Lie Groups and Lie Algebra

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Abstract

Potentially good books: Humphreys, Erdmann and Wildson

Contents

1 Introduction to Lie Algebras

1.1 Lie Brackets and Lie Algebras

Definition 1.1.1: Lie Brackets

Let V be a vector space over a field k. Let $[-,-] : V \times V \to V$ be a bilinear map. We say that $[-,-]$ is a Lie bracket if the following are true.

- The Alternating Property: $[X, X] = 0$
- Jacobi identity: $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Consider the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ in $\mathbb{R}^3$. It is easy to see that it is a Lie bracket.

Definition 1.1.2: Lie Algebras

A Lie algebra is a vector space V over a field K together with a Lie bracket

$$[-,-] : V \times V \to V$$

For k a field, $M_n(k)$ for any $n \geq 1$ is a Lie algebra with Lie bracket defined as $[A,B] = AB - BA$ for $A, B \in M_n(k)$.

Lemma 1.1.3

Let L be a Lie Algebra. Then for all $x, y \in L$, we have that

$$[x,y] = -[y,x]$$

In other words, the Lie bracket is anti-commutative.

Proof. We have that

$$\begin{aligned} [x,y] + [y,x] &= [x,x] + [x, y-x] + [y,y] + [y, x-y] && \text{(Bilinearity)} \\ &= [x,x] + [y,y] - [x-y, x-y] && \text{(Bilinearity)} \\ &= 0 && \text{(Alternating)} \end{aligned}$$

and so we conclude. □

Lie Algebras are not algebras (in the sense of Rings and Modules) because the Lie bracket fails associativity. Therefore we have to redefine all the standard notions one has in algebra.

While Lie Algebras are not in general algebras, every associative algebra can be equipped with a Lie algebra. For A an associative algebra over a field, we can define a bilinear map on A by

$$[a,b] = ab - ba$$

for all $a, b \in A$. There may also be more than one way to equip an algebra with a Lie algebra structure. One should not think that Lie Algebras encompasses associative algebras because of the different Lie algebras one can equip. Instead, we think of the Lie bracket as an extra structure on associative algebras such that they become Lie algebras.

Definition 1.1.4: Structure Constants

Let L be a Lie algebra such that its underlying vector space has basis $e_1, \dots, e_n$. Define the structure constants of L to be the elements $c_{ij}^k \in \mathbb{F}$ such that

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$$

for all $1 \leq i, j \leq n$.

The structure constants are useful in the following sense. Let L be a Lie algebra and let $a = \sum_{k=1}^n a_k e_k$ and $b = \sum_{k=1}^n b_k e_k$ be elements of L. Then there Lie bracket can be written as

$$[a, b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)[e_i, e_j]$$

by bilinearity. Plugging in the structure constants, we obtain

$$[a, b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i) \sum_{k=1}^n c_{ij}^k e_k$$

Thus we can write $[a, b]$ in terms of the basis $e_1, \dots, e_n$ using structure constants.

1.2 Lie Subalgebras and Ideals

Definition 1.2.1: Lie Subalgebra

Let V be a Lie algebra over K. A lie subalgebra of V is a subset $W \subseteq V$ such that
- W is a vector subspace of V
- $[w_1, w_2] \in W$ for all $w_1, w_2 \in W$

It is clear that a Lie subalgebra is also a Lie algebra in its own right.

Definition 1.2.2: Ideal

Let V be a Lie algebra over K. Let I be a subset of V. Then I is an ideal of V if the following are true.
- I is a vector subspace of V
- $[v, i] \in I$ for all $v \in V$ and $i \in I$.

It is clear from definitions that every ideal of a Lie algebra is a Lie subalgebra. However, the converse is not always true.

Proposition 1.2.3

Let V be a Lie algebra and I, J ideals of V. Then the following are also ideals of V.
- The intersection $I \cap J$
- The sum $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$

Proof.
- It is clear that $I \cap J$ is a vector subspace of V. Since I is an ideal, $[V, I \cap J] \subseteq [V, I] \subseteq I$. Similarly, $[V, I \cap J] \subseteq [V, J] \subseteq J$. Hence $[V, I \cap J] \subseteq I \cap J$.
- It is clear that $I + J$ is a vector subspace of V. Let $i + j \in I + J$. Let $v \in V$. Then we have
$$[v, i + j] = [v, i] + [v, j] \in I + J$$
Hence $I + J$ is an ideal of V.

□

Definition 1.2.4: The Lie Bracket of Ideals

Let V be a Lie algebra. Let I, J be ideals of V. Define the Lie bracket of I and J to be

$$[I, J] = \langle [i, j] \mid i \in I \text{ and } j \in J \rangle$$

Lemma 1.2.5

Let V be a Lie algebra. Let I, J be ideals of V. Then the Lie bracket $[I, J]$ is an ideal of V.

Proof. By definition, $[I, J]$ is a vector subspace of V. Let $v \in V$ and $\sum_{k=1}^n a_k[i_k, j_k] \in [I, J]$. Then we have

$$\begin{aligned}\left[v, \sum_{k=1}^n a_k[i_k, j_k]\right] &= \sum_{k=1}^n a_k[v, [i_k, j_k]] \\ &= \sum_{k=1}^n a_k[v, [i_k, j_k]] \\ &= -\sum_{k=1}^n a_k[[i_k, j_k], v] \\ &= -\sum_{k=1}^n a_k\left(-[[j_k, v], i_k] - [[v, i_k], j_k]\right) \\ &= -\sum_{k=1}^n a_k\left([[v, j_k], i_k] - [[v, i_k], j_k]\right) \\ &= \sum_{k=1}^n a_k\left([[v, i_k], j_k] - [[v, j_k], i_k]\right)\end{aligned}$$

Since I and J are ideals, $s_k = [v, i_k] \in I$ and $t_k = [v, j_k] \in J$. We now have

$$\left[v, \sum_{k=1}^n a_k[i_k, j_k]\right] = \sum_{k=1}^n a_k\left([s_k, j_k] - [t_k, i_k]\right)$$

But $[s_k, j_k], [t_k, i_k]$ are generators of $[I, J]$. Hence the sum also lies in $[I, J]$. □

1.3 Products and Quotients of Lie Algebras

Definition 1.3.1: Direct Sum of Lie Algebras

Let L_1 and L_2 be Lie algebras. Define the direct sum of L_1 and L_2 by

$$L_1 \oplus L_2 = \{(a_1, a_2) \mid a_1 \in L_1, a_2 \in L_2\}$$

together with component wise addition and scalar multiplication and Lie bracket operation

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1], [a_2, b_2])$$

which is component wise application of the Lie bracket for $(a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2$.

Proposition 1.3.2

Let L_1 and L_2 be Lie algebras. Then the following are true.

- $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$
- $\{(x, 0) \mid x \in L_1\} \cong L_1$ is an ideal of $L_1 \oplus L_2$
- $\{(0, y) \mid y \in L_2\} \cong L_2$ is an ideal of $L_1 \oplus L_2$

Definition 1.3.3: Quotient Lie Algebra

Let V be a Lie algebra. Let U be an ideal of V. Define the quotient Lie algebra to be the set

$$V/U = \{v + U \mid v \in V\}$$

together with the Lie bracket defined by $[v + U, w + U] = [v, w] + U$.

1.4 The Centers and Centralizers of Lie Algebras

Definition 1.4.1: Center of a Lie Algebra

Let L be a Lie algebra. Define the center of L by

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$$

Lemma 1.4.2

Let L be a Lie algebra. Then $Z(L)$ is an ideal of L.

Proposition 1.4.3

Let L_1, L_2 be Lie algebras over the same field K. Then

$$Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$$

Definition 1.4.4: The Centralizer of a Subset

Let L be a Lie algebra. Let $A \subseteq L$ be a subset. Define the centralizer of A in L to be the set

$$C_L(A) = \{x \in L \mid [x, a] = 0 \text{ for all } a \in A\}$$

Lemma 1.4.5

Let L be a Lie algebra. Let $A \subseteq L$ be a subset. Then $C_L(A)$ is a Lie subalgebra of L.

1.5 The Lie Algebra of Endomorphisms

Definition 1.5.1: The Lie Algebra of Endomorphisms

Let V be a vector space over a field k. Define the Lie algebra of endomorphisms of V to be the vector space

$$\mathfrak{gl}_k(V) = \text{End}_k(V) = \{T : V \to V \mid T \text{ is linear}\}$$

over k together with Lie bracket $[-, -] : \mathfrak{gl}_k(V) \to \mathfrak{gl}_k(V)$ given by

$$[T, S] = T \circ S - S \circ T$$

A priori one needs to check that the above map is indeed a Lie bracket. Let $A\text{End}_k(V)$. Then $[A, A] = A^2 - A^2 = 0$ so that the alternating property is satisfied. For the Jacobi identity, we have that

$$\begin{aligned}
[[A, B], C] + [[B, C], A] + [[C, A], B] &= [A, B]C - C[A, B] + [B, C]A - A[B, C] + [C, A]B - B[C, A] \\
&= ABC - BAC - CAB + CBA + BCA - CBA \\
&\quad - ABC + ACB + CAB - ACB - BCA + BAC \\
&= 0
\end{aligned}$$

for all $A, B, C \in \text{End}_k(V)$.

Definition 1.5.2: The Lie Algebra of the Matrix Ring

Let k be a field. Let $n \in \mathbb{N} \setminus \{0\}$. The Lie algebra of the matrix ring $\mathfrak{gl}_n(k) = M_n(k)$ is given by the Lie bracket $[-,-] : \mathfrak{gl}_n(k) \to \mathfrak{gl}_n(k)$ defined by

$$[A, B] = AB - BA$$

Proposition 1.5.3

Let k be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Choose a basis $\{e_1, \dots, e_n\}$ of k^n. Then the map $\mathfrak{gl}_n(k) \to M_n(k)$ defined by

$$T \mapsto \begin{pmatrix} T(e_1) & \cdots & T(e_n) \end{pmatrix}$$

is a Lie algebra isomorphism.

Lemma 1.5.4

Let V be a vector space over a field k. Let $T, S, R \in \mathfrak{gl}_k(V)$. Then

$$\text{tr}([T, S] \circ R) = \text{tr}(T \circ [S, R])$$

Definition 1.5.5: Lie Sub-algebra of 0 trace

Let k be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Define

$$\mathfrak{sl}_n(k) = \{A \in \mathfrak{gl}_n(k) \mid \text{tr}(A) = 0\}$$

Definition 1.5.6: Lie Sub-algebra of Upper Triangular Matrices

Let k be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Define the Lie sub-algebra of upper triangular matrices to be

$$\mathfrak{b}_n(k) = \{A \in \mathfrak{gl}_n(k) \mid \text{tr}(A) = 0\}$$

Definition 1.5.7: Lie Sub-algebra of Strictly Upper Triangular Matrices

Let k be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Define the Lie sub-algebra of strictly upper triangular matrices to be

$$\mathfrak{u}_n(k) = \{A \in \mathfrak{gl}_n(k) \mid \text{tr}(A) = 0\}$$

Definition 1.5.8: Sympletic Lie Algebra

Let k be a field. Let $n \in \mathbb{N}$. Define the sympletic Lie algebra to be

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{gl}_{2n}(k) \;\middle|\; A, B, C \in \mathfrak{gl}_n(k), B = B^T, C = C^T \right\}$$

Definition 1.5.9: Special Orthogonal Lie Algebra

Let k be a field. Let $n \in \mathbb{N}$. Define the even dimensional special orthogonal Lie algebra to be

$$\mathfrak{so}_{2n} = \left\{ \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \in \mathfrak{gl}_{2n}(k) \;\middle|\; A, B, C \in \mathfrak{gl}_n(k), B = -B^T, C = -C^T \right\}$$

Define the odd dimensional special orthogonal Lie algebra to be

$$\mathrm{so}_{2n+1}=\left\{\begin{pmatrix}0 & E^T & -D^T\\ D^T & A & B\\ -E & C & -A^T\end{pmatrix}\in\mathfrak{gl}_{2n}(k)\;\middle|\; A,B,C\in\mathfrak{gl}_n(k), D,E\in M_{n\times 1}(k), B=-B^T, C=-C^T\right\}$$

Proposition 1.5.10

Let k be a field. Let $n\in\mathbb{N}\setminus\{0\}$. Then the following are true.

- $\mathfrak{sl}_n(k)$ is a Lie sub-algebra of $\mathfrak{gl}_n(k)$ of dimension n^2-1.
- $\mathfrak{b}_n(k)$ is a Lie sub-algebra of $\mathfrak{gl}_n(k)$ of dimension $\frac{n^2+n}{2}$.
- $\mathfrak{u}_n(k)$ is a Lie sub-algebra of $\mathfrak{gl}_n(k)$ of dimension $\frac{n^2-n}{2}$.
- $\mathfrak{sp}_{2n}(k)$ is a Lie sub-algebra of dimension $2n^2+n$.
- $\mathfrak{so}_{2n}(k)$ is a Lie sub-algebra of dimension $2n^2+n$
- $\mathfrak{so}_{2n+1}(k)$ is a Lie sub-algebra of dimension $2n^2+3n$

Example 1.5.11

Write $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The structural constants of $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$[e,f]=h\quad [e,h]=-2e\quad [f,h]=2f$$

Proof. We have

$$\begin{aligned}[e,f]=ef-fe&=\begin{pmatrix}1&0\\0&0\end{pmatrix}-\begin{pmatrix}0&0\\0&1\end{pmatrix}=h\\ [e,h]=eh-he&=\begin{pmatrix}0&-1\\0&0\end{pmatrix}-\begin{pmatrix}0&1\\0&0\end{pmatrix}=-2e\\ [f,h]=fh-hf&=\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\-1&0\end{pmatrix}=2f\end{aligned}$$

□

2 Lie Algebra Homomorphisms

2.1 Lie Algebra Homomorphisms

Definition 2.1.1: Homomorphism of Lie algebra

Let V and W be Lie algebras over a field K. A homomorphism from V to W is an K-linear map $F : V \to W$ such that
$$[F(a), F(b)] = F([a, b])$$
for all $a, b \in V$.

Definition 2.1.2: Kernel of a Lie Algebra Homomorphism

Let V and W be Lie algebras over a field K. Let $F : V \to W$ be a Lie algebra homomorphism. Define the kernel of F to be
$$\ker(F) = \{v \in V \mid F(v) = 0_W\}$$

Lemma 2.1.3

Let V and W be Lie algebras over a field K. Let $F : V \to W$ be a Lie algebra homomorphism. Then $\ker(F)$ is a Lie subalgebra of V.

Proof. It is clear that $\ker(F)$ is a vector subspace of V. Let $k_1, k_2 \in \ker(F)$. Then we have
$$\begin{aligned} F([k_1, k_2]) &= [F(k_1), F(k_2)] && (F \text{ is a Lie algebra homomorphism}) \\ &= [0, 0] \\ &= 0 \end{aligned}$$
Hence $[k_1, k_2] \in \ker(F)$ and $\ker(F)$ is a Lie subalgebra of V. □

Definition 2.1.4: Isomorphisms of Lie Algebras

Let V and W be Lie algebras over a field k. Let $\phi : V \to W$ be a Lie algebra homomorphism. We say that F is a Lie algebra isomorphism if there exists a Lie algebra homomorphism $\Phi : W \to V$ such that $\Phi \circ \phi = \text{id}_V$ and $\phi \circ \Phi = \text{id}_W$.

Proposition 2.1.5

Let V and W be Lie algebras over a field k. Let $\phi : V \to W$ be a Lie algebra isomorphism. Then ϕ is a vector space isomorphism.

Theorem 2.1.6: First Isomorphism Theorem

Let $\phi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the following are true.
- $\ker(\phi)$ is an ideal of L_1
- $\text{im}(\phi)$ is a Lie subalgebra of L_2

Moreover, we have an isomorphism
$$\frac{L_1}{\ker(\phi)} \cong \text{im}(\phi)$$

Theorem 2.1.7: Second Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.

- I and J are ideals of $I+J$
- $I\cap J$ is an ideal of I and J

Moreover, we have an isomorphism

$$\frac{I+J}{J}\cong\frac{I}{I\cap J}$$

Theorem 2.1.8: Third Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L such that $I\subseteq J$. Then J/I is an ideal of L/I. Moreover, there is an isomorphism

$$\frac{L/I}{J/I}\cong\frac{L}{J}$$

Theorem 2.1.9: Correspondence Theorem

Let L be a Lie algebra with ideal I. Then there exists a bijective correspondence

$$\{J\mid J\text{ is an ideal of }L\text{ and }I\subseteq J\}\quad\stackrel{1:1}{\longleftrightarrow}\quad\{K\mid K\text{ is an ideal of }L/I\}$$

2.2 The Adjoint Homomorphism

Definition 2.2.1: The Adjoint Homomorphism

Let V be a Lie algebra. Define the adjoint homomorphism $\text{ad}:V\to\text{End}(V)$ to be the map given by

$$\text{ad}(x)(y)=[x,y]$$

Lemma 2.2.2

Let V be a Lie algebra. Then then the adjoint homomorphism $\text{ad}:V\to\text{End}(V)$ is a Lie algebra homomorphism.

Proof.

- Linearity: Let $x,y\in V$ and let $a,b\in F$. For any $z\in V$, we have

$$\begin{aligned}\text{ad}(ax+by)(z)&=[ax+by,z]\\&=a[x,z]+b[y,z]\\&=a\text{ad}(x)(z)+b\text{ad}(y)(z)\\&=(a\text{ad}(x)+b\text{ad}(y))(z)\end{aligned}$$

  so that $\text{ad}:V\to\text{End}(V)$ is a linear map.

- Preserving the Lie bracket: Let $x, y \in V$. For any $z \in V$, we have

$$\begin{aligned}[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= (\mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x))\,(z) \\ &= \mathrm{ad}(x)\,(\mathrm{ad}(y)(z)) - \mathrm{ad}(y)\,(\mathrm{ad}(x)(z)) \\ &= \mathrm{ad}(x)([y, z]) - \mathrm{ad}(y)([x, z]) \\ &= [x, [y, z]] - [y, [x, z]] \\ &= -[[y, z], x] + [[x, z], y] \\ &= -[[y, z], x] - [[z, y], x] - [[y, x], z] \\ &= -[[y, z], x] + [[y, z], x] - [[y, x], z] \\ &= [[x, y], z] \\ &= \mathrm{ad}([x, y])(z)\end{aligned}$$

Thus we have showed that $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}([x, y])$.

$\square$

Lemma 2.2.3

Let V be a Lie algebra. Then the kernel of the adjoint homomorphism is equal to

$$\ker(\mathrm{ad}) = Z(V)$$

the center of V.

Proof. Let $k \in \ker(\mathrm{ad})$. Let $v \in V$. Then $[k, v] = \mathrm{ad}(k)(v) = 0$ since $\mathrm{ad}(k) = 0 \in \mathrm{End}(V)$. Hence $k \in Z(V)$. Conversely, if $z \in Z(V)$ then we have

$$\mathrm{ad}(z)(v) = [z, v] = 0$$

for all $v \in V$. Hence $\mathrm{ad}(z) = 0 \in \mathrm{End}(V)$ and $z \in \ker(\mathrm{ad})$. $\square$

Definition 2.2.4: Ad-Nilpotency

Let L be a Lie algebra. Let $x \in L$. We say that x is ad-nilpotent if $\mathrm{ad}(x)$ is a nilpotent in $\mathfrak{gl}(V)$. (as an element of a ring).

Lemma 2.2.5

Let V be a vector space. Let $L \subseteq \mathrm{End}(V)$ be a Lie subalgebra. If $x \in L$ is nilpotent, then x is ad-nilpotent.

Let V be a vector space over a field k. Let $T \in \mathrm{End}(V)$. Recall from Linear Algebra that a Jordan-Chevalley decomposition is two linear maps $D, S \in \mathrm{End}(V)$ such that the following are true:

- $T = D + S$
- D is diagonal and S is nilpotent.
- $DS = SD$

We showed that such a decomposition always exists and is unique.

Proposition 2.2.6

Let V be a finite dimensional vector space over a field k. Let $T \in \mathrm{End}_k(V)$. Let $T = D + S$ be the unique Jordan-Chevalley decomposition. Then

$$\mathrm{ad}(T) = \mathrm{ad}(D) + \mathrm{ad}(S)$$

is the Jordan-Chevalley decomposition of $\text{ad}(T) \in \text{End}(\text{End}(V))$.

Proof. Let $T \in \text{End}_k(V)$ be an endomorphism. Let $T = D + S$ be the Jordan-Chevalley decomposition of T. Since ad is linear, we have that

$$\text{ad}(T) = \text{ad}(D) + \text{ad}(S)$$

For any $C \in \text{End}_k(V)$, $\text{ad}(D)$ is defined by $\text{ad}(D)(C) = DC - CD$. Since D is diagonalizable, we can choose a basis $B = \{b_1, \dots, b_n\}$ of V such that the matrix representing D given by $D_B = \text{diag}(\alpha_1, \dots, \alpha_r)$ is diagonal on the basis. For the standard basis $\{e_{i,j} \mid 1 \leq i, j \leq n\}$ on $\text{End}_k(V)$, we have that

$$\text{ad}(D)(e_{i,j}) = [D_B, e_{i,j}] = (\alpha_i - \alpha_j)e_{i,j}$$

which shows that every standard basis vector is an eigenvector of $\text{ad}(D)$. Hence $\text{ad}(D)$ is diagonal. On the other hand, since S is nilpotent, by the above $\text{ad}(S)$ is nilpotent.

Finally, we have that

$$(\text{ad}(D) \circ \text{ad}(S) - \text{ad}(S) \circ \text{ad}(D))(C) = [\text{ad}(D), \text{ad}(S)](C) = \text{ad}([D, S])(C) = \text{ad}(0)(C) = 0$$

for all $C \in \text{End}_k(V)$. Hence $\text{ad}(D)$ and $\text{ad}(S)$ commutes. Thus $\text{ad}(T) = \text{ad}(D) + \text{ad}(S)$ is a Jordan-Chevalley decomposition. $\square$

3 Weights and Weight Spaces

3.1 Weights

Definition 3.1.1: Common Eigenvectors

Let k be a field. Let V be a vector space over k. Let W be a vector subspace of $\text{End}_k(V)$. A common eigenvector of W is a vector $v \in V$ such that for all $T \in W$, v is an eigenvector of T.

Definition 3.1.2: Weights

Let k be a field. Let V be a vector space over k. Let W be a vector subspace of $\text{End}_k(V)$. Let $v \in V$ be an eigenvector of W. A weight of W for v is an assignment $\lambda : W \to k$ such that the eigenvalue of T corresponding to v is $\lambda(T)$.

Lemma 3.1.3

Let k be a field. Let V be a vector space over k. Let W be a vector subspace of $\text{End}_k(V)$. Let $\lambda : W \to k$ be a weight of W. Then $\lambda \in W^*$.

3.2 Weight Spaces

Definition 3.2.1: Weight Spaces

Let k be a field. Let V be a vector space over k. Let W be a vector subspace of $\text{End}_k(V)$. Let $\lambda \in W^*$ be a weight of W. Define the weight space of λ to be

$$V_\lambda = \{v \in V \mid T(v) = \lambda(T)v \text{ for all } T \in W\}$$

Lemma 3.2.2

Let k be a field. Let V be a vector space over k. Let W be a vector subspace of $\text{End}_k(V)$. Let $\lambda \in W^*$ be a weight of W. Then V_λ is a vector subspace of V.

Proposition 3.2.3

Let k be a field. Let V be a vector space over k. Let L be a Lie-subalgebra of $\mathfrak{gl}_k(V)$. Let I be an ideal of L. Then

$$V_0 = \{v \in V \mid T(v) = 0 \text{ for all } T \in I\}$$

is an L-invariant subspace of V.

Proof. Let $y \in L$ and $m \in V_0$. We want to show that $y(m) \in V_0$. So let $T \in I$ be arbitrary. We compute that

$$T(y(m)) = T(y(m)) - y(T(0)) = [T, y](m)$$

since the bracket of I is the bracket of L, which is the bracket of $\mathfrak{gl}(V)$. Also, $[T, y](m) = 0$ since I is an ideal implies that $[T, y] \in I$ so that $m \in \ker([T, y])$ since $m \in V_0$. Thus V_0 is an invariant subspace of L. $\square$

Proposition 3.2.4

Let k be a field. Let V be a finite dimensional vector space over k. Let L be a Lie-subalgebra of $\mathfrak{gl}_k(V)$. Let I be an ideal of L. Let λ be a weight of I. Then V_λ is an L invariant subspace of V.

Proof. Let $x \in L$ and $w \in V_\lambda$. We want to show that $x(w) \in V_\lambda$. So let $T \in I$. We compute that
$$T(x(w)) = [T,x](w) + x(T(w)) = \lambda([T,x])w + \lambda(T)x(w)$$
since $[T,x] \in I$. Therefore it suffices to show that $\lambda([T,x]) = 0$.

Consider the subspace
$$W = \mathbb{C}\langle x^k(w) \mid k \in \mathbb{N}\rangle \subseteq V$$
Since V is finite dimensional, there exists $k \in \mathbb{N}$ such that $\{w, x(w), \dots, x^k(w)\}$ is a basis for W. I claim that for any W is I-invariant. Let $T \in I$. We inductively show that $T(x^i(w)) \in W$. When $i = 0$, $T(w) = \lambda(T)(w) \in W$. Then we also have

$$T(x^i(w)) = [T,x](x^{i-1}(w)) + x(T(x^{i-1}(w))) = \lambda([T,x])x^{i-1}(w) + \lambda(T)x^i(w)$$

by applying the inductive hypothesis. Then $x^{i-1}(w), x^i(w) \in W$ implies that $T(x^i(w)) \in W$. Then the matrix of the restriction of $T \in I$ to W with respect to our basis of W is given by

$$T = \begin{pmatrix} \lambda(T) & * & * & \cdots & * \\ 0 & \lambda(T) & * & \cdots & * \\ 0 & 0 & \lambda(T) & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \lambda(T) \end{pmatrix}$$

Since this is true for any $T \in I$, we can apply this to any $z = [T,x] \in [L,I] \subseteq I$ to get that $\text{tr}(z) = (k+1)\lambda(z)$. Now $\text{tr}(z) = \text{tr}([T,x]) = \text{tr}(Tx - xT) = 0$ so that $(k+1)\lambda(z) = 0$. Hence $\lambda([T,x]) = \lambda(z) = 0$ as required. $\square$

4 Types of Lie Algebras

4.1 Abelian Lie Algebras

Lie algebras that are Abelian are the simplest Lie algebra there is to study.

Definition 4.1.1: Abelian Lie Algebras

Let L be a Lie algebra. We say that L is abelian if

$$[x, y] = 0$$

for all $x, y \in L$.

Lemma 4.1.2

Let L be a Lie algebra. Then $Z(L)$ is abelian.

Proof. True by definition of $Z(L)$. □

Lemma 4.1.3

Let L be a Lie algebra. Let I be an ideal of L. Then L/I is abelian if and only if

$$[L, L] \subseteq I$$

Proof. Let L/I be abelian. Let $v, w \in L$. Since L/I is abelian, we have that $[v + I, w + I] = I$. But $[v + I, w + I] = [v, w] + I$ implies that $[v, w] \in I$. Conversely, suppose that $[L, L] \subseteq I$. Then for any $v + I, w + I \in L/I$, $[v + I, w + I] = [v, w] + I = I$. Hence L/I is abelian. □

We can think of this as saying $I = [L, L]$ is the smallest ideal of L for which L/I is abelian.

Proposition 4.1.4

Let L be a Lie algebra of dimension 1. Then L is abelian.

Proof. If L is 1-dimensional over a field k, let x be a spanning element of L. Then for $s, t \in L$, there exists $a, b \in k$ such that $s = ax$ and $t = bx$. Now we have that

$$[ax, bx] = ab[x, x] = 0$$

Hence L is abelian. □

4.2 Nilpotent Lie Algebras

Definition 4.2.1: Lower Central Series

Let L be a Lie algebra. Define the lower central series $L^0, L^1, \dots, L^n, \dots$ as follows.

- For $n = 0$, define $L^0 = L$
- For $n \in \mathbb{N} \setminus \{0\}$, define

$$L^n = [L, L^{n-1}]$$

Lemma 4.2.2

Let L be a Lie algebra. Then the following are true.

- For all $n \in \mathbb{N}$, $L^{n+1} \subseteq L^n$.
- For all $n \in \mathbb{N}$, L^n is an ideal of L.

Proof. When $n = 0$, both statements are clearly true. Suppose that they are true for some $k \in \mathbb{N}$. Then by lemma 1.2.5, $L^{k+1} = [L, L^k]$ is an ideal of L, and therefore $L^{k+1} \subseteq L^k$. By induction, both statements are true for all $n \in \mathbb{N}$. □

Lemma 4.2.3

Let L_1, L_2 be Lie algebras. Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then

$$\phi(L^k) = (\phi(L))^k$$

for all $k \in \mathbb{N}$.

Proof. We prove by induction. The base case $k = 0$ is clear. Suppose that $\phi(L^k) = (\phi(L))^k$. Then we have that

$$\begin{aligned} \phi(L^{k+1}) &= \phi([L, L^k]) \\ &= [\phi(L), \phi(L^k)] \\ &= [\phi(L), \phi(L)^k] \\ &= (\phi(L))^{k+1} \end{aligned}$$

By induction, we conclude. □

Definition 4.2.4: Nilpotent Lie Algebras

Let L be a Lie algebra. We say that L is nilpotent if there exists $n \in \mathbb{N}$ such that

$$L^n = 0$$

Lemma 4.2.5

Let L be a Lie algebra. If L is abelian, then L is nilpotent.

Proof. Let L be abelian. Then $L^1 = [L, L] = 0$. □

Example 4.2.6

Consider the following Lie algebras.

- $SL(2, \mathbb{C})$ is nilpotent.
- $b_n(\mathbb{C})$ is not nilpotent for all $n \geq 2$.

Example 4.2.7

Let $n \in \mathbb{N}$ such that $n \geq 2$. Let k be a field. Then $\mathfrak{u}_n(k)$ is nilpotent.

Proof. Write $e_{i,j}$ the matrix with 1 at the (i, j)th position and 0 everywhere else. Notice that a basis for $\mathfrak{u}_2(\mathbb{C})$ is given by $\{e_{i,j} \mid 1 \leq i < j \leq n\}$. Moreover, one can compute that $[e_{i,j}, e_{k,l}] = \delta_{j,k} e_{i,l} - \delta_{i,l} e_{k,j}$.

I claim that

$$\mathfrak{u}_n(k)^p = \langle e_{i,j} \mid 1 \le i, i+p < j \le n \rangle$$

for all $0 \le p \le n-1$. We proceed by induction. The base case is trivial.

Assume the result is true for some $p \le n-2$. Let $a, b \in \mathbb{N}$ be such that $a+p+1 < b$. Then notice that $e_{a,b} = [e_{a,a+1}, e_{a+p+1,b}] \in [\mathfrak{u}_n(k), \mathfrak{u}_n(k)^p] = \mathfrak{u}_n(k)^{p+1}$ However, notice that $e_{a,a+p+1}$ does not lie in $\mathfrak{u}_n(k)^{p+1}$. By anti-symmetry notice that we just need to consider the case $j = k$ in $[e_{i,j}, e_{k,l}]$. But $i+p < j+p < l$ implies that $l - i > p+1$. But $(a+p+1) - a = p+1$ means that no $i, l \in \mathbb{N}$ is such that $e_{a,a+p+1} = [e_{i,j}, e_{j,l}]$. Hence we are done.
The it is clear that $\mathfrak{u}_n(k)^{n-1} = 0$. So $\mathfrak{u}_n(k)$ is nilpotent. □

Lemma 4.2.8

Let L be a Lie algebra. Then the following are true.

- Let M be a Lie subalgebra of L. If L is nilpotent, then M is nilpotent.
- If $L \neq 0$ is nilpotent, then $Z(L) \neq 0$
- If $L/Z(L)$ is nilpotent, then L is nilpotent.

Proof.

- Let M be a Lie subalgebra of L. I claim that $M^k \subseteq L^k$ for all $k \in \mathbb{N}$. The base case $k = 0$ is clearly true. Suppose that $M^k \subseteq L^k$. Let $x \in [M, M^k] = M^{k+1}$. Then $x = [m, t]$ for some $m \in M$ and $t \in M^k$. Then $t \in L^k$. Also $m \in L$ implies that $x = [m, t] \in [L, L^k] = L^{k+1}$. Thus $M^{k+1} \subseteq L^{k+1}$. Now since L is nilpotent, there exists $n \in \mathbb{N}$ such that $L^n = 0$. Then $M^n \subseteq L^n = 0$ so that M is also nilpotent.
- Suppose that $n \in \mathbb{N}$ is the smallest natural number such that $L^n = 0$. Then $[L, L^{n-1}] = 0$. Let $x \in L$. Then for all $y \in L^{n-1}$, we have that $[x, y] = 0$. Thus $x \in Z(L)$.
- Since $L/Z(L)$ is nilpotent, there exists $n \in \mathbb{N}$ such that $(L/Z(L))^n = 0$. Let $\pi : L \to L/Z(L)$ be the quotient homomorphism. Since π is surjective, we use the above lemma to find that

  $$\pi(L^n) = \pi(L)^n = \left(\frac{L}{Z(L)}\right)^n = \frac{L^n + Z(L)}{Z(L)} = 0$$

  This means that $L^n \subseteq Z(L)$. It follows that $L^{n+1} = [L, L^n] \subseteq [L, Z(L)] = 0$ and we conclude.

□

Proposition 4.2.9

Let k be a field. Let V be a vector space over k. Let $L \subseteq \mathfrak{gl}(V)$ be a Lie sub-algebra such that for all $T \in L$, T is nilpotent. Then

$$V_0 = \{v \in V \mid T(v) = 0 \text{ for all } T \in L\} \neq \{0\}$$

Proof. We induct on the dimension of V. When $\dim(L) = 1$, we have $L = \langle T \rangle$ for some non-zero $T \in \mathfrak{gl}(V)$. Suppose that $T^n = 0$ but $T^{n-1} \neq 0$. Suppose that $x \in V$ is such that $T^{n-1}(x) = y \neq 0$. Then for any $S \in L = \langle T \rangle$, write $S = \lambda T$ for some $\lambda \in k$. Then we have

$$S(y) = \lambda T(T^{n-1}(x)) = 0$$

since $T^n = 0$. Thus $0 \neq x \in V_0 \neq \{0\}$.

Suppose the result is true for all Lie sub-algebras of $\mathfrak{gl}(V)$ of dimension $< k$. Let L be a Lie sub-algebra of $\mathfrak{gl}(V)$ of dimension k. Let M be a maximal proper Lie sub-algebra of L. Such a Lie sub-algebra exists since L is finite dimensional and $\{0\}$ is a proper Lie sub-algebra.

Step 1: Apply inductive hypothesis ro homomorphic image of M.
Define $f : M \to \mathfrak{gl}(L/M)$ by
$$f(m)(x+M) \mapsto [m,x]+M$$
I claim that f is a Lie algebra homomorphism. Such a map is well defined since $x \in M$ implies that $[m,x] \in M$. Moreover, since $[-,-]$ is bilinear, the map $f(m)$ lies in $\mathfrak{gl}(L/M)$ and that f is linear. Finally, we check that

$$\begin{aligned}
[f(m_1), f(m_2)](x+M) &= (f(m_1)f(m_2) - f(m_2)f(m_1))(x+M) \\
&= f(m_1)([m_2,x]+M) - f(m_2)([m_1,x]+M) \\
&= [m_1,[m_2,x]] - [m_2,[m_1,x]] + M \\
&= -[[m_2,x],m_1] + [[m_1,x],m_2] + M \\
&= -[[m_2,x],m_1] - [[x,m_1],m_2] + M \\
&= [[m_1,m_2],x] + M && \text{(Jacobi Identity)} \\
&= f([m_1,m_2])(x+M)
\end{aligned}$$

for all $m_1, m_2 \in M$ and $x \in L$. Thus f is a Lie algebra homomorphism. Let $m \in M \subseteq L$. Then m is nilpotent, and so m is ad-nilpotent by lemma 2.2.5. This means that $\text{ad}(m)^n = 0$ for some $n \in \mathbb{N} \setminus \{0\}$. Then we have

$$f(m)^n(x+M) = \text{ad}(m)^n(x) + M = M$$

and that $f(m)$ is nilpotent. Now we have that $\dim(f(M)) \leq \dim(M) < \dim(L) = k$ and all elements of $f(M)$ are nilpotent. We can apply the induction hypothesis to conclude that there exists $x \in L \setminus M$ such that

$$x + M \in \{y \in L/M \mid T(y) = 0 \text{ for all } T \in f(M)\}$$

Step 2: Apply inductive hypothesis to M.
I claim that $M \oplus \langle x \rangle$ is a Lie sub-algebra of L. To see this, let $n_1, n_2 \in M \oplus \langle x \rangle$, which we can write as $n_1 = m_1 + \lambda_1 x$ and $n_2 = m_2 + \lambda_2 x$ for some $m_1, m_2 \in M$ and $\lambda_1, \lambda_2 \in k$. Then we check that

$$[n_1, n_2] = [m_1 + \lambda_1 x, m_2 + \lambda_2 x] = [m_1 + m_2] + \lambda_1 [x, m_2] + \lambda_2 [m_1, x] \in M$$

so that $M \oplus \langle x \rangle$ is indeed a Lie sub-algebra of L. Since M is maximal and $\dim(M \oplus \langle x \rangle) > \dim(M)$, we conclude that $M \oplus \langle x \rangle = L$. Moreover, M is an ideal of L since we can check that for any $z \in L$ and $m \in M$, we can write $z = m + \lambda x$ for $m \in M$ and $\lambda \in k$ and compute that
$$[z, m'] = [m, m'] + \lambda [x, m'] \in M$$
for any $m' \in M$. Now apply the inductive hypothesis on M to get that

$$W = \{v \in V \mid T(v) = 0 \text{ for all } T \in M\}$$

is a non-trivial subspace of L.
Step 3: Find the element in L satisfying the hypothesis.
By 3.2.3, W is an L-invariant subspace. This means that $x(w) \in W$ for all $w \in W$ and $x \in L$. Since $x \in L$, x is nilpotent. Suppose that $x^t = 0$ but $x^{t-1} \neq 0$. This means that there exists $w \in W$ such that $q = x^{t-1}(w) \neq 0$, and $x(q) = 0$. Let $T \in L$ be arbitrary. Since $L = M \oplus \langle x \rangle$, we can write $T = m + \lambda x$ for some $m \in M$ and $\lambda \in k$. Then we have

$$T(q) = m(q) + \lambda x(q) = 0$$

because $q \in W$ implies that $m(q) = 0$ for all $m \in M$. Thus $0 \neq q \in V_0$ so that we conclude. □

Proposition 4.2.10: Engel's Theorem for Lie Sub-algebras of $\mathfrak{gl}(V)$

Let k be a field. Let V be a finite dimensional vector space over k. Let L be a Lie subalgebra of End(V). If all $x \in L$ are nilpotent, then the following are true.

- There exists a basis for V such that all elements of L are strictly upper triangular.
- L is nilpotent.

Proof. We induct on the dimension of V. When $\dim(V) = 1$, then $\mathfrak{gl}(V) \cong k$ as vector spaces. The only nilpotent element of k is 0. Thus $L = \{0\}$ and we are done.

Assume the results are true for all vector spaces V with $\dim(V) < k$. Let V be a k dimensional vector space. By prp4.2.9, there exists $v \in V$ such that $0 \neq v \in V_0$. Let $W = V/\langle v\rangle$. Define a map $\phi : L \to \mathfrak{gl}(W)$ by

$$\phi(x)(u + \langle v\rangle) = x(u) + \langle v\rangle$$

Notice that this is well defined since $u \in \langle v\rangle$ implies that $x(u) = 0$ by definition of v. Moreover, $\phi(x)$ is a well defined linear map since x is a linear map. Finally, ϕ is a linear map since addition in $\mathfrak{gl}(V)$ is defined pointwise. It is also a Lie algebra homomorphism since

$$\begin{aligned}\phi([x, y])(u + \langle v\rangle) &= [x, y](u) + \langle v\rangle \\ &= x(y(u)) - y(x(u)) + \langle v\rangle \\ &= \phi(x)(\phi(y)(u)) - \phi(y)(\phi(x)(u)) + \langle v\rangle \\ &= [\phi(x), \phi(y)](u + \langle v\rangle)\end{aligned}$$

Hence $\phi(L)$ is a Lie sub-algebra of $\mathfrak{gl}(W)$.

Recall that all $x \in L$ are nilpotent, say $x^n = 0$. Then we have

$$\phi(x)^n(u + \langle v\rangle) = x^n(u) + \langle v\rangle = \langle v\rangle$$

so that all elements of $\phi(L)$ are nilpotent. Since $\dim(W) = \dim(V) - 1$, we can apply the induction hypothesis to get a basis $u_1 + \langle v\rangle, \dots, u_{k-1} + \langle v\rangle$ such that all elements of $\mathfrak{gl}(W)$ are strictly upper triangular. I claim that $v, u_1, \dots, u_{k-1}$ is a basis for L so that all elements of L are strictly upper triangular. Let $x \in L$. The matrix of x is given by the matrix of $x|_{\langle v\rangle}$ and the block sum of the matrix of $x|_W = \phi(x)$. By assumption $\phi(x)$ is upper triangular with respect to $u_1 + \langle v\rangle, \dots, u_{k-1} + \langle v\rangle$. Since $x(v) = 0$ by construction, we conclude that x is upper triangular with respect to the basis $v, u_1, \dots, u_{k-1}$ and so the first part is complete.

For the second part, we know that V has a basis for which all elements of L are strictly upper triangular. This means that L is a Lie subalgebra of $\mathfrak{u}_n(k)$ for $n = \dim(V)$. By 3.2.7 and 3.2.8 we conclude that L is nilpotent. $\square$

Theorem 4.2.11: Engel's Theorem

Let L be a finite dimensional Lie algebra. Then L is nilpotent if and only if all $x \in L$ are ad-nilpotent.

Proof. Let L be nilpotent. Then $L^n = 0$ for some $n \in \mathbb{N}$. Since ad$(x)^n(y) \in L^n$ for all x and y, we conclude that x is ad-nilpotent.

Conversely, suppose that all $x \in L$ are ad-nilpotent. Consider the adjoint homomorphism ad $: L \to \mathfrak{gl}(L)$. Since all $x \in L$ are ad-nilpotent, this means that ad$(x) \in \mathfrak{gl}(L)$ is nilpotent for all ad$(x) \in$ ad(L). Then by Engel's theorem for Lie sub-algebras of $\mathfrak{gl}(V)$, we conclude that ad(L) is nilpotent. Since ad$(L) \cong \frac{L}{\text{ad}} = \frac{L}{Z(L)}$, by 3.2.8 we conclude that L is nilpotent. $\square$

4.3 Soluble Lie Algebras

Definition 4.3.1: Derived Series

Let L be a Lie algebra. Define the derived series $L^{(n)}$ of L to be the sequence recursively defined as follows.

- For $n = 0$, define $L^{(0)} = L$
- When $n \in \mathbb{N} \setminus \{0\}$, define
$$L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$$

Lemma 4.3.2

Let L be a Lie algebra. Then the following are true.

- For all $n \in \mathbb{N}$, $L^{(n+1)} \subseteq L^{(n)}$.
- For all $n \in \mathbb{N}$, $L^{(n)}$ is an ideal of L.

Proof. Both are clear when $n = 0$. Suppose that they are true for some $k \in \mathbb{N}$. Then by lemma 1.2.5, $L^{(k+1)} = [L^{(k)}, L^{(k)}]$ is an ideal of L, and hence $L^{(k+1)} \subseteq L^{(k)}$. By induction, both cases are true for all $n \in \mathbb{N}$. □

Lemma 4.3.3

Let L_1, L_2 be Lie algebras. Let $\phi : L_1 \to L_2$ be a Lie algebra homomorphism. Then

$$\phi(L_1^{(k)}) = \phi(L_1)^{(k)}$$

Proof. When $k = 0$, the lemma is trivial. Assume that it is true for some k. Then we have

$$\phi(L^{(k+1)}) = \phi([L^{(k)}, L^{(k)}]) = [\phi(L^{(k)}), \phi(L^{(k)})] = [\phi(L)^{(k)}, \phi(L)^{(k)}] = \phi(L)^{(k+1)}$$

By induction we conclude. □

Definition 4.3.4: Soluble Lie Algebras

Let L be a Lie algebra. We say that L is soluble if there exists $n \in \mathbb{N}$ such that

$$L^{(n)} = 0$$

Lemma 4.3.5

Let L be a Lie algebra. Then the following are true.

- If L is abelian, then L is soluble.
- If L is nilpotent, then L is soluble.

Proof. Let L be abelian. Then $L^{(1)} = [L, L] = 0$.

I claim that $L^{(n)} \subseteq L^n$. When $n = 0$, the case is trivial. Suppose that $L^{(k)} \subseteq L^k$ for some $k \in \mathbb{N}$. Then we have

$$L^{(k+1)} = [L^{(k)}, L^{(k)}] \subseteq [L, L^{(k)}] \subseteq [L, L^k] = L^{k+1}$$

By induction, we conclude that $L^{(n)} = L^n$ for all $n \in \mathbb{N}$. If L is nilpotent, then $L^n = 0$ for some n so that $L^{(n)} = 0$. Then L is soluble. □

Proposition 4.3.6

Let L be a Lie algebra of dimension 2. Then L is soluble.

Proof. If L is abelian, then lmm 2.2.4 implies that L is soluble. So suppose that L is non-abelian. Then $L^{(1)} = [L, L] \neq 0$. Suppose for a contradiction that $[L, L] = L$. Let $v, w \in L$ be non-zero. Define $x = [v, w]$ and extend it to a basis $\{x, y\}$ of L. Suppose that $v = ax + by$ and $w = cx + dy$. Then we have that

$$x = [v, w] = [ax + by, cx + dy] = (ad - bc)[x, y]$$

Since $[L, L] = L$, there exists $f, g \in L$ such that $[f, g] = y$. Let $f = px + qy$ and $g = rx + sy$. Then we have that

$$y = [f, g] = [px + qy, rx + sy] = (ps - qr)[x, y]$$

Then we have that

$$\frac{1}{ad - bc}x = [x, y] = \frac{1}{(ps - qr)}y$$

which means that x and y are linearly dependent. This is a contradiction. Hence $[L, L] \neq L$.

Since $[L, L]$ is a Lie subalgebra of L, $[L, L]$ must be 1-dimensional. By the classification of 1-dimensional Lie algebras, $L^{(1)} = [L, L]$ is abelian. Hence $L^{(2)} = [L^{(1)}, L^{(1)}] = 0$. Thus L is soluble. $\square$

Example 4.3.7

The Lie algebras $\mathfrak{gl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ are not soluble.

Proof. Using a similar method as Example3.2.7, recall that a basis for $\mathfrak{gl}_2(\mathbb{C})$ is given by $\{e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}\}$. Then we compute that $\mathfrak{gl}_2(\mathbb{C})^{(1)}$ is spanned by the elements

- $[e_{1,1}, e_{1,2}] = e_{1,2}$
- $[e_{1,1}, e_{2,1}] = -e_{2,1}$
- $[e_{1,1}, e_{2,2}] = 0$
- $[e_{1,2}, e_{2,1}] = e_{1,1} - e_{2,2}$
- $[e_{1,2}, e_{2,2}] = e_{1,2}$
- $[e_{2,1}, e_{2,2}] = -e_{2,1}$

So that $\mathfrak{gl}_2(\mathbb{C})^{(1)} = \mathfrak{sl}_2(\mathbb{C})$. We can then compute $\mathfrak{gl}_2(\mathbb{C})^{(2)} = \mathfrak{sl}_2(\mathbb{C})^{(1)}$ which is spanned by

- $[e_{1,2}, e_{2,1}] = e_{1,1} - e_{2,2}$
- $[e_{1,2}, e_{1,1} - e_{2,2}] = -2e_{1,2}$
- $[e_{2,1}, e_{1,1} - e_{2,2}] = e_{2,1} + e_{2,2}$

So that $\mathfrak{gl}_2(\mathbb{C})^{(2)} = \mathfrak{sl}_2(\mathbb{C})^{(1)} = \mathfrak{sl}_2(\mathbb{C})$. Since both of the derived series stabilizes, we conclude that they are not soluble. $\square$

Example 4.3.8

Let $n \geq 2$. Then $\mathfrak{b}_2(\mathbb{C})$ is soluble.

Proof. Using a similar method as Example3.2.7, we conclude that $\mathfrak{b}_n(\mathbb{C})^{(1)} = [\mathfrak{b}_n(\mathbb{C}), \mathfrak{b}_n(\mathbb{C})] = \mathfrak{u}_n(\mathbb{C})$. Assume that $\mathfrak{b}_n(\mathbb{C})^{(k)} = \mathfrak{u}_n(\mathbb{C})^{(k-1)}$. Then we have

$$\mathfrak{b}_n(\mathbb{C})^{(k+1)} = [\mathfrak{b}_n(\mathbb{C})^{(k)}, \mathfrak{b}_n(\mathbb{C})^{(k)}] = [\mathfrak{u}_n(\mathbb{C})^{(k-1)}, \mathfrak{u}_n(\mathbb{C})^{(k-1)}] = \mathfrak{u}_n(\mathbb{C})^{(k)}$$

So by induction we have $\mathfrak{b}_n(\mathbb{C})^{(n+1)} = \mathfrak{u}_n(\mathbb{C})^{(n)}$ for all $n \in \mathbb{N}$. Since $\mathfrak{u}_n(\mathbb{C})$ is nilpotent, it is also soluble. Hence $\mathfrak{b}_n(\mathbb{C})$ is also soluble. $\square$

Proposition 4.3.9

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.
- Let $\phi : L \to K$ be a Lie algebra homomorphism. If L is soluble then $\phi(L)$ is soluble.
- Let M be a Lie subalgebra of L. If L is soluble, then M is soluble.
- If L is soluble, then L/I is soluble.
- If I and L/I are soluble, then L is soluble.
- If I and J are soluble, then $I+J$ is soluble.

Proof.
- Suppose that $L^{(n)} = 0$ for some n. Then $0 = \phi(L^{(n)}) = \phi(L)^{(n)}$ so we are done.
- When $k = 0$, we have $M^{(0)} = M \subseteq L = L^{(0)}$. Assume that $M^{(k)} \subseteq L^{(k)}$ for some $k \in \mathbb{N}$. Then we have
$$M^{(k+1)} = [M^{(k)}, M^{(k)}] \subseteq [L^{(k)}, L^{(k)}] = L^{(k+1)}$$
So by induction we have $M^{(n)} \subseteq L^{(n)}$ for all n. If L is soluble, then $L^{(n)}) = 0$ for some n so that $M^{(n)} \subseteq L^{(n)} = 0$.
- Suppose that $L^{(n)} = 0$ for some $n \in \mathbb{N}$. Let $\pi : L \to L/I$ be the projection map. Then we have
$$0 = \pi(L^{(n)}) = (L/I)^{(n)}$$
so we are done.
- Suppose that $(L/I)^{(n)} = 0$ for some $n \in \mathbb{N}$. Let $\pi : L \to L/I$ be the projection map. Since π is surjective, we have
$$\pi(L^{(n)}) = (L/I)^{(n)} = 0$$
This means that $L^{(n)} \subseteq I$. We know that I being soluble means that $I^{(k)} = 0$ for some k. Therefore we have
$$L^{(n+k)} \subseteq I^{(k)} = 0$$
so that L is soluble.
- By the third isomorphism theorem, we have a Lie algebra isomorphism $\frac{I+J}{J} \cong \frac{I}{I\cap J}$. Since I is soluble, $\frac{I}{I\cap J}$ is soluble by the above. Hence $\frac{I+J}{J}$ is soluble. Since J is also soluble, from the above we conclude that $I+J$ is soluble.

□

Proposition 4.3.10

Let V be a vector space over $\mathbb{C}$. Let L be a Lie sub-algebra of $\mathfrak{gl}(V)$. If L is soluble, then there exists $0 \neq v \in V$ such that v is an eigenvector for T for all $T \in L$.

Proof. We proceed by induction on the dimension of L. When $\dim(L) = 1$ it is clear. Suppose that the result is true for any Lie algebra of dimension $< n$. Let L be a Lie algebra of dimension n.

Step 1: Find an ideal M of dimension $\dim(L) - 1$ containing $[L, L]$.
Since L is soluble, $[L, L] \neq L$. Let M be a vector space containing $[L, L]$ of dimension $n-1$. I claim that M is an ideal of L. Indeed, let $x \in L$ and $m \in M$. Then we have
$$[x, m] \in [L, L] \subseteq M$$
Let $z \in L/M$ such that $L = M \oplus \langle z \rangle$.

Step 2: Apply induction hypothesis to M.
Since $\dim(M) = \dim(L) - 1$, we can apply the inductive hypothesis to see that there exists $w \in V$ that is a common eigvevector for all $m \in M$. Let $\lambda : M \to \mathbb{C}$ be the corresponding

weight of m. By 3.2.4, V_λ is L-invariant and also $\langle z\rangle$-invariant. This means that there exists $0 \neq v \in V_\lambda$ that is an eigenvector for both z and hence $\langle z\rangle$, and M. □

Theorem 4.3.11: Lie's Theorem

Let V be a vector space over $\mathbb{C}$. Let L be a soluble Lie sub-algebra of $\mathfrak{gl}(V)$. Then there exists a basis B of V such that for all $T \in L$, T is upper triangular.

Proposition 4.3.12

Let L be a Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $[L, L]$ is nilpotent.

Proof. Suppose that $[L, L]$ is nilpotent. Then $[L, L]$ is soluble by 3.3.5. Since $L/[L, L]$ is abelian, it is also soluble. Hence by 3.3.9, L is soluble.

Suppose that L is soluble. Then $\text{ad}(L) \leq \mathfrak{gl}(L)$ is soluble by 3.3.9. By Lie's theorem, there exists a basis B of V such that for all $T \in \text{ad}(L)$, T is upper triangular. Hence $\text{ad}(L)$ is isomorphic as Lie algebras to a Lie sub-algebra of $\mathfrak{b}_n(\mathbb{C})$. Since $\text{ad}(L)^1 = \mathfrak{b}_n(\mathbb{C})^1 = \mathfrak{u}_n(\mathbb{C})$ and the latter is nilpotent, we conclude that $\text{ad}(L)$ is nilpotent. Then $\text{ad}([L, L]) = [\text{ad}(L), \text{ad}(L)]$ implies that $\text{ad}(x)$ is nilpotent for all $x \in [L, L]$. By Engel's theorem, $[L, L]$ is nilpotent □

4.4 Low Dimensional Lie Algebras

Proposition 4.4.1

Let L be a Lie algebra over a field k such that $\dim(L) = 1$. Then L is isomorphic to k equipped with the trivial Lie bracket. In particular, L is abelian.

Proposition 4.4.2

Let L be a Lie algebra over a field k such that $\dim(L) = 2$. Then L is isomorphic to one of the following.

- The vector space k^2 together with the trivial Lie bracket.
- The vector space $k^2 = \langle x, y\rangle$ together with the Lie bracket defined by $[x, y] = x$.

In both cases, L is soluble.

Proof. Let L be non-abelian. Then $[L, L]$ must have dimension at least 1. It must also have dimension at most 1, otherwise $[L, L] = L$ implies that L is not soluble. Thus $\dim([L, L]) = 1$. Suppose that x spans $[L, L]$. Let $y \in L$ be such that $\{x, y\}$ is a basis for L. Then $[x, y] = \lambda x$ for some $\lambda \neq 0$. Then let $y' = \frac{1}{\lambda} y$ so that $[x, y'] = x$. Hence L has a basis $\{x, y\}$ such that $[x, y] = x$ and we are done. □

Proposition 4.4.3

Let L be a Lie algebra over $\mathbb{C}$ such that $\dim(L) = 3$. Then L is isomorphic to one of the following.

- The vector space $\mathbb{C}^3$ together with the trivial Lie bracket.
- $\mathfrak{u}_3(\mathbb{C})$. (Case $\dim([L, L]) = 1$ and $[L, L] \subseteq Z(L)$)
- The direct sum of the non-abelian $\mathbb{C}^2$ Lie algebra and $\mathbb{C}$. (Case $\dim([L, L]) = 1$ and $[L, L] \not\subseteq Z(L)$)
- (Case $\dim([L, L]) = 2$)
- $\mathfrak{sl}_2(\mathbb{C})$ (Case $\dim([L, L]) = 3$)

Lemma 4.4.4

Let L be a Lie algebra such that $\dim(L) = 3$. If $Z(L) \neq \emptyset$ then L is nilpotent.

4.5 Semisimple Lie Algebras

Lemma 4.5.1

Let L be a Lie algebra. Then there exists a unique soluble ideal S of L such that for any soluble ideal $J \subseteq L$, we have $J \subseteq S$.

Proof. Every Lie algebra has a soluble ideal $Z(L)$. Hence the set of all soluble ideals of L is non-empty. Then it has a maximal element S. For any other soluble ideal I, $S + I$ is soluble and is either $S + I = L$ or $S + I = S$. Hence S is the unique maximal soluble ideal containing all other soluble ideals. □

Definition 4.5.2: Radical Ideals

Let L be a Lie algebra. Define the radical ideal $\text{rad}(L) \subseteq L$ of L to be the unique soluble ideal of L that contains all other soluble ideals.

Definition 4.5.3: Semisimple Lie Algebras

Let L be a Lie algebra. We say that L is semisimple if

$$\text{rad}(L) = \{0\}$$

Lemma 4.5.4

Let L be a Lie algebra. If L is semisimple, then $\dim(L) \geq 3$.

Proof. Any Lie algebra of dimension 1 or 2 must be soluble, and cannot be semisimple. □

Lemma 4.5.5

Let L be a Lie algebra. Then $L/\text{rad}(L)$ is semisimple.

Proof. Let K be a soluble ideal of $L/\text{rad}(L)$. By the correspondence theorem, there exists an ideal I of L such that $\text{rad}(L) \subseteq I$ and $K = I/\text{rad}(L)$. Since $\text{rad}(L)$ and K are soluble, we conclude that I is soluble. Hence $I \subseteq \text{rad}(L)$. We conclude that $I = \text{rad}(L)$. Hence $K = \{0\}$. Thus $L/\text{rad}(L)$ is semisimple. □

Lemma 4.5.6

Let L be a Lie algebra. Then L is not semisimple if and only if L contains a non-trivial abelian ideal.

Proof. If L is not semisimple, then $\text{rad}(L) \neq \{0\}$ is a non-trivial soluble ideal. This means that there exists a smallest $n \in \mathbb{N}$ such that $\text{rad}(L)^{(n)} = 0$. But this is the same as saying that

$$[\text{rad}(L)^{(n-1)}, \text{rad}(L)^{(n-1)}] = \text{rad}(L)^{(n)} = 0$$

This means that $\text{rad}(L)^{(n-1)}$ is an abelian ideal. In particular, it is non-trivial since n is the smallest number for which $\text{rad}(L)^{(n)}$ is zero.

If L contains a non-trivial abelian ideal I, then by lmm 2.2.4 we have that I is soluble. Hence $I \subseteq \text{rad}(L)$ and $\text{rad}(L)$ is non-zero. Hence L is not semisimple. □

Example 4.5.7

The following are true.

- $\mathfrak{gl}_2(\mathbb{C})$ is not semisimple.
- $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

Proof. Since $Z(\mathfrak{gl}_2(\mathbb{C}))$ is abelian, it is soluble. Hence $Z(\mathfrak{gl}_2(\mathbb{C})) \subseteq \text{rad}(\mathfrak{gl}_2(\mathbb{C}))$. Thus $\mathfrak{gl}_2(\mathbb{C})$ is not semisimple.

I claim that

$$\text{rad}(\mathfrak{gl}_2(\mathbb{C})) = Z(\mathfrak{gl}_2(\mathbb{C})) = \{aI_2 \mid a \in \mathbb{C}\}$$

The second equality is clear. Notice that if the dimension of $\text{rad}(\mathfrak{gl}_2(\mathbb{C}))$ is 2 or 3, then $\frac{\mathfrak{gl}_2(\mathbb{C})}{\text{rad}(\mathfrak{gl}_2(\mathbb{C}))}$ has dimension 1 or 2. In both cases, we have seen that it must be soluble. This is a contradiction since $\text{rad}(\mathfrak{gl}_2(\mathbb{C}))$ being soluble implies that $\mathfrak{gl}_2(\mathbb{C})$ is soluble. Therefore $\dim_{\mathbb{C}}(\text{rad}(\mathfrak{gl}_2(\mathbb{C}))) = 1$. Since $Z(\mathfrak{gl}_2(\mathbb{C}))$ also has dimension 1, we conclude that $Z(\mathfrak{gl}_2(\mathbb{C})) = \text{rad}(\mathfrak{gl}_2(\mathbb{C}))$.

Consider the composite Lie algebra homomorphism given by the inclusion $\mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathfrak{gl}_2(\mathbb{C})$ composed with the projection $\mathfrak{gl}_2(\mathbb{C}) \to \frac{\mathfrak{gl}_2(\mathbb{C})}{Z(\mathfrak{gl}_2(\mathbb{C}))}$. They both have dimension 3. Moreover, it is injective because if $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in Z(\mathfrak{gl}_2(\mathbb{C}))$, then we conclude that $a = b = c = 0$. Therefore the composite map is a Lie algebra isomorphism. By the above theorem, we have that

$$\text{rad}(\mathfrak{sl}_2(\mathbb{C})) \cong \text{rad}\left(\frac{\mathfrak{gl}_2(\mathbb{C})}{Z(\mathfrak{gl}_2(\mathbb{C}))}\right) = \text{rad}\left(\frac{\mathfrak{gl}_2(\mathbb{C})}{\text{rad}(\mathfrak{gl}_2(\mathbb{C}))}\right) = 0$$

Thus $\mathfrak{sl}_2(\mathbb{C})$ is semisimple. □

4.6 Simple Lie Algebras

Definition 4.6.1: Simple Lie Algebras

Let L be a Lie algebra. We say that L is simple if L is non-abelian and has no proper non-zero ideals.

Lemma 4.6.2

Let L be a Lie algebra. If L is simple, then L is semisimple.

Proof. Since L has no non-trivial proper ideals, $\text{rad}(L) = \{0\}$ or L. If $\text{rad}(L) = L$, then L is abelian and hence a contradiction. □

Example 4.6.3

$\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra.

Proof. We know that $\dim_{\mathbb{C}}(\mathfrak{sl}_2(\mathbb{C})) = 3$. Let I be a proper non-zero ideal of $\mathfrak{sl}_2(\mathbb{C})$. Then I has dimension 1 or 2. As a Lie algebra I must be soluble. Then $I \subseteq \text{rad}(\mathfrak{sl}_2(\mathbb{C})) \neq \{0\}$ so that $\mathfrak{sl}_2(\mathbb{C})$ is not semisimple, a contradiction. $\square$

5 The Killing Form of a Lie Algebra

5.1 The Killing Form

Let $A = (a_{i,j})$ be an $n \times n$ matrix. Recall that the trace of A is defined as

$$\text{tr}(A) = \sum_{k=1}^{n} a_{k,k}$$

Now let $T : V \to V$ be a linear map. Then we can also define the trace of T abstractly so that any choice of representation of T with a matrix gives coinciding trace. This is also given in Linear Algebra. The formula is

$$\text{tr}(T) = \sum_{i=1}^{n} \langle T(e_i), e_i \rangle$$

Definition 5.1.1: The Killing Form

Let L be a Lie algebra over $\mathbb{C}$. Define the killing form of L to be the function

$$k_L : L \times L \to \mathbb{C}$$

given by $k_L(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y))$.

Lemma 5.1.2

Let L be a Lie algebra. Then then following are true.
- The killing form on L is a symmetric bilinear form.
- $k_L([x, y], z) = k_L(x, [y, z])$ for all $x, y, z \in L$.

Proof. Let L be a Lie algebra over a field $\mathbb{F}$. Let $x, y \in L$. Then we have that

$$k_L(x, y) = \text{tr}(\text{ad}(x) \circ \text{ad}(y)) = \text{tr}(\text{ad}(y) \circ \text{ad}(x)) = k_L(y, x)$$

since the trace function preserve commutation. Thus k_L is symmetric. Because it is symmetric, it is sufficient to show that k_L is linear in the first variable for bilinearity. But the adjoint homomorphism is linear and composition preserves linearity. Hence k_L is bilinear.

Finally, we have that

$$\begin{aligned} k_L([x, y], z) &= \text{tr}(\text{ad}([x, y]) \circ \text{ad}(z)) \\ &= \text{tr}([\text{ad}(x), \text{ad}(y)] \circ \text{ad}(z)) && (\text{ad is a Lie algebra Hom}) \\ &= \text{tr}(\text{ad}(x) \circ [\text{ad}(y), \text{ad}(z)]) && (\text{lmm1.6.2}) \\ &= k_L(x, [y, z]) \end{aligned}$$

□

Lemma 5.1.3

Let L be a finite dimensional Lie algebra. Let I be an ideal of L. Then we have

$$k_I = k_L|_{I \times I}$$

Proof. Let I be an ideal of L so that I is also a Lie subalgebra of L. Let B_I be a basis for I. Extend it to a basis B_L of L. Let $x \in I$. Then we have a Lie algebra homomorphism $\text{ad}(x) : L \to I$ since $[x, z] \in I$ for all $z \in L$. We can then represent $\text{ad}(x)$ in the basis B using the matrix

$$T = \begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}$$

where A_x is the matrix representing the linear map $\text{ad}(x)|_I : I \to I$. For any $x, y \in I$, we have that

$$\begin{aligned} k_L(x,y) &= \text{tr}(\text{ad}(x) \circ \text{ad}(y)) \\ &= \text{tr}\left(\begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_y & B_y \\ 0 & 0 \end{pmatrix}\right) \\ &= \text{tr}\left(\begin{pmatrix} A_xA_y & A_xB_y \\ 0 & 0 \end{pmatrix}\right) \\ &= \text{tr}(A_xA_y) \\ &= k_I(x,y) \end{aligned}$$

□

5.2 Cartan's Two Criteria

Lemma 5.2.1

Let V be a vector space over a field k. Let $T : V \to V$ be a linear map. Let $T = D + N$ be the Jordan-Chevalley decomposition of T. Then there exists $p, q \in \mathbb{C}[t]$ such that $p(T) = D$ and $q(T) = N$.

Proposition 5.2.2

Let V be a vector space over a field k. Let $L \leq \mathfrak{gl}(V)$ be a Lie sub-algebra. If $\text{tr}(xy) = 0$ for all $x, y \in L$, then L is soluble.

Proof. Let $x \in [L, L]$. Let $x = d + n$ be the Jordan decomposition of $x \in \mathfrak{gl}(V)$. Choose a basis B for V such that d is diagonal. Then x is in its Jordan canonical form. Since $x \in [L, L]$, there exists $y_1, \dots, y_r, z_1, \dots, z_r \in L$ such that $x = \sum_{i=1}^r [y_i, z_i]$. Then we have

$$\text{tr}(\overline{d}x) = \text{tr}\left(\overline{d}\sum_{i=1}^r [y_i, z_i]\right) = \sum_{i=1}^r \text{tr}(\overline{d}[y_i, z_i]) = \sum_{i=1}^r \text{tr}([\overline{d}, y_i]z_i)$$

By 2.2.6, the Jordan decomposition of $\text{ad}(x)$ is $\text{ad}(x) = \text{ad}(d) + \text{ad}(n)$. By the above lemma and ???, there exists a polynomial $p \in \mathbb{C}[t]$ such that $p(\text{ad}(x)) = \text{ad}(\overline{d})$. Since $\text{ad}(x) : L \to L$, we have that $\text{ad}(\overline{d}) = p(\text{ad}(x)) : L \to L$. Hence $\text{ad}(\overline{d})(y_i) = [\overline{d}, y_i] \in L$. Then $z_i \in L$ and $\text{ad}(\overline{d}) \in L$ implies that $\text{tr}([\overline{d}, y_i]z_i) = 0$. Since $\text{tr}(\overline{d}x)$ is the sum of the square of the absolute values of the elements in $\overline{d}$ and it is equal to 0, we conclude that $\overline{d} = 0$ and hence $d = 0$. So that x is nilpotent.

Then x is ad-nilpotent so by Engel's theorem, $[L, L]$ is nilpotent. Then by 4.3.12, L is soluble. □

Theorem 5.2.3: Cartan's First Criterion

Let L be a Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $k(x, y) = 0$ for all $x \in L$ and $y \in [L, L]$.

Proof. Let L be soluble. Then $\text{ad}(L) = [L, L]$ is a Lie subalgebra of L and is soluble by 4.3.9. By Lie's theorem, there exists a basis B of L such that every element of $\text{ad}(L) \leq \text{End}_{\mathbb{C}}(L)$ is upper triangular. Hence every element of $\text{ad}([L, L]) = [\text{ad}(L), \text{ad}(L)]$ is represented by a strictly upper triangular matrix since $[\mathfrak{b}_n(k), \mathfrak{b}_n(k)] = \mathfrak{u}_n(k)$. Hence $\text{tr}(\text{ad}(x) \circ \text{ad}(y)) = 0$ for all $x \in L$ and $y \in [L, L]$.

Now suppose that $\text{tr}(\text{ad}(x) \circ \text{ad}(y)) = 0$ for all $x \in L$ and $y \in [L, L]$. This means that $\text{ad}([L, L]) = [\text{ad}(L), \text{ad}(L)]$ is soluble by the above proposition. By 4.1.3 we know that $\text{ad}(L)/([\text{ad}(L), \text{ad}(L)])$ is abelian. By prp4.3.9, we conclude that $\text{ad}(L)$ is soluble. Since $Z(L)$ is abelian, $Z(L)$ is soluble. Hence by the same proposition we conclude that L is soluble since $\text{ad}(L) = L/Z(L)$. □

Let L be a Lie algebra. The killing form allows L to be an inner product space. Recall from Linear Algebra that the orthogonal complement of a subspace W of L is given by

$$W^\perp = \{x \in L \mid k(x, w) = 0 \text{ for all } w \in W\}$$

Lemma 5.2.4

Let L be a Lie algebra. Let I be an ideal of I. Then $I^\perp$ is an ideal of L.

Proof. We already know that $I^\perp$ is a vector space. We want to show that $[x, i] \in I^\perp$ for all $x \in L$ and $i \in I^\perp$. Let $j \in I$ be arbitrary. Then we have

$$k([x, i], j) = k(x, [i, j]) = 0$$

since $[i, j] \in I$. Hence $[x, i] \in I^\perp$ and we are done. □

Recall from Linear Algebra that a bilinear form is non-degenerate if

$$\{v \in V \mid \tau(v, w) = 0 \text{ for all } w \in V\} = \{0\}$$

Because the killing form is symmetric, this is the same as saying that the orthogonal complement $V^\perp = \{0\}$ is trivial.

Theorem 5.2.5: Cartan's Second Criterion

Let L be a Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if k is non-degenerate.

Proof. Assume that k is degenerate. Then $L^\perp \neq \{0\}$ and hence a non-trivial ideal of L. Let $x \in L^\perp$ and $y \in [L^\perp, L^\perp] \subseteq L$. Then $k(x, y) = 0$ by definition of $L^\perp$. Then $k_{L^\perp} = 0$. By Cartan's first criterion, $L^\perp$ is soluble. Hence $L^\perp \subseteq \text{rad}(L)$ and L is not semisimple.

Now suppose that L is not semisimple. Then L contains a non-trivial abelian ideal I. Let $x \in L$ and $i \in I$. Then we have that

$$(\text{ad}(i) \circ \text{ad}(x) \circ \text{ad}(i))\,(y) = [i, [x, [i, y]]] = 0$$

for any $y \in L$ because $[x, [i, y]] \in I$ and I is abelian. Applying $\text{ad}(x)$ on both sides show that

$$(\text{ad}(x) \circ \text{ad}(i))^2 = 0$$

Thus $\text{ad}(x) \circ \text{ad}(i)$ is nilpotent in $\text{End}(L)$. By Engel's theorem, we can find a basis of L such that $\text{ad}(x) \circ \text{ad}(i)$ is strictly upper triangular. This means that $k(x, i) = \text{tr}(\text{ad}(x) \circ \text{ad}(i)) = 0$. But this means that $I \subseteq L^\perp$ since $i \in I \subseteq L$ is such that $k(i, x) = 0$ for all $x \in L$. Since I is non-trivial, $L^\perp$ is non-trivial. Thus k is degenerate. □

Lemma 5.2.6

Let L be a Lie algebra over $\mathbb{C}$. Let I be an ideal of L. If L is semisimple, then the following are true.

- $I \cap I^{\perp} = \{0\}$
- $L = I \oplus I^{\perp}$ as Lie algebras
- I and $I^{\perp}$ are semisimple

Proof.

- We know that $I \cap I^{\perp}$ is also an ideal of L. For $x, y \in I \cap I^{\perp}$, $x \in I$ and $y \in I^perp$ implies that $k_L(x, y) = 0$. Moreover, we know that $k_{I\cap I^{\perp}} = k_L|_{I\cap I^{\perp}\times I\cap I^{\perp}}$. Hence $k_{I\cap I^{\perp}} = 0$. By Cartan's first criterion, $I \cap I^{\perp}$ is soluble, and so $I \cap I^{\perp} \subseteq \text{rad}(L) = \{0\}$. Hence $I \cap I^{\perp} = \{0\}$.
- By the same reasoning, $V = I \cap I^{\perp}$ as vector spaces. But this is also true as Lie algebras because $[I, I^{\perp}] \subseteq I \cap I^{\perp} = \{0\}$.
- Assume that I is not semisimple. Then k_I is degenerate. So there exists $0 \neq i \in I$ such that $k_I(i, x) = 0$ for all $x \in I$. Then $k_L(i, x) = k_I(i, x) = 0$ for all $x \in I$ implies that $i \in I^{\perp}$. Hence $i \in I \cap I^{\perp} = \{0\}$, a contradiction. The proof is identical for $I^{\perp}$ since $I^{\perp}$ is also an ideal of L.

$\square$

Theorem 5.2.7

Let L be a Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if there exists simple ideals $I_1, \dots, I_k$ of L such that

$$L = I_1 \oplus \cdots \oplus I_k$$

Proof. Suppose that L is semisimple. We induct on the dimension of L. We have seen that 1-dimensional and 2-dimensional Lie algebras cannot be semisimple. So we begin with the base case $\dim(L) = 3$. If L has a proper non-trivial ideal I, then I is either 1-dimensional or 2-dimensional. Then I is soluble by the classification theorems. This means that $I \subseteq \text{rad}(L)$ is non-zero. This is a contradiction since we assumed that L is semisimple. Hence L has no proper non-trivial ideals. Hence L is simple.

Now suppose the result holds for all Lie algebras of dimension $< n$. Let $\dim(L) = n$. Let I be a non-zero minimal ideal of L. By lmm 4.2.6, $L = I \oplus I^{\perp}$ where I and $I^{\perp}$ are both semisimple. If $L = I$ then we are done. If $L \neq I$ then I has dimension strictly less than n. By inductive hypothesis, there exists simple ideals $I_1, \dots, I_k$ of I and simple ideals $J_1, \dots, J_s$ such that

$$I = \bigoplus_{i=1}^{k} I_i \quad \text{and} \quad I^{\perp} = \bigoplus_{j=1}^{s} J_j$$

Now we want to show that I_i and J_j are ideals of L. Let $x \in L$. Then $x = a + b$ for $a \in I$ and $b \in I^{\perp}$. If $z \in I_i$, then

$$[x, z] = [a, z] + [b, z] = [a, z] \in I_i$$

since $[b, z] \in I \cap I^{\perp} = \{0\}$. Similarly, one can show that $[x, r] \in J_j$ whenever $r \in J_j$. Thus I_i and J_j are ideals of L. Hence we obtain a direct sum decomposition

$$L = I \oplus I^{\perp} = \bigoplus_{i=1}^{k} I_i \oplus \bigoplus_{j=1}^{s} J_j$$

Thus the induction is complete.

Now let $L = \bigoplus_{j=1}^{k} I_j$. Assume that $\text{rad}(L) \neq \{0\}$. This means that $R_j = \text{rad}(L) \cap I_j$ is a soluble ideal of L and I_j. Since I_j is simple, I_j is semisimple. Hence $R_j = \{0\}$ so that

$$[R, I_j] \subseteq R \cap I_j = R_j = \{0\}$$

Let $x \in R$ and $y \in L$. Then we can write y as $y = y_1 + \cdots + y_k$ for $y_j \in I_j$. Since $[R, I_j] = \{0\}$, we have that $[x, y_j] = 0$ so that $[x, y] = 0$. Thus $x \in Z(L)$. This means that $R \subseteq Z(L)$. But $Z(L)$ is abelian so $Z(L) \subseteq R$. Hence $R = Z(L)$. By prp 1.5.3, we have that

$$Z(L) = \bigoplus_{j=1}^{k} Z(I_j) = \{0\} \oplus \cdots \oplus \{0\} = \{0\}$$

Hence $R = \{0\}$ and L is semisimple. □

Given a Lie algebra L over $\mathbb{C}$, the following are now equivalent.

- L is semisimple ($\text{rad}(L) = \{0\}$)
- L contains no non-zero abelian ideals.
- The killing form $k : L \times L \to L$ is non-degenerate.
- L decomposes into a finite direct sum of simple Lie algebras.

Example 5.2.8

For all $n \geq 2$, $\mathfrak{sl}_n(\mathbb{C})$ is semi-simple.

Proof. Recall that a basis for $\mathfrak{sl}_n(\mathbb{C})$ is given by

$$\{e_{i,j} \mid 1 \leq i \neq j \leq n\} \cup \{e_{i,i} - e_{i+1,i+1} \mid 1 \leq i \leq n-1\}$$

Notice that the killing form can be written as

$$k_{\mathfrak{sl}_2(\mathbb{C})}(x, y) = 2n\text{tr}(xy)$$

Suppose that k is degenerate. Then there exists $0 \neq x \in \mathfrak{sl}_n(\mathbb{C})$ such that $k(x, e_{i,j}) = 0$ for $i \neq j$. Write x in terms of the basis vectors $\{e_{i,j} \mid 1 \leq i, j \leq n\}$ by

$$x = \sum_{1 \leq i,j \leq n} a_{i,j} e_{i,j}$$

Then we have

$$0 = k(x, e_{k,l}) = 2n\text{tr}\left(\sum_{1 \leq i,j \leq n} a_{i,j} e_{i,j}, e_{k,l}\right) = 2n\text{tr}\left(\sum_{1 \leq i \leq n} a_{i,k} e_{i,l}\right) = 2n a_{l,k}$$

This implies $a_{l,k} = 0$ for all $l \neq k$ and so x is diagonal. Now write $x = \sum_{i=1}^n a_i e_{i,i}$. Then we have

$$\begin{aligned} k(x, e_{k,k} - e_{k+1,k+1}) &= 2n\text{tr}\left(\sum_{i=1}^n a_i e_{i,i} e_{k,k} - \sum_{i=1}^n a_i e_{i,i} e_{k+1,k+1}\right) \\ &= 2n\text{tr}(a_k e_{k,k} - a_{k+1} e_{k+1}) \\ &= 2n(a_k - a_{k+1}) \end{aligned}$$

Since $k(x, e_{k,k} - e_{k+1,k+1}) = 0$, we have that $a_k = a_{k+1}$ for $1 \leq k \leq n-1$. Since the trace of x is 0, We can only have $a_1 = a_2 = \cdots = a_n = 0$. Hence $x = 0$, a contradiction. By Cartan's second criterion, $\mathfrak{sl}_n(\mathbb{C})$ is semi-simple. □

6 Special Subalgebras of a Lie Algebra

6.1 Derivations of Lie Algebras

Let $\mathbb{F}$ be a field. Let V be an algebra over $\mathbb{F}$. Recall that a derivation of V is a linear map $\phi : V \to V$ such that the Leibniz rule holds true:

$$\phi(ab) = a\phi(b) + \phi(a)b$$

Clearly it is a $\mathbb{F}$-subalgebra of $\text{End}(V)$. Now that we know that $\text{End}(V)$ can be equipped with a Lie algebra structure, we can also give the $\mathbb{F}$-subalgebra a Lie bracket.

Definition 6.1.1: The Lie Subalgebra of All Derivations

Let $\mathbb{F}$ be a field. Let V be an algebra over $\mathbb{F}$. Define the Lie subalgebra of derivations to be the set

$$\text{Der}_{\mathbb{F}}(V) = \{\phi \in \mathfrak{gl}(V) \mid \phi \text{ is a derivation }\}$$

together with Lie algebra structure inherited from $\mathfrak{gl}_{\mathbb{F}}(V)$.

Lemma 6.1.2

Let $\mathbb{F}$ be a field. Let V be an algebra over $\mathbb{F}$. Then $\text{Der}_{\mathbb{F}}(V)$ is a Lie sub-algebra of $\mathfrak{gl}(V)$.

Proof. We know that $\text{Der}_{\mathbb{F}}(V)$ is a vector subspace of $\mathfrak{gl}(V)$. Let $f_1, f_2 \in \text{Der}_{\mathbb{F}}(V)$. For any $a, b \in V$ we have

$$\begin{aligned}
[f_1, f_2](ab) &= f_1(f_2(ab)) - f_2(f_1(ab)) \\
&= f_1(f_2(a)b + af_2(b)) - f_2(f_1(a)b + af_1(b)) \\
&= f_1(f_2(a))b + f_2(a)f_1(b) + f_1(a)f_2(b) + af_1(f_2(b)) \\
&\quad - f_2(f_1(a))b - f_1(a)f_2(b) - f_2(a)f_1(b) - af_2(f_1(b)) \\
&= f_1(f_2(a))b + af_1(f_2(b)) - f_2(f_1(a))b - af_2(f_1(b)) \\
&= a[f_1, f_2](b) + [f_1, f_2](a)b
\end{aligned}$$

so that the bracket operator is closed. □

Lemma 6.1.3

Let $\mathbb{F}$ be a field. Let L be a Lie algebra over $\mathbb{F}$. Then $\text{ad}(L)$ is an ideal of $\text{Der}_{\mathbb{F}}(L)$.

Proof. Let $x \in L$. I claim that $\text{ad}(x)$ is a derivation. Indeed, for $y, z \in L$, we have

$$\begin{aligned}
\text{ad}(x)([y, z]) &= [x, [y, z]] \\
&= -[[y, z], x] \\
&= [[z, x], y] + [[x, y], z] \\
&= -[\text{ad}(x)z, y] + [\text{ad}(x)(y), z] \\
&= [\text{ad}(x)(y), z] + [y, \text{ad}(x)(z)]
\end{aligned}$$

Thus $\text{ad}(L) \subseteq \text{Der}_{\mathbb{C}}(L)$.

Let $x \in L$ and $\phi \in \text{Der}_{\mathbb{F}}(L)$. It remains to show that $[\phi, \text{ad}(x)]$ is the adjoint homomorphism of some element. We compute that

$$\begin{aligned}[\phi, \mathrm{ad}(x)](y) &= \phi(\mathrm{ad}(x)(y)) - \mathrm{ad}(x)(\phi(y)) \\ &= \phi([x,y]) - [x,\phi(y)] \\ &= [x,\phi(y)] + [\phi(x),y] - [x,\phi(y)] \\ &= [\phi(x),y] \\ &= \mathrm{ad}(\phi(x))(y)\end{aligned}$$

for any $y \in L$. Hence we are done. □

Proposition 6.1.4

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then

$$\mathrm{Der}_{\mathbb{C}}(L) = \{\mathrm{ad}(x) : L \to L \mid x \in L\} = \mathrm{ad}(L)$$

In other words, the only derivations of L is given exactly by the adjoints.

Proof. We know that $\mathrm{ad}(L)$ is an ideal of $\mathrm{Der}_{\mathbb{F}}(L)$. By lemma 4.1.3 we know that the killing form of $\mathrm{ad}(L)^\perp$ is just the restriction of $k_{\mathrm{Der}_{\mathbb{F}}(L)}$ to the ideal. So we compute that

$$\begin{aligned}\mathrm{ad}(L) \cap \mathrm{ad}(L)^\perp &= \mathrm{ad}(L) \cap \{\phi \in \mathrm{Der}_{\mathbb{F}}(L) \mid k_{\mathrm{Der}_{\mathbb{F}}(L)}(\phi, \Phi) = 0 \text{ for all } \Phi \in M\} \\ &= \{\phi \in \mathrm{ad}(L) \mid k|_{\mathrm{ad}(L)}(\phi, \Phi) = 0 \text{ for all } \Phi \in \mathrm{ad}(L)\}\end{aligned}$$

But this is precisely the orthogonal complement of $\mathrm{ad}(L)$ in $\mathrm{ad}(L)$. Since $\mathrm{ad}(L) \cong L$ is semisimple, by Cartan's second criterion $k|_{\mathrm{ad}(L)}$ is non-degenerate. Hence the orthogonal complement of $\mathrm{ad}(L)$ in $\mathrm{ad}(L)$ is 0, and hence $\mathrm{ad}(L) \cap \mathrm{ad}(L)^\perp = \{0\}$ in $\mathrm{Der}_{\mathbb{F}}(V)$. Let $\phi \in \mathrm{ad}(L)^\perp$. For any $x \in L$, we have that $\mathrm{ad}(\phi(x)) = [\phi, \mathrm{ad}(x)]$ by the above lemma. Since $\mathrm{ad}(L)^\perp$ and $\mathrm{ad}(L)$ are both ideals, this element lies in $\mathrm{ad}(L) \cap \mathrm{ad}(L)^\perp = \{0\}$. Hence $\phi(x) \in \ker(\mathrm{ad}) = \{0\}$ for all $x \in L$. This implies that ϕ is the zero map, and so $\mathrm{ad}(L)^\perp = \{0\}$.

There are now two cases to consider. If $\mathrm{Der}_{\mathbb{F}}(V)$ is degenerate, then $\mathrm{Der}_{\mathbb{F}}(V)^\perp \neq \{0\}$. Then $\{0\} = \mathrm{ad}(L)^\perp \supseteq \mathrm{Der}_{\mathbb{F}}(V)^\perp \neq \{0\}$ is a contradiction. Then $\mathrm{Der}_{\mathbb{F}}(V)$ must be non-degenerate. From Linear Algebra we have that $\mathrm{Der}(L) = \mathrm{ad}(L) \oplus \mathrm{ad}(L)^\perp$ since $\mathrm{ad}(L) \cap \mathrm{ad}(L)^\perp = \{0\}$. Since $\mathrm{ad}(L)^\perp = \{0\}$, we conclude that $\mathrm{ad}(L) \oplus \{0\} = \mathrm{Der}_{\mathbb{F}}(V)$. □

Recall the we have seen from Linear Algebra the following fact: If $T \in GL(V)$ is a linear map on V with minimal polynomial factorizing as

$$\mu_T = (x - \lambda_1)^{a_1} \cdots (x - \lambda_k)^{a_k}$$

where the eigenvalues λ_i are distinct and $a_i \geq 1$, then V decomposes as a direct sum of T-invariant subspaces

$$V = V_1 \oplus \cdots \oplus V_k$$

where $V_i = \ker(T - \lambda_i I)^{a_i}$ is the generalized eigenspace.

Lemma 6.1.5

Let L be a Lie algebra over $\mathbb{C}$. Let $\phi \in \mathrm{Der}_{\mathbb{C}}(L)$ be a derivation with Jordan-Chevalley decomposition given by $\phi = d_\phi + n_\phi$ as an element of $\mathfrak{gl}(L)$. Then $d_\phi, n_\phi \in \mathrm{Der}_{\mathbb{C}}(L)$.

Proof. Let λ be an eigenvalue of ϕ. Recall that the generalized eigenspace of d_ϕ is given by

$$L_\lambda = \{x \in L \mid x \in \ker(\phi - \lambda I)^n \text{ for some } n \in \mathbb{N}\}$$

Since d_ϕ is the diagonal part of the decomposition of ϕ, ϕ and d_ϕ share the same eigenvalues. The primary decomposition from Linear Algebra also yields

$$L = \bigoplus_{\lambda \text{ an eigenvalue of } \phi} L_\lambda$$

I claim that $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$. Let $x \in L_\lambda$ and $y \in L_\mu$. We induct on $k \in \mathbb{N}$ to show that

$$(\phi - (\lambda+\mu)I)^k([x,y]) = \sum_{i=0}^{k} \binom{k}{i} \left[(\phi - \lambda I)^k(x), (\phi - \mu I)^{k-i}(y)\right]$$

When $k = 0$, it is trivial. Suppose that it is true for some $k \in \mathbb{N}$. Then we have

$$\begin{aligned}
&(\phi - (\lambda+\mu)I)^{k+1}([x,y]) \\
=&(\phi - (\lambda+\mu)I)^k((\phi - (\lambda+\mu)I)([x,y])) \\
=&(\phi - (\lambda+\mu)I)^k([(\phi - \lambda I)(x), y] + [x, (\phi - \mu I)(y)]) \\
=&\sum_{i=0}^{k} \binom{k}{i} \left[(\phi - \lambda I)^{i+1}(x), (\phi - \mu I)^{k-i}(y)\right] + \sum_{i=0}^{k} \binom{k}{i} \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right] \\
=&\sum_{i=1}^{k+1} \binom{k}{i-1} \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right] + \sum_{i=0}^{k} \binom{k}{i} \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right] \\
=&\left[x, (\phi - \mu I)^{k+1}(y)\right] + \sum_{i=1}^{k} \left(\binom{k}{i-1} + \binom{k}{i}\right) \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right] + \left[(\phi - \lambda I)^{k+1}(x), y\right] \\
=&\left[x, (\phi - \mu I)^{k+1}(y)\right] + \sum_{i=1}^{k} \binom{k+1}{i} \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right] + \left[(\phi - \lambda I)^{k+1}(x), y\right] \\
=&\sum_{i=0}^{k+1} \binom{k+1}{i} \left[(\phi - \lambda I)^{i}(x), (\phi - \mu I)^{k+1-i}(y)\right]
\end{aligned}$$

so the induction is complete. Now since $x \in L_\lambda$, there exists $p \in \mathbb{N}$ such that $(\phi - \lambda I)^p(x) = 0$. Similarly, there exists $q \in \mathbb{N}$ such that $(\phi - \mu I)^q(y) = 0$. Then choose $k = p + 1$, we see that $[x, y] \in L_{\lambda+\mu}$ by the above formula.

Since ϕ and d_ϕ share the same eigenvalues and d_ϕ is diagonalizable, the above generalized eigenspaces collapses to the usual eigenspace. So every vector $v \in L_\lambda$ is an eigenvector of d_ϕ with eigenvalue λ. To check that d_ϕ is a derivation, we see that for $x \in L_\lambda$ and $y \in L_\mu$, we have

$$d_\phi([x,y]) = (\lambda+\mu)([x,y]) = [x, \mu y] + [\lambda x, y] = [x, d_\phi(y)] + [d_\phi(x), y]$$

By bilinearity this extends to all elements of L. Hence $d_\phi \in \mathrm{Der}_{\mathbb{F}}(L)$. Finally, ϕ and d_ϕ in $\mathrm{Der}_{\mathbb{F}}(L)$ implies that $n_\phi = \phi - d_\phi \in \mathrm{Der}_{\mathbb{F}}(L)$ hence we are done. □

6.2 Cartan Subalgebras

Let V be a vector space. Then any element of $\mathfrak{gl}(V)$ admits a Jordan-Chevalley decomposition using the Jordan decomposition form. This means that for any $T \in \mathfrak{gl}(V)$, there exists $D, N \in \mathfrak{gl}(V)$ such that the following are true.

- $T = D + N$.
- D is diagonalizable and N is nilpotent.
- $DN = ND$.

However let $L \leq \mathfrak{gl}(V)$ be a Lie subalgebra. Then consider an element $T \in L$. There is no guarantee that components of the Jordan-Chevalley decomposition lies in L. Therefore we have the notion of abstract Jordan decomposition.

Definition 6.2.1: Abstract Jordan Decomposition

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $x \in L$. Define an abstract Jordan decomposition of x to be a pair of elements $d, n \in L$ such that the following are true.

- $x = d + n$
- $\text{ad}(d) \in \text{End}_{\mathbb{C}}(L)$ is diagonalizable and $\text{ad}(n) \in \text{End}_{\mathbb{C}}(L)$ is nilpotent.
- $[d, n] = 0$

Lemma 6.2.2

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $x \in L$ be an element. Then there exists a unique abstract Jordan decomposition of x.

Proof. Let $x \in L$. Since $\text{ad}(x) \in \mathfrak{gl}(L)$, there exists a unique Jordan-Chevalley decomposition

$$\text{ad}(x) = D + N$$

for some diagonalizable map D and nilpotent map N, such that they commute. By 5.1.5, we know that $\text{ad}(x) \in \text{ad}(L) = \text{Der}_{\mathbb{C}}(L)$ implies that $D, N \in \text{Der}_{\mathbb{C}}(L)$. This means that there exists $d, n \in L$ such that $D = \text{ad}(d)$ and $N = \text{ad}(n)$. Since ad is a linear map, we have that $\text{ad}(x) = \text{ad}(d) + \text{ad}(n)$ implies that $\text{ad}(x) = \text{ad}(d + n)$. Since L and $\text{ad}(L)$ are isomorphic (L is semisimple over $\mathbb{C}$). we thus have $x = d + n$. It remains to show that $[d, n] = 0$. We have that

$$\text{ad}([d, n]) = [\text{ad}(d), \text{ad}(n)] = 0$$

Since ad is an isomorphism, we have $[d, n] = 0$. □

Proposition 6.2.3

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let V be a vector space over $\mathbb{C}$. Let $\phi : L \to \mathfrak{gl}(V)$ be a Lie algebra homomorphism. Let $x = d + n$ be the abstract Jordan decomposition of $x \in L$. Then

$$\phi(x) = \phi(d) + \phi(n)$$

is the Jordan-Chevalley decomposition of $\phi(x)$.

Definition 6.2.4: Cartan Subalgebra

Let L be a Lie algebra. Let $H \leq L$ be a Lie subalgebra of L. We say that H is a Cartan subalgebra of L if the following are true.

- H is abelian.
- For each $h \in H$, $\text{ad}(h)$ is diagonalizable.
- H is maximal with respect to these two properties.

Lemma 6.2.5

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then there exists a non-trivial Cartan subalgebra.

Proof. If for all $x \in L$, the diagonalizable part of the abstract Jordan decomposition is 0, then x is ad-nilpotent. By Engel's theorem, L is nilpotent. Then L is soluble and so $\text{rad}(L) \neq \{0\}$, a contradiction.

Hence there exists at least one element $x \in L$ whose diagonalizable part is non-zero. Let the diagonalizable part be h. Then $\mathbb{C}\langle h \rangle$ is an abelian subalgebra of L consisting entirely of diagonalizable elements. Hence the set of all abelian subalgebras consisting only of diagonalizable elements is non-empty, and has a maximal element. □

7 Representation Theory of Lie Algebras

7.1 Modules over Lie Algebras

Definition 7.1.1: Modules over Lie Algebras

Let L be a Lie algebra over k. An L-module consists of a vector space V over k together with an action
$$\cdot : L \times V \to V$$
such that the following are true.

- Linearity in L: For all $a, b \in k$ and $x, y \in L$, we have that
$$(ax + by) \cdot v = a(x \cdot v) + b(y \cdot v)$$
for all $v \in V$.
- Linearity in V: For all $x \in L$, we have that
$$x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$$
for all $a, b \in k$ and $v, w \in V$.
- Commutes with the Lie bracket: $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$.

Definition 7.1.2: Submodule of Modules over Lie Algebras

Let L be a Lie algebra. Let V be an L-module. A submodule of V is a vector subspace W that is a module over L in its own right.

Definition 7.1.3: L-Module Homomorphisms

Let L be a Lie algebra. Let V, W be L-modules. An L-module homomorphism is a linear transformation $\phi : V \to W$ such that
$$\phi(x \cdot v) = x \cdot \phi(v)$$
for all $x \in L$ and $v \in V$.

Definition 7.1.4: L-Module Isomorphisms

Let L be a Lie algebra. Let V, W be L-modules. Let $\phi : V \to W$ be an L-module homomorphism. We say that ϕ is an L-module isomorphism if ϕ is an isomorphism of vector spaces.

7.2 Reducibility of Modules over Lie Algebras

Definition 7.2.1: Direct Sum of Modules over Lie Algebras

Let L be a Lie algebra. Let V be an L-module. Let U and W be vector subspaces of V. We say that V is the direct sum of U and W if
$$V = U \oplus W$$
as vector spaces and U and W are L-submodules of V.

Definition 7.2.2: Irreducible Modules over Lie Algebras

Let L be a Lie algebra. Let V be an L-module. We say that V is irreducible (simple) if V has no proper non-trivial L-submodules.

Definition 7.2.3: Completely Reducible

Let L be a Lie algebra. Let V be an L-module. We say that V is completely reducible if for all L-submodules W, there exists an L-submodule U of V such that

$$V = W \oplus U$$

Simple modules are thus vacuously completely reducible.

Theorem 7.2.4: Weyl's Theorem

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let V be an L-module. Then V is completely reducible.

Corollary 7.2.5

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let V be an L-module such that V is a finite dimensional vector space. Then there exists irreducible V-submodules $W_1, \dots, W_n$ such that

$$V = \bigoplus_{i=1}^{n} W_i$$

7.3 Representations of Lie Algebras

Definition 7.3.1: Representations of a Lie Algebra

Let L be a Lie algebra. Let V be a vector space. A representation of L is a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(V)$$

Proposition 7.3.2

Let L be a Lie algebra over k. Then representations of L and L-modules are in bijection

$$\{V \text{ an } L\text{-module}\} \overset{1:1}{\leftrightarrow} \{L \to \mathfrak{gl}(V) \text{ a Lie algebra representation}\}$$

This bijection is given by sending an L-module $\cdot : L \times V \to V$ to the Lie algebra homomorphism $\phi : L \to \mathfrak{gl}(V)$ defined by $\phi(l)(v) = l \cdot v$.

These two ways two think about the same thing is natural. Recall that a representation of a group can be thought of as either group homomorphism $G \to GL(V)$ or a $k[G]$-module.

Idea???? This bijection is a ???-homomorphism.

Lemma 7.3.3

Let L be a Lie algebra over k. Then above bijection restricts to a bijection:

$$\{W \text{ an } L\text{-submodule of } L\} \overset{1:1}{=} \{\text{Ideals of } L\}$$

Proof. Let W be an L-submodule of L. Then W is a vector subspace of L. Moreover, for any $x \in L$ and $w \in W$, we have $[x, w] = x \cdot w \in W$ so that W is an ideal of L.

Conversely, let I be an ideal of L. Then I is a vector subspace of L. Moreover, for any $x \in L$ and $i \in I$, we have that $x \cdot i = [x, i] \in I$. Hence I is an L-submodule of L. $\square$

7.4 The Case of $\mathfrak{sl}(2,\mathbb{C})$

We wish to classify all $\mathfrak{sl}_2(\mathbb{C})$-modules. Since $\mathfrak{sl}_2(\mathbb{C})$ is semisimple, Weyl's theorem implies that it suffices to classify all irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules.

Recall that the polynomial ring $\mathbb{C}[x,y]$ is graded by total degree. We write $\mathbb{C}[x,y]_d$ for the dth homogeneous component of the graded ring.

A basis for $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\left\langle e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\rangle$$

Proposition 7.4.1

Let $d \in \mathbb{N}$. The map $\phi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(\mathbb{C}[x,y]_d)$ defined on basis vectors by

$$\phi_d(e) = x\frac{\partial}{\partial y} \quad \text{and} \quad \phi_d(f) = y\frac{\partial}{\partial x} \quad \text{and} \quad \phi_d(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

is an irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ (Equivalently $\mathbb{C}[x,y]_d$ is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module).

Proof. Linearity of ϕ_d is clear since $\phi_d(e), \phi_d(f)$ and $\phi_d(h)$ are linear combinations of partial derivatives. Also, we have

$$\begin{aligned}
[\phi_d(e), \phi_d(f)] &= \phi_d(e) \circ \phi_d(f) - \phi_d(f) \circ \phi_d(e) \\
&= \phi_d(e)\left(y\frac{\partial}{\partial x}\right) - \phi_d(f)\left(x\frac{\partial}{\partial y}\right) \\
&= x\frac{\partial}{\partial x} + xy\frac{\partial^2}{\partial y \partial x} - y\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial x \partial y} \\
&= \phi_d(h) \\
&= \phi_d([e,f])
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
[\phi_d(e), \phi_d(h)] &= \phi_d(e) \circ \phi_d(h) - \phi_d(h) \circ \phi_d(e) \\
&= \phi_d(e)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) - \phi_d(h)\left(x\frac{\partial}{\partial y}\right) \\
&= x^2\frac{\partial^2}{\partial y \partial x} - x\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial y^2} - x\frac{\partial}{\partial y} - x^2\frac{\partial^2}{\partial x \partial y} + xy\frac{\partial^2}{\partial y^2} \\
&= -x\frac{\partial}{\partial y} - x\frac{\partial}{\partial y} \\
&= \phi_d(-2e) \\
&= \phi_d([e,h])
\end{aligned}$$

The proof that $[\phi_d(f), \phi_d(h)] = \phi_d([f,h])$ is similar. Hence ϕ_d is a representation of $\mathfrak{sl}_2(\mathbb{C})$.

We are left to show that ϕ_d is an irreducible representation. Using the basis $\{x^i y^{d-i} \mid 0 \le i \le d\}$, we see that the matrix representation of $\phi_d(e), \phi_d(f), \phi_d(h)$ is given by

$$\phi_d(e) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & \\ 0 & 0 & 2 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & d & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \phi_d(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \\ d & 0 & \cdots & 0 & 0 & \\ 0 & d-1 & \cdots & 0 & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and

$$\phi_d(h) = \begin{pmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -(d-2) & 0 \\ 0 & 0 & \cdots & 0 & -d \end{pmatrix}$$

Let $U \subseteq \mathfrak{gl}(\mathbb{C}[x,y]_d)$ be a sub-module of $\mathfrak{sl}_2(\mathbb{C})$. Then $\phi_d(h)(U) \subseteq U$. Since $\phi_d(h)$ is diagonal, $\phi_d(h)|_U$ is also diagonalizable. Since all eigenvalues of $\phi_d(h)$ are different, all its eigenspaces are one dimensional and is spanned by $x^i y^{d-i}$ for each i. Hence U contains some monomial $x^i y^{d-i}$. I claim that U contains all other monomials and hence $U = \mathfrak{gl}_2(\mathbb{C}[x,y]_d)$. Indeed, if $x^i y^{d-i} \in U$, then $\phi_d\left(\frac{f}{d-i}\right)(x^i y^{d-i}) = x^{i-1}y^{d-i+1} \in U$. Hence
$x^i y^{d-i}, \dots, x^{i-1}y^{d-i+1}, \dots, y^d \in U$. Then $\phi_d(e^{d+1}) \in \mathbb{C}\langle x^d \rangle$. Hence $x^d \in U$ and so all monomials lie in U. □

Lemma 7.4.2

Let $p : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation. Let $v \in V$ be an eigenvector of $p(h)$ with eigenvalue λ. Then the following are true.

- Either $p(e)(v) = 0$ or $p(e)(v)$ is an eigenvector of $p(h)$ with eigenvalue $\lambda + 2$.
- Either $p(f)(v) = 0$ or $p(f)(v)$ is an eigenvector of $p(h)$ with eigenvalue $\lambda - 2$.

Proof.

- If $p(e)(v) \neq 0$, then we have

$$p(h)(p(e)(v)) = p([h,e])(v) + p(e)(p(h)(v)) = p(2e)(v) + p(e)(\lambda v) = (\lambda+2)p(e)(v)$$

- If $p(f)(v) \neq 0$, then we have

$$p(h)(p(f)(v)) = p([h,f])(v) + p(f)(p(h)(v)) = p(-2f)(v) + p(f)(\lambda v) = (\lambda-2)p(f)(v)$$

□

Lemma 7.4.3

Let $p : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V)$ be a finite dimensional representation. Then there exists an eigenvector $v \in V$ for $p(h)$ such that the following are true.

- $p(e)(v) = 0$
- $p(f)^d(v) \neq 0$ and $p(f)^{d+1}(v) = 0$ for some $d \in \mathbb{N}$.

Proof. Since V is a vector space over $\mathbb{C}$, $p(h)$ must contain at least one eigenvector w with eigenvalue λ. Consider the set $\{p(e)^n(w) \mid n \in \mathbb{N}\}$. This set must be finite, otherwise by the above lemma $p(h)$ has an infinite number of eigenvalues, and hence infinitely many eigenvectors which are linearly independent. Hence there exists the smallest $k \in \mathbb{N}$ such that $p(e)^k(w) \neq 0$ but $p(e)^{k+1}(w) = 0$. Setting $v = p(e)^{k+1}(w)$ gives us the first part. The second part is similar. □

Theorem 7.4.4: Classification of Irreducible Representations of $\mathfrak{sl}_2(\mathbb{C})$

Let V be an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module. Then V is isomorphic to $\mathbb{C}[x,y]_d$ as $\mathfrak{sl}_2(\mathbb{C})$-modules for some $d \in \mathbb{N}$.

Proof. Let $v \in V$ be an eigenvalue with the properties in the above lemma. I claim that $\{v, \dots, f^d \cdot v\}$ is a basis for V, and that v has eigenvalue $\lambda = d$. We know that they are linearly independent since they have different eigenvalues. By construction, U is invariant under f and h. It suffices to show that the basis vectors are invariant under r. We proceed by induction. When $i = 0$, $e \cdot (f^i \cdot v) = e \cdot v = 0$ by the above lemma. Suppose that $e \cdot (f^i \cdot v) \in U$ for some i. Then we have

$$e \cdot (f^{i+1} \cdot v) = [e, f] \cdot (f^i \cdot v) + f \cdot (e \cdot (f^i \cdot v)) = h \cdot (f^i \cdot v) + f \cdot (e \cdot (f^i \cdot v))$$

The first term lies in U since U is invariant under h and f. The second term also lies in U since U is invariant under f by induction hypothesis, U is invariant under e. Hence U is a sub-module of V. But V is an irreducible module. Hence $U = V$.

Now $\{v, \dots, f^d \cdot v\}$ are distinct eigenvectors of h because they have distinct eigenvalues. The matrix of $h \cdot -$ with respect to the basis is diagonal, and its trace is

$$\text{tr}(h \cdot -) = \lambda + (\lambda - 2) + \cdots + (2 - \lambda) + (-\lambda) = (d + 1)(\lambda - d)$$

by lemma 7.4.2. Since $h \cdot - = [e \cdot -, f \cdot -] \in \mathfrak{gl}(V)$ is a commutator of some matrices, $h \cdot -$ must have zero trace. Hence $\lambda = d$.

Define $\Phi : V \to \mathbb{C}[x, y]_d$ by the formula

$$\Phi(f^i \cdot v) = \phi_d(f)^i(x^d) = f^i \cdot x^d = (d - i - 1)!x^{d-i}y^i$$

Clearly it sends the basis to basis, hence it is a vector space isomorphism. It remains to show that it is a $\mathfrak{sl}_2(\mathbb{C})$-module homomorphism. Clearly we have

$$\Phi(f \cdot (f^i \cdot v)) = f^{i+1} \cdot (x^d) = f \cdot \Phi(f^i \cdot v)$$

Also, we have

$$\Phi(h \cdot (f^i \cdot v)) = \Phi((\lambda - 2i)(f^i \cdot v)) = (\lambda - 2i)f^i(v) = h \cdot (f^i \cdot v) = h \cdot \Phi(f^i \cdot v)$$

and

$$\begin{aligned}
\Phi(e \cdot (f^i \cdot v)) &= \Phi([e, f] \cdot (f^{i-1} \cdot v) + f \cdot (e \cdot (f^{i-1} \cdot v))) \\
&= \Phi(h \cdot (f^{i-1} \cdot v) + f \cdot (e \cdot (f^{i-1} \cdot v))) \\
&= h \cdot \Phi(f^{i-1} \cdot v) + f \cdot \Phi(e \cdot (f^{i-1} \cdot v)) \\
&= h \cdot \Phi(f^{i-1} \cdot v) + f \cdot (e \cdot \Phi(f^{i-1} \cdot v)) && \text{(Induction)} \\
&= [e, f] \cdot \Phi(f^{i-1} \cdot v) + f \cdot (e \cdot \Phi(f^{i-1} \cdot v)) \\
&= e \cdot \Phi(f^i \cdot v)
\end{aligned}$$

by induction. Hence we are done. $\square$

8 Introduction to Lie Groups

8.1 Lie Groups

Definition 8.1.1: Lie Groups

A Lie group G is a group G that is also a smooth manifold such that the following are true.

- The multiplication map $\cdot : G \times G \to G$ defined by
$$(g, h) \mapsto gh$$
is a smooth map of manifolds.
- The inverse map $(-)^{-1} : G \times G$ defined by
$$g \mapsto g^{-1}$$
is a smooth map of manifolds.

Some immediate examples of Lie groups include the following:

- Any finite group is discrete and hence are zero-dimensional manifolds. So they are Lie groups.
- $(\mathbb{R}^n, +)$ and $(\mathbb{R}^n \setminus \{0\}, \times)$.
- The torus $(\mathbb{R}^n/\mathbb{Z}^n, +)$.
- $\text{GL}(V)$ for any finite dimensional real vector space V.
- $U(n), SU(n), O(n), SO(n), SL(n), PSL(n)$ for any $n \in \mathbb{N}$.

Proposition 8.1.2

Let G be a Lie group. Let H be a subgroup of G. If H is closed in G, then H inherits the structure of a Lie group from G.

Definition 8.1.3: Lie Group Homomorphism

Let G, H be Lie groups. A Lie group homomorphism is a map of sets
$$\phi : G \to H$$
that is a group homomorphism and a smooth map.

8.2 Relation between Lie Groups and Lie Algebras

For a group G, denote the left multiplication map of $h \in G$ by l_h. If G is a Lie group, we have seen that l_h is a smooth map, and so it induces a differential $(l_h)_*$.

Definition 8.2.1: Left Invariant Vector Field

Let G be a Lie group and X a vector field on G. We say that X is left invariant if
$$(l_h)_*(X_g) = X_{hg}$$
for all $X_g \in T_g(G)$.

Proposition 8.2.2

Let G be a Lie group. The vector space of left invariant vector fields of G is a Lie algebra of dimension $\dim(G)$.

Proposition 8.2.3

Let G be a Lie group. Let $v \in T_{1_G}(G)$ be a tangent vector at the identity. Then there exist a unique left invariant vector field $X : G \to TG$ on G such that $X(1_G) = v$.

Definition 8.2.4: Lie Algebra of a Lie Group

Let G be a Lie group. Define the Lie algebra of G to be the Lie algebra $T_{1_G}(G)$.

Recall that given a homomorphism of Lie groups $\phi : G \to H$, it induces a differential $\phi_* : T_g(G) \to T_{\phi(g)}(H)$.

Proposition 8.2.5

Let $\phi : G \to H$ be a homomorphism of Lie groups with Lie algebras V and W respectively. Then the induced map from the differential

$$(\phi_*)_{1_G} : T_{1_G}G \to T_{1_H}H$$

is a Lie algebra homomorphism.

In other words, we constructed a functor from Lie groups to Lie algebras sending a Lie group to the tangent space at the identity.

8.3 The Exponential Map

9 Root Systems of Vector Spaces

9.1 Properties of Root Systems

Definition 9.1.1: Root Systems of an Inner Product Space over $\mathbb{R}$

Let $(E, (-,-))$ be a finite dimensional inner product space over $\mathbb{R}$. Let $R \subseteq E$. Write

$$\langle x, y \rangle = \frac{2(x,y)}{(y,y)}$$

for $x, y \in E$. We say that R is a root system in E if the following are true.

- R is finite and $0 \notin R$
- $\mathbb{R}\langle R \rangle = E$ (R spans E)
- If $a \in R$, the $ca \in R$ if and only if $c = \pm 1$
- If $a, b \in R$, then
$$\sigma_a(b) = b - 2\frac{(a,b)}{(a,a)}a = b - \langle a, b \rangle a$$
is an element of R.
- If $a, b \in R$, then $\langle a, b \rangle \in \mathbb{Z}$

Example 9.1.2

Let $\{e_1, \dots, e_{n+1}\}$ be the standard basis vectors of $\mathbb{R}^{n+1}$. Let E be the inner product space

$$E = \{(c_1, \dots, c_{n+1}) \subseteq \mathbb{R}^{n+1} \mid c_1 + \cdots + c_n = 0\} \cong \mathbb{R}^n$$

equipped with the standard inner product of $\mathbb{R}^{n+1}$. Then the set

$$R = \{e_i - e_j \mid 1 \leq i \neq j \leq n+1\}$$

is a root system of E.

Lemma 9.1.3

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let $\alpha, \beta \in R$. Then the following are true.

- The value of $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ must lie in the set $\{0, 1, 2, 3\}$
- The angle between α and β must lie in the set
$$\left\{ \frac{\pi}{2}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{5\pi}{6} \right\}$$

We can draw a table of the possible values of $\langle \alpha, \beta \rangle$ and the corresponding angle between α and β:

$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$	$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ
0	0	0	$\frac{\pi}{2}$
1	1	1	$\frac{\pi}{3}$
	-1	-1	$\frac{2\pi}{3}$
2	1	2	$\frac{\pi}{4}$
	-1	-2	$\frac{3\pi}{4}$
3	1	3	$\frac{\pi}{6}$
	-1	-3	$\frac{5\pi}{6}$

Here we assume $\|\alpha\| \leq \|\beta\|$ so we exclude for instance the case $\langle \alpha, \beta \rangle = 2$ and $\langle \beta, \alpha \rangle = 1$.

Proposition 9.1.4

The only root system in $\mathbb{R}^1$ is given by $R = \{\alpha, -\alpha\}$ for any $\alpha \in \mathbb{R}$.

Proposition 9.1.5

There are exactly four possible root systems in $\mathbb{R}^2$. Let $\alpha, \beta \in \mathbb{R}^2$ be linearly independent such that $\|\alpha\| \leq \|\beta\|$. Let θ be the angle between α and β.

- $A_1 \times A_1 = \{\pm\alpha, \pm\beta\}$ and $\theta = \frac{\pi}{2}$.
- $A_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\}$ and $\theta = \frac{2\pi}{3}$.
- $B_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\}$ and $\theta = \frac{3\pi}{4}$.
- $G_2 = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta), \pm(3\alpha + \beta), \pm(3\alpha + 2\beta)\}$

9.2 Decomposing Root Systems into Irreducibles

Definition 9.2.1: Irreducible Root Systems

Let $(E, (-,-))$ be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. We say that R is reducible if there exists two disjoint root systems R_1 and R_2 of E such that $(r_1, r_2) = 0$ for $r_1 \in R_1$ and $r_2 \in R_2$. Otherwise, we say that R is irreducible.

Proposition 9.2.2

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Then the following are true.

- There is a decomposition
$$R = R_1 \amalg \cdots \amalg R_k$$
where each R_i is an irreducible root system of $E_i = \mathbb{R}\langle R_i \rangle$
- There is a decomposition
$$E = E_1 \oplus \cdots \oplus E_k$$
where E_i is orthogonal to E_j for $i \neq j$.

Definition 9.2.3: Isomorphic Root Systems

Let E, F be a finite dimensional inner product spaces over $\mathbb{R}$. Let R and S be root systems of E and F respectively. We say that R and S are isomorphic if there exists an isomorphism

$$\phi : E \to F$$

of vector spaces such that the following are true.

- $\phi(R) = S$
- For all $a, b \in R$, $\langle \phi(a), \phi(b) \rangle = \langle a, b \rangle$.

Example 9.2.4

The following are true.

- $A_1 \times A_1 = \{\pm\alpha, \pm\beta\}$ is reducible into the product of A_1 and A_1.
- A_2, B_2, G_2 are irreducible.

9.3 Bases for Root Systems

Definition 9.3.1: Bases for Root Systems

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let $B \subseteq R$. We say that B is a base of R if the following are true.

- B is a basis for E
- For any $r \in R$, there is a decomposition

$$r = \sum_{b \in B} k_b \cdot b$$

where $k_b \in \mathbb{Z}$ and either all k_b are positive or negative.

Proposition 9.3.2

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Then R admits a base.

9.4 The Weyl Group of a Root System

Recall that $\sigma_a(b) = b - 2\frac{\langle a,b\rangle}{\langle a,a\rangle}a$ for any a, b in a root system R of a finite dimensional inner product space E. In particular, they are elements of the general linear group $GL(E)$.

Definition 9.4.1: The Weyl Group of a Root System

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Define the Weyl group of R to be the subgroup

$$W(R) = \langle \sigma_a \mid a \in R \rangle \leq GL(E)$$

Example 9.4.2

The Weyl group of A_2 is given by

$$W(A_2) \cong S_3$$

Proposition 9.4.3

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let B be a base of R. Then we have

$$W(R) = \langle \sigma_a \mid a \in B \rangle$$

Lemma 9.4.4

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let B be a base of R. Then for all $\alpha \in R$, there exists $w \in W(R)$ and $\beta \in B$ such that

$$w(\beta) = \alpha$$

Proposition 9.4.5

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let B, B' be bases of R. Then there exists $w \in W$ such that

$$B' = \{g(\alpha) \mid \alpha \in B\}$$

9.5 Dynkin Diagrams and Cartan Matrices

Definition 9.5.1: The Dynkin Diagram of a Root System

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let B be a base of R. Define a graph $\Delta(R)$ as follows.

- There is one vertex v_b for each $b \in B$
- For any two vertices v_a and v_b, there are $d_{a,b} = \langle a, b\rangle\langle b, a\rangle$ number of undirected edges between v_a and v_b.

Proposition 9.5.2

Let E, F be a finite dimensional inner product spaces over $\mathbb{R}$. Let R and S be root systems of E and F respectively. Then $R \cong S$ if and only if $\Delta(R) \cong \Delta(S)$.

Proposition 9.5.3

Let E be a finite dimensional inner product space. Let R be a root system of E. Then R is irreducible if and only if $\Delta(R)$ is a connected graph.

Definition 9.5.4: The Cartan Matrix

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Let $B = \{\alpha_1, \dots, \alpha_r\}$ be a base of R. Define the Cartan matrix of R to be

$$\begin{pmatrix} \langle \alpha_1, \alpha_1\rangle & \cdots & \langle \alpha_1, \alpha_r\rangle \\ \vdots & \ddots & \vdots \\ \langle \alpha_r, \alpha_1\rangle & \cdots & \langle \alpha_r, \alpha_r\rangle \end{pmatrix}$$

Proposition 9.5.5

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be a root system of E. Then the Cartan matrix of R is unique up to reordering.

9.6 Classification of Irreducible Root Systems

Theorem 9.6.1: Classification of Irreducible Root Systems

Let E be a finite dimensional inner product space over $\mathbb{R}$. Let R be an irreducible root system of E. Then $\Delta(R)$ is isomorphic to one of the following:

- A_n for $n \geq 1$.
- B_n for $n \geq 2$.
- C_n for $n \geq 3$.
- D_n for $n \geq 4$.
- G_2, F_4, E_6, E_7, E_8.

Conversely, any of the above types can be realized as the Dynkin diagram of some irreducible root system.

10 Classification of Semisimple Lie Algebras over $\mathbb{C}$

10.1 Root Space Decomposition

Let V be a vector space. We denote the dual space of V by V^*.

Let L be semisimple over $\mathbb{C}$. Let H be a Cartan subalgebra. By lmm 5.1.2, we can choose a basis $v_1, \dots, v_m$ of $GL(V)$ such that $\text{ad}(h)$ is diagonal for all $h \in H$. In particular, such a basis consists of common eigenvectors of $\text{ad}(h)$. Fix such a common eigenvector v and write its eigenvalue by $\alpha(\text{ad}(h))$ (dependent on h). Then we have

$$\text{ad}(h)(v) = \alpha(\text{ad}(h))(v)$$

and in particular $\alpha : \text{ad}(H) \to \mathbb{C}$ is a weight of $\text{ad}(H)$. Since L is semisimple, we have an isomorphism $\text{ad}(H) \cong H$. So we can think of the weight as an element

$$\alpha : H \cong \text{ad}(H) \to \mathbb{C}$$

of the dual space H^*. This motivates the following definition.

Definition 10.1.1: Roots of Lie Algebra relative to a Cartan Subalgebra

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan subalgebra of L. A root of L relative to H is an element $\alpha \in H^*$ such that $\alpha \neq 0$ and the weight space

$$L_\alpha = \{x \in L \mid [h, x] = \alpha(h)(x) \text{ for all } h \in H\}$$

is non-zero. In this case, we call L_α the root space of α.

For $x \in L_\alpha$, notice that x is a common eigenvector for $\text{ad}(h)$ for all $h \in H$.

Definition 10.1.2: The Set of Roots relative to a Cartan Subalgebra

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan subalgebra of L. Define the set of roots of L relative to H to be

$$\Phi = \{\alpha \in H^* \mid \alpha \text{ is a root of } L \text{ with respect to } H\}$$

Note that Φ is finite if L is finite dimensional.

Lemma 10.1.3

Let V be a vector space over k. Suppose that $T_1, \dots, T_n \in GL(V)$ is diagonalizable. Then there exists a basis of $GL(V)$ such that $T_1, \dots, T_n$ are diagonal if and only if $T_1, \dots, T_n$ pair-wise commute. In particular, the basis are precisely common eigenvectors of $T_1, \dots, T_n$.

Lemma 10.1.4

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let H be a Cartan sub-algebra of L. Let Φ be the set of roots of L relative to H. Then there is a decomposition of direct sums

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

given by the primary decomposition (eigenspace decomposition) of any $\text{ad}(h) \in \text{ad}(H)$.

Proof. Let $h \in H$. Recall that $\text{ad}(h)$ is diagonalizable by definition. By the primary decomposition theorem (or eigenspace decomposition), we can decompose L as one-dimensional invariant subspaces:

$$L = \bigoplus_{i=1}^{r} \ker(\text{ad}(h) - \lambda_i(h)I)$$

where $\lambda_1(h), \dots, \lambda_r(h)$ is the complete list of distinct eigenvalues of ad(h).

Since H is abelian, elements of H pairwise commute. By the above lemma, all elements of ad(H) share a common set of eigenvectors. In particular, for any $h' \in H$, ad(h) we have $\ker(\text{ad}(h) - \lambda_i(h)I) = \ker(\text{ad}(h') - \lambda_i(h')I)$. Then the linear functions λ_i are precisely roots of L relative to H, and so we can rewrite the decomposition as

$$L = L_0 \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$$

□

Proposition 10.1.5

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan sub-algebra of L. Let α, β be roots of L relative to H. Then the following are true.

- $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$
- If $\alpha + \beta \neq 0$, then $k(L_\alpha, L_\beta) = 0$ where k is the killing form of L.
- $L_0 \cap L_0^\perp = \{0\}$. In particular, $k|_{L_0}$ is non-degenerate.

Proof.

- Let $x \in L_\alpha$ and $y \in L_\beta$. Then we have

$$\begin{aligned}[h, [x, y]] &= -[[x, y], h] \\ &= [[y, h], x] + [[h, x], y] \\ &= -[\beta(h)y, x] + [\alpha(h)x, y] \\ &= \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y]\end{aligned}$$

  so that $[x, y] \in L_{\alpha+\beta}$.
- Let $x \in L_\alpha$ and $y \in L_\beta$. Since $\alpha + \beta \neq 0$, there exists $h \in H$ such that $\alpha(h) + \beta(h) \neq 0$. Then

$$(\alpha+\beta)(h)k(x, y) = k(\alpha(h)x, \beta(h)y) = k([h, x], [h, y]) = -k([x, h], [h, y]) = -k(x, [h, [h, y]]) = 0$$

  implies that $k(x, y) = 0$.
- Assume that $a \in L_0 \cap L_0^\perp$. Then $k(a, b) = 0$ for all $b \in L_0$. For any $v \in L$, we can use the above decomposition to get $v = v_0 + \sum_{\alpha\in\Phi} v_\alpha$ for $v_0 \in L_0$ and $v_\alpha \in L_\alpha$. Then we have

$$k(a, v) = k(a, v_0) + \sum_{\alpha\in\Phi} k(a, v_\alpha) = 0$$

  since $a \in L_0$ and $v_\alpha \in L_\alpha$ implies $k(a, v_\alpha) = 0$ by the above. This proves that $a \in L_0^\perp$. Since L is semisimple, k is non-degenerate so that $L_0^\perp = \{0\}$. Hence $a = 0$ and $L_0 \cap L_0^\perp = \{0\}$.

□

Lemma 10.1.6

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let $H \leq L$ be a Cartan sub-algebra of L. Let Φ be the set of roots of L relative to H. Then the following are true.

- For all $0 \neq h \in H$, there exists $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.
- $H^* = \mathbb{C}\langle\Phi\rangle$.
- If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

Proof. Suppose that $-\alpha \notin \Phi$. Then $k(L_\alpha, L_\beta) = 0$ for all $\beta \in \Phi$. Hence $L_\alpha \subseteq L^\perp = \{0\}$. This is a contradiction since L_α is assumed to be non-zero. □

Proposition 10.1.7

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then we have

$$H = C_L(H) = C_L(h_0)$$

for any $h_0 \in H$ such that $\dim(C_L(h_0)) \leq \dim(C_L(H))$ for all $h \in H$.

Proof. It is clear that $H \subseteq C_L(H) \subseteq C_L(h_0)$ since H is abelian.

Suppose for a contradiction that $a \in H$ but $a \notin Z(C_L(h_0))$. Since $C_L(a) \cap C_L(h_0)$ is abelian, choose a basis $\{e_1, \dots, e_k\}$ of common eigenvectors for $C_L(a) \cap C_L(h_0)$. □

Theorem 10.1.8: Root Space Decomposition

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let H be a Cartan sub-algebra of L. Let Φ be the set of roots of L relative to H. Then there is a decomposition of direct sums

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

given by the primary decomposition (eigenspace decomposition) of any $\text{ad}(h) \in \text{ad}(H)$.

Proof. Since $L_0 = C_L(H) = H$, we conclude by 9.1.5. □

Notice that 9.1.6 then implies that the killing form on H restricted to L is non-degenerate.

Example 10.1.9

A Cartan sub-algebra of $\mathfrak{sl}_2(\mathbb{C})$ is $\mathbb{C}\langle h\rangle$, and we obtain a decomposition

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}\langle h\rangle \oplus L_\alpha \oplus L_{-\alpha}$$

where $\alpha(h) = 2$.

Proof. Notice that since $\mathbb{C}\langle h\rangle$ is one-dimensional, it is abelian and is contained in some Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$. Since $[e, h] = -2e$ and $[f, h] = 2f$ we see that the Cartan subalgebra containing h is $\mathbb{C}\langle h\rangle$. Moreover, from the two equations we see that $\alpha \in H^*$ defined by $\alpha(h) = 2$ is a root of $\mathfrak{sl}_2(\mathbb{C})$ since $L_\alpha \supseteq \mathbb{C}\langle e\rangle$ and $L_{-\alpha} \supseteq \mathbb{C}\langle f\rangle$. These make up at least a three dimensional subspace of $\mathfrak{sl}_2(\mathbb{C})$, hence we deduce the desired decomposition. □

10.2 Copies of $\mathfrak{sl}_2(\mathbb{C})$ as Sub-algebras

Let V be a vector space over $\mathbb{C}$. Let $\tau : V \to V \to \mathbb{C}$ be a bilinear map. Recall that τ is non-degenerate if and only if $v \mapsto \tau(v, -)$ is an isomorphism between V and V^*. In this case, every linear map $T \in V^*$ determines a unique element $v \in V$ such that $T(-) = \tau(v, -)$.

Applying the above, we know that the killing form on H is non-degenerate, so any root $\alpha \in \Phi \subseteq H^*$ can be represented using the killing form.

Lemma 10.2.1

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let H be a Cartan sub-algebra of L. Let Φ be a set of roots of L relative to H. Let $\alpha \in \Phi$. There exists

$$t_\alpha \in H \setminus \{0\}$$

such that the following are true.

- $\alpha(-) = k(t_\alpha, -) : H \to \mathbb{C}$ as elements of H^*.
- $[x, y] = k(x, y)t_\alpha$ for all $x \in L_\alpha$ and $y \in L_{-\alpha}$.
- $[L_\alpha, L_{-\alpha}] = \mathbb{C}\langle t_\alpha \rangle$.
- $\alpha(t_\alpha) \neq 0$.
- $t_\alpha = -t_{-\alpha}$

Proof. By 10.1.6, $k_H = k_L|_{H \times H}$ is non-degenerate. By Linear Algebra, there is an isomorphism $H \cong H^*$ given by $h \mapsto k_H(h, -)$. Since $\alpha \in H^*$, there exists $t_\alpha \in H$ such that $\alpha(-) = k_H(t_\alpha, -)$, proving the first item. t_α is non-zero otherwise $k_H(t_\alpha, -) = \alpha(-)$ is the zero map.

Since $x \in L_\alpha$ and $y \in L_{-\alpha}$, we know that $[x, y] \in L_0 = H$. Since $\alpha \neq 0$, there exists $t_\alpha \in H \setminus \{0\}$ such that $\alpha(t_\alpha) \neq 0$. Then for any $h \in H$ we have

$$k(h, [x, y]) = k([h, x], y) = k(\alpha(h)x, y) = \alpha(h)k(x, y) = k(t_\alpha, h)k(x, y)$$

Similarly, we compute that

$$k(h, k(x, y)t_\alpha) = k(x, y)k(t_\alpha, h)$$

This means that $k(h, [x, y] - k(x, y)t_\alpha) = 0$ for all $h \in H$. Since $h, [x, y] - k(x, y)t_\alpha \in H$, this implies that $[x, y] - k(x, y)t_\alpha \in H^*$. Proposition 10.1.6 implies that $H \cap H^\perp = \{0\}$. Hence $[x, y] = k(x, y)t_\alpha$, proving the second item.

For arbitrary $x \in L_\alpha$ and $y \in L_{-\alpha}$ we proved that $[x, y] \in \mathbb{C}\langle t_\alpha \rangle$. Hence $[L_\alpha, L_{-\alpha}] \subseteq \mathbb{C}\langle t_\alpha \rangle$. It remains to show that there exists $x \in L_\alpha$ and $y \in L_{-\alpha}$ such that $[x, y] \neq 0$. If this is not the case, then $[x, y] = 0$ for some x and y. Then $k(x, y) = 0$. By 10.1.6, $k(L_\alpha, L_\beta) = 0$ for $\alpha \neq -\beta$. This shows that $k(x, z) = 0$ for all $z \in L$. Hence $x \in L^\perp = \{0\}$. Then $L_\alpha = \{0\}$ is a contradiction. This proves the third item.
For the fourth item, suppose for a contradiction that $\alpha(t_\alpha) = 0$. For all $x \in L_\alpha$, we have $[t_\alpha, x] = \alpha(t_\alpha)x = 0$. Similarly, for all $y \in L_{-\alpha}$ we have $[t_\alpha, y] = 0$. Then $\alpha([x, y]) = k(x, y)\alpha(t_\alpha) = 0$. Hence $t_\alpha \subseteq Z(\mathbb{C}\langle x, y, [x, y]\rangle)$. By 4.4.4, $\mathbb{C}\langle x, y, [x, y]\rangle$ is nilpotent and hence soluble. Since L is semisimple, ad is an isomorphism. Hence $\text{ad}(\mathbb{C}\langle x, y, [x, y]\rangle) \cong \mathbb{C}\langle x, y, [x, y]\rangle$. By Lie's theorem, there exists a basis such that every element of $\text{ad}(\mathbb{C}\langle x, y, [x, y]\rangle)$ is upper triangular. Since $t_\alpha \in [\mathbb{C}\langle x, y, [x, y]\rangle, \mathbb{C}\langle x, y, [x, y]\rangle]$, $\text{ad}(t_\alpha)$ is strictly upper triangular. Thus $\text{ad}(t_\alpha)$ is nilpotent and so t_α is nilpotent. This is a contradiction since every element of H has non-zero diagaonalizable part.

For any $h \in H$, $k(t_\alpha, h) = \alpha(h)$. So we have

$$k(t_\alpha + t_{-\alpha}, h) = k(t_\alpha, h) + k(t_{-\alpha}, h) = \alpha(h) - \alpha(h) = 0$$

Since this is true for all $h \in H$, we have that $t_\alpha + t_{-\alpha} \in H^\perp$. Since $H \cap H^\perp = \{0\}$, we conclude. □

Definition 10.2.2: Lie Sub-algebra associated to a Root

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let α be a root of L. Let $x \in L_\alpha$ and $y \in L_{-\alpha}$ be non-zero. Define the Lie sub-algebra associated to α by

$$M_\alpha = \mathbb{C}\langle x, y, [x, y] = k(x, y)t_\alpha \rangle$$

Proposition 10.2.3

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let α be a root of L. Then the following are true. Denote

$$e_\alpha = x \quad \text{and} \quad f_\alpha = \frac{2}{k(t_\alpha, t_\alpha)k(x, y)}y \quad \text{and} \quad h_\alpha = \frac{2}{k(t_\alpha, t_\alpha)}t_\alpha$$

Then the following are true.

- $h_\alpha = -h_{-\alpha}$.
- $\alpha(h_\alpha) = 2$.
- There is a Lie algebra isomorphism

$$M_\alpha \cong \mathbb{C}\langle e_\alpha, f_\alpha, h_\alpha \rangle$$

Proposition 10.2.4

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let α be a root of L. Then

$$M_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$$

Proof. We check the structural constants:

$$[e_\alpha, f_\alpha] = \left[x, \frac{2}{k(t_\alpha, t_\alpha)k(x, y)}y\right] = \frac{2}{k(t_\alpha, t_\alpha)k(x, y)}[x, y] = \frac{2}{k(t_\alpha, t_\alpha)}t_\alpha = h_\alpha$$
$$[e_\alpha, h_\alpha] = \frac{2}{k(t_\alpha, t_\alpha)}[x, t_\alpha] = -\frac{2}{k(t_\alpha, t_\alpha)}\alpha(t_\alpha)x = -\frac{2}{k(t_\alpha, t_\alpha)}k(t_\alpha, t_\alpha)x = 2e_\alpha$$

and

$$\begin{aligned}
[f_\alpha, h_\alpha] &= \frac{4}{k(t_\alpha, t_\alpha)^2k(x, y)}[y, t_\alpha] \\
&= -\frac{4}{k(t_\alpha, t_\alpha)^2k(x, y)}\alpha(t_\alpha)y \\
&= -\frac{4}{k(t_\alpha, t_\alpha)^2k(x, y)}k(t_\alpha, t_\alpha)y \\
&= -\frac{4}{k(t_\alpha, t_\alpha)k(x, y)}y \\
&= -2f_\alpha
\end{aligned}$$

It has the same structural constants as $\mathfrak{sl}_2(\mathbb{C})$ so M_α is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. □

Proposition 10.2.5

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let α be a root of L. Let V be an M_α-submodule of L, then the eigenvalues of the map $\mathrm{ad}(h_\alpha)$ are integers.

10.3 The Dimension of the Subspaces in Root Space Decompositions

Lemma 10.3.1

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let Φ be the set of roots relative to some Cartan sub-algebra of L. Let $\alpha \in \Phi$. Then there is a decomposition

$$H \oplus \bigoplus_{c\alpha \in \Phi} L_{c\alpha} = \ker(\alpha) \oplus M_\alpha$$

Proof. Write $U_\alpha = H \oplus \bigoplus_{c\alpha\in\Phi} L_{c\alpha}$. Write $e_\alpha, f_\alpha, h_\alpha$ as the basis of M_α. Recall that M_α is spanned by $\{x, y, [x, y]\}$ where $x \in L_\alpha$, $y \in L_{-\alpha}$. Since $[L_\alpha, L_{-\alpha}] \subseteq L_0 = H$, we have that $M_\alpha \subseteq U_\alpha$.

Step 1: U_α is an M_α-submodule of L.
Let $h \in H$. Then we have

$$\begin{aligned} [e_\alpha, h] &= -[h, e_\alpha] = -\alpha(h)e_\alpha \in M_\alpha \subseteq U_\alpha \\ [f_\alpha, h] &= -[h, f_\alpha] = \alpha(h)f_\alpha \in M_\alpha \subseteq U_\alpha \\ [h_\alpha, h] &= 0 && \text{(Since } H = C_L(H)\text{)} \end{aligned}$$

Hence $[M_\alpha, U_\alpha] \subseteq U_\alpha$. Now for any $c\alpha \in \Phi$, we have

$$\begin{aligned} [e_\alpha, L_{c\alpha}] &\subseteq L_{(c+1)\alpha} \subseteq U_\alpha \\ [f_\alpha, L_{c\alpha}] &\subseteq L_{(c-1)\alpha} \subseteq U_\alpha \\ [h_\alpha, L_{c\alpha}] &\subseteq L_{c\alpha} \subseteq U_\alpha \end{aligned}$$

so we are done.

Step 2: $\ker(\alpha)$ is a trivial M_α-submodule of U_α.
Let $h \in \ker(\alpha) \subseteq H$. Then we compute that

$$\begin{aligned} [e_\alpha, h] &= -[h, e_\alpha] = -\alpha(h)e_\alpha = 0 \\ [f_\alpha, h] &= -[h, f_\alpha] = \alpha(h)f_\alpha = 0 \\ [h_\alpha, h] &= 0 && \text{(Since } H = C_L(H)\text{)} \end{aligned}$$

So we are done.

Step 3: M_α is an irreducible M_α-submodule of U_α.
Clearly M_α is a submodule of U_α. Since sub-modules of M_α correspond bijectively to ideals of M_α and $M_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$ is simple, we conclude that M_α is irreducible.

Step 4: $\ker(\alpha) \oplus M_\alpha$ is an M_α-submodule of U_α.
It suffices to show that $\ker(\alpha) \cap M_\alpha = \{0\}$. Since $\ker(\alpha) \cap M_\alpha \subseteq H$ and $H \cap M_\alpha = \mathbb{C}\langle h_\alpha \rangle$, as well as $e_\alpha, f_\alpha \notin L_0 = H$, we conclude that $\ker(\alpha) \cap M_\alpha = \{0\}$.

By Weyl's theorem, there exists an M_α-submodule of U_α so that

$$U_\alpha = M_\alpha \oplus \ker(\alpha) \oplus W$$

Step 5: $W = \{0\}$.
Suppose that W is non-zero. Then there exists an irreducible M_α-submodule V of W. By the classification of irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules, we know that $V \cong W_d$ for some $d \in \mathbb{N}$. By 6.4.1 and 6.4.2, we knwo that the eigenvalues of $\text{ad}(h_\alpha)$ is given by $d, d-2, \dots, -d$. They are either all even or all odd.

If d is even, then $\text{ad}(h_\alpha)$ has an eigenvector $0 \neq v \in V$ whose eigenvalue is 0. Now the eigenspace of $\text{ad}(h)$ corresponding to the eigenvalue 0 is H because $H = L_0 = C_L(H)$. Hence $v \in H$ so that $v \in H \cap V$. At the same time, $H \subseteq \ker(\alpha) \oplus M_\alpha$ and $(\ker(\alpha) \oplus M_\alpha) \cap W = \{0\}$ implies that $H \cap V = \{0\}$, a contradiction.

If d is odd, then $\text{ad}(h_\alpha)$ has an eigenvector $0 \neq v \in V$ whose eigenvalue is 1. By 9.2.4, $\alpha(h_\alpha) = 2$. The linear map $\text{ad}(h) : L_{c\alpha} \subseteq U_\alpha L_{c\alpha} \subseteq U_\alpha$ by definition satisfies the equation $\text{ad}(h_\alpha)(w) = c\alpha(h_\alpha)w = 2cw$. When $v = w$, its eigenvalue is 1 and this implies that $c = 1/2$ so that both $\alpha/2$ and α are roots. I claim that this is a contradiction. By 9.2.4, $\frac{\alpha}{2}(h_{\alpha/2}) = 2$ so that $\text{ad}(\alpha/2)$ has eigenvalue 2 on $L_\alpha \subseteq U_{\alpha/2}$. Then the eigenvalues of $\text{ad}(h_{\alpha/2}$ on $\ker(\alpha/2) \oplus M_{\alpha/2}$ is 0 and ± 2. This implies that W_d is a submodule of W for some even d, and by the even case this is impossible. Hence we reached a contradiction.

The fact that $W = \{0\}$ then implies that $U_\alpha = \ker(\alpha) \oplus M_\alpha$ as required. □

Proposition 10.3.2

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let Φ be the set of roots relative to some Cartan sub-algebra of L. Let $\alpha \in \Phi$. If $c\alpha \in \Phi$, then $c = \pm 1$.

Proof. We know that $H \oplus \bigoplus_{c\alpha} L_{c\alpha} = \ker(\alpha) \oplus M_\alpha$. Analyzing the right hand side, the fact that $\alpha(h_\alpha) = 2$ implies that the only eigenvalues for $\text{ad}(h_\alpha)$ on $\ker(\alpha) \oplus M_\alpha$ can only be 0 or ± 2. Analyzing the left hand side, the eigenvalue of $\text{ad}(h_\alpha)$ on H is 0 and the eigenvalue of $\text{ad}(h)$ on $L_{c\alpha}$ is $c\alpha$ by construction. This proves that $c = \pm 1$. □

Proposition 10.3.3

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let Φ be the set of roots relative to some Cartan sub-algebra of L. Let $\alpha \in \Phi$. Then we have

$$\dim(L_\alpha) = 1$$

Proof. We have that

$$\dim(H) + 2 = \dim\left(H \oplus L_\alpha \oplus L_{-\alpha}\right) = \dim(\ker(\alpha) \oplus M_\alpha) = \dim(\ker \alpha) + 3$$

$\dim(\ker(\alpha) = 1$ since $\alpha : H \to \mathbb{C}$ is a root. Since both L_α and $L_{-\alpha}$ are non-trivial vector spaces, we are left with $\dim(L_\alpha) + \dim(L_{-\alpha}) = 2$ which implies that $\dim(L_\alpha) = 1$. □

10.4 Root Strings

Definition 10.4.1: Root Strings

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let Φ be a root system of L. Let $\alpha \in \Phi$ and $\beta \in \Phi \cup \{0\}$. Define the α-root string through β to be

$$S_{\alpha,\beta} = \bigoplus_{\substack{c \in \mathbb{Z} \\ \beta + c\alpha \in \Phi}} L_{\beta + c\alpha}$$

Lemma 10.4.2

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let Φ be a root system of L. Let $\alpha, beta \in \Phi$ such that $\beta \neq \pm\alpha$. Then the following are true.

- There exists $k_1, k_2 \in \mathbb{Z}$ such that for any $c \in \mathbb{Z}$, $\beta + c\alpha \in \Phi$ if and only if $k_2 < c < k_1$. Moreover, $\beta(h_\alpha) = k_1 - k_2$.
- $\beta - \beta(h_\alpha)\alpha \in \Phi$.

Proposition 10.4.3

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let Φ be a root system of L. Let $\alpha \in \Phi$ and $\beta \in \Phi \cup \{0\}$. Then $S_{\alpha,\beta}$ is an irreducible $M_\alpha \cong \mathfrak{sl}_2(\mathbb{C})$-module.

Example 10.4.4

Let $\varepsilon_i \in H^*$ be the map defined by $\varepsilon_i(e_{j,j}) = \delta_{i,j}$. Recall that a root system of $\mathfrak{sl}_3(\mathbb{C})$ is given by $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq 3\}$ and a base for the root system is given by

$$\{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3\}$$

- The α_2-root string through α_1 is given by
$$L_{\alpha_1} \oplus L_{\alpha_1+\alpha_2} = \mathbb{C}\langle e_{1,2}, e_{1,3}\rangle$$
- The $-\alpha_1$-root string through $-\alpha_2$ is given by
$$L_{-\alpha_2} \oplus L_{-\alpha_2-\alpha_1} = \mathbb{C}\langle e_{3,2}, e_{3,1}\rangle$$
- The α_1-root string through 0 is given by
$$L_{-\alpha_1} \oplus L_0 \oplus L_{-\alpha_1} = \mathbb{C}\langle e_{1,1} - e_{2,2}, e_{1,2}, e_{2,1}\rangle$$

10.5 The Set of Roots as a Root System

Proposition 10.5.1

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let H be a Cartan subalgebra of L. Let Φ be the set of roots relative to H. Let $\alpha_1, \dots, \alpha_r \in \Phi$ be a basis for H^*. Then $\mathbb{R}[\alpha_1, \dots, \alpha_r] \subset H^*$ is an inner product space whose symmetric bilinear form is given by

$$B(\phi, \psi) = k(t_\phi, t_\psi)$$

(as in the notation of 9.2.1).

Proposition 10.5.2

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let H be a Cartan subalgebra of L. Let Φ be the set of roots relative to H. Let $\alpha_1, \dots, \alpha_r \in \Phi$ be a basis for H^*. Then Φ is a root system for the inner product space $(\mathbb{R}[\alpha_1, \dots, \alpha_r], B)$.

Proposition 10.5.3

Let L be a semi-simple Lie algebra over $\mathbb{C}$. Let H and K be two Cartan subalgebras of L. Let Φ_H and Φ_K be the set of roots relative to H and K respectively. Then $\Phi_H \cong \Phi_K$.

This proves that although the set of roots come from a chosen Cartan subalgebra of the Lie algebra L, it is an invariant of L and not the Cartan subalgebra. Up until this point, we have constructed an invariant for Lie algebras following the below schematic:

$$\begin{tikzcd}
 & & & \text{Associated Weyl Group} \\
\text{Semi-Simple Lie Algebras} \arrow[r] & \text{Set of Cartan Sub-Algebras} \arrow[r] & \text{Associated Root System} \arrow[ur] \arrow[dr] & \\
 & & & \text{Dynkin Diagram}
\end{tikzcd}$$

We will now show that Simple Lie algebras give rise to irreducible Dynkin diagrams and vice versa.

Example 10.5.4

The following are true regarding $\mathfrak{sl}_n(\mathbb{C})$.

- A Cartan sub-algebra of $\mathfrak{sl}_n(\mathbb{C})$ is given by $H = \mathfrak{d}_n(\mathbb{C}) \cap \mathfrak{sl}_n(\mathbb{C})$
- Let $\varepsilon_i \in H^*$ be the map defined by $\varepsilon_i(e_{j,j}) = \delta_{i,j}$. Then the set of roots of L with respect to H is given by
$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n\}$$
- A copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_n(\mathbb{C})$ is given by
$$\mathbb{C}\langle e_{i,j}, e_{j,i}, e_{i,i} - e_{j,j}\rangle$$
for $i \neq j$.
- A base for Φ is given by
$$\{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\}$$

10.6 The Classification Theorem

Proposition 10.6.1

Let L be a semisimple Lie algebra over $\mathbb{C}$. Let Φ be a root system of L with base $B = \{\alpha_1, \dots, \alpha_r\}$. Let $\mathfrak{sl}_2(\mathbb{C}) \overset{\Phi_i}{\cong} M_{\alpha_i}$ be the corresponding Lie subalgebra of L generated by the elements
$$e_i = \Phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \quad f_i = \Phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \quad h_i = \Phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$
Then the following are true.

- $L = \mathbb{C}\langle e_i, f_i \mid 1 \leq i \leq r\rangle$.
- $[h_s, h_t] = 0$ for $1 \leq s, t \leq l$.
- $[h_s, e_t] = \langle \alpha_t, \alpha_s\rangle e_t$ for $1 \leq s, t \leq l$.
- $[h_s, f_t] = -\langle \alpha_t, \alpha_s\rangle f_t$ for $1 \leq s, t \leq l$.
- $[e_s, f_t] = \begin{cases} h_s & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$
- $(\text{ad}(e_s))^{1-\langle \alpha_t, \alpha_s\rangle}(e_t) = 0 = (\text{ad}(f_s))^{1-\langle \alpha_t, \alpha_s\rangle}(f_t)$ for $1 \leq s, t \leq l$.

Theorem 10.6.2: Serre's Theorem

Let R be a root system over $\mathbb{R}^n$ with base $B = \{\alpha_1, \dots, \alpha_l\}$. Let L be the Lie algebra defined by
$$L = \mathbb{C}\langle e_i, f_i, h_i, 1 \leq i \leq l \mid S\rangle$$
where S are the following relations:

- $[h_s, h_t] = 0$ for $1 \leq s, t \leq l$.
- $[h_s, e_t] = \langle \alpha_t, \alpha_s\rangle e_t$ for $1 \leq s, t \leq l$.

- $[h_s, f_t] = -\langle \alpha_t, \alpha_s \rangle f_t$ for $1 \leq s, t \leq l$.
- $[e_s, f_t] = \begin{cases} h_s & \text{if } s = t \\ 0 & \text{otherwise} \end{cases}$
- $(\text{ad}(e_s))^{1-\langle \alpha_t, \alpha_s \rangle}(e_t) = 0 = (\text{ad}(f_s))^{1-\langle \alpha_t, \alpha_s \rangle}(f_t)$ for $1 \leq s, t \leq l$.

Then the following are true.
- L is finite dimensional and semi-simnple.
- $H = \mathbb{C}\langle h_1, \dots, h_l \rangle$ is a Cartan sub-algebra of L.
- The root system Φ_H of H is equal to R.

Corollary 10.6.3

There is a bijection

$$\frac{\text{Semisimple Lie algebras over } \mathbb{C}}{\cong} \overset{1:1}{\longleftrightarrow} \frac{\text{Root Systems over } \mathbb{R}^n \text{ for all } n}{\cong}$$

given as follows.
- For a semisimple Lie algebra over $\mathbb{C}$, choose a Cartan sub-algebra H of L. Then H has an associated root system.
- For a root system R, Serre's theorem gives a semisimple Lie algebra over $\mathbb{C}$.

Proof. The forward map is well defined since the associated root system is independent of the chosen Cartan sub-algebra by 9.3.4. The backwards map is well defined by construction.

Let L be a semisimple Lie algebra over $\mathbb{C}$. Then using its root system, we can construct a new semisimple Lie algebra over $\mathbb{C}$ by Serre's theorem. We need to show that these two semisimple Lie algebras over $\mathbb{C}$ are isomorphic.

Given a root system R, Serre's theorem gives a semisimple Lie algebra over $\mathbb{C}$ whose associated root system is equal to R. Hence the backward maps composed with the forward map is the identity. $\square$

10.7 Simple Lie Algebras and Irreducible Dynkin Diagrams

We now want to enrich the above diagram:

$$\begin{tikzcd}
 & & & \text{Associated Weyl Group} \\
\text{Semi-Simple Lie Algebras} \arrow[r] & \text{Set of Cartan Sub-Algebras} \arrow[r] & \text{Associated Root System} \arrow[ur] \arrow[dr] & \\
 & & & \text{Dynkin Diagram} \\
\text{Simple Lie Algebras} \arrow[rr] & & \text{Irreducible Root System} \arrow[r] & \text{Connected Dynkin Diagram}
\end{tikzcd}$$

Proposition 10.7.1

Let L be a simple Lie algebra over $\mathbb{C}$. Let H be a Cartan sub-algebra of L. Let Φ be the root system associated to H. Then Φ is irreducible.

Corollary 10.7.2

Let Φ be an irreducible root system. Then there exists a simple Lie algebra over $\mathbb{C}$ such that its associated root system is Φ.

Root System	Rank	Simple Lie Algebra	Dimension	Number of Roots
$A_n \quad (n \geq 1)$	n	$\mathfrak{sl}_{n+1}(\mathbb{C})$	$n^2 + 2n$	$n^2 + n$
$B_n \quad (n \geq 2)$	n	$\mathfrak{so}_{2n+1}(\mathbb{C})$	$2n^2 + n$	$2n^2$
$C_n \quad (n \geq 3)$	n	$\mathfrak{sp}_{2n}(\mathbb{C})$	$2n^2 + n$	$2n^2$
$D_n \quad (n \geq 4)$	n	$\mathfrak{so}_{2n}(\mathbb{C})$	$2n^2 - n$	$2n^2 - 2n$
G_2	2	$\mathfrak{g}_2(\mathbb{C})$	14	12
F_4	4	$\mathfrak{f}_4(\mathbb{C})$	52	48
E_6	6	$\mathfrak{e}_6(\mathbb{C})$	78	72
E_7	7	$\mathfrak{e}_7(\mathbb{C})$	133	126
E_8	8	$\mathfrak{e}_8(\mathbb{C})$	248	240

Corollary 10.7.3

There are no semisimple Lie algebras of dimension $1, 2, 4, 5, 7$.

Another table of simple Lie algebras, indexed by the dimension.

Dimension	Simple Lie Algebras	Root System
3	$\mathfrak{sl}_2(\mathbb{C})$	A_1
8	$\mathfrak{sl}_3(\mathbb{C})$	A_2
10	$\mathfrak{so}_5(\mathbb{C})$	B_2
14	$\mathfrak{g}_2(\mathbb{C})$	G_2
15	$\mathfrak{sl}_4(\mathbb{C})$	A_3
21	$\mathfrak{so}_7(\mathbb{C}), \mathfrak{sp}_6(\mathbb{C})$	B_3, C_3
24	$\mathfrak{sl}_5(\mathbb{C})$	A_4
28	$\mathfrak{so}_8(\mathbb{C})$	D_4

Finally, a table of semisimple Lie algebras.

Dimension	Lie Algebras	Total Number
3	$\mathfrak{sl}_2(\mathbb{C})$	1
6	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 2}$	1
8	$\mathfrak{sl}_3(\mathbb{C})$	1
9	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 3}$	1
10	$\mathfrak{so}_5(\mathbb{C})$	1
11	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})$	1
12	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 4}$	1
13	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_5(\mathbb{C})$	1
14	$\mathfrak{g}_2(\mathbb{C}), \quad \mathfrak{sl}_2(\mathbb{C})^{\oplus 2} \oplus \mathfrak{sl}_3(\mathbb{C})$	2
15	$\mathfrak{sl}_4(\mathbb{C}), \quad \mathfrak{sl}_2(\mathbb{C})^{\oplus 5}$	2
16	$\mathfrak{sl}_3(\mathbb{C})^{\oplus 2}, \quad \mathfrak{sl}_2(\mathbb{C})^{\oplus 2} \oplus \mathfrak{so}_5(\mathbb{C})$	2
17	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 3} \oplus \mathfrak{sl}_3(\mathbb{C}), \quad \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{g}_2(\mathbb{C})$	2
18	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 6}, \quad \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{so}_5(\mathbb{C}), \quad \mathfrak{sl}_4(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$	3
19	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C})^{\oplus 2}, \quad \mathfrak{sl}_2(\mathbb{C})^{\oplus 3} \oplus \mathfrak{so}_5(\mathbb{C})$	2
20	$\mathfrak{sl}_2(\mathbb{C})^{\oplus 4} \oplus \mathfrak{sl}_3(\mathbb{C}), \quad \mathfrak{sl}_2(\mathbb{C})^{\oplus 2} \oplus \mathfrak{g}_2(\mathbb{C}), \quad \mathfrak{so}_5(\mathbb{C})^{\oplus 2}$	3