Mathematical Finance

Labix

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1 Mathematical Finance

Assets: Items Stocks: Makes you own a small percentage of a company. Shares: The unit for stocks. (Owning share = Owning a certain number of stocks) Bonds: Buying debt from a loan issuer. Options: Contract that gives the ability to buy / sell assets at a certain time in the future. Futures: Contract to buy / sell assets at a certain time in the future. Value: Total gain / loss from the whole ordeal.

Derivatives: Just think of them as options / future for now. Strike price: the cost of exercising the derivative at the expiration date.

Idea: Derivatives themselves can also be traded.

Call: Looking to buy. Put: Looking to sell.

European options: can only exercise at the specified date. American options: can exercise any time before the specified date.

Derivative Traders: Hedger: Actually exercises the derivative. (To minimize loss) Speculator: Actually bets on the ups and downs of the derivatives. Arbitrageurs: Finding certain entry point to guarantee some winnings by calling / putting a combination of derivatives. (Too powerful and rare so we usually assume no arbitrage opportunities occur)

The price of an option is a function of

- S_0: Current stock price.
- K: Strike price.
- T: Time to expiration.
- r: Risk free interest rate.
- Dividends that are expected to be paid.

More notation:

- S_T: Stock price at expiration day.
- C/P: American call / put.
- c/p: European call / put.

We assume no arbitrage opportunities (if there is then it disappears quickly).

We have

- $C \leq S_0$
- $S_0 - Ke^{rT} \leq c \leq S_0$.
- $P \leq K$
- $Ke^{-rT} - S_0 \leq p \leq Ke^{-rT}$

More:

- $c + Ke^{-rT} = p + S_0$ (idea: European call and put options worth the same price at time T)
- $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$

2 Mathematical Models for Option Pricing

2.1 Background Terminology

Risk-neutral: A situation where investors do not expect an increase in return when the risk increases. This has two consequences.

- The expected return of a stock is the risk free rate.
- The discount rate for the expected pay off of an option is the risk free.

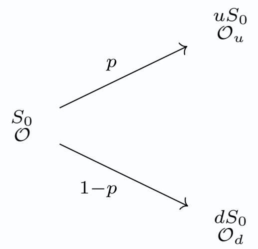
Risk-less: A portfolio is risk-less when regardless of the outcome, the net total is constant.

2.2 One-Period Binomial Models

Definition 2.2.1: The Binomial Model

Suppose that t is the current time. An asset has a current stock price of S_0 with an unknown option price $\mathcal{O}$. After a time step of dt, the stock price either increase by a multiplicative factor of $u > 1$ or the stock price decreases by a multiplicative factor of $d < 1$. Denote the corresponding option value (the net gain from the option) by $\mathcal{O}_u$ and $\mathcal{O}_d$ respectively.

$$t \xrightarrow{dt} t + dt$$



The number $0 < p < 1$ is an unknown quantity denoting the probability that the stock price increases.

Lemma 2.2.2

Assume there is no arbitrage opportunity and the world is risk-neutral. Assume the binomial model. Suppose a portfolio of buying $\Delta \in \mathbb{N}$ stocks and shorting a call option. Then the portfolio is risk-less when

$$\Delta = \frac{\mathcal{O}_u - \mathcal{O}_d}{uS_0 - dS_0}$$

Proposition 2.2.3

Assume there is no arbitrage opportunity and the world is risk-neutral. Assume the binomial model. Denote $V(t) = S_0\Delta - \mathcal{O}$ the value of the portfolio of buying Δ stocks and shorting a call option. Then the following are true.

- The probability p that the stock price increases is given by

$$p = \frac{e^{rdt} - d}{u - d}$$

- The option price is given by

$$\mathcal{O} = e^{-rdt}\left(p\mathcal{O}_u + (1-p)\mathcal{O}_d\right)$$

Proof. Accounting for the discount rate, we have

$$V(t+dt) = (uS_0\Delta - \mathcal{O}_u)e^{-rdt} = (dS_0\Delta - \mathcal{O}_d)e^{-rdt}$$

by definition of Δ. Equating $V(t)$ and $V(t+dt)$ gives the desired result. $\square$

Lemma 2.2.4

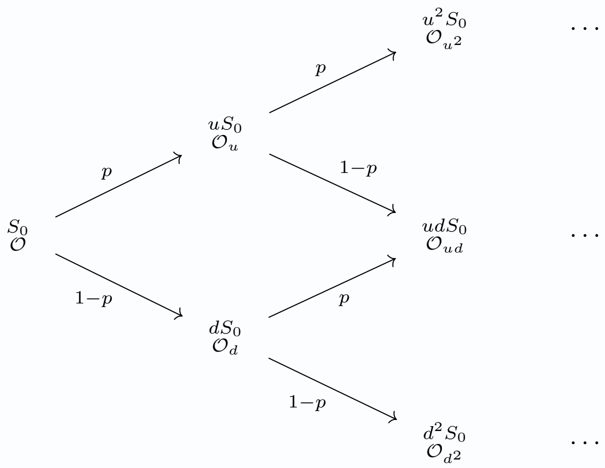
Assume there is no arbitrage opportunity and the world is risk-neutral. Assume the binomial model. Suppose that the implied volatility is σ. Then we have

$$u = e^{\sigma\sqrt{dt}} \quad \text{and} \quad d = e^{-\sigma\sqrt{dt}}$$

2.3 Multi-Period Binomial Models

Definition 2.3.1: The Multi-Period Binomial Model

Suppose that t is the current time. An asset has a current stock price of S_0 with an unknown option price $\mathcal{O}$. After a time step of dt, the stock price either increase by a multiplicative factor of $u > 1$ or the stock price decreases by a multiplicative factor of $d < 1$. Denote the corresponding option value (the net gain from the option) by $\mathcal{O}_u$ and $\mathcal{O}_d$ respectively.



The number $0 < p < 1$ is an unknown quantity denoting the probability that the stock price increases. The process iterates for n-times.

Notice at each time-step, the process is identical except with a different current stock and option price. Therefore we can iterate the one-period binomial model n-times to recover the option value.

Proposition 2.3.2

Assume there is no arbitrage opportunity and the world is risk-neutral. Assume the mutli-period binomial model with period n. Then the European call option price is given by

$$\mathcal{O} = e^{-nrdt}\left(\sum_{i=0}^{n}\binom{n}{i}p^{n-i}(1-p)^i\mathcal{O}_{u^{n-i}d^i}\right)$$

2.4 Black-Scholes-Merton Model

Definition 2.4.1: Black-Scholes-Merton Model

Let T be a set time (in years). Denote $S : \Omega \times [0, T] \to \mathbb{R}$ the stochastic processes that is the value of a certain stock. Let μ be the annual expected return. Let σ be the annual volatility of the stock price. The Black-Scholes-Merton Model makes the following assumptions

- The percentage in stock is normally distributed:

$$\frac{S_{t+dt} - S_t}{S_t} \sim N(\mu dt, \sigma^2 dt)$$

- μ and σ are constants over time.
- The risk free interest rate is constant over time.
- There are no dividends during the lifetime of the stock.
- There are no arbitrage opportunities.

Lemma 2.4.2

Let T be a set time (in years). Denote $S : \Omega \times [0, T] \to \mathbb{R}$ the stochastic processes that is the value of a certain stock. Let μ be the annual expected return. Let σ be the volatility of the stock price. Assume the Black-Scholes-Merton model. Then we have

$$\ln(S_t) \sim N\left(\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

Rate of return: The percentage return averaged over a certain amount of time.

Lemma 2.4.3

Let T be a set time (in years). Denote $S : \Omega \times [0, T] \to \mathbb{R}$ the stochastic processes that is the value of a certain stock. Assume the Black-Scholes-Merton model. Let r be the continuously compounded rate of return of S. Then we have

$$r \sim N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$$

Volatility: a measure of uncertainty of return. So it is precisely the variance of the rate of return. Hence the annual volatility is σ. Checks out.

Proposition 2.4.4

Let T be a set time (in years). Denote $S : \Omega \times [0, T] \to \mathbb{R}$ the stochastic processes that is the value of a certain stock. Assume the Black-Scholes-Merton model. Then the European call option price according to the model is

$$c = S_0 F_N(d_1) - Ke^{-rT} F_N(d_2)$$

where K is the strike price, r is the risk-free interest rate and F_N is the cumulative distribution function of $N(0, 1)$ and

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$