Differential Forms in n-dimensional Real Space

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Abstract

Before diving deep into the abstract Differential Geometry, iit is crucial to understand those methods we apply to non-euclidean surfaces by first studying it on a more familiar setting.

Contents

1 Introduction to Multilinear Algebra

1.1 Basic Definitions

Definition 1.1.1 (Multilinear Function)**.** Let V be a vector space over $\mathbb{R}$. A function $f : V^k \to \mathbb{R}$ is k-linear if it is linear in each of its k arguments

$$f(v_1, \dots, av_i + bw_i, \dots, v_k) = af(v_1, \dots, v_i, \dots, v_k) + bf(v_1, \dots, w_i, \dots, v_k)$$

for $i \in \{1, \dots, k\}$ nd $a, b \in \mathbb{R}$. It is also called a k-tensor on V. Denote the set of all k-tensors on V by $L_k(V)$

Definition 1.1.2 (Symmetric)**.** Let V be a vector space over $\mathbb{R}$. $f : V^k \to \mathbb{R}$ is symmetric if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = f(v_1, \dots, v_k)$$

for all $\sigma \in S_k$

Definition 1.1.3 (Alternating)**.** Let V be a vector space over $\mathbb{R}$. $f : V^k \to \mathbb{R}$ is alternating if

$$f(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \text{sign}(\sigma) f(v_1, \dots, v_k)$$

for all $\sigma \in S_k$. Alternating k-tensors are also called k-covectors. Denote the set of all k-covectors $\Lambda_k(V)$. Thus we have $\Lambda_k(V) \subseteq L_k(V)$

Definition 1.1.4. Let $f : V^k \to \mathbb{R}$ be a k-linear function. Define

$$(Sf)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \sigma(f)$$

Define

$$(Af)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \text{sign}(\sigma)\sigma(f)$$

Proposition 1.1.5. Let $f : V^k \to \mathbb{R}$ be a k-linear function. Then Sf is symmetric and Af is alternating.

Proof. We have

$$\begin{aligned} \tau(Sf) &= \sum_{\sigma \in S_k} (\tau\sigma) f \\ &= Sf \end{aligned}$$

and

$$\begin{aligned} \tau(Af) &= \sum_{\sigma \in S_k} \text{sign}(\sigma)(\tau\sigma) f \\ &= \text{sign}(\tau) \sum_{\sigma \in S_k} \text{sign}(\tau\sigma)(\tau\sigma) f \\ &= \text{sign}(\tau)(Af) \end{aligned}$$

□

Lemma 1.1.6. If f is an alternating k-linear function on a vector space V, then $Af = (k!)f$.

Proof. We have

$$\begin{aligned} Af &= \sum_{\sigma\in S_k} \operatorname{sign}(\sigma)(\sigma f) \\ &= \sum_{\sigma\in S_k} \operatorname{sign}(\sigma)\operatorname{sign}(\sigma) f \\ &= \sum_{\sigma\in S_k} f \\ &= (k!)f \end{aligned}$$

□

1.2 Tensor Product and Wedge Product

Definition 1.2.1 (Tensor Product)**.** Let f be k-linear on V and g be l linear on V. Their tensor product is defined to be the $k+l$ linear function

$$(f\otimes g)(v_1,\dots,v_{k+l}) = f(v_1,\dots,v_k)g(v_{k+1},\dots,v_{k+l})$$

Proposition 1.2.2. Let f, g, h be multilinear functions on V. Then

$$f\otimes(g\otimes h) = (f\otimes g)\otimes h$$

Definition 1.2.3 (Wedge Product)**.** Let $f\in\Lambda_k(V)$ and $g\in\Lambda_l(V)$. Their wedge product is defined to be the $k+l$ linear function

$$f\wedge g = \frac{1}{k!l!}A(f\otimes g)$$

Proposition 1.2.4. Let $f\in\Lambda_k(V)$ and $g\in\Lambda_l(V)$. Then

$$f\wedge g = (-1)^{kl} g\wedge f$$

Corollary 1.2.5. Let $f\in\Lambda_k(V)$ and k is odd. Then $f\wedge f = 0$

Proposition 1.2.6. Let f, g, h be multilinear functions on V. Then

$$f\wedge(g\wedge h) = (f\wedge g)\wedge h$$

Proposition 1.2.7. Let $f_k\in\Lambda_{d_k}(V)$ for $k\in\{1,\dots,n\}$. Then

$$f_1\wedge\cdots\wedge f_n = \frac{1}{(d_1)!\cdots(d_n)!}A(f_1\otimes\cdots\otimes f_n)$$

Definition 1.2.8 (Multi-index Notation)**.** Suppose that V is a vector space and $\alpha^1,\dots,\alpha^n$ the dual basis of V. Define $I=(i_1,\dots,i_k)$ and write α^I for $\alpha^{i_1}\wedge\cdots\wedge\alpha^{i_k}$. We usually want $i_1<\cdots<i_k$.

Lemma 1.2.9. Let $e_1,\dots,e_n$ be a basis for V and $\alpha^1,\dots,\alpha^n$ be the dual basis of V. Then

$$\alpha^I(e_J) = \delta^I_J\begin{cases} 1 & \text{if } I=J \\ 0 & \text{if } I\neq J \end{cases}$$

Proposition 1.2.10. The set of all α^I where $I = (i_1 < \cdots < i_k)$ form a basis for the space $\Lambda_k(V)$. The dimension of $\Lambda_k(V)$ is $\binom{n}{k}$

Corollary 1.2.11. If $k > \dim(V)$, then $\Lambda_k(V) = 0$

2 Tangent Vectors in $\mathbb{R}^n$

2.1 Tangent Space

Definition 2.1.1 (Tangent Space)**.** The set of all vectors with tail at $p \in \mathbb{R}^n$ is denoted $T_p(\mathbb{R}^n)$. We write a point in $\mathbb{R}^n$ as $p = (p_1, \dots, p_n)$ and a vector v in $T_p(\mathbb{R}^n)$ as $\langle v_1, \dots, v_n \rangle$

Definition 2.1.2 (Line Through a Point)**.** The line through a point $p \in \mathbb{R}^n$ with direction v has parametrization

$$c(t) = (p_1 + tv_1, \dots, p_n + tv_n)$$

with its i-component $c_i(t) = p_i + tv_i$

Definition 2.1.3 (Directional Derivative)**.** Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^\infty$. Let $v \in T_p(\mathbb{R}^n)$. The directional derivative of f in the direction v at p is defined to be

$$D_v(f) = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t))$$

Proposition 2.1.4. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^\infty$. Then

$$D_v(f) = \sum_{k=1}^n v_k \left.\frac{\partial f}{\partial x_k}\right|_p$$

and D_v is a map from $\mathcal{C}_p^\infty(\mathbb{R}^n) \to \mathbb{R}$

Proposition 2.1.5. The map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ given by $\phi(v) = D_v$ is an isomorphism of vector spaces.

Proposition 2.1.6. The standard basis of $T_p(\mathbb{R}^n)$ corresponds to

$$\left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\}$$

Definition 2.1.7 (Vector Fields)**.** A vector field X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector denoted $X_p \in T_p(R^n)$. This means that $X : \mathbb{R}^n \to T_p(\mathbb{R}^n)$

Proposition 2.1.8. For every vector field X,

$$X_p = \sum_{k=1}^n a_k(p) \left.\frac{\partial}{\partial x_k}\right|_p$$

where $a_k(p) \in \mathbb{R}$

3 Differential Forms on $\mathbb{R}^n$

3.1 Differential 1-forms

Definition 3.1.1 (Cotangent Space)**.** Define the cotangent space to $\mathbb{R}^n$ at p to be $T_p^*(\mathbb{R}^n)$, the dual space of $T_p(\mathbb{R}^n)$.

Definition 3.1.2 (Differential 1-form)**.** A differential 1-form is a function $\omega : U \subseteq \mathbb{R}^n \to \bigcup_{p\in U} T_p^*(\mathbb{R}^n)$ from $p \in \mathbb{R}^n$ to $\omega_p \in T_p^*(\mathbb{R}^n)$

Proposition 3.1.3. Fix $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Define $df_p : T_p(\mathbb{R}^n) \to \mathbb{R}$ by

$$(df)_p(X_p) = X_p(f)$$

Then the mapping $(df)(p) = (df)_p$ from p to $(df)_p$ is a differential 1-form.

Proposition 3.1.4. Suppose that $x_1, \dots, x_n$ are the standard coordinate for $\mathbb{R}^n$. Then for each point $p \in \mathbb{R}^n$,

$$\{(dx_1)_p, \dots, (dx_n)_p\}$$

is the basis for $T_p^*(\mathbb{R}^n)$ dual to

$$\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$$

in $T_p(\mathbb{R}^n)$

Proposition 3.1.5. If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^\infty$, then

$$df = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} dx_k$$

3.2 Differential k-forms

Definition 3.2.1 (Differential k-forms)**.** A differential k-form ω on $U \subseteq \mathbb{R}^n$ is a function that assigns to each point $p \in U$ an alternating k-linear function. This means $\omega : \mathbb{R}^n \to \Lambda_k(T_p(\mathbb{R}^n))$ Denote $\Omega^k(U)$ the vector space of $\mathcal{C}^\infty$ k-forms on U.

Proposition 3.2.2. A differential k-form ω is of the form

$$\omega = \sum_I \alpha_I dx^I$$

with $a_I : U \subseteq \mathbb{R}^n \to \mathbb{R}$

3.3 Exterior Derivative

Definition 3.3.1 (Exterior Derivative of 0-forms)**.** Let $f \in \mathcal{C}^\infty(U)$. Then f is a 0-form. Define its exterior derivative to be its differential $df \in \Omega^1(U)$.

Definition 3.3.2 (Exterior Derivative of k-forms)**.** Let $\omega = \sum_I \alpha_I dx^I \in \Omega^k(U)$. Define

$$d\omega = \sum_I d\alpha_I \wedge dx^I = \sum_I \left(\sum_j \frac{\partial \alpha_I}{\partial x_j} dx_j\right) \wedge dx^I \in \Omega^{k+1}(U)$$

Proposition 3.3.3. Let $\omega \in \Omega^k(\mathbb{R}^n)$. Then $d^2\omega = 0$

Definition 3.3.4 (Closed Forms)**.** A k-form ω on U is closed if $d\omega = 0$

Definition 3.3.5 (Exact Forms)**.** A k-form ω on U is exact if there exists a $k-1$ form τ such that $\omega = d\tau$.