Algebraic Geometry 2

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Abstract

Algebraic Geometry is such a messy subject in a sense that a different books and lecture notes introduce different materials in a different orders, as well as having different prerequisites. After understanding a bit more in the subject, I believe that there is the need to give a clear distinction between traditional algebraic geometry and contemporary algebraic geometry. Although there are undoubtedly many overlapping between the two, I attempt to separate them to make clear their motivations as well as their results.

This book will mainly cover traditional algebraic geometry in the sense that the construction of affine and projective varieties will be covered, as well as the Hilbert Nullstellensatz theorems, morphisms, tangent maps and smoothness as well as classical constructions of some varieties. Affine schemes and sheaf theory are left for another time where they attempt to reinvent the fundamentals of algebraic geometry.

Knowledge on commutative algebra is required as a prerequisite. These set of notes make use of

- Algebraic Geometry I by I. R. Shafarevich and V. I. Danilov
- Algebraic Geometry by R. Hartshorne
- An Invitation to Algebraic Geometry by Karen. S, Pekka. K, Lauri .K, William .T

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1 The Tangent Space and Smooth Points

1.1 The Tangent Space of Affine Varieties

Definition 1.1.1: The Tangent Space of an Affine Variety

Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_r)$ be an affine variety over k. Define the tangent space of V at $p \in V$ to be the zero set

$$T_pV = \mathbb{V}\left(\sum_{k=1}^n \frac{\partial f_1}{\partial x_k}\bigg|_p (x_k - p_k), \dots, \sum_{k=1}^n \frac{\partial f_r}{\partial x_k}\bigg|_p (x_k - p_k)\right)$$

It should first be made sense that the definition is independent of the choice of polynomials $f_1, \dots, f_r$ of the zero set.

Proposition 1.1.2

Let V be a closed affine variety over $\mathbb{C}$. Let $p \in V$. Let m_p denote the corresponding maximal ideal. Then there is an isomorphism

$$T_pV \cong \left(\frac{m_p}{m_p^2}\right)^*$$

given by ?????. In particular, we have the identity

$$\dim(T_pV) = \dim_{\mathbb{C}[V]/m_p}(m_p/m_p^2)$$

Definition 1.1.3: The Jacobian Matrix

Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_m) \subseteq \mathbb{A}_k^n$ be an affine variety. Let $p \in V$. Define the Jacobian matrix of V at p to be the $m \times n$ matrix

$$J_{V,p} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_p \end{pmatrix}$$

Proposition 1.1.4

Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_m) \subseteq \mathbb{A}_k^n$ be an affine variety. Let $p = (p_1, \dots, p_n) \in V$. Then

$$T_pV = \left\{ (x_1, \dots, x_n) \in \mathbb{A}_k^n \;\middle|\; J_{V,p} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix} = 0 \right\}$$

1.2 Smooth Points of an Affine Variety

We continue to restrict our attention to affine varieties.

Definition 1.2.1: Smooth and Singular Points of Affine Varieties

Let k be a field. Let X be an irreducible affine variety over k. Let $p \in X$ be a point. We say that p is a smooth point of X if

$$\dim(T_p(X)) = \dim(X)$$

Otherwise, we say that p is a singular point of X.

Proposition 1.2.2

Let $V = \mathbb{V}(f_1, \dots, f_m) \subseteq \mathbb{A}^n_{\mathbb{C}}$ be an irreducible affine variety. Let $p \in V$. Then the following are equivalent.

- p is a smooth point of V.
- $\text{rank}(J_{V,p}) = n - \dim(V)$.
- $\mathcal{O}_{V,p}$ is a regular local ring.

In particular, this shows that smoothness is independent of the choice of generators of V, because we have given a characterization in terms of a property of the local ring $\mathcal{O}_{V,p}$.

Hard to prove: smoothness is preserved by isomorphisms.

1.3 The Tangent Space of Varieties in General

Definition 1.3.1: The Tangent Space of a Quasi-Projective Variety

Let k be a field. Let V be a quasi-projective variety over k. Let $p \in V$. Define the tangent space of V at p to be

$$T_pV = \left(\frac{m_p}{m_p^2}\right)^*$$

where m_p is the unique maximal ideal of the local ring $\mathcal{O}_{V,p}$.

1.4 Smooth Points of a Variety in General

We can now motivate the definition of a smooth point using the purely algebraic characterization.

Definition 1.4.1: Smooth and Singular Points of A General Variety

Let X be a variety. We say $p \in X$ is a smooth point of X if the local ring $\mathcal{O}_{X,p}$ is a regular local ring. Otherwise, we say that p is a singular point of X.

Theorem 1.4.2

Let X be a variety. Then the set of singular points of X is a proper closed subset of X.

Proposition 1.4.3

Let X be a variety. If $p \in X$ is a smooth point, then $\mathcal{O}_{X,p}$ is a UFD.

Proposition 1.4.4

Let X be a variety and let $Y \subseteq X$ be an irreducible subvariety of X. If $p \in X$ is non-singular, then there exists an affine neighbourhood $U \subseteq X$ of x together with $f_1, \dots, f_k \in k[U]$

2 The Algebra of Rational Functions

2.1 Rational Functions and The Function Field

Definition 2.1.1: Function Field

Let k be a field. Let V be a variety over k. Define the set of rational functions on V to be

$$K(V) = \{(U, f) \mid U \subseteq V \text{ open and } f : U \to k \text{ is regular }\}/\sim$$

where we say that $(U, f) \sim (V, g)$ if there exists $W \subseteq U \cap V$ open such that $f|_W = g|_W$. Elements of the function field are called rational functions.

Lemma 2.1.2

Let k be a field. Let V be an variety over k. Define the operations

$$(U, f) + (V, g) = (U \cap V, f + g) \quad \text{and} \quad (U, f) \cdot (V, g) = (U \cap V, fg)$$

Then they induce well defined operations on $K(V)$ so that it is a k-algebra. Moreover, it is a field.

Lemma 2.1.3

Let V be a variety over $\mathbb{C}$. Then the following are true.

- The map $\mathcal{O}_V(V) \to \mathcal{O}_{V,p}$ given by $f \mapsto (V, f)$ is injective for any $p \in V$.
- The map $\mathcal{O}_{V,p} \to k(V)$ given by $(V, f) \mapsto (V, f)$ is injective for any $p \in V$.

Proposition 2.1.4

Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}^n_k$ be an irreducible affine variety. Then there is an isomorphism

$$K(V) \cong \text{Frac}(k[V])$$

Moreover, $K(V)$ is a finitely generated field extension of $\mathbb{C}$.

Proposition 2.1.5

Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}^n_k$ be an irreducible affine variety. Then we have

$$\text{trdeg}_k(K(V)) = \dim(V)$$

Proposition 2.1.6

Let k be an algebraically closed field. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety over k. Then there is a k-algebra isomorphism

$$K(X) \cong (k[V]_{(0)})_0$$

where the zero refers to taking the degree zero part of the graded ring.

Proof. □

2.2 Rational Maps between Varieties

Definition 2.2.1: Equivalent Maps

Let X, Y be irreducible varieties. Let $U_1, U_2 \subseteq X$ be open. Let $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ be morphisms of varieties. We say that f_1 and f_2 are equivalent if there exists an open subset $W \subseteq U_1 \cap U_2$ such that

$$f_1|_W = f_2|_W : W \to Y$$

Definition 2.2.2: Rational Maps

Let X, Y be irreducible varieties. A rational map $f : X \to Y$ is an equivalent class of morphisms of varieties $f : U \to Y$ for some open subset $U \subseteq X$.

Since open subsets of a variety dense, rational maps are maps that are defined almost entirely on X. In particular, notice that rational functions on an irreducible variety V is the same as a rational map $V \to k$.

Definition 2.2.3: Dominant Maps

Let X, Y be irreducible varieties. Let $f : X \to Y$ be a rational map defined on $U \subseteq X$. We say that f is dominant if $f(U)$ contains an open subset.

It only makes sense to compose rational maps if the former one is dominant.

Proposition 2.2.4

Let X, Y, Z be irreducible varieties. Let $f : X \to Y$ and $g : Y \to Z$ be rational maps. If f is dominant, then $g \circ f$ is rational.

Definition 2.2.5: Induced Map on Rational Functions

Let k be a field. Let X, Y be irreducible varieties over k. Let $\phi : X \to Y$ be a dominant rational map. Define the induced map on rational functions to be

$$\phi^* : K(Y) \to K(X)$$

given by $(U, f) \mapsto (\phi^{-1}(U), f \circ \phi)$.

Proposition 2.2.6

Let k be a field. Let X, Y be irreducible varieties over k. Let $\phi : X \to Y$ be a dominant rational map. Then the induced map $\phi^* : K(Y) \to K(X)$ is a k-algebra homomorphism.

Proposition 2.2.7

Let k be a field. Let X, Y be irreducible varieties over k. Then there is a one-to-one correspondence

$$\left\{\substack{\text{Dominant Rational Maps}\\ X \to Y}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{k\text{-algebra homomorphisms}\\ K(Y) \to K(X)}\right\}$$

given by $\phi \mapsto \phi^*$.

TBA: Equivalence of categories between irreducible varieties and dominant rational maps, and field extensions of k and k-algebra homomorphisms.

2.3 Birational Equivalence

Definition 2.3.1: Birational Maps

Let X, Y be irreducible varieties. Let $f : X \to Y$ be a dominant rational map defined on $U \subseteq X$. We say that f is a birational map if there exists a dominant rational map $g : Y \to X$ such that

$$g \circ f = \text{id}_U \quad \text{and} \quad f \circ g = \text{id}_V$$

for some open subsets $U \subseteq X$ and $V \subseteq Y$. In this case, we say that X and Y are birational.

Proposition 2.3.2

Let k be a field. Let X, Y be irreducible varieties over k. The the following conditions are equivalent.

- X and Y are birationally equivalent
- There exists open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V
- $K(X)$ and $K(Y)$ are isomorphic k-algebras

Lemma 2.3.3

Let $n \in \mathbb{N}$. Then $\mathbb{A}^n$ is birationally equivalent to $\mathbb{P}^n$.

Proof. The function field of $\mathbb{A}^n$ is given by $K(\mathbb{A}^n) = K(x_1, \dots, x_n)$. On the other hand, we can compute the function field of $\mathbb{P}^n$ to get

$$\begin{aligned} K(\mathbb{P}^n) &= \left(k[x_0, \dots, x_n]_{(0)}\right)_0 \\ &= (k(x_0, \dots, x_n))_0 \end{aligned}$$

The zeroth graded component of $k(x_0, \dots, x_n)$ is given by

$$(k(x_0, \dots, x_n))_0 = \left\{ \frac{f}{g} \in k(x_0, \dots, x_n) \mid \deg(f) = \deg(g) \right\}$$

Define a map $\phi : (k(x_0, \dots, x_n))_0 \to k(x_1, \dots, x_n)$ by the map

$$f(x_0, \dots, x_n)/g(x_0, \dots, x_n) \mapsto f(1, x_1, \dots, x_n)/g(1, x_1, \dots, x_n)$$

Evaluation of the zeroth variable with 1 is a well defined $\mathbb{C}$-algebra homomorphism. I claim that this is a bijection. Define another map $\psi : k(x_1, \dots, x_n) \to (k(x_0, \dots, x_n))_0$ by

$$h/k \mapsto x_0^d h(x_1/x_0, \dots, x_n/x_0)/x_0^d k(x_1/x_0, \dots, x_n/x_0)$$

where $d = \max\{\deg(f), \deg(g)\}$. By construction we see that $x_0^d h(x_1/x_0, \dots, x_n/x_0)$ has the same degree as $x_0^d k(x_1/x_0, \dots, x_n/x_0)$ and that they are homogeneous polynomials (similar to the homogenization of a polynomial). Moreover it is clear that ψ and ϕ are inverses of each other. Hence we obtain an isomorphism of $\mathbb{C}$-algebras. □

3 Normal Varieties

3.1 Normal Varieties

Definition 3.1.1: Normal Varieties

Let k be a field. Let X be a variety over k. We say that X is normal if $\mathcal{O}_{X,p}$ is a normal domain for all $p \in X$.

Proposition 3.1.2

Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then X is normal if and only if $k[X]$ is a normal domain.

3.2 Normalization

Definition 3.2.1: Normalization of an Affine Variety

Let k be a field. Let $X \subseteq \mathbb{A}^n$ be an affine variety over k. Define the normalization of X to be unique variety $\widetilde{X}$ such that $\overline{k[X]} = k[\widetilde{X}]$

Proposition 3.2.2

Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine variety over k. Then the following are true regarding the induced map

$$\nu : \widetilde{X} \to X$$

from the inclusion $k[X] \hookrightarrow k[\widetilde{X}] = \overline{k[X]}$.

- The map is birational.
- The map is surjective.
- $\nu^{-1}(p)$ is finite for any $p \in X$.

4 Resolution of Singularities

4.1 Blowing Ups

Definition 4.1.1: Blowing Up at $\mathbb{A}^n$

Let $n \in \mathbb{N}$. Define the blowing up of $\mathbb{A}^n$ at the point $0 \in \mathbb{A}^n$ to be

$$\text{BL}_0(\mathbb{A}^n) = \mathbb{V}\{(x_i y_j - x_j y_i \mid 1 \leq i, j \leq n\}) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

together with the projection map

$$\varphi : \text{BL}_0(\mathbb{A}^n) \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \overset{\text{proj.}}{\to} \mathbb{A}^n$$

defined by $(x_1, \dots, x_n, [y_1 : \dots : y_n]) \mapsto (x_1, \dots, x_n)$.

Proposition 4.1.2

The following are true regarding the blowing up $\text{BL}_0(\mathbb{A}^n)$ of $\mathbb{A}^n$ at 0 and the projection map $\varphi : \text{BL}_0(\mathbb{A}^n) \to \mathbb{A}^n$.

- $\varphi^{-1}(p)$ is a single point for $0 \neq p \in \mathbb{A}^n$.
- $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$.
- $\text{BL}_0(\mathbb{A}^n)$ is irreducible.

Definition 4.1.3: The Blowing Up of a Variety

Let $X \subseteq \mathbb{A}^n$ be a closed variety such that $0 \in X$. Define the blowing up of X at 0 to be the set

$$\widetilde{X} = \overline{\varphi^{-1}(X \setminus \{0\})}$$

If X passes through $p \in X$, then the blowing up of X at p is obtained by translating p to the origin and blowing up.

Note: if X is a curve we call this the exceptional divisor.

4.2 Normalization

5 Theory of Divisors

5.1 Divisors of a Variety

Definition 5.1.1: Divisors of a Variety

Let X be a variety. Let $C_1, \dots, C_r$ be irreducible closed subvarieties of X of codimension 1. A divisor of X is of the form
$$D = \sum_{i=1}^{r} k_i C_i$$
for $k_i \in \mathbb{Z}$. We say that k_i is the multiplicity of C_i in D. Define the free group of all divisors of X by
$$\text{Div}(X) = \mathbb{Z}\left\langle C \mid C_{\substack{\text{is an irreducible closed} \\ \text{subvariety of codimension } 1}} \right\rangle$$
Generators of $\text{Div}(X)$ are called prime divisors.

Definition 5.1.2: Effective Divisor

Let X be a variety. We say that a divisor
$$D = \sum_{i=1}^{r} k_i C_i$$
of X is effective if $k_i \geq 0$ for all i. In this case we write $D > 0$.

Definition 5.1.3: Divisor of a Function

Let X be a variety such that the set of singular points of X has codimension ≥ 2. Let $f \in K(X)$. Let C be a prime divisor of X.

Definition 5.1.4: Principal Divisors

Let X be a variety. A divisor of the form $D = \text{div}(f)$ for some $f \in K(X)$ is called a principal divisor.

Define the set of all principal divisors by $P(X)$.

Proposition 5.1.5

Let X be a variety. The set of all principal divisors $P(X)$ is a group.

Definition 5.1.6: Divisor Class Group

Let X be a variety. Define the divisor class group of X to be
$$\text{Cl}(X) = \frac{\text{Div}(X)}{P(X)}$$
We say that two divisors D_1 and D_2 are linearly equivalent if they lie in the same coset of $\text{Cl}(X)$, written as $D_1 \sim D_2$.

Definition 5.1.7: Degree of a Divisor

Proposition 5.1.8

Let X be a variety. Then D is a principal divisor if and only if $\deg(D) = 0$.

5.2 The Linear System of a Divisor

Definition 5.2.1: Associated Vector Space of a Divisor

Let X be a nonsingular variety. Define the associated vector space of a divisor D of X to be

$$\mathcal{L}(D) = \{f \in K(X) \mid \text{div}(f) + D \geq 0\} \cup \{0\}$$

Lemma 5.2.2

Let X be a nonsingular variety. Then $\mathcal{L}(D)$ is a vector space over the field k.

Definition 5.2.3: Dimension of the Associated Vector Space

Let X be a nonsingular variety. Denote $\ell(D)$ the dimension of $\mathcal{L}(D)$, which is also called the dimension of D.

Theorem 5.2.4

Linearly equivalent divisors have the same dimension.

6 Intersection Theory