Commutative Algebra 2

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Abstract

Contents

1 Completion

1.1 Direct and Inverse Systems

1.2 The Artin-Rees Lemma

Theorem 1.2.1: Artin-Rees Lemma

Assume A is Noetherian and I is an ideal of A. Let M be a finite module and $N \subset M$ a submodule. Then there exists $c > 0$ such that

$$I^n M \cap N = I^{n-c}(I^c M \cap N)$$

for every $n > c$.

1.3 Hensel's Lemma

2 Dimension Theory

2.1 Dimension and Height

Definition 2.1.1: Krull Dimension

Let R be a commutative ring. Define the Krull dimension of R to be

$$\dim(R) = \sup\{t \in \mathbb{N} | p_0 \subset \cdots \subset p_t \text{ for } p_0, \dots, p_t \text{ prime ideals }\}$$

Definition 2.1.2: Height of a Prime Ideal

Let p be a prime ideal in a ring R. Define the height of p to be

$$\text{ht}(p) = \sup\{t \in \mathbb{N} | p_0 \subset \cdots \subset p_t = p \text{ for } p_0, \dots, p_t \text{ prime ideals }\}$$

Lemma 2.1.3

Let p be a prime ideal in a ring R. Then $\text{ht}(p) = \dim(R_p)$.

2.2 The Hilbert Polynomial

2.3 Fundamental Theorem on Local Rings

Theorem 2.3.1

Proposition 2.3.2

Let (R, m) be a Noetherian local ring and let $k = R/m$ be the residue field. Then

$$\dim(R) \leq \dim_k(m/m^2)$$

Theorem 2.3.3: Krull's Principal Ideal Theorem

3 Regular Local Rings

3.1 Regular Local Rings

Regularity is an important concept in algebraic geometry to detecting singularities. We motivate the definition by the following proposition.

Definition 3.1.1: Regular Local Rings

A local ring R is said to be regular if $\dim_k(m/m^2) = \dim(R)$ for k the residue field of R.

Theorem 3.1.2

Let A be a Noetherian local ring of dimension 1 with maximal ideal m. Then the following are equivalent:

- A is regular
- m is principal
- A is an integral domain, and all ideals are of the form m^n for $n \geq 0$ or (0)
- A is a principal ideal domain