Higher Algebra

Labix

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Abstract

Contents

1 Pushouts and Pullbacks

1.1 Pointed Infinity Categories

Upshot: stable infinity categories are the infinity categorical version of stable model categories. Protypical example: category of spectra.

Definition 1.1.1: Zero Objects

Let $\mathcal{C}$ be an infinity category. A zero object of $\mathcal{C}$ is an object 0 of $\mathcal{C}$ such that 0 is both initial and final. We say that $\mathcal{C}$ is pointed if it contains a zero object.

Lemma 1.1.2

Let $\mathcal{C}$ be an infinity category. The zero object of $\mathcal{C}$ is unique up to equivalence if it exists.

Proof. Let 0 and $0'$ be two zero objects of $\mathcal{C}$. Then they are both final objects of $\mathcal{C}$. But final objects are unique up to equivalence. Hence 0 and $0'$ are equivalent. □

Lemma 1.1.3

Let $\mathcal{C}$ be an infinity category. Then $\mathcal{C}$ is pointed if and only if the following are true.

- $\mathcal{C}$ has an initial object $\emptyset$
- $\mathcal{C}$ has a final object $*$
- There exists a morphism $* \to \emptyset$ in $\mathcal{C}$

1.2 Fibers and Cofiber Sequences

Definition 1.2.1: Triangles

Let $\mathcal{C}$ be a pointed infinity category. A triangle in $\mathcal{C}$ consists of a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\exists!"'] & Y \arrow[d, "g"] \\ 0 \arrow[r, "\exists!"'] & Z \end{tikzcd}$$

where X, Y, Z are objects and f, g are morphisms.

Definition 1.2.2: Fiber and Cofiber Sequences

Let $\mathcal{C}$ be a pointed infinity category.

- A triangle in $\mathcal{C}$ is called a fiber sequence if it is a pullback square
- A triangle in $\mathcal{C}$ is called a cofiber sequence if it is a pushout square.

Upshot: Fiber Sequences $\subseteq$ Pullbacks and Cofiber Sequences $\subseteq$ Pushouts therefore we want to talk about pushouts and pullbacks.

1.3 Excisive and Reduced Functors

Definition 1.3.1: Excisive Functors

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Suppose that $\mathcal{C}$ admits all pushouts. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is excisive if F sends pushout squares to pullback squares.

Upshot: Cofiber sequences are sent to fiber sequences under excisive functors.

Definition 1.3.2: Reduced Functors

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Suppose that $*$ is the final object of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is reduced if $F(*)$ is the final object of $\mathcal{D}$.

Definition 1.3.3: Full Subcategory of Excisive and Reduced Functors

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Suppose that $\mathcal{C}$ admits all pushouts and admits a final object $*$. Define
$$\text{Exc}_*(\mathcal{C}, \mathcal{D}) \subseteq \text{Hom}_{\mathcal{C}_\infty}(\mathcal{C}, \mathcal{D})$$
to be the full sub infinity category of $\text{Hom}_{\mathcal{C}_\infty}(\mathcal{C}, \mathcal{D})$ consisting of excisive functors and reduced functors.

1.4 Suspension and Loop Functors

Definition 1.4.1: Subcategory of Pushout Squares

Let $\mathcal{C}$ be a pointed infinity category. Define M^Σ to be the full sub infinity category of $\text{Func}_{\infty-\mathbf{Cat}}(\Delta^1 \times \Delta^1, \mathcal{C})$ consisting of pushout squares of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Definition 1.4.2: Projection Maps

Let $\mathcal{C}$ be a pointed infinity category. Define the following two projection maps.

- $\text{proj}_1 : M^\Sigma \to \mathcal{C}$ is the functor defined by sending an object
$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$
in M^Σ to X.
- $\text{proj}_2 : M^\Sigma \to \mathcal{C}$ is the functor defined by sending an object
$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$
in M^Σ to Y.

Lemma 1.4.3

Let $\mathcal{C}$ be a pointed infinity category. If every morphism in $\mathcal{C}$ admits cofibers, then the evaluation functor $\text{proj}_1 : M^\Sigma \to \mathcal{C}$ is a trivial fibration.

Recall that every trivial fibration admits a section.

Definition 1.4.4: The Suspension Functor

Let $\mathcal{C}$ be a pointed infinity category such that every morphism of $\mathcal{C}$ admits cofibers. Let $s_1 : \mathcal{C} \to M^\Sigma$ be a section of the trivial fibration $\text{proj}_1 : M^\Sigma \to \mathcal{C}$. Define the suspension functor

$\Sigma : \mathcal{C} \to \mathcal{C}$ to be the composite
$$\Sigma : \mathcal{C} \overset{s_1}{\to} M^\Sigma \overset{\text{proj}_2}{\to} \mathcal{C}$$
where $\text{proj}_2 : M^\Sigma \to \mathcal{C}$ is the projection

Upshot: For every object X, there is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

for some Y an object of $\mathcal{C}$. We define this Y to be precisely the suspension. Indeed, classically the homotopy pushout of the diagram $* \leftarrow X \to *$ is a suspension.

Definition 1.4.5: Subcategory of Pullback Squares

Let $\mathcal{C}$ be a pointed infinity category. Define M^Ω to be the full sub infinity category of $\text{Func}_{\infty-\textbf{Cat}}(\Delta^1 \times \Delta^1, \mathcal{C})$ consisting of pullback squares of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Definition 1.4.6: Projection Maps

Let $\mathcal{C}$ be a pointed infinity category. Define the following two projection maps.

- $\text{proj}_1 : M^\Omega \to \mathcal{C}$ is the functor defined by sending an object
$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$
in M^Σ to X.
- $\text{proj}_2 : M^\Omega \to \mathcal{C}$ is the functor defined by sending an object
$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$
in M^Ω to Y.

Lemma 1.4.7

Let $\mathcal{C}$ be a pointed infinity category. If morphism in $\mathcal{C}$ admits fibers, then the evaluation functor $\text{proj}_2 : M^\Omega \to \mathcal{C}$ is a trivial fibration.

Recall that every trivial fibration admits a section.

Definition 1.4.8: The Loop Functor

Let $\mathcal{C}$ be a pointed infinity category such that every morphism in $\mathcal{C}$ admits fibers. Let $s_2 : \mathcal{C} \to M^\Omega$ be a section of the trivial fibration $\text{proj}_2 : M^\Omega \to \mathcal{C}$. Define the suspension functor $\Omega : \mathcal{C} \to \mathcal{C}$ to be the composite
$$\Omega : \mathcal{C} \overset{s_2}{\to} M^\Omega \overset{\text{proj}_1}{\to} \mathcal{C}$$

where $\mathrm{proj}_2 : M^\Omega \to \mathcal{C}$ is the projection

Proposition 1.4.9

Let $\mathcal{C}$ be a pointed infinity category. Then there is an adjunction given by $\Sigma : \mathcal{C} \rightleftarrows \mathcal{C} : \Omega$.

Proposition 1.4.10

Let $\mathcal{C}, \mathcal{D}$ be pointed infinity categories. Suppose that $\mathcal{C}$ admits all finite colimits and $\mathcal{D}$ admits all finite limits. Then the following are equivalent.

- F is reduced and excisive.
- F is reduced and it satisfies the following. For all $X \in \mathcal{C}$, the comparison map

  $$\eta_X : F(X) \to \Omega_{\mathcal{D}}(F(\Sigma_{\mathcal{C}} X))$$

  is an equivalence in $\mathcal{D}$, where η_X is given as follows. For any $X \in \mathcal{C}$, the diagram

  $$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma_{\mathcal{C}} X \end{tikzcd}$$

  is a pushout in $\mathcal{C}$. Sending the diagram through F gives the wanted comparison map η_X by the universal property of limits.

Proof. Let F be reduced and excisive. Notice that following diagram on the left

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma_{\mathcal{C}} X \end{tikzcd} \quad \overset{F}{\Longrightarrow} \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma_{\mathcal{C}} X) \end{tikzcd}$$

is a pushout diagram in $\mathcal{C}$. Applying F gives a pullback diagram on the right. On the other hand, we know that

$$\begin{tikzcd} \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & F(\Sigma_{\mathcal{C}} X) \end{tikzcd}$$

is a pullback diagram. Since limits in infinity category are unique up to equivalence, the comparison map $F(X) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} X)$ is an equivalence.

Now suppose that F satisfies the second conditions. Let

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

be a pushout square. Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] & \\
Y \arrow[r] \arrow[d] & X \coprod_W Y \arrow[r] \arrow[d] & 0 \coprod_W Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X \coprod_W 0 \arrow[r] \arrow[d] & \Sigma_{\mathcal{C}} W \arrow[r] \arrow[d] & \Sigma_{\mathcal{C}} Y \arrow[d] \\
 & 0 \arrow[r] & \Sigma_{\mathcal{C}} X \arrow[r] & \Sigma_{\mathcal{C}}(X \coprod_W Y)
\end{tikzcd}$$

Label the small squares 1 to 7 from left to right and top to bottom. By definition, 1 is a pushout diagram. Since $1+2$ is a pushout diagram, by the pasting law 2 is a diagram. Similarly, 3 is a pushout diagram. Now $1+2+3+4$ is a pushout by definition. Since $1+3$ is a pushout, $2+4$ is a pushout diagram. Since 2 is a pushout diagram, 4 is a pushout diagram. Now $2+4+6$ is a pushout diagram by definition. Since $2+4$ is a pushout, 6 is a pushout. Similarly,$3+4+5$ is a pushout. Since $3+4$ is a pushout then so is 5. Finally, $4+5+6+7$ is a pushout diagram. Since $4+6$ is a pushout, so is $5+7$. Since 5 is a pushout, then 7 is a pushout. This proves that all squares 1 to 7 are pushouts. By applying F, we obtain the following diagram:

$$\begin{tikzcd}
F(W) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
F(Y) \arrow[r] \arrow[d] & F(X \coprod_W Y) \arrow[r] \arrow[d] & F(0 \coprod_W Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & F(X \coprod_W 0) \arrow[r] \arrow[d] & F(\Sigma_{\mathcal{C}} W) \arrow[r] \arrow[d] & F(\Sigma_{\mathcal{C}} Y) \arrow[d] \\
 & 0 \arrow[r] & F(\Sigma_{\mathcal{C}} X) \arrow[r] & F(\Sigma_{\mathcal{C}}(X \coprod_W Y))
\end{tikzcd}$$

Since Z is equivalent to $X \coprod_W Y$, we can remove the top left object and replace it with the pullback so that we obtain a commutative diagram:

$$\begin{tikzcd}
F(W) \arrow[dr, "\mu"] \arrow[drr, bend left] \arrow[ddr, bend right] & & & & \\
 & F(X) \times_{F(Z)} F(Y) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & F(Y) \arrow[r] \arrow[d] & F(Z) \arrow[r] \arrow[d] & F(0 \coprod_W Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & 0 \arrow[r] & F(X \coprod_W 0) \arrow[r] \arrow[d] & F(\Sigma_{\mathcal{C}} W) \arrow[r] \arrow[d] & F(\Sigma_{\mathcal{C}} Y) \arrow[d] \\
 & & 0 \arrow[r] & F(\Sigma_{\mathcal{C}} X) \arrow[r] & F(\Sigma_{\mathcal{C}} Z)
\end{tikzcd}$$

By considering the large square on the left, we obtain a comparison map $F(X) \times_{F(Z)} F(Y) \to \Omega_{\mathcal{D}} F(\Sigma_{\mathcal{C}} W)$ which will be called θ. Let μ be the comparison map $F(W) \to F(X) \times_{F(Z)} F(Y)$. Finally, notice that the following diagram on the left sits in the bottom right of the above square, and that we can add 0s to the map so that the limits of the two diagrams remain equivalent (coinitial):

$$\begin{tikzcd}
 & & 0 \arrow[d] \arrow[ddl] \\
 & & F(\Sigma_{\mathcal{C}}Y) \arrow[d] \\
0 \arrow[r] \arrow[urr] & F(\Sigma_{\mathcal{C}}X) \arrow[r] & F(\Sigma_{\mathcal{C}}Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \arrow[r] \arrow[ddl] & 0 \arrow[ddl] & 0 \arrow[l] \arrow[ddl] \\
0 \arrow[r] \arrow[dr] & 0 \arrow[dr] & 0 \arrow[l] \arrow[dr] & & \\
 & F(\Sigma_{\mathcal{C}}X) \arrow[r] & F(\Sigma_{\mathcal{C}}Z) & F(\Sigma_{\mathcal{C}}Y) \arrow[l] &
\end{tikzcd}$$

Their limit is precisely computed vertically on each slice and hence is the pullback

$$\Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}X) \times_{\Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}Z)} \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}Y)$$

Since all the diagrams map to each other, we obtain a commutative diagram:

$$\begin{tikzcd}
F(W) \arrow[r, "\mu"] \arrow[dr, "\simeq"'] & F(X) \times_{F(Z)} F(Y) \arrow[d, "\theta"] \arrow[dr, "\simeq"] & \\
 & \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}W) \arrow[r] & \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}X) \times_{\Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}Z)} \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}Y)
\end{tikzcd}$$

where by assumption we have equivalences $F(W) \simeq \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}W)$ hence there are also equivalences on pullbacks. Notice that this implies θ has a left and right homotopy inverse, and hence is an equivalence. By the two out of three property, μ is also an equivalence. Hence F sends pushouts to pullbacks. □

Proposition 1.4.11

Let $\mathcal{C}$ be a pointed infinity category that admits all finite limits and colimits. Then the following are true.

- If the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is fully faithful, then every pushout square in $\mathcal{C}$ is a pullback square in $\mathcal{C}$.
- If the loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ is fully faithful, then every pullback square in $\mathcal{C}$ is a pushout square in $\mathcal{C}$.

Proof. Let Σ be a fully faithful functor. Then the comparison map $X \to \Omega\Sigma X$ is an equivalence (Why ????). By the above proposition, the identity map is reduced and excisive. Hence given a pushout diagram, it is also a pullback square. The other statement follows from the dual. □

2 Stable Infinity Categories

2.1 Properties of Stable Infinity Categories

Definition 2.1.1: Stable Infinity Categories

Let $\mathcal{C}$ be an infinity category. We say that $\mathcal{C}$ is stable if the following are true.

- $\mathcal{C}$ has a zero object 0
- Every morphism in $\mathcal{C}$ admits a fiber and a cofiber
- A triangle in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence

Motivation: In the category of spectra, a square is a homotopy pushout if and only if it is a homotopy pullback.

Proposition 2.1.2

Let $\mathcal{C}$ be an infinity category. Then $\mathcal{C}$ is a stable infinity category if and only if the following are true.

- $\mathcal{C}$ has a zero object 0
- $\mathcal{C}$ admits all finite limits and colimits
- A square in $\mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pushout if and only if it is a pullback.

Proof. Suppose that $\mathcal{C}$ satisfies the three conditions. It is clear that $\mathcal{C}$ has a zero object. Since fiber and cofiber sequences are special cases of pullbacks and pushouts respectively, the fact that $\mathcal{C}$ admits all finite limits and colimits imply that every morphism in $\mathcal{C}$ admits a fiber and a cofiber. Finally, since pushouts diagrams and pullback diagrams coincide, fiber and cofiber sequences also coincide.

Suppose now that $\mathcal{C}$ is stable. It is clear that $\mathcal{C}$ has a zero object. □

Proposition 2.1.3

Let $\mathcal{C}$ be a stable infinity category. Then the following are true regarding the stability of suspension and looping.

- $M^\Sigma = M^\Omega$
- There is an equivalence of infinity categories given by

$$\Sigma : \mathcal{C} \leftrightarrow \mathcal{C} : \Omega$$

Proof. Let $\mathcal{C}$ be stable. By the above prp, pushouts coincide with pullbacks. Hence $M^\Sigma = M^\Omega$. Let $X \in \mathcal{C}$. Then

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

is a pushout in $\mathcal{C}$. Then it is also a pullback in $\mathcal{C}$ hence applying Ω to ΣX shows that $\Omega\Sigma X$ is equivalent to X. Similarly, let $Y \in \mathcal{C}$. Then

$$\begin{tikzcd} \Omega Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is a pullback and a pushout in $\mathcal{C}$. Applying Σ to ΩY shows that $\Sigma\Omega Y$ is equivalent to Y. This shows that Σ is a homotopy inverse of Ω and vice versa. □

Proposition 2.1.4

Let $\mathcal{C}$ be an infinity category. Then $\mathcal{C}$ is a stable infinity category if and only if the following are true.

- $\mathcal{C}$ has a zero object 0
- $\mathcal{C}$ admits all finite limits and colimits
- The loop functor $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence of infinity categories.

Proof. Let $\mathcal{C}$ be stable. By prp2.1.2, $\mathcal{C}$ has a zero object and admits all finite limits and colimits. By prp2.1.3, Ω is an equivalence of infinity categories hence we are done.

Conversely, suppose that the above conditions are satisfied. Since Σ is adjoint to Ω, this means that Σ is also an equivalence of infinity categories. In particular, Σ and Ω are both fully faithful. By prp1.4.10, pushout squares are the same as pullback squares and vice versa. By the above prp, we conclude that $\mathcal{C}$ is stable. □

Definition 2.1.5: Distinguished Triangles

Let $\mathcal{C}$ be a stable infinity category. Let the following

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_{\mathcal{C}} X$$

be a diagram in $h(\mathcal{C})$. We say that it is a distinguished triangle if there exists a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r, "\widetilde{f}"] \arrow[d] & Y \arrow[r] \arrow[d, "\widetilde{g}"] & 0 \arrow[d] \\ 0' \arrow[r] & Z \arrow[r, "\widetilde{h}"] & W \end{tikzcd}$$

in $\mathcal{C}$ such that the following are true.

- 0 and $0'$ are zero objects of $\mathcal{C}$.
- Both the left and the right squares are pushout diagrams.
- The morphisms $\widetilde{f}$ and $\widetilde{g}$ in $\mathcal{C}$ represents f and g in $h(\mathcal{C})$ respectively.
- $h : Z \to \Sigma_{\mathcal{C}} X$ is the composition of the homotopy class $\widetilde{h}$ with the equivalence $W \simeq \Sigma_{\mathcal{C}} X$ determined by the outer rectangle that is a pushout.

Proposition 2.1.6

Let $\mathcal{C}$ be a stable infinity category. Then $h(\mathcal{C})$ is a triangulated category with the following data.

- The shift functor is given by $\Sigma_{\mathcal{C}}$
- The class of distinguished triangles are given as the above.

We give a summary of some of the properties of stable infinity categories.

- $\mathcal{C}$ is pointed, admits all fibers and cofibers and fibers and cofibers coincide (def2.1.1)
- $\mathcal{C}$ admits all finite limits and colimits (prp2.1.2)

- Pullbacks and pushouts in $\mathcal{C}$ coincide (prp2.1.2)
- Σ and Ω gives an equivalence of categories (prp2.1.3)
- $h(\mathcal{C})$ is triangulated (prp2.1.6)

2.2 Exact Functors and its Equivalent Criteria

Definition 2.2.1: Exact Functors

Let $\mathcal{C}, \mathcal{D}$ be two stable infinity categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is exact if the following are true.

- The zero object is preserved: $F(0) = 0$
- F sends fiber sequences to fiber sequences.

Definition 2.2.2: Left and Right Exact Sequences

Let $\mathcal{C}, \mathcal{D}$ be two infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

- F is left exact if F commutes with finite limits. Explicitly, this means that for all finite simplicial sets K and functors $X : K \to \mathcal{C}$,

$$F\left(\lim_K X\right) = \lim_K F \circ X$$

- F is right exact if F commutes with finite colimits. Explicitly, this means that for all finite simplicial sets K and functors $X : K \to \mathcal{C}$,

$$F\left(\operatorname*{colim}_K X\right) = \operatorname*{colim}_K F \circ X$$

Proposition 2.2.3

Let $\mathcal{C}, \mathcal{D}$ be two stable infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then the following are equivalent.

- F is exact.
- F is left exact.
- F is right exact.
- F is reduced and excisive
- F sends the zero object to the zero object, and for all $X \in \mathcal{C}$ the map

$$\Sigma_{\mathcal{D}} F(X) \to F(\Sigma_{\mathcal{C}} X)$$

is an equivalence in $\mathcal{D}$.

Proof.

- $(1) \implies (2)$: Suppose that F is exact. ????
- $(2) \implies (1)$: Suppose that F is left exact. Then the zero object 0 of $\mathcal{C}$ is a final object. But the final object is the limit of the empty diagram. Since F commutes with finite limits, $F(0)$ is a final object of $\mathcal{D}$. But the zero object of $\mathcal{D}$ is also a final object of $\mathcal{D}$. Since final objects are unique up to equivalence, we conclude that $F(0)$ is the zero object of $\mathcal{D}$. Given a fiber sequence, it is in particular a pullback, which is a finite limit. Since F preserves finite limits, F sends the fiber sequence to a pullback. But $F(0) = 0$ hence the pullback is a fiber sequence. Hence F sends fiber sequences to fiber sequences.
- $(1) \implies (3)$ and $(3) \implies (1)$ has a dual argument.

- (1) $\implies$ (4): Suppose that F is exact. By the above we know that F commutes with finite limits. Given a pushout diagram in $\mathcal{C}$, since $\mathcal{C}$ is stable by prp2.1.3 we know that it is a pullback. Since F commutes with finite limits, F sends the pullback to a pullback. Hence F is excisive. Let $*$ be a final object of $\mathcal{C}$. Since F is left exact and $*$ is the limit of the empty diagram, $F(*)$ is a final object of $\mathcal{D}$. Hence F is reduced.

- (4) $\implies$ (1): Suppose that F is reduced and excisive. Let 0 be a zero object of $\mathcal{C}$. Since F is reduced, $F(0)$ is a final object of $\mathcal{D}$. But the zero object of $\mathcal{D}$ is also a final object of $\mathcal{D}$. Since final objects are unique up to equivalence, we conclude that $F(0)$ is the zero object of $\mathcal{D}$. Given a fiber sequence in $\mathcal{C}$, by definition it is also a cofiber sequence in $\mathcal{C}$. In particular it is a pushout. Since F is excisive, the pushout diagram is sent to a pullback. Moreover, since $F(0) = 0$, the pullback diagram is in fact a fiber sequence. Hence F sends fiber sequences to fiber sequences.

- (4) $\implies$ (5): Let F be reduced and exicisive. Clearly it sends zero objects to zero objects. By prp1.4.10 this implies that there is an equivalence $F(X) \to \Omega_{\mathcal{D}}(F(\Sigma_{\mathcal{C}}X))$. Passing to $\Sigma_{\mathcal{D}}$ gives an equivalence $\Sigma_{\mathcal{D}}F(X) \to \Sigma_{\mathcal{D}}\Omega_{\mathcal{D}}F\Sigma_{\mathcal{C}}X$. Since $\Sigma_{\mathcal{D}}$ and $\Omega_{\mathcal{D}}$ is an equivalence of infinity categories by prp2.1.3, this implies that $\Sigma_{\mathcal{D}}\Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}X)$ and $F(\Sigma_{\mathcal{C}}X)$ are equivalent. Hence we obtain an equivalence $\Sigma_{\mathcal{D}}F(X) \to F(\Sigma_{\mathcal{C}}X)$.

- (5) $\implies$ (4): We already know that F is reduced. Suppose that we have equivalences $\Sigma_{\mathcal{D}}F(X) \to F(\Sigma_{\mathcal{C}}X)$ for all $X \in \mathcal{C}$. Since $\mathcal{D}$ is stable, applying $\Omega_{\mathcal{D}}$ to both sides give an equivalence $F(X) \to \Omega_{\mathcal{D}}\Sigma_{\mathcal{D}}F(X) \to \Omega_{\mathcal{D}}F(\Sigma_{\mathcal{C}}X)$. By prp1.4.10 we conclude that F is excisive.

□

Proposition 2.2.4

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Suppose that $\mathcal{C}$ is pointed and admits all finite colimits. Suppose that $\mathcal{D}$ admits all finite limits. Then

$$\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$$

is a stable infinity category.

Proof. We first show that $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is pointed. Let $*$ denote a final object of $\mathcal{C}$. Since $\mathcal{D}$ admits all finite limits and final objects are limits of the empty diagram, $\mathcal{D}$ admits a final object $*'$. Let $X : \mathcal{C} \to \mathcal{D}$ be the constant functor landing in $\Delta^0 \cong \{*'\} \subset \mathcal{D}$. It is clear that X is reduced and excisive. Moreover, X is a final object of $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ (Why ????).

Now let $Y \in \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$. Since X and Y are reduced, the mapping space $\mathrm{Hom}_{\mathcal{D}}(X(*), Y(*))$ is contractible. Consider the restriction map

$$\mathrm{Hom}_{\mathrm{Func}(\mathcal{C},\mathcal{D})}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(X(*), Y(*))$$

given by sending a natural transformation $F : X \Rightarrow Y$ to $F(*) : X(*) \to Y(*)$. Admit finite limits (Why ????) □

2.3 The Infinity Category of Stable Infinity Categories

Definition 2.3.1: The Infinity Category of Stable Infinity Categories

Define the infinity category $\mathcal{C}_\infty^{\mathrm{Ex}}$ of stable infinity categories as follows.
- The objects are stable infinity categories.
- For $\mathcal{C}, \mathcal{D}$ stable infinity categories, $\mathrm{Hom}_{\mathcal{C}_\infty^{\mathrm{Ex}}}(\mathcal{C}, \mathcal{D})$ is the full sub infinity category of

$\mathrm{Hom}_{\mathcal{C}_\infty}(\mathcal{C}, \mathcal{D})$ consisting of exact functors.

3 Spectrum Objects

3.1 The Stable Infinity Category of Spectrum Objects

The infinity category of spectrum objects is the prototypical example of a stable infinity category. Recall that we denote

$$\mathcal{S} = N_\bullet^{\text{hc}}(\mathbf{Kan})$$

as the infinity category of spaces.

Definition 3.1.1: The Infinity Category of Finite Spaces

Define the infinity category of finite pointed spaces

$$\mathcal{S}_*^{\text{fin}}$$

to be the smallest full subcategory of pointed spaces $\mathcal{S}_*$ that contains the final object $*$ and is stable under finite colimits.

Heuristic: $\mathcal{S}_*^{\text{fin}}$ is freely generated by by $*$ under finite colimits.

Lemma 3.1.2

Let $\mathcal{C}$ be an infinity category which admits finite colimits. Then the evaluation functor gives an equivalence of infinity categories

$$\text{ev}_* : \text{Func}^{\text{Rex}}(\mathcal{S}^{\text{fin}}, \mathcal{C}) \to \mathcal{C}$$

where $\text{Func}^{\text{Rex}}(\mathcal{S}^{\text{fin}}, \mathcal{C})$ refers to the full sub infinity category of right exact functors.

Definition 3.1.3: The Category of Spectrum Objects

Let $\mathcal{C}$ be an infinity category that admits all finite limits. A spectrum object of $\mathcal{C}$ is a reduced and excisive functor

$$F : \mathcal{S}_*^{\text{fin}} \to \mathcal{C}$$

Define the infinity category of spectrum objects of $\mathcal{C}$ to be

$$\text{Sp}(\mathcal{C}) = \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$$

Heuristic: Spectra in the usual sense is weakly equivalent to reduced and excisive functors from $\mathbf{Top}_*$ to $\mathbf{Top}_*$.

Lemma 3.1.4

Let $\mathcal{C}$ be an infinity category that admits all finite limits. Then $\text{Sp}(\mathcal{C})$ is a stable infinity category.

Proof. Follows from prp2.2.4 □

3.2 The Delooping Functor

Definition 3.2.1: Delooping

Let $\mathcal{C}$ be an infinity category that admits all finite limits. Define the delooping of $\mathcal{C}$ to be the evaluation functor

$$\Omega^\infty : \text{Sp}(\mathcal{C}) \to \mathcal{C}$$

given by $(F : \mathcal{S}_*^{\text{fin}} \to \mathcal{C}) \mapsto F(S^0)$.

Proposition 3.2.2

Let $\mathcal{C}$ be an infinity category that admits all finite limits. Then $\mathcal{C}$ is stable if and only if

$$\Omega^\infty : \text{Sp}(\mathcal{C}) \to \mathcal{C}$$

is an equivalence of infinity categories.

Proof. Suppose first that $\mathcal{C}$ is stable. Let $F : \mathcal{S}^{\text{fin}} \to \mathcal{S}_*^{\text{fin}}$ be a left adjoint to the forgetful functor $\mathcal{S}_*^{\text{fin}} \to \mathcal{S}^{\text{fin}}$ which is obtained by adding a disjoint base point. Now consider the functor

$$\text{Sp}(\mathcal{C}) = \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C}) \xrightarrow{-\circ F} \text{Exc}'(\mathcal{S}^{\text{fin}}, \mathcal{C}) \xrightarrow{\text{ev}_*} \mathcal{C}$$

where $\text{Exc}'(\mathcal{S}^{\text{fin}}, \mathcal{C})$ is the infinity category of functors that send initial objects to final objects. This functor sends a reduced and excisive functor $G \in \text{Sp}(\mathcal{C})$ to $(G \circ F)(*) = G(S^0)$. Hence this composition is in fact equivalent to Ω^∞.

Consider the composite functor given by

$$\text{Func}(\mathcal{S}^{\text{fin}}, \mathcal{C}) \times \mathcal{S}_*^{\text{fin}} \longhookrightarrow \text{Func}(\mathcal{S}^{\text{fin}}, \mathcal{C}) \times \text{Func}(\Delta^1, \mathcal{S}^{\text{fin}}) \xrightarrow{\circ} \text{Func}(\Delta^1, \mathcal{C}) \xrightarrow{\text{cofiber}} \mathcal{C}$$

Using the product-hom adjunction, this is the same data as a map
$\theta : \text{Func}(\mathcal{S}^{\text{fin}}, \mathcal{C}) \to \text{Func}(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$. Let $T : \mathcal{S}^{\text{fin}} \to \mathcal{C}$ be a functor sending initial objects to final objects. Consider the functor $\theta(T) : \mathcal{S}_*^{\text{fin}} \to \mathcal{C}$. ??????? Hence θ restricts to a map
$\psi : \text{Exc}'(\mathcal{S}^{\text{fin}}, \mathcal{C}) \to \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{C})$. ????????? Thus ψ is a homotopy inverse of $- \circ F$.

Now I claim that a functor $T : \mathcal{S}^{\text{fin}} \to \mathcal{C}$ sends initial objects to final objects if and only if F is right exact. Suppose that T sends initial objects to final objects. ?????
It follows from lmm3.1.2 that ev_* is an equivalence of infinity categories. Hence Ω^∞ is an equivalence of infinity categories.

The other direction follows from lmm3.1.4. □

HA 1.4.2.21

Proposition 3.2.3

Let $\mathcal{C}$ be a pointed infinity category that admits all finite colimits. Let $\mathcal{D}$ be an infinity category that admits all finite limits. Then post composition with Ω^∞ gives an equivalence of infinity categories

$$\Omega^\infty_{\mathcal{D}} \circ - : \text{Exc}_*(\mathcal{C}, \text{Sp}(\mathcal{D})) \xrightarrow{\simeq} \text{Exc}_*(\mathcal{C}, \mathcal{D})$$

Proof. Notice that there is a canonical isomorphism

$$\begin{aligned} \text{Exc}_*(\mathcal{C}, \text{Sp}(\mathcal{D})) &= \text{Exc}_*(\mathcal{C}, \text{Exc}_*(\mathcal{S}_*^{\text{fin}}, \mathcal{D})) \\ &\simeq \text{Exc}_*(\mathcal{C} \times \mathcal{S}_*^{\text{fin}}, \mathcal{D}) \\ &\simeq \text{Exc}_*(\mathcal{S}^{\text{fin}}, \text{Exc}_*(\mathcal{C}, \mathcal{D})) \\ &= \text{Sp}(\text{Exc}_*(\mathcal{C}, \mathcal{D})) \end{aligned}$$

Under this identification, the functor $\Omega^\infty \circ -$ corresponds to the functor $\Omega^\infty_{\text{Exc}_*(\mathcal{C},\mathcal{D})}$ since Ω are computed term wise (like all limits). By prp2.2.4 $\text{Exc}_*(\mathcal{C}, \mathcal{D})$ is stable. Hence by prp3.2.2 we conclude that $\Omega^\infty_{\text{Exc}_*(\mathcal{C},\mathcal{D})}$ is an equivalence of infinity categories. Hence $\Omega^\infty_{\mathcal{D}} \circ -$ is an equivalence of infinity categories. □

Proposition 3.2.4

Let $\mathcal{C}$ be a pointed infinity category that admits all finite limits. Then there is an equivalence of infinity categories

$$\mathrm{Sp}(\mathcal{C}) \simeq \lim\left(\cdots \to \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C}\right)$$

given by the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$.

Proof. Step 1: $\overline{\mathcal{C}} = \lim\left(\cdots \to \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C} \stackrel{\Omega}{\to} \mathcal{C}\right)$ is stable.

Step 2:
Consider the canonical map $G : \overline{\mathcal{C}} \to \mathcal{C}$ sending $\{X_n\}$ to X_0. □

Proposition 3.2.5

Let $\mathcal{C}$ be an infinity category. Then the following are equivalent.

- $\mathcal{C}$ is a stable infinity category.
- $\mathcal{C}$ is pointed, admits all finite limits and $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence of infinity categories.
- $\mathcal{C}$ is pointed, admits all finite colimits and $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence of infinity categories.

Proof.

- $(1) \implies (2), (3)$: If $\mathcal{C}$ is stable, then by prp2.1.4 Ω is an equivalence. Since Ω is right adjoint to Σ, Σ is also an equivalence. Hence (2) and (3) follows.
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□