Lie Groups and Lie Algebra

Labix

October 1, 2024

Abstract

Potentially good books: Humphreys, Erdmann and Wildson

Contents

1 Introduction to Lie Algebras

1.1 Lie Algebras

Definition 1.1.1: Lie Brackets

Let V be a vector space over a field k. Let $[-,-] : V \times V \to V$ be a bilinear map. We say that $[-,-]$ is a Lie bracket if the following are true.

- The Alternating Property: $[X, X] = 0$
- Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

Consider the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ in $\mathbb{R}^3$. It is easy to see that it is a Lie bracket.

Definition 1.1.2: Lie Algebras

A Lie algebra is a vector space V over a field K together with a Lie bracket $[-,-] : V \times V \to V$.

For k a field, $M_n(k)$ for any $n \geq 1$ is a Lie algebra with Lie bracket defined as $[A, B] = AB - BA$ for $A, B \in M_n(k)$.

Lemma 1.1.3

Let L be a Lie Algebra. Then for all $x, y \in L$, we have that

$$[x, y] = -[y, x]$$

In other words, the Lie bracket is anti-commutative.

Proof. We have that

$$\begin{aligned} [x, y] + [y, x] &= [x, x] + [x, y - x] + [y, y] + [y, x - y] && \text{(Bilinearity)} \\ &= [x, x] + [y, y] - [x - y, x - y] && \text{(Bilinearity)} \\ &= 0 && \text{(Alternating)} \end{aligned}$$

and so we conclude. □

Lie Algebras are not algebras (in the sense of Rings and Modules) because the Lie bracket fails associativity. Therefore we have to redefine all the standard notions one has in algebra.

While Lie Algebras are not in general algebras, every associative algebra can be equipped with a Lie algebra. For A an associative algebra over a field, we can define a bilinear map on A by

$$[a, b] = ab - ba$$

for all $a, b \in A$. There may also be more than one way to equip an algebra with a Lie algebra structure. One should not think that Lie Algebras encompasses associative algebras because of the different Lie algebras one can equip. Instead, we think of the Lie bracket as an extra structure on associative algebras such that they become Lie algebras.

Definition 1.1.4: Structure Constants

Let L be a Lie algebra such that its underlying vector space has basis $e_1, \dots, e_n$. Define the structure constants of L to be the elements $c_{ij}^k \in \mathbb{F}$ such that

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$$

for all $1 \leq i, j \leq n$.

The structure constants are useful in the following sense. Let L be a Lie algebra and let $a = \sum_{k=1}^{n} a_k e_k$ and $b = \sum_{k=1}^{n} b_k e_k$ be elements of L. Then there Lie bracket can be written as

$$[a,b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i)[e_i, e_j]$$

by bilinearity. Plugging in the structure constants, we obtain

$$[a,b] = \sum_{1 \leq i < j \leq n} (a_i b_j - a_j b_i) \sum_{k=1}^{n} c_{ij}^k e_k$$

Thus we can write $[a,b]$ in terms of the basis $e_1, \dots, e_n$ using structure constants.

1.2 Homomorphisms and Ideals

Definition 1.2.1: Homomorphism of Lie algebra

Let V and W be Lie algebras over K. A homomorphism from V to W is an K-linear map $F : V \to W$ such that

$$[F(a), F(b)] = [a,b]$$

for all $a, b \in V$.

Definition 1.2.2: Lie Subalgebra

Let V be a Lie algebra over K. A lie subalgebra of V is a subset $W \subseteq V$ such that

- W is a vector subspace of V
- $[w_1, w_2] \in W$ for all $w_1, w_2 \in W$

It is clear that a Lie subalgebra is also a Lie algebra in its own right. Moreover, the inclusion $mapW \to V$ is a Lie algebra homomorphism.

Definition 1.2.3: Ideal

Let V be a Lie algebra over K. Let I be a subset of V. Then I is an ideal of V if the following are true.

- I is a vector subspace of V
- $[v, i] \in I$ for all $v \in V$ and $i \in I$.

It is clear from definitions that every ideal of a Lie algebra is a Lie subalgebra. However, the converse is not always true.

Proposition 1.2.4

Let V be a Lie algebra and I, J ideals of V. Then the following are also ideals of V.

- The intersection $I \cap J$
- The sum $I + J = \{i + j \mid i \in I \text{ and } j \in J\}$
- The Lie bracket $[I, J] = \langle [i, j] \mid i \in I \text{ and } j \in J \rangle$

Proposition 1.2.5

Let V be a Lie algebra over K and U an ideal of V. Then V/U has a unique Lie algebra structure such that the quotient map $V \to V/U$ is a Lie algebra homomorphism.

Definition 1.2.6: Center

Let L be a Lie algebra. Define the center of L by

$$Z(L) = \{z \in L \mid [z, x] = 0 \text{ for all } x \in L\}$$

Lemma 1.2.7

Let L be a Lie algebra. Then $Z(L)$ is an ideal of L.

Definition 1.2.8: Direct Sum of Lie Algebras

Let L_1 and L_2 be Lie algebras. Define the direct sum of L_1 and L_2 by

$$L_1 \oplus L_2 = \{(a_1, a_2) \mid a_1 \in L_1, a_2 \in L_2\}$$

together with component wise addition and scalar multiplication and Lie bracket operation

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1], [a_2, b_2])$$

which is component wise application of the Lie bracket for $(a_1, a_2), (b_1, b_2) \in L_1 \oplus L_2$.

Proposition 1.2.9

Let L_1 and L_2 be Lie algebras. Then the following are true.

- $[L_1 \oplus L_2, L_1 \oplus L_2] = [L_1, L_1] \oplus [L_2, L_2]$
- $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$
- $\{(x, 0) \mid x \in L_1\} \cong L_1$ is an ideal of $L_1 \oplus L_2$
- $\{(0, y) \mid y \in L_2\} \cong L_2$ is an ideal of $L_1 \oplus L_2$

1.3 The Isomorphism Theorems

Theorem 1.3.1: First Isomorphism Theorem

Let $\phi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then the following are true.

- $\ker(\phi)$ is an ideal of L_1
- $\operatorname{im}(\phi)$ is a Lie subalgebra of L_2

Moreover, we have an isomorphism

$$\frac{L_1}{\ker(\phi)} \cong \operatorname{im}(\phi)$$

Theorem 1.3.2: Second Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L. Then the following are true.

- I and J are ideals of $I + J$
- $I \cap J$ is an ideal of I and J

Moreover, we have an isomorphism

$$\frac{I + J}{J} \cong \frac{I}{I \cap J}$$

Theorem 1.3.3: Third Isomorphism Theorem

Let L be a Lie algebra. Let I and J be ideals of L such that $I \subseteq J$. Then J/I is an ideal of

L/I. Moreover, there is an isomorphism

$$\frac{L/I}{J/I} \cong \frac{L}{J}$$

Theorem 1.3.4: Correspondence Theorem

Let L be a Lie algebra with ideal I. Then there exists a bijective correspondence

$$\{J \mid J \text{ is an ideal of } L \text{ and } I \subseteq J\} \quad \stackrel{1:1}{\longleftrightarrow} \quad \{K \mid K \text{ is an ideal of } L/I\}$$

2 Types of Lie Algebras

2.1 Nilpotent Lie Algebras

2.2 Soluble Lie Algebras

Let L be a Lie algebra. We have seen that $\text{rad}(L)$ is soluble and $L/\text{rad}(L)$ is semisimple. Therefore to study a general Lie algebra, we need to understand soluble Lie algebras and semisimple Lie algebras. If we restrict the case to Lie algebras over $\mathbb{C}$, Lie's theorem will solve the first part of the problem, while the study of semisimple Lie algebras is postponed until section 5.

Theorem 2.2.1: Lie's Theorem

Let V be a vector space over $\mathbb{C}$. Let L be a soluble Lie subalgebra of $GL(V)$. Then there exists a basis B of V such that for all $M \in L$, M is upper triangular.

2.3 Simple and Semisimple Lie Algebras

3 The Killing Form and Cartan's Criteria

4 Introduction to Lie Groups

4.1 Lie Groups

Definition 4.1.1: Lie Groups

A Lie group G is a smooth manifold which is also a group such that the multiplication map $G \times G \to G$ given by $(g, h) \mapsto gh$ and the inverse map $i : G \to G$ given by $g \mapsto g^{-1}$ are smooth maps.

Proposition 4.1.2

Let G be a Lie group. A subgroup H of G has the unique structure of a Lie subgroup if H is closed in G.

4.2 Relation between Lie Groups and Lie Algebras

For a group G, denote the left multiplication map of $h \in G$ by l_h. If G is a Lie group, we have seen that l_h is a smooth map, and so it induces a differential $(l_h)_*$.

Definition 4.2.1: Left Invariant Vector Field

Let G be a Lie group and X a vector field on G. We say that X is left invariant if

$$(l_h)_*(X_g) = X_{hg}$$

for all $X_g \in T_g(G)$.

Proposition 4.2.2

Let G be a Lie group. The vector space of left invariant vector fields of G is a Lie algebra of dimension $\dim(G)$. Moreover, if $X_e \in T_e(G)$ is a tangent vector at e the identity, then there is a unique left invariant vector field X on G such that its identity is X_e.

Definition 4.2.3: Lie Algebra of a Lie Group

Let G be a Lie group. Define the Lie algebra V of G to be the vector space $T_e(G)$.

Recall that given a homomorphism of Lie groups $\phi : G \to H$, it induces a differential $\phi_* : T_g(G) \to T_{\phi(g)}(H)$.

Proposition 4.2.4

Let $\phi : G \to H$ be a homomorphism of Lie groups with Lie algebras V and W respectively. Then the induced map from the differential $\phi_* : V \to W$ is a Lie algebra homomorphism.

5 Root Systems

6 Representation of Lie Algebras

7 Classification of Semisimple Lie Algebras over $\mathbb{C}$