

# Topological Manifolds

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**Abstract**

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# 1 Point Set Topology of Topological Manifolds

## 1.1 Triangulation of Manifolds

### Definition 1.1.1: Triangulable Manifolds

Let  $M$  be a  $k$ -manifold. We say that  $M$  is triangulable if  $M$  is a  $\delta$ -complex structure consisting of a finite number of top simplices.

## 1.2 Covering Spaces of Manifolds

### Proposition 1.2.1

Let  $M$  be a manifold. Let  $p : \tilde{M} \rightarrow M$  be a covering space. Then  $\tilde{M}$  is also a manifold.

## 2 Orientability of a Topological Manifold

### 2.1 Classical Orientability

The key observation in defining orientation through homology is the following proposition, which shows that the local homology groups on a manifold are isomorphic to  $\mathbb{Z}$  on the top dimension.

#### Proposition 2.1.1

Let  $M$  be a  $k$ -dimensional topological manifold and  $x \in M$  a point. Then

$$H_n(M | \{x\}) \cong \begin{cases} \mathbb{Z} & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let  $x \in M$ . Since  $M$  is a manifold, there exists an open neighbourhood of  $x$  such that  $U \cong \mathbb{R}^k$  via some transition map  $\varphi : U \rightarrow \mathbb{R}^k$ . Since  $M \setminus U$  is closed, we can apply excision to obtain an isomorphism

$$H_n(M | \{x\}) \cong H_n(U | \{x\})$$

By the homeomorphism  $(U, U \setminus \{x\}) \cong (\mathbb{R}^k, \mathbb{R}^k \setminus \{\varphi(x)\})$  and 6.3.2 in AT2, we obtain the desired result.  $\square$

We can now consider locally what it means to have an orientation (because we excised the data to that of  $\mathbb{R}^k$ ), and then try and glue all the choices of orientation into a coherent global orientation.

#### Definition 2.1.2: Local Orientation

Let  $M$  be a  $k$ -dimensional topological manifold and let  $x \in M$ . A local orientation of  $M$  at  $x$  is a choice of generator of

$$H_k(M | \{x\}) \cong \mathbb{Z}$$

#### Definition 2.1.3: Open and Closed Ball in Manifolds

Let  $M$  be a  $k$ -dimensional topological manifold and let  $(U, \varphi)$  be a chart of  $M$ . We say that  $B \subset U$  is an open / closed ball if  $\varphi(B) \subseteq \mathbb{R}^k$  is an open / closed ball of  $\mathbb{R}^k$ .

The point of the definition is that we have the following. We have homotopy equivalences

$$\begin{array}{ccc} & (\mathbb{R}^k, \mathbb{R}^k \setminus B) & \\ \swarrow \simeq & & \nwarrow \simeq \\ (\mathbb{R}^k, \mathbb{R}^k \setminus \{x\}) & & (\mathbb{R}^k, \mathbb{R}^k \setminus \{y\}) \end{array}$$

given by deformation retracts. If  $M$  is a  $k$ -manifold and  $U \subseteq M$  is an open ball, we can use excision to obtain isomorphisms

$$\begin{array}{ccc} & H_k(M | U) & \\ \swarrow \cong & & \searrow \cong \\ H_k(M | \{x\}) & & H_k(M | \{y\}) \end{array}$$

All of the above groups are just  $\mathbb{Z}$ , and a choice of local orientation is a choice of generators of the lower two homology groups. If we want the choice to be consistent, then we better have the two generators coincide to the same generator in  $H_n(M | U)$  under the above isomorphisms. We note here that the isomorphism

$$H_n(M | U) \xrightarrow{\cong} H_n(M | \{x\})$$

for any  $x \in U$  came from the inclusion map  $(M, M \setminus U) \hookrightarrow (M, M \setminus \{x\})$ .

**Definition 2.1.4: Consistent Local Orientations**

Let  $M$  be a  $k$ -manifold. Let  $B$  be an open ball in  $M$ . For each  $x \in B$ , let  $\omega_x$  be a choice of local orientation at  $x$ . We say that the choices of local orientations at  $B$  is consistent if there exists a generator  $\omega_B \in H_k(M | B)$  such that for any  $x, y \in B$ , under the isomorphisms

$$H_k(M | \{x\}) \xrightarrow{\cong} H_k(M | B) \xleftarrow{\cong} H_k(M | \{y\})$$

$$\omega_x \longmapsto \omega_B \longleftarrow \omega_y$$

the choice of local orientation maps to the same generator  $\omega_B$ .

With this, we can now formally define orientations in a manifold.

**Definition 2.1.5: Orientation of a Manifold**

Let  $M$  be a  $k$ -dimensional topological manifold. An orientation of  $M$  is a function

$$x \mapsto \omega_x \in H_k(M, M \setminus \{x\})$$

assigning every point to a local orientation such that for every  $x \in M$ , there exists an open ball  $x \in B$  such that  $(\omega_x)_{x \in B}$  a consistent local orientation.

**2.2 The Orientation Double Cover**

In order to deduce orientability of a manifolds, we appeal to the theory of vector bundles.

**Definition 2.2.1: Orientation Bundle**

Let  $M$  be a topological manifold. Define the orientation bundle  $\widetilde{M}$  to be the set of pairs

$$\widetilde{M} = \left\{ (x, \omega_x) \mid x \in M, \omega_x \text{ is a generator of } H_k(M | x) \right\}$$

together the projection map  $\pi : \widetilde{M} \rightarrow M$  defined by  $\pi(x, \omega_x) = x$ .

**Definition 2.2.2: Topology on the Orientation Bundle**

Let  $M$  be a topological manifold. Define the topology on the orientation bundle  $\widetilde{M}$  as follows. Let  $B$  be an open ball in  $M$ . Since there are exactly two distinct orientation classes on  $B$  we have that

$$\pi^{-1}(B) = B_+ \amalg B_-$$

where  $B_+$  and  $B_-$  are homeomorphic to  $B$ . Define the topology of  $\widetilde{M}$  to be generated by sets of the form  $B_+$  and  $B_-$ .

**Lemma 2.2.3**

For any topological manifold  $M$ ,  $\widetilde{M}$  is a manifold and is a 2-sheeted covering.

*Proof.* Let  $(x, \omega_x)$  and  $(y, \omega_y)$  in  $\widetilde{M}$  be distinct. If  $x = y$  then  $\omega_x = -\omega_y$ . We know that there are two distinct orientation classes so  $\pi^{-1}$  is a disjoint union consisting of those with positive orientation and those with negative. Since  $\omega_x$  and  $\omega_y$  are opposite, they lie in the disjoint union separately so that they are disjoint. If  $x \neq y$ , then since  $M$  is Hausdorff then we can choose  $U_1$  and  $U_2$  disjoint neighbourhoods of  $x$  and  $y$  respectively. Then this means that  $\pi^{-1}(U_1)$  and  $\pi^{-1}(U_2)$  are disjoint. Thus we have shown that  $\widetilde{M}$  is Hausdorff.

Now let  $(x, \omega_x) \in \widetilde{M}$ . Then since  $M$  is manifold, there is an open ball  $B$  around  $x$  so that  $B$  is homeomorphic to  $\mathbb{R}^k$ .  $\pi^{-1}(B)$  is then a disjoint union of two copies of  $B$ , one such copy contains  $(x, \omega_x)$ . Then we have found a neighbourhood for  $(x, \omega_x)$  that is homeomorphic to  $\mathbb{R}^k$ . Thus we are done.

It is clear that it is a two sheeted covering because for any open set  $B \subseteq M$ ,  $\pi^{-1}(B) = B_+ \amalg B_-$ . □

#### Lemma 2.2.4

Let  $M$  be a topological  $k$ -manifold. Then the orientation bundle  $\widetilde{M}$  is orientable.

*Proof.* Suppose that  $(x, \omega_x)$  and  $(y, \omega_y)$  in  $\widetilde{M}$  share an open ball  $\widetilde{B}$  of  $\widetilde{M}$ . By definition of  $\widetilde{M}$ , the topology is generated by  $\pi^{-1}(B) = B_+ \amalg B_-$  for any open ball  $B$  of  $M$ . Hence any open ball of  $\widetilde{M}$  must be equal to some  $B_+$  or  $B_-$ . Without loss of generality, suppose that  $B$  is an open ball of  $M$  such that  $\widetilde{B}$  is one of  $B_+$  or  $B_-$  in  $\pi^{-1}(B)$ . Then  $x$  and  $y$  share an open ball  $B$ . We have seen the following isomorphisms induced by inclusions:

$$\begin{array}{ccccc}
 & & H_k(M | B) & & \\
 & \nearrow \cong & & \nwarrow \cong & \\
 H_k(M | x) & & & & H_k(M | y) \\
 & \nearrow \cong & H_k(\widetilde{M} | \widetilde{B}) & \nwarrow \cong & \\
 H_k(\widetilde{M} | (x, \omega_x)) & & & & H_k(\widetilde{M} | (y, \omega_y))
 \end{array}$$

By excision, we can connect the above diagram with isomorphisms:

$$\begin{array}{ccccc}
 & & H_k(M | B) & & \\
 & \nearrow \cong & \uparrow \cong & \nwarrow \cong & \\
 H_k(M | x) & & H_k(\widetilde{M} | \widetilde{B}) & & H_k(M | y) \\
 \uparrow \cong & \nearrow \cong & & \nwarrow \cong & \uparrow \cong \\
 H_k(\widetilde{M} | (x, \omega_x)) & & & & H_k(\widetilde{M} | (y, \omega_y))
 \end{array}$$

By definition of  $\pi^{-1}(B) = B_+ \amalg B_-$ , if  $(x, \omega_x)$  and  $(y, \omega_y)$  lie in the same ball, they have a consistent local orientation. Hence there exists a generator  $\omega_B$  of  $H_k(M | B)$  such that the above diagram maps elements in the following way:

$$\begin{array}{ccc}
 & \omega_B & \\
 \swarrow & & \searrow \\
 \omega_x & & \omega_y
 \end{array}$$

Under the isomorphism  $H_k(M | x) \cong H_k(\widetilde{M} | (x, \omega_x))$  let  $\omega_x$  be sent to the generator  $\mu_{(x, \omega_x)}$ . Define  $\mu_{(y, \omega_y)}$  and  $\mu_{\widetilde{B}}$  similarly. Then using the above diagram with 6 local homology groups, we can see that elements are sent in the following way:

$$\begin{array}{ccccc}
 & & \omega_B & & \\
 & \swarrow & \downarrow & \searrow & \\
 \omega_x & & \mu_{\widetilde{B}} & & \omega_y \\
 \uparrow & \nearrow \text{dashed} & & \nwarrow \text{dashed} & \uparrow \\
 \mu_{(x, \omega_x)} & & & & \mu_{(y, \omega_y)}
 \end{array}$$

This show that we made a choice of generators for the local homology groups of  $\widetilde{M}$  for which they are consistent (they both map to the same generator  $\mu_{\widetilde{B}}$ ). Hence  $\widetilde{M}$  is orientable.  $\square$

### Lemma 2.2.5

Let  $M$  be a  $k$ -manifold. Then  $M$  is orientable if and only if there exists a section  $M \rightarrow \widetilde{M}$ . In particular, the given section is then the assignment required in the definition of orientability.

*Proof.* Let  $M$  be orientable. Then there exists an assignment  $x \mapsto \omega_x \in H_k(M | x)$  for each  $x \in M$ . We can rewrite the assignment into  $x \mapsto (x, \omega_x)$  so that the codomain is now  $\widetilde{M}$ . It is clear that composing with the projection map gives the identity. It remains to show that the assignment is continuous. Since the topology of  $\widetilde{M}$  is generated by open balls, it suffices to check continuity on open balls. So let  $\widetilde{B}$  be an open ball of  $\widetilde{M}$ . It is clear that the preimage of  $\widetilde{B}$  is given by  $x \in M$  such that  $(x, \omega_x) \in \widetilde{B}$ . But this is the same set as  $B = \pi(\widetilde{B})$ , which by definition is an open ball. Hence  $s$  is continuous.

Now let  $s : M \rightarrow \widetilde{M}$  be a section. By restricting to the second factor we obtain an assignment  $x \mapsto \omega_x \in H_k(M | x)$ . I claim that defines an orientation. By continuity of  $s$ , the preimage of each open ball  $\widetilde{B}$  of  $\widetilde{M}$  by  $s$  is also an open ball  $B$  of  $M$ . For  $x, y \in B$ ,  $\omega_x$  and  $\omega_y$  is in  $\widetilde{B}$ . But  $\widetilde{B}$  is one of the factors of the disjoint union  $\pi^{-1}(B) = B_+ \amalg B_-$ , which by definition consists of consistent local orientations. Hence  $\omega_x$  and  $\omega_y$  are consistent. Thus we conclude.  $\square$

### Theorem 2.2.6

Let  $M$  be a connected topological manifold. Then the following are true.

- $M$  is orientable if and only if  $\widetilde{M} \cong M \amalg M$ . In this case,  $M$  admits exactly two possible orientations.
- $M$  is non-orientable if and only if  $\widetilde{M} \rightarrow M$  is a non-trivial two sheeted cover.

*Proof.* Let  $M$  first be orientable. Then there exists a section  $s : M \rightarrow \widetilde{M}$  to the covering space. Assume for a contradiction that  $\widetilde{M}$  is connected. Let  $\gamma$  be a path from  $(x, \omega_x)$  to  $(x, -\omega_x)$ . Then notice that  $\gamma$  and  $s \circ \pi \circ \gamma$  are distinct lifts of the path  $\pi \circ \gamma$  in  $M$ . This contradicts the uniqueness of path lifting. Thus  $\widetilde{M}$  is disconnected. The unique disconnected two sheeted cover of a space is precisely the disjoint union of the space. So we are done.

Now let  $\widetilde{M} \cong M \amalg M$ . Then it is easy to see that there exists a section  $s : M \rightarrow \widetilde{M}$  simply be mapping homeomorphically to one of the disjoint components.

When  $M$  is orientable, we have that  $\widetilde{M} \cong M \amalg M$ . By the above lemma, each section  $M \rightarrow \widetilde{M}$  corresponds to one choice of orientation. There can only be two choices of distinct sections  $M \rightarrow M \amalg M$ . Hence  $M$  has exactly two orientations.

The second statement is precisely the contrapositive of the equivalent characterization of orientability.  $\square$

An overview of what is happening: Let  $M$  be a manifold. Then the orientation sheaf is a locally constant sheaf with constant value  $\mathbb{Z}$ . Since  $M$  is locally connected, there is an equivalence between locally constant sheaves and covering spaces induced by the presheaf-bundle adjunction. The orientation sheaf then corresponds to the orientation bundle. (Why does the existence of global sections imply orientability?)

## Corollary 2.2.7

Any simply connected manifold is orientable.

*Proof.* By Galois theory of covering spaces, any 2-sheeted cover of a simply connected space is disconnected.  $\square$

## Proposition 2.2.8

Let  $k \geq 1$ . Then  $\mathbb{RP}^k$  is orientable if and only if  $k$  is odd.

*Proof.* The quotient map  $q : S^k \rightarrow \mathbb{RP}^k$  is the unique connected two-sheeted cover of  $\mathbb{RP}^k$  by Galois theory for covering spaces. The non-trivial deck transformation is given by the antipodal map which has degree  $(-1)^{k+1}$ . If  $k$  is odd then this degree is 1 so that the deck transformation is orientation preserving. Since deck transformations of the orientation bundle must be orientation reversing, we conclude that  $S^k \neq \widetilde{\mathbb{RP}^k}$ . This means that the orientation bundle of  $\mathbb{RP}^k$  is disconnected.

Now assume that  $k$  is even. By the lifting criterion, there exists a lift of  $q$  called  $\tilde{q}$  such that

$$\begin{array}{ccc} & & \widetilde{\mathbb{RP}^k} \\ & \nearrow \tilde{q} & \downarrow p \\ S^k & \xrightarrow{q} & \mathbb{RP}^k \end{array}$$

where  $p$  is the covering map. Then  $\tilde{q}$  must also be a covering space. Assume that  $q$  is not injective. This means that  $\tilde{q} \circ (-\text{id}) = \tilde{q}$  since  $-\text{id}$  is the only other deck transformation of  $S^k$  over  $\mathbb{RP}^k$ . This means that for any  $x \in S^k$ , we have that

$$H_k(S^k) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}) \longrightarrow H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

where the second map is given by the long exact sequence in relative homology. Denoting this entire map by  $\alpha$ , we have that  $\alpha \circ (-\text{id})_* = \alpha$  since  $\tilde{q} \circ (-\text{id}) = \tilde{q}$ . But  $\alpha$  is a map from  $\mathbb{Z}$  to  $\mathbb{Z}$ . Since  $\alpha \circ (-\text{id})_* = \alpha$  this implies that  $\alpha = 0$ . But  $\alpha$  also factors as

$$H_k(S^k) \xrightarrow{\cong} H_k(S^k, S^k \setminus \{x\}) \xrightarrow{\tilde{q}} H_k(\widetilde{\mathbb{RP}^k}, \widetilde{\mathbb{RP}^k} \setminus \{\tilde{q}(x)\})$$

by the long exact sequence in relative homology and naturality. But the second map is also an isomorphism since covering spaces of manifolds induces an isomorphism in local homology groups.

Now  $S^k$  being compact and  $\mathbb{RP}^k$  being Hausdorff together with  $\tilde{q}$  being injective implies that  $\tilde{q}$  is a homeomorphism onto an open and closed subspace of  $\mathbb{RP}^k$ . Assume that  $\tilde{q}$  is not surjective, then we have that  $\widetilde{\mathbb{RP}^k} \cong S^k \amalg X$  for some other space  $X$ . But this is impossible thus  $q$  is surjective and  $\tilde{q}$  gives a homeomorphism between  $S^k$  and  $\widetilde{\mathbb{RP}^k}$ . Since  $S^k$  is connected,  $\mathbb{RP}^k$  is thus non orientable.  $\square$

One has to be careful that homotopy equivalence does not preserve orientability. For example, the Möbius strip is homotopy equivalent to  $S^1$  but the former is non-orientable while the latter is.



## 2.3 Orientability in Arbitrary Coefficient Ring

### Proposition 2.3.1

Let  $M$  be a  $k$ -dimensional topological manifold and  $x \in M$  a point. Let  $R$  be a ring. Then

$$H_n(M \setminus \{x\}; R) \cong \begin{cases} R & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let  $x \in M$ . Since  $M$  is a manifold, there exists an open neighbourhood of  $x$  such that  $U \cong \mathbb{R}^k$  via some transition map  $\varphi : U \rightarrow \mathbb{R}^k$ . Since  $M \setminus U$  is closed, we can apply excision to obtain an isomorphism

$$H_n(M \setminus \{x\}; R) \cong H_n(U \setminus \{x\}; R)$$

By the homeomorphism  $(U, U \setminus \{x\}) \cong (\mathbb{R}^k, \mathbb{R}^k \setminus \{\varphi(x)\})$  and 6.3.2 in AT2, we obtain the desired result.  $\square$

### Definition 2.3.2: Local Orientation

Let  $M$  be a  $k$ -dimensional topological manifold and let  $x \in M$ . A local orientation of  $M$  at  $x$  is a choice of generator of

$$H_k(M \setminus \{x\}; R) \cong R$$

Notice that being a generator of  $R$  is the same as saying that it is a unit of  $R$ .

If  $M$  is a  $k$ -manifold and  $U \subseteq M$  is an open ball, a similar argument as the case of  $R = \mathbb{Z}$  shows that there are isomorphisms

$$\begin{array}{ccc} & H_k(M \setminus U; R) & \\ \swarrow \cong & & \searrow \cong \\ H_k(M \setminus \{x\}; R) & & H_k(M \setminus \{y\}; R) \end{array}$$

where all of the above groups are just  $R$ . We also obtain the same definition for consistent local orientations.

### Definition 2.3.3: Consistent Local Orientations

Let  $M$  be a  $k$ -manifold. Let  $B$  be an open ball in  $M$ . For each  $x \in B$ , let  $\omega_x$  be a choice of local orientation at  $x$ . We say that the choices of local orientations at  $B$  is consistent if there exists a generator  $\omega_B \in H_k(M \setminus B; R)$  such that for any  $x, y \in B$ , under the isomorphisms

$$H_k(M \setminus \{x\}; R) \xrightarrow{\cong} H_k(M \setminus B; R) \xleftarrow{\cong} H_k(M \setminus \{y\}; R)$$

$$\omega_x \longmapsto \omega_B \longleftarrow \omega_y$$

the choice of local orientation maps to the same generator  $\omega_B$ .

### Definition 2.3.4: R-Orientation of a Manifold

Let  $M$  be a  $k$ -dimensional topological manifold. An orientation of  $M$  is a function

$$x \mapsto \omega_x \in H_k(M, M \setminus \{x\}; R)$$

assigning every point to a local orientation such that for every  $x \in M$ , there exists an open ball  $x \in B$  such that  $(\omega_x)_{x \in B}$  a consistent local orientation.

In order to deduce interesting results, we need to define a more general version than that of the orientation double cover.

**Definition 2.3.5: Generalized Orientation Bundle**

Let  $M$  be a  $k$ -manifold. Let  $R$  be a ring. Define the generalized orientation bundle  $M_R$  of  $M$  to be the set

$$M_R = \{(x, \mu_r) \mid x \in M, r \in H_k(M \mid x; R) \cong R\}$$

together with the topology generated by each  $B_r$  in

$$\pi^{-1}(B) = \coprod_{r \in R \text{ is a unit}} B_r$$

When  $R = \mathbb{Z}$ , we notice that  $M_{\mathbb{Z}} \rightarrow M$  is infinite sheeted, and contains a copy of  $M$  as the subspace of  $M_{\mathbb{Z}}$  by choosing  $\mu_r = 0 \in \mathbb{Z}$ . More generally, if we write

$$M_k = \{(x, \mu_x) \in M_{\mathbb{Z}} \mid \mu_x = \pm k\}$$

$M_{\mathbb{Z}}$  contains a copy of  $M_k$  for  $k \in \mathbb{N} \setminus \{0\}$ , and each copy  $M_k$  is homeomorphic to the orientation double cover  $\widetilde{M}$ .

If we instead consider an arbitrary ring  $R$ , then we can similarly define

$$M_r = \left\{ (x, \mu_x) \in M_R \mid \begin{array}{l} \mu_x \otimes r \in H_k(M \mid x) \otimes R \cong H_k(M \mid x; R) \\ \mu_x \text{ is a generator } H_k(M \mid x) \cong \mathbb{Z} \end{array} \right\}$$

If  $2r = 0$  in the ring then  $M_r$  becomes only one copy of  $M$ . Otherwise for each  $r \in R$ ,  $M_r$  is homeomorphic to  $\widetilde{M}$ . Hence the covering space  $M_R$  is a disjoint union of  $M_r$  for  $r \in R$ , except that  $M_r$  and  $M_{-r}$  are not disjoint.

**Lemma 2.3.6**

Let  $M$  be a topological manifold. Let  $R$  be a ring. Then  $M$  is  $R$ -orientable if and only if there exists a section  $M \rightarrow M_R$ . In particular, the section is precisely the assignment required in the definition of  $R$ -orientability.

*Proof.* Let  $M$  be  $R$ -orientable. Then there exists an assignment  $x \mapsto \omega_x \in H_k(M \mid x; R)$  for each  $x \in M$ . We can rewrite the assignment into  $x \mapsto (x, \omega_x)$  so that the codomain is now  $M_R$ . It is clear that composing with the projection map gives the identity. It remains to show that the assignment is continuous. Since the topology of  $M_R$  is generated by open balls, it suffices to check continuity on open balls. So let  $\tilde{B}$  be an open ball of  $M_R$ . It is clear that the preimage of  $\tilde{B}$  is given by  $x \in M$  such that  $(x, \omega_x) \in \tilde{B}$ . But this is the same set as  $B = \pi(\tilde{B})$ , which by definition is an open ball. Hence  $s$  is continuous.

Now let  $s : M \rightarrow M_R$  be a section. By restricting to the second factor we obtain an assignment  $x \mapsto \omega_x \in H_k(M \mid x; R)$ . I claim that defines an orientation. By continuity of  $s$ , the preimage of each open ball  $\tilde{B}$  of  $M_R$  by  $s$  is also an open ball  $B$  of  $M$ . For  $x, y \in B$ ,  $\omega_x$  and  $\omega_y$  is in  $\tilde{B}$ . But  $\tilde{B}$  is one of the factors of the disjoint union  $\pi^{-1}(B) = \coprod_{r \in R \text{ is a unit}} B_r$ , which by definition consists of consistent local orientations. Hence  $\omega_x$  and  $\omega_y$  are consistent. Thus we conclude.  $\square$

**Lemma 2.3.7**

Let  $M$  be a topological manifold. Let  $R$  be a ring. Then the following are true.

- If  $M$  is orientable, then  $M$  is  $R$ -orientable.
- If  $M$  is non-orientable, then  $M$  is  $R$ -orientable if and only if  $R$  contains a unit of order 2.

## 2.4 Implications of R-Orientability

### Proposition 2.4.1

Let  $M$  be a compact  $k$ -manifold. Then the following are true.

- Let  $\Gamma(M, M_R)$  denote the set of sections from  $M$  to  $M_R$ . If  $M$  is  $R$ -orientable, then there is a bijection

$$\Gamma(M, M_R) \cong \text{Hom}(\pi_0(M), R)$$

of sets. This assignment is given by sending  $s : \pi_0(M) \rightarrow R$  to the map  $x \mapsto (x, s([x]))$ .

- If  $M$  is connected and not  $R$ -orientable, then there are no global sections.

### Proposition 2.4.2

Let  $M$  be a  $k$ -manifold. Let  $K \subseteq M$  be compact. Then the following are true.

- If  $s : M \rightarrow M_R$  is a section sending  $x$  to  $(x, \mu_x)$ , there exists a unique  $\mu_K \in H_k(M | K; R)$  such that under induced map of inclusions

$$H_k(M | K; R) \rightarrow H_k(M | x; R)$$

$\mu_K$  is mapped to  $\mu_x$ .

- The local homology groups  $H_i(M | K; R) = 0$  for all  $i > k$ .

### Theorem 2.4.3

Let  $M$  be a compact and connected  $k$ -dimensional manifold. Let  $R$  be a ring. Then the following are true.

- If  $M$  is  $R$ -orientable, then the map

$$H_k(M; R) \rightarrow H_k(M | x; R) \cong R$$

is an isomorphism for all  $x \in M$ .

- If  $M$  is not  $R$ -orientable, then the map

$$H_k(M; R) \rightarrow H_k(M | x; R) \cong R$$

is injective and has image  $\{r \in R \mid 2r = 0\}$  for all  $x \in M$

Notice this theorem is not a definitive criterion for orientability in general. However, if  $R = \mathbb{Z}$ , then it becomes a sufficient criterion.

### Corollary 2.4.4

Let  $M$  be a compact and connected  $k$ -dimensional manifold. Then the following are true.

- $M$  is orientable if and only if  $H_k(M) \cong \mathbb{Z}$
- $M$  is non-orientable if and only if  $H_k(M) = 0$
- In any case,  $H_k(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

*Proof.* Let  $M$  be orientable. Then by the above theorem it is clear that  $H_k(M) \cong \mathbb{Z}$ . Now let  $H_k(M) \cong \mathbb{Z}$ . Suppose for a contradiction that  $M$  is not orientable. Then the map

$$\mathbb{Z} \cong H_k(M) \rightarrow H_k(M | x) \cong \mathbb{Z}$$

is injective and has image  $\{k \in \mathbb{Z} \mid 2k = 0\} = \{0\}$ . But if  $\mathbb{Z} \rightarrow \mathbb{Z}$  is injective, its image must be non-trivial. Hence we have a contradiction, so that  $M$  is orientable.

Let  $M$  be non-orientable. Then the map  $H_k(M) \rightarrow \mathbb{Z}$  must be injective with trivial image. Hence  $H_k(M) = 0$ . Now let that  $H_k(M) = 0$ . Then 0 and  $\mathbb{Z}$  are not isomorphic, by the

contrapositive of the first statement of the above theorem, we conclude that  $M$  is not orientable.

If  $M$  is orientable then by the above theorem we are done. If  $M$  is not orientable, then the image of the map  $H_k(M; \mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z}$  is equal to  $\{k \in \mathbb{Z}/2\mathbb{Z} \mid 2k = 0\} = \mathbb{Z}/2\mathbb{Z}$ . Since the map is also injective, we conclude that  $H_k(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ .  $\square$

We can summarize the situation as follows:

$$M \text{ is orientable} \implies \begin{array}{c} \text{TF AE:} \\ M \text{ is } R\text{-orientable} \\ \text{there exists a section } M \rightarrow M_R \end{array} \implies \begin{array}{c} \Gamma(M, M_R) \cong \text{Hom}(\pi_0(M), R) \\ H_k(M; R) \cong H_k(M \mid x) \cong R \end{array}$$

$$\begin{array}{c} \text{TF AE:} \\ M \text{ is not } R\text{-orientable} \\ \text{there are no sections } M \rightarrow M_R \end{array} \implies H_k(M; R) \longrightarrow H_k(M \mid x) \cong R \text{ is injective}$$

#### Corollary 2.4.5

Let  $M$  be a compact, connected and orientable  $k$ -dimensional manifold. Let  $R$  be a ring. Then the following are true.

- There is an isomorphism

$$H_k(M; R) \cong R$$

- The higher homology groups

$$H_i(M; R) = 0$$

for all  $i > k$ .

#### Corollary 2.4.6

Let  $M$  be a compact and connected  $k$ -dimensional manifold. Then the following are true.

- If  $M$  is orientable, then the torsion part of  $H_{k-1}(M)$  is trivial.
- If  $M$  is not-orientable, then the torsion part of  $H_{k-1}(M)$  is  $\mathbb{Z}/2\mathbb{Z}$ .

#### Corollary 2.4.7

Let  $M$  be a non-compact connected  $k$ -dimensional manifold. Let  $R$  be a ring. Then we have

$$H_i(M; R) = 0$$

for all  $i \geq k$ .

## 2.5 Fundamental Class

### Definition 2.5.1: Fundamental Class

Let  $M$  be a compact and connected manifold of dimension  $n$ . Let  $R$  be a ring. A fundamental class for  $M$  with coefficients in  $R$  is an element  $[c] \in H_n(M; R)$  such that the element  $[c]$  is sent to a generator under the induced map of inclusion

$$H_n(M; R) \rightarrow H_n(M \mid x; R) \cong R$$

for any  $x \in M$ .

### Lemma 2.5.2

Let  $M$  be a compact and connected manifold of dimension  $n$ . Let  $R$  be a ring. Then  $M$  is  $R$ -orientable if and only if  $M$  has a fundamental class for  $M$  with coefficients in  $R$ .

When  $M$  is a  $\delta$ -complex, we can represent the fundamental class as a linear combination of top-simplices satisfying some conditions.

**Proposition 2.5.3**

Let  $M$  be a compact and connected  $n$ -manifold. Let

$$\rho = \sum_{\sigma_i \text{ is an } n \text{ simplex}} k_i \sigma_i \in C_n(M)$$

be an  $n$ -chain. Then  $[\rho]$  is a fundamental class of  $M$  if and only if each  $k_i = \pm 1$  and  $\rho$  is an  $n$ -cycle. Moreover,  $M$  is orientable if and only if there exists such a  $\rho$ .

## 2.6 Relation to Orientability of Smooth Manifolds

### 3 Poincare Duality

#### 3.1 The Cap Product

##### Definition 3.1.1: The Cap Product

Let  $\sigma = [v_0, \dots, v_k] \in C_k(X)$  and  $\phi \in C^l(X)$  where  $k \geq l$  with coefficients in a ring  $R$ . Define the cap product to be

$$\sigma \frown \phi = \phi(\sigma|_{[v_0, \dots, v_l]})\sigma|_{[v_l, \dots, v_k]} \in C_{k-l}(X)$$

##### Lemma 3.1.2

The cap product  $\frown: C_k(X) \times C^l(X) \rightarrow C_{k-l}(X)$  with coefficients in a ring  $R$  induces a cap product in homology  $\frown: H_k(X) \times H^l(X, R) \rightarrow H_{k-l}(X)$  for  $k \geq l$ .

#### 3.2 Cohomology with Compact Support

#### 3.3 The Duality Theorem

##### Theorem 3.3.1: Poincare Duality

Let  $M$  be a compact and oriented topological  $n$ -manifold. Then the homomorphism

$$D: H^p(M) \rightarrow H_{n-p}(M)$$

is an isomorphism.

## 4 The Theory of Surfaces

### 4.1 Connected Sums

Recall that a compact surface is a connected topological manifold of dimension 2 that is compact.

#### Definition 4.1.1: Connected Sum

Let  $S_1$  and  $S_2$  be two compact surfaces. Let  $D_i \subseteq S_i$  be two small closed disks for  $i = 1, 2$ . Define the connected sum to be

$$S_1 \# S_2 = \frac{(S_1 \setminus D_1^\circ) \amalg (S_2 \setminus D_2^\circ)}{\partial D_1 \cong \partial D_2}$$

#### Lemma 4.1.2

The connected sum of two compact surfaces is again a compact surface.

#### Proposition 4.1.3

The connected sum is invariant under the choice of homeomorphism and the location of the small discs.

### 4.2 Classification of Compact Surfaces

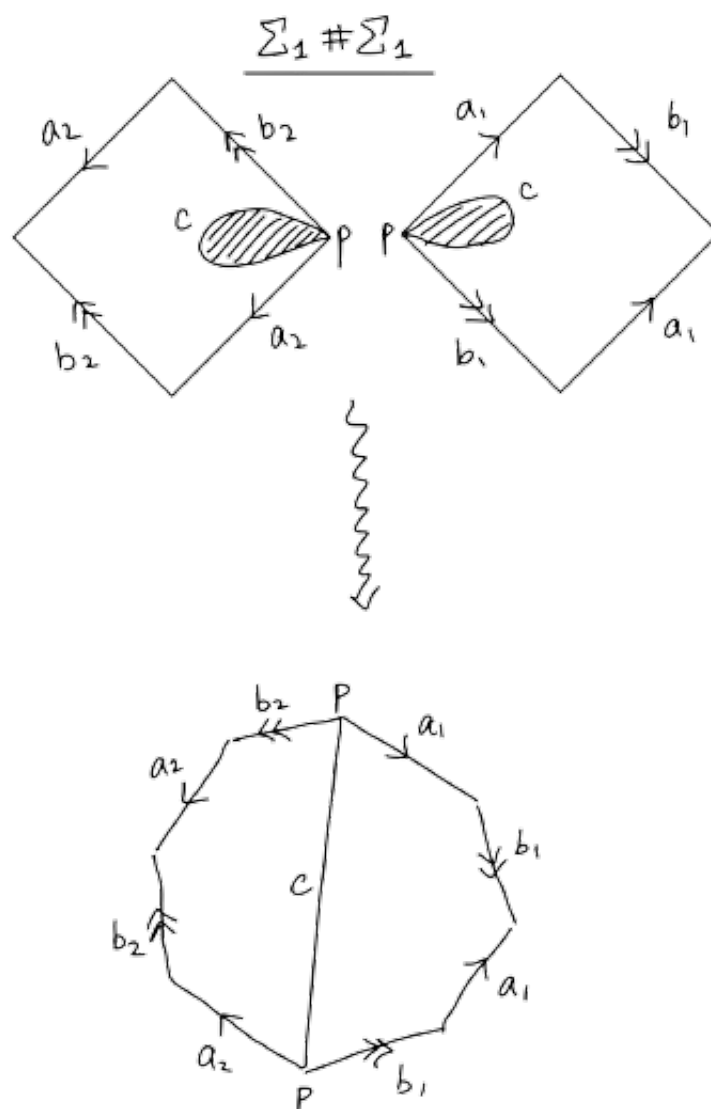
#### Definition 4.2.1: $g$ -Holed Torus

For  $g \geq 0$ , define the  $g$ -holed torus to be

$$\Sigma_g = \mathbb{T} \# \cdots \# \mathbb{T}$$

the connected sum of  $g$  toruses. By convention when  $g = 0$ ,  $\Sigma_g$  is the 2-sphere.

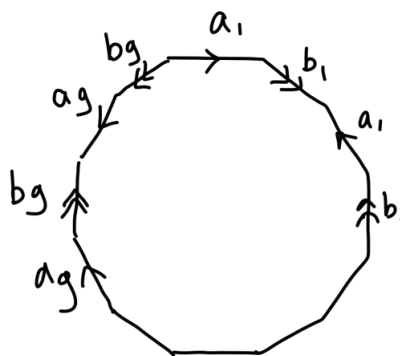
Recall the CW complex of the torus. We can visualize the connected sum of two toruses using the CW complex.



This is done by cutting a hole at the CW complex at the point  $p$ , and then pushing the boundary  $c$  out, and then connecting them together. The cut-out hole is exactly a disc in the torus. By gluing the two toruses along the boundary  $c$ , we are effectively gluing the two toruses along the discs.

The new heptagon obtained is precisely then the CW complex of  $\Sigma_2$ . In general, we can perform the operation of connected sum on a  $(4g - 4)$ -gon and a square. We then obtain the CW complex of the  $g$ -holed torus.





Another class of compact surfaces is the connected sum of projective spaces  $\mathbb{RP}^2$ .

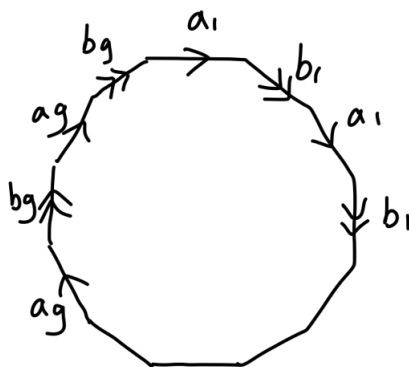
**Definition 4.2.2: Non-Orientable Surface**

For  $h \geq 1$ , define

$$N_h = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$$

the connected sum of  $h$  projective spaces.

We can do the same process of gluing the CW complexes just like that of the torus to obtain the  $4h$ -gon that represents  $N_h$ :



It is also meaningful to ask what would happen if we perform connected sums through the two class of compact surfaces. We obtain the following.

**Proposition 4.2.3**

Let  $N_3$  denote the connected sum of three projective spaces  $\mathbb{RP}^2$ . Then we have that

$$T \# \mathbb{RP}^2 = N_3$$

The above two classes of compact surfaces together with the sphere exhausts all possible cases for compact surfaces.

**Theorem 4.2.4**

Every compact surface is homeomorphic to exactly one of the following.

- $\Sigma_g$  for  $g \geq 0$
- $N_h$  for  $h \geq 1$

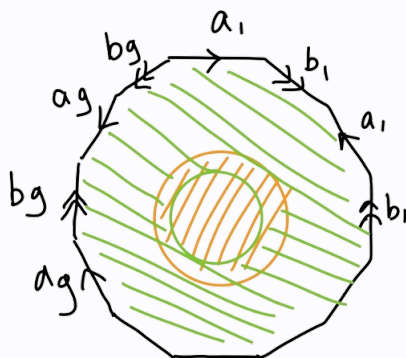
### 4.3 Algebraic Invariants of the Orientable Surfaces

#### Proposition 4.3.1

Let  $g \geq 0$ . The homology of the  $g$ -holed torus  $\Sigma_g$  is given by

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Cut an open disc along the middle of the CW complex as follows



and label it  $V$  (the orange part). Label the green part as  $U$ . It is clear that  $U \cap V \simeq S^1$ ,  $U$  is contractible and  $V$  deformation retracts to the boundary, which is actually just a wedge sum of  $2g$  circles. By the formula for the homology of wedge sums we have that

$$H_n(V) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

By the reduced Mayer-Vietoris sequence, the only non-trivial homology groups in the sequence are

$$0 \longrightarrow \tilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \tilde{H}_1(\Sigma_g) \longrightarrow 0$$

and the exact sequence

$$0 \longrightarrow \tilde{H}_0(\Sigma_g) \longrightarrow 0$$

in which the latter immediately shows that  $H_0(\Sigma_g) \cong \mathbb{Z}$ . Now the map  $\mathbb{Z} \rightarrow \mathbb{Z}^{2g}$  sends a generator of the first homology of  $U \cap V \simeq S^1$  to the loop

$$a_1 + b_1 - a_1 - b_1 + \cdots + a_g + b_g - a_g - b_g$$

Since  $\mathbb{Z}^{2g}$  is abelian, we conclude that this map is actually the zero map. It follows that  $H_2(\Sigma_g) \cong \mathbb{Z}$  and  $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$ .  $\square$

We can immediately deduce the orientability of  $\Sigma_g$  using the machinery in section 1.

## Corollary 4.3.2

The surfaces  $\Sigma_g$  for  $g \geq 0$  is orientable.

*Proof.* By the above, we have that  $H_2(\Sigma_g) \cong \mathbb{Z}$ . The long exact sequence for relative homology groups give

$$\cdots \longrightarrow H_2(\Sigma_g \setminus \{x\}) \longrightarrow H_2(\Sigma_g) \longrightarrow H_2(\Sigma_g, \Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g \setminus \{x\}) \longrightarrow H_1(\Sigma_g) \longrightarrow \cdots$$

Let  $U$  be as the proof above. Then the inclusion from  $U$  to  $\Sigma \setminus \{x\}$  is a homotopy equivalence. Moreover,  $\Sigma \setminus \{x\}$  is a  $2g$ -fold wedge of circles labelled  $a_1, b_1, \dots, a_g, b_g$  and  $H_2(\Sigma_g \setminus \{x\}) = 0$ . Also, we have that  $H_1(U) \cong H_1(\Sigma_g)$  from above and hence  $H_1(\Sigma_g \setminus \{x\}) \cong H_1(\Sigma_g)$ . The last map is invertible so that by exactness, the third map is the zero map. Then what remains is an isomorphism

$$H_2(\Sigma_g) \cong H_2(\Sigma_g, \Sigma_g \setminus \{x\})$$

Now since this isomorphism factors through  $H_2(\Sigma_g, \Sigma_g \setminus B)$  for any ball  $B$  containing  $x$ , we thus have a consistent local orientation throughout all of  $\Sigma_g$ .  $\square$

## Proposition 4.3.3

Let  $g \geq 0$ . The singular cohomology of the  $g$ -holed torus  $\Sigma_g$  with coefficients in  $\mathbb{Z}$  is given by

$$H^n(\Sigma_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Applying the universal coefficient theorem easily gives all the required cohomology groups.  $\square$

We use the cohomology of  $\Sigma_g$  to illustrate generators of cohomology. Ref: Hatcher Ex3.7.

## Proposition 4.3.4

Let  $g \geq 0$ . The integral cohomology ring of  $\Sigma_g$  is given by

$$H^*(\Sigma_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_1, \beta_1, \dots, \alpha_g, \beta_g]}{(\alpha_i^2, \beta_i^2, \alpha_i \alpha_j, \beta_i \beta_j, \alpha_i \beta_j, \beta_i \alpha_j, \alpha_i \beta_i + \beta_i \alpha_i \mid 1 \leq i \neq j \leq g)} \cong \Lambda^2(\mathbb{Z}^{2g})$$

where each  $\alpha_i$  and  $\beta_i$  are of degree 1 in the graded ring.

*Proof.* Consider the following CW complex of  $\Sigma_g$ . Recall that a basis for  $H_1(\Sigma_g)$  is given by  $a_1, b_1, \dots, a_g, b_g$ . Consider the dual basis  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$ . By definition these are precisely the generators of  $H^1(\Sigma_g; \mathbb{Z})$ . By definition the value of  $\alpha_i$  is 1 on  $a_i$  and 0 otherwise. This is similar for  $\beta_i$ . We want to find elements of  $Z^1(\Sigma_g; \mathbb{Z})$  that represent  $\alpha_i$  and  $\beta_i$ . Consider the following diagram: Define  $\phi_i \in C^1(\Sigma_g; \mathbb{Z})$  to be the map that gives 1 for any edge that intersects with  $s_i$  and 0 otherwise. Similarly, define  $\psi_i \in C^1(\Sigma_g; \mathbb{Z})$  to be the map that gives 1 for any edge that intersects with  $t_i$  and 0 otherwise. It is easy to see that  $\delta(\phi_i) = 0$  and  $\delta(\psi_i) = 0$  so that  $\phi$  and  $\psi$  are indeed cocycles. Moreover, they represent  $\alpha_i$  and  $\beta_i$  respectively.

Now notice that I have indicated orientations for each 2-simplices in  $\Sigma_g$  depending on whether they are oriented clockwise or anti-clockwise. Define an element of  $C_2(\Sigma_g)$  by the

sum of the 2-simplices with  $\pm 1$  as their coefficient depending on their orientation. It is easy to see that the sum is a 2-cycle that generates  $H_2(\Sigma_g)$ . Let  $\gamma$  be its dual generator.

It is easy to check that

$$\phi_i \smile \psi_j = -\psi_j \smile \phi_i = \begin{cases} \gamma & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and  $\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0$  for  $1 \leq i, j \leq g$ . We conclude that the cohomology ring of  $\Sigma_g$  is given by desired form.  $\square$

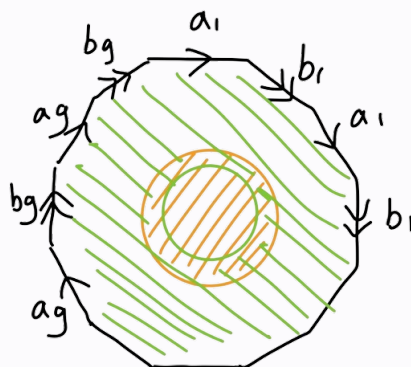
## 4.4 Algebraic Invariants of the Non-Orientable Surfaces

### Proposition 4.4.1

Let  $h \geq 1$ . The homology of  $N_h$  is given by

$$H_n(N_h) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Similar to the proof in that of  $\Sigma_g$ , cut an open disc along the middle of the CW complex of  $N_h$  as follows



and again label the green part  $U$  and the orange part  $V$ . Then apply Mayer-Vietoris sequence to acquire a similar exact sequence

$$0 \longrightarrow \tilde{H}_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \tilde{H}_1(\Sigma_g) \longrightarrow 0$$

together with  $\tilde{H}_0(N_h) \cong 0$ . Notice that the third non-zero term counting from the left is now  $\mathbb{Z}^h$  instead of  $\mathbb{Z}^{2g}$  as in the torus because the boundary circle is the wedge sum of  $h$  circles labelled  $a_1b_1, \dots, a_hb_h$ . The map  $\mathbb{Z}$  to  $\mathbb{Z}^h$  is now given by sending the generator 1 to

$$2(a_1 + b_1 + \dots + a_h + b_h)$$

The Smith Normal form of the matrix is an  $h \times 1$  matrix with 2 at the first entry and 0 everywhere else. In particular, it means that this map is injective so that  $\tilde{H}_2(N_h) \rightarrow \mathbb{Z}$  is the 0 map so that  $\tilde{H}_2(N_h) \cong 0$ . Now it remains an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^h \longrightarrow \tilde{H}_1(N_h) \longrightarrow 0$$

The image of the matrix is  $2\mathbb{Z}$  and by exactness this is the kernel of the map  $\mathbb{Z}^h \rightarrow \tilde{H}_1(N_h)$ . Thus we have an isomorphism

$$\tilde{H}_1(N_h) \cong \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}$$

and so we conclude.  $\square$

#### Corollary 4.4.2

The surfaces  $N_h$  for  $h \geq 1$  is non-orientable.

*Proof.* Notice that removing a small closed disk from  $\mathbb{RP}^2$  yields a space homeomorphic to the open Mobius strip. It follows that for  $h > 0$ , the space  $N_h$  contains the open Mobius strip as a subspace. Since the Mobius strip is non-orientable,  $N_h$  is also non-orientable.  $\square$

#### Proposition 4.4.3

Let  $h \geq 1$ . The singular cohomology of non-orientable surface  $N_h$  with coefficients in  $\mathbb{Z}$  is given by

$$H^n(N_h; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} & \text{if } n = 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* We use the universal coefficient theorem in all dimensions. When  $n = 0$ , we have that

$$\begin{aligned} H^0(N_h; \mathbb{Z}) &\cong \text{Hom}(H_0(N_h; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_{-1}(N_h; \mathbb{Z}), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus 0 \\ &\cong \mathbb{Z} \end{aligned}$$

When  $n = 1$ , we have that

$$\begin{aligned} H^1(N_h; \mathbb{Z}) &\cong \text{Hom}(H_1(N_h), \mathbb{Z}) \oplus \text{Ext}(H_0(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}^{h-1} \oplus 0 \\ &\cong \mathbb{Z}^{h-1} \end{aligned}$$

When  $n = 2$ , we have that

$$\begin{aligned} H^2(N_h; \mathbb{Z}) &\cong \text{Hom}(H_2(N_h), \mathbb{Z}) \oplus \text{Ext}(H_1(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &\cong 0 \oplus \text{Ext}(\mathbb{Z}^{h-1}, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \\ &\cong 0 \oplus 0 \oplus \mathbb{Z}/2\mathbb{Z} \\ &\cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

When  $n \geq 3$ , we have that

$$\begin{aligned} H^n(N_h; \mathbb{Z}) &\cong \text{Hom}(H_n(N_h), \mathbb{Z}) \oplus \text{Ext}(H_{n-1}(N_h), \mathbb{Z}) \\ &\cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(0, \mathbb{Z}) \\ &\cong 0 \end{aligned}$$

and so we conclude.  $\square$

## 4.5 The Euler Characteristic

Recall that if  $X$  is a CW complex such that  $U \cap V = X$  and  $U$  and  $V$  are open subsets, then we have the formula

$$\chi(X) = \chi(U) + \chi(V) - \chi(U \cap V)$$

### Corollary 4.5.1

Let  $S_1 \# S_2$  be the connected sum of two compact surfaces, then we have that

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

*Proof.* Let  $D_i$  be the gluing discs for  $S_i$  for  $i = 1, 2$ . Using the above formula, we have that

$$\chi(S_i) = \chi(D_i) + \chi(S_i \setminus D_i^\circ) - \chi(S^1)$$

since the intersection of the disc and  $S_i$  is  $S^1$ . It follows that

$$\begin{aligned} \chi(S_1 \# S_2) &= \chi(S_1 \setminus D_1^\circ) + \chi(S_2 \setminus D_2^\circ) - \chi(S^1) \\ &= \chi(S_1) + \chi(S_2) - 2 \end{aligned}$$

and so we conclude. □

### Corollary 4.5.2

For  $g \geq 0$  and  $h > 1$ , the Euler characteristic of any compact surface is given by

$$\chi(\Sigma_g) = 2 - 2g \quad \text{and} \quad \chi(N_h) = 2 - h$$

*Proof.* It follows directly by repeated applications of the above corollary. □

Recall that if  $p : \tilde{X} \rightarrow X$  is a  $d$ -sheeted covering and  $X$  is a finite CW complex, then we have the formula

$$\chi(\tilde{X}) = d \cdot \chi(X)$$