# Higher Category Theory

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**Abstract**

# Contents

# 1 Introduction to Infinity Categories

## 1.1 Infinity Categories as Simplicial Sets

We recall some basic facts about simplicial sets. If $S : \Delta \to \textbf{Set}$ is a simplicial set, then by Yoneda's emebdding we know that the $n$-simplices of $S$ are given by

$$S([n]) = \text{Hom}_{\textbf{sSet}}(\Delta^n, S)$$

In other words, specifying an $n$-simplex is the same as specifying a map of simplicial sets

$$\Delta^n \to S$$

The foundations of infinity categories lay on the simpicial sets. Intuitively, any face $\partial_k\Delta$ of an $n$-simplex $\Delta$ captures a homotopy of the faces of $\partial_k\Delta$.

**Definition 1.1.1: Infinity Categories**

An infinity category is a simplicial set $C$ such that each inner horn admits a filler. In other words, for all $0 < i < n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

**Definition 1.1.2: Objects and Morphisms**

Let $\mathcal{C}$ be an infinity category. Define the following notions for $\mathcal{C}$.
- Define the objects of $\mathcal{C}$ to be the 0-simplices of $\mathcal{C}$.
- Define the morphisms of $\mathcal{C}$ to be the 1-simplices of $\mathcal{C}$.

**Theorem 1.1.3**

Let $\mathcal{C}$ be a category. Every inner horn of the nerve $N(C)$ of $\mathcal{C}$ admits a filler and hence is an infinity category.

## 1.2 The Homotopy Category of Infinite Categories

Let $S$ be a simplicial set. Recall that we have functorially assigned a category $h(S)$ to $S$ called the homotopy category of $S$. This is given together with the universal functor $u : S \to N(h(S))$ by the universal property: For category $\mathcal{D}$ and a functor $F : S \to N(\mathcal{D})$, there exists a unique morphism $F : h(S) \to \mathcal{D}$ such that $F = N(G) \circ u$. When $S$ is an infinity category, compositions of morphisms forming $n$-simplexes can be shortened to one by the filler-admitting property.

**Definition 1.2.1: Homotopic Morphisms**

Let $\mathcal{C}$ be an infinity category. Two morphisms $f, g : C \to D$ are said to be homotopic if there exists a 2-simplex $\sigma$ such that
- $d_0(\sigma) = \text{id}_D$
- $d_1(\sigma) = g$
- $d_2(\sigma) = f$

In this case we write $f \simeq g$.

Pictorially, we denote the existence of such a $\sigma$ by

$$\begin{tikzcd} & D \arrow[dr, "\mathrm{id}_D"] & \\ C \arrow[ur, "f"] \arrow[rr, "g"'] & \sigma & D \end{tikzcd}$$

This diagram here does not denote commutative, but instead denotes the existence of a 2-simplex $\sigma$ that has the above as vertices and edges. Rewriting the above definition, we can say that $g \circ f : C \to E$ is homotopic to $h : C \to E$ if there exists a 2-simplex of the form

$$\begin{tikzcd} & D \arrow[dr, "g"] & \\ C \arrow[ur, "f"] \arrow[rr, "h"'] & \sigma & E \end{tikzcd}$$

By definition of an infinity category, every inner horn admits a filler. This means that for any composable morphisms $f$ and $g$ giving $g \circ f$, we can always find a morphism $h$ such that $g \circ f$ is homotopic to $h$. However, this $h$ may not be unique, so we cannot conclude that infinity categories have a well defined notion of composition.

**Proposition 1.2.2**

Let $C$ be an infinity category. Let $f, f' : C \to D$ and $g, g' : D \to E$ be morphisms in $C$. If $f \simeq f'$ and $g \simeq g'$, then

$$g \circ f \simeq g' \circ f'$$

**Lemma 1.2.3**

Homotopy is an equivalence relation in any infinity category.

We can explicitly write out the homotopy category of an infinity category as follows.

**Proposition 1.2.4**

Let $\mathcal{C}$ be an infinity category. Then the homotopy category $h(\mathcal{C})$ is isomorphic (as categories) to the category defined as follows.

- The objects of $h(\mathcal{C})$ are the objects of $\mathcal{C}$
- For $A, B \in \mathcal{C}$ two objects, the morphisms are equivalent classes of morphisms $[f]$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.
- Composition is defined by

  $$[g] \circ [f] = [g \circ f]$$

  which is well defined by .2

**Definition 1.2.5: Isomorphisms in Infinity Categories**

Let $\mathcal{C}$ be an infinity category. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. We say that $f$ is an isomorphism if $[f]$ is an isomorphism in $h(\mathcal{C})$.

## 1.3 The Infinity Category of Morphisms

Let $\mathcal{C}$ and $\mathcal{D}$ be infinity categories. Recall that the nerve functor is fully faithful. This means that there is a bijection

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathbf{sSet}}(N(\mathcal{C}), N(\mathcal{D}))$$

We generalize this bijection to define functors for infinity categories.

**Definition 1.3.1: Functors between Infinity Categories**

Let $\mathcal{C},\mathcal{D}$ be infinity categories. A functor $F:\mathcal{C}\to\mathcal{D}$ is a morphism of simplicial sets.

In other words, there is no extra structure for morphisms between infinity categories and between simplicial sets.

**Lemma 1.3.2**

Let $\mathcal{C},\mathcal{D}$ be infinity categories. Let $F:\mathcal{C}\to\mathcal{D}$ be a functor. Then the following are true.

- $F$ sends an object of $\mathcal{C}$ to an object of $\mathcal{D}$.
- $F$ sends a morphism in $\mathcal{C}$ to a morphism in $\mathcal{D}$.
- $F$ sends the identity morphism of $X\in\mathcal{C}$ to the identity morphism of $F(X)\in\mathcal{D}$.
- If $f:X\to Y$ and $g:Y\to Z$ are morphisms in $\mathcal{C}$, then $F(g\circ f)=F(g)\circ F(f)$

Explicitly, morphisms of infinity categories behave exactly what we want it to be like: A generalization of functors between ordinary categories. However, note that it is not enough to specify a morphism of infinity categories just from specifying it on objects. This is because we also need to tell the functor where to map the $n$-simplices. In other words, we need to tell the functor where to send the homotopy data.

Because the data of a functor between infinity categories carry 2-simplicies to 2-simplicies, we can easily deduce the following.

**Lemma 1.3.3**

Let $\mathcal{C},\mathcal{D}$ be infinity category. Let $F:\mathcal{C}\to\mathcal{D}$ be a functor. Then the following are true.

- If $f\simeq g$ are homotopic in $\mathcal{C}$, then $F(f)\simeq F(g)$ are homotopic in $\mathcal{D}$.
- If $f$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.

When $\mathcal{C},\mathcal{D}$ are ordinary categories, we can talk about diagrams of shape $\mathcal{C}$ in $\mathcal{D}$. This just means that we only care about the shape of $\mathcal{C}$, and we consider this shape inside $\mathcal{D}$. This was the foundations for limits and colimits of a category. We can also do this for infinity categories, but recall that a functor between infinity categories carries much more data than just the shape of the domain infinity category: it also carries homotopy information.

Now recall that for $S,T$ two simplicial sets, we can canonically identify the internal hom $[S,T]$ with the external hom $\text{Hom}_{\textbf{sSet}}(S,T)$ (What is the identification?). This gives the structure of a simplicial set with $\text{Hom}_{\textbf{sSet}}(S,T)$. When $S$ and $T$ are infinity categories, we can show that the Hom set is also an infinity category.

**Proposition 1.3.4**

Let $\mathcal{C},\mathcal{D}$ be infinity categories. Then

$$\text{Hom}_{\textbf{sSet}}(\mathcal{C},\mathcal{D})$$

is an infinity category.

## 1.4 Natural Transformations

**Definition 1.4.1: Natural Transformations**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G \in \text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{D})$ be functors. A natural transformation $\alpha : F \Rightarrow G$ from $F$ to $G$ is a morphism in $\text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{D})$.

**Proposition 1.4.2**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G \in \text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{D})$ be functors. Then $\alpha : F \Rightarrow G$ is a natural transformation if and only if $\alpha$ is a homotopy of simplicial sets from $F$ to $G$.

**Lemma 1.4.3**

Let $\mathcal{C}, \mathcal{D}$ be categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Then $\alpha : F \Rightarrow G$ is a natural transformation if and only if $N(\alpha) : N(\mathcal{C}) \to N(\mathcal{D})$ is a natural transformation of infinity categories.

**Definition 1.4.4: Natural Isomorphisms**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F, G \in \text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{D})$ be functors. A natural isomorphism from $F$ to $G$ is a natural transformation $\alpha : F \Rightarrow G$ such that $\alpha$ is an isomorphism in $\text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{D})$. In this case, we say that $F$ and $G$ are naturally isomorphic.

## 1.5 Equivalence of Infinity Categories

**Definition 1.5.1: Equivalence of Infinity Categories**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. We say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent infinity categories if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that the following are true.

- $G \circ F$ is isomorphic to $\text{id}_{\mathcal{C}}$ in $\text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{C})$
- $F \circ G$ is isomorphic to $\text{id}_{\mathcal{D}}$ in $\text{Hom}_{\textbf{sSet}}(\mathcal{D}, \mathcal{D})$

Recall that two objects in an infinity category $\mathcal{C}$ is isomorphic if they are isomorphic in $h(\mathcal{C})$ in the ordinary sense. In our case, this means that we consider $G \circ F$ and $\text{id}_{\mathcal{C}}$ to be objects of the infinity category $\text{Hom}_{\textbf{sSet}}(\mathcal{C}, \mathcal{C})$, and they are isomorphic if $[G \circ F] = [\text{id}_{\mathcal{C}}]$. This is the same as saying that $G \circ F$ and $\text{id}_{\mathcal{C}}$ are homotopic. (It is also the same as saying $\mathcal{C}$ and $\mathcal{D}$ are homotopy equivalent as simplicial sets)

**Lemma 1.5.2**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. If $\mathcal{C}$ and $\mathcal{D}$ are naturally isomorphic, then $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

**Proposition 1.5.3**

Let $\mathcal{C}, \mathcal{D}$ be ordinary categories. Let $F : \mathcal{C} \to \mathcal{D}$ be functor. Then $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories if and only if $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$ induces an equivalence of categories.

**Proposition 1.5.4**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $F$ is an equivalence of infinity categories, then $h(F) : h(\mathcal{C}) \to h(\mathcal{D})$ is an equivalence of ordinary categories.

**Proposition 1.5.5**

Let $\mathcal{C}, \mathcal{D}$ be infinity categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then $F$ is an equivalence of infinity categories if and only if

$$F \circ - : \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C}) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{D})$$

is an equivalence of infinity categories for all simplicial sets $K$.

# 2 Simplicial Categories

## 2.1 Infinity Categories as Simplicial Categories

**Definition 2.1.1: Simplicial Categories**

A simplicial category is a category $\mathcal{C}$ enriched over **sSet**. A simplicial functor is a functor $F : \mathcal{C} \to \mathcal{D}$ that is **sSet**-enriched. Denote the category of simplicial categories by

$$\mathbf{Cat}_{\mathbf{sSet}}$$

**Proposition 2.1.2**

Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a simplicial category if and only if $\mathcal{C}$ is a simplicial object in **Cat** such that the underlying simplicial set of objects is constant.

1.1.4.2 HTT

**Definition 2.1.3: Weakly Equivalent Simplicial Categories**

Let $\mathcal{C}, \mathcal{D}$ be simplicial categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a simplicial functor. We say that $F$ is a weak equivalence if the following are true.

- For all $A, B \in \mathcal{C}$, the induced map of simplicial sets
$$F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$$
is weakly equivalent.
- For all $D \in \mathcal{D}$, there exists some $C \in \mathcal{C}$ such that $F(C) \cong D$

Note: Markus land says this is weak equivalence, HTT says that this equivalence.

**Definition 2.1.4: Topological Categories**

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a topological category if $\mathcal{C}$ is enriched over **CGWH**.

Recall that two enriched categories are equivalent if $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective. Being fully faithful as $\mathcal{S}$-functor means that $F$ induces an isomorphism on Hom sets. However this notion is too strong for us because we only want to consider spaces up to homotopy equivalence.

# 3 Kan Complexes

**Lemma 3.0.1**

Let $X$ be a space. Then applying the singular functor $S(X)$ gives an infinity category.

**Proposition 3.0.2**

Let $X$ be a space. Then the homotopy category of the singular set of $X$ is equal to $h(S(X)) = \prod_1(X)$ the fundamental groupoid of $X$.

## 3.1 Kan Complexes

**Definition 3.1.1: Kan Complexes**

A Kan complex is a simplicial set $C$ such that each horn (inner and outer) admits a filler. In other words, for all $0 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \Lambda_i^n \arrow[r, "\forall"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Since infinity catregories require only inner horns to admit a filler, we have the following inclusion relation:

$$\substack{\text{Kan}\\\text{Complexes}} \subset \substack{\text{Infinity}\\\text{Categories}}$$

**Proposition 3.1.2**

Let $X$ be a space. Then $S(X)$ is a Kan complex.

**Theorem 3.1.3**

Let $\mathcal{C}$ be a small category. Then the simplicial set $N(\mathcal{C})$ is a Kan complex if and only if $\mathcal{C}$ is a groupoid.

More: Kan complexes = infinity groupoids (quillen equivalence in model category), and we should think of spaces as Kan complexes / infinity groupoids from now on.

# 4 Infinity Categorical Constructions

## 4.1 Joins and Slices

We begin by rewriting the definition of a simplex category as follows. Instead of having distinguished names $[n]$ for the objects, we instead just think of the simplex category with objects as finite and totally ordered sets. Indeed any of these sets will be in bijection to $[n]$ for some $n \in \mathbb{N}$. This language will help us define the join.

**Definition 4.1.1**

Let $J$ be a finite and totally ordered set. A cut of $J$ consists of two subsets $I, I' \subseteq J$ such that
$$J = I \amalg I'$$
and $i < i'$ for all $i \in I$ and $i' < I'$.

**Definition 4.1.2: Joins**

Let $X, Y$ be simplicial sets. Define the join of $X$ and $Y$ to be the simplicial set $X * Y$ as follows.

- Denote $J \neq \emptyset$ any finite and totally ordered set. Define
$$X * Y(J) = \coprod_{\substack{I \amalg I' = J \\ i<i' \text{ for } i \in I, i' \in I'}} X(I) \times Y(I') \quad \coprod_{I,I' \text{ cuts of } J} X(I) \times Y(I')$$
where by convention, $X(\emptyset) = Y(\emptyset) = *$.
- For two finite and totally ordered sets $J$ and $J'$ and a morphism $J \to J'$ preserving order, the map
$$(X * Y)[J'] \to (X * Y)[J]$$
is defined as follows. Let $K, K'$ be a cut of $J'$. Then $\alpha$ restricts to two well defined maps
$$\alpha|_{\alpha^{-1}(K)} : \alpha^{-1}(K) \to K \quad \text{and} \quad \alpha|_{\alpha^{-1}(K')} : \alpha^{-1}(K') \to K'$$
In particular these are order preserving, and each are morphisms in the simplex category $\Delta$. Thus this gives us a unique morphism
$$X(K) \times X(K') \to X(\alpha^{-1}(K)) \times X(\alpha^{-1}(K'))$$
By taking the product of these maps, we thus obtain a morphism $(X * Y)[J'] \to (X * Y)[J]$, turning the above definition into a simplicial set.

Concrete examples:

- When $J = [0]$, we have that
$$\begin{aligned}(X * Y)[0] &= X[0] \times Y(\emptyset) \amalg X(\emptyset) \times Y[0] \\ &= X_0 \amalg Y_0\end{aligned}$$
which means that the vertices of $X * Y$ are the vertices of $X$ and $Y$ combined disjointly.
- When $J = [1]$, we have that
$$\begin{aligned}(X * Y)[1] &= X[1] \times Y(\emptyset) \amalg X(\{0\}) \times Y(\{1\}) \amalg X(\emptyset) \times Y[1] \\ &= X_1 \amalg X_0 \times Y_0 \amalg Y_1\end{aligned}$$

TBA: The join of ordinary categories.

**Lemma 4.1.3**

Let $X$ and $Y$ be simplicial sets. Then $N(X * Y) \cong N(X) * N(Y)$

TBA: functoriality of join

**Proposition 4.1.4**

Let $X, Y$ be simplicial sets. Then $X * Y$ is an infinity category if and only if $X$ and $Y$ are infinity categories.

Recall that the over category $\mathcal{C}/X$ consists of pairs $(Y, f : Y \to X)$ and morphism are given by commutative diagrams. Let us rephrase the definition as follows. The over category is the unique category such that if $\mathcal{D}$ is another category, there is a bijection

$$\mathrm{Hom}_{\mathbf{CAT}}(\mathcal{D}, \mathcal{C}/X) \cong \mathrm{Hom}_X(\mathcal{D} * [0], \mathcal{C})$$

where the right hand side indicates that we only consider morphisms $\mathcal{D} * [0] \to \mathcal{C}$ in which $[0]$ is mapped to $X$. This characterization is due to the fact that a morphism $[0] \to \mathcal{C}$ is essentially a choice of object in $\mathcal{C}$, in which case we choose to be $X$.

**Definition 4.1.5: Over Category for Infinity Categories**

Let $K, X$ be simplicial sets. Let $f : K \to X$ be a map. Define the over category (which is a simplicial set)

$$f/X : \Delta \to \mathbf{Set}$$

as follows.

- For each $n$, we have
$$(f/X)_n = \mathrm{Hom}_{K/\mathbf{sSet}}(K * \Delta^n, X)$$

TBA: Adjunction of join and slice.

## 4.2 Mapping Spaces

**Definition 4.2.1: Mapping Spaces**

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the mapping space from $x$ to $y$ to be the pullback

$$\mathrm{Hom}_{\mathcal{C}}(x, y) = \{x\} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{0\}, \mathcal{C})} \times \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathcal{C}) \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{1\}, \mathcal{C})} \{y\}$$

Note: $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathcal{C}) \cong \mathcal{C}$ via the map $\mathrm{Ev} : \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, \mathcal{C}) \times \Delta^0 \to \mathcal{C}$.

Note: Land 1.3.47, Kerodon 4.6

## 4.3 Left and Right (Pinched) Mapping Spaces

For an ordinary category $\mathcal{C}$, we have the notion of Hom sets (at least for locally small categories). We would like to reproduce this notion for infinity categories.

Recall that a an $n$-simplex $x$ is degenerate if any two of its consecutive vertices are given by the same element. Explicitly, this means that $x$ lies in the image of some degeneracy map $s_k$.

**Definition 4.3.1: The Right Mapping Space**

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the right mapping space from $x$

to $y$ to be the simplicial set defined by

$$\mathrm{Hom}^R_{\mathcal{C}}(x,y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(x) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = y \right\}$$

for each $n \in \mathbb{N}$.

In plain English, the hom set from $x$ to $y$ on the $n$th level consists of $n+1$-simplices $h$ for which the face of $h$ with the first $n$-vertices are given by the $n$ simplex $[x, \dots, x]$, while the last vertex of $h$ is given by $y$.

**Definition 4.3.2: The Left Mapping Space**

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$ be objects. Define the left mapping space from $x$ to $y$ to be the simplicial set defined by

$$\mathrm{Hom}^L_{\mathcal{C}}(x,y)([n]) = \left\{ h \in \mathcal{C}_{n+1} \;\middle|\; d_{n+1}(h) = (\underbrace{s_0 \circ \cdots \circ s_0}_{n \text{ times}})(y) \text{ and } (d_0 \circ \cdots \circ d_n)(h) = x \right\}$$

for each $n \in \mathbb{N}$.

These two notions are equivalent up to homotopy (Land) Also pullbacks (Land)

**Proposition 4.3.3**

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$. Then both mapping spaces $\mathrm{Hom}^R_{\mathcal{C}}(x,y)$ and $\mathrm{Hom}^L_{\mathcal{C}}(x,y)$ are Kan complexes.

**Proposition 4.3.4**

Let $\mathcal{C}$ be an infinity category. Let $x, y \in \mathcal{C}$. Then the following are true.

- The right mapping space is isomorphic to the pullback

$$\mathrm{Hom}^R_{\mathcal{C}}(x,y) \cong \{x\} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{0\},\mathcal{C})} \mathcal{C}/y$$

- The left mapping space is isomorphic to the pullback

$$\mathrm{Hom}^L_{\mathcal{C}}(x,y) \cong x/\mathcal{C} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{1\},\mathcal{C})} \{y\}$$

## 4.4 Composition of Morphisms in Infinity Categories

**Definition 4.4.1**

Let $\mathcal{C}$ be an infinity category. Let $x, y, z \in \mathcal{C}$ be objects. Define the ???

$$\mathrm{Map}_{\mathcal{C}}(x,y,z)$$

of $x, y, z$ to be the pullback of the diagram:

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{sSet}}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(x,y,z)}"] & \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0 \times \Delta^0 \times \Delta^0, \mathcal{C}) \simeq \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{tikzcd}$$

in $\mathbf{sSet}$, where the vertical map is given by the inclusion of each $\Delta^0$ to a vertex of $\Delta^2$.

Rewriting notation: $\text{Map}_{\mathcal{C}}(x,y) = \text{Hom}_{\mathcal{C}}(x,y)$.

**Lemma 4.4.2**

Let $\mathcal{C}$ be an infinity category. Let $x,y,z \in \mathcal{C}$ be objects. Then the map

$$\text{Map}_{\mathcal{C}}(x,y,z) \xrightarrow{d_0 \times d_2} \text{Map}_{\mathcal{C}}(y,z) \times \text{Map}_{\mathcal{C}}(x,y)$$

is a trivial Kan fibration.

**Definition 4.4.3: Composition of Morphisms**

Let $\mathcal{C}$ be an infinity category. Let $x,y,z \in \mathcal{C}$ be objects. Let $f : x \to y$ and $g : y \to z$ be morphisms in $\mathcal{C}$. Define the composite of $f$ and $g$ to be the image of $(g,f)$ in the following:

$$\text{Map}_{\mathcal{C}}(y,z) \times \text{Map}_{\mathcal{C}}(x,y) \xrightarrow{k} \text{Map}_{\mathcal{C}}(x,y,z) \xrightarrow{d_1} \text{Map}_{\mathcal{C}}(x,z)$$

where $k$ is a choice of the homotopy inverse of $d_0 \times d_2 : \text{Map}_{\mathcal{C}}(x,y,z) \to \text{Map}_{\mathcal{C}}(y,z) \times \text{Map}_{\mathcal{C}}(x,y)$.

Upshot: Composition of morphisms are only well defined up to an equivalence class of homotopic maps.

# 5 Limits and Colimits

## 5.1 Terminal and Initial Objects

**Definition 5.1.1: Initial and Terminal Objects**

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object.

- We say that $x$ is initial if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\text{Hom}_{\mathcal{C}}(x, y) \simeq \Delta^0$$
- Dually, we say that $x$ is terminal if for all objects $y \in \mathcal{C}$, there is a homotopy equivalence
$$\text{Hom}_{\mathcal{C}}(y, x) \simeq \Delta^0$$

**Proposition 5.1.2**

Let $\mathcal{C}$ be an infinity category. Let $x \in \mathcal{C}$ be an object. Then the following are equivalent.

- $x$ is terminal.
- For all $n \geq 1$, every lifting problem of the form
$$\begin{tikzcd} \Delta^{\{n\}} \arrow[r, hook] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$
has a solution.

initial / terminal carries over by equivalence

initial in i-cat imply initial in hCat

## 5.2 Limits and Colimits

**Definition 5.2.1: Constant Diagrams**

Let $K$ be a simplicial set. Let $\mathcal{C}$ be an infinity category. Let $X \in \mathcal{C}$. Define the constant diagram at $X$ to be the unique morphism
$$\Delta X \in \text{Hom}_{\mathbf{sSet}}(K, \mathcal{C})0$$
given by the unique morphism sending $K$ to the simplicial set $\{X\}$.

**Definition 5.2.2: Induced Functor of Constant Diagrams**

Let $K$ be a simplicial set. Let $\mathcal{C}$ be an infinity category. Define the induced functor of constant diagrams
$$\Delta : \mathcal{C} \to \text{Hom}_{\mathbf{sSet}}(K, \mathcal{C})$$
as follows.

- For each $X \in \mathcal{C}$, $\Delta X$ is the constant diagram at $X$
- For each morphism $f : X \to Y$ in $\mathcal{C}$, $\Delta f : \Delta X \to \Delta Y$ is the natural transformation defined by ????

Note: Adjunct with the projection map $K \times \mathcal{C} \to \mathcal{C}$.

**Definition 5.2.3: Limits in Infinity Categories**

Let $K, \mathcal{C}$ be infinity categories. Let $F : K \to \mathcal{C}$ be a morphism. We say that $X$ is the limit of $F$ exhibited by $u : \Delta X \Rightarrow F$ if for all $Z \in \mathcal{C}$, the following composite:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C})}(\Delta X, \Delta Y) \xrightarrow{[u]\circ -} \mathrm{Hom}_{\mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C})}(\Delta X, F)$$

is a homotopy equivalence of simplicial sets.

TBA: If $K, \mathcal{C}$ are nerves of ordinary category, then the above definition recovers the usual notion of limits.

## 5.3 Equivalent Formulations of the (Co)Limit

**Proposition 5.3.1**

Let $K$ be a Kan complex. Let $\mathcal{C}$ be an infinity category. Let $F : K \to \mathcal{C}$ be a functor. Let $X \in \mathcal{C}$ be an object.

- $X$ the limit of $F$ exhibited by $u : \Delta X \Rightarrow F$ if and only if $u$ is the final object in

$$\mathcal{C} \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{0\}, \mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C}))} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^1, \mathrm{Hom}_{\mathbf{sSet}}(K, \mathcal{C})) \times_{\mathrm{Hom}_{\mathbf{sSet}}(\{1\}, \mathrm{Hom}_{\mathbf{sSet}}(K,\mathcal{C}))} \{u\}$$

# 6 Relation to Model Categories

## 6.1 Inverting Morphisms in an Infinity Category

**Definition 6.1.1**

Let $\mathcal{C}$ be an infinity category. Let $W$ be a collection of morphisms in $\mathcal{C}$. Define the category

$$\mathcal{C}[W^{-1}]$$

together with its canonical functor $F : \mathcal{C} \to \mathcal{C}[W^{-1}]$ by the following universal property.

For every infinity category $\mathcal{D}$ together with a functor $G : \mathcal{C} \to \mathcal{D}$ such that $G(f)$ is an equivalence for $f \in W$, there exists a unique functor $H : \mathcal{C}[W^{-1}] \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}[W^{-1}] \arrow[d, dashed, "\exists! H"] \\ & \mathcal{D} \end{tikzcd}$$

**Proposition 6.1.2**

Let $\mathcal{C}$ be an infinity category. Let $W$ be a collection of morphisms in $\mathcal{C}$. Then $\mathcal{C}[W^{-1}]$ exists and is unique up to equivalence of infinity categories.

Given a category $\mathcal{C}$ with weak equivalences $\mathcal{W}$, we now have a way to systematically construct an infinity category associated to $\mathcal{C}$. Namely,

$$(\mathcal{C}, \mathcal{W}) \mapsto N(\mathcal{C})[\mathcal{W}^{-1}]$$

## 6.2 Exhibiting a Model Category as an Infinity Category

Up until now, we have two ways of associating different types of categories with its homotopy category. Namely, if $\mathcal{C}$ is a model category, then we can associate to it the homotopy category $\text{Ho}(\mathcal{C})$. Similarly, if $\mathcal{D}$ is an infinity category, we can also associate to it a homotopy category $\text{Ho}(\mathcal{D})$. This constructions are highly related. In particular, there is a functor sending every model category to an infinity category such that the most important notions such as homotopy limits and colimits coincide.

Recall that for a model category $\mathcal{C}$, we denote the full subcategory spanned by cofibrant objects by $\mathcal{C}_c$.

**Definition 6.2.1**

Let $(\mathcal{C}, W)$ be a model category. Let $\mathcal{D}$ be an infinity category. Let $F : N(\mathcal{C}_c) \to \mathcal{D}$ be a functor. We say that $F$ exhibits the underlying category $\mathcal{C}$ as $\mathcal{D}$ if the functor induces an equivalence of categories

$$N(\mathcal{C}_c)[W^{-1}] \simeq \mathcal{D}$$

Ref:1.3.4.20 HA

**Theorem 6.2.2: [Dwyer-Kan]**

Let $(\mathcal{C}, \mathcal{W})$ be a model category. ??? determines a map $N(\mathcal{C}_c) \to N(\mathcal{C}_{cf})$ that induces an equivalence of infinity categories

$$N(\mathcal{C}_c)[\mathcal{W}^{-1}] \simeq N(\mathcal{C}_{cf})$$

TBA: Left Quillen equivalence implies equivalence of infinity categories.

### 6.3

Presentable iff $\mathcal{D} \simeq N(\mathcal{C}_c f)$ where $\mathcal{C}$ is a combinatorial simplicial model category.