Lie Groups and Lie Algebra

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Abstract

Potentially good books: Humphreys, Erdmann and Wildson

Contents

1 Introduction to Lie Algebras

1.1 Lie Algebras

Definition 1.1.1: Lie Brackets

Let V be a vector space over a field k. Let $[-,-]:V\times V\to V$ be a bilinear map. We say that $[-,-]$ is a Lie bracket if the following are true.

- The Alternating Property: $[X,X]=0$
- Jacobi identity: $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$

Consider the cross product $\times:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ in $\mathbb{R}^3$. It is easy to see that it is a Lie bracket.

Definition 1.1.2: Lie Algebras

A Lie algebra is a vector space V over a field K together with a Lie bracket $[-,-]:V\times V\to V$.

For k a field, $M_n(k)$ for any $n\geq 1$ is a Lie algebra with Lie bracket defined as $[A,B]=AB-BA$ for $A,B\in M_n(k)$.

Lemma 1.1.3

Let L be a Lie Algebra. Then for all $x,y\in L$, we have that

$$[x,y]=-[y,x]$$

In other words, the Lie bracket is anti-commutative.

Proof. We have that

$$\begin{aligned}
[x,y]+[y,x]&=[x,x]+[x,y-x]+[y,y]+[y,x-y] && \text{(Bilinearity)}\\
&=[x,x]+[y,y]-[x-y,x-y] && \text{(Bilinearity)}\\
&=0 && \text{(Alternating)}
\end{aligned}$$

and so we conclude. $\square$

Lie Algebras are not algebras (in the sense of Rings and Modules) because the Lie bracket fails associativity. Therefore we have to redefine all the standard notions one has in algebra.

While Lie Algebras are not in general algebras, every associative algebra can be equipped with a Lie algebra. For A an associative algebra over a field, we can define a bilinear map on A by

$$[a,b]=ab-ba$$

for all $a,b\in A$. There may also be more than one way to equip an algebra with a Lie algebra structure. One should not think that Lie Algebras encompasses associative algebras because of the different Lie algebras one can equip. Instead, we think of the Lie bracket as an extra structure on associative algebras such that they become Lie algebras.

Definition 1.1.4: Structure Constants

Let L be a Lie algebra such that its underlying vector space has basis $e_1,\dots,e_n$. Define the structure constants of L to be the elements $c_{ij}^k\in\mathbb{F}$ such that

$$[e_i,e_j]=\sum_{k=1}^n c_{ij}^k e_k$$

for all $1\leq i,j\leq n$.

The structure constants are useful in the following sense. Let L be a Lie algebra and let $a = \sum_{k=1}^n a_k e_k$ and $b = \sum_{k=1}^n b_k e_k$ be elements of L. Then there Lie bracket can be written as

$$[a, b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)[e_i, e_j]$$

by bilinearity. Plugging in the structure constants, we obtain

$$[a, b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i) \sum_{k=1}^n c_{ij}^k e_k$$

Thus we can write $[a, b]$ in terms of the basis $e_1, \dots, e_n$ using structure constants.

1.2 Homomorphisms and Ideals

Definition 1.2.1: Homomorphism of Lie algebra

Let V and W be Lie algebras over K. A homomorphism from V to W is an K-linear map $F : V \to W$ such that

$$[F(a), F(b)] = [a, b]$$

for all $a, b \in V$.

Definition 1.2.2: Lie Subalgebra

Let V be a Lie algebra over K. A lie subalgebra of V is a subset $W \subseteq V$ such that
- W is a vector subspace of V
- $[w_1, w_2] \in W$ for all $w_1, w_2 \in W$

It is clear that a Lie subalgebra is also a Lie algebra in its own right. Moreover, the inclusion $mapW \to V$ is a Lie algebra homomorphism.

Definition 1.2.3: Ideal

Let V be a Lie algebra over K. Let I be a subset of V. Then I is an ideal of V if the following are true.
- I is a vector subspace of V
- $[v, i] \in I$ for all $v \in V$ and $i \in I$.

It is clear from definitions that every ideal of a Lie algebra is a Lie subalgebra. However, the converse is not always true.

Proposition 1.2.4

Let V be a Lie algebra and I, J ideals of V. Then the following are also ideals of V.
- The intersection $I \cap J$
- The sum $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$

Definition 1.2.5: The Lie Bracket

Let V be a Lie algebra. Let I, J be ideals of V. Define the Lie bracket of I and J to be

$$[I, J] = \langle [i, j] \mid i \in I \text{ and } j \in J \rangle$$

Lemma 1.2.6

Let V be a Lie algebra. Let I, J be ideals of V. Then the Lie bracket $[I, J]$ is an ideal of V.

1.3 Products and Quotients of Lie Algebras

Definition 1.3.1: Direct Sum of Lie Algebras

Let L_1 and L_2 be Lie algebras. Define the direct sum of L_1 and L_2 by

$$L_1 \oplus L_2 = \{(a_1, a_2) \mid a_1 \in L_1, a_2 \in L_2\}$$

together with component wise addition and scalar multiplication and Lie bracket operation

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1], [a_2, b_2])$$

which is component wise application of the Lie bracket for $(a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2$.

Proposition 1.3.2

Let L_1 and L_2 be Lie algebras. Then the following are true.

- $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$
- $\{(x, 0) \mid x \in L_1\} \cong L_1$ is an ideal of $L_1 \oplus L_2$
- $\{(0, y) \mid y \in L_2\} \cong L_2$ is an ideal of $L_1 \oplus L_2$

Proposition 1.3.3

Let V be a Lie algebra over K and U an ideal of V. Then V/U has a unique Lie algebra structure such that the quotient map $V \to V/U$ is a Lie algebra homomorphism.

1.4 The Center and Centralizers of a Lie Algebra

Definition 1.4.1: Center of a Lie Algebra

Let L be a Lie algebra. Define the center of L by

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$$

Lemma 1.4.2

Let L be a Lie algebra. Then $Z(L)$ is an ideal of L.

Proposition 1.4.3

Let L_1, L_2 be Lie algebras over the same field K. Then

$$Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$$

Definition 1.4.4: The Centralizer of a Subset

Let L be a Lie algebra. Let $A \subseteq L$ be a subset. Define the centralizer of A in L to be the set

$$C_L(A) = \{x \in L \mid [x, a] = 0 \text{ for all } a \in A\}$$

Lemma 1.4.5: L

t L be a Lie algebra. Let $A \subseteq L$ be a subset. Then $C_L(A)$ is a Lie subalgebra of L.

1.5 The Adjoint Homomorphism

Definition 1.5.1: The Adjoint Homomorphism

Let V be a Lie algebra. Define the adjoint homomorphism $\text{ad} : V \to GL(V)$ to be the map given by
$$\text{ad}(x)(y) = [x, y]$$

Lemma 1.5.2

Let V be a Lie algebra. Then then the adjoint homomorphism $\text{ad} : V \to GL(V)$ is a Lie algebra homomorphism.

Lemma 1.5.3

Let V be a Lie algebra. Then the kernel of the adjoint homomorphism is equal to
$$\ker(\text{ad}) = Z(V)$$
the center of V.

1.6 The Isomorphism Theorems

Theorem 1.6.1: First Isomorphism Theorem

Let $\phi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the following are true.

- $\ker(\phi)$ is an ideal of L_1
- $\text{im}(\phi)$ is a Lie subalgebra of L_2

Moreover, we have an isomorphism
$$\frac{L_1}{\ker(\phi)} \cong \text{im}(\phi)$$

Theorem 1.6.2: Second Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.

- I and J are ideals of $I + J$
- $I \cap J$ is an ideal of I and J

Moreover, we have an isomorphism
$$\frac{I+J}{J} \cong \frac{I}{I \cap J}$$

Theorem 1.6.3: Third Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L such that $I \subseteq J$. Then J/I is an ideal of L/I. Moreover, there is an isomorphism
$$\frac{L/I}{J/I} \cong \frac{L}{J}$$

Theorem 1.6.4: Correspondence Theorem

Let L be a Lie algebra with ideal I. Then there exists a bijective correspondence
$$\{J \mid J \text{ is an ideal of } L \text{ and } I \subseteq J\} \quad \overset{1:1}{\longleftrightarrow} \quad \{K \mid K \text{ is an ideal of } L/I\}$$

2 Types of Lie Algebras

2.1 Abelian Lie Algebras

Lie algebras that are Abelian are the simplest Lie algebra there is to study.

Definition 2.1.1: Abelian Lie Algebras

Let $(L, [-,-])$ be a Lie algebra. We say that L is abelian if

$$[x, y] = 0$$

for all $x, y \in L$.

Lemma 2.1.2

Let L be a Lie algebra. Let I be an ideal of L. Then L/I is abelian if and only if $[L, L] \subseteq I$

Corollary 2.1.3

Let L be a Lie algebra. Then the smallest ideal I such that L/I is abelian is given by

$$I = [L, L]$$

2.2 Soluble Lie Algebras

Let L be a Lie algebra. We have seen that rad(L) is soluble and $L/\text{rad}(L)$ is semisimple. Therefore to study a general Lie algebra, we need to understand soluble Lie algebras and semisimple Lie algebras. If we restrict the case to Lie algebras over $\mathbb{C}$, Lie's theorem will solve the first part of the problem, while the study of semisimple Lie algebras is postponed until section 5.

Definition 2.2.1: Derived Series

Let L be a Lie algebra. Define the derived series $L^{(n)}$ of L to be the sequence recursively defined as follows.

- For $n = 0$, define $L^{(0)} = L$
- When $n \in \mathbb{N} \setminus \{0\}$, define
$$L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$$

Lemma 2.2.2

Let L_1, L_2 be Lie algebras. Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then

$$\phi(L_1^{(k)}) = \phi(L_1)^{(k)}$$

Definition 2.2.3: Soluble Lie Algebras

Let L be a Lie algebra. We say that L is soluble if there exists $n \in \mathbb{N}$ such that

$$L^{(n)} = 0$$

Lemma 2.2.4

Let L be a Lie algebra. If L is abelian, then L is soluble.

Example 2.2.5

Consider the following Lie lalgebras.

- $b_n(\mathbb{C})$ the set of all upper triangular $n \times n$ matrices is soluble
- $SL(2, \mathbb{C})$ is not soluble
- $GL(2, \mathbb{C})$ is not soluble

Proposition 2.2.6

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.

- Let $\phi : L \to K$ be a Lie algebra homomorphism. If L is soluble then $\phi(L)$ is soluble.
- Let M be a Lie subalgebra of L. If L is soluble, then M is soluble.
- If I and L/I are soluble, then L is soluble.
- If I and J are soluble, then $I + J$ is soluble.

Theorem 2.2.7: Lie's Theorem

Let V be a vector space over $\mathbb{C}$. Let L be a soluble Lie subalgebra of $GL(V)$. Then there exists a basis B of V such that for all $M \in L$, M is upper triangular.

2.3 Nilpotent Lie Algebras

Definition 2.3.1: Lower Central Series

Let L be a Lie algebra. Define the lower central series $L^0, L^1, \dots, L^n, \dots$ as follows.

- For $n = 0$, define $L^0 = L$
- For $n \in \mathbb{N} \setminus \{0\}$, define
$$L^n = [L, L^{n-1}]$$

Lemma 2.3.2

Let L be a Lie algebra. Then there is an isomorphism

$$[L, L^n] = [L^n, L]$$

for all $n \in \mathbb{N}$ given by the opposite map $x \mapsto -x$.

Lemma 2.3.3

Let L be a Lie algebra. Then the following are true.

- For all $n \in \mathbb{N}$, L^n is an ideal of L.
- $L^{n+1} \subseteq L^n$.

Definition 2.3.4: Nilpotent Lie Algebras

Let L be a Lie algebra. We say that L is nilpotent if there exists $n \in \mathbb{N}$ such that

$$L^n = 0$$

Lemma 2.3.5

Let L be a Lie algebra. If L is abelian, then L is nilpotent.

Example 2.3.6

Consider the following Lie algebras.

- $SL(2,\mathbb{C})$ is nilpotent.
- $b_n(\mathbb{C})$ is not nilpotent for all $n \geq 2$.
- $U_3(\mathbb{C})$ the Heisenberg Lie algebra is nilpotent (3×3 strictly upper triangular matrices)
- $U_n(\mathbb{C})$ is nilpotent for all $n \geq 3$.

Lemma 2.3.7

Let L_1, L_2 be Lie algebras. Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then

$$\phi(L^k) = (\phi(L))^k$$

for all $k \in \mathbb{N}$.

Proof. We prove by induction. The base case $k = 0$ is clear. Suppose that $\phi(L^k) = (\phi(L))^k$. Then we have that

$$\begin{aligned}\phi(L^{k+1}) &= \phi([L, L^k]) \\ &= [\phi(L), \phi(L^k)] \\ &= [\phi(L), \phi(L)^k] \\ &= (\phi(L))^{k+1}\end{aligned}$$

By induction, we conclude. □

Lemma 2.3.8

Let L be a Lie algebra. Then the following are true.

- Let M be a Lie subalgebra of L. If L is nilpotent, then M is nilpotent.
- If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$
- If $L/Z(L)$ is nilpotent, then L is nilpotent.

Proof.

- Let M be a Lie subalgebra of L. I claim that $M^k \subseteq L^k$ for all $k \in \mathbb{N}$. The base case $k = 0$ is clearly true. Suppose that $M^k \subseteq L^k$. Let $x \in [M, M^k] = M^{k+1}$. Then $x = [m, t]$ for some $m \in M$ and $t \in M^k$. Then $t \in L^k$. Also $m \in L$ implies that $x = [m, t] \in [L, L^k] = L^{k+1}$. Thus $M^{k+1} \subseteq L^{k+1}$. Now since L is nilpotent, there exists $n \in \mathbb{N}$ such that $L^n = 0$. Then $M^n \subseteq L^n = 0$ so that M is also nilpotent.
- Suppose that $n \in \mathbb{N}$ is the smallest natural number such that $L^n = 0$. Then $[L, L^{n-1}] = 0$. Let $x \in L$. Then for all $y \in L^{n-1}$, we have that $[x, y] = 0$. Thus $x \in Z(L)$.
- Since $L/Z(L)$ is nilpotent, there exists $n \in \mathbb{N}$ such that $(L/Z(L))^n = 0$. Let $\pi : L \to L/Z(L)$ be the quotient homomorphism. Since π is surjective, we use the above lemma to find that

  $$\pi(L^n) = \pi(L)^n = \left(\frac{L}{Z(L)}\right)^n = \frac{L^n + Z(L)}{Z(L)} = 0$$

  This means that $L^n \subseteq Z(L)$. It follows that $L^{n+1} = [L, L^n] \subseteq [L, Z(L)] = 0$ and we conclude.

□

2.4 Engel's Theorem and Lie's Theorem

Definition 2.4.1: Ad-Nilpotency

Let L be a Lie algebra. Let $x \in L$. We say that x is ad-nilpotent if there exists $n \in \mathbb{N}$ such that
$$\text{ad}(x)^n = 0 \in GL(L)$$

Lemma 2.4.2

Let L be Lie algebra. If L is nilpotent, then all elements $x \in L$ are ad-nilpotent.

Lemma 2.4.3

Let V be a vector space. Let $L \subseteq GL(V)$ be a Lie subalgebra. If $T \in L$ is nilpotent, then T is ad-nilpotent.

Proposition 2.4.4

Let V be a vector space. Let $L \subseteq GL(V)$ be a Lie subalgebra such that for all $T \in L$, T is nilpotent. Then there exists $v \in V$ such that $T(v) = 0$ for all $T \in L$.

Proof. We induct on the dimension of L. □

Theorem 2.4.5: Engel's Theorem

Let V be a vector space. Let L be a Lie subalgebra of $GL(V)$. Suppose that for all $x \in L$, x is ad-nilpotent. Then the following are true.

- There exists a basis B of V such that every $T \in L$ is upper triangular.
- L is nilpotent.

Theorem 2.4.6: Lie's Theorem

Let V be a vector space over $\mathbb{C}$. Let L be a Lie subalgebra of $GL(V)$. If L is soluble, then there exists a basis B of V such that every $T \in L$ is upper triangular.

Corollary 2.4.7

Let L be a Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $[L, L]$ is nilpotent.

Corollary 2.4.8

Let L be a Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $[L, L]$ is nilpotent.

2.5 Semisimple Lie Algebras

Proposition 2.5.1

Let L be a Lie algebra. Then there exists a unique soluble ideal I of L such that for any soluble ideal $J \subseteq L$, we have $J \subseteq I$.

Definition 2.5.2: Radical Ideals

Let L be a Lie algebra. Define the radical ideal $\text{rad}(L) \subseteq L$ of L to be the unique soluble ideal of L that contains all other soluble ideals.

Definition 2.5.3: Semisimple Lie Algebras

Let L be a Lie algebra. We say that L is semisimple if

$$\frac{L}{\text{rad}(L)} = \{0\}$$

Example 2.5.4

Consider the following Lie algebras.

- $\{0\}$ is semisimple.
- $SL(2, \mathbb{C})$ is semisimple.
- $GL(2, \mathbb{C})$ is not semisimple.

Lemma 2.5.5

Let L be a Lie algebra. Then $L/\text{rad}(L)$ is semisimple.

Proof. Let K be a soluble ideal of $L/\text{rad}(L)$. By the correspondence theorem, there exists an ideal I of L such that $\text{rad}(L) \subseteq I$ and $K = I/\text{rad}(L)$. Since $\text{rad}(L)$ and K are soluble, we conclude that I is soluble. Hence $I \subseteq \text{rad}(L)$. We conclude that $I = \text{rad}(L)$. Hence $K = \{0\}$. Thus $L/\text{rad}(L)$ is semisimple. $\square$

2.6 Simple Lie Algebras

Definition 2.6.1: Simple Lie Algebras

Let L be a Lie algebra. We say that L is simple if L is non-abelian and has no proper non-zero ideals.

Lemma 2.6.2

Let L be a Lie algebra. If L is simple, then L is semisimple.

Definition 2.6.3: Simple Ideals

Let L be a Lie algebra. Let I be an ideal of L. We say that I is simple if I is simple as a Lie algebra.

3 The Killing Form and Jordan Decompositions

3.1 The Jordan-Chevalley Decompositions

Definition 3.1.1: Jordan-Chevalley Decompositions

Let k be a field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. A Jordan-Chevalley decomposition of V consists of $D, S \in \text{End}(V)$ such that the following are true.

- $T = D + S$
- D is diagonalizable
- S is nilpotent
- $SD = DS$

We note here that if we consider vector spaces as a k-module, then saying V semisimple is the same as saying V is diagonalizable.

Proposition 3.1.2

Let k be a field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. Then T admits a unique Jordan-Chevalley decomposition.

Proposition 3.1.3

Let k be an algebraically closed field. Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. Let $D, S \in \text{End}(V)$ be the Jordan-Chevalley decomposition of T. Then there exists $p, q \in k[x]$ such that $p(T) = D$ and $q(T) = S$.

Lemma 3.1.4

Let V be a finite dimensional vector space over k. Let $T \in \text{End}(V)$. Let $D, S \in \text{End}(V)$ be the Jordan-Chevalley decomposition of T. Then

$$\text{ad}(T) = \text{ad}(D) + \text{ad}(S)$$

is the Jordan decomposition of $\text{ad}(T)$.

3.2 The Killing Form

Definition 3.2.1: The Killing Form

Let L be a Lie algebra over $\mathbb{C}$. Define the killing form of L to be the function

$$k : L \times L \to \mathbb{C}$$

given by $k(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y))$.

Lemma 3.2.2

Let $\mathbb{F}$ be a field. Let L be a Lie algebra over $\mathbb{C}$. Then the killing form on L is a symmetric bilinear form.

Recall from Linear algebra that L is now an inner product space when equipped with the killing form.

Lemma 3.2.3

Let L be a Lie algebra. Let I be an ideal of L. If k is the killing form of L and $k|_I$ is the killing form of I, then

$$k|_I = k|_{I\times I}$$

3.3 Cartan's Criterion

Proposition 3.3.1

Let V be a finite dimensional vector space over $\mathbb{C}$. Let L be a Lie subalgebra of $GL(V)$. If $\text{tr}(xy) = 0$ for all $x, y \in L$, then L is soluble.

Theorem 3.3.2: Cartan's First Criterion

Let L be a Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $k(x, y) = 0$ for all $x \in L$ and $y \in [L, L]$.

Let L be a Lie algebra. The killing form allows L to be an inner product space. Recall from Linear Algebra that the orthogonal complement of a subspace W of L is given by

$$W^\perp = \{x \in L \mid k(x, w) = 0 \text{ for all } w \in W\}$$

Lemma 3.3.3

Let L be a Lie algebra over $\mathbb{C}$. Let I be an ideal of I. Then $I^\perp$ is an ideal of L.

Recall from Linear Algebra that a bilinear form is non-degenerate if if

$$\{v \in V \mid \tau(v, w) = 0 \text{ for all } w \in V\} = 0$$

Because the killing form is symmetric bilinear, this is the same as saying the orthogonal complement $V^\perp = 0$.

Theorem 3.3.4: Cartan's Second Criterion

Let L be a Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if k is non-degenerate.

Lemma 3.3.5

Let L be a Lie algebra over $\mathbb{C}$. Let I be an ideal of L. If L is semisimple, then the following are true.

- $L = I \oplus I^\perp$ as Lie algebras
- I and $I^\perp$ are semisimple.

Theorem 3.3.6

Let L be a Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if there exists simple ideals $I_1, \dots, I_k$ of L such that

$$L = I_1 \oplus \cdots \oplus I_k$$

3.4 Derivations of Lie Algebras

Definition 3.4.1: Derivations

Let L be a Lie algebra over $\mathbb{F}$. Let $\phi : L \to L$ be a Lie algebra homomorphism. We say that ϕ is a derivation if

$$\phi([a, b]) = [a, \phi(b)] + [\phi(a), b]$$

for all $a, b \in L$.

Definition 3.4.2: The Lie Subalgebra of All Derivations

Let L be a Lie algebra over $\mathbb{F}$. Define the Lie subalgebra of derivations

$$\text{Der}_{\mathbb{F}}(L) = \{\phi : L \to L \mid \phi \text{ is a derivation } \}$$

of $GL(L)$ to be the set of all derivations of L together with pointwise addition and composition.

Lemma 3.4.3

Let L be a Lie algebra. Then the adjoint homomorphism $\text{ad}(x) : L \to L$ for each $x \in L$ is a derivation.

Recall the we have seen from Linear Algebra the following fact: If $T \in GL(V)$ is a linear map on V with minimal polynomial factorizing as

$$\mu_T = (x - \lambda_1)^{a_1} \cdots (x - \lambda_k)^{a_k}$$

where the eigenvalues λ_i are distinct and $a_i \geq 1$, then V decomposes as a direct sum of T-invariant subspaces

$$V = V_1 \oplus \cdots \oplus V_k$$

where $V_i = \ker(T - \lambda_i I)^{a_i}$ is the generalized eigenspace.

Proposition 3.4.4

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then

$$\text{Der}_{\mathbb{C}}(L) = \{\text{ad}(x) : L \to L \mid x \in L\}$$

In other words, the only derivations of L is given exactly by the adjoints.

3.5 Abstract Jordan Decomposition

Definition 3.5.1

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $x \in L$. Define an abstract Jordan decomposition of x to be a pair of elements $d, n \in L$ such that the following are true.

- $x = d + n$
- $\text{ad}(d) \in GL(L)$ is diagonalizable and $\text{ad}(n) \in GL(L)$ is nilpotent.
- $[d, n] = 0$

Corollary 3.5.2

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $x \in L$ be an element. Then there exists a unique abstract Jordan decomposition of x.

Overloading notation: notes says that $x \in L$ is nilpotent if $\text{ad}(x) \in GL(L)$ is nilpotent. Similarly for semisimplicity (which equals diagonalizable in $GL(L)$).

4 Introduction to Lie Groups

4.1 Lie Groups

Definition 4.1.1: Lie Groups

A Lie group G is a smooth manifold which is also a group such that the multiplication map $G \times G \to G$ given by $(g, h) \mapsto gh$ and the inverse map $i : G \to G$ given by $g \mapsto g^{-1}$ are smooth maps.

Proposition 4.1.2

Let G be a Lie group. A subgroup H of G has the unique structure of a Lie subgroup if H is closed in G.

4.2 Relation between Lie Groups and Lie Algebras

For a group G, denote the left multiplication map of $h \in G$ by l_h. If G is a Lie group, we have seen that l_h is a smooth map, and so it induces a differential $(l_h)_*$.

Definition 4.2.1: Left Invariant Vector Field

Let G be a Lie group and X a vector field on G. We say that X is left invariant if

$$(l_h)_*(X_g) = X_{hg}$$

for all $X_g \in T_g(G)$.

Proposition 4.2.2

Let G be a Lie group. The vector space of left invariant vector fields of G is a Lie algebra of dimension $\dim(G)$. Moreover, if $X_e \in T_e(G)$ is a tangent vector at e the identity, then there is a unique left invariant vector field X on G such that its identity is X_e.

Definition 4.2.3: Lie Algebra of a Lie Group

Let G be a Lie group. Define the Lie algebra V of G to be the vector space $T_e(G)$.

Recall that given a homomorphism of Lie groups $\phi : G \to H$, it induces a differential $\phi_* : T_g(G) \to T_{\phi(g)}(H)$.

Proposition 4.2.4

Let $\phi : G \to H$ be a homomorphism of Lie groups with Lie algebras V and W respectively. Then the induced map from the differential $\phi_* : V \to W$ is a Lie algebra homomorphism.

5 Representation Theory of Lie Algebras

5.1 Representations and Modules

Definition 5.1.1: Representations of a Lie Algebra

Let L be a Lie algebra. Let V be a vector space. A representation of L is a Lie algebra homomorphism
$$\rho : L \to GL(V)$$

Lemma 5.1.2

Let L be a Lie algebra. Then the adjoint homomorphism
$$\text{ad} : L \to GL(L)$$
is a representation of L.

Definition 5.1.3: Modules over Lie Algebras

Let L be a Lie algebra over k. An L-module consists of a vector space V over k together with an action
$$\cdot : L \times V \to V$$
such that the following are true.

- Linearity in L: For all $a, b \in k$ and $x, y \in L$, we have that
$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$$
for all $v \in V$.
- Linearity in V: For all $x \in L$, we have that
$$x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$$
for all $a, b \in k$ and $v, w \in V$.
- Commutes with the Lie bracket: $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

The third item may seem unnatural. Indeed the first two axioms are the usual ones for a module over an associative algebra. We think about the third axiom in the following way: Giving a map $\cdot : L \times V \to V$ is the same as giving a map $L \to GL(V)$ (curling / internal hom adjunction). The third item then means that the map $L \to GL(V)$ commutes with the Lie bracket. This is the content of the following lemma.

Proposition 5.1.4

Let L be a Lie algebra over k. Then representations of L and L-modules are in bijection
$$\{\cdot : L \times V \to V \text{ an } L\text{-module}\} \overset{1:1}{\leftrightarrow} \{L \to GL(V) \text{ a Lie algebra representation}\}$$
This bijection is given by sending an L-module $\cdot : L \times V \to V$ to the Lie algebra homomorphism $\phi : L \to GL(V)$ defined by $\phi(l)(v) = l \cdot v$.

These two ways two think about the same thing is natural. Recall that a representation of a group can be thought of as either group homomorphism $G \to GL(V)$ or a $k[G]$-module.

Idea???? This bijection is a ???-homomorphism.

Definition 5.1.5: Submodule of Modules over Lie Algebras

Definition 5.1.6: L-Module Homomorphisms

Definition 5.1.7: L-Module Isomorphisms

Definition 5.1.8: Completely Reducible

Theorem 5.1.9: Weyl's Theorem

Every non-trivial semisimple Lie algebra over $\mathbb{C}$ is completely reducible.

5.2 Weights

Definition 5.2.1: Eigenvectors of GL(V)

Let V be a vector space over a field k. Let M be a Lie subalgebra of $GL(V)$. We say that $v \in V$ is an eigenvector of M if for all $T \in M$, v is an eigenvector of T in the sense of Linear Algebra.

This notion of eigenvectors for Lie algebras is different to the standard notion of eigenvectors in linear algebra. Notice that an eigenvector of a Lie algebra M is a vector $v \in V$ that is simultaneously an eigenvector of all linear maps $T \in M \leq GL(V)$. Now we can rephrase this in another way.

Lemma 5.2.2

Let V be a vector space over a field k. Let M be a Lie subalgebra of $GL(V)$. Then $v \in V$ is an eigenvector of M if and only if there exists a linear map $\lambda : M \to k$ such that

$$T(v) = \lambda(T)v$$

for all $T \in M$.

We use the existence of a linear map $\lambda : M \to k$, conditional on $v \in V$, to determine whether v is an eigenvector of M.

Definition 5.2.3: Subspace of Eigenvectors of GL(V)

Let V be a vector space over a field k. Let M be a Lie subalgebra of $GL(V)$. Define the subspace of eigenvectors of M by

$$V_\lambda = \{v \in V \mid T(v) = \lambda(T)v \text{ for all } T \in M\}$$

Definition 5.2.4: Weights

Let V be a vector space over a field k. Let M be a Lie subalgebra of $GL(V)$. We say that a linear map $\lambda : M \to k$ is a weight of M if $V_\lambda \neq 0$.

Lemma 5.2.5

Let V be a vector space over a field k. Let M be a Lie subalgebra of $GL(V)$. Let I be an ideal of M. Let $W = \{w \in V \mid T(w) = 0 \text{ for all } T \in I\}$. Then W is an M-invariant subspace of V.

6 Root Systems

6.1 Cartan Subalgebras

Definition 6.1.1: Cartan Subalgebra

Let L be a Lie algebra. Let $H \leq L$ be a Lie subalgebra of L. We say that H is a Cartan subalgebra of L if the following are true.

- H is abelian
- For each $h \in H$, the abstract Jordan decomposition $h = d + n$ of h is such that $n = 0$ (In other words ad(h) is semisimple????)
- H is maximal with respect to these two properties

The reason we would like to consider such a subalgebra is due to the following result in Linear Algebra:

Lemma 6.1.2

Let V be a vector space over k. Suppose that $T_1, \dots, T_n \in GL(V)$ is diagonalizable. Then there exists a basis of $GL(V)$ such that $T_1, \dots, T_n$ are diagonal if and only if $T_1, \dots, T_n$ pairwise commute.

Lemma 6.1.3

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then there exists a non-trivial Cartan subalgebra.

6.2 Roots and Root Systems

Let V be a vector space. We denote the dual space of V by V^*.

Let L be semisimple over $\mathbb{C}$. Let H be a Cartan subalgebra. By lmm 5.1.2, we can choose a basis $v_1, \dots, v_m$ of $GL(V)$ such that ad(h) is diagonal for all $h \in H$. In particular, such a basis consists of common eigenvectors of ad(h). Fix such a common eigenvector v and write its eigenvalue by $\alpha(\text{ad}(h))$ (dependent on h). Then we have

$$\text{ad}(h)(v) = \alpha(\text{ad}(h))(v)$$

and in particular $\alpha : \text{ad}(H) \to \mathbb{C}$ is a weight of ad(H). Since L is semisimple, we have an isomorphism ad$(H) \cong H$. So we can think of the weight as an element

$$\alpha : H \cong \text{ad}(H) \to \mathbb{C}$$

of the dual space H^*. This motivates the following definition.

Definition 6.2.1: Roots of Lie Algebra with respect to a Cartan Subalgebra

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan subalgebra of L. A root of L relative to H is an element $\alpha \in H^*$ such that $\alpha \neq 0$ and the weight space

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)(x) \text{ for all } h \in H\}$$

is non-zero. In this case, we call L_α the root space of α.

Definition 6.2.2: The Set of Roots relative to a Cartan Subalgebra

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan subalgebra of L. Define the set of roots of L relative to H to be

$$\Psi = \{\alpha \in H^* \mid \alpha \text{ is a root of } L \text{ with respect to } H\}$$

Note that Ψ is finite since we assume that L is finite dimensional.

Lemma 6.2.3

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then there is a decomposition of direct sums

$$L = L_0 \oplus \bigoplus_{\alpha\in\Psi} L_\alpha$$

Proposition 6.2.4

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Lie subalgebra of L. Then the following are true.

- $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$
- If $\alpha + \beta \neq 0$, then $k(L_\alpha, L_\beta) = 0$ where k is the killing form of L.
- $L_0 \cap L_0^\perp = \{0\}$. In particular, $k|_{L_0}$ is non-degenerate.

Proposition 6.2.5

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Lie subalgebra of L. Then

$$H = C_L(H)$$

Lemma 6.2.6

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Lie subalgebra of L. Then the following are true.

- Ψ span H^*.
- If $\alpha \in \Psi$, then $-\alpha \in \Psi$.

Lemma 6.2.7

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Lie subalgebra of L. Then the following are true.

- For each $\alpha \in \Psi$, there exists $t_\alpha \in H \setminus \{0\}$ such that
$$[x, y] = k(x, y)t_\alpha$$
for all $x \in L_\alpha$ and $y \in L_{-\alpha}$. Moreover, $k(t_\alpha, h) = \alpha(h)$ for all $h \in H$.
- For each $\alpha \in \Psi$, $[L_\alpha, L_{-\alpha}] = \mathbb{C}\langle t_\alpha\rangle$.
- For each $\alpha \in \Psi$, $\alpha(t_\alpha) \neq 0$.

7 Classification of Semisimple Lie Algebras over $\mathbb{C}$