Selected Topics

Labix

February 9, 2025

Abstract

Contents

1 Excisive Functors between Spaces

1.1 Homotopy Pushouts and Homotopy Pullbacks

Why we want this: pushouts dont preserve homotopies, as with any limits / colimts (therefore we have homotopy limits / colimits in model category)

Definition 1.1.1: Standard Model for Homotopy Pushouts

Definition 1.1.2: Standard Model for Homotopy Pullbacks

Definition 1.1.3: Homotopy Pushouts

Definition 1.1.4: Homotopy Pullbacks

Example 1.1.5

Suspension and loopspace.

Proposition 1.1.6

Definition 1.1.7: Excisive Functors

1.2 The Failure of the Identity Functor to be Excisive

1.3 Excisive Functors Coming From Spectra

2 Spectra as Reduced and Excisive Functors

2.1 Stable Infinity Categories

Definition 2.1.1: Infinity Pushouts

Let $\mathcal{C}$ be an infinity category. Let $F : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a morphism of simplicial sets. Let $X \in \mathcal{C}$ be an object. We say that X is a pushout in $\mathcal{C}$ if there exists a natural transformation $u : \Delta X \Rightarrow F$ such that there is a homotopy equivalence of Kan complexes:

Definition 2.1.2: Infinity Pullbacks

Why are these the correct analogue?

Definition 2.1.3: Stable Infinity Categories

Example in mind: spectra in ordinary categories: pushout=pullback.

Definition 2.1.4: Excisive Functors

2.2 Suspension and Loop Functors

Own notes: Higher algebra 1.4

trivial kan fibration -> section (Kerodon 1.5.5.5)

2.3 Stable Infinity Categories

Recall that $\mathcal{S} = N_\bullet^{\text{hc}}(\mathbf{Top}_*)$ is the infinity category of spaces.

Proposition 2.3.1

Let $\mathcal{C}$ be a pointed infinity category that admits all finite colimits. Then $\text{Exc}_*(\mathcal{C}, \mathcal{S})$ is stable.

Proof. Let $F : \mathcal{C} \to \mathcal{S}$ be excisive and reduced. Then $\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F) = F \circ \Sigma_{\mathcal{C}}$. By definition of the suspension functor,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma_{\mathcal{C}}(X) \end{tikzcd}$$

is a pushout in $\mathcal{C}$. Since F is excisive,

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & (F \circ \Sigma_{\mathcal{C}})(X) \end{tikzcd}$$

is a pullback in $\mathcal{S}$. On the other hand, $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F) = \Omega_{\mathcal{S}} \circ F$. By definition of the loop functor,

$$\begin{tikzcd}
(\Omega_{\mathcal{S}} \circ F \circ \Sigma_{\mathcal{C}})(X) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & (F \circ \Sigma_{\mathcal{C}})(X)
\end{tikzcd}$$

is a pullback in $\mathcal{S}$ for any $X \in \mathcal{C}$. Therefore $F(X)$ and $(\Omega_{\mathcal{S}} \circ F \circ \Sigma_{\mathcal{C}})(X)$ are equivalent. Hence F and $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F))$ are equivalent. $\square$

Theorem 2.3.2

There is an equivalence of infinity categories

$$\text{Sp}(\mathcal{S}) \simeq \lim(\cdots \to \mathcal{S} \xrightarrow{\Omega} \mathcal{S} \xrightarrow{\Omega} \mathcal{S}) =: \overline{\mathcal{S}}$$

induced by the evaluation map $\text{ev}_{S^0} : \overline{\mathcal{S}} \to \mathcal{S}$.

Proof.
Since $\mathcal{S}$ is presentable and the infinity category of presentable infinity categories admit all small limits, $\overline{\mathcal{S}}$ is also presentable. Every presentable infinity category admits all small limits and colimits. Since $\mathcal{S}$ is pointed, $\overline{\mathcal{S}}$ is also pointed. Since all limits are computed term-wise, we have that in particular $\Omega_{\overline{\mathcal{S}}}$ is computed term wise. given $\{X_n \mid n \in \mathbb{N}\}$ an object of $\overline{\mathcal{S}}$, $\{\Omega X_n \mid n \in \mathbb{N}\}$ is equivalent to $\{X_n \mid n \in \mathbb{N}\}$ because we have that ΩX_{n+1} is equivalent to X_n for all n. By a prp we conclude that $\overline{\mathcal{S}}$ is stable.

Consider the canonical functor $G : \overline{\mathcal{S}} \to \mathcal{S}$ defined by recovering the first factor: $(X_0, X_1, \dots) \mapsto X_0$. It is clear that it commutes with finite limits since limits are computed term-wise.

Let $\mathcal{C}$ be an arbitrary stable infinity category. Any functor $\mathcal{C} \to \mathcal{S}$ is left exact if and only if it is exact so that $\text{Exc}_*(\mathcal{C},\mathcal{S}) = \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$. 1.4.2.16 implies that $\text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$ is a stable infinity category. Thus $\Omega_{\mathcal{S}} \circ -$ is an equivalence.

On the other hand, since Ω are computed term-wise (like all limits) and since $\text{Func}(\mathcal{C},\overline{\mathcal{S}})$ is right adjoint to products we know that Func commutes with finite limits . Thus we have that

$$\text{Exc}_*^{\text{L}}(\mathcal{C},\overline{\mathcal{S}}) = \lim(\cdots \to \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S}) \xrightarrow{\Omega\circ-} \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S}) \xrightarrow{\Omega\circ-} \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S}))$$

Since each $\Omega_{\overline{\mathcal{S}}} \circ -$ is an equivalence of infinity categories, we conclude that $\text{Exc}_*^{\text{L}}(\mathcal{C},\overline{\mathcal{S}}) \simeq \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$. Thus evaluation on the first factor $G \circ - : \text{Exc}_*^{\text{L}}(\mathcal{C},\overline{\mathcal{S}}) \to \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S})$ is an equivalence of infinity categories.

By a previous corollary, there is an equivalence of infinity categories given by

$$\Omega^\infty \circ - : \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}},\text{Sp}(\mathcal{S})) \to \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}},\mathcal{S})$$

The fact that G is left exact means that there is a factorization

$$\begin{tikzcd}
\overline{\mathcal{S}} \arrow[rr, "G"] \arrow[dr, "G' \circ -"'] & & \mathcal{S} \\
& \text{Sp}(\mathcal{S}) \arrow[ur, "\Omega^\infty"'] &
\end{tikzcd}$$

By functoriality we obtain a similar factorization:

$$\begin{tikzcd}
\text{Exc}_*^{\text{L}}(\mathcal{C},\overline{\mathcal{S}}) \arrow[rr, "G\circ-"] \arrow[dr, "G' \circ -"'] & & \text{Exc}_*^{\text{L}}(\mathcal{C},\mathcal{S}) \\
& \text{Exc}_*^{\text{L}}(\mathcal{C},\text{Sp}(\mathcal{S})) \arrow[ur, "\Omega^\infty \circ -"'] &
\end{tikzcd}$$

Since $G \circ -$ and $\Omega^\infty \circ -$ are both equivalence of infinity categories, we conclude that $G' \circ -$ is an equivalence of infinity categories.

Since this is true for all stable infinity categories, the fact that

$$\text{Exc}_*(\mathcal{C}, \overline{\mathcal{S}}) = \text{Exc}_*^L(\mathcal{C}, \overline{\mathcal{S}}) \simeq \text{Exc}_*^L(\mathcal{C}, \text{Sp}(\mathcal{S})) = \text{Exc}_*(\mathcal{C}, \text{Sp}(\mathcal{S}))$$

is an equivalence for all stable $\mathcal{C}$ together with the Yoneda embedding implies that $\overline{\mathcal{S}}$ and $\text{Sp}(\mathcal{S})$ is an equivalence of infinity categories. □

Beware that in the proof we also showed that $G \circ -$ is an equivalence of infinity categories for any stable infinity category $\mathcal{C}$. But this does not imply that $\overline{\mathcal{S}}$ and $\mathcal{S}$ are equivalent because we are applying the Yoneda embedding on the category of stable infinity categories, and a priori $\mathcal{S}$ is not stable.

3 From Functors to Excisive Functors

3.1 Goodwillie Calculus

Definition 3.1.1

T1 and P1

Theorem 3.1.2

P1 is excisive.

3.2 Excisive Approximations

Example 3.2.1

Id -> Infinite loop suspension

3.3 Spectra and (Co)Homology Theories

Theorem 3.3.1: Brown's Representability Theorem

Definition 3.3.2: Cohomology Theory Associated to Spectra

Definition 3.3.3: Spectra Associated to Cohomology Theory

Example 3.3.4: Singular Cohomology

Example 3.3.5: K theory

Example 3.3.6: Landweber-exact Spectra

Theorem 3.3.7: Landweber exact functor theorem

3.4 A Map From Functors to (Co)Homology Theories

Example 3.4.1

Identity Functor -> stable homotopy theory (it is a homology theory)

Example 3.4.2

Excisive functor F -> F(Sn) -> corresponding cohomolog theory