Bundle Structures in Topology

Labix

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Abstract

- Notes on Algebraic Topology by Oscar Randal-Williams

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1 Fibrations and Cofibrations

1.1 Fibrations and The Homotopy Lifting Property

Definition 1.1.1: The Homotopy Lifting Property

Let $p : E \to B$ be a map and let X be a space. We say that p has the homotopy lifting property with respect to X if for every homotopy $H : X \times I \to B$ and a lift $\widetilde{H(-,0)} : X \to E$ of $H(-,0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \widetilde{H}"] & B \end{tikzcd}$$

Definition 1.1.2: Fibrations

We say that a map $p : E \to B$ is a fibration if it has the homotopy lifting property with respect to all topological spaces X. We call B the base space and E the total space.

Definition 1.1.3: The Hopf Fibration

Define the Hopf fibration $h : S^3 \to S^2$ as follows. Consider S^2 as the one point compactification of $\mathbb{C}$. Also consider $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Define the map h by

$$(z_1, z_2) \to \frac{z_2}{z_1}$$

Example 1.1.4

The Hopf fibration $h : S^3 \to S^2$ is a fibration. Moreover, the fibers of the Hopf fibration are circles S^1.

Proof. We can rewrite the coordinates of S^3 by $r_j e^{i\theta_j}$. Then

$$h(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)}$$

Fix $re^{i\theta} \in S^2$. Then there exists a unique pair (r_1, r_2) that solves the simultaneous equation $rr_1 = r_2$ and $r_1^2 + r_2^2 = 1$. □

We will see that fibrations are a very well behaved class of maps in **Top**.

Lemma 1.1.5

Let X, Y, Z be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Let $h : X \to Y$ be a map over Z. If f and g are fibrations, then h is a homotopy equivalence if and only if it is a fiber homotopy equivalence.

1.2 Cofibrations and The Homotopy Extension Property

Definition 1.2.1: The Homotopy Extension Property

Let $i : A \to X$ be a map and let Y be a space. Denote i_0 the inclusion map $A \times \{0\} \hookrightarrow A \times I$. We say that i has the homotopy extension property with respect to Y if for every homotopy $H : A \times I \to Y$ and every map $f : X \to Y$ such that

$$H \circ i_0 = f \circ i$$

there exists a homotopy $\widetilde{H} : X \times I \to Y$ such that the following diagram commute:

$$\begin{tikzcd}
A \cong A \times \{0\} \arrow[r, "\iota_0"] \arrow[d, hook, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}_I"] \arrow[ddr, bend left, "H"] & \\
X \cong X \times \{0\} \arrow[r, "\iota_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
& & Y
\end{tikzcd}$$

Definition 1.2.2: Cofibrations

Let A, X be spaces. Let $i : A \to X$ be a map. We say that i is a cofibration if it has the homotopy extension property for all spaces Y.

Proposition 1.2.3

Let A, X be spaces. Let $i : A \to X$ be a cofibration. Then $i : A \to i(A)$ is a homeomorphism.

There is actually an easier way to write out cofibrations when (X, A) is a pair of spaces.

Lemma 1.2.4

Let (X, A) be a pair of spaces with A closed in X. Let $\iota : A \to X$ be the inclusion. Then ι is a cofibration if and only if for all spaces Y and maps $f : X \to Y$ and $H : A \times I \to Y$, there exists a map $\tilde{H} : X \times I \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "f \cup H"] \arrow[d, "\iota"'] & Y \\
X \times I \arrow[ur, "\tilde{H}"'] &
\end{tikzcd}$$

1.3 Basic Properties of Fibrations and Cofibrations

Proposition 1.3.1

Let $X_1, X_2, Y_{,1}, Y_2 \in \mathbf{CGWH}$. Let $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ be maps. Then the following are true.

- If p_1 and p_2 are fibrations then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a fibration.
- If p_1 and p_2 are cofibrations then $p_1 \coprod p_2 : X_1 \coprod X_2 \to Y_1 \coprod Y_2$ is a cofibration.

Proposition 1.3.2

Let $X, Y, Z \in \mathbf{CGWH}$. Let $f : X \to Y$ be a map.

- Let f be a fibration. Consider the following lifting problem:

$$\begin{tikzcd}
Z \times \{0\} \arrow[r, "g"] \arrow[d, hook, "i_0"'] & X \arrow[d, "f"] \\
Z \times I \arrow[r, "h"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

If h_0 and h_1 are both solutions to the lifting problem, then h_0 and h_1 are homotopic relative to $Z \times \{0\}$.

- Let f be a cofibration. Consider the following extension problem:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"'] & Z \times \{0\} \arrow[d, hook, "\mathrm{ev}_0"] \\
Y \arrow[r, "h"'] \arrow[ur, dashed] & Z \times I
\end{tikzcd}$$

If h_0 and h_1 are both solutions to the extension problem, then h_0 and h_1 are homotopic relative to Z.

1.4 Serre Fibrations

Definition 1.4.1: Serre Fibration

We say that a map $p : E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

Lemma 1.4.2

Every (Hurewicz) fibration is a Serre fibration.

Proof. This is true since Hurewicz fibrations satisfies the homotopy lifting property with respect to all topological spaces, including CW complexes. $\square$

2 Vector Bundles

2.1 Basic Definitions

Definition 2.1.1: Vector Bundles

Let F be a field. A vector bundle (E, B, p) consists of two topological spaces E and B, a continuous surjection $p : E \to B$ such that

- For every $b \in B$, the fibre $E_b = p^{-1}(b)$ is an F-vector space of dimension k.
- For every $b \in B$, there exists an open neighbourhood $U \subseteq B$ of p and a homeomorphism $\phi : p^{-1}(U) \to U \times F^k$ such that for $\pi : U \times F^k \to U$ the projection map, the following diagram commutes

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times K^r \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  and the map
  $$E_b \xrightarrow{\phi|_{E_b}} \{b\} \times F^k \xrightarrow{\pi} F^k$$
  is a vector space isomorphism.

B is said to be the base space and E the total space. Each (U, ϕ) is said to be a local trivialization.

The local trivialization means that locally at a neighbourhood, the vector bundle looks the same the open set times F^k. In particular, there is also a notion of trivial bundle which means that the bundle is globally just $B \times \mathbb{R}^r$.

Definition 2.1.2: Sections

A section of a vector bundle $p : E \to B$ is a map $s : B \to E$ assigning to each $b \in B$ a vector space $s(b)$ in the fiber $p^{-1}(b)$.

Proposition 2.1.3

Let $p : E \to B$ be a vector bundle. Let s, s_1, s_2 be sections of E. Then $s_1 + s_2$ and λs are also vector bundles for any $\lambda \in \mathbb{R}$. Moreover, the set of all sections $s(E)$ is a vector space.

Definition 2.1.4: Morphism of Vector Bundles

Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be vector bundles. A morphism of these vector bundles is given by is a pair of continuous maps $f : E_1 \to E_2$ and $g : B_1 \to B_2$ such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

If $B = B_1 = B_2$ then the diagram collapses:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

Definition 2.1.5: Isomorphism of Vector Bundles

A bundle homomorphism from E_1 to E_2 is an isomorphism if there exists an inverse bundle homomorphism from E_2 to E_1. In this case, we say that E_1 and E_2 are isomorphic.

2.2 The Cocycle Conditions

Given two charts (U_α, ϕ_α) and (U_β, ϕ_β) of a vector bundle,

$$\phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times F^k \to (U_\alpha \cap U_\beta) \times F^k$$

is a well defined function. In particular, by fixing a point in $U_\alpha \cap U_\beta$, we obtain a linear map.

Definition 2.2.1: Transition Functions

Let $p : E \to B$ be an F-vector bundle of rank r. Let (U_α, ϕ_α) and (U_β, ϕ_β) be local trivialization. For each $x \in U_\alpha \cap U_\beta$, $\phi_\beta \circ \phi_\alpha^{-1}(x, -) : F^k \to F^k$ is a linear map. Define $g_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, F)$ by

$$x \mapsto \phi_\beta \circ \phi_\alpha^{-1}(x, -) : F^k \to F^k$$

In other words, $g_{U_\alpha U_\beta}$ is such that

$$\phi_\beta \circ \phi_\alpha^{-1}(x, v) = (x, g_{U_\alpha U_\beta}(x)v)$$

For $x \in U_\alpha \cap U_\beta$ and $v \in F^k$.

Proposition 2.2.2

Let $p : E \to B$ be a K-vector bundle of rank r. The transition functions of the vector bundle satisfies the following.

- Cocycle condition: $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I_r$ on $U_\alpha \cap U_\beta \cap U_\gamma$
- $g_{\alpha\alpha} = I_r$ on U_α

2.3 Operations on Vector Bundles

Definition 2.3.1: Whitney Sum

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be two vector bundles. Define the direct sum of the vector bundles to be

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 \mid p_1(v_1) = p_2(v_2)\}$$

together with the projection $p : E_1 \oplus E_2 \to B$ defined by $(v_1, v_2) \mapsto p_1(v) = p_2(v)$.

Lemma 2.3.2

The Whitney sum $E_1 \oplus E_2$ of two vector bundles is again a vector bundle.

Proposition 2.3.3: Tensor Product Bundle

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be vector bundles. Define the tensor product bundle of it to be

$$E_1 \otimes E_2 = \{p_1^{-1}(x) \otimes p_2^{-1}(x) | x \in B\}$$

The construction $E_1 \otimes E_2$ is a vector bundle over B.

Theorem 2.3.4: Pullback Bundle

Let $p : E \to Y$ be a vector bundle. Let $f : X \to Y$ be a continuous map. Then there exists E' and p' such that $p' : E' \to X$ is a vector bundle.

Theorem 2.3.5: Dual Bundle

Let $p : E \to B$ be a K-vector bundle. Then the dual bundle $p^* : E^* \to B$ defined by

$$E_b^* = \mathrm{Hom}_K(E_b, K)$$

is a vector bundle over B.

3 The Topology of Fiber Bundles

3.1 Fiber Bundles

Fiber bundles serve as somewhat of a generalization of both vector bundles and covering spaces, while being a special case of a fibration. It therefore has the properties of a fibration.

Definition 3.1.1: Fiber Bundles

Let E, B, F be spaces with B connected, and $p : E \to B$ a continuous map. We say that p is a fiber bundle over F if the following are true.

- $p^{-1}(b) \cong F$ for all $b \in B$
- $p : E \to B$ is surjective
- Local Triviality: For every $x \in B$, there is an open neighbourhood $U \subset B$ of x and a homeomorphism $\phi_U : p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi"] \\ & U & \end{tikzcd}$$

  where π is the projection by forgetting the second variable.

We say that B is the base space, E the total space. It is denoted as (F, E, B)

Intuitively, we would like a fiber bundle to locally look like the product $B \times F$. The condition is also equivalent to the following form: There exists an open cover $\{U_i \mid i \in I\}$ and a collection of homeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times F$ for which the same diagram commutes.

Vector bundles generalizes vector bundles in the sense that the fibers are no longer vector spaces but instead arbitrary spaces.

Lemma 3.1.2

Every vector bundle is a fiber bundle.

Proof. Indeed if $p : E \to B$ is a vector bundle, then each fiber $p^{-1}(b)$ is an n-dimensional vector spaces over a field F. Moreover, by definition the local triviality condition is also satisfied. □

A lot of examples of fiber bundles therefore come from vector bundles. Another familiar collection of examples come from covering space theory.

Lemma 3.1.3

Every covering space is a fiber bundle.

Proof. If $p : \tilde{X} \to X$ is a covering space, then we have seen that $p^{-1}(x)$ remains constant as $x \in X$ varies. Moreover, $p^{-1}(x)$ has the discrete topology with countable fiber since each $p^{-1}(U)$ is a disjoint union for $U \subseteq X$ open. Thus they must all be homeomorphic.

Finally, for any $U \subseteq X$, recall that

$$p^{-1}(U) = \coprod_{i \in I} V_i$$

where each $V_i \cong U$. It is clear by definition that $|p^{-1}(x)||I|$ for any $x \in X$. By giving I the discrete topology, we obtain a homeomorphism $p^{-1}(x) \cong I$. The homeomorphism

$p^{-1}(U) = \coprod_{i\in I} V_i$ translates to

$$p^{-1}(U) = \coprod_{i\in I} V_i \cong \coprod_{i\in I} U \cong U \times I$$

defined by $\tilde{x} \in V_i \mapsto (p(\tilde{x}) = x, i)$. It is thus clear that the local triviality condition is satisfied. □

Proposition 3.1.4

Every fiber bundle is a Serre fibration.

We can provide a partial converse for the fact that every fiber bundle is a Serre fibration.

Proposition 3.1.5

Let $p : E \to B$ be a fiber bundle. If B is paracompact, then p is a (Hurewicz) fibration.

We there fore have inclusions

$$\substack{\text{Fiber}\\\text{Bundles}} \subset \substack{\text{Serre}\\\text{Fibrations}} \subset \substack{\text{(Hurewicz)}\\\text{Fibrations}}$$

Definition 3.1.6: Map of Fiber Bundles

Let (F_1, E_1, B_1) and (F_2, E_2, B_2) be fiber bundles. A map of fiber bundles is a pair of base-point preserving continuous maps $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ such that the following diagram commutes:

$$\begin{tikzcd}
E_1 \arrow[r, "\tilde{f}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\
B_1 \arrow[r, "f"'] & B_2
\end{tikzcd}$$

Such a map of fiber bundles determine a continuous of the fibers $F_1 \cong p_1^{-1}(b_1) \to p_2^{-1}(b_2) \cong F_2$.

A map of fiber bundles $(\tilde{f}, f)$ is said to be an isomorphism if there is a map $(\tilde{g} : E_2 \to E_1, g : B_2 \to B_1)$ such that $\tilde{g}$ is the inverse of $\tilde{f}$ and g is the inverse of f.

Notice that a morphism of fiber bundles preserves fibers. Indeed, If $p_1^{-1}(b)$ is a fiber of B, then using the commutativity of the diagram we have that

$$p_2(\overline{f}(p_1^{-1}(b))) = f(p_1(p_1^{-1}(b))) = f(b)$$

which implies that

$$p_2^{-1}(f(b)) = \overline{f}(p^{-1}(b))$$

or in other words, the fiber at $f(b)$ is the same as the fiber at b applied with $\overline{f}$.

Definition 3.1.7: Equivalent Fiber Bundles

Let $p : E_1 \to B_1$ and $p : E_2 \to B_2$ be two fiber bundles. We say that they are equivalent if there exists an isomorphism $(\tilde{f} : E_1 \to E_2, f : B_1 \to B_2)$ of fiber bundles.

There are two important special cases of fiber bundles that will appear time and time again.

Definition 3.1.8: Trivial Bundles

We say that a fiber bundle (F, E, B) is trivial if (F, E, B) is isomorphic to the trivial fibration $B \times F \to B$.

Definition 3.1.9: The Pullback Bundle

Let $p : E \to B$ be a fiber bundle with fiber F. Let $f : B' \to B$ be a continuous function. Define the pullback of p by f to be the space

$$f^*(E) = \{(b', e) \in B' \times E \mid p(e) = f(b')\}$$

Theorem 3.1.10

Let $p : E \to B$ be a fiber bundle. Suppose that $f, g : X \to B$ are homotopic maps. Then the pull back bundles

$$f^*(E) \cong g^*(E)$$

are equivalent.

3.2 Sections of a Bundle

Definition 3.2.1: Sections

Let (F, E, B) be a fiber bundle. A section on the fiber bundle is a map $s : B \to E$ such that

$$p \circ s = \mathrm{id}_B$$

Definition 3.2.2: Local Sections

Let (F, E, B) be a fiber bundle. Let $U \subset B$ be an open set. A local section of the fiber bundle on U is a map $s : U \to B$ such that

$$p \circ s = \mathrm{id}_U$$

3.3 Sphere Bundles

We now consider a special type of fibrations where the fibers are given by S^1. When we pick $n = 1$ we obtain the classical object of study in algebraic topology called the Hopf fibration.

Definition 3.3.1: Sphere Bundles

A sphere bundle is a fiber bundle $p : E \to B$ for which its fibers are the n-sphere S^n.

Theorem 3.3.2

Let $n \in \mathbb{N}$. Consider S^{2n+1} lying inside $\mathbb{C}^{n+1}$. Then canonical map $\mathbb{C}^n \to \mathbb{CP}^n$ given by

$$(z_0, \dots, z_n) \mapsto [z_0 : \cdots : z_n]$$

is a fiber bundle with fiber S^1.

Definition 3.3.3: Hopf Fibration / Hopf Bundle

The fiber bundle $p : S^3 \to S^2$ with fiber S^1 is called the Hopf fibration / Hopf bundle.