Vector Bundles

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Abstract

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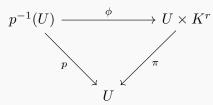
1 Vector Bundles

1.1 Basic Definitions

Definition 1.1.1: Vector Bundles

Let B, E be topological spaces and $p: E \to B$ a map. A K-vector bundle of rank r is a triple (B, E, p) such that p is a continuous surjection and that

- For every $b \in B$, the fibre $E_b = p^{-1}(b)$ is a K-vector space of dimension r.
- For every $b \in B$, there exists an open neighbourhood $U \subseteq B$ of p and a homeomorphism $((U, \phi)$ is called local trivialization) $\phi : p^{-1}(U) \to U \times K^r$ such that the map
 - The following diagram commutes



where π is the projection.

- The map

$$E_p \stackrel{\phi|_{E_b}}{\longrightarrow} \{b\} \times K^r \stackrel{\pi}{\longrightarrow} K^r$$

is a vector space isomorphism.

Definition 1.1.2: Transition Functions

Let $p: E \to B$ be a K-vector bundle of rank r. Let $(U_{\alpha}, \phi_{\alpha})$ and $(U_{\beta}, \phi_{\beta})$ be local trivialization. Define the induced map from $\phi_{\alpha} \circ \phi_{\beta}^{-1}$ to be the transition function $g_{\alpha\beta}: U_{\alpha} \cap U_{\beta} \to GL(r,K)$ where

$$g_{\alpha\beta}(p):K^r\to K^r$$

Proposition 1.1.3

Let $p:E\to B$ be a K-vector bundle of rank r. The transition functions of the vector bundle satisfies the following.

- $g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = I_r \text{ on } U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$
- $g_{\alpha\alpha} = I_r$ on U_{α}

Definition 1.1.4: Sections

A section of a vector bundle $p: E \to B$ is a map $s: B \to E$ assigning to each $b \in B$ a vector space s(b) in the fiber $p^{-1}(b)$.

Proposition 1.1.5

Let $p: E \to B$ be a vector bundle. Let s, s_1, s_2 be sections of E. Then $s_1 + s_2$ and λs are also vector bundles for any $\lambda \in \mathbb{R}$. Moreover, the set of all sections s(E) is a vector space.

Definition 1.1.6: Morphism of Vector Bundles

Let $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$ be vector bundles. A morphism of these vector bundles is given by is a pair of continuous maps $f: E_1 \to E_2$ and $g: B_1 \to B_2$ such that the following

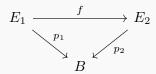
diagram commutes

$$E_1 \xrightarrow{f} E_2$$

$$\downarrow^{p_1} \qquad \downarrow^{p_2}$$

$$B_1 \xrightarrow{g} B_2$$

If $B = B_1 = B_2$ then the diagram collapses:



Definition 1.1.7: Isomorphism of Vector Bundles

A bundle homomorphism from E_1 to E_2 is an isomorphism if there exists an inverse bundle homomorphism from E_2 to E_1 . In this case, we say that E_1 and E_2 are isomorphic.

1.2 Operations on Vector Bundles

Theorem 1.2.1: Whitney Sum

Let $p_1: E_1 \to B$ and $p_2: E_2 \to B$ be two vector bundles. Define the direct sum of the vector bundles to be

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 | p_1(v_1) = p_2(v_2) \}$$

together with the projection $p: E_1 \oplus E_2 \to B$ defined by $(v_1, v_2) \mapsto p_1(v) = p_2(v)$.

The construction $E_1 \oplus E_2$ is a vector bundle over B.

Proposition 1.2.2: Tensor Product Bundle

Let $p_1: E_1 \to B$ and $p_2: E_2 \to B$ be vector bundles. Define the tensor product bundle of it to be

$$E_1 \otimes E_2 = \{p_1^{-1}(x) \otimes p_2^{-1}(x) | x \in B\}$$

The construction $E_1 \otimes E_2$ is a vector bundle over B.

Theorem 1.2.3: Pullback Bundle

Let $p: E \to Y$ be a vector bundle. Let $f: X \to Y$ be a continuous map. Then there exists E' and p' such that $p': E' \to X$ is a vector bundle.

Theorem 1.2.4: Dual Bundle

Let $p: E \to B$ be a K-vector bundle. Then the dual bundle $p^*: E^* \to B$ defined by

$$E_b^* = (E_b)^* = \operatorname{Hom}(E_b, K)$$

is a vector bundle over B.

1.3 Trivial Bundles

Definition 1.3.1: The Trivial Bundle

Let B be a base space. Define the trivial rank n bundle to be vector bundle with total space $E = B \times \mathbb{R}^n$. We say that a vector bundle is trivial if it is isomorphic to $B \times \mathbb{R}^n$.

Proposition 1.3.2

Let $p: E \to B$ be a vector bundle over a paracompact base B and $E_0 \subset E$ is a vector subbundle, then there exists a vector subbundle E_0^{\perp} such that $E_0 \otimes E_0^{\perp} = E$.

Proposition 1.3.3

For each vector bundle $E \to B$ over a compact Hausdorff space B, there exists a vector bundle $E' \to B$ such that $E \otimes E'$ is the trivial bundle