Category Theory

Labix

July 17, 2024

Abstract

Contents

1 Categories and Functors

1.1 The Concept of a Category

Categories are introduced by Eileenberg-Maclane in 1945 to formally define the concept of naturality, and to lay foundations for homological algebra. Note that there are some set-theoretic subtleties with the following definition.

Definition 1.1.1: Categories

A category $\mathcal{C}$ consists of the following data.

- A collection $\mathrm{Obj}\,\mathcal{C}$ called the objects of $\mathcal{C}$.
- For every $C, D \in \mathrm{Obj}(\mathcal{C})$, a collection $\mathrm{Hom}_{\mathcal{C}}(C, D)$ called the morphisms from C to D. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ is denoted by $f : C \to D$. We call C the source, D the target.
- For every $C, D, E \in \mathcal{C}$, there is a map of sets
$$\circ : \mathrm{Hom}_{\mathcal{C}}(D, E) \times \mathrm{Hom}_{\mathcal{C}}(C, D) \to \mathrm{Hom}_{\mathcal{C}}(C, E)$$
called composition satisfying
  - Associativity:
$$(h \circ g) \circ f = h \circ (g \circ f)$$
for all $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} A$
  - Unitality: For every $C \in \mathcal{C}$, there is a morphism $\mathrm{id} : C \to C$ such that
$$\begin{aligned} \mathrm{id} \circ f &= f \quad \text{for } f : D \to C \\ g \circ \mathrm{id} &= g \quad \text{for } g : C \to D \end{aligned}$$

We write
$$\mathcal{C} = \left(\mathrm{Obj}\,\mathcal{C}, \mathrm{Hom}_{\mathcal{C}} = \bigcup_{C, D \in \mathrm{Obj}\,\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(C, D), \circ \right)$$
to abbreviate notation. Sometimes we write $\mathrm{Hom}_{\mathcal{C}}(C, D) = \mathrm{Hom}(C, D)$ if context is clear.

Definition 1.1.2: Small Categories

A category $\mathcal{C}$ is said to be small if all its morphisms $\mathrm{Hom}(-, -)$ form a set.

A category $\mathcal{C}$ is said to be locally small if between any pair of objects $C, D \in \mathrm{Obj}\,\mathcal{C}$, all the arrows between the pair of objects $\mathrm{Hom}_{\mathcal{C}}(C, D)$ form a set.

We now define different types of morphisms in a category. The idea is that a category is a formal context to compare objects via morphisms.

Definition 1.1.3: Types of Morphisms

Let $\mathcal{C}$ be a category. Let $f : C \to D$ be a morphism in $\mathcal{C}$. Then f is said to be a

- Isomorphism if there exists $g : D \to C$ such that $g \circ f = 1_C$ and $f \circ g = 1_D$
- Endomorphism if $C = D$
- Automorphism if it is endomorphic and isomorphic
- Monomorphism if for any pair of morphisms $h, k : B \to C$, $f \circ h = f \circ k$ implies $h = k$
- Epimorphism if for any pair of morphisms $h, k : D \to E$, $h \circ f = k \circ f$ implies $h = k$

It is important to note that monomorphisms and epimorphisms in a category whose objects have underlying sets translates roughly to the notion of injections and surjections. However not every monomorphisms are an injection and vice versa.

In particular, it is easy to see that every isomorphism is a monomorphism and an epimorphism, but not every morphism that is monic and epic is an isomorphism. For example, the inclusion $Z \hookrightarrow \mathbb{Q}$ is not an isomorphism in the category of rings, but it is monic and epic.

Proposition 1.1.4

Let $\mathcal{C}$ be a category and let $f : C \to D$ and $g : D \to E$ be morphisms in $\mathcal{C}$. Then the following are true.

- If f and g are monic then $g \circ f$ is monic
- If f and g are epic then $g \circ f$ is epic
- If $g \circ f$ is monic then f is monic
- If $g \circ f$ is epic then g is epic.
- If f is an isomorphism then f is monic and epic.

Proof.

- Suppose that $g \circ f \circ h = g \circ f \circ k$. Since g is monic, we conclude that $f \circ h = f \circ k$. Since f is monic, we conclude that $h = k$.
- Suppose that $h \circ g \circ f = k \circ g \circ f$. Since f is epic, we conclude that $h \circ g = k \circ g$. Since g is epic, we conclude that $h = k$.
- Suppose that $f \circ h = f \circ k$. Then $g \circ f \circ h = g \circ f \circ k$. Since $g \circ f$ is monic, we conclude that $h = k$.
- Suppose that $h \circ g = k \circ g$. Then $h \circ g \circ f = k \circ g \circ f$. Since $g \circ f$ is epic, we conclude that $h = k$.
- Suppose that f is an isomorphism with inverse $g : D \to C$. Suppose that $f \circ h = f \circ k$. Then applying g on the left gives $h = k$. Similarly if $h \circ f = k \circ f$, then applying g on the right gives $h = k$.

$\square$

Definition 1.1.5: Opposite Categories

Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\text{op}}$ is another category where

- $\text{Obj}\,\mathcal{C}^{\text{op}} = \text{Obj}\,\mathcal{C}$
- $\text{Hom}_{\mathcal{C}^{\text{op}}}(C, D) = \text{Hom}_{\mathcal{C}}(D, C)$ where for $f \in \text{Hom}_{\mathcal{C}}(D, C)$, write f^{op} for the corresponding morphism in $\text{Hom}_{\mathcal{C}^{\text{op}}}(C, D)$
- For every $C, D, E \in \mathcal{C}$, the composition

$$\circ : \text{Hom}_{\mathcal{C}^{\text{op}}}(D, E) \times \text{Hom}_{\mathcal{C}^{\text{op}}}(C, D) \to \text{Hom}_{\mathcal{C}^{\text{op}}}(C, E)$$

where for $f^{\text{op}} \in \text{Hom}_{\mathcal{C}^{\text{op}}}(C, D)$ and $g^{\text{op}} \in \mathcal{C}^{\text{op}}(D, E)$, define

$$g^{\text{op}} \circ_{\mathcal{C}^{\text{op}}} f^{\text{op}} = f \circ_{\mathcal{C}} g$$

One can also dualize notions and statements. For example monic and epic are dual notions. This

means that f is monic in $\mathcal{C}$ if and only if f is epic in $\mathcal{C}^{\text{op}}$.
In particular, this means that every notion we define for categories has a categorical dual.

Lemma 1.1.6

Let $\mathcal{C}$ be a category and $f : C \to D$ be a morphism. Then the following are equivalent.

- f is an isomorphism
- For all $E \in \operatorname{Obj}\mathcal{C}$, the map of sets
$$f_* : \operatorname{Hom}_{\mathcal{C}}(E, C) \to \operatorname{Hom}_{\mathcal{C}}(E, D)$$
defined by $g \mapsto f \circ g$ is a bijection
- For all $E \in \operatorname{Obj}\mathcal{C}$, the map of sets
$$f^* : \operatorname{Hom}_{\mathcal{C}}(D, E) \to \operatorname{Hom}_{\mathcal{C}}(C, E)$$
defined by $g \mapsto g \circ f$ is a bijection.

Proof.

- (1) $\implies$ (2): Let $f^{-1} : D \to C$ be the inverse of f. Then
$$(f^{-1})_* : \operatorname{Hom}_{\mathcal{C}}(E, D) \to \operatorname{Hom}_{\mathcal{C}}(E, C)$$
defined by $h \mapsto f^{-1} \circ h$ is an inverse of f_* since
$$\begin{aligned}(f^{-1})_*(f_*(g)) &= (f^{-1})(f \circ g)\\ &= f^{-1}(f \circ g)\\ &= f^{-1} \circ f \circ g && \text{(Associativity of morphisms)}\\ &= g\end{aligned}$$
And $f_*((f^{-1})_*(h)) = h$ similarly.
- (2) $\implies$ (1): Choose $E = D$. Then $f_* : \operatorname{Hom}_{\mathcal{C}}(D, C) \to \operatorname{Hom}_{\mathcal{C}}(D, D)$ is a bijection. Then $f^{-1} = (f_*)^{-1}(\operatorname{id}_D)$ is the inverse of f. But by definition $f_*(f^{-1}) = \operatorname{id}_D$ implies $f \circ f^{-1} = \operatorname{id}_D$. Now we just need to show that $f^{-1} \circ f = \operatorname{id}_C$. Now choose $E = C$. Then $f_* : \operatorname{Hom}_{\mathcal{C}}(C, C) \to \operatorname{Hom}_{\mathcal{C}}(C, D)$ is a bijection. Then $f^{-1} \circ f = \operatorname{id}_C$ if and only if $f_*(f^{-1} \circ f) = f_*(\operatorname{id}_C)$. Indeed, we have on the left hand side
$$f_*(f^{-1} \circ f) = f \circ f^{-1} \circ f = \operatorname{id}_D \circ f = f$$
and on the right, we have
$$f_*(\operatorname{id}_C) = f \circ \operatorname{id}_C = f$$
- (2) $\iff$ (3): Let $f : C \to D$ be a morphism in $\mathcal{C}$. We want $f^* : \operatorname{Hom}_{\mathcal{C}}(D, E) \to \operatorname{Hom}_{\mathcal{C}}(C, E)$ to be a bijection. But using the dual, we have that $\operatorname{Hom}_{\mathcal{C}}(D, E) = \operatorname{Hom}_{\mathcal{C}^{\text{op}}}(E, D)$ and $\operatorname{Hom}_{\mathcal{C}}(C, E) = \operatorname{Hom}_{\mathcal{C}^{\text{op}}}(E, C)$ which means that f_* actually maps g^{op} to
$$(f^*(g))^{\text{op}} = (g \circ_{\mathcal{C}} f)^{\text{op}} = f^{\text{op}} \circ_{\mathcal{C}^{\text{op}}} g^{\text{op}} = (f^{\text{op}})_*(g^{\text{op}})$$
This shows that f^* is actually $(f^{\text{op}})_*$. Then f^* is a bijection if and only if $(f^{\text{op}})_*$ is a bijection.

$\square$

Definition 1.1.7: Subcategories

Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D}$ of $\mathcal{C}$ consists of

- A subcollection $\operatorname{Obj}\mathcal{D} \subseteq \operatorname{Obj}\mathcal{C}$ of objects
- For each $C, D \in \operatorname{Obj}\mathcal{D}$, a subcollection $\operatorname{Hom}_{\mathcal{D}}(C, D) \subseteq \operatorname{Hom}_{\mathcal{C}}(C, D)$ closed under composition, and containing the identities of each objects in $\operatorname{Obj}\mathcal{D}$

We say that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ if $\operatorname{Hom}_{\mathcal{D}}(C, D) = \operatorname{Hom}_{\mathcal{C}}(C, D)$ for all $C, D \in \operatorname{Obj}\mathcal{D}$

1.2 Functoriality

The idea is that a good construction should also tell you what to do on morphisms.

Definition 1.2.1: Functors

Let $\mathcal{C}, \mathcal{D}$ be categories. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ consists of

- The object part of F where $F : \operatorname{Obj}\mathcal{C} \to \operatorname{Obj}\mathcal{D}$
- The arrow part of F where $F : \operatorname{Hom}_{\mathcal{C}}(C, D) \to \operatorname{Hom}_{\mathcal{D}}(F(C), F(D))$ for all $C, D \in \operatorname{Obj}\mathcal{C}$ satisfying:
  - For all $C \xrightarrow{f} D \xrightarrow{g} E$ in $\mathcal{C}$, we have
  $$F(g \circ_{\mathcal{C}} f) = F(g) \circ_{\mathcal{D}} F(f)$$
  - For all $C \in \operatorname{Obj}\mathcal{C}$,
  $$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$$

A contravariant functor is a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. This consists of

- $F : \operatorname{Obj}\mathcal{C} \to \operatorname{Obj}\mathcal{D}$
- $F : \operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}(D, C) \to \operatorname{Hom}_{\mathcal{D}}(F(C), F(D))$ for all $C, D \in \operatorname{Obj}\mathcal{C}$ satisfying:
  - For all $E \xrightarrow{g^{\mathrm{op}}} D \xrightarrow{f^{\mathrm{op}}} C$ in $\mathcal{C}$, we have
  $$F(f^{\mathrm{op}} \circ_{\mathcal{C}^{\mathrm{op}}} g^{\mathrm{op}}) = F(g) \circ_{\mathcal{D}} F(f)$$
  - For all $C \in \operatorname{Obj}\mathcal{C}$,
  $$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$$

Lemma 1.2.2

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $f : C \to D$ is an isomorphism in $\mathcal{C}$. Then $F(f) : F(C) \to F(D)$ is an isomorphism in $\mathcal{D}$.

Proof. Let $f^{-1} : D \to C$ be the inverse of f. Then $F(f^{-1}) : F(D) \to F(C)$ is the inverse of $F(f)$. Indeed, we have

$$F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(\mathrm{id}_D) = \mathrm{id}_{F(D)}$$

and $F(f^{-1}) \circ F(f) = \mathrm{id}_{F(C)}$ similarly. □

Proposition 1.2.3

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. Define $G \circ F : \mathcal{C} \to \mathcal{E}$ where

- On objects, $(G \circ F)(C) = G(F(C))$ for all $C \in \operatorname{Obj}\mathcal{C}$
- On morphisms, for $f : C \to D$ in $\mathcal{C}$, $(G \circ F)(f) = G(F(f))$.

Then $G \circ F$ is also a functor.

Proof. It clearly satisfies the requirements of a functor. $\square$

Definition 1.2.4: Types of Functors

A functor $F : \mathcal{C} \to \mathcal{D}$ is

- full if for each $A, B \in \mathcal{C}$, the map $\operatorname{Hom}_C(A, B) \to \operatorname{Hom}_D(FA, FB)$ is surjective
- faithful if for each $A, B \in \mathcal{C}$, the map $\operatorname{Hom}_C(A, B) \to \operatorname{Hom}_D(FA, FB)$ is injective
- fully faithful if it is full and faithful
- is an embedding of $\mathcal{C}$ on $\mathcal{D}$ if it is fully faithful and the object part of F is injective
- is essentially surjective if for every $D \in \mathcal{D}$ there exists $C \in \mathcal{C}$ such that $FC \cong D$

1.3 Split Morphisms

Functors in general do not preserve monomorphisms and epimorphisms. However if there is a morphism “witnessing” the monomorphism or epimorphism, the functor does preserve these type of morphisms. We have also seen that while isomorphisms must be monic and epic, the converse need not be true. There is a class of morphisms (and their dual) that resolve all these issues.

Definition 1.3.1: Split Morphisms

Let $\mathcal{C}$ be a category and $f : C \to D$ a morphism in $\mathcal{C}$. Then f is said to be a

- split monomorphism if there exists $r : D \to C$ such that $r \circ f = \mathrm{id}_C$.
- split epimorphism if there exists $s : D \to C$ such that $f \circ s = \mathrm{id}_D$

We say that r is a retract of f and s is a section of f.

One can easily see that split monomorphisms are dual to split epimorphisms similar to how the concept of monomorphisms and epimorphisms are dual to each other.

Proposition 1.3.2

Let $\mathcal{C}$ be a category and $f : C \to D$ a morphism in $\mathcal{C}$. Then the following are true.

- If f is a split monomorphism then f is a monomorphism
- If f is a split epimorphism then f is an epimorphism

Proof. Suppose that f has a retract r and that $f \circ h = f \circ k$. Then $r \circ f \circ h = r \circ f \circ k$ implies that $h = k$ since $r \circ f = \mathrm{id}_C$.

Now suppose that f has a section s and that $h \circ f = k \circ f$. Then $h \circ f \circ s = k \circ f \circ s$ implies that $h = k$ since $f \circ s = \mathrm{id}_D$. $\square$

Proposition 1.3.3

Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to \mathcal{D}$ a functor. Let $f : C \to D$ be a morphism in $\mathcal{C}$. Then the following are true.

- If f is a split monomorphism then $F(f)$ is a split monomorphism
- If f is a split epimorphism then $F(f)$ is a split epimorphism

Proof. Suppose that f has a retract r. Then by functoriality, we have that

$$F(r) \circ F(f) = F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$$

hence $F(f)$ has a retract $F(r)$.

Suppose that f has a section s. Then by functoriality, we have that

$$F(f) \circ F(s) = F(\mathrm{id}_D) = \mathrm{id}_{F(D)}$$

hence $F(f)$ has a section $F(s)$. □

Proposition 1.3.4

Let $\mathcal{C}$ be a category and $f : C \to D$ a morphism in $\mathcal{C}$. Then the following conditions are equivalent.

- f is an isomorphism
- f is a monomorphism and a split epimorphism
- f is a split monomorphism and an epimorphism

Proof.

- (1) $\implies$ (2): Suppose that f is an isomorphism. We have already seen that f is a monomorphism by **??**. Now there exists a morphism $s : D \to C$ such that $f \circ s = \mathrm{id}_D$. Hence f is a split epimorphism.
- (2) $\implies$ (3): Suppose that f is a monomorphism and a split epimorphism. We have seen that f is then an epimorphism. Since f is a split epimorphism, there exists a section $s : D \to C$ such that $f \circ s = \mathrm{id}_D$. We can then pre compose with f on both sides to obtain
$$f \circ s \circ f = f$$
Since f is a monomorphism, we conclude that $s \circ f = \mathrm{id}_C$. Thus f is a split monomorphism.
- (3) $\implies$ (1): Suppose that f is a split monomorphism and an epimorphism. Then f has a retract r for which $r \circ f = \mathrm{id}_C$. We can then post compose with f to obtain
$$f \circ r \circ f = f$$
Since f is an epimorphism, we conclude that $f \circ r = \mathrm{id}_D$. Thus f has a two sided inverse r and hence an isomorphism.

□

1.4 Categories as Objects

Definition 1.4.1: The Category of Locally Small Categories

Let **CAT** be the category where

- $\operatorname{Obj} \mathbf{CAT} =$ "all" the locally small categories
- For $\mathcal{C}, \mathcal{D} \in \operatorname{Obj} \mathrm{CAT}$, define $\operatorname{Hom}_{\mathbf{CAT}}(\mathcal{C}, \mathcal{D}) = \{\text{Functors } F : \mathcal{C} \to \mathcal{D}\}$
- Composition is defined as the composition of functors as seen in 1.2.3

This is a very large category.

Definition 1.4.2: The Category of Small Categories

Define **Cat** to be the full subcategory of **CAT** consisting of small categories.

Lemma 1.4.3

The category **Cat** of small categories is a locally small category.

Now that we have defined **Cat**, we can talk about isomorphisms of categories.

In general, few categories are isomorphic. There are categories which are not isomorphic but that we want to consider "the same".

1.5 Diagrams and Concrete Categories

Definition 1.5.1: Commutative Diagram

Let $\mathcal{J}$ be a category. A commutative diagram in a category $\mathcal{C}$ of shape $\mathcal{J}$ is a functor $X : \mathcal{J} \to \mathcal{C}$. A diagram is said to be small if I is small.

In nice cases, such a functor can be visualized by drawing the images of the objects and morphism of I in $\mathcal{C}$. For example, if I is the preorder

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 2 \arrow[r] & 3 \end{tikzcd}$$

then a functor $I \to \mathcal{C}$ consists of the following data in $\mathcal{C}$:

$$\begin{tikzcd} C_0 \arrow[r, "f"] \arrow[d, "g"] & C_1 \arrow[d, "\bar{g}"] \\ C_2 \arrow[r, "\bar{f}"] & C_3 \end{tikzcd}$$

such that $\bar{g} \circ f = \bar{f} \circ g$.

Note that functors also preserve diagrams. If $X : I \to \mathcal{C}$ is a commutative diagram ad $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F \circ X : I \to \mathcal{D}$ is again a commutative diagram.

Definition 1.5.2: The Category of Sets

The category **Set** of sets where

- $\operatorname{Obj}(\mathbf{Set}) =$ "all" sets

- For two set X and Y, the morphisms $\text{Hom}_{\textbf{Set}}(X,Y)$ consists of functions $f : X \to Y$.
- Composition is given by the composition of functions.

Definition 1.5.3: Concrete Category

A concrete category is a category $\mathcal{C}$ together with a faithful functor $\mathcal{C} \to \text{Set}$.

It turns out that it is difficult to construct examples of non-concrete categories. In fact, every small category admits a faithful functor to sets.

A non-examples was given by Freyd which says that the category of homotopic spaces is not concrete.

Proposition 1.5.4

Let $F : \mathcal{C} \to \text{Set}$ be a concrete category. Then $g \circ f = h$ in $\mathcal{C}$ is true if and only if $F(g) \circ F(f) = F(h)$.

Proof. Notice that $F(g) \circ F(f) = F(h)$ if and only if $F(g \circ f) = F(h)$ since F is a functor. Also $F(g \circ f) = F(h)$ if and only if $g \circ f = h$ since F is faithful. □

This in particular means that a diagram commutes in $\mathcal{C}$ if and only if F of the diagram commutes in Set. Thus it is very convenient to work in concrete categories.

1.6 New Categories From Old

Definition 1.6.1: Slice Category

Let $\mathcal{C}$ be a category and C an object. Define the slice category $\mathcal{C}/C$ to consist of the following data.

- The objects consists of morphisms $f : D \to C$ for $D \in \mathcal{C}$. This means that they are a pair (D, f).
- For two objects (D, f) and (E, g), a morphism in the slice category is a morphism $h : D \to E$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

- Composition of morphisms are defined as the composition in $\mathcal{C}$.

Dually, there is the notion of a coslice category.

Definition 1.6.2: Coslice Category

Let $\mathcal{C}$ be a category and C an object. Define the coslice category $C/\mathcal{C}$ to consist of the following data.

- The objects consists of morphisms $f : C \to D$ for $D \in \mathcal{C}$. This means that they are a pair (D, f).
- For two objects (D, f) and (E, g), a morphism in the slice category is a morphism $h : D \to E$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & C & \\ D \arrow[ur, "f"] \arrow[rr, "h"] & & E \arrow[ul, "g"'] \end{tikzcd}$$

- Composition of morphisms are defined as the composition in $\mathcal{C}$.

Proposition 1.6.3

Let $\mathcal{C}$ be a category and C an object. Then the two categories

$$(\mathcal{C}/C)^{\text{op}} = C/(\mathcal{C}^{\text{op}})$$

are equal.

Proof. An object in $C/(\mathcal{C}^{\text{op}})$ is a morphism $f^{\text{op}} : D \to C$ in $\mathcal{C}^{\text{op}}$. This is precisely an object in $(\mathcal{C}/C)^{\text{op}}$. Thus the objects of the two categories are the same.

A morphism in $C/(\mathcal{C}^{\text{op}})$ consists of two objects $f^{\text{op}} : D \to C$ and $g^{\text{op}} : E \to C$ together with a morphism $h^{\text{op}} : D \to E$ such that the following diagram commutes:

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

This is precisely the morphisms in $(\mathcal{C}/C)^{\text{op}}$. Indeed (D, f^{op}) and (E, g^{op}) are objects in $(\mathcal{C}/C)^{\text{op}}$ and the morphism $h^{\text{op}} : D \to E$ also gives a commutative diagram:

$$\begin{tikzcd} D \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

which is exactly the same diagram.

Finally, composition of morphisms in either categories are just compositions in $\mathcal{C}$ so the composition laws for both categories are identical. $\square$

Definition 1.6.4: Arrow Category

Let $\mathcal{C}$ be a category. Define the arrow category $\text{Arr}(\mathcal{C})$ to consist of the following data.

- The objects are morphisms $f : C \to D$ in $\mathcal{C}$.
- For two objects $f : C \to D$ and $g : E \to F$, a morphism from f to g is a commutative square:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "h"'] & D \arrow[d, "k"] \\ E \arrow[r, "g"'] & F \end{tikzcd}$$

  This means that a morphism from f to g is a pair $(h : C \to E, k : D \to F)$.
- Composition is given by placing commutative squares side by side to obtain a commutative rectangle.

2 Examples of Categories and Functors

2.1 The Big List of Categories

Definition 2.1.1: The Category of Groups

The category **Grp** of groups where

- The objects consists of groups $(G, *)$
- For two groups G and H, the morphisms $\text{Hom}_{\textbf{Grp}}(G, H)$ consists of group homomorphisms $f : G \to H$
- Composition is given by composition of functions

Definition 2.1.2: Category of Algebraic Objects

Most of the algebraic objects can be made into a category.

- **Ab** is the category of Abelian groups and group homomorphisms
- **Mon** is the category of monoids and monoid homomorphisms
- **Rings** is the category of rings and ring homomorphisms
- ${}_R\textbf{Mod}$ is the category of (left) R-modules and R-module homomorphisms for a ring R
- $\textbf{Vect}_k$ is the category of vector spaces over a field k and linear maps

The category **Ab** of abelian groups is a full subcategory of **Grp**.

Definition 2.1.3: The Category of Topological Spaces

The category **Top** of topological spaces where

- The objects consists of topological spaces $(X, \mathcal{T})$
- For two spaces X and Y, the morphisms $\text{Hom}_{\textbf{Top}}(X, Y)$ consists of continuous maps $f : G \to H$
- Composition is given by composition of functions

Definition 2.1.4: Category of G-Sets

Let G be a group. Define the category G-Set to consist of the following data.

- The objects are sets which have a group action G.
- For two G-sets X and Y, a morphism is a G-equivariant function $f : X \to Y$. This means that
$$f(g \cdot x) = g \cdot f(x)$$
for all $g \in G$ and $x \in X$.
- Composition is given by the composition of functions.

These examples are of the same type. They are sets with additional structures and morphisms are functions which preserve the additional structure. They are in particular, concrete categories. What follows is that the categories no longer follow the same way we construct the ones above.

Definition 2.1.5: The Category of Matrices Mat_k

The category $\textbf{Mat}_k$ of manifolds where

- $\text{Obj}(\textbf{Mat}_k) = \mathbb{N}$
- $\text{Hom}(n, k) = M_{k\times n}(k)$
- Given $n \xrightarrow{A} k \xrightarrow{B} l$, define
$$B \circ A = B \cdot A \in M_{l\times n}(k)$$
Associativity and unitality follows from matrix multiplications.

Definition 2.1.6: The Category of a Preorder

Let $(P, \leq)$ be a preorder. Define a category associated with a preorder as follows.

- The objects are the elements of P
- For $x, y \in P$, define the morphisms to be
$$\text{Hom}(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise} \end{cases}$$
- For two morphisms $x \to y$ and $y \to z$, define their composition to be the unique morphism $x \to z$ which exists by transitivity.

2.2 Important Functors

Definition 2.2.1: The Forgetful Functor

Given a category whose objects are sets $(\textbf{Grp}, \textbf{Ab}, {}_R\textbf{Mod}, \textbf{Top}, \dots)$, define the forgetful functor $u : \textbf{Grp} \to \textbf{Set}$ as follows.

- u sends the object to the underlying set
- Each morphism is sent to itself $u(f) = f$ since every morphism in these categories are also functions of sets.

Definition 2.2.2: The Free Group Functor

The free group functor $F : \textbf{Set} \to \textbf{Grp}$ is defined as follows.

- On objects, define $F(X) = \langle X \rangle$ the free group on X
- If $f : X \to Y$ is a morphism in $\textbf{Set}$, define $F(f) : \langle X \rangle \to \langle Y \rangle$ by
$$F(f)(x_1^{\pm 1} \cdots x_m^{\pm 1}) = f(x_1)^{\pm 1} \cdots f(x_m)^{\pm 1}$$

It is easy to check that $F(f)$ is a well defined group homomorphism. Indeed we have that

$$\begin{aligned} &F(f)(x_1^{\pm 1} \cdots x_m^{\pm 1}) \cdot F(f)(y_1^{\pm 1} \cdots y_n^{\pm 1}) \\ &= f(x_1)^{\pm 1} \cdots f(x_m)^{\pm 1} \cdot f(y_1)^{\pm 1} \cdots f(y_n)^{\pm 1} \\ &= F(f)(x_1^{\pm 1} \cdots x_m^{\pm 1} \cdot y_1^{\pm 1} \cdots y_n^{\pm 1}) \end{aligned}$$

Moreover, it is also to see that composition of functions are respective by the free functor F.

Definition 2.2.3: The Opposite Functor

There is a opposite functor $(-)^{\text{op}} : \text{CAT} \to \text{CAT}$ defined as follows.

- $(-)^{\text{op}}(\mathcal{C}) = \mathcal{C}^{\text{op}}$
- For a morphism $F : \mathcal{C} \to \mathcal{D}$ in CAT, $(-)^{\text{op}}$ sends the morphism to

$$F^{\text{op}} : \mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$$

  defined as follows.

  - F^{op} sends $C \in \mathcal{C}^{\text{op}}$ to $F(C) \in \mathcal{D}^{\text{op}}$
  - A morphism $f^{\text{op}} : C \to D$ in $\mathcal{C}^{\text{op}}$ to

$$F(f)^{\text{op}} : F(C) \to F(D)$$

    in $\mathcal{D}^{\text{op}}$

It is clear that $(-)^{\text{op}}$ is a functor since it respects composition of functors by definition.

3 Naturality

3.1 Naturality and Equivalence of Categories

Definition 3.1.1: Natural Transformation

Let $\mathcal{C}, \mathcal{D}$ be categories with two functors $F, G : \mathcal{C} \to \mathcal{D}$ that are both covariant. A natural transformation τ is a family of arrows of $\mathcal{D}$,

$$\tau = \{\tau_A : FA \to GA | A \in \mathcal{C}\}$$

such that for each morphism $f : A \to B$ in $\mathcal{C}$, we have

$$G(f) \circ \tau_A = \tau_B \circ F(f)$$

In other words, the square on the right side of the following diagram is required to commute:

$$\begin{tikzcd}
 & & F(A) \arrow[r, "Ff"] \arrow[d, "\tau_A"] & F(B) \arrow[d, "\tau_B"] \\
A \arrow[r, "f"] & B \arrow[ur, Rightarrow, "F"] \arrow[dr, Rightarrow, "G"'] & & \\
 & & G(A) \arrow[r, "Gf"] & G(B)
\end{tikzcd}$$

Proposition 3.1.2

The composition of nartural transformation with appropriate domain and range is again a natural transformation.

Definition 3.1.3: Natural Isomorphism

A natural isomorphism between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\tau : F \to G$ such that $\tau_C : F(C) \to G(C)$ is an isomorphism in $\mathcal{D}$ for every $\tau_C \in \tau$.

Lemma 3.1.4

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors and $\alpha : F \to G$ be a natural isomorphism. Then the inverses $\alpha_C^{-1} : G(C) \to F(C)$, for $C \in \operatorname{Obj} \mathcal{C}$ defines a natural isomorphism $\alpha^{-1} : G \to F$.

Proof. The naturality condition of α implies that there is a commutative diagram:

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\cong"'] & F(D) \arrow[d, "\cong"] \\
G(C) \arrow[r, "G(f)"'] & G(D)
\end{tikzcd}$$

By changing one's point of view, one can easily use the isomorphism to define inverses from $G(C)$ to $F(C)$ for each $C \in \mathcal{C}$ such that the naturality condition holds. $\square$

Definition 3.1.5: Isomoprhic Categories

Two categories are said to be isomorphic if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = I_{\mathcal{C}}$ and $F \circ G = I_{\mathcal{D}}$.

In other words, F and G are inverses in the category CAT.

We relax the notion of isomorphic categories to obtain the following.

Definition 3.1.6: Equivalence of Categories

Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that $\mathcal{C}$ and $\mathcal{D}$ is are equivalent categories if there exists natural isomorphisms

$$\eta : \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Rightarrow} G \circ F \quad \text{ and } \quad \tau : F \circ G \overset{\cong}{\Rightarrow} \mathrm{id}_{\mathcal{D}}$$

In this case we say that F and G are an equivalence of categories.

Proposition 3.1.7

A functor $F : \mathcal{C} \to \mathcal{D}$ defines an equivalence of categories if and only if F is fully faithful and essentially surjective.

Proof. Suppose that F and G are an equivalence of categories with natural isomorphisms $\eta : \mathrm{id}_{\mathcal{C}} \overset{\cong}{\Rightarrow} G \circ F$ and $\tau : \mathrm{id}_{\mathcal{D}} \overset{\cong}{\Rightarrow} F \circ G$. Consider the following map between morphisms:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C, C') \arrow[r, "F"] & \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')) \arrow[d, "G"] \\
& \mathrm{Hom}_{\mathcal{C}}(G(F(C)), G(F(C'))) \arrow[ul, "\eta_*"]
\end{tikzcd}$$

where η_* sends $g \in \mathrm{Hom}_{\mathcal{C}}(G(F(C)), G(F(C')))$ to $\eta_{C'}^{-1} \circ g \circ \eta_C : C \to C'$. I claim that $\eta_* \circ G$ is an inverse of F.

We have that

$$\begin{aligned}
\eta_*(G(F(f : C \to C'))) &= \eta_{C'}^{-1} \circ (G(F(f : C \to C'))) \circ \eta_C \\
&= \eta_{C'}^{-1} \circ \eta_{C'} \circ \mathrm{id}(f : C \to C') && (\nu \text{ is natural}) \\
&= f : C \to C'
\end{aligned}$$

Similarly $F(\nu_*(G))$ is the identity map. Thus F is fully faithful. F is also essentially surjective. For all $D \in \mathrm{Obj}\,\mathcal{D}$, we can choose $G(D)$ such that

$$F(G(D)) \cong D$$

by the naturality of τ.

Now suppose that F is fully faithful and essentially surjective. Define $G : \mathcal{D} \to \mathcal{C}$ as follows:

- For every $D \in \mathrm{Obj}\,\mathcal{D}$, choose $C_D \in \mathrm{Obj}\,\mathcal{C}$ and isomorphism $\tau_D : F(C_D) \cong D$. This is possible since F is essentially surjective and by the axiom of choice.
- For $g : D \to D'$ in $\mathcal{D}$, define

$$G(g : D \to D') = F^{-1}(\tau_{D'} \circ g \circ \tau_D)$$

We check that G is a functor.

- Associativity: Let $g : D \to D'$ and $g' : D' \to D''$ be morphisms in $\mathcal{D}$. Since F is fully faithful, associativity holds if and only if

$$F(G(g' \circ g)) = F(G(g') \circ G(g))$$

We have that on the left hand side,

$$\begin{aligned} F(G(g' \circ g)) &= F(F^{-1}(\tau_{D''}^{-1} \circ (g' \circ g) \circ \tau_D)) \\ &= \tau_{D''}^{-1} \circ (g' \circ g) \circ \tau_D \end{aligned}$$

On the right hand side, we have

$$\begin{aligned} F(G(g') \circ G(g)) &= F(G(g')) \circ F(G(g)) \\ &= \tau_{D''}^{-1} \circ g' \circ \tau_{D'} \circ \tau_{D'}^{-1} \circ g \circ \tau_D \\ &= \tau_{D''}^{-1} \circ (g' \circ g) \circ \tau_D \end{aligned}$$

Thus we are done.

- We also have that

$$F(G(\mathrm{id}_D)) = F(F^{-1}(\tau_D^{-1} \circ \mathrm{id}_D \circ \tau_D)) = \mathrm{id}_{F(G(D))} = F(\mathrm{id}_{G(D)})$$

Thus G is a functor.

We also need natural isomorphisms. We show that $\tau_D : F(G(D)) \overset{\cong}{\to} D$ is natural. For every $g : D \to D'$ in $\mathcal{D}$, we have

$$\begin{aligned} \tau_{D'} \circ F(G(g)) = \tau_{D'} \circ F(F^{-1}(\tau_{D'} \circ g \circ \tau_D)) \\ &= g \circ \tau_D \\ &= \mathrm{id}_{\mathcal{D}}(g) \circ \tau_D \end{aligned}$$

Finally, define $\eta_C = F^{-1}(\tau_{F(C)}^{-1}) : C \to G(F(C))$. For $f : C \to C'$ in $\mathcal{C}$, $G(F(f)) \circ \eta_C = \eta_{C'} \circ f$. But this is true if and only if $F(G(F(f)) \circ \eta_C) = F(\eta_{C'} \circ f)$ since F is fully faithful. On the left hand side, we have

$$\begin{aligned} F(G(F(f)) \circ \eta_C) &= F\left(F^{-1}\left(\tau_{F(C')}^{-1} \circ F(f) \circ \tau_{F(C)}\right) \circ F^{-1}\left(\tau_{F(C)}^{-1}\right)\right) \\ &= \tau_{F(C')}^{-1} \circ F(f) \circ \tau_{F(C)} \circ \tau_{F(C)}^{-1} \\ &= \tau_{F(C')}^{-1} \circ F(f) \end{aligned}$$

On the right hand side, we have

$$\begin{aligned} F(\eta_{C'} \circ f) &= F(F^{-1}(\tau_{F(C')}) \circ f) \\ &= \tau_{F(C')}^{-1} \circ F(f) \end{aligned}$$

Finally, η_C is an isomorphism with inverse $F^{-1}(\tau_{F(C)})$ since □

3.2 The Skeleton of a Category

There is an explicit way of understanding what equivalence of categories mean.

Definition 3.2.1: Skeletal Category

A category $\mathcal{C}$ is said to be skeletal if each isomorphism class in $\mathcal{C}$ contains exactly one object.

Using the axiom of choice, any category contains a skeletal category.

Definition 3.2.2: Skeleton of a Category

Let $\mathcal{C}$ be a category. The skeleton $\text{sk}(\mathcal{C})$ of $\mathcal{C}$ is the skeletal category equivalent to $\mathcal{C}$ that is unique up to isomorphism.

For a category $\mathcal{C}$, one can construct $\text{sk}(\mathcal{C})$ as follows. Just choose an object in each isomorphism class in $\mathcal{C}$ and take the full subcategory of the chosen objects. It is then easy to see that the inclusion functor $\text{sk}(\mathcal{C}) \hookrightarrow \mathcal{C}$ defines an equivalence of categories.

In particular, one can think of equivalence of categories being a comparison on isomorphism classes of objects instead of a comparison of the objects themselves. For instance, some significantly "larger" categories can be equivalent to "smaller" categories while they can never be isomorphic.

3.3 The 2-Category of Categories

Proposition 3.3.1: Vertical Composition

Let $F, G, H : \mathcal{C} \to \mathcal{D}$ be functors and let $\lambda : F \Rightarrow G$ and $\tau : G \Rightarrow H$ be natural transformations. For each object $C \in \mathcal{C}$, the collection of morphisms

$$\tau_C \circ \lambda_C : F(C) \to H(C)$$

assemble into a natural transformation denoted

$$\tau \circ \lambda : F \Rightarrow H$$

Proof. The naturality conditions of λ and τ implies that there is a commutative diagram

$$\begin{tikzcd}
F(C) \arrow[r, "F(f)"] \arrow[d, "\lambda_C"'] & F(D) \arrow[d, "\lambda_D"] \\
G(C) \arrow[r, "G(f)"] \arrow[d, "\tau_C"'] & G(D) \arrow[d, "\tau_D"] \\
H(C) \arrow[r, "H(f)"] & H(D)
\end{tikzcd}$$

if $f : C \to D$ is a morphism in $\mathcal{C}$. It is easy to see that a naturality condition between F and H via the morphisms $\tau_C \circ \lambda_C$ for each $C \in \mathcal{C}$. □

Definition 3.3.2: The Functor Category

Let $\mathcal{C}, \mathcal{D}$ be categories. The category of functors from $\mathcal{C}$ to $\mathcal{D}$ is the category $\mathcal{D}^{\mathcal{C}}$ where

- The objects are functors $F : \mathcal{C} \to \mathcal{D}$
- For two functors $F, G : \mathcal{C} \to \mathcal{D}$, the morphisms $\text{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$ are natural transformations $\lambda : F \Rightarrow G$
- For $\lambda : F \Rightarrow G$ and $\tau : G \Rightarrow H$ natural transformations, define their composition as $\tau \circ \lambda$

It is easy to see that composition of natural transformations satisfy associativity since morphisms in $\mathcal{D}$ satisfy associativity. It is also clear that there is an identity natural transformation.

Proposition 3.3.3: Horizontal Composition

Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ be functors such that there are natural transformations $\lambda : F_1 \Rightarrow F_2$ and $\tau : G_1 \Rightarrow G_2$. Then there exists a natural transformation from $G_1 \circ F_1$ to $G_2 \circ F_2$.

Proof. For each $C \in \mathcal{C}$, consider the morphism $\lambda_C : F_1(C) \to F_2(C)$ in $\mathcal{D}$. The natruality condition of τ implies that there is a commutative diagram in $\mathbb{E}$:

$$\begin{tikzcd}
 & & & G_1(F_1(C)) \arrow[r, "G_1(\lambda_C)"] \arrow[dd, "\tau_{F_1(C)}"'] & G_1(F_2(C)) \arrow[dd, "\tau_{F_2(C)}"] \\
F_1(C) \arrow[r, "\lambda_C"] & F_2(C) \arrow[urr, Rightarrow] \arrow[drr, Rightarrow] & & & \\
 & & & G_2(F_1(C)) \arrow[r, "G_2(\lambda_C)"'] & G_2(F_2(C))
\end{tikzcd}$$

In particular we now have a morphism $G_1(F_1(C)) \to G_2(F_2(C))$ given by

$$\eta_C = G_2(\lambda_C) \circ \tau_{F_1(C)} = \tau_{F_2(C)} \circ G_1(\lambda_C)$$

for each $C \in \mathcal{C}$. I claim that these morphisms assemble into a natural transformation from $G_1 \circ F_1$ to $G_2 \circ F_2$. This means that we must check that the following diagram commutes:

$$\begin{tikzcd}
G_1(F_1(C)) \arrow[r, "G_1(F_1(f))"] \arrow[d, "\eta_C"'] & G_1(F_1(D)) \arrow[d, "\eta_D"] \\
G_2(F_2(C)) \arrow[r, "G_2(F_2(f))"'] & G_2(F_2(D))
\end{tikzcd}$$

if $f : C \to D$ is a morphism in $\mathcal{C}$. Consider the following diagram in $\mathcal{E}$:

$$\begin{tikzcd}
G_1(F_1(C)) \arrow[r, "G_1(F_1(f))"] \arrow[d, "G_1(\lambda_C)"'] & G_1(F_1(D)) \arrow[d, "G_1(\lambda_D)"] \\
G_1(F_2(C)) \arrow[r, "G_1(F_2(C))"] \arrow[d, "\tau_{F_2(C)}"'] & G_1(F_2(D)) \arrow[d, "\tau_{F_2(D)}"] \\
G_2(F_2(C)) \arrow[r, "G_2(F_2(f))"'] & G_2(F_2(D))
\end{tikzcd}$$

The bottom square commutes since τ is a natural transformation. The top square commutes since λ is a natural transformation and functors preserve commutative diagrams. Thus we conclude. $\square$

Proposition 3.3.4: Middle Four Interchange

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories and let $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$ be functors.

4 Universality

4.1 Representable Functors

Given a category $\mathcal{C}$, we can regard $\text{Hom}_{\mathcal{C}}$ as a functor by fixing one of the two objects and allowing the other to vary.

Definition 4.1.1: Hom Functor

Let $\mathcal{C}$ be a locally small category. For every $C \in \text{Obj}\,\mathcal{C}$, define the Hom functor to be

$$\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \to \text{Set}$$

where

- On objects, sends $D \in \mathcal{C}$ to the set $\text{Hom}_{\mathcal{C}}(C, D)$
- On morphisms, sends $f : D \to E$ in $\mathcal{C}$ to the map of sets

$$f_* : \text{Hom}_{\mathcal{C}}(C, D) \to \text{Hom}_{\mathcal{C}}(C, E)$$

  defined by $g \mapsto f \circ g$.

Similarly, there is a functor $\text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\text{op}} \to \text{Set}$ which is the same as $\text{Hom}_{\mathcal{C}^{\text{op}}}(C, -) : \mathcal{C} \to$ Set.

Explicitly, the functor $\text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{\text{op}} \to \text{Set}$ is defined as follows.

- On objects, sends $D \in \text{Obj}\,\mathcal{C}$ to the set $\text{Hom}_{\mathcal{C}}(D, C) = \text{Hom}_{\mathcal{C}^{\text{op}}}(C, D)$
- On morphisms, sends $f : D \to E$ in $\mathcal{C}^{\text{op}}$ to the map of sets

$$f^* : \text{Hom}_{\mathcal{C}}(E, C) \to \text{Hom}_{\mathcal{C}}(D, C)$$

  defined by $g \mapsto g \circ f$.

Given morphisms $f : D \to D'$ and $g : C' \to C$ in a category $\mathcal{C}$, it is easy to see that the following diagram commutes:

$$\begin{tikzcd}
\text{Hom}_{\mathcal{C}}(C, D) \arrow[r, "g^*"] \arrow[d, "f_*"'] & \text{Hom}_{\mathcal{C}}(C', D) \arrow[d, "f_*"] \\
\text{Hom}_{\mathcal{C}}(C, D') \arrow[r, "g^*"] & \text{Hom}_{\mathcal{C}}(C', D')
\end{tikzcd}$$

The compositions $g^* \circ f_*$ and $f_* \circ g^*$ is the map sending $h : C \to D$ to the morphism $f \circ h \circ g : C' \to D'$.

Definition 4.1.2: Representable Functor

Let $\mathcal{C}$ be a locally small category. A functor $F : \mathcal{C} \to \text{Set}$ is called representable if there is an object $C \in \mathcal{C}$ and a natural isomorphism

$$\alpha : F \overset{\cong}{\Rightarrow} \text{Hom}_{\mathcal{C}}(C, -)$$

In this case (C, α) is called a representation.

Sometimes, these functors are called copresentable, and $F : \mathcal{C}^{\text{op}} \to \text{Set}$ with $F \cong \text{Hom}_{\mathcal{C}}(-, C)$ is called representable. There is no distinction (replace $\mathcal{C}$ by $\mathcal{C}^{\text{op}}$) and we only talk about representable functors.

Definition 4.1.3: Initial and Terminal Objects

Let $\mathcal{C}$ be a category.

- $A \in Obj(\mathcal{C})$ is initial if for any object $C \in \mathcal{C}$, there is a unique morphism $A \to C$
- $B \in Obj(\mathcal{C})$ is terminal if for any object $C \in \mathcal{C}$, there is a unique morphism $C \to B$.
- $A \in \text{Obj}\,\mathcal{C}$ is a zero object if it is both initial and terminal.

Lemma 4.1.4

Let $\mathcal{C}$ be a category. Then A is initial in $\mathcal{C}$ if and only if A is terminal in $\mathcal{C}^{\text{op}}$.

Proposition 4.1.5

Let $\mathcal{C}$ be a category. Then initial an terminal objects (if it exists) are unique up to unique isomorphism.

Proof. Suppose that A, B are two terminal objects of $\mathcal{C}$. Then there is a unique morphism $f : A \to B$ and $g : B \to A$ respectively since B and A are terminal. Again since B and A are terminal, there is only one unique morphism $B \to B$ and $A \to A$ which is the identity. Thus $f \circ g : A \to A$ and $g \circ f : B \to B$ are both the identity. □

Initial and terminal objects has an equivalent characterization via representability. This means that there must be some functor for which the initial / terminal object represents. It turns out that the following functor is precisely the required functor.

Definition 4.1.6: Constant Functor

Let $\mathcal{J}, \mathcal{C}$ be categories. The constant functor with value $C \in \text{Obj}\,\mathcal{C}$ is the functor $\Delta C : \mathcal{J} \to \mathcal{C}$ defined by

- $(\Delta C)(J) = C$ on objects $J \in \text{Obj}\,\mathcal{J}$
- $(\Delta C)(f : I \to J) = \text{id}_C$ on morphisms $f : I \to J$ in $\mathcal{J}$

Proposition 4.1.7

Let $\mathcal{C}$ be a locally small category. An object $C \in \mathcal{C}$ is initial if and only if there is a natural isomorphism

$$\Delta\{*\} \cong \text{Hom}_{\mathcal{C}}(C, -)$$

where $\Delta\{*\} : \mathcal{C} \to \textbf{Set}$ is the constant functor to the one point set. Dually, C is terminal if and only if there is a natural isomorphism

$$\Delta\{*\} \cong \text{Hom}_{\mathcal{C}}(-, C)$$

where $\Delta\{*\} : \mathcal{C}^{\text{op}} \to \textbf{Set}$ is the constant functor to the one point set.

In other words, an initial object in $\mathcal{C}$ exists if and only if the constant functor is representable.

Theorem 4.1.8

The following functors are all representable.

- The identity functor $\text{id} : \textbf{Set} \to \textbf{Set}$ is representable with the object $\{*\}$
- The forgetful functor $u : \textbf{Grp} \to \textbf{Set}$ is representable with the object $\mathbb{Z}$

- The functor $\mathcal{P} : \mathbf{Set}^{\text{op}} \to \mathbf{Set}$ is representable with the object $\{0, 1\}$.

4.2 The Yoneda Lemma

Theorem 4.2.1: Yoneda's Lemma

Let $F : \mathcal{C} \to \text{Set}$ be a covariant functor where $\mathcal{C}$ is locally small. Then for every object $C \in \text{Obj}\,\mathcal{C}$, the map

$$\Phi : \text{Hom}_{\text{Set}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(C, -), F) \xrightarrow{\cong} F(C)$$

defined by

$$(\alpha : \text{Hom}_{\mathcal{C}}(C, -) \Rightarrow F) \mapsto (\alpha_C(\text{id}_C))$$

is a bijection. Moreover, this bijection is natural in F on $\mathcal{C}$. This means that by allowing $F \in \mathbf{Set}^{\mathcal{C}}$ and $C \in \mathcal{C}$ to vary, the functor

$$\text{Hom}_{\text{Set}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(-, -), -) \Rightarrow \text{ev}$$

is a natural isomorphism of functors from $\mathcal{C} \times \text{Set}^{\mathcal{C}}$ to Set, where $\text{ev}(C, F) = F(C)$.

Proof. Define a function of sets

$$\Psi : F(C) \to \text{Hom}_{\text{Set}^{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(C, -), F)$$

as follows. For each $x \in F(C)$, $\Psi(x)$ is a natural transformation

$$\Psi(x) : \text{Hom}_{\mathcal{C}}(C, -) \Rightarrow F$$

This is defined as for each $D \in \mathcal{C}$, $\Psi(x)_D : \text{Hom}_{\mathcal{C}}(C, D) \to F(D)$ is defined by sending $f : C \to D$ to $F(f)(x)$.

We first show that $\Psi(x)$ for each x is indeed a natural transformation. This means that we need to show the commutativity of the following diagram:

$$\begin{tikzcd}
\text{Hom}_{\mathcal{C}}(C, D) \arrow[r, "g_*"] \arrow[d, "\Psi(x)_D"'] & \text{Hom}_{\mathcal{C}}(C, D') \arrow[d, "\Psi(x)_{D'}"] \\
F(D) \arrow[r, "F(g)"] & F(D')
\end{tikzcd}$$

where $g : D \to D'$ is a morphism in $\mathcal{C}$. Let $f : C \to D$ be a morphism in $\text{Hom}_{\mathcal{C}}(C, D)$. Then we have that

$$\begin{aligned}
(F(g) \circ \Psi(x)_D)(f) &= F(g)(F(f)(x)) \\
&= (F(g) \circ F(f))(x) \\
&= F(g \circ f)(x) \\
&= \Psi(x)_{D'}(g \circ f) \\
&= (\Psi(x)_{D'} \circ g_*)(f)
\end{aligned}$$

And thus Ψ is well defined. Next step is to show that Ψ and Φ are inverses of each other. We have that

$$\Phi(\Psi(x)) = \Psi(x)_C(\text{id}_C) = F(\text{id}_C)(x) = x$$

and

$$\begin{aligned}
\Psi(\Phi(\alpha))_D(f : C \to D) &= F(f)(\Phi(\alpha)) \\
&= F(f)(\alpha_C(\text{id}_C)) \\
&= \alpha_D(f_*(\text{id}_C)) \\
&= \alpha_D(f)
\end{aligned}$$

Thus we are done. Finally, we show naturality in $\mathcal{C}$ and $\mathsf{Set}^{\mathcal{C}}$. To show naturality in $\mathcal{C}$, we want to show that the square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),F) \arrow[r, "f_*"] \arrow[d, "\Phi_C"'] & \mathrm{Hom}_{\mathsf{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(C',-),F) \arrow[d, "\Phi_{C'}"] \\
F(C) \arrow[r, "F(f)"'] & F(C')
\end{tikzcd}$$

commutes for $f : C \to C'$ a morphism in $\mathcal{C}$. In this case, f_* is defined as follows. For a natural transformation $\alpha : \mathrm{Hom}_{\mathcal{C}}(C,-) \Rightarrow F$ with components

$$\alpha_D : \mathrm{Hom}_{\mathcal{C}}(C,D) \to F(D)$$

define a natural transformation $f_*(\alpha)$ with components

$$(f_*(\alpha))_D : \mathrm{Hom}_{\mathcal{C}}(C',D) \to F(D)$$

defined by

$$(g : C' \to D) \mapsto ((F(g) \circ \alpha_{C'})(f))$$

If $f_*(\alpha)$ is indeed a natural transformation, then we have that

$$\begin{aligned} F(f)(\Phi_C(\alpha)) &= F(f)(\alpha_C(\mathrm{id}_C)) \\ &= \alpha_{C'}(f) \end{aligned}$$

by the naturality of α. Also we have

$$\begin{aligned} \Phi_{C'}(f_*(\alpha)) &= f_*(\alpha)(\mathrm{id}_{C'}) \\ &= F(\mathrm{id}_{C'})(\alpha_{C'}(f)) \\ &= \mathrm{id}_{F(C')}(\alpha_{C'}(f)) \\ &= \alpha_{C'}(f) \end{aligned}$$

which shows that naturality condition. Now it remains to show that $f_*(\alpha)$ is a natural transformation. This amounts to showing that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(C',D) \arrow[r, "h_*"] \arrow[d, "(f_*(\alpha))_D"'] & \mathrm{Hom}_{\mathcal{C}}(C',D') \arrow[d, "(f_*(\alpha))_{D'}"] \\
F(D) \arrow[r, "F(h)"'] & F(D')
\end{tikzcd}$$

for $h : D \to D'$ a morphism in $\mathcal{C}$. Let $u \in \mathrm{Hom}_{\mathcal{C}}(C',D)$ By following the arrows on the bottom left, we have that

$$(F(h) \circ (f_*(\alpha)_D))(u) = F(h)\left(F(u) \circ \alpha_{C'}(f)\right) = \left(F(h \circ u) \circ \alpha_{C'}\right)(f)$$

By following the arrows on the top right, we have that

$$((f_*(\alpha)_{D'}) \circ h_*)(u) = (f_*(\alpha))_D(h \circ u) = \left(F(h \circ u) \circ \alpha_{C'}\right)(f)$$

Thus we conclude.

It remains to show naturality on $\mathsf{Set}^{\mathcal{C}}$. Naturality condition means the following diagram must commute:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),F) \arrow[r, "\beta_*"] \arrow[d, "\Phi_F"'] & \mathrm{Hom}_{\mathbf{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),G) \arrow[d, "\Phi_G"] \\
F(C) \arrow[r, "\beta_C"] & G(C)
\end{tikzcd}$$

for a natural transformation β. Let $\alpha : \mathrm{Hom}_{\mathcal{C}}(C,-) \Rightarrow F$ be a natural transformation. Then following the bottom left of the diagram gives

$$(\beta_C \circ \Phi_F)(\alpha) = \beta_C(\alpha_C(1_C))$$

Following the top right of the diagram gives

$$(\Phi_G \circ \beta_*)(\alpha) = \Phi_G(\beta \circ \alpha) = (\beta \circ \alpha)_C(1_C) = \beta_C(\alpha_C(1_C))$$

and so we conclude. $\square$

Note that $\mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),F)$ is a priori large, but it is small as a consequence of the Yoneda lemma.

Corollary 4.2.2: Yoneda's Embedding

Let $\mathcal{C}$ be a locally small category. Define a functor

$$y : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathcal{C}}$$

as follows.

- On objects, sends C to $\mathrm{Hom}_{\mathcal{C}}(C,-) : \mathcal{C} \to \mathrm{Set}$
- On morphisms, sends $f : C \to D$ to a natural transformation

  $$f^* : \mathrm{Hom}_{\mathcal{C}}(D,-) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(C,-)$$

  with components

  $$f^*_X : \mathrm{Hom}_{\mathcal{C}}(D,X) \to \mathrm{Hom}_{\mathcal{C}}(C,X)$$

  for varying X. These maps are defined by $\varphi \mapsto \varphi \circ f$.

Then y is fully faithful. Moreover, two objects C, D in $\mathrm{Obj}\,\mathcal{C}$ are isomorphic if and only if the functors $\mathrm{Hom}_{\mathcal{C}}(C,-)$ and $\mathrm{Hom}_{\mathcal{C}}(D,-)$ are naturally isomorphic.

Proof. On the level of morphisms, fully faithful means that we want to show that

$$y : \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C,D) \to \mathrm{Hom}_{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),\mathrm{Hom}_{\mathcal{C}}(D,-)))$$

is bijective. Using the fact that $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(C,D) = \mathrm{Hom}_{\mathcal{C}}(D,C)$ and Yoneda's lemma, we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(D,C) \arrow[r, "y"] & \mathrm{Hom}_{\mathcal{C}^{\mathrm{Set}}}(\mathrm{Hom}_{\mathcal{C}}(C,-),\mathrm{Hom}_{\mathcal{C}}(D,-))) \arrow[d, "\Psi"] \\
 & \mathrm{Hom}_{\mathcal{C}}(D,C)
\end{tikzcd}$$

We want to show that $\Psi(y(f)) = f$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(D,C)$. But

$$\begin{aligned}
\Psi(y(f)) &= y(f)_D(\mathrm{id}_D) \\
&= \mathrm{id}_D \circ f \\
&= f
\end{aligned}$$

Thus y is fully faithful.

Clearly if $C \cong C'$, Then $y(C) \cong y(C')$ since y is a functor. Conversely, suppose $\alpha : y(C) \overset{\cong}{\Rightarrow} y(C')$ is an isomorphism. Since y is fully faithful, there is a unique $f : C \to C'$ in $\mathcal{C}^{\text{op}}$ such that $y(f) = \alpha$. Similarly, $\alpha^{-1} : y(C') \to y(C)$ gives a unique morphism $g : C' \to C$. in $\mathcal{C}^{\text{op}}$. Notice that we have

$$\begin{aligned} y(f \circ g) &= y(f) \circ y(g) \\ &= \alpha \circ \alpha^{-1} \\ &= \text{id}_{y(C')} \\ &= y(\text{id}_{C'}) \end{aligned}$$

Since y is faithful, this implies that $f \circ g = \text{id}_{C'}$. A similar argument shows that $g \circ f = \text{id}_C$ and thus g is an inverse of f. $\square$

4.3 Universal Properties

Essentially the universal property is a way of saying the phrase "unique up to isomorphism". Much of the later categorical constructs as we see will have this universal property.

Definition 4.3.1: Universal Property

Let $\mathcal{C}$ be a locally small category. Let $C \in \text{Obj}\,\mathcal{C}$. A universal property for C consists of a representable functor $F : \mathcal{C} \to \text{Set}$, where the representation is given by

$$\alpha : \text{Hom}_{\mathcal{C}}(C, -) \overset{\cong}{\Rightarrow} F$$

(Yoneda lemma implies that α corresponds to a unique $x \in F(C)$). The element $x \in F(C)$ in this case is called the universal element, and (C, x) is called a representation of F.

The name universality is given because a universal property is a description of the maps out of C.

This universal property of an object C of $\mathcal{C}$ uniquely determines C up to isomorphism. If C, D have the same universal property, this means that there are isomorphisms

$$\alpha : \text{Hom}_{\mathcal{C}}(C, -) \xRightarrow{\cong} F \xLeftarrow{\cong} \text{Hom}_{\mathcal{C}}(D, -) : \beta$$

Then since $\text{Hom}_{\mathcal{C}}(C, -) \cong \text{Hom}_{\mathcal{C}}(D, -)$, by the Yoneda embedding we have $C \cong D$. Verbally speaking, this says that if two objects has all maps going out of it being the same, then the objects are isomorphic.

5 Limits and Colimits

Limits and colimits are objects constructed from diagrams by means of certain universal properties. They formalize the notion of subobjects in objects and gluing of objects.

5.1 The Category of Cones

Recall that a commutative diagram in $\mathcal{C}$ of shape $\mathcal{J}$ is a functor $X : \mathcal{J} \to \mathcal{C}$. Though there are no restriction on the following definition, we will eventually take $\mathcal{J}$ to be a small category.

Definition 5.1.1: Cone Over and Under

Let $X : \mathcal{J} \to \mathcal{C}$ be a commutative diagram. A cone over X with summit $C \in \operatorname{Obj} \mathcal{C}$ is a natural transformation $\lambda : \Delta C \Rightarrow X$ from the constant functor to the diagram. Explicitly, λ is a collection of morphisms $\lambda_J : C \to X(J)$ for each $J \in \operatorname{Obj} \mathcal{J}$ such that

$$\begin{tikzcd} & C \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] & \\ X(I) \arrow[rr, "X(f)"] & & X(J) \end{tikzcd}$$

commutes for $f : I \to J$ a morphism in $\mathcal{J}$.

Dually, a cone under X, or a cocone, is a cone over $X^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$. Explicitly, it is a natural transformation $\lambda : \mathcal{J} \Rightarrow \Delta C$ such that the following diagram commutes:

$$\begin{tikzcd} X(I) \arrow[rr, "X(f)"] \arrow[dr, "\lambda_I"'] & & X(J) \arrow[dl, "\lambda_J"] \\ & C & \end{tikzcd}$$

where $f : I \to J$ is a morphism in $\mathcal{J}$.

Notice that the cones over X form a category where a morphism from $\lambda : \Delta C \Rightarrow X$ to $\lambda' : \Delta C' \Rightarrow X$ is a morphism $f : C \to C'$ such that the diagram

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\lambda_J"'] & & C' \arrow[dl, "\lambda'_J"] \\ & X(J) & \end{tikzcd}$$

commutes for each $J \in \mathcal{J}$.

Definition 5.1.2: Category of Cones

Let $\mathcal{C}$ be a category and $X : \mathcal{J} \to \mathcal{C}$ a diagram. Let $C \in \mathcal{C}$. Define the category of cones over X to consist of the following data.

- The objects consists of cones $\lambda : \Delta C \Rightarrow X$ over X for varying $C \in \mathcal{C}$.
- Let $\lambda : C \Rightarrow X$ and $\lambda' : C' \Rightarrow X$ be cones over X. A morphism of the cones is a morphism $f : C \to C'$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[rr, "f"] \arrow[dr, "\lambda_J"'] & & C' \arrow[dl, "\lambda'_J"] \\ & X(J) & \end{tikzcd}$$

  for each $J \in \mathcal{J}$.
- Composition is defined as the composition in $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{tikzcd}
C \arrow[r, "f"] \arrow[dr, "\lambda_J"'] & C' \arrow[r, "g"] \arrow[d, "\lambda'_J"] & C'' \arrow[dl, "\lambda''_J"] \\
& X(J) &
\end{tikzcd}
$$

for $f : C \to C'$ and $g : C' \to C''$ morphisms in the category of cones.

Definition 5.1.3: Limits and Colimits

Let $X : \mathcal{J} \to \mathcal{C}$ be a commutative diagram. A limit of X is an object $\lim_{\mathcal{J}} X \in \operatorname{Obj} \mathcal{C}$ together with a natural transformation $\lambda : \Delta(\lim_{\mathcal{J}} X) \Rightarrow X$ which is terminal in the category of cones over X.

Explicitly, this consists of an object $\lim_{\mathcal{J}} X$ of $\mathcal{C}$ with maps $\lambda_J : \lim_{\mathcal{J}} X \to X(J)$ such that for any other cone $\mu : \Delta C \Rightarrow X$, there is a unique map $u : C \to \lim_{\mathcal{J}} X$ such that the following diagram commutes:

$$
\begin{tikzcd}
& C \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "\mu_I"'] \arrow[ddr, bend left, "\mu_J"] & \\
& \lim_{\mathcal{J}} X \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] & \\
X(I) \arrow[rr, "X(f)"] & & X(J)
\end{tikzcd}
$$

Dually, a colimit of X is a limit of the diagram $X : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}}$ with the following diagram:

$$
\begin{tikzcd}
X(I) \arrow[rr, "X(f)"] \arrow[dr, "\lambda_I"'] \arrow[ddr, bend right, "\psi_I"'] & & X(J) \arrow[dl, "\lambda_J"] \arrow[ddl, bend left, "\psi_J"] \\
& \operatorname{colim}_{\mathcal{J}} X \arrow[d, dashed, "\exists! u"] & \\
& C &
\end{tikzcd}
$$

Theorem 5.1.4: Uniqueness of (Co)Limits

Let C and D be two limits of a diagram $X : \mathcal{J} \to \mathcal{C}$, then there exists a unique isomorphism $C \cong D$ defining an isomorphism of cones.

Proof. Since limits and colimits are terminal and initial respectively, by 2.1.5 we are done. □

Proposition 5.1.5

Let $\mathcal{C}$ be a category and let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. If $\mathcal{J}$ has an initial object $0 \in \mathcal{J}$, then

$$\lim_{\mathcal{J}} X = X(0)$$

Dually, if $\mathcal{J}$ has a terminal object $1 \in \mathcal{J}$, then

$$\operatorname*{colim}_{\mathcal{J}} X = X(1)$$

Recall that initial and terminal objects has an equivalent characterization using representability.

Therefore by admitting a suitable functor, we should be able to convert the definition of limits and colimits in to representability criteria. It turns out that the following functor gives the characterization.

Definition 5.1.6: Cone Functor

Let $\mathcal{C}$ be a locally small category and $X : \mathcal{J} \to \mathcal{C}$ a small diagram. Define

$$\text{Cone}(C, X) = \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta C, X)$$

to be the cones over X with summit C. Define the functor

$$\text{Cone}(-, X) : \mathcal{C}^{\text{op}} \to \textbf{Set}$$

as follows.

- It sends an object $C \in \mathcal{C}$ to $\text{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta C, X)$
- It sends a morphism $f : C \to D$ in $\mathcal{C}$ to a morphism of cones $\lambda : \Delta D \Rightarrow X$ to $\lambda' : \Delta C \Rightarrow X$.

Dually, define the cones under X as

$$\text{Cocone}(X, C) = \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(X, \Delta C)$$

and a functor

$$\text{Cocone}(X, -) : \mathcal{C} \to \textbf{Set}$$

as in the case of cones.

Note that this is not the same as the category of cones over X.

Theorem 5.1.7

Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. Then $\lambda : \Delta C \Rightarrow X$ is a limit of X if and only if the functor $\text{Cone}(-, X) : \mathcal{C}^{\text{op}} \to \textbf{Set}$ is representable by C. This means that there is a natural isomorphism

$$\text{Hom}_{\mathcal{C}}(-, C) \cong \text{Cone}(-, X) = \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(-), X)$$

Dually, $\lambda : X \Rightarrow \Delta C$ is a colimit of X if and only if there is a natural isomorphism

$$\text{Hom}_{\mathcal{C}}(C, -) \cong \text{Cocone}(X, -) = \text{Hom}_{\mathcal{C}^{\mathcal{J}}}(X, \Delta(-))$$

Proof. Define $\Psi : \text{Hom}_{\mathcal{C}}(-, C) \Rightarrow \text{Cone}(-, X)$ as follows. For each $D \in \mathcal{C}$, $\Psi_D : \text{Hom}_{\mathcal{C}}(D, C) \to \text{Cone}(D, X)$ is defined by

$$(f : D \to C) \mapsto \left(D \xrightarrow{f} C \xrightarrow{\lambda_J} X(J)\right)$$

for each $J \in \mathcal{J}$.

We first show that Ψ is an isomorphism in **Set** (A bijection). We know that $\lambda : \Delta C \Rightarrow X$ is a limit of X if and only if it is terminal. This means that λ is a limit if and only if for all $\delta : \Delta D \Rightarrow X$, there exists a unique $u : D \to C$ a map of cone such that

$$\begin{tikzcd} D \arrow[rr, "u"] \arrow[dr, "\delta_J"'] & & C \arrow[dl, "\lambda'_J"] \\ & X(J) & \end{tikzcd}$$

for all $J \in \text{Obj}\,\mathcal{J}$. But this precisely means that every δ has a unique preimage u by Ψ. Thus Ψ is a bijection.

It remains to show that Ψ is a natural isomorphism. $\square$

Using the notation for limits and colimits, the above theorem translates to natural isomorphisms

$$\mathrm{Cone}(-,X) \cong \mathrm{Hom}_{\mathcal{C}}(-, \lim_{\mathcal{J}} X) \quad \text{and} \quad \mathrm{Cocone}(X,-) \cong \mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_{\mathcal{J}} X, -)$$

Definition 5.1.8: Complete and Cocomplete Categories

Let $\mathcal{C}$ be a category. $\mathcal{C}$ is said to be (co)complete if every small diagram $\mathcal{J} \to \mathcal{C}$ admits a (co)limit.

The smallness in the definition is key! If completeness is defined by requiring all limits to exists, then any complete category $\mathcal{C}$ that has the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(C, D)$ between any two objects C, D being larger than 1, then $\mathcal{C}$ will be a preorder!

Theorem 5.1.9

The category Set is complete and cocomplete.

Proof. Denote 1 the one element set. Then since we are working with the category of sets, we have the isomorphism

$$L(X) = \lim_{\mathcal{J}} X \cong \mathrm{Hom}_{\mathrm{Set}}(1, \lim_{\mathcal{J}} X) \cong \mathrm{Cone}(1, X)$$

for any small diagram $X : \mathcal{J} \to \mathcal{C}$. This is equal to

$$\lim_{\mathcal{J}} X = \left\{ \{x_J \in X(J)\}_{J \in \mathrm{Obj}\,\mathcal{J}} \in \prod_{J \in \mathrm{Obj}\,\mathcal{J}} X(J) \;\middle|\; \forall (f : I \to J) \in \mathcal{J}, f_*(x_I) = x_J \right\}$$

where the maps of the cones are $\pi_J : \lim_{\mathcal{J}} X \to X(J)$ defined by $\{x_I\}_{I \in \mathrm{Obj}\,\mathcal{J}} \to x_J$. Dually, we have that

$$\lim_{\mathcal{J}} X = \frac{\coprod_{J \in \mathrm{Obj}\,\mathcal{J}} X(J)}{\sim}$$

where $\sim$ is the equivalence relation generated by $x_I \in X(I) \sim x_J \in X(J)$ if and only if there exists $f : I \to K$ and $g : J \to K$ in $\mathcal{J}$ such that $f_*(x_I) = g_*(x_J)$. The maps are the inclusions

$$\iota_J : X(J) \hookrightarrow \frac{\coprod_{I \in \mathrm{Obj}\,\mathcal{J}} X(I)}{\sim}$$

Note that these are sets since $\mathcal{J}$ is small.

We now prove the statement for the limit, the dual statement for colimit will follow. Claim: the π_J assemble into a natural transformation $\pi : \Delta(L(X)) \Rightarrow X$. Indeed, for $f : I \to J$ in $\mathcal{J}$, we have that

$$\begin{aligned} f_*(\pi_I(\{x_K\}_{K \in \mathrm{Obj}\,\mathcal{J}})) &= f_*(x_I) \\ &= x_J \\ &= \pi_J(\{x_K\}_{K \in \mathrm{Obj}\,\mathcal{J}}) \end{aligned}$$

showing that this is true. Now let $\alpha : \Delta Y \Rightarrow X$ be a cone over X. We need to show that there is a unique map

$$u : Y \to L(X)$$

such that

$$\begin{tikzcd} Y \arrow[rr, dashed, "\exists ! u"] \arrow[dr, "\alpha_I"'] & & L(X) \arrow[dl, "\pi'_I"] \\ & X(I) & \end{tikzcd}$$

Let us define $u(y) = \{\alpha_J(y)\}_{J \in \mathrm{Obj}\,\mathcal{J}}$. If this is well defined, it is unique. (Two points in $prod_{J \in \mathrm{Obj}\,\mathcal{J}} X(J)$) if and only if they have the same components). We need to see that $u(y) \in L(X)$. Let $f : I \to J$, then $f_*(\alpha_I(y)) = \alpha_J(y)$ since α is natural. Thus we are done. $\square$

Limits and colimits generalizes all the important concepts in category theory, beginning with initial and terminal objects.

5.2 Products and Coproducts

Definition 5.2.1: Discrete Categories

Let K be a set. Define the discrete category K^S to consist of the following data.

- The objects of K^S are the elements of K
- The only morphisms are the identity morphisms
- Composition is trivial since there are only identity morphisms.

Definition 5.2.2: Products and Coproducts

Let K be a set, and $C_k \in \operatorname{Obj}\mathcal{C}$ for every $k \in K$ (K becomes an indexing set). Define the product of the objects $\{C_k\}_{k\in K}$ to be the limit of the diagram $X : K^S \to \mathcal{C}$ sending k to C_k. We denote the limit in this case as

$$\prod_{k\in K} C_k = \lim_{K^S} X$$

Dually, the coproduct of the objects $\{C_k\}_{k\in K}$ is the colimit of the same diagram. We denote the colimit in this case as

$$\coprod_{k\in K} C_k = \operatorname*{colim}_{K^S} X$$

if they exists.

The product satisfies the following universal property: For any $D \in \operatorname{Obj}\mathcal{C}$ with maps $\lambda_k : D \to C_k$ for all $k \in K$, there exists a unique $u : D \to \prod_{k\in K} C_k$ such that $\pi_k \circ u = \lambda_k$ (Unravel the universal property of the limit in this specific case). This means that

$$\operatorname{Hom}_{\mathcal{C}}\left(D, \prod_{k\in K} C_k\right) = \operatorname{Hom}_{\mathcal{C}^{K^S}}(\Delta D, X) = \prod_{k\in K} \operatorname{Hom}_{\mathcal{C}}(D, C_k)$$

In the case that $K = \{1, 2\}$ has two elements, we have the following:

A product of $C_1, C_2 \in \mathcal{C}$ is an object

$$C_1 \times C_2 \in \mathcal{C}$$

equipped with a pair of morphisms $\pi_1 : C_1 \times C_2 \to C_1$ and $\pi_2 : C_1 \times C_2 \to C_2$ such that for every object $D \in \mathcal{C}$ and every pair of morphisms $f_1 : D \to C_1$ and $f_2 : D \to C_2$, there exists a unique morphism

$$f : D \to C_1 \times C_2$$

Together with the dual argument, we have the following two diagrams:

$$\begin{tikzcd}
D \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "f_2"'] & & \\
& C_1 \times C_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & C_1 \\
& C_2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
D & & \\
& C_1 \amalg C_2 \arrow[ul, dashed, "\exists! u"'] & C_1 \arrow[l, "\iota_1"'] \arrow[ull, bend right, "g_1"'] \\
& C_2 \arrow[u, "\iota_2"] \arrow[uul, bend left, "g_2"] &
\end{tikzcd}$$

for maps $g_1 : C_1 \to X$ and $g_2 : C_2 \to X$.

Proposition 5.2.3

The following categories exhibit products and coproducts.

- In $\mathcal{C} = \text{Set}$ and $X, Y \in \text{Obj}\,\mathcal{C}$, products and coproducts are cartesian products and disjoint union
$$X \times Y \quad \text{and} \quad X \amalg Y$$
with projection maps and inclusion maps respectively.

- In $\mathcal{C} = \text{Top}$ and $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y) \in \text{Obj}\,\mathcal{C}$, products and coproducts are the product space and the disjoint union
$$(X \times Y, \mathcal{T}_X \times \mathcal{T}_Y) \quad \text{and} \quad (X \amalg Y, \mathcal{T}_X \amalg \mathcal{T}_Y)$$
with projection maps and inclusion maps respectively. The topology in coproducts is defined as $U \subseteq \mathcal{T}_X \amalg \mathcal{T}_Y$ if and only if $U \cap X \in \mathcal{T}_X$ and $U \cap Y \in \mathcal{T}_Y$.

- In $\mathcal{C} = \text{Grp}$ and $G, H \in \text{Obj}\,\mathcal{C}$, products and coproducts are direct products and free product
$$(G \times H, \cdot) \quad \text{and} \quad (G * H, *)$$
with projection maps and inclusion maps respectively.

- In $\mathcal{C} = \text{Ab}$ and $A, B \in \text{Obj}\,\mathcal{C}$, products and coproducts are both direct products of groups
$$(A \oplus B, +)$$
but with projection maps and inclusion maps respectively. In this case we call the direct product the direct sum and instead denote it as $A \oplus B$.

Following this we have a stronger notion of products which restricts the product to an even stronger sense by forcing them to be related by a commutative square.

Definition 5.2.4: Pullbacks and Pushouts

Let $\mathcal{C}$ be a category. A pullback is the limit of a diagram $X : \mathcal{J} \to \mathcal{C}$ where $\mathcal{J} = (\cdot \to \cdot \leftarrow \cdot)$. We denote the limit in this case as
$$\lim_{\mathcal{J}} X = C \times_D B$$
Dually, a pushout is the colimit of a diagram $X : \mathcal{J}^{\text{op}} \to \mathcal{C}$, where $\mathcal{J} = (\cdot \leftarrow \cdot \to \cdot)$. We denote the colimit in this case as
$$\operatorname*{colim}_{\mathcal{J}} X = C \amalg_D B$$

In particular, the universal product of limits means that for any $X \in \text{Obj}\,\mathcal{C}$ together with maps $\lambda_C : X \to C$ and $\lambda_B : X \to B$ such that $f \circ \lambda_C = g \circ \lambda_D$, there exists a unique $u : X \to C \times_D B$ such that the following diagram commutes (Dually, the diagram on the right):

$$
\begin{tikzcd}
X \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "\lambda_C"] \arrow[ddr, bend right, "\lambda_B"'] & & \\
& C \times_D B \arrow[r, "\pi_C"] \arrow[d, "\pi_B"] & C \arrow[d, "f"] \\
& B \arrow[r, "g"] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
X & & \\
& C \amalg_D B \arrow[ul, dashed, "\exists! u"'] & C \arrow[l, "\iota_C"'] \arrow[ull, bend right, "\lambda_C"'] \\
& B \arrow[u, "\iota_B"] \arrow[uul, bend left, "\lambda_B"] & D \arrow[l, "g"] \arrow[u, "f"']
\end{tikzcd}
$$

Proposition 5.2.5

The following categories exhibit pullbacks and pushouts.

- For $\mathcal{C} = \text{Set}$, the pullback is precisely
$$C \times_D B \cong \{(x, y) \in C \times B \mid f(x) = g(y)\}$$
and the pushout is precisely
$$C \amalg_D B \cong \frac{C \amalg B}{\sim}$$
where $x \in C \sim y \in B$ if and only if $f(x) = g(y)$ for C, D, B sets.
- For $\mathcal{C} = \text{Top}$, pullbacks and pushouts are the same as in Set.
- For $\mathcal{C} = \text{Grp}$, pullbacks are the same as pullbacks in Set. The pushout is precisely
$$\operatorname{colim}\left(G \xleftarrow{f} K \xrightarrow{g} H\right) = G *_K H$$
the amalgamated product of the groups G and H over K.

  (Recall the amalgamated product is defined by $G *_K H = \frac{G*H}{N}$ for N the normalizer of $\{f(k) \cdot (g(k))^{-1} \mid k \in K\}$

5.3 Inverse and Direct Limits

For diagram constructing purposes we define the category of natural numbers.

Definition 5.3.1: Category of Natural Numbers

The category of natural numbers $(\mathbb{N}, \leq)$ is the category where

- the objects are $\mathbb{N}$
- the morphisms are $f_{ij} : i \to j$ for $i \leq j$
- Composition is defined such that $f_{jk} \circ f_{ij} = f_{ik}$

Definition 5.3.2: Inverse and Direct Limits

Let $\mathcal{C}$ be a category. A direct limit is the limit of a diagram $X : (\mathbb{N}, \leq) \to \mathcal{C}$. Dually, the inverse limit is the limit of a diagram $X : \mathcal{J}^{\text{op}} \to \mathcal{C}$ where $\mathcal{J}^{\text{op}} = (\mathbb{N}, \leq)^{\text{op}}$.

The universal property of the inverse limit is given by the following: For $C \in \text{Obj}\,\mathcal{C}$ and maps $\lambda_i : C \to X(i)$, there exists a unique map $u : C \to \lim_{(\mathbb{N}, \leq)^{\text{op}}} X$ such that the following diagram commutes:

$$\begin{tikzcd}
 & C \arrow[d, dashed, "\exists ! u"] \arrow[ddl, bend right, "\lambda_i"'] \arrow[ddr, bend left, "\lambda_j"] & \\
 & \lim X \arrow[dl, "\pi_j"'] \arrow[dr, "\pi_i"] & \\
X(i) \arrow[rr, "f_{ij}"] & & X(j)
\end{tikzcd}$$

Proposition 5.3.3

The following categories exhibit inverse and direct limits.

- For $\mathcal{C} = \text{Set}$, the direct limit is
$$\lim_{(\mathbb{N},\leq)} X = \left\{ (x_0, x_1, \dots) \in \prod_{i\in\mathbb{N}} X(i) \;\middle|\; x_j = f_{ij}(x_i) \text{ for all } i \leq j \right\} = \varinjlim_{n\in\mathbb{N}} X(n)$$
and the inverse limit is
$$\lim_{(\mathbb{N},\leq)^{\text{op}}} X = \frac{\coprod_{i\in\mathbb{N}} X(i)}{\sim} = \varprojlim_{n\in\mathbb{N}} X(n)$$
where $x_i \sim x_j$ for $i < j$ and $x_i \in X(i)$ and $x_j \in X(j)$ if and only if there exists some $k \in K$ such that $f_{ik}(x_i) = f_{jk}(x_j)$.

Proposition 5.3.4

Let $X : (\mathbb{N}, \leq) \to \text{Set}$ such that each map $X_i \to X_{i+1}$ is a subset inclusion. Then
$$\lim_{(\mathbb{N},\leq)} X = \bigcup_{i=0}^{\infty} X_i$$

5.4 Equalizers and Coequalizers

Definition 5.4.1: Equalizers and Coequalizers

Let $\mathcal{C}$ be a category and $f, g : C \to D$ be two morphisms. Let $\mathcal{J}$ be the category
$$\begin{tikzcd} I \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & J \end{tikzcd}$$
and let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. The equalizer of f and g is defined to be the limit
$$\text{Eq}(f, g) = \lim_{\mathcal{J}} X$$
of the diagram X. Dually, the coequalizer is
$$\text{coeq}(f, g) = \operatorname*{colim}_{\mathcal{J}} X$$
should they exists.

The universal property of the equalizer is given as follows: For $A \in \text{Obj}\,\mathcal{C}$ together with a map $\lambda : A \to C$ for which $f \circ \lambda = g \circ \lambda$, there exists a unique map $u : A \to N$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, dashed, "\exists! u"'] \arrow[dr, "a"] & & \\ \lim_{\mathcal{J}} X \arrow[r, "h"'] & C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \end{tikzcd}$$

Dually, the coequalizer has the following universal property:

$$\begin{tikzcd} & & & A \\ C \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & D \arrow[urr, "a"] \arrow[rr, "h"'] & & \operatorname{colim}_{\mathcal{J}} X \arrow[u, dashed, "\exists! u"'] \end{tikzcd}$$

Definition 5.4.2: Kernels and Cokernels

Let $f : A \to B$ be a morphism in a category $\mathcal{C}$. Then a kernel of f is an equalizer of f and the zero morphism from A to B. In other words, it is the following diagram:

$$\begin{tikzcd} C \arrow[d, dashed, "\exists!"] \arrow[dr, "a"] & & \\ N \arrow[r, "i"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Dually, the cokernel of f is the kernel of f in the dual category.

5.5 Completeness and Cocompleteness of more Categories

Definition 5.5.1: Small Products and Small Coproducts

Let $\mathcal{C}$ be a category. We say that a (co)product in $\mathcal{C}$ (if it exists) is small if the diagram for the (co)product is small.

The equalizer gives us a necessary and sufficient criterion for a category to be complete and cocomplete.

Theorem 5.5.2

Let $\mathcal{C}$ be a category with small products, and $X : \mathcal{J} \to \mathcal{C}$ be a small diagram. Then there exists maps

$$f, g : \prod_{J \in \operatorname{Obj} \mathcal{J}} X_J \to \prod_{(\alpha : J \to I) \in \operatorname{Hom} \mathcal{J}} X_I$$

such that if $\operatorname{Eq}(f, g)$ exists, it is the limit of the diagram $\mathcal{J}$.

In particular, $\mathcal{C}$ is complete if and only if it admits small products and equalizers. Dually, $\mathcal{C}$ is cocomplete if and only if it admits small coproducts and coequalizers.

Proof. Let $f, g : \prod_{J \in \operatorname{Obj} \mathcal{J}} X_J \to \prod_{(\alpha : J \to I) \in \operatorname{Hom} \mathcal{J}} X_I$ be defined as follows. f is determined by the maps

$$\pi_I = \pi_\alpha \circ f : \prod_{K \in \operatorname{Obj} \mathcal{J}} X_K \stackrel{\pi_I}{\to} X_I$$

Similarly, g is determined by the maps

$$\pi_\alpha \circ g : \prod_{K \in \operatorname{Obj} \mathcal{J}} X_K \stackrel{\alpha_* \circ \pi_J}{\to} X_I$$

Now define a cone over X by $\pi_I^E : \mathsf{Eq}(f,g) \to X_I$ for all $I \in \mathrm{Obj}\,\mathcal{J}$ by

$$\mathsf{Eq}(f,g) \xrightarrow{\pi_0} \prod_{J\in\mathrm{Obj}\,\mathcal{J}} X_J \xrightarrow{\pi_I} X_I$$

where π_0 is the projection onto the source of f and g. Now we show that this cone is indeed the limit using 3.1.4. We have that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(D, \mathsf{Eq}(f,g)) &\cong \mathsf{Cone}\left(D, \prod_{J\in\mathrm{Obj}\,\mathcal{J}} X_J \overset{f}{\underset{g}{\rightrightarrows}} \prod_{(\alpha:J\to I)\in\mathrm{Hom}\,\mathcal{J}} X_I\right) \\
&= \left\{(a \in \mathrm{Hom}_{\mathcal{C}}(D, \prod_{J\in\mathrm{Obj}\,\mathcal{J}} X_J)) \;\middle|\; f\circ a = g\circ a\right\} \\
&= \left\{a \in \mathrm{Hom}_{\mathcal{C}}(D, \prod_{J\in\mathrm{Obj}\,\mathcal{J}} X_J) \;\middle|\; \forall \alpha : J \to I \in \mathcal{J}, \pi_\alpha \circ f \circ \alpha \cong \pi_\alpha \circ g \circ a\right\} \\
&\cong \left\{\{a_J \in \mathrm{Hom}_{\mathcal{C}}(D, X_J)\}_{J\in\mathrm{Obj}\,\mathcal{J}} \;\middle|\; a_i = \alpha_* \circ a_j\right\} \\
&\qquad\qquad\qquad\qquad (\text{Universal property of } \textstyle\prod_{J\in\mathrm{Obj}\,\mathcal{J}} X_J) \\
&= \mathsf{Cone}(D, X) \\
&= \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta D, X)
\end{aligned}$$

where the third equality and the fourth isomorphism is due to the fact that maps into products is defined by their components.

It remains to show that the bijection is the map defined from the cone maps π_I^E. $\square$

Corollary 5.5.3

The category **Top** of topological spaces is complete and cocomplete.

Proof. We first show that small products in **Top** exists. It is clear that the finite products are the product space with projection maps. We have seen that they satisfy the universal property in Point Set Topology. We have to define countable products.
Now we show that equalizers exists. Let $f, g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be continuous maps. We define a topology on

$$\mathsf{Eq}(f,g) = \{x \in X \mid f(x) = g(x)\}$$

using the subspace topology. Now suppose that there is a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y \\
& Z \arrow[ul, bend left, "\lambda_X"] \arrow[ur, bend right, "\lambda_Y"'] &
\end{tikzcd}$$

We want to show that there exists a unique morphism $u : Z \to \mathsf{Eq}(f,g)$ that fits into the limit diagram. It is clear that by constraints of the subspace topology, the only possible map is $u(z) = \lambda_X(z)$. Since λ_X is continuous and takes values in $\mathsf{Eq}(f,g)$, u is continuous.

It remains to prove that colimits exists. $\square$

Note that while we could prove the above corollary by directing defining the subspace topology on the limit in Set, we will see that it will be harder to work directly through the definitions in the case for other categories.

Corollary 5.5.4

The category **Grp** of groups is complete and cocomplete.

Proof. Let $X : \mathcal{J} \to \textbf{Grp}$ be a functor. Then the limit of the diagram $\mathcal{J} \xrightarrow{X} \textbf{Grp} \xrightarrow{u} \textbf{Set}$ for u the forgetful functor has a group structure (as a subgroup of $\prod_{I \in \text{Obj}(\mathcal{J})} X_I$) which is the limit in **Grp**.

Small coproducts clearly exists since they are simple free products on an arbitrary set of generators. We prove that coproducts exists. Let $\varphi, \psi : G \to H$ be group homomorphisms. Then the quotient
$$\frac{H}{N(\langle \varphi(g) \cdot (\psi(g))^{-1} \mid g \in G \rangle)}$$
with the normalizer, forms a commutative diagram as follows:

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] \arrow[dr, "\pi\circ\varphi=\pi\circ\psi"'] & H \arrow[d, "\pi"] \\ & \frac{H}{N(\langle \varphi(g) \cdot (\psi(g))^{-1} \mid g \in G \rangle)} \end{tikzcd}$$

Notice that $\pi \circ \varphi = \pi \circ \psi$ is true if and only if $\varphi(g)(\psi(g))^{-1} \in N(\langle \varphi(g) \cdot (\psi(g))^{-1} \mid g \in G \rangle)$ which is true by definition. We now show that this cone is a colimit. Write $N = N(\langle \varphi(g) \cdot (\psi(g))^{-1} \mid g \in G \rangle)$. Let K be an arbitrary group. We have that

$$\begin{aligned} \text{Hom}_{\textbf{Grp}}\left(\frac{H}{N}, K\right) &\cong \{f : H \to K \in \textbf{Grp} \mid f(n) = 1 \text{ for all } n \in N\} \\ &\cong \{f : H \to K \in \textbf{Grp} \mid f(\varphi(g) \cdot (\psi(g))^{-1}) \text{ for all } g \in G\} \\ &= \{f : H \to K \in \textbf{Grp} \mid f(\varphi(g)) = f(\psi(g)) \text{ for all } g \in G\} \end{aligned}$$

But this is exactly the cone
$$\text{Cone}\,(X, K)$$
where X is the diagram:

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & H \end{tikzcd}$$

and so we conclude. $\square$

Proposition 5.5.5

The category **Cat** is complete and cocomplete.

There are also non-examples. For example, **Set** $\setminus \{\emptyset\}$ is not cocomplete. The category **Man** of manifolds is not cocomplete.

6 Adjunction

6.1 Adjoint Functors

Definition 6.1.1: Adjunction

An adjunction consists of two functors $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ together with a natural isomorphisms

$$a_{-,-} : \text{Hom}_{\mathcal{D}}(L(-), -) \overset{\cong}{\to} \text{Hom}_{\mathcal{C}}(-, R(-))$$

In this case, L is said to be the left adjoint of R and R is said to be the right adjoint of L.

Note that naturality of a means the following: The two functors

$$\text{Hom}_{\mathcal{D}}(L(-), -) \quad \text{and} \quad \text{Hom}_{\mathcal{C}}(-, R(-)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \to \text{Set}$$

is natural on both variables. This means that the following two diagrams commute:

$$\begin{tikzcd}
\text{Hom}_{\mathcal{D}}(L(C), D) \arrow[r, "g_*"] \arrow[d, "\cong"'] & \text{Hom}_{\mathcal{D}}(L(C), D') \arrow[d, "\cong"] \\
\text{Hom}_{\mathcal{C}}(C, R(D)) \arrow[r, "R(g)_*"'] & \text{Hom}_{\mathcal{C}}(C, R(D'))
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{Hom}_{\mathcal{D}}(L(C), D) \arrow[r, "L(f)^*"] \arrow[d, "\cong"'] & \text{Hom}_{\mathcal{D}}(L(C'), D) \arrow[d, "\cong"] \\
\text{Hom}_{\mathcal{C}}(C, R(D)) \arrow[r, "f^*"'] & \text{Hom}_{\mathcal{C}}(C', R(D))
\end{tikzcd}$$

for morphisms $g : D \to D'$ in $\mathcal{D}$ and $f : C' \to C$ in $\mathcal{C}$. Both diagrams can be simplified into one commutative diagram:

$$\begin{tikzcd}
\text{Hom}_{\mathcal{D}}(L(C), D) \arrow[r, "a_{C,D}"] \arrow[d, "g_* \circ L(f)^*"] & \text{Hom}_{\mathcal{C}}(C, R(D)) \arrow[d, "R(g)_* \circ f^*"] \\
\text{Hom}_{\mathcal{D}}(L(C'), D') \arrow[r, "a_{C',D'}"] & \text{Hom}_{\mathcal{C}}(C', R(D'))
\end{tikzcd}$$

This means that for all $\alpha : L(C) \to D$, we have that

$$R(g) \circ (a_{C,D}(\alpha)) \circ f = a_{C',D'}(g \circ \alpha \circ L(f))$$

Lemma 6.1.2

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be left and right adjoint functors of each other such that there is a natural isomorphism

$$a_{C,D} : \text{Hom}_{\mathcal{D}}(L(C), D) \overset{\cong}{\to} \text{Hom}_{\mathcal{C}}(C, R(D))$$

The left diagram below commutes if and only if the right diagram commutes:

$$\begin{tikzcd}
L(C) \arrow[r, "f"] \arrow[d, "L(h)"'] & D \arrow[d, "k"] \\
L(C') \arrow[r, "g"'] & D'
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "a(f)"] \arrow[d, "h"'] & R(D) \arrow[d, "R(k)"] \\
C' \arrow[r, "a(g)"'] & R(D')
\end{tikzcd}$$

Proof. □

Theorem 6.1.3

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be functors. There is a bijection between the natural isomorphisms

$$a_{(-),(-)} : \text{Hom}_{\mathcal{D}}(L(-), -) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(-, R(-))$$

and the pair of natural transformations

$$\varepsilon : L \circ R \Rightarrow \mathrm{id}_{\mathcal{D}} \qquad \eta : \mathrm{id}_{\mathcal{C}} \Rightarrow R \circ L$$

satisfying the triangle conditions (i) and (ii):

$$\begin{tikzcd} L(C) \arrow[r, "L(\eta_C)"] \arrow[dr, "\mathrm{id}_{L(C)}"'] & LRL(C) \arrow[d, "\varepsilon_{L(C)}"] \\ & L(C) \end{tikzcd} \qquad \begin{tikzcd} R(D) \arrow[r, "\eta_{R(D)}"] \arrow[dr, "\mathrm{id}_{R(D)}"'] & RLR(D) \arrow[d, "R(\varepsilon_D)"] \\ & R(D) \end{tikzcd}$$

This means that there is a one-to-one correspondence:

$$\left\{ \substack{a:\mathrm{Hom}_{\mathcal{D}}(L(-),-)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{C}}(-,R(-)) \\ a \text{ is a natural isomorphism}} \right\} \overset{1:1}{\longleftrightarrow} \left\{ (\varepsilon, \eta) \;\middle|\; \substack{\eta:\mathrm{id}_{\mathcal{C}}\Rightarrow RL \text{ and } \varepsilon:LR\Rightarrow\mathrm{id}_{\mathcal{D}} \\ \text{are natural transformations} \\ \text{satisfying the triangle conditions}} \right\}$$

Proof. Suppose that there is a natural isomorphism

$$a : \mathrm{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, R(D))$$

Then by choosing $D = L(C)$, we obtain an isomorphism

$$a : \mathrm{Hom}_{\mathcal{D}}(L(C), L(C)) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, R(L(C)))$$

Define $\eta_C : C \to RL(C)$ by $\eta_C = a(\mathrm{id}_{L(C)})$. Similarly, define $\varepsilon_D : LR(D) \to D$ by $\varepsilon_D = a^{-1}(\mathrm{id}_{R(D)})$. We wish to show that the η_C and the ε_C assembles into natural transformations $\eta : \mathrm{id}_C \Rightarrow RL$ and $\varepsilon : LR \Rightarrow \mathrm{id}_D$ respectively. This is clear by lemma 5.1.2 by applying $D = L(C)$ for η and applying $C = R(D)$ for ε respectively. We now show that the triangle conditions are satisfied. We have that

$$\begin{aligned} a(\varepsilon_{L(C)} \circ L(\eta_C)) &= a(\varepsilon_{L(C)}) \circ \eta && \text{(Naturality of } a\text{)} \\ &= a(a^{-1}(\mathrm{id}_{RL(C)})) \circ \eta_C \\ &= \eta_C \end{aligned}$$

This means that

$$\varepsilon_{L(C)} \circ L(\eta_C) = a^{-1}(\eta_C) = \mathrm{id}_{L(C)}$$

a similar argument shows that the second triangle condition is also satisfied.

Conversely, suppose that we are given the natural transformations η and ε. Define a collection of isomorphisms

$$a_{C,D} : \mathrm{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{C}}(C, R(D))$$

for varying $C \in \mathcal{C}$ and $D \in \mathcal{D}$ as follows. The morphism $f : L(C) \to D$ in $\mathcal{D}$ is sent to

$$C \overset{\eta_C}{\to} RL(C) \overset{R(f)}{\to} R(D)$$

We need to show that these assemble into a natural transformation.

Now each $a_{C,D}$ has inverse given by

$$L(C) \overset{L(g)}{\to} LR(D) \overset{\varepsilon_D}{\to} D$$

for $g : C \to R(D)$ in $\mathcal{C}$. We also need to show that these assemble into a natural transformation. □

The above theorem implies that given two functors, if one wants to specify an adjunction between them, we just have to exhibit two natural transformations $\eta : \text{id}_{\mathcal{C}} \Rightarrow RL$ and $\varepsilon : LR \Rightarrow \text{id}_{\mathcal{D}}$ satisfying the triangle conditions.

Definition 6.1.4: Units and Counits

Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be adjoint functors with a natural isomorphism

$$a_{C,D} : \text{Hom}_{\mathcal{D}}(L(C), D) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(C, R(D))$$

for each $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Define the unit of the adjunction to be the corresponding natural transformation

$$\eta : \text{id}_{\mathcal{C}} \Rightarrow R \circ L$$

Define the counit of the adjunction to be the corresponding natural transformation

$$\varepsilon : L \circ R \Rightarrow \text{id}_{\mathcal{D}}$$

Evidently, the data of an adjunction is not only the two functors exhibiting the adjunction, but also the choice of the natural isomorphism.

Proposition 6.1.5

Let $U : \mathbf{Top} \to \mathbf{Set}$ be the forgetful functor. Then U admits a left adjoint and a right adjoint.

Proof. The data of a left adjoint of U consists of isomorphisms

$$\text{Hom}_{\mathbf{Top}}(L(X), Y) \cong \text{Hom}_{\mathbf{Set}}(X, U(Y))$$

that assemble into a natural isomorphism. Consider the functor $L : \mathbf{Set} \to \mathbf{Top}$ sending a set X to the space X with its discrete topology. Then any map $f : X \to Y$ on the level sets must also be a continuous map since X has the discrete topology. Thus we have constructed a left adjoint.

Similarly, the data of a right adjoint of U consists of isomorphisms

$$\text{Hom}_{\mathbf{Set}}(U(X), Y) \cong \text{Hom}_{\mathbf{Top}}(X, R(Y))$$

that assemble into a natural isomorphism. Consider the functor $R : \mathbf{Set} \to \mathbf{Top}$ sending Y to the space Y with the indiscrete topology. Then any map $f : X \to Y$ on the level of sets is also a continuous map since Y has the indiscrete topology. Thus we have constructed a right adjoint. $\square$

Proposition 6.1.6

Let $U : \mathbf{Grp} \to \mathbf{Set}$ be the forgetful functor. Then U admits a left adjoint.

Proof. The data of a left adjoint of U consists of isomorphisms

$$\text{Hom}_{\mathbf{Grp}}(L(X), Y) \cong \text{Hom}_{\mathbf{Set}}(X, U(Y))$$

that assemble into a natural isomorphism. Consider the functor $L : \mathbf{Set} \to \mathbf{Grp}$ sending a set X to the free group $F(X)$ on X. Then any map $f : X \to Y$ on the level sets extend to a group homomorphism. Thus we have constructed a left adjoint. $\square$

We end the section with a theorem that connects (co)limits to adjunctions.

Theorem 6.1.7

Let $\mathcal{C}$ be a category. Let $\mathcal{J}$ be a small category and let $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ be the diagonal functor sending C to $\Delta(C)$. Then the following are true.

- If $\mathcal{C}$ is complete, then $\Delta(-)$ admits a right adjoint $\lim_{\mathcal{J}}(-)$.
- Dually, if X is cocomplete, then $\Delta(-)$ admits a left adjoint $\operatorname{colim}_{\mathcal{J}}(-)$.

6.2 (Co)Limit Preserving Functors

Let $\mathcal{J} \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathcal{D}$ be functors where $\mathcal{J}$ is small. Notice that if X has a limit cone $\pi : \Delta(\lim_{\mathcal{J}} X) \Rightarrow X$. Then $F(\pi_I) : F(\lim_{\mathcal{J}} X) \to F(X_I)$ defines a cone

$$F(\pi) : \Delta\left(F\left(\lim_{\mathcal{J}} X\right)\right) \to F \circ X$$

The question remains that whether this is a limit cone in $\mathcal{D}$.

Definition 6.2.1: (Co)Limit Preserving

Let $\mathcal{J} \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathcal{D}$ be functors, where $\mathcal{J}$ is small. We say that F preserves limits if for every limit cone $\pi : \Delta(\lim_{\mathcal{J}} X) \Rightarrow X$, the cone

$$F(\pi) : \Delta\left(F\left(\lim_{\mathcal{J}} X\right)\right) \to F \circ X$$

is a limit cone. This means that

$$\lim_{\mathcal{J}} F \circ X = F\left(\lim_{\mathcal{J}} X\right)$$

Dually, F preserves colimits if it preserve colimit cones.

Proposition 6.2.2

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is representable, then F preserves limits.

Notice that it is not necessarily true that F preserves colimits. This is true when the representability condition is provided by a contravariant functor.

Lemma 6.2.3

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with a left adjoint L. Then for every small category $\mathcal{J}$, the functor

$$F \circ (-) : \mathcal{C}^{\mathcal{J}} \to \mathcal{D}^{\mathcal{J}}$$

has a left adjoint by $L \circ (-)$.

Proof. We need a natural bijection

$$\operatorname{Hom}_{\mathcal{C}^{\mathcal{J}}}(L \circ X, Y) \cong \operatorname{Hom}_{\mathcal{D}^{\mathcal{J}}}(X, F \circ Y)$$

for all $X : \mathcal{J} \to \mathcal{D}$ and for all $Y : \mathcal{J} \to \mathcal{C}$. By assumption, we have a natural bijection $a : \operatorname{Hom}_{\mathcal{C}}(L(D), C) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{D}}(D, F(C))$. Let $\alpha : L \circ X \Rightarrow Y$. Then

$$a(\alpha_J : L(X_J) \to Y_J) : X_J \to F(Y_J)$$

We claim that $a(\alpha_J)$ defines a natural transformation $A(\alpha) : X \Rightarrow F \circ Y$. We need to show that for all $f : I \to J$ in $\mathcal{J}$, we have that the square

$$\begin{tikzcd}
X_I \arrow[r, "a(\alpha_I)"] \arrow[d, "X(f)"'] & F(Y_I) \arrow[d, "F(Y(f))"] \\
X_J \arrow[r, "a(\alpha_J)"] & F(Y_J)
\end{tikzcd}$$

commutes. This means that we want

$$F(Y(f)) \circ a(\alpha_I) = a(\alpha_J) \circ X(f)$$

Recall that since a is natural, we have that

$$F(Y(f)) \circ a(\alpha_I) \circ \mathrm{id} = a(Y(f) \circ \alpha_I \circ \mathrm{id})$$

Again by naturality (Swap the places of f and id), we have that

$$a(\alpha_J) \circ X(f) = a(\alpha_J \circ L(X(f)))$$

Since a is a bijection, it remains to show that $\alpha_J \circ L(X(f)) = Y(f) \circ \alpha_I$ which holds by naturality of α. Clearly A is bijective since a is. It remains to show that A is natural. □

Proposition 6.2.4

Suppose that $F : \mathcal{C} \to \mathcal{D}$ admits a left adjoint. Then F preserves limits. Dually, F preserves colimits if F admits a right adjoint.

Proof. Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram with limit $\Delta \lim_{\mathcal{J}} X \to X$. Let L be the left adjoint of F. We want to show that

$$\mathrm{Hom}_{\mathcal{D}}\left(D, F\left(\lim_{\mathcal{J}} X\right)\right) \cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(\Delta(D), F \circ X)$$

But this is true sine

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}\left(D, F\left(\lim_{\mathcal{J}} X\right)\right) &\cong \mathrm{Hom}_{\mathcal{C}}\left(L(D), \lim_{\mathcal{J}} X\right) && \text{(Natural bijection of adjunction)} \\
&\cong \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(\Delta(L(D)), X) && \text{(Universal property of limits)} \\
&= \mathrm{Hom}_{\mathcal{C}^{\mathcal{J}}}(L(\Delta(D)), X) \\
&\cong \mathrm{Hom}_{\mathcal{D}^{\mathcal{J}}}(\Delta(D), F \circ X) && \text{(Lemma 4.2.2)}
\end{aligned}$$

This is moreover the same bijection defined from the cone maps of $F(\lim_{\mathcal{J}} X)$ since □

6.3 The Adjoint Functor Theorem

In the previous section we saw that functors that admit left adjoints preserve limits. We may also ask the converse: Does functors that preserve limits admit left adjoints? There is a partial converse to this and we will work towards it in this section.

Lemma 6.3.1

Let $U : \mathcal{A} \to \mathcal{S}$ be a functor. Then U admits a left adjoint if and only if for any $S \in \mathrm{Obj}\,\mathcal{S}$, there is a map $S \xrightarrow{i} U(A)$ such that for all $f : S \to U(B)$ in $\mathcal{S}$, there is a unique map $\lambda : B \to A$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & U(A) \\ & U(B) \arrow[u, dashed, "U(\exists!\lambda)"'] \end{tikzcd}$$

This lemma is almost tautological. In practise it is not helpful. We want to relax this condition.

Definition 6.3.2: Weakly Initial Elements

Let $\mathcal{C}$ be a category. An object $C \in \operatorname{Obj}\mathcal{C}$ is weakly initial if for every $D \in \operatorname{Obj}\mathcal{C}$, there is a map $C \to D$. A set of objects $\{C_i \mid i \in I\}$ is jointly weakly initial if for every $D \in \operatorname{Obj}\mathcal{C}$, there is a map $C_i \to D$ for some $i \in I$.

Recall the definition of slice categories S/U for the functor $U : \mathcal{A} \to \mathcal{S}$. Then the condition in lemma 4.3.1 becomes exactly that S/U admits an initial object.

Theorem 6.3.3: General Adjoint Functor Theorem

Let $\mathcal{A}$ be a locally small and complete category. Let $U : \mathcal{A} \to \mathcal{S}$ be a functor which preserves limits. Suppose that U satisfies the solution set condition: For any $S \in \operatorname{Obj}\mathcal{S}$, there is a set of maps

$$\Psi_S = \{f_i : S \to U(A_i)\}$$

such that for every morphism $f : S \to U(B)$ there exists an $f_i \in \Psi_i$ and $\lambda : A_i \to B$ such that

$$\begin{tikzcd} S \arrow[r, "f_i"] \arrow[dr, "f"'] & U(A_i) \arrow[d, dashed, "U(\exists\lambda)"'] \\ & U(B) \end{tikzcd}$$

Then U has a left adjoint.

Proof. Notice that the solution set condition is equal to Ψ_S being a set of jointly weakly initial object in S/U. So now we need to prove that if U preserves limits and every S/U has a set of jointly weakly initial objects, then S/U has an initial object. By lemma 4.3.1 this would imply that U has a left adjoint.

First notice that since U preserves limits, S/U is complete: Given $X : \mathcal{J} \to S/U$, then

$$X_J = (A_J \in \operatorname{Obj}\mathcal{A}, f_J : S \to U(A_J))$$

and one can check that

$$\lim_{\mathcal{J}} X = \left(\lim_{J \in \mathcal{J}} A_J, S \overset{\{f_J\}}{\to} \lim_{J \in \operatorname{Obj}\mathcal{J}} U(A_J) \overset{\cong}{\to} U\left(\lim_{J} A_J\right) \right)$$

It is clear that S/U has a set of jointly weakly initial objects $\Psi = \Psi_J$.

Let $\mathcal{J} \in S/U$ be the full subcategory of S/U on the objects in Ψ. $\lim(\mathcal{J} \hookrightarrow S/U)$ exists since Ψ is small and S/U is complete. Let us prove that it is initial in S/U. Let $C \in S/U$. First, there is a map $\lambda_C : \lim(\mathcal{J} \hookrightarrow S/U) \to C$ defined as follows: Choose a $J \in \Psi$ and a map $h_C : J \to C$ and define

$$\lambda_C : \lim(\mathcal{J} \hookrightarrow S/U) \overset{\pi_J}{\to} J \overset{h_C}{\to} C$$

where π_J is the projection map of the limit. We need to see that this map is unique.

First let us prove that $\lambda_C = \mathrm{id}_C$ when $C = \lim(\mathcal{J} \hookrightarrow S/U)$. By the universal property of the limit, this is the case if and only if

$$\pi_I \circ \lambda_C = \pi_I \circ \mathrm{id}$$

for all $I \in \Psi$. And indeed, we have that

$$\pi_I \circ \lambda_C = \left(\lim(\mathcal{J} \hookrightarrow S/U) \xrightarrow{\pi_J} J \xrightarrow{h_C} \lim(\mathcal{J} \hookrightarrow S/U) \xrightarrow{\pi_I} I\right)$$

Since J is a full subcategory of S/U, $\pi_I \circ h_c : J \to I$ lies in J. The cone condition of π implies that π_J post composed with the map from J to I is equal to π_I so that the map $\lim(\mathcal{J} \hookrightarrow S/U)$ to I is π_I. Thus we have that $\pi_I \circ \lambda_C = \pi_I$.

Now let $C \in S/U$ be any object of S/U. Let $f : \lim(\mathcal{J} \hookrightarrow S/U) \to C$ be any morphism. Since Ψ consists of the weakly initial objects, choose J' together with a map $h_l : J \to \lim(\mathcal{J} \hookrightarrow S/U)$. Then since S/U is complete, we can take the pullback of J and J' to form a diagram

$$\begin{tikzcd}
P \arrow[rr] \arrow[d] & & J \arrow[d, "h_C"'] \\
J' \arrow[r, "h_l"] & \lim(\mathcal{J} \hookrightarrow S/U) \arrow[r, "f"] & C
\end{tikzcd}$$

which commutes. Now choose $J'' \in \Psi$ and a map $h_P : J'' \to P$. This is possible since Ψ is weakly initial. Since $\mathcal{J}$ is a full subcategory, we have maps $J'' \to J$ and $J'' \to J$ so that we have the following diagram

$$\begin{tikzcd}
J'' \arrow[dr, "h_P"] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& P \arrow[rr] \arrow[d] & & J \arrow[d, "h_C"'] \\
& J' \arrow[r, "h_l"] & \lim(\mathcal{J} \hookrightarrow S/U) \arrow[r, "f"] & C
\end{tikzcd}$$

that is commutative. Finally, since $l = \lim(\mathcal{J} \hookrightarrow S/U)$ is a limit, we can find maps to $\pi_J : l \to J$, $\pi_{J'} : l \to J'$ and $\pi_{J''} : l \to J''$ so that the following diagram

$$\begin{tikzcd}
\lim(\mathcal{J} \hookrightarrow S/U) \arrow[dr, "\pi_{J''}"] \arrow[ddrrrr, bend left, "\pi_J"] \arrow[dddrr, bend right, "\pi_{J'}"'] & & & & \\
& J'' \arrow[dr, "h_P"] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& & P \arrow[rr] \arrow[d] & & J \arrow[d, "h_C"'] \\
& & J' \arrow[r, "h_l"] & \lim(\mathcal{J} \hookrightarrow S/U) \arrow[r, "f"] & C
\end{tikzcd}$$

is commutative. This implies that $h_C \circ \pi_J = f \circ h_l \circ \pi_{J'}$. Since $h_C \circ \pi_J = \lambda_C$ and $h_l \circ \pi_{J'} = \lambda_l = \mathrm{id}$, we have that $\lambda_C = f$ which shows uniqueness of λ_C. $\square$

7 Groups and Categories

7.1 Groupoids and Groups

Definition 7.1.1: Groupoids

A groupoid is a category $\mathcal{G}$ such that every morphism has an inverse.

Definition 7.1.2: BG of a Group

Let G be a group. Then define the category BG as follows.

- BG consists of one object $\{*\}$
- The morphisms of BG are precisely

$$\text{Hom}_{BG}(*,*) = G$$

- Composition is defined as the group operation of G.

Lemma 7.1.3

Let G be a group. Then BG is a groupoid.

Proposition 7.1.4

Let G and H be groups and $f : G \to H$ a group homomorphism. Then there is a functor $Bf : BG \to BH$ defined by sending the unique object to the unique object, and sending each morphism $g \in G$ to $f(g) \in H$.

Proposition 7.1.5

The construction BG for a group G defines a functor $B : \mathbf{Grp} \to \mathbf{Cat}$ defined by $B(G) = BG$ and for a group homomorphism $f : G \to H$, the functor B sends f to $Bf : BG \to BH$.

Definition 7.1.6: Connected Groupoids

Let $\mathcal{G}$ be a groupoid. We say that $\mathcal{G}$ is connected if any two objects of $\mathcal{G}$ are isomorphic.

Proposition 7.1.7

Let $\mathcal{G}$ be a connected groupoid. Let $c_0 \in \mathcal{G}$ be an object. Then there is an equivalence of categories

$$B\text{Aut}_{\mathcal{G}}(c_0) \cong \mathcal{G}$$

In particular, by regarding the isomorphic class of the connected groupoid as one object this means that the groupoids with one objects are in one-to-one correspondence with groups.

Theorem 7.1.8

There is a one to one correspondence between the functors $F : BG \to \mathbf{Set}$ and the left G-sets for a group G.

7.2 Transition Groupoids and Group Actions

Definition 7.2.1: Transition Groupoid

Let G be a group and X a G-set. Define the transition groupoid $T_G(X)$ of X to consist of the following data.

- The objects of $T_G(X)$ are the elements of X.
- For $x, y \in X$, define the morphisms of $T_G(X)$ to be

$$\text{Hom}_{T_G(X)}(x, y) = \{g \in G \mid g \cdot x = y\}$$

- Composition is defined by the group operation of G.

Lemma 7.2.2

Let G be a group and let X be a G-set. Let $T_G(X)$ be the transition groupoid of X. Then

$$\text{Aut}_{T_G(X)}(x) = \{g \in G \mid g \cdot x = x\} = \text{Stab}_G(x)$$

for any $x \in X$. Moreover, $T_G(x)$ is a connected groupoid if and only if the action of G is transitive.

Definition 7.2.3: Fixed Points and Orbits

Let G be a group and $\mathcal{C}$ be a category such that there is a diagram $X : BG \to \mathcal{C}$. Define the fixed points of X to be the limit

$$X^G = \lim_{BG} X$$

Define the orbits of X to be the colimit

$$X_G = \operatorname*{colim}_{BG} X$$

Theorem 7.2.4

Let G be a group. Then there is a natural group action on the functor $X : BG \to \textbf{Set}$. Moreover, the following are true regarding the diagram X.

- The fixed points (in the categorical sense) are

$$X^G = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$$

  which is the same as the usual sense in group theory.

- The orbits (in the categorical sense) are

$$X_G = \{\text{Orb}_G(x) \mid x \in X\}$$

  which is the usual orbits in group theory.

Proposition 7.2.5

Let G be a group and $X : BG \to \mathcal{C}$ be a diagram to an arbitrary category $\mathcal{C}$. Then

$$\lim_{BG} X = \{x \in X \mid g \cdot x = x \text{ for all } g \in G\}$$

However, the analogy no longer works for the orbits (colimits of $X : BG \to \mathcal{C}$). For example, if

$\mathcal{C} = \mathbf{Ab}$, then

$$X_G = \frac{X}{\langle g \cdot x - x \mid g \in G, x \in X \rangle}$$

8 Monoid and Group Objects

8.1 Monoidal Categories

Definition 8.1.1: Strict Monoidal Categories

A strict monoidal category is a category $\mathcal{A}$ consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

- Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- Identity: $I \otimes A = A$ and $A \otimes I = A$

Notice that we require strict equality in the associativity and identity laws. Since we usually only consider objects up to isomorphism in a category, strict monoidal categories may seem quite rare in practise.

Definition 8.1.2

A weak monoidal category is a category $\mathcal{A}$ consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

- Associativity: There are isomorphisms

$$(A \otimes B) \otimes C \overset{\alpha_{A,B,C}}{\cong} A \otimes (B \otimes C)$$

  that is natural in A, B and C

- Identity: There are isomorphisms

$$I \otimes A \overset{\lambda_A}{\cong} A \quad \text{and} \quad A \otimes I \overset{\rho_A}{\cong} A$$

  that are both natural in A

Such natural isomorphisms must also satisfy the following commutative laws:

- The pentagon identity:

$$\begin{tikzcd} & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\ ((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes 1_D"'] & & A \otimes (B \otimes (C \otimes D)) \\ (A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A\otimes\alpha_{B,C,D}"'] \end{tikzcd}$$

- The triangle identity:

$$\begin{tikzcd} (A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A\otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A\otimes\lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

It is clear that every strict monoidal category is also a weak monoidal category.

Lemma 8.1.3

Every category $\mathcal{C}$ with finite products is a monoidal category with product $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and identity $*$ the terminal object.

Proposition 8.1.4

For any commutative ring R, the category $\mathbf{Mod}_R$ of R-modules is a monoidal category with the tensor product $\otimes$ and the identity object R.

8.2 Monoidal Object

8.3 Group Objects

Definition 8.3.1: Group Objects

Let $\mathcal{C}$ be a category with finite products. We say that $G \in \mathcal{C}$ is a group object if there exists three morphisms

- Multiplication: $m : G \times G \to G$
- Identity: $e : * \to G$ where $*$ is the terminal object
- Inverse: $\text{inv} : G \to G$

such that the following diagrams commute.

- Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- Identity:

$$\begin{tikzcd} G \arrow[r, "(e, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, e)"'] \arrow[dr, "\mathrm{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- Inverse:

$$\begin{tikzcd} G \arrow[r, "(\mathrm{inv}, \mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G, \mathrm{inv})"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Proposition 8.3.2

A group object in the category $\mathbf{Set}$ of sets is a group in the usual sense.

Proposition 8.3.3

A group object in the category $\mathbf{Grp}$ of groups is an abelian group.