1.3: Support of a Module

Let A be a commutative ring. Let M be an A-module. The support of M is the subset

$$\text{Supp}(M) = \{P \text{ a prime ideal of } A \mid M_P \neq 0\}$$

Let R be a commutative ring. Let M be an R-module. Recall that the annihilator of an element $m \in M$ is the ideal

$$\text{Ann}_R(m) = \{r \in R \mid r \cdot m = 0\}$$

Moreover, we define

$$\text{Ann}_R(M) = \{r \in R \mid r \cdot m = 0 \text{ for all } m \in M\} = \bigcap_{m \in M} \text{Ann}_R(m)$$

Proposition 6.1.4

Let R be a commutative ring. Let M be an R-module. Then

$$\{P \in \text{Spec}(R) \mid \text{Ann}_R(M) \subseteq P\} = \text{Supp}(M)$$

We can write the set on the left as a vanishing set so the proposition can be read as

$$\mathbb{V}(\text{Ann}_R(M)) = \text{Supp}(M)$$

6.2 Associated Prime

Definition 6.2.1: Associated Prime

Let R be a commutative ring. Let M be an R-module. Let P be a prime ideal of R. We say that P is an associated prime of M if

$$\text{Ann}_R(m) = P$$

for some $m \in M$.

Definition 6.2.2: Set of Associated Prime

Let R be a commutative ring. Let M be an R-module. Define the set of associated primes of M to be

$$\text{Ass}(M) = \{P \in \text{Spec}(R) \mid P \text{ is an associated prime of } M\}$$

Proposition 6.2.3

Let R be a commutative ring. Let M be an R-module. Then

$$\text{Ass}(M) \subseteq \text{Supp}(M)$$

Proposition 6.2.4

Let R be a commutative ring. Let M be an R-module. Then the following are true.

- $\text{Ass}(M)$ is a finite set.
- For $P \in \text{Ass}(M)$, $\text{Ann}_R(M) \subseteq P$.
- We have

$$\text{Ass}(M) = \{P \in \text{Spec}(R) \mid \text{ For any prime ideal } Q \subseteq P, Q \text{ does not contain } \text{Ann}_R(M)\}$$

Proof.

-
- We have seen that every $P \in \text{Supp}(M)$ is such that $\text{Ann}_R(M) \subseteq P$. Since $\text{Ass}(M) \subseteq \text{Supp}(M)$, we are done.

□

Proposition 6.2.5

Let R be a commutative ring. Let M be an R-module. Then

$$\bigcup_{P \in \text{Ass}(M)} P = \{m \in M \mid m \text{ is a zero divisor of } M\} \cup \{0\}$$

Theorem 6.2.6: Disassembly of an R-Module

Let R be a Noetherian commutative ring. Let M be a finitely generated R-module. Then there exists a chain of R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that

$$\frac{M_{i+1}}{M_i} \cong \frac{R}{P_i}$$

for some prime ideal P_i of R.

6.3 Primary Ideals

Definition 6.3.1: Primary Ideals

Let R be a commutative ring. Let Q be a proper ideal of R. We say that Q is a primary ideal of R if $fg \in Q$ implies $f \in Q$ or $g^m \in Q$ for some $m > 0$.

Proposition 6.3.2

Let R be a commutative ring. Let Q be a proper ideal of R. Then Q is primary if and only if every zero divisor in R/Q is nilpotent.

Lemma 6.3.3

Let R be a commutative ring. Let P be a prime ideal of R. Then P is a primary ideal.

Lemma 6.3.4

Let R be a commutative ring. Let Q be a primary ideal of R. Then the following are true.
- $\sqrt{Q}$ is a prime ideal.
- $\sqrt{Q}$ is minimal among primes that contain Q.

Definition 6.3.5: P-Primary Ideals

Let R be a commutative ring. Let P be a prime ideal. Let Q be an ideal. We say that Q is a P-primary ideal of R if the following are true.
- Q is a primary ideal.
- $Q = \sqrt{P}$.

Proposition 6.3.6

Let R be a commutative ring. Let I be an ideal of R. If $\sqrt{I}$ is maximal, then I is an $\sqrt{I}$-primary ideal.

Proposition 6.3.7

Let R be a Noetherian commutative ring. Let P be a prime ideal of R. Let Q be a proper ideal. Then the following are equivalent.
- Q is P-primary.
- $\text{Ann}(A/Q) = \{P\}$
- There exists $n \in \mathbb{N}$ such that $P^n \subseteq Q \subseteq P$.

6.4 Primary Decomposition

We want to express ideal I in R as $I = P_1^{e_1} \cdots P_n^{e_n}$ similar to a factorization of natural numbers, for some prime ideals $P_1, \dots, P_n$. However this notion fails and thus we have the following new type of ideal.

Definition 6.4.1: Primary Decompositions

Let A be a commutative ring. Let I be an ideal of A. A primary decomposition I consists of primary ideals $Q_1,\dots,Q_r$ of A such that

$$I=Q_1\cap\cdots\cap Q_r$$

Definition 6.4.2: Minimal Primary Decompositions

Let A be a commutative ring. Let I be an ideal of A. Let

$$I=Q_1\cap\cdots\cap Q_r$$

be a primary decomposition of I. We say that the decomposition is minimal if the following are true.

- Each $\sqrt{Q_i}$ are distinct for $1\leq i\leq r$
- Removing a primary ideal changes the intersection. This means that for any i, $I\neq\bigcap_{j\neq i}Q_j$

Lemma 6.4.3

Let $\phi:R\to S$ be a ring homomorphism and Q be a primary ideal in S. Then $\phi^{-1}(Q)$ is primary in R.

6.5 The Noetherian Case

Theorem 6.5.1

Let R be a Noetherian commutative ring. Let I be a proper ideal of R. Then I admits a primary decomposition.

Proposition 6.5.2

Let R be a Noetherian commutative ring. Let m be maximal ideal of R. Let I be an ideal of R. Then the following are equivalent.

- I is m-primary.
- $\sqrt{I}=m$.
- There exists $n\in\mathbb{N}$ such that $m^n\subseteq I\subseteq m$.

7 Integral Dependence

7.1 Integral Elements

Definition 7.1.1: Integral Elements

Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b \in B$. We say that b is integral over A if there exists a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$ such that $p(b) = 0$.

When A and B are field, this is a familiar notion in Field and Galois theory.

Lemma 7.1.2

Let K be a field. Let $F \subseteq K$ be a subfield. Let $k \in K$. Then k is integral over F if and only if k is algebraic over F.

Proposition 7.1.3

Let B be a commutative ring and let $A \subseteq B$. Let $b \in B$. Then the following are equivalent.

- b is integral over A
- $A[b] \subseteq B$ is finitely generated A-submodule.
- There exists an A sub-algebra $A' \subseteq B$ such that $A[b] \subseteq A'$ and A' is finitely generated as an A-module.

Proposition 7.1.4

Let $A \subseteq B$ be commutative rings. Then B is a finitely generated A-module if and only if $B = A[x_1, \dots, x_n]$ for some $x_1, \dots, x_n \in B$ that is integral over A.

Proposition 7.1.5

Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b_1, b_2 \in B$ be integral over A. Then $b_1 + b_2$ and $b_1 b_2$ are both integral over A.

7.2 Integral Closure

Definition 7.2.1: Integral Closure

Let B be a commutative ring. Let $A \subseteq B$ be a subring. Define the subring

$$\overline{A} = \{b \in B \mid b \text{ is integral over } A\}$$

to be the integral closure of A in B.

Proposition 7.2.2

Let B be a commutative ring. Let $A \subseteq B$ be a subring. Let S be a multiplicatively closed subset of A. Then

$$\overline{S^{-1}A} = S^{-1}\overline{A}$$

Definition 7.2.3: Integral Extensions

Let B be a commutative ring and let $A \subseteq B$ be a subring. We say that B is integral over A if $\overline{A} = B$. We also say that B is the integral extension of A.

Lemma 7.2.4

Let $A \subseteq B \subseteq C$ be commutative rings. Then C is integral over B and B is integral over A if and only if C is integral over A.

Proposition 7.2.5

Let A, B be commutative rings such that $A \subset B$ is an integral extension. Let J be an ideal of B. Then $\frac{B}{J}$ is integral over $\frac{A}{J \cap A}$.

Proposition 7.2.6

Let A, B be commutative rings such that $A \subset B$ is an integral extension. Let S be a multiplicative subset of B. Then $S^{-1}B$ is integral over $S^{-1}A$.

Lemma 7.2.7

Let A, B be integral domains such that $A \subset B$ is an integral extension. Then A is a field if and only if B is a field.

Definition 7.2.8: Integrally Closed

Let B be a commutative ring. Let $A \subseteq B$ be a subring. We say that A is integrally closed in B if $\overline{A} = A$.

7.3 The Going-Up and Going-Down Theorems

We want to compare prime ideals between integral extensions.

Proposition 7.3.1

Let A, B be rings such that $A \subset B$ is an integral extension. Let Q be a prime ideal of B. Then $Q \cap A$ is a maximal ideal of A if and only if Q is a maximal ideal of B.

Proposition 7.3.2

Let A, B be rings such that $A \subset B$ is an integral extension. Let P be a prime ideal of A. Then the following are true.

- There exists a prime ideal Q of B such that $P = Q \cap A$
- If Q_1, Q_2 are prime ideals of B such that $Q_1 \cap A = P = Q_2 \cap B$ and $Q_1 \subseteq Q_2$, then $Q_1 = Q_2$.

Theorem 7.3.3: The Going-Up Theorem

Let A, B be rings such that $A \subset B$ is an integral extension. Let $0 \leq m < n$. Consider the following situation

$$\begin{array}{llll} B & Q_1 \subseteq \cdots \subseteq Q_m & & \text{(Prime ideals of } B\text{)} \\ \big\uparrow & & & \\ A & P_1 \subseteq \cdots \subseteq P_m & \subseteq P_{m+1} \subseteq \cdots \subseteq P_n & \text{(Prime ideals of } A\text{)} \end{array}$$

where $Q_i \cap A = P_i$ for $1 \leq i \leq m$. Then there exists prime ideals $Q_{m+1}, \dots, Q_n$ of B such that the following are true.

- $Q_{m+1} \subseteq \cdots \subseteq Q_n$
- $Q_i \cap A = P_i$ for $m + 1 \leq i \leq n$

7.4 Normal Domains

We now concern ourselves with integral domains. Let R be an integral domain. A special fact about R is that the canonical homomorphism $R \to R_{(0)} = \text{Frac}(R)$ is an injection. This means that we can we can think of R as living inside of $\text{Frac}(R)$ while preserving all the structure of R.

Definition 7.4.1: Normal Domains

Let R be an integral domain. We say that R is normal if R is integrally closed in $\text{Frac}(R)$.

Proposition 7.4.2

Let R be a normal domain. Let S be a multiplicative subset of R. Then $S^{-1}R$ is a normal domain.

Proof. We want to show that $S^{-1}R$ is integrally closed in $\text{Frac}(R) = \text{Frac}(S^{-1}R)$. This means that we want to show $\overline{S^{-1}R} = S^{-1}R$. It is clear that $S^{-1}R \subseteq \overline{S^{-1}R}$. So let $g \in \overline{S^{-1}R}$. Suppose that $p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in (S^{-1}R)[x]$ such that $p(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for $0 \leq i \leq n-1$. Then notice that $sg \in S^{-1}R$ satisfies the monic polynomial

$$q(x) = x^n + \sum_{k=0}^{n-1} s^{n-k} a_k x^k$$

since $q(sg) = s^n g^n + \sum_{k=0}^{n-1} s^n a_k x^k = s^n p(g) = 0$. But q is a polynomial in R since $s^{n-k} a_k \in R$. Thus we have that $sg \in \overline{R} = R$ since R is normal. This means that $g \in S^{-1}R$ and hence we conclude. ∎

Proposition 7.4.3

Let R be a commutative ring. If R is a UFD, then R is normal.

Proposition 7.4.4: Normal is a Local Property

Let R be an integral domain. Then the following are equivalent.

- R is normal
- R_P is normal for all prime ideals P
- R_m is normal for all maximal ideals m.

Proof. Notice that an integral domain R is normal if and only if the canonical inclusion map $R \hookrightarrow \overline{R}$ is surjective. Since surjectivity is a local property, this map is surjective if and only if for all prime ideals P of R, $R_P \hookrightarrow \overline{R_P}$ is surjective. But $\overline{R}_P = \overline{R_P}$ by the above. Hence $R \hookrightarrow \overline{R}$ is surjective if and only if $R_P \to \overline{R_P}$ is surjective. Hence R is normal if and only if R_P is normal for all prime ideals P of R. The similar holds for all maximal ideals. ∎

Atiyah-Macdonald

Proposition 7.4.5

Let R be a normal domain. Then $R[x]$ is a normal domain.

Proposition 7.4.6

Let R be a normal domain. Let $K/\text{Frac}(R)$ be an algebraic extension. Let $f \in K$. Then f is integral over R if and only if the minimal polynomial $\min(K, f) \in R[x]$.

8 Introduction to Dimension Theory for Rings

8.1 Krull Dimension

Definition 8.1.1: Krull Dimension

Let R be a commutative ring. Define the Krull dimension of R to be

$$\dim(R) = \max\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t \text{ for } p_0, \dots, p_t \text{ prime ideals}\}$$

In particular, notice that a commutative ring R has $\dim(R) = 0$ if and only if every prime ideal is maximal.

Lemma 8.1.2

Let R, S be commutative rings such that $R \subseteq S$ is an integral extension. Then $\dim(R) = \dim(S)$.

Proposition 8.1.3

Let F be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Then the following are true.

- $\dim(F[x_1, \dots, x_n]) = n$.
- Every maximal chain prime ideals in $F[x_1, \dots, x_n]$ is of length n.

Lemma 8.1.4

Let R be a commutative ring. Then the following are true.

- If R is a field, then $\dim(R) = 0$
- If R is Artinian, then $\dim(R) = 0$

Proof. Let R be a field. Then the only proper prime ideal of R is (0). In particular, (0) forms the only chain of prime ideals in R. Hence $\dim(R) = 0$.

Now let R be Artinian. Let P be a prime ideal of R. Then R/P is an integral domain. Moreover, every quotient of an Artinian ring is Artinian. Hence R/P is Artinian. By prp1.3.1, we conclude that R/P is a field. Hence P is a maximal ideal. Any chain of prime ideals of R must terminate at the first prime ideal since it is maximal. Hence $\dim(R) = 0$. □

8.2 Height of Prime Ideals

Definition 8.2.1: Height of a Prime Ideal

Let R be a commutative ring. Let p be a prime ideal of R. Define the height of p to be

$$\text{ht}(p) = \max\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t = p \text{ for } p_0, \dots, p_t \text{ prime ideals }\}$$

Lemma 8.2.2

Let R be a commutative ring. Then

$$\dim(R) = \max\{\text{ht}(P) \mid P \in \text{Spec}(R)\}$$

Lemma 8.2.3

Let R be a commutative ring. Let P be a prime ideal of R. Then

$$\text{ht}(P) = \dim(R_P)$$

Proof. Let $\dim(R_P) = n$. Then there exists a strict chain of prime ideals of R_P of length n (and no chain of prime ideals of length $> n$). By prp5.4.6, prime ideals of R_P are in bijection with prime ideals of R that P contains. Hence the maximal chain of prime ideals of length n correspond to a chain of prime ideals in R that contain P, of length n. Hence $\dim(R_p) = n \leq \text{ht}(P)$. Conversely, let $m = \text{ht}(P)$. Then there exists a strict chain of prime ideals that are subsets of P, that are of length m. By the same correspondence, the chain of prime ideals correspond to a chain of prime ideals in R_P of length m. Hence $\text{ht}(P) = m \leq \dim(R_P)$.

The two inequalities combine to show that $\dim(R_P) = \text{ht}(P)$. ☐

Lemma 8.2.4

Let R be a commutative ring. Let P be a prime ideal of R. Then

$$\dim(R) \geq \dim(R/P) + \text{ht}_R(P)$$

Proposition 8.2.5

Let k be a field. Let A be an integral domain that is a finitely generated k-algebra. Then the following are true.

- $\dim(A) = \text{trdeg}_k(\text{Frac}(A))$
- For any prime ideal P of A, we have

$$\dim(A) = \dim(A/P) + \text{ht}_A(P)$$

Proposition 8.2.6: Dimension is a Local Concept

Let R be a commutative ring. Then the following numbers are equal.

- The Krull dimension $\dim(R)$
- The supremum $\sup\{\dim(R_m) \mid m \text{ is a maximal ideal of } R\}$
- The supremum $\sup\{\text{ht}_R(m) \mid m \text{ is a maximal ideal of } R\}$

Corollary 8.2.7

Let (R, m) be a local ring. Then

$$\dim(R) = \dim(R_m) = \text{ht}_R(m)$$

Theorem 8.2.8: Krull's Principal Ideal Theorem

Let R be a Noetherian ring. Let I be a proper and principal ideal of R. Let p be the smallest prime ideal containing I. Then

$$\text{ht}_R(p) \leq 1$$

8.3 The Length of Modules over Commutative Rings

Let R be a ring. Recall that the length of an R-module M is defined to be the supremum

$$l_R(M) = \sup\{n \in \mathbb{N} \mid 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M\}$$

Lemma 8.3.1

Let (A, m) be a local ring and let M be an A-module. If $mM = 0$, then

$$l_A(M) = \dim_{A/m}(M)$$

Proposition 8.3.2

Let R be a commutative ring and let M be an R-module. Then the following are equivalent.

- M is simple
- $l_R(M) = 1$
- $M \cong R/m$ for some maximal ideal m of R

8.4 Structure Theorem for Artinian Rings

Let R be a ring. Let M be an R-module. Recall that a composition series for M is a sequence of R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that $\frac{M_{i+1}}{M_i}$ is a simple R-module for $1 \leq i < k$.

Proposition 8.4.1

Let $R \neq 0$ be a commutative ring. Then R is Artinian if and only if R is Noetherian and $\dim(R) = 0$.

Proof. Let R be Artinian. In Rings and Modules, the Akizuki-Hopkins-Levitzki theorem proves that R is Noetherian. Moreover, lmm8.1.4 shows that $\dim(R) = 0$.

Now let R be Notherian and $\dim(R) = 0$. This means that every prime ideal of R is maximal. Let S be the set of all ideals of R that admit a composition series. I claim that S is non-empty. Let $T = \{\text{Ann}(x) \mid 0 \neq x \in R\}$. Clearly T is non-empty. Let $Y_1 \subseteq Y_2 \subseteq \cdots$ be a chain in T. Since R is Noetherian, the chain terminates at finitely many sets with $Y = \text{Ann}(x) \subseteq R$ for some $x \in R$. I claim that Y is a prime ideal. By definition $R = \text{Ann}(0) \notin T$ hence $R \notin T$. This means that $Y \neq R$. Let $ab \in Y = \text{Ann}(x)$. Suppose that $b \notin Y$. We know that $abx = 0$ so $a \in \text{Ann}(bx)$. Since $bx \neq 0$, we have $\text{Ann}(bx) \in T$. Since R is commutative, we also have that $\text{Ann}(x) \subseteq \text{Ann}(bx)$. Since $\text{Ann}(x)$ is maximal, we have that $\text{Ann}(x) = \text{Ann}(bx)$. Hence $a \in \text{Ann}(x)$. Thus $\text{Ann}(x)$ is prime. Since $\dim(R) = 0$ we have $\text{Ann}(x)$ is a maximal ideal. $R/\text{Ann}(x)$ is a field (and hence a simple R-module). The multiplication map $r \mapsto rx$ has kernel $\text{Ann}(x)$. Hence the induced map $R/\text{Ann}(x) \to R$ is injective, and we can consider $R/\text{Ann}(x)$ as a subring of R. Together with the fact that it is a simple R-module makes it an R-submodule with composition series length of 1. Hence S is non-empty.

Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain in S. Since R is Noetherian, the chain terminates with some ideal $I \in S$. If $I = R$, then R has a composition series. If $I \neq R$, then R/I is non-zero. Choose a prime ideal P of R such that $I \subseteq P \neq R$ (this always exists since we can choose maximal ideals). Then we have $0 \neq R/P \subseteq R/I$. Let $p : R \to R/I$ be the projection map. Let $T = p^{-1}(R/P)$. Then we have that $N \subset T \subseteq M$ and $T/N \cong R/P$. Since $\dim(R) = 0$, P is maximal hence R/P is a field (and a simple R-module). This proves that $T \in S$. But this contradicts the maximality of N. Hence $N = R \in T$. Thus R has a composition series. From Rings and Modules we know that this implies R is Noetherian. Hence we conclude. $\square$

Recall from Rings and Modules that we have seen that Artinian rings have finitely many maximal ideals.

Theorem 8.4.2: Structure Theorem for Commutative Artinian Rings

Let R be an Artinian commutative ring. Then R decomposes into a direct product of Artinian local rings

$$R \cong \bigoplus_{i=1}^{k} R_i$$

Moreover, the decomposition is unique up to reordering of the direct product.

Proof. Let $m_1, \dots, m_k$ be the full list of distinct maximal ideals of R. Then

$$\prod_{i=1}^{k} m_i^n = 0$$

for some $n \in \mathbb{N} \setminus \{0\}$. The ideals m_i^n and m_j^n are pairwise coprime for $i \neq j$. Hence by the Chinese Remainder Theorem we obtain ring isomorphisms

$$\begin{aligned} R &\cong \frac{R}{0} \\ &\cong \frac{R}{\prod_{i=1}^{k} m_i^n} \\ &\cong \frac{R}{\bigcap_{i=1}^{k} m_i^n} && (m_i^n \text{ and } m_j^n \text{ pairwise coprime}) \\ &\cong \bigoplus_{i=1}^{k} \frac{R}{m_i^n} && (\text{CRT}) \end{aligned}$$

By the correspondence of maximal ideals, R/m_i^n has a unique maximal ideal m_i/m_i^n. Hence it is local. Also since R is Artinian, R/m_i^n is Artinian. Thus we are done. $\square$

9 Valuation and Valuation Rings

9.1 Valuation Rings

Definition 9.1.1: Valuation Rings

Let R be an integral domain. We say that R is a valuation ring if for all $x \in \text{Frac}(R)$ and $x \neq 0$, then either x or x^{-1} is in R.

Lemma 9.1.2

Let R be a valuation ring. Then the following are true.
- R is a local ring.
- R is normal.

Proof. Let R be a valuation ring. The set of units of R are precisely $S = \{x \in \text{Frac}(R) \mid x \in R \text{ and } x^{-1} \in R\}$. Let $m = R \setminus S$. Let $x \in m$ and $r \in R$. Then rx is not a unit because if $arx = 1$, then $ar \in R$ is an inverse of x, which is a contradiction since $x \in S$. Hence $rx \in R$.

Let $x, y \in m$. If one of them are zero then their sum lies in m. If both are not zero, then xy^{-1} is an element of $\text{Frac}(R)$. Since R is a valuation ring, either xy^{-1} or yx^{-1} is in R. In either case, we have
$$x + y = y(y^{-1}x + 1) = x(1 + x^{-1}y) \in m$$
(one factor is in m and the other in R). Hence m is an ideal. By prp2.1.3 we conclude that R is a local ring with unique maximal ideal m.

Let $x \in \text{Frac}(R)$ be integral over R. Then
$$x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 = 0$$
for some $r_0, \dots, r_{n-1} \in R$. If $x \in R$ then we are done. If $x \notin R$ then since R is a valuation ring, $x^{-1} \in R$. Then
$$x = -(r_1 + r_2x^{-1} + \cdots + r_nx^{1-n}) \in R$$
so that R is normal. □

9.2 Valuations on a Field

Definition 9.2.1: Totally Ordered Group

Let G be an abelian group. We say that G is a totally ordered group if there is a total order "$\leq$" on G such that $a \leq b$ implies $ca \leq cb$ for all $a, b, c \in G$.

Definition 9.2.2: Valuation on a Field

Let K be a field. Let G be a totally ordered abelian group. A valuation on K with values in G is a group homomorphism $v : K^\times \to G$ such that for all $x, y \in K^*$, we have
- $v(xy) = v(x) + v(y)$
- $v(x + y) \geq \min\{v(x), v(y)\}$

We use the convention that $v(0) = \infty$.

Definition 9.2.3: Associated Valuation Ring

Let K be a field and $v : K \to \mathbb{Z}$ a discrete valuation. Define the associated valuation ring of

K to be the subring

$$R_v = \{x \in K \mid v(x) \geq 0\}$$

Lemma 9.2.4

Let K be a field. Let v be a discrete valuation on K. Then R_v is a valuation ring.

9.3 Discrete Valuations and Normalizations

Definition 9.3.1: Discrete Valuations

Let K be a field. A discrete valuation on K is a valuation $v : K^\times \to \mathbb{Z}$.

Definition 9.3.2: Normalized Discrete Valuations

Let (K, v) be a discrete valuation ring. We say that it is normalized if v is surjective.

Lemma 9.3.3

Let K be a field with a discrete valuation v. Then $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$.

Lemma 9.3.4: Normalization of a Discrete Valuation

Let K be a field with a discrete valuation v such that $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Define the normalization of v to be the valuation $v_N : K^\times \to \mathbb{Z}$ defined by

$$v_N(k) = \frac{1}{n}v(k)$$

for all $k \in K^\times$.

Therefore we always work on normalized discrete valuation rings.

Definition 9.3.5: Discrete Valuation Rings

Let R be a commutative ring. We say that R is a discrete valuation ring if there exists a field K and a discrete valuation v on K such that

$$R = R_v$$

is the associated valuation ring of K.

Proposition 9.3.6

Let R be a discrete valuation ring with valuation v. Let $t \in R$ be such that $v(t) = 1$. Then the following are true.

- A nonzero element $u \in R$ is a unit if and only if $v(u) = 0$
- $\dim(R) = 1$

Proof.

- Let R be a discrete valuation ring. Suppose that $x \in R$ is a unit. Then $v(x^{-1}) = -v(x)$. Then $-v(x), v(x) \geq 0$ implies $v(x) = 0$. Now if $v(y) > 0$, suppose for contradiction that $u \in R$ is an inverse of y, then

  $$0 = v(1) = v(uy) = v(u) + v(y)$$

  But $v(y) > 0$ implies that $v(u) < 0$ which implies that $u \notin R$, a contradiction.

□

9.4 Uniformizing Parameters

Definition 9.4.1: Uniformizing Parameter

Let R be a discrete valuation ring with valuation v. A uniformizing parameter for R is an element $t \in R$ such that $v(t) = 1$.

Proposition 9.4.2

Let R be a discrete valuation ring with valuation v. Let $t \in R$ be a uniformizing parameter of R. Then the following are true.

- Every non-zero ideal of R is a principal ideal of the form (t^n) for some $n \geq 0$
- Every $r \in R \setminus \{0\}$ can be written in the form $r = ut^n$ for some unit u and $n \geq 0$.

Proof.

- Let $t \in R$ such that $v(t) = 1$. Let $x \in m$ where $v(x) = n > 0$. Then $v(x) = nv(t) = v(t^n)$ means that every $x \in m$ is of the form t^n. Thus $m = (t)$. Since every ideal I is a subset of this maximal ideal, any ideal is of the form $I = (t^n)$ for some $n > 0$.
- Follows from the fact that (t^n) is the unique maximal ideal.

□

Proposition 9.4.3

Let R be a discrete valuation ring. Let t be a uniformizing parameter of R. Let u be a unit of R. Then

$$v(ut^n) = n$$

for all $n \in \mathbb{N}$.

9.5 Recognizing Discrete Valuation Rings

The rest of the section devotes efforts to recognizing discrete valuation rings.

Proposition 9.5.1

Let R be a valuation ring. Then the following are equivalent.

- R is a discrete valuation ring.
- R is a principal ideal domain.
- R is Noetherian.

Proposition 9.5.2: Equivalent Characterizations of DVRs I

Let R be an integral domain. Then the following are equivalent.

- R is a discrete valuation ring
- R is Noetherian, local, $\dim(R) = 1$ and normal.
- R is local, a PID and not a field.
- R is a UFD with a unique irreducible element up to multiplication of a unit

Proof.

- (1) $\implies$ (3): We have seen that the set of non-units is precisely the set $m = \{x \in K | v(x) > 0\}$. We show that this is an ideal. Clearly $x, y \in m$ implies $v(x + y) = \min\{v(x), v(y)\} > 0$. Let $u \in R$. Then $v(ux) = v(u) + v(x) > 0$ since $v(x) > 0$

and $v(u) \geq 0$.

We have seen that every ideal is of the form (t^n) for some $n > 0$. Thus every ascending chains of ideal must be of the form

$$(t^{n_1}) \subset (t^{n_2}) \subset \dots$$

for $n_1 > n_2 > \dots$. Since $n_1, n_2, \dots$ is strictly decreasing, the chain must eventually stabilizes. This proves that R is Noetherian and has principal maximal ideal.

- (1) $\implies$ (3):

□

Proposition 9.5.3: Equivalent Characterizations of DVRs II

Let R be an integral domain that is Noetherian and local with unique maximal ideal m. Then the following are equivalent.

- R is a discrete valuation ring.
- $\dim(R) = 1$ and R is normal.
- $\dim(R) = 1$ and $\dim_{R/m}(m/m^2) = 1$ (R is a regular local ring)
- R is not a field and m is principal.
- $I = m^k$ for all non-zero ideals I of R
- There exists $t \in R$ and $k > 0$ such that $I = (t^k)$ for all non-zero ideal I of R

Proof. □

Proposition 9.5.4

Let R be a Noetherian integral domain and $\dim(R) = 1$. Then R is normal if and only if R_m is a discrete valuation ring for all maximal ideals m.

In summary, if R is a discrete valuation ring, then R has the following properties.

- R is integrally closed and in particular is normal.
- R is a PID and in particular is a UFD and an integral domain.
- R is Noetherian and local
- R has Krull dimension 1.
- $\dim_{R/m}(m/m^2) = 1$ (these are called regular local rings as we will see in Commutative Algebra 2)
- Every ideal I of R is equal to the power m^k of the maximal ideal m. In particular if m is generated by the uniformizing parameter t, then $= I = (t^k)$ in this case.
- Such a t is an irreducible element (that is unique up to multiplication by a unit), and every element of R can be written as ut^n for u a unit and $n \in \mathbb{N}$.

There is a simple diagram of relationships between DVRs and some other standard types of commutative rings.

$$\text{DVRs} \subset \text{PIDs} \subset \text{UFDs} \subset \text{Normal Domains} \subset \text{Integral Domains}$$

10 Dedekind Domains

10.1 Fractional Ideals

Definition 10.1.1: Fractional Ideal

Let R be an integral domain. Let I be a R-submodule of Frac(R). We say that I is a fractional ideal of R if there exists $r \in R \setminus \{0\}$ such that $rI \subseteq R$.

While I is not exactly an ideal of R, we can think of it as if it were an ideal because it is isomorphic to an actual ideal of R.

Lemma 10.1.2

Let R be an integral domain. Let I be a fractional ideal of R where $rI \subseteq R$ for some $r \in R \setminus \{0\}$. Then there is an R-module isomorphism

$$I \cong rI \subseteq R$$

given by $i \mapsto ri$.

Proof. I claim that there is an R-module isomorphism $I \cong rI$ for $rI \subseteq R$ given by $i \mapsto ri$. The kernel of this R-module homomorphism is given by $\{i \in I \mid ri = 0\}$. But $ri = 0$ if and only if $r = 0$ or $i = 0$. Since $r \neq 0$ we must have $i = 0$ so that the kernel is trivial. Moreover, this R-module homomorphism is surjective since for any $k \in rI$ it can be written as $k = ri$ for some i. Then $i \in I$ maps to ri under the morphism. Hence $I \cong rI$ as R-modules. □

Lemma 10.1.3

Let R be an integral domain. Let I be a fractional ideal of R. If R is Noetherian, then I is finitely generated.

Proof. Let R be Noetherian. Since I is isomorphic to rI for some non-zero $r \in R$, and rI is an ideal of R, R being Noetherian implies that rI is finitely generated and hence I is finitely generated. □

10.2 Invertible Ideals

Definition 10.2.1: Invertible Ideals

Let R be an integral domain. Let I be an R-submodule of Frac(R). We say that I is invertible if there exists an ideal J of R such that $JI = R$.

Lemma 10.2.2

Let R be an integral domain. Let I be an R-submodule of Frac(R). Then I is invertible if and only if $I^{-1}I = R$ where we define

$$I^{-1} = \{s \in \text{Frac}(R) \mid sI \subseteq R\}$$

Proposition 10.2.3

Let R be an integral domain. Let I be an R-submodule of Frac(R). Then the following are true.

- If I is a non-zero principal ideal of R, then I is invertible.
- If I is invertible, then I is fractional.

Proposition 10.2.4

Let R be an integral domain. Let I be a fractional ideal. Then I is invertible if and only if I is finitely generated, and for any maximal ideal m of R, IR_m is a principal ideal of R_m.

Proposition 10.2.5

Let R be an integral domain. Let P be a non-zero prime ideal of R. If R is Noetherian and P is invertible, then R_P is a discrete valuation ring.

Proof. Let R be a Noetherian integral domain and P a non-zero invertible prime ideal. We know that PR_P is the unique maximal ideal of the local ring R_P. By the above prp, PR_P is a principal ideal. Thus R_P is now a Noetherian local ring with principal maximal ideal. By prp10.4.6 in Commutative Algebra 1, we conclude that R_P is a discrete valuation ring. $\square$

10.3 Dedekind Domains

Definition 10.3.1: Dedekind Domains

Let R be an integral domain. We say that R is a dedekind domain if every non-zero ideal can be expressed uniquely as a direct product of finitely many prime ideals of R.

Dedekind sought for an integral domain whose ideals can be factorized uniquely as a product of primes.

Proposition 10.3.2

Let R be an integral domain that is not a field. Then the following are equivalent.

- R is a Dedekind domain.
- Every non-zero fractional ideal I of R is invertible ($I^{-1}I = R$).
- R is Noetherian, $\dim(R) = 1$ and normal
- R is Noetherian, $\dim(R) = 1$ and for any non-zero maximal ideal m of R, R_m is a discrete valuation ring.
- R is Noetherian, $\dim(R) = 1$ and every primary ideal in R is a prime power.

Proof.

- (2) $\implies$ (3): Let I be an ideal of R. Since I is invertible, by 1.1.5 we conclude that I is finitely generated. Hence R is Noetherian. Let P be a prime ideal of R. By assumption, P is invertible. prp1.2.5 implies that R_P is a DVR. In particular, it is integrally closed and $\dim(R_P) = 1$. This means that $\text{ht}_R(P) = 1$. Thus R is either a field or $\dim(R) = 1$. By assumption R is not a field. Hence $\dim(R) = 1$. We know that $R = \bigcap_{m \text{ a maximal ideal}} R_m$. Since prime ideals are maximal ideals in one dimensional rings, we can rewrite the intersection as

$$R = \bigcap_{P \text{ a prime ideal}} R_P$$

But each R_P is a DVR. Hence R is a DVR and we conclude that R is normal.
- (3) $\implies$ (2): m be a maximal ideal of R. We have seen from Commutative Algebra 1 that R_m is a Noetherian local ring. By 7.4.2 in Commutative Algebra 1 we also conclude that R_m is normal. By 9.3.2 of Commutative Algebra 1 we know that $\dim(R_m) = \text{ht}_R(m) = 1$. By 10.4.6 of Commutative Algebra 1, R_m is a DVR and in particular m is a principal ideal.

  Let I be a fractional ideal of R. We know by 1.1.3 that I is finitely generated. Since R_m is a normal Noetherian local ring of dimension 1, the ideal I_m of R_m must be principal. By 1.1.5 we conclude that I is invertible.

- $(3) \implies (4)$:
- $(4) \implies (3)$: Let m be a maximal ideal of R. We know that R_m is a DVR. In particular, it is a normal domain.

□

By virtue of the fourth item, we can think of Dedekind domains as a patching up of local discrete valuation rings.

Proposition 10.3.3

Let R be a Dedekind domain. Let I and J be ideals of R whose prime factorization is given by
$$I = P_1^{a_1} \times \cdots \times P_n^{a_n} \quad \text{and} \quad J = P_1^{b_1} \times \cdots \times P_n^{b_n}$$
for $P_1, \dots, P_n$ distinct prime ideals of R. Then the following are true.
- $I + J = P_1^{\min\{a_1,b_1\}} \times \cdots \times P_n^{\min\{a_n,b_n\}}$
- $I \cap J = P_1^{\max\{a_1,b_1\}} \times \cdots \times P_n^{\max\{a_n,b_n\}}$
- $IJ = P_1^{a_1+b_1} \times \cdots \times P_n^{a_n+b_n}$

Proposition 10.3.4

Let R be a Dedekind domain. Let I be an ideal of R. Then the following are true.
- For any $a \in I$, there exists $b \in R$ such that $I = (a, b)$.
- I is can be finitely generated by two elements.

We summarize the relation between Dedekind domains and other types of domains in the following diagram:

$$\begin{tikzcd}
 & & \text{Dedekind Domains} \arrow[dr, phantom, sloped, "\subset"] & & \\
\text{DVRs} \arrow[r, phantom, "\subset"] & \text{PIDs} \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & \text{Normal Domains} \arrow[r, phantom, "\subset"] & \text{Integral Domains} \\
 & & \text{UFDs} \arrow[ur, phantom, sloped, "\subset"] & &
\end{tikzcd}$$

In particular, DVRs, PIDs and Dedekind domains are 1-dimensional. Moreover, notice that the only difference between DVRs and Dedekind domains is that DVRs are local rings. They both share the fact that they are Noetherian, $\dim(R) = 1$ and normal.