Hochschild Homology

Labix

August 7, 2024

Abstract

Contents

1 Hochschild Homology

1.1 Hochschild Homology

Definition 1.1.1: Hochschild Complex

Let M be an R-module. Define the Hoschild complex to be the chain complex $C(R, M)$ given as follows.

$$\cdots \longrightarrow M \otimes R^{\otimes n+1} \xrightarrow{d} M \otimes R^{\otimes n} \xrightarrow{d} M \otimes R^{\otimes n-1} \longrightarrow \cdots \longrightarrow M \otimes R \longrightarrow M \longrightarrow 0$$

The map d is defined by $d = \sum_{i=0}^{n}(-1)^i d_i$ where $d_i : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1}$ is given by the following formula.

- If $i = 0$, then $d_0(m \otimes r_1 \otimes \cdots \otimes r_n) = mr_1 \otimes r_2 \otimes \cdots \otimes r_n$
- If $i = n$, then $d_n(m \otimes r_1 \otimes \cdots \otimes r_n) = r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1}$
- Otherwise, then $d_i(m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{n-1}$

Definition 1.1.2: Hochschild Homology

Let M be an R-module. Define the Hochschild homology of M to be the homology groups of the Hochschild complex $C(R, M)$:

$$H_n(R, M) = \frac{\ker(d : M \otimes R^{\otimes n} \to M \otimes R^{\otimes n-1})}{\operatorname{im}(d : M \otimes R^{\otimes n+1} \to M \otimes R^{\otimes n})} = H_n(C(R, M))$$

If $M = R$ then we simply write

$$HH_n(R) = H_n(R, R) = H_n(C(R, R))$$

TBA: Functoriality.

Proposition 1.1.3

Let A be an R-algebra. Then $HH_n(A)$ is a $Z(A)$-module.

Proposition 1.1.4

Let A be an R-algebra. Then the following are true regarding the 0th Hochschild homology.

- Let M be an A-module. Then $H_0(A, M) = \frac{M}{\{am - ma \mid a \in A, m \in M\}}$
- The 0th Hochschild homology of A is given by $HH_0(A) = \frac{A}{[A,A]}$
- If A is commutative, then the 0th Hochschild homology is given by $HH_0(A) = A$.

Theorem 1.1.5

Let A be a commutative R-algebra. Then there is a canonical isomorphism

$$HH_1(A) \cong \Omega^1_{A/R}$$

1.2 Bar Complex

Definition 1.2.1: Enveloping Algebra

Let A be an R-algebra. Define the enveloping algebra of A to be

$$A^e = A \otimes A^{\text{op}}$$

Proposition 1.2.2

Let A be an R-algebra. Then any A, A-bimodule M equal to a left (right) A^e-module.

Definition 1.2.3: Bar Complex

Proposition 1.2.4

Let A be an R-algebra. The bar complex of A is a resolution of the A viewed as an A^e-module.

Theorem 1.2.5

Let A be an R-algebra that is projective as an R-module. If M is an A-bimodule, then there is an isomorphism

$$H_n(A, M) = \text{Tor}_n^{A^e}(M, A)$$

1.3 Relative Hochschild Homology

1.4 The Trace Map

Definition 1.4.1: The Generalized Trace Map

Let R be a ring and let M be an R-module. Define the generalized trace map

$$\text{tr} : M_r(M) \otimes M_r(A)^{\oplus n} \to M \otimes A^{\otimes n}$$

by the formula

$$\text{tr}((m_{i,j}) \otimes (a_{i,j})_1 \otimes \cdots \otimes (a_{i,j})_n) = \sum_{0 \leq i_0, \dots, i_n \leq r} m_{i_0,i_1} \otimes (a_{i_1,i_2})_1 \otimes \cdots \otimes (a_{i_n,i_0})_n$$