Kähler Manifolds

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Abstract

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1.1 Kähler Manifolds

Definition 1.1.1: Kähler Manifolds

A Kähler metric is a Hermitian metric h whose associated $(1,1)$-form ω is closed.

Definition 1.1.2: Kähler Manifolds

A Kähler manifold is a complex manifold M with a Hermitian metric h whose associated $(1,1)$-form ω is closed.

Proposition 1.1.3

Every Kähler manifold M is a Riemannian manifold.

Proof. We have seen that every hermitian metric induces a Riemannian metric. □

Let M be a Kähler manifold with associated $(1,1)$-form ω. Recall that we can write ω in local coordinates in $(U, \phi = (z_1, \dots, z_n))$ as

$$\omega = \frac{i}{2}\sum_{i,j=1}^{n} h_{ij} dz_i \wedge d\overline{z}_j$$

where $h_{ji} = \overline{h}_{ij}$. This is the case even when M is just a Hermitian manifold. With the Kähler structure, we can do more.

$$\omega = \frac{i}{2}\sum_{i=1}^{n} \chi_i \wedge \overline{\chi}_i$$

Proposition 1.1.4

Let M be a Kähler manifold with associated $(1,1)$-form ω. Then $\omega^d/d!$ is the volume element of the Riemannian metric g defined by the Kähler form ω.

Proposition 1.1.5

Let M be a Kähler manifold of complex dimension n. Then

$$\dim(H^{2k}(M, \mathbb{R})) > 0$$

for all $k = 0, \dots, n$.