Classical Homotopy Theory

Labix

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Abstract

- Notes on Algebraic Topology by Oscar Randal-Williams

Contents

1 A Convenient Category of Spaces

Reason:

- Want $- \wedge -$ associative and unital and commutative (so that the category is symmetric monoidal)

- Want adjunction $- \times X$ and $\mathrm{Hom}_{\mathcal{C}}(X, -)$ (non pointed)

- Want adjunction $- \wedge X$ and $\mathrm{Map}_*(X, -)$ (pointed) (Intuitively, $X \wedge Y$ represents maps from $X \times Y$ that are base point preserving separately in each variable)

1.1 Compactly Generated Spaces

Definition 1.1.1: Compactly Generated Spaces

Let X be a space. We say that X is compactly generated (k-space) if for every set $A \subseteq X$, A is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Definition 1.1.2: Category of Compactly Generated Spaces

Define the category of compactly generated spaces **CG** to be the full subcategory of **Top** consisting of spaces that are compactly generated. In other words, **CG** consists of the following data:
- $\mathrm{Obj}(\mathbf{CG})$ consists of all spaces that are compactly generated.
- For $X, Y \in \mathrm{Obj}(\mathbf{CG})$, the morphisms are

$$\mathrm{Hom}_{\mathbf{CG}}(X, Y) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces $\mathbf{CG}_*$.

Definition 1.1.3: New k-space from Old

Let X be a space. Define $k(X)$ to be the set X together with the topology defined as follows: $A \subseteq X$ is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Lemma 1.1.4

Let X be a space. Then $k(X)$ is a compactly generated space.

Unfortunately $X \times Y$ may not be compactly generated even when X and Y are. But as it turns out, products do exists in **CG** and are given by $X \times_{\mathbf{CG}} Y = k(X \times_{\mathbf{Top}} Y)$.

Proposition 1.1.5

Let X, Y be compactly generated spaces. Then the categorical product of X and Y in the category of compactly generated spaces is given by

$$X \times_{\mathbf{CG}} Y = k(X \times_{\mathbf{Top}} Y)$$

Proposition 1.1.6

Every CW complex is compactly generated.

Definition 1.1.7: Category of Compactly Generated and Weakly Hausdorff Spaces

Define the category of compactly generated and weakly Hausdorff spaces **CGWH** to be the full subcategory of **Top** consisting of spaces that are compactly generated and weakly Hausdorff. In other words, **CGWH** consists of the following data:
- Obj(**CGWH**) consists of all spaces that are compactly generated and weakly Hausdorff.
- For $X, Y \in \text{Obj}(\mathbf{CGWH})$, the morphisms are

$$\text{Hom}_{\mathbf{CGWH}}(X, Y) = \text{Hom}_{\mathbf{Top}}(X, Y)$$

- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces **CGWH$_*$**.

Proposition 1.1.8

A compactly generated space X is weakly Hausdorff if and only if the diagonal subspace $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Proposition 1.1.9

Product of CGWH is CGWH

CGWH is complete and cocomplete

1.2 The Cartesian Product and the Mapping Space

Definition 1.2.1: The Mapping Space

Let $X, Y \in \mathbf{CG}$. Define the mapping space of X and Y by

$$\text{Map}(X, Y) = k(\text{Hom}_{\mathbf{CG}}(X, Y))$$

where $\text{Hom}_{\mathbf{CG}}(X, Y)$ is equipped with the compact open topology. If (X, x_0) and (Y, y_0) are pointed spaces, define the mapping space to be

$$\text{Map}_*((X, x_0), (Y, y_0)) = k(\text{Hom}_{\mathbf{CG}}((X, x_0), (Y, y_0)))$$

By restricting to also weakly Hausdorff spaces, we obtain an adjunction.

Theorem 1.2.2

Let $X, Y, Z \in \mathbf{CGWH}$. Then the functors $- \times_{\mathbf{CGWH}} Y : \mathbf{CGWH} \to \mathbf{CGWH}$ and $\text{Map}(Y, -) : \mathbf{CGWH} \to \mathbf{CGWH}$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathbf{CGWH}}(X \times_{\mathbf{CGWH}} Y, Z) \cong \text{Hom}_{\mathbf{CGWH}}(X, \text{Map}(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism
$$\text{Map}(X \times_{\mathbf{CGWH}} Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

1.3 The Smash Product and the Pointed Mapping Space

Aside from the adjunction between the product space and the mapping space, another major reason one considers compactly generated spaces is that the smash product gives another adjunction.

Definition 1.3.1: The Smash Product

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Define the smash product of the two pointed spaces to be the pointed space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

together with the point (x_0, y_0).

Proposition 1.3.2

Let X, Y, Z be compactly generated spaces with a chosen base point. Then the following are true.
- $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
- $X \wedge Y \cong Y \wedge X$

Theorem 1.3.3

The category $\mathbf{CG}$ of compactly generated spaces is a symmetric monoidal category with operator the smash product $\wedge : \mathbf{CG} \times \mathbf{CG} \to \mathbf{CG}$ and the unit S^0.

Note that this is not true if we do not restrict the spaces to the category of compactly generated spaces.

Lemma 1.3.4

Let X be a pointed space. Then the reduced suspension and the smash product with the circle

$$\Sigma X \cong X \wedge S^1$$

are homeomorphic spaces.

Theorem 1.3.5

Let X, Y, Z be compactly generated with a chosen basepoint. Then the functors $- \wedge Y : \mathcal{K}_* \to \mathcal{K}_*$ and $\mathrm{Map}_*(Y, -) : \mathcal{K}_* \to \mathcal{K}_*$ are adjoint functors with the adjunction formula

$$\mathrm{Hom}_{\mathcal{K}_*}(X \wedge Y, Z) \cong \mathrm{Hom}_{\mathcal{K}_*}(X, \mathrm{Map}_*(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\mathrm{Map}_*(X \wedge Y, Z) \cong \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

By choosing $Y = I$ in the adjunction, we recover the usual suspension-loopspace adjunction in $\mathbf{Top}_*$.

Corollary 1.3.6

Let X be a compactly generated space with a chosen basepoint. Then there is a natural homeomorphism

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

given by adjunction of the functors $- \wedge S^1 : \mathcal{K}_* \to \mathcal{K}_*$ and $\mathrm{Map}_*(S^1, -) : \mathcal{K}_* \to \mathcal{K}_*$.

1.4 The Mapping Cylinder and the Mapping Path Space

Equipped with the Cartesian closed structure in $\mathbf{CG}$ together with a canonical topology on the mapping space Y^X, we can now talk about the duality between the mapping cylinder and the mapping path space.

Definition 1.4.1: Mapping Cylinder

Let X, Y be spaces and let $f : X \to Y$ a map. Define the mapping cylinder of f to be

$$M_f = \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)} = (X \times I) \amalg_f Y$$

for $f : X \times \{1\} \cong X \to Y$ together with the quotient topology. It is the push forward of f and the inclusion map $i_0 : X \cong X \times \{0\} \hookrightarrow X \times I$.

Lemma 1.4.2

Let X, Y be spaces and let $f : X \to Y$ be a map. Then Y is a deformation retract of M_f.

Definition 1.4.3: The Mapping Path Space

Let X, Y be spaces and let $f : X \to Y$ be a map. Define the mapping path space of f to be

$$P_f = \{(x, \gamma) \in X \times \text{Map}(I, Y) \mid \gamma(0) = f(x)\}$$

It is the pull back of f and $\pi_0 : \text{Map}(I, Y) \to Y$ given by $\pi_0(\gamma) = \gamma(0)$ in **CG**.

2 (Co)Fibers of (Co)Fibrations

2.1 The Relative Point of View

Definition 2.1.1: Fibers of a Map

Let X, Y be spaces. Let $f : X \to Y$ be a map. Define the fiber of f at $y \in Y$ to be

$$\text{Fib}_y(f) = f^{-1}(y)$$

Definition 2.1.2: Cofibers of a Map

Let X, Y be spaces. Let $f : X \to Y$ be a map. Define the cofiber of f to be

$$\text{Cofib}(f) = \frac{Y}{f(X)}$$

Up until this point, in algebraic topology we have asked questions relating to two spaces and tried to answer them. For instance, we can ask whether two spaces are homeomorphic, homotopy equivalent or weakly equivalent. We can also ask these questions in a relative setting, this involves considering maps of spaces as objects themselves, instead of just the spaces.

Definition 2.1.3: Maps of Maps

Let X, Y, A, B be spaces. Let $f : X \to Y$ and $g : A \to B$ be maps. A map from f to g is a pair of maps $(\alpha : X \to A, \beta : Y \to B)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\alpha \downarrow & & \downarrow \beta \\
A & \xrightarrow{g} & B
\end{array}
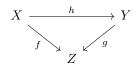$$

Definition 2.1.4: Homotopy from Maps to Maps

Let X, Y, A, B be spaces. Let $f : X \to Y$ and $g : A \to B$ be maps. Let (a, b) and (c, d) be two maps from f to g. We say that (a, b) and (c, d) are homotopic if there exists maps $H : X \times I \to Y$ and $K : A \times I \to B$ such that the following diagram

$$
\begin{array}{ccc}
X \times I & \xrightarrow{f \times \text{id}_I} & Y \times I \\
H \downarrow & & \downarrow K \\
A & \xrightarrow{g} & B
\end{array}
$$

commutes and the following are true.
- H is a homotopy from a to c
- K is a homotopy from b to d

We now restrict to the point of view where maps are over a fixed space Z (In other words we are considering objects of $\textbf{Top}_Z$). A map of maps becomes the following data: Let $f : X \to Z$ and $g : Y \to Z$ be maps. A map from f to g is a map $h : X \to Y$ such that $g \circ h = f$. In other words the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{h} & Y \\
& f \searrow \quad \swarrow g & \\
& Z &
\end{array}
$$

A homotopy then becomes the following data: Let $f : X \to Z$ and $g : Y \to Z$ be two maps. Let $h, k : X \to Y$ be two maps from f to g. We say that h and k are homotopic if there exists a homotopy $H : X \times I \to Y$ from h to k such that $H(-, t)$ for each t is a map over Z. This means that the following diagram commutes:

$$
\begin{array}{ccc}
X \times I & \xrightarrow{\quad H \quad} & Y \\
& {\scriptstyle f \circ p_X} \searrow \quad \swarrow {\scriptstyle g} & \\
& Z &
\end{array}
$$

We can now discuss homotopy equivalences on these maps.

Definition 2.1.5: Fiber Homotopy Equivalence

Let X, Y, Z be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. We say that f and g are homotopy equivalent if there exists two maps $h : X \to Y$ and $k : Y \to X$ such that $k \circ h$ and $h \circ k$ are both homotopic to the identity over Z.

The reason that it is called a fiber homotopy equivalence is because it gives homotopy equivalences on fibers.

Proposition 2.1.6

Let X, Y, Z be spaces. Let $f : X \to B$ and $g : Y \to B$ be maps. If f and g are homotopy equivalent, then for any $b \in B$, the fibers

$$
\mathrm{Fib}_b(f) \simeq \mathrm{Fib}_b(g)
$$

are homotopy equivalent.

- -

Proof. Suppose that f and g are homotopy equivalent via two maps $h : X \to Y$ and $k : Y \to X$. This means that there exists a homotopy $H : X \times I \to X$ such that $H(-, 0) = h \circ k$ and $H(-, 1) = \mathrm{id}_X$. Similarly, there exists a homotopy $K : Y \times I \to Y$ such that $K(-, 0) = k \circ h$ and $K(-, 1) = \mathrm{id}_Y$. Consider the map $h|_{f^{-1}(b)} : f^{-1}(b) \to g^{-1}(b)$ and similarly for $k|_{g^{-1}(b)}$. Define two maps $\overline{H} : H|_{f^{-1}(b) \times I}$ and $\overline{K} = K|_{g^{-1}(b) \times I}$. To show that they are homotopies, we just need to show that $\overline{H} \subseteq f^{-1}(b)$ and similarly for $\overline{K}$. Now by definition, each $H(-, t) : X \to Y$ is such that $g(H(-, t)) = f$. Choose $x \in f^{-1}(b)$. Then

$$
g(H(x, t)) = f(x) = b
$$

Hence the entire homotopy $\overline{H}$ stays in the fiber $f^{-1}(b)$. Hence $\overline{H}$ is a well defined homotopy on $f^{-1}(b)$. Similarly for $\overline{K}$. Hence the two fibers are homotopy equivalent. □

Unfortunately for most maps $f : X \to Y$, the fibers themselves are not homeomorphic, and not even homotopy equivalent.

Example 2.1.7

The fibers of the projection map $S^1 \to \mathbb{R}$ to the x-axis are not homotopy equivalent.

- -

Proof. It is clear that the fiber of $S^1 \to \mathbb{R}$ is either empty, consist of one point, or of two points. Neither two of the three are homotopy equivalent. □

There are two ways to proceed from here. We first try to find a set of maps in which all fibers are homotopy equivalent. This is the content of this section. Otherwise, we try and define a new notion of fiber so that we obtain homotopy equivalence. This is the content of the next section.
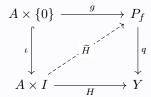
2.2   (Co)Fibration Replacements

Proposition 2.2.1

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f : X \to Y$ be a map. Then the map

$$q : P_f \to Y$$

given by $q(x, \gamma) = \gamma(1)$ is a fibration.

Proof. Suppose that we are given a homotopy lifting problem:

$$
\begin{array}{ccc}
A \times \{0\} & \xrightarrow{\ \ g\ \ } & P_f \\
{\scriptstyle \iota}\downarrow & \overset{\tilde{H}}{\nearrow} & \downarrow{\scriptstyle q} \\
A \times I & \xrightarrow[\ \ H\ \ ]{} & Y
\end{array}
$$

We write $g(a) = (g_1(a), g_2(a))$ for the components of g. Now recall that the definition of the mapping path space implies that $f(g_1(a)) = g_2(a)(0)$. By commutativity of the diagram and definition of q we also have $g_2(a)(1) = H(a, 0)$. Define $\tilde{H} : A \times I \to P_f$ by the fomula

$$\tilde{H}(a, t) = (g_1(a), h_2(a, t))$$

where

$$h_2(a, t)(s) = \begin{cases} g_2(a)(1 + t)(s) & \text{if } 0 \leq s \leq \frac{1}{1+t} \\ H(a, (1 + t)s - 1) & \text{if } \frac{1}{1+t} \leq s \leq 1 \end{cases}$$

The definition of h_2 makes sense because $g_2(a)(1) = H(a, 0)$. By the gluing lemma $h_2(a, t)$ is continuous. h_2 is also continuous in a and t because $g_2(a)$ is a path and g_2 is continuous and H is continuous in both variables and the composite of continuous functions are continuous. Hence $\tilde{H}$ is continuous. Now $\tilde{H}(-, 0) = (g_1(-), g_2(-)) = g(-)$. Thus $\tilde{H}(-, 0)$ is a lift of g. It remains to show that $\tilde{H}$ is a lift of H. We have that

$$
\begin{aligned}
q(\tilde{H}(a, t)) &= q(g_1(a), h_2(a, t)) \\
&= h_2(a, t)(1) \\
&= H(a, t)
\end{aligned}
$$

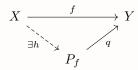and so we conclude. $\square$

We can factorize any continuous map into a fibration and a homotopy equivalence through the mapping path space. Because we are working with the mapping path space here, we need to restrict our attention to compactly generated space.

Theorem 2.2.2

Let $X, Y \in \mathbf{CGWH}$. Let $f : X \to Y$ be a map. Then there exists a homotopy equivalence $h : X \to P_f$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ f\ \ } & Y \\
{\scriptstyle \exists h}\searrow & & \nearrow{\scriptstyle q} \\
& P_f &
\end{array}
$$

where $q : P_f \to Y$ is the above defined map.

Proof. Define the map $h : X \to P_f$ by $h(x) = (x, e_{f(x)})$. It is easy to see that $q \circ h = f$. I claim

that the projection map $p_X : P_f \to X$ gives the homotopy inverse of h. Define a map $H : P_f \times I \to P_f$ by

$$H(x, \gamma, t) = (x, \gamma_t)$$

where γ_s is the path $s \mapsto \gamma(st)$. It is continuous since the composition of continuous functions are continuous and each component of H is continuous. Also, we have that $h(p_X(x, \gamma)) = h(x) = (x, e_{f(x)})$ and $H(x, \gamma, 0) = (x, \gamma_0) = (x, e_{f(x)})$ so that $H(-, 0) = h \circ p_X$. When $t = 1$ we also have

$$H(-, 1) = (x, \gamma, 1) = (x, \gamma_1) = (x, \gamma) = \mathrm{id}_{P_f}(x, \gamma)$$

so that H is a homotopy. $\qquad\square$

Proposition 2.2.3

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f : X \to Y$ be a map. Let $h : X \to P_f$ be the map that gives a factorization $q \circ h = f$. If f is a fibration, then h is a fiber homotopy equivalence.

Cofibrations and fibrations are dual in the following sense. Recall from section 1 that if X and Y are in **CGWH**, then there is a bijection

$$\mathrm{Hom}_{\mathbf{CGWH}}(X \times I, Y) \cong \mathrm{Hom}_{\mathbf{CGWH}}(X, \mathrm{Map}(I, Y))$$

Now under this bijection, we can rewrite the diagram in the homotopy lifting property:

$$
\begin{array}{ccc}
X & \longrightarrow & E^I \\
\mathrm{id}_X \downarrow & \overset{\exists \tilde{H}}{\nearrow} & \downarrow p_* \\
X & \underset{H}{\longrightarrow} & B^I
\end{array}
$$

Proposition 2.2.4

Let $A, X \in \mathbf{CGWH}$ be spaces. Let $f : A \to X$ be a map. Then the map

$$q : A \to M_f$$

given by $q(a) = [a, 0]$ is a cofibration.

- -

Proof. Suppose that we are given a homotopy lifting problem:

$$
\begin{array}{ccc}
A \cong A \times \{0\} & \xrightarrow{\ \iota_0\ } & A \times I \\
i \downarrow & & \downarrow i \times \mathrm{id}_I \qquad {\searrow}^{H} \\
X \cong X \times \{0\} & \xrightarrow[\ \iota_0\ ]{} & X \times I \\
& {\searrow}_f & \quad \dashrightarrow^{\tilde{H}} \qquad \downarrow \\
& & \qquad\qquad Y
\end{array}
$$

$\qquad\square$

Dual to the factorization through the mapping path space, we can factorize a map into a homotopy equivalence and a cofibration through the mapping cylinder

$$M_f = \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)} = (X \times I) \amalg_f Y$$

Theorem 2.2.5

Let $f : A \to X$ be a map. Then the inclusion map $i : A \to M_f$ defined by $i(a) = [a, 0]$ is a cofibration. Moreover, there exists a homotopy equivalence $h : M_f \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ f\ \ } & X \\
 & \searrow{\scriptstyle i} \quad \nearrow{\scriptstyle \exists h} & \\
 & M_f &
\end{array}
$$

2.3 Long Exact Sequences from (Co)Fibrations

Theorem 2.3.1: Homotopy Long Exact Sequence in Fibration

Let $p : E \to B$ be a fibration over a path connected space B with fiber F. Let $\iota : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\ \partial\ } \pi_n(F, e_0) \xrightarrow{\ \iota_*\ } \pi_n(E, e_0) \xrightarrow{\ p_*\ } \pi_n(B, b_0) \xrightarrow{\ \partial\ } \pi_{n-1}(F, e_0) \longrightarrow \cdots \longrightarrow \pi_1(E, e_0) \xrightarrow{\ p_*\ } \pi_1(B, b_0)$$

for $e_0 \in E$ and $b_0 = p(e_0)$. Moreover, p_* is an isomorphism.

Theorem 2.3.2: Homology Long Exact Sequence in Cofibration

Let $p : X \to Y$ be a cofibration with cofiber $C = \frac{Y}{p(X)}$. Let $\text{proj} : Y \to C$ be the projection map. Then there is a long exact sequence in homology groups:

$$\cdots \longrightarrow \widetilde{H}_{n+1}(C) \xrightarrow{\ \partial\ } \widetilde{H}_n(X) \xrightarrow{\ f_*\ } \widetilde{H}_n(Y) \xrightarrow{\ \text{proj}_*\ } \widetilde{H}_n(C) \xrightarrow{\ \partial\ } \widetilde{H}_{n-1}(X \longrightarrow \cdots \longrightarrow \widetilde{H}_0(Y) \xrightarrow{\ \text{proj}_*\ } \widetilde{H}_0(B, b_0)$$

2.4 (Co)Fibers of a (Co)Fibration are Homotopic

The following definition is a supporting notion for our proof that fibers of a fibration are homotopy equivalent.

Definition 2.4.1: Induced Map of Fibers

Let $p : E \to B$. Let $\gamma : I \to B$ be a path from b_1 to b_2. Define the induced map of fibers of γ as follows: The map $H : E_{b_1} \times I \to B$ defined by $H(x, t) = \gamma(t)$ is a homotopy. Using the HLP of p, we obtain a lift:

$$
\begin{array}{ccc}
E_{b_1} \times \{0\} & \xrightarrow{\ \widetilde{H(-,0)}\ } & E \\
\downarrow & \quad \nearrow{\scriptstyle \widetilde{H}} & \downarrow{\scriptstyle p} \\
E_{b_1} \times I & \xrightarrow{\ H\ } & B
\end{array}
$$

Since $p \circ \widetilde{H}(x, t) = \gamma(t)$, we have that $\widetilde{H}(x, 1) \in E_{b_2}$. The induced map of fibers is then the map

$$L_\gamma : E_{b_1} \to E_{b_2}$$

defined by $L_\gamma = \widetilde{H(-, 1)}$

Lemma 2.4.2

Let $p : E \to B$ be a fibration. Let $\gamma : I \to B$ be a path from b_1 to b_2. Then the following are true regarding L_γ.
- If $\gamma \simeq \gamma'$ relative to boundary, then $L_\gamma \simeq L_{\gamma'}$.
- If $\gamma : I \to B$ and $\gamma' : I \to B$ are two composable paths, there is a homotopy equivalence $L_{\gamma \cdot \gamma'} \simeq L_{\gamma'} \circ L_\gamma$

Proof.
- Let $F : I \times I \to B$ be a homotopy equivalence from γ to γ'. Now consider the map $G : E_{b_1} \times I \times I \to B$ defined by $G(x, s, t) = F(s, t)$. Notice that $G(x, s, 0) = F(s, 0) = \gamma(s)$ and $G(x, s, 1) = F(s, 1) = \gamma'(s)$. Thus, we proceed as above by lifting $G(x, s, 0)$ and $G(x, s, 1)$ to obtain respectively $\widetilde{G(x, s, 0)}$ and $\widetilde{G(x, s, 1)}$ for which $\widetilde{G(x, 1, 0)} = L_\gamma$ and $\widetilde{G(x, 1, 1)} = L_{\gamma'}$. Now define $K : E_{b_1} \times I \times \partial I \to E$ by

$$K(x, s, t) = \begin{cases} \widetilde{G(x, s, 1)} & \text{if } t = 0 \\ G(x, s, 1) & \text{if } t = 1 \end{cases}$$

We now obtain a homotopy called $\widetilde{G} : E_{b_1} \times I \times I \to E$ by the homotopy lifting property:

$$
\begin{array}{ccc}
X \times I \times \partial I & \xrightarrow{K} & E \\
\downarrow & \nearrow_{\widetilde{G}} & \downarrow{\scriptstyle p} \\
X \times I \times I & \xrightarrow{G} & B
\end{array}
$$

Now $\tilde{G}(-, 1, -) : E_b \times I \to E$ is then a homotopy equivalence from $\widetilde{G}(x, 1, 0) = L_\gamma$ to $\widetilde{G}(x, 1, 1) = L_{\gamma'}$.
- We can repeat the above construction for γ and γ' to obtain homotopies $G : E_{b_1} \times I \to E$ and $G' : E_{b_1} \times I \to E$ such that when $t = 1$ we recover $\tilde{\gamma}$, $\tilde{\gamma}'$ and $\gamma \tilde{\cdot} \gamma'$ respectively. Now the composition of G and G' by traversing along $t \in I$ with twice the speed gives precisely a lift of $\gamma \cdot \gamma'$ (one can check the boundary conditions). Thus $L_{\gamma \cdot \gamma'}$ obtained in this manner coincides up to homotopy equivalence to $L_{\gamma'} \circ L_\gamma$ by invoking part a). $\qquad\square$

Theorem 2.4.3

Let $p : E \to B$ be a fibration. Let b_1 and b_2 lie in the same path component of B. Then there is a homotopy equivalence

$$E_{b_1} \simeq E_{b_2}$$

given by the lift of any path $\gamma : I \to B$ from b_1 to b_2.

Proof. Let $\gamma : I \to B$ be a path from b_1 to b_2. From the above, it follows that $L_{\overline{\gamma}} \circ L_\gamma \simeq \mathrm{id}_{E_b}$ for any loop $\gamma : I \to B$ with basepoint b. We conclude that L_γ is a homotopy equivalence and so the fibers of $p : E \to B$ are homotopy equivalent. $\qquad\square$

3 Homotopy Fibers and Homotopy Cofibers

3.1 Basic Definitions

> **Definition 3.1.1: Homotopy Fibers and Cofibers**
>
> Let $f : X \to Y$ be a map. Define the homotopy fiber of f at $y \in Y$ to be
>
> $$\text{hofiber}_y(f) = \{(x, \phi) \in X \times \text{Map}(I, Y) \mid f(x) = \phi(0), \phi(1) = y\} = \text{Fib}_y(P_f \to Y)$$
>
> Define the homotopy cofiber of f to be
>
> $$\text{hocofiber} = \frac{(X \times I) \amalg Y}{(x, 1) \sim f(x), (x, 0) \sim (x', 0)} = \text{Cofib}(X \to M_f) = C_f$$

TBA: hofiber = pullback $P_f \to Y \leftarrow *$ (time $t = 1$ and $* \mapsto y$).

Since the map $P_f \to Y$ is a fibration, the fibers of $P_f \to Y$, and hence the homotopy fibers of f are all homotopy equivalent.

> **Proposition 3.1.2**
>
> Let $p : E \to B$ be a fibration. Then the there is a homotopy equivalence
>
> $$\text{Fib}_b(f) \simeq \text{Hofib}_b(f)$$
>
> for each $b \in B$, given by the inclusion map $x \mapsto (x, e_x)$.
>
> -
>
> *Proof.* Consider the following diagram
>
> $$
> \begin{array}{ccc}
> \text{Fib}_b(f) & \xrightarrow{\;\simeq\;} & \text{Hofib}_b(f) \\
> \downarrow & & \downarrow \\
> X & \xrightarrow{\;h, \simeq\;} & P_f \\
> {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\
> Y & \xrightarrow[\;\text{id}_Y\;]{} & Y
> \end{array}
> $$
>
> and apply 2.4.6 to conclude. $\square$

> **Proposition 3.1.3**
>
> Let X, Y be spaces. Let $f, g : X \to Y$ be maps. If f, g are homotopic, then for all $y \in Y$, there is a homotopy equivalence
> $$\text{Hofib}_y(f) \simeq \text{Hofib}_y(g)$$
> induced by the homotopy from f to g.

> **Example 3.1.4**
>
> Let X, Y be spaces. Let $f : X \to Y$ be a map.
> - The homotopy fiber of $\{y\} \hookrightarrow Y$ is given by $\text{Hofib}_y(\{y\} \hookrightarrow Y) \cong \Omega Y$
> - If f is null-homotopic, then the homotopy fiber of f is given by $\text{Hofib}_y(f) \cong X \times \Omega Y$

3.2 The Fiber and Cofiber Sequences

> **Definition 3.2.1: Path Spaces**
>
> Let (X, x_0) be a pointed space. Define the path space of (X, x_0) to be
> $$PX = \{\phi : (I, 0) \to (X, x_0) \mid \phi(0) = x_0\} = \mathrm{Map}_*((I, 0), (X, x_0))$$
> together with the topology of the mapping space.

> **Theorem 3.2.2**
>
> Let X be a space. Then the following are true.
> - The map $\pi : PX \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber ΩX
> - The map $\pi : X^I \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber homeomorphic to PX.

We now write a fibration as a sequence $F \to E \to B$ for F the fiber of the fibration $p : E \to B$. This compact notation allows the following theorem to be formulated nicely.

> **Theorem 3.2.3**
>
> Let $f : X \to Y$ be a fibration with homotopy fiber F_f. Let $\iota : \Omega Y \to F_f$ be the inclusion map and $\pi : F_f \to X$ the projection map. Then up to homotopy equivalence of spaces, there is a sequence
> $$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega F_f \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$
> where any two consecutive maps form a fibration. Moreover, $-\Omega f : \Omega X \to \Omega Y$ is defined as
> $$(-\Omega f)(\zeta)(t) = (f \circ \zeta)(1 - t)$$
> for $\zeta \in \Omega X$.

There is then the dual notion of loop spaces and the corresponding sequence. Write a cofibration $f : A \to X$ with homotopy cofiber B as $B \to A \to X$.

> **Theorem 3.2.4**
>
> Let $f : X \to Y$ be a cofibration with homotopy cofiber C_f. Let $i : Y \to C_f$ be the inclusion map and $\pi : C_f \to C_f / Y \cong \Sigma X$ be the projection map. Then up to homotopy equivalence of spaces, there is a sequence
> $$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$
> where any two consecutive maps form a cofibration. Moreover, $-\Sigma f : \Sigma X \to \Sigma Y$ is defined by
> $$(-\Sigma f)(x \wedge t) = f(x) \wedge (1 - t)$$

3.3 n-Connected Maps

> **Definition 3.3.1: n-Connected Maps**
>
> Let X, Y be pointed spaces. Let $f : X \to Y$ be a map. We say that f is n-connected if the induced map
> $$\pi_k(f) : \pi_k(X) \to \pi_k(Y)$$
> is an isomorphism for $0 \le k < n$ and a surjection for $k = n$.

We can rephrase some of the corner stone theorems of homotopy theory using n-connected maps.

- The homotopy excision theorem can be rephrased into the following. For X a CW-complex and A, B sub complexes of X such that $X = A \cup B$ and $A \cap B \neq \emptyset$. If $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected for $m, n \geq 0$, then the inclusion

$$\iota : (A, A \cap B) \to (X, B)$$

is $(m + n)$-connected.

- The Freudenthal suspension theorem says that if X is an n-connected CW complex, then the map

$$\Omega\Sigma : X \to \Omega(\Sigma(X))$$

is a $(2n + 1)$-connected map.

Proposition 3.3.2

Let X, Y be pointed spaces. Let $f : X \to Y$ be a map. Then the following are equivalent.
- f is k-connected.
- $\mathrm{Hofib}_y(f)$ is $(k - 1)$-connected for all $y \in Y$.
- (M_f, X) is a k-connected space.

Lemma 3.3.3

Let (X, x_0) be a pointed space. Then (X, x_0) is n-connected if and only if the inclusion map $\iota : \{x_0\} \hookrightarrow X$ is n-connected.

4 Homotopy Pullbacks and Pushouts

Homotopy pullbacks and pushouts are a special case of homotopy limits and colimits. It would be fruitful for us to first consider this case also because of how it is related to maps of spaces and (co)fibrations.

4.1 The Standard Homotopy Pullback and Pushout

Consider the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Z & \xleftarrow{\ g\ } & Y \\ {\scriptstyle e_X}\downarrow & & {\scriptstyle e_Z}\downarrow & & \downarrow{\scriptstyle e_Y} \\ X' & \xrightarrow{\ f'\ } & Z' & \xleftarrow{\ g'\ } & Y' \end{array}$$

We can form pullbacks of the upper and lower horizontals and obtain an induced map. However, when e_X, e_Y, e_Z are homotopy equivalences, the induced map is not a homotopy equivalence. We remedy this by introducing a homotopic notion of pullbacks.

Definition 4.1.1: The Standard Homotopy Pullback

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Define the standard homotopy pullback of f and g to be the subspace

$$\mathrm{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) = \{(x, \alpha, y) \in X \times \mathrm{Map}(I, Z) \times Y \mid \alpha(0) = f(x), \alpha(1) = g(y)\}$$

The idea is that normally in pullbacks, we require that under f and g the elements of the pullback must arrive at the same point in Z. But here we relax the requirement by simply allowing elements of the homotopy pullback to arrive at the same path component of Z (so up to the existence of an homotopy of the two points in Z).

Definition 4.1.2: The Canonical Map of Homotopy Pullbacks

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : X \to Y$ and $g : Y \to Z$ be maps. Define the canonical map from the pullback to the homotopy pullback

$$c : \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \mathrm{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

to be given by $(x, y) \mapsto (x, e_{f(x)=g(y)}, y)$ where e refers to the constant loop at $f(x) = g(y)$.

Recall that we motivated the definition of a homotopy pullback from the fact that pullbacks does not work well with homotopy. We can now show that homotopy pullbacks remedy the issue.

Theorem 4.1.3: The Matching Lemma

Suppose that we have a commutative diagram of spaces

$$\begin{array}{ccccc} X & \xrightarrow{\ f\ } & Z & \xleftarrow{\ g\ } & Y \\ {\scriptstyle e_X}\downarrow & & {\scriptstyle e_Z}\downarrow & & \downarrow{\scriptstyle e_Y} \\ X' & \xrightarrow{\ f'\ } & Z' & \xleftarrow{\ g'\ } & Y' \end{array}$$

in $\mathbf{CGWH}$. Define the map

$$\phi_{X,Z,Y}^{X',Z',Y'} : \mathrm{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \mathrm{holim}(X' \xrightarrow{f'} Z' \xleftarrow{g'} Y')$$

by the formula $(x, \gamma, y) \mapsto (e_X(x), e_Z \circ \gamma, e_Y(y))$. Then the following are true.
- If each e_X, e_Y, e_Z are homotopy equivalences, then ϕ is a homotopy equivalence.
- If each e_X, e_Y, e_Z are weak equivalences, then ϕ is a weak equivalence.

Proof. We first prove the case for homotopy equivalence. Consider the following commutative diagram:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Z & \xleftarrow{\ g\ } & Y \\
{\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle e_Z}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_Y} \\
X & \xrightarrow{\ e_Z \circ f\ } & Z' & \xleftarrow{\ e_Z \circ g\ } & Y \\
{\scriptstyle e_X}\downarrow & & {\scriptstyle \mathrm{id}_{Z'}}\downarrow & & \downarrow{\scriptstyle e_Y} \\
X' & \xrightarrow{\ f'\ } & Z' & \xleftarrow{\ g'\ } & Y'
\end{array}
$$

We prove that the homotopy pullback of the first row is homotopy equivalent to that of the second, and we prove that the homotopy pullback of the second row is homotopy equivalent to that of the third.

Since e_Z is a homotopy equivalence, we can find a homotopy inverse k for e_Z and a homotopy $H : Z \times I \to Z$ such that $H(-, 0) = \mathrm{id}_Z$ and $H(-, 1) = k \circ e_Z$. Define a map

$$\rho : \mathrm{holim}(X \xrightarrow{f} Z' \xleftarrow{g} Y) \to \mathrm{holim}(X \xrightarrow{e_Z \circ f} Z \xleftarrow{e_Z \circ g} Y)$$

by the formula

$$(x, \gamma', y) \mapsto (x, H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)} : I \to Z, y)$$

where $*$ denotes concatenation of paths. The path concatenation is well defined because we have that $H(f(x), 1) = (k \circ e_Z \circ f)(x) = (k \circ \gamma')(0)$ and $k(\gamma'(1)) = k(e_Z(g(y))) = H(g(y), 1)$. This is well defined on the homotopy pullback because we have that

- $H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)}(0) = H(f(x), 0) = \mathrm{id}_Z(f(x)) = f(x)$
- $H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)}(1) = H(g(y), 0) = \mathrm{id}_Z(g(y)) = g(y)$

I claim that this map is inverse to the map $\phi = \phi^{X,Z',Y}_{X,Y,Z}$ where we take $e_X = \mathrm{id}_X$ and $e_Y = \mathrm{id}_Y$. We have that

$$\rho(\phi(x, \gamma, y)) = \rho(x, e_Z \circ \gamma, y)$$
$$= (x, H(f(x), -) * k(e_Z(\gamma(-)) * \overline{H(g(y), -)}, y)$$

Now I claim that the middle path is homotopic to γ. For the first part, the path $H(f(x), t) : I \to Z$ can be contracted to $H(f(x), 0) = f(x) = \gamma(0)$ so you can homotope the traversal along $H(f(x), -)$ to the single point $f(x) = \gamma(0)$. For the third part, this is similar so we can homotope the traversal of $\overline{H(g(y), -)}$ to the single point $g(y) = \gamma(1)$. The middle part of the path is homotopic to γ because $k \circ e_Z$ is homotopic to id_Z. Thus we conclude. $\square$

Dually, we define the notion of standard homotopy pushouts.

Definition 4.1.4: The Standard Homotopy Pushout

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. Define the standard homotopy pushout of the diagram to be the quotient space

$$\mathrm{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y) = \frac{X \amalg (Z \times I) \amalg Y}{\sim}$$

where $\sim$ is the equivalence relation generated by $f(z) \sim (z, 0)$ and $g(z) \sim (z, 1)$ for $z \in Z$.

TBA: Mimic the univ property up to homotopy

Definition 4.1.5: The Canonical Map of Homotopy Pushouts

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. Define the canonical map of the homotopy pushout of the diagram to be the map

$$s : \mathrm{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y) \to \mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$$

given by the formula

$$u \mapsto \begin{cases} u & \text{if } u \in X \\ f(z) = g(z) & \text{if } u = (z, t) \in Z \times I \\ u & \text{if } u \in Y \end{cases}$$

Theorem 4.1.6: The Gluing Lemma

Suppose that we have a commutative diagram of spaces

$$
\begin{array}{ccccc}
X & \xleftarrow{\;\;f\;\;} & Z & \xrightarrow{\;\;g\;\;} & Y \\
{\scriptstyle e_X}\downarrow & & {\scriptstyle e_Z}\downarrow & & \downarrow{\scriptstyle e_Y} \\
X' & \xleftarrow{\;\;f'\;\;} & Z' & \xrightarrow{\;\;g'\;\;} & Y'
\end{array}
$$

in $\mathbf{CGWH}$. If each e_X, e_Y, e_Z are (homotopy) weak equivalences, then the induced map

$$\mathrm{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y) \to \mathrm{hocolim}(X' \xleftarrow{f'} Z' \xrightarrow{g'} Y')$$

defined by the formula

$$u \mapsto \begin{cases} e_X(u) & \text{if } u \in X \\ (e_Z(v), t) & \text{if } u = (v, t) \in Z \times I \\ e_Y(u) & \text{if } u \in Y \end{cases}$$

is a (homotopy) weak equivalence.

4.2 Homotopy Pullbacks and Pushout Squares

We use the standard homotopy pullback and pushout to recognize homotopy pullbacks and pushouts up to weak equivalence.

Definition 4.2.1: Homotopy Pullback Squares

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that there is a (not necessarily commutative) diagram

$$
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

- We say that the diagram is a homotopy pullback if the map

$$\alpha : W \to \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \xrightarrow{\;c\;} \mathrm{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

  is a weak equivalence.
- We say that the diagram is k-cartesian if α is k-connected.

It is clear that the standard homotopy pullback indeed is a homotopy pullback.

Definition 4.2.2: Homotopy Pushout Squares

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that there is a (not necessarily commutative) diagram

$$
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

- We say that the square is a homotopy pushout square if the map

$$\beta : \operatorname{hocolim}(X \xleftarrow{f} W \xrightarrow{g} Y) \xrightarrow{s} \operatorname{colim}(X \xleftarrow{f} W \xrightarrow{g} Y) \to Z$$

  is a weak equivalence.
- We say that the diagram is k-cocartesian if β is k-connected.

4.3   Recognizing Homotopy Pullbacks and Pushouts

Proposition 4.3.1

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Then the following spaces are homeomorphic to

$$\operatorname{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

via the canonical map.
- $\lim(P_f \to Z \xleftarrow{g} Y)$
- $\lim(X \xrightarrow{f} Z \leftarrow P_g)$
- $\lim(P_f \to Z \leftarrow P_g)$

In model theoretic terms: the homotopy pullback can be computed by the standard pullback of fibrant replacements.

Proposition 4.3.2

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. Then the following spaces are homeomorphic to

$$\operatorname{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$$

via the canonical map.
- $\operatorname{colim}(M_f \leftarrow Z \to Y)$
- $\operatorname{colim}(X \leftarrow Z \to M_g)$
- $\operatorname{colim}(M_f \leftarrow Z \to M_g)$

When one of the maps f or g is a fibration, then the notion of a pullback coincides with that of homotopy pullback.

Proposition 4.3.3

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. If f or g is a fibration, then there is a homotopy equivalence

$$\lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \simeq \operatorname{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

induced by the canonical map.

Proposition 4.3.4

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. If f or g is a cofibration, then there is a homotopy equivalence

$$\mathrm{colim}(X \xleftarrow{f} Z \xrightarrow{g} Y) \simeq \mathrm{hocolim}(X \xleftarrow{f} Z \xrightarrow{g} Y)$$

induced by the canonical map.

Recall that the mapping path space P_f of a map $f : X \to Y$ is defined to be

$$P_f = f^*(\mathrm{Map}(I, Y)) = \{(x, \phi) \subseteq X \times \mathrm{Map}(I, Y) \mid f(x) = \pi_0(\phi) = \phi(0)\}$$

we can now prove that P_f is a homotopy invariance.

Proposition 4.3.5

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f, g : X \to Y$ be maps. If f and g are homotopic, then the following are true.
- $P_f \simeq P_g$
- $M_f \simeq M_g$

4.4   Relation to Homotopy (Co)Fibers

Proposition 4.4.1

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f, g : X \to Y$ be maps. If f and g are homotopic, then the following are true.
- $\mathrm{hofiber}_y(f) \simeq \mathrm{hofiber}_y(g)$ for any $y \in Y$.
- $\mathrm{hocofiber}(f) \simeq \mathrm{hocofiber}(g)$

Recall that the fiber of a map $f : X \to Y$ behaves poorly because the fibers are not homeomorphic and not even homotopy equivalent. However, we can now prove that the homotopy fibers are the correct notion of a fiber to study because they are homotopy equivalent.

Corollary 4.4.2

Let $X, Y \in \mathbf{CGWH}$ be space. Let $f : X \to Y$ be a map. If $y_1, y_2 \in Y$ are path connected, then there is a homotopy equivalence

$$\mathrm{hofiber}_{y_1}(f) = \mathrm{hofiber}_{y_2}(f)$$

We can we interpret homotopy pullbacks and pushouts using homotopy (co)fibers.

Proposition 4.4.3

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that following is a (not necessarily commutative) square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Then the following are true.
- The square is a homotopy pullback if and only if for all $x \in X$, the map

$$\mathrm{hofiber}_x(W \to X) \to \mathrm{hofiber}_{f(x)}(Y \to Z)$$

  is a weak equivalence.

- The square is k-cartesian if and only if for all $x \in X$, the map

$$\text{hofiber}_x(W \to X) \to \text{hofiber}_{f(x)}(Y \to Z)$$

  is k-connected.

4.5 Connectedness of Homotopy Squares

5   Blakers-Massey Theorem

5.1   The Blakers-Massey Theorem for Squares

The Blakers-Massey theorem is a direct generalization of the homotopy excision theorem. Its proof takes a similar form to the homotopy excision theorem. Let us recall some definitions used.

> **Definition 5.1.1: (Degenerative) Cubes**
>
> Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Let $\delta > 0$. Let $L \subseteq \{1, \ldots, n\}$. A cube in $\mathbb{R}^n$ is a set of the form
> $$W = W(a, \delta, L) = \{x \in \mathbb{R}^n \mid a_i \leq x \leq a_i + \delta \text{ for } i \in L \text{ and } x_i = a_i \text{ for } i \notin L\}$$

The notation is making the object more complicated than what it should look like. $a \in \mathbb{R}^n$ refers to the bottom left coordinate of the cube. δ is the length of the cube and L refers to the number of non-degenerate faces of the cube. In particular, any cube in $\mathbb{R}^n$ is homeomorphic to the standard cube I^k for some $k \leq n$.

- When $n = 3$, $W(0, 1, \{1, 2\})$ is the unit square on the xy-plane.

- When $n = 3$, $W(0, 1, \{1, 2, 3\})$ is the unit cube.

- When $n = 4$, $W(0, 1, \{1, 2, 3\})$ is the unit cube with nonzero first three coordinates and zero otherwise.

> **Definition 5.1.2: Special Sub-cube of a Cube**
>
> Let $W = W(a, \delta, L)$ be a cube in $\mathbb{R}^n$. Let $j = 1$ or 2. Suppose that $1 \leq p \leq |L|$. Define
> $$K_p^j(W) = \left\{(x_1, \ldots, x_n) \in W \ \middle| \ \frac{\delta(j-1)}{2} + a_i < x_i < \frac{\delta j}{2} + a_i \text{ for at least } p \text{ values of } i \in L\right\}$$

Again the notation is making the object more complicated. Taking $W = W(0, 1, \{1, 2, 3\}) = I^3$ in $\mathbb{R}^3$, we have

- $K_3^1(W) = W(0, 1/2, \{1, 2, 3\})$ is one eighth of the cube I^3 with bottom left corner at the origin.

- $K_3^2(W) = W((1/2, 1/2, 1/2), 1/2, \{1, 2, 3\})$ is one eighth of the cube I^3 with bottom left corner at $(1/2, 1/2, 1/2)$

- $K_2^1(W)$ allows for one coordinate to go beyond the bottom left one eighth of the cube, and is the union four of the 1/8-sub-cubes that are adjacent to the xy-plane, the yz-plane and the xz-plane.

- $K_1^1(W)$ allows for two coordinate to go beyond the bottom left one eighth of the cube, and is equal to $W \setminus K_3^2(W)$.

- $K_0^1(W)$ allows for all coordinate to go beyond the bottom left one eighth of the cube, so the condition becomes vacuous and is equal to W.

Summarizing, we think of $K_p^j(W)$ as follows. Subdivide the $|L|$-dimensional cube into $2^{|L|}$ sub-cubes of equal volume. K_p^1 is the union of a number of sub-cubes closest to the bottom left sub-cube. K_p^2 is the union of a number of sub-cubes closed to the upper right sub-cube.

> **Lemma 5.1.3**
>
> Let Y be a space. Let $B \subseteq Y$ be a subspace of Y. Let $W = W(a, \partial, L)$ be a cube in $\mathbb{R}^n$. Let $f : W \to Y$ be a map. Let $j = 1$ or 2. Suppose that there exists some $p \leq |L|$ such that
> $$f^{-1}(B) \cap C \subset K_p^j(C)$$
> for all cubes $C \subset \partial W$. Then there exists a map $g : W \to Y$ such that $g \overset{\partial W}{\simeq} f$ and
> $$g^{-1}(B) \subset K_p^j(W)$$

Proof. (Proof by Munson in Cubical Homotopy Theory)Firstly, notice that any cube W is homeomorphic to I^n for some n, so we can just prove the statement for when $W = I^n$. In this case, our parameters of the cube is given by $I^n = W(a = 0, \delta = 1, L = \{1, \ldots, n\})$ and our $K_p^j(W)$ is given by

$$K_p^j(W) = \left\{ (x_1, \ldots, x_n) \in C \ \middle| \ \frac{j-1}{2} < x_i < \frac{j}{2} \text{ for at least } p \text{ values of } i \in \{1, \ldots, n\} \right\}$$

Let p_j be the center of the sub-cube $\left[\frac{j-1}{2}, \frac{ji}{2}\right]^n$ inside I^n for $j = 1, 2$. Let R be a ray with starting point p_j. Let $P(R, p_j)$ be the intersection of R and $\partial \left[\frac{j-1}{2}, \frac{ji}{2}\right]^n$. Let $Q(R, p_j)$ be the intersection of R and ∂I^n. By construction, the points p_j, $P(R, p_j)$ and $Q(R, p_j)$ are collinear with $P(R, p_j)$ always being the mid point and $P(R, p_j)$ is possibly equal to $Q(R, p_j)$. Being a line, we can define a linear homotopy from the line $[p_j, P(R, p_j)]$ to the line $[p_j, Q(R, p_j)]$ that fixes the point p_j and sends $P(R, p_j)$ to $Q(R, p_j)$. Denote the homotopy by $h(y, t)$ for $y \in [p_j, P(R, p_j)]$ and t the time variable.

Now we can define a homotopy $H_j : I^n \times I \to I^n$ as follows: For each $y \in I^n$, there exists a unique ray R starting at p_j and passing through y. Then we obtain a homotopy h from $[p_j, P(R, p_j)]$ to $[p_j, Q(R, p_j)]$ as above. Define $H(y, t) = h(y, t)$. It is clear that $H_j(q, 1) = q$ for all $q \in \partial I^n$ so that H_j is a homotopy from the identity, relative to the boundary ∂I^n.

Let $g = f \circ H_j(-, 1)$. From the properties of the homotopy H_j, we notice that $f \circ H_j : I^n \times I \to Y$ is a homotopy from $f \circ H(-, 0) = f \circ \text{id} = f$ to $f \circ H(-, 1) = g$, relative to the boundary ∂I^n. Thus we now have a homotopy from f to a map g relative to the boundary. It remains to show that $g^{-1}(B) \subset K_p^j(C)$.

Let $z = (z_1, \ldots, z_n) \in g^{-1}(B)$. If $z \in \left[\frac{j-1}{2}, \frac{j}{2}\right]^n$ then clearly $z \in K_p^j(C)$ is true. So suppose instead that $z = (z_1, \ldots, z_n) \in g^{-1}(B)$ satisfies the fact that either $z_a \geq \frac{j}{2}$ or $z_b \leq \frac{j-1}{2}$ for $1 \leq a, b \leq n$. Let R be the ray from p_j passing through z. Then the condition on z means that $z \in [P(R, p_j), Q(R, p_j)]$. Hence under the homotopy H, z is mapped to ∂I^n. But ∂I^n is a union $n - 1$ dimensional faces of I^n which are cubes. So $H(z, 1)$ lies in some cube $C \subseteq \partial I^n$. By construction of g, $g(z) = f(H(z, 1))$ and $g(z) \in B$ implies that $H(z, 1) \in f^{-1}(B)$. Then $H(z, 1) \in f^{-1}(B)$ and $H(z, 1) \in C$ implies that

$$H(z, 1) \in f^{-1}(B) \cap C \subseteq K_p^j(C)$$

by the assumption on f. Write $H(z, 1) = (w_1, \ldots, w_n) \in \partial I^n$. This means that $\frac{j-1}{2} < w_i < \frac{j}{2}$ for at least p of the coordinates of $H(z, 1)$.

Now the ray starting at p_j and passing through z is parametrized by the line $p_j + t(z - p_j)$ for $t \geq 0$. Since $H(z, 1)$ lies behind the two points z and p_j, we can write $H(z, 1) = p_j - t_0(z - p_j)$ for some $t_0 \geq 1$. By definition, p_j is the point given in coordinates by $\left(\frac{2j-1}{4}, \ldots, \frac{2j-1}{4}\right)$. Hence the ith coordinate of $H(z, 1)$ can be written as

$$w_i = \frac{2j-1}{4} + t_0 \left(z_i - \frac{2j-1}{4} \right)$$

Recall that in the previous paragraph we found that $\frac{j-1}{2} < w_i < \frac{j}{2}$ for at least p of the coordinates of $H(z, 1)$. Substituting w_i into the inequality and simplifying gives

$$-\frac{1}{4t_0} + \frac{2j-1}{4} < z_i < \frac{1}{4t_0} + \frac{2j-1}{4}$$

Since $t_0 \geq 1$, we get

$$-\frac{1}{4} + \frac{2j-1}{4} < -\frac{1}{4t_0} + \frac{2j-1}{4} < z_i < \frac{1}{4t_0} + \frac{2j-1}{4} < \frac{1}{4} + \frac{2j-1}{4}$$

The leftmost and rightmost terms bound z_i between $\frac{j-1}{2}$ and $\frac{j}{2}$ for at least p amount of coordinates z_i of z. Hence $z \in K_p^j(C)$. This completes the proof. $\square$

Lemma 5.1.4

Let X be a space. Let $X_0, X_1, X_2 \subseteq X$ be subspaces of X such that

$$X = X_1 \cup X_2$$

and $X_0 = X_1 \cap X_2$ is non-empty. Assume that for each $i = 1, 2$, (X_i, X_0) is k_i-connected with $k_i \geq 0$. Let $f : I^n \to X$ be a map. Let

$$I^n = \bigcup_k W_k$$

be the decomposition of I^n into cubes W_k such that $f(W_k) \subseteq X_i$ for one of $i = 0, 1, 2$ by the Lebesgue covering lemma. Then there exists a homotopy

$$H : I^n \times I \to X$$

such that the following are true.
- $f(-) = H(-, 0)$
- If $f(W) \subset X_i$, then $H(W, t) \subset X_i$ for all $t \in I$.
- If $f(W) \subset X_0$, then $H(W, t) = f(W)$ for all $t \in I$.
- If $f(W) \subset X_i$, then $\left((H(-, 1))^{-1}(X_i \setminus X_0)\right) \cap W \subset K_{k_i+1}^i(W)$.

Proof. Let C^d be the union of all cubes of dimension $\leq d$. We induct on d, the existence of such a homotopy $H : C^d \times I \to X$ that holds the required conditions true for all cubes W with dimension $\leq d$.

We first construct the homotopy for all cubes of dimension 0. When $\dim(W) = 0$, there are two cases:
- If $f(W) \subset X_0$, define $H|_{W \times I}$ by $H(w, t) = f(w)$
- If $f(W) \subset X_j$ and $f(W) \not\subset X_i$ for $1 \leq i \neq j \leq 2$, (X_j, X_0) is $(k_j \geq 0)$-connected implies that there exists a path $\gamma : I \to X$ from $f(W)$ to a point in X_0. Define $H|_{W \times I}$ by $H(w, t) = \gamma(t)$ (again $W = \{w\}$ is a one point set).

Thus we now have a well defined map $H : C^0 \times I \to X$. We need to show that this map satisfies the required conditions.
- For each $z \in C^0$, either $H(z, 0) = f(z)$ from the first case or $H(z, 0) = \gamma(0) = f(z)$.
- If $f(W) \subset X_i$, then by construction $H(W, t) \subset X_i$ from the second case.
- If $f(W) \subset X_0$, then $H(W, t) = f(W)$ by the first case.
- $K_{k_i+1}^i(W) = \{w\}$ is a one point set and $(H(-, 1))^{-1}(X_i \setminus X_0) \cap W \subseteq W$ means that this condition is satisfied.

Now H is built on three pieces: the union of cubes landing in X_i for $i = 0, 1, 2$. The second and third conditions guarantee that each of the three pieces define a homotopy on each piece respectively. Since $\partial W \hookrightarrow W$ is a cofibration, we can extend these pieces of homotopy from 0-dimensional cubes to 1-dimensional. Recursively we are able to define a homotopy for all cubes of all dimensions inside I^n that satisfy the first three conditions.

Therefore now we can invoke the inductive hypothesis, so that there exists a homotopy from f so that the new function satisfy all our required conditions for all cubes of dimension $< d$. With abuse of notation, call the restriction of our newly acquired function to C^{d-1} also by the name f. Let W be a cube of dimension d.
- If $f(W) \subseteq X_0$, define $H|_{W \times I}$ by $H(w, t) = f(w)$
- If $f(W) \subset X_1$ and $f(W) \not\subset X_2$ and $\dim(W) = d \leq k_1$, (X_j, X_0) is $(k_j \geq 0)$-connected implies there exists a homotopy $K : W \times I \to X$ from f relative to ∂W such that $K(W, 1) \subseteq X_0$. Define $H|_{W \times I}$ by $H = K$.

- If $f(W) \subset X_1$ and $f(W) \not\subset X_2$ and $\dim(W) = d > k_1$, then by induction we have

$$f^{-1}(X_1 \setminus X_0) \cap W' \subset K_d^1(W') \subset K_{k_1+1}^1(W')$$

  for all $W' \subset \partial W$ (induction is applicable since $\dim(W') < \dim(W)$). By the above lemma, there exists a map $g : W \to X$ such that g and f are homotopic relative to ∂W such that $g^{-1}(X_1 \setminus X_0) \subset K_{k_1+1}^1(W)$. Call this homotopy from f to g by $R : W \times I \to X$. Then we define $H|_{W \times I}$ by $H = R$.

(WLOG the cases where X_1 is swapped with X_2 and k_1 is swapped with k_2 and $K_p^1(W)$ is swapped with $K_p^2(W)$ in the last two sub-cases has a symmetrical argument). Finally we show that our required conditions are satisfied.

- In all cases, $H(-, 0) = f$ as one can immediately see.
- The second condition holds for all cubes of dimension $< d$ by inductive hypothesis. It also holds for our first and second case since $X_0 \subset X_1, X_2$. For the third case, H is a homotopy relative to ∂W. Since by induction hypothesis $f(\partial W) \subseteq X_1$, we also have $g(\partial W) \subseteq X_1$. Since g is continuous then $H(W, 1) = g(W) \subseteq X_1$.
- The third condition holds for all cubes of dimension $< d$ by inductive hypothesis, and holds true for all cubes of dimension d by the first case.
- The fourth condition holds true by our argument in the third case, and is vacuously true in the second case since $H(W, 1) \subseteq X_0$ implies that $(H(-, 1))^{-1}(X_1 \setminus X_0) = \emptyset$.

Thus the proof is complete. $\qquad\square$

We can now prove a weaker version of Blakers-Massey theorem.

Proposition 5.1.5

Let X be a space. Let e^{d_i} be a cell of dimension d_i for $i = 1, 2$. Then the following diagram

$$
\begin{array}{ccc}
X & \longrightarrow & X \cup e^{d_1} \\
\downarrow & & \downarrow \\
X \cup e^{d_2} & \longrightarrow & X \cup e^{d_1} \cup e^{d_2}
\end{array}
$$

given by inclusion maps is $(d_1 + d_2 - 3)$-cartesian.

- -

Proof. (Proof is by Munson in Cubical Homotopy Theory) Let $p_1 \in e^{d_1}$ and $p_2 \in e^{d_2}$ be interior points. Since $X \cup e^{d_2}$ is weakly equivalent to $X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1\}$ by inclusion (and similarly for $X \cup e^{d_1}$), the above square admits a weak equivalence to the following square:

$$
\begin{array}{ccc}
X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1, p_2\} & \longhookrightarrow & X \cup e^{d_1} \cup e^{d_2} \setminus \{p_2\} \\
\downarrow & & \downarrow \\
X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1\} & \longrightarrow & X \cup e^{d_1} \cup e^{d_2}
\end{array}
$$

Let $Y = X \cup e^{d_1} \cup e^{d_2}$. By thm???? to show that the square is $(d_1 + d_2 - 3)$-cartesian is the same as showing the map

$$\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \to \text{hofiber}_y(Y \setminus \{p_2\} \to Y)$$

is $(d_1 + d_2 - 3)$-cartesian. Let

$$C = \text{hofiber}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\}) \simeq *$$

Now the intersections below can be calculated to give

$$C \cap \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) = \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1, p_2\})$$

and

$$C \cap \text{hofiber}_y(Y \setminus \{p_2\} \to Y) = \text{hofiber}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\}) \simeq *$$

(because C and the homotopy fibers are all subspaces of $Y \times \text{Map}(I, Y)$). By thm??? we conclude that the square

$$\begin{array}{ccc} C \cap \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) & \longrightarrow & \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \\ \downarrow & & \downarrow \\ C & \lhook\joinrel\longrightarrow & C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \end{array}$$

is a homotopy pullback. Since the left two terms are contractible, it is a weak equivalence. By thm??? we conclude that the map on the right is a weak equivalence. Similarly, we can consider the diagram

$$\begin{array}{ccc} C \cap \text{hofiber}_y(Y \setminus \{p_2\} \to Y) & \longrightarrow & \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \\ \downarrow & & \downarrow \\ C & \lhook\joinrel\longrightarrow & C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \end{array}$$

which is a homotopy pullback. And the fact that the terms on the left are contractible imply that the map on the right (which is in fact equal)

$$\text{hofiber}_y(Y \setminus \{p_2\} \to Y) \overset{=}{\lhook\joinrel\to} \text{hofiber}_y(Y \setminus \{p_2\} \to Y)$$

is a weak equivalence by the same theorem. We now have a chain of maps

$$\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \xrightarrow{\text{weak eq.}} C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\})$$

$$C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \xrightarrow{\quad \text{weak eq.} \quad} \text{hofiber}_y(Y \setminus \{p_2\} \to Y)$$

where the middle map is inclusion. It remains to show that the middle map is $(d_1 + d_2 - 3)$-connected. This is the same as showing that the pair

$$(C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y), C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}))$$

is $(d_1 + d_2 - 3)$-connected.

To simplify notations let us write the pair as (A, B). Let $\phi : (I^n, \partial I^n) \to (A, B)$ be a map. Recall that $A = \{(x, \phi) \in Y \times \text{Map}(I, Y) \mid \phi(0) = y, \phi(1) = x\}$. The first variable is determined by the end point of ϕ so giving a map $I^n \to A$ is the same as giving a map $I^n \to \text{Map}(I, Y)$ for which all paths in the image has starting point y and ending point in $Y \setminus \{p_2\}$. By the hom-product adjunction, this is equivalent to giving a map $\psi : I^n \times I \to Y$ such that $\psi(z, 0) = y$ is the base point and $\psi(z, 1) \in Y \setminus \{p_2\}$. Similarly, we can consider the map $\phi : \partial I^n \to B$ and deduce that $\phi(z)$ is a path lying entirely in the codomain of the map of the homotopy fiber $C = \text{hofib}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\})$ or it is a path lying entirely in the codomain of the map of the homotopy fiber $\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}))$. By the same adjunction we conclude that our ψ above must also satisfy for any fixed $z \in \partial I^n$, $\psi(z, t)$ lies entirely in either $Y \setminus \{p_1\}$ or $Y \setminus \{p_2\}$ (and conversely these information give a map $(I^n, \partial I^n) \to (A, B)$ by the adjunction).

To summarize: we have a map

$$\psi : I^n \times I \to Y$$

such that
- $\psi(z, 0) = y$ is the base point
- $\psi(z, 1) \in Y \setminus \{p_2\}$
- For any fixed $z \in \partial I^n$, $\psi(z, t)$ lies entirely in $Y \setminus \{p_1\}$ or $Y \setminus \{p_2\}$ for varying t.

The goal is to make a homotopy from ψ to a map whose third condition holds for any $z \in I^n$ when $n \leq d_1 + d_2 - 3$. Then passing through the adjunction again we see that our original map $(I^n, \partial I^n) \to (A, B)$ is homotopic to the constant map as required. $\qquad \square$

Theorem 5.1.6: Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_{12}
\end{array}
$$

is a homotopy pushout. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cartesian. Explicitly, this means that

$$\alpha : X_0 \to \mathrm{holim}(X_1 \to X_{12} \leftarrow X_2)$$

is $(k_1 + k_2 - 1)$-connected.

- -

Proof. $\square$

This theorem directly generalizes the homotopy excision theorem in the following way. For X a CW complex and A, B two subcomplexes with non-empty intersection and $X = A \cup B$, consider the following square of inclusions:

$$
\begin{array}{ccc}
A \cap B & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & X
\end{array}
$$

We have seen that such a square diagram is a homotopy pushout diagram. Now any inclusion map $W \hookrightarrow Z$ is k-connected if and only if (Z, W) is k-connected. So $(A, A \cap B)$ is k_1-connected and (X, B) is k_2-connected. Blaker's-Massey theorem implies that

$$\mathrm{hofiber}(A \cap B \to A) \to \mathrm{hofiber}(B, X)$$

is $(k_1 + k_2 - 1)$-connected. But by definition we have an isomorphism $\pi_k(\mathrm{hofiber}(U \to V) \cong \pi_{k+1}(V, U)$. So we are really just saying that $\pi_k(A, A \cap B) \to \pi_k(X, B)$ given by the inclusion is $(k_1 + k_2)$-connected.

5.2   The Dual Blakers-Massey Theorem for Squares

Theorem 5.2.1: Dual Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$
\begin{array}{ccc}
X_0 & \longrightarrow & X_1 \\
\downarrow & & \downarrow \\
X_2 & \longrightarrow & X_{12}
\end{array}
$$

is a homotopy pullback. Suppose the map $X_i \to X_{12}$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cocartesian. Explicitly, this means that

$$\beta : \mathrm{hocolim}(X_1 \leftarrow X_0 \to X_2) \to X_{12}$$

is $(k_1 + k_2 - 1)$-connected.

6 Homotopy n-Cubes

In algebraic topology, we have learnt about spaces, maps of spaces and maps of maps of spaces. We can say this in a more compact way. Namely, if we think of maps of maps of space as a square (2-cube), we can think of spaces as 0-cubes and maps of spaces as 1-cube. We have studied 2-cubes extensively under the guise of homotopy pullbacks and pushouts. We can now take this further and consider general n-cubes.

6.1 Maps between Homotopy Squares

6.2 n-Cubes of Spaces

Definition 6.2.1: n-Cubes of Spaces

Let $n \in \mathbb{N}$. Let $P(n)$ denote the category of posets of the set $\{1, \ldots, n\}$. An n-cube of spaces is a functor

$$X : P(n) \to \mathbf{CGWH}$$

An n-cube of based spaces is a functor $X : P(n) \to \mathbf{CGWH}_*$.

Explicitly, an n-cube of spaces $X : P(n) \to \mathbf{CGWH}$ consists of the following data.

- For each $S \subseteq \{1, \ldots, n\}$ a space X_S

- For each $S \subseteq T$, a map $f_{S \subseteq T} : X_S \to X_T$ such that $f_{S \subseteq S} = 1_{X_S}$ and for all $R \subseteq S \subseteq T$, we have a commutative diagram

$$
\begin{array}{ccc}
X_R & \xrightarrow{f_{R \subseteq S}} & X_S \\
 & \searrow^{f_{R \subseteq T}} & \downarrow{f_{S \subseteq T}} \\
 & & X_T
\end{array}
$$

Omit drawing composite arrows and omit drawing identities.
Also: punctured cubes def

Definition 6.2.2: Cube of Cubes

An n-cube of m-cubes is a functor

$$X : P(n) \times P(m) \to \mathbf{CGWH}$$

Lemma 6.2.3

An n-cube of m-cubes X is precisely an $(n + m)$-cube.

Definition 6.2.4: Map of n-Cubes

Let $X, Y : P(n) \to \mathbf{CGWH}$ be n-cubes. A map of n-cubes is a natural transformation $F : X \to Y$ such that the assignment $Z : P(n+1) \to \mathbf{CGWH}$ given by

$$
Z(S) = \begin{cases} X(S) & \text{if } S \subseteq \{1, \ldots, n\} \\ Y(S \setminus \{1, \ldots, n+1\}) & \text{if } \{1, \ldots, n+1\} \subseteq S \end{cases}
$$

defines an $(n + 1)$-cube.

---

objectwise (co)fibration, homotopy (weak) equivalence. homeomorphism

Definition 6.2.5: Strongly Homotopy Cartesian

Let X be an n-cube of spaces. We say that X is strongly homotopy cartesian if each of its faces of dimension $n \geq 2$ is homotopy cartesian.

7   Homotopy Limits and Colimits

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Denote the constant functor of the one point space by $\Delta* : \mathbf{X} \to \mathbf{Top}$. The data of the constant functor is given as follows.

- For each $I \in \mathcal{J}$, $\Delta * (I) = *$
- For each morphism $f : I \to J$ in X, define $\Delta * (f) = \mathrm{id}_*$.

Consider the set of all natural transformations $\Delta* \Rightarrow X$ denoted by $\mathrm{Nat}(\Delta*, X)$. Now this set can inherit a subspace topology via the isomorphism (of sets)

$$\mathrm{Nat}(\Delta*, X) \cong \prod_{I \in \mathcal{J}} \mathrm{Hom}_{\mathbf{Top}}(*, X_I) \subset \prod_{I \in \mathcal{J}} X_I$$

There is in fact a canonical homeomorphism between the set of natural transformations and the limit of X.

Theorem 7.0.1

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Then there is a canonical homeomorphism

$$\lim_{\mathcal{J}} X \cong \mathrm{Nat}(\Delta*, X)$$

Ref: Cubical diagrams

Theorem 7.0.2

Let $X : \mathcal{J} \to \mathbf{Top}$ be a diagram of spaces. Then there is a canonical homeomorphism

$$\mathrm{colim}_{\mathcal{J}} X \cong \frac{\coprod_{I \in \mathcal{J}} X_I}{\sim}$$

where $x \in X_I \sim y \in X_J$ if and only if there exists $f : I \to J$ such that $X(f)(x) = y$.

7.1   Homotopy Limits and Colimits

Let $\mathcal{J}$ be a small diagram. Let $I \in \mathcal{J}$ and denote $\mathcal{J}/I$ to be the overcategory with the distinguished object I. We can turn it into a topological space by constructing its classifying space

$$\mathcal{B}(\mathcal{J}/I) = |N(\mathcal{J}/I)|$$

which is the geometric realization of the nerve of $\mathcal{J}/I$. We aim to use the overcategory $\mathcal{J}/I$ to record homotopy information. Recall that the limit is canonically isomorphic to the equalizer of

$$f, g : \prod_{J \in \mathrm{Obj}\,\mathcal{J}} X_J \to \prod_{(\alpha : J \to I) \in \mathcal{J}} X_I$$

as follows.

- Define f to be the unique map such that $\pi_I \circ f = \pi_{\mathrm{cod}(\alpha)}$ where both π are projections.
- Define g to be the unique map such that $\pi_{\mathrm{cod}(\alpha)} \circ g = F(\alpha) \circ \pi_{\mathrm{dom}(\alpha)}$

We aim to replace each space X_I by the space $\mathrm{Map}(\mathcal{B}(\mathcal{J}/I), X_I)$ (which is why we work with $\mathbf{CGWH}$). Indeed the hom space and X_I are homotopy equivalent since $\mathcal{B}(\mathcal{J}/I)$ is contractible. Notice that this is no longer a functor in I, but rather a bifunctor. This all works with a simplicial model category.

Definition 7.1.1: Homotopy Limits

Let $\mathcal{C}$ be a simplicial model category. Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. Define two maps

$$f, g : \prod_{J \in \mathrm{Obj}\,\mathcal{J}} \mathrm{Map}(\mathcal{B}(\mathcal{J}/J), X_J) \to \prod_{(\alpha : J \to I) \in \mathcal{J}} \mathrm{Map}(\mathcal{B}(\mathcal{J}/J), X_I)$$

as follows.
- Define f to be the unique map such that

$$\pi_I \circ f = \mathrm{Map}(\mathcal{B}(\mathcal{J}/J), X(\alpha) : X_J \to X_I)$$

  for any $\alpha : J \to I$ a morphism in $\mathcal{J}$.
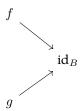- Define g to be the unique map such that

$$\pi_I \circ g = \mathrm{Map}\left(\mathcal{B}\left(\mathcal{J}/X : \mathcal{J}/I \to \mathcal{J}/J\right), X_I\right)$$

  for any $\alpha : J \to I$ a morphism in $\mathcal{J}$.
Define the homotopy limit $\mathrm{holim} X$ of X to be the equalizer of the maps

$$\mathrm{holim} X = \mathrm{Eq}(f, g)$$

Let $\mathcal{J}$ be the small diagram consisting of two objects A, B and two non-trivial morphisms $f, g : A \to B$. Let us illustrate the homotopy limit of $\mathcal{J}$. The slice category $\mathcal{J}/A$ consists only of a single object corresponding to the identity map $\mathrm{id}_A : A \to A$. The slice category $\mathcal{J}/B$ consists of 3 objects corresponding to id_B, f and g. Non-trivial morphisms are given as follows:

$$f \searrow$$
$$\mathrm{id}_B$$
$$g \nearrow$$

Passing to the classifying space, $\mathcal{B}(\mathcal{J}/A)$ is just the one point space and $\mathcal{B}(\mathcal{J}/B)$ becomes the interval I.

Expanding things out show that

$$f, g : \mathrm{Map}(, A) \times \mathrm{Map}(, B)$$

Proposition 7.1.2

Let $X : \mathcal{J} \to \mathbf{CGWH}$ be a diagram. Then there is a natural transformation

$$\mathrm{holim}_{\mathcal{J}} X \cong \mathrm{Nat}(\mathcal{B}(\mathcal{J}/-) : \mathcal{J} \to \mathbf{CGWH}, X)$$

Definition 7.1.3: Homotopy Colimits

Let $\mathcal{C}$ be a simplicial model category. Let $X : \mathcal{J} \to \mathcal{C}$ be a diagram. Define two maps

$$f, g : \coprod_{(\alpha : J \to I) \in \mathcal{J}} X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \to \coprod_{J \in \mathrm{Obj}\,\mathcal{J}} X_J \times \mathcal{B}\left((J/\mathcal{J})^{\mathrm{op}}\right)$$

as follows.
- On each summand of the domain of f, define f to be the map

$$X(\alpha) \times \mathrm{id}_{\mathcal{B}((I/\mathcal{J})^{\mathrm{op}})} : X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \longrightarrow X_I \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right)$$

  and then injecting the into the coproduct.

- On each summand of the domain of g, define g to be the map

$$\mathrm{id}_{X_J} \times \mathcal{B}(\mathcal{J}/X)^{\mathrm{op}} : X_J \times \mathcal{B}\left((I/\mathcal{J})^{\mathrm{op}}\right) \longrightarrow X_J \times \mathcal{B}\left((J/\mathcal{J})^{\mathrm{op}}\right)$$

and then injecting into the coproduct.
Define the homotopy colimit $\mathrm{hocolim}\, X$ of X to be the coequalizer of the maps

$$\mathrm{hocolim}\, X = \mathrm{Coeq}(f, g)$$