Introduction to Graph Theory

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Abstract

References:

Contents

1 Basic Definition of Graphs

1.1 Graphs and Graph Homomorphisms

Definition 1.1.1: Graphs

A graph G consists of the following data.

- A set V called the vertices of G
- A set E is called the edges of G
- A function $\phi_G : E \to \{\{v, w\} \mid v, w \in V\}$ that assigns to each edge two vertices.

Definition 1.1.2: Finite Graphs

Let $G = (V, E)$ be a graph. We say that G is a finite graph if $|V|$ and $|E|$ are finite.

Definition 1.1.3: Graph Homomorphism

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. A graph homomorphism

$$f : G \to H$$

consists of the following data:

- A map of sets $f : V(G) \to V(H)$
- A map of sets $f : E(G) \to E(H)$ such that $\phi_G(e) = \{v, w\}$ implies $\phi_H(f(e)) = \{f(v), f(w)\}$

Definition 1.1.4: Graph Isomorphism

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. Let $\phi : G \to H$ be a graph homomorphism. We say that ϕ is a graph isomorphism if there exists a graph homomorphism $\psi : H \to G$ such that $\psi \circ \phi = \mathrm{id}_G$ and $\phi \circ \psi = \mathrm{id}_H$.

Lemma 1.1.5

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. Let $\phi : G \to H$ be a graph isomorphism. Then

$$|V(G)| = |V(H)| \quad \text{and} \quad |E(G)| = |E(H)|$$

1.2 Subgraphs of a Graph

Definition 1.2.1: Subgraphs

Let G, H be graphs. We say that H is a subgraph of G if the following are true.

- $V(H) \subseteq V(G)$
- $E(H) \subseteq E(G)$
- $\phi_H = \phi_G|_{E(H)}$

Definition 1.2.2: Spanning Subgraph

Let G be a graph. Let $H \subseteq G$ be a subgraph of G. We say that H is a spanning subgraph of G if $V(H) = V(G)$.

Definition 1.2.3: Induced Subgraphs of Vertices

Let $G = (V(G), E(G))$ be a graph. Let $W \subseteq V(G)$ be a subset of vertices of G. Define the induced subgraph $G[W]$ by the following data.

- $V(G[W]) = W$
- $E(G[W]) = \{e \in E(G) \mid \phi_G(e) \subseteq W\}$
- $\phi_{G[W]} = \phi_G|_W$

Definition 1.2.4: Induced Subgraphs of Edges

Let $G = (V(G), E(G))$ be a graph. Let $F \subseteq E(G)$ be a subset of edges of G. Define the induced subgraph $G[F]$ by the following data.

- $V(G[W]) = \{v \in V \mid v \in \phi_G(e) \text{ for some } e \in E\}$
- $E(G[W]) = F$
- $\phi_{G[F]} = \phi_G|_{V(G[W])}$

1.3 The Degree of Vertices

Definition 1.3.1: Degree of a Vertex

Let G be a graph. Let $v \in V$ be a vertex. Define the degree of v to be

$$\deg(v) = |\{e \in E \mid v \in \phi(e)\}|$$

In other words, the degree of a vertex is the number of edges incident with v. Notice that loops count for two such incidents.

Proposition 1.3.2

Let G be a finite graph. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

Corollary 1.3.3

Let G be a finite graph. Then the number of vertices with odd degree is even.

Definition 1.3.4: Neighbouring Vertices

Let G be a graph. Let $S \subseteq V$ be a set of vertices of G. Define the neighbours of the vertices of S to be

$$N(S) = \{v \in V \mid \{v, s\} \in \phi_G(e) \text{ for some } e \in E \text{ and some } s \in S\}$$

2 Walking in a Graph

2.1 Walks, Path and Cycles

Definition 2.1.1: Walks in a Graph

Let G be a graph. A walk W in G is a finte sequence

$$W = v_1, e_1, v_2, e_2, \dots, v_{k-1}e_{k-1}v_k$$

of alternating vertices and edges that start and end with vertices, such that $\phi_G(e_i) = \{v_{i-1}, v_i\}$ for all $1 \leq i < k$. In this case we say that the walk has length k.

Definition 2.1.2: Paths in a Graph

Let G be a graph. Let $W = v_1, e_1, v_2, e_2, \dots, v_{k-1}e_{k-1}v_k$ be a walk. We say that W is a path if the following are true.

- Each of $v_1, \dots, v_k \in V$ are distinct.
- Each of $e_1, \dots, e_{k-1} \in E$ are distinct.

Definition 2.1.3: Closed Walks in a Graph

Let G be a graph. Let $W = v_1, e_1, v_2, e_2, \dots, v_{k-1}e_{k-1}v_k$ be a walk in G. We say that W is closed if $v_1 = v_k$.

Definition 2.1.4: Cycles in a Graph

Let G be a graph. Let $W = v_1, e_1, v_2, e_2, \dots, v_{k-1}e_{k-1}v_k$ be a walk in G. We say that W is a cycle in G if the following are true.

- $v_1 = v_k$.
- Each $v_1, \dots, v_{k-1} \in V$ are distinct.

We say that W is odd (even) if k is odd (even).

Lemma 2.1.5

Let G be a graph. Let W be a cycle in G. Then W is closed.

2.2 Connected Components of a Graph

Definition 2.2.1: Connected Vertices

Let G be a graph. Let $v, w \in V$ be vertices. We say that v and w are connected if there exists a path in G from v to w.

Lemma 2.2.2

Let G be a graph. Then connectedness of vertices is an equivalence relation in the vertices V of G.

2.3 Euler Tours

Definition 2.3.1: Tours in a Graph

Let G be a graph. Let W be a walk in G. We say that W is a tour in G if every edge in G is contained in the sequence W.

Definition 2.3.2: Euler Tours in a Graph

Let G be a graph. Let W be a walk in G. We say that W is an Euler tour in G if every edge in G is contained in the sequence W exactly once.

Definition 2.3.3: Eulerian Graphs

Let G be a graph. We say that G is Eulerian if G contains an Euler tour.

Proposition 2.3.4

Let G be a non-empty connected graph. Then G is Eulerian if and only if G contains no vertices of odd degree.

2.4 Hamiltonian Cycles

Definition 2.4.1: Hamiltonian Path

Let G be a graph. Let W be a walk in G. We say that W is a Hamiltonian path if W is a path and every vertex in G is contained in the sequence W.

Definition 2.4.2: Hamiltonian Cycle

Let G be a graph. Let W be a walk in G. We say that W is a Hamiltonian cycle if W is a cycle and every vertex in G is contained in the sequence W.

Definition 2.4.3: Hamiltonian Graphs

Let G be a graph. We say that G is Hamiltonian if G contains an Hamiltonian cycle.

3 Matchings of Edges

3.1 Different Types of Matchings

Definition 3.1.1: Matchings

Let G be a graph. Let $M \subseteq E$ be a subset of edges. We say that M is a matching if no two edges share a common vertex.

Definition 3.1.2: Maximal Matchings

Let G be a graph. Let $M \subseteq E$ be a matching. We say that M is maximal if $M \subseteq T \subseteq E$ is another matching, then $T = M$.

Definition 3.1.3: Maximum Matchings

Let G be a graph. Let $M \subseteq E$ be a matching. We say that M is a maximum matching if M contains the largest number of edges possible.

Definition 3.1.4: Alternating Paths

Let G be a graph. Let $M \subseteq E$ be a matching. Let P be a path in G. We say that P is M-alternating if the subsequence of edges in P alternates between M and $E \setminus M$.

Definition 3.1.5: Augmenting Paths

Let G be a graph. Let $M \subseteq E$ be a matching. Let P be a path in G. We say that P is M-augmenting if it is M-alternating and the sequence of edges begins and ends in $E \setminus M$.

Proposition 3.1.6

Let G be a graph. Let $M \subseteq E$ be a matching. Then M is maximum if and only if G contains no M-augmenting paths.

3.2 Coverings

Definition 3.2.1: Coverings

Let G be a graph. Let $T \subseteq E$ be a subset of edges. A covering of T is a subset $W \subseteq V$ of vertices such that for all $e \in T$, there exists $v \in W$ such that $v \in \phi_G(e)$.

Lemma 3.2.2

Let G be a graph. Let $M \subseteq E$ be a matching. Let $W \subseteq V$ be a covering of M. Then $|M| \leq |W|$.

3.3 Maximum Matchings and Minimum Coverings

Definition 3.3.1: Minimum Coverings

Let G be a graph. Let $T \subseteq E$ be a subset of edges. Let $W \subseteq V$ be a covering for T. We say that W is a minimum covering if for any covering X of T, $|W| \leq |X|$.

Lemma 3.3.2

Let G be a graph. Let $M \subseteq E$ be a matching. Let $W \subseteq V$ be a covering of M. If $|M| = |W|$, then M is a maximum matching and W is a minimum covering.

3.4 Perfect Matchings

Definition 3.4.1: Perfect Matchings

Let G be a graph. Let $M \subseteq E$ be a matching. We say that M is perfect if for all $v \in V$, there exists some $e \in M$ for which $v \in \phi_G(e)$.

Proposition 3.4.2

Let G be a graph. Then G contains a perfect matching if and only if

$$O(G \setminus S) \leq |S|$$

for all $S \subseteq V$ vertices, where $O(G \setminus S)$ refers to the number of odd vertices of $G \setminus S$.

4 Special Types of Graphs

4.1 Simple Graphs

Definition 4.1.1: Loops

Let (V, E) be a graph. Let $e = \{v_1, v_2\} \in E$ be an edge. We say that e is a loop if $v_1 = v_2$.

Definition 4.1.2: Simple Graphs

Let $G = (V, E)$ be a graph. We say that G is simple if the following are true.
- G has no loops.
- For any $v, w \in V$ with $v \neq w$, $\left|\phi^{-1}(\{v, w\})\right| \leq 1$.

The condition says that for any two distinct vertices, there is at most one edge connecting the two.

4.2 Complete Graphs

Definition 4.2.1: Complete Graphs

Let $G = (V, E)$ be a graph. We say that G is complete if the following are true.
- G has no loops.
- For any $v, w \in V$ with $v \neq w$, $\left|\phi^{-1}(\{v, w\})\right| = 1$.

Lemma 4.2.2

Every complete graph is simple.

Proposition 4.2.3

Let $n \in \mathbb{N}$. Then there exists a unique (up to isomorphism) complete graph with n vertices.

Lemma 4.2.4

Let $G = (V, E)$ be a simple finite graph. Then $|E| = \binom{|V|}{2}$ if and only if G is a complete graph.

Definition 4.2.5: Standard Complete Graph

Let $n \in \mathbb{N} \setminus \{0\}$. Define the standard complete graph K_n to be graph consisting of the following data.
- $V(K_n) = \{1, \dots, n\}$.
- $E = \{e_{i,j} \mid 1 \leq i < j \leq n\}$
- $\phi_{K_n}(e_{i,j}) = \{i, j\}$

4.3 Regular Graphs

Definition 4.3.1: Regular Graphs

Let G be a graph. We say that G is regular if there exists $k \in \mathbb{N}$ such that

$$\deg(v) = k$$

for all $v \in V$.

Lemma 4.3.2

Let G be a graph. If G is complete, then G is regular.

4.4 Trees

Definition 4.4.1: Acyclic Graphs

Let G be a graph. We say that G is acyclic if G contains no cycles.

Definition 4.4.2: Trees

Let G be a graph. We say that G is a tree if G is connected and acyclic.

Proposition 4.4.3

Let G be a graph. Then the following are equivalent.

- G is tree.
- G is acyclic and for any graph H such that $G \subseteq H$ and $E(H) > E(G)$, then H contains a cycle.
- G is connected and any proper subgraph of G is disconnected.
- Any two vertices in G are connected by a unique path.

Lemma 4.4.4

Let G be a finite graph. Then G is tree if and only if G is connected and $|E| = |V| - 1$.

Proposition 4.4.5

Let G be a finite connected graph. Then G contains a spanning tree.

Lemma 4.4.6

Let G be a finite connected graph. Then $|E| \geq |V| - 1$.

4.5 Bipartite Graphs

Definition 4.5.1: Bipartite Graphs

Let $G = (V, E)$ be a graph. We say that G is bipartite if there exist a partition $X \amalg Y = V$ of V such that for each $e \in E$, $\phi_G(e)$ has one element in X and one element in Y.

Proposition 4.5.2

Let G be a graph. Then G is bipartite if and only if it contains no odd cycles.

Proposition 4.5.3

Let G be a bipartite graph. Let $M \subseteq E$ be a matching. Let $W \subseteq V$ be a covering of M. Then $|M| = |W|$ if and only if M is a maximum matching and W is a minimum covering.

Theorem 4.5.4: Hall's Theorem

Let G be a bipartite graph with partition $V = X \amalg Y$. Then G contains a matching $M \subseteq E$ for which for all $v \in V$, there exists $e \in E$ such that $v \in \phi_G(e)$ if and only if

$$|N(S)| \geq |S|$$

for all $S \subseteq X$.

Theorem 4.5.5: Marriage Theorem

Let G be k-regular bipartite graph for $k > 0$. Then G contains a perfect matching.