Fiber Bundles

Labix

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Abstract

- Notes on Algebraic Topology by Oscar Randal-Williams

Contents

1 A Convenient Category of Spaces

Reason:

- Want $- \wedge -$ associative and unital and commutative (so that the category is symmetric monoidal)
- Want adjunction $- \times X$ and $\text{Hom}_{\mathcal{C}}(X, -)$ (non pointed)
- Want adjunction $- \wedge X$ and $\text{Map}_*(X, -)$ (pointed) (Intuitively, $X \wedge Y$ represents maps from $X \times Y$ that are base point preserving separately in each variable)

1.1 Compactly Generated Spaces

Definition 1.1.1: Compactly Generated Spaces

Let X be a space. We say that X is compactly generated (k-space) if for every set $A \subseteq X$, A is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Definition 1.1.2: Category of Compactly Generated Spaces

Define the category of compactly generated spaces $\mathbf{CG}$ to be the full subcategory of $\mathbf{Top}$ consisting of spaces that are compactly generated. In other words, $\mathbf{CG}$ consists of the following data:

- Obj($\mathbf{CG}$) consists of all spaces that are compactly generated.
- For $X, Y \in$ Obj($\mathbf{CG}$), the morphisms are

$$\text{Hom}_{\mathbf{CG}}(X, Y) = \text{Hom}_{\mathbf{Top}}(X, Y)$$

- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces $\mathbf{CG}_*$.

Definition 1.1.3: New k-space from Old

Let X be a space. Define $k(X)$ to be the set X together with the topology defined as follows: $A \subseteq X$ is open if and only if $A \cap K$ is open in K for every compact subspace $K \subseteq X$.

Lemma 1.1.4

Let X be a space. Then $k(X)$ is a compactly generated space.

Unfortunately $X \times Y$ may not be compactly generated even when X and Y are. But as it turns out, products do exists in $\mathbf{CG}$ and are given by $X \times_{\mathbf{CG}} Y = k(X \times_{\mathbf{Top}} Y)$.

Proposition 1.1.5

Let X, Y be compactly generated spaces. Then the categorical product of X and Y in the category of compactly generated spaces is given by

$$X \times_{\mathbf{CG}} Y = k(X \times_{\mathbf{Top}} Y)$$

Proposition 1.1.6

Every CW complex is compactly generated.

Definition 1.1.7: Category of Compactly Generated and Weakly Hausdorff Spaces

Define the category of compactly generated and weakly Hausdorff spaces $\mathbf{CGWH}$ to be the full subcategory of $\mathbf{Top}$ consisting of spaces that are compactly generated and weakly Hausdorff. In other words, $\mathbf{CGWH}$ consists of the following data:

- $\text{Obj}(\mathbf{CGWH})$ consists of all spaces that are compactly generated and weakly Hausdorff.
- For $X, Y \in \text{Obj}(\mathbf{CGWH})$, the morphisms are

$$\text{Hom}_{\mathbf{CGWH}}(X, Y) = \text{Hom}_{\mathbf{Top}}(X, Y)$$

- Association is given by composition of functions.

Define similarly the category of pointed compactly generated spaces $\mathbf{CGWH}_*$.

Proposition 1.1.8

A compactly generated space X is weakly Hausdorff if and only if the diagonal subspace $\Delta = \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

Proposition 1.1.9

Product of CGWH is CGWH

CGWH is complete and cocomplete

1.2 The Cartesian Product and the Mapping Space

Definition 1.2.1: The Mapping Space

Let $X, Y \in \mathbf{CG}$. Define the mapping space of X and Y by

$$\text{Map}(X, Y) = k(\text{Hom}_{\mathbf{CG}}(X, Y))$$

where $\text{Hom}_{\mathbf{CG}}(X, Y)$ is equipped with the compact open topology. If (X, x_0) and (Y, y_0) are pointed spaces, define the mapping space to be

$$\text{Map}_*((X, x_0), (Y, y_0)) = k(\text{Hom}_{\mathbf{CG}}((X, x_0), (Y, y_0)))$$

By restricting to also weakly Hausdorff spaces, we obtain an adjunction.

Theorem 1.2.2

Let $X, Y, Z \in \mathbf{CGWH}$. Then the functors $- \times_{\mathbf{CGWH}} Y : \mathbf{CGWH} \to \mathbf{CGWH}$ and $\text{Map}(Y, -) : \mathbf{CGWH} \to \mathbf{CGWH}$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathbf{CGWH}}(X \times_{\mathbf{CGWH}} Y, Z) \cong \text{Hom}_{\mathbf{CGWH}}(X, \text{Map}(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}(X \times_{\mathbf{CGWH}} Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$$

1.3 The Smash Product and the Pointed Mapping Space

Aside from the adjunction between the product space and the mapping space, another major reason one considers compactly generated spaces is that the smash product gives another adjunction.

Definition 1.3.1: The Smash Product

Let (X, x_0) and (Y, y_0) be pointed topological spaces. Define the smash product of the two pointed spaces to be the pointed space

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

together with the point (x_0, y_0).

Proposition 1.3.2

Let X, Y, Z be compactly generated spaces with a chosen base point. Then the following are true.
- $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$
- $X \wedge Y \cong Y \wedge X$

Theorem 1.3.3

The category **CG** of compactly generated spaces is a symmetric monoidal category with operator the smash product $\wedge : \mathbf{CG} \times \mathbf{CG} \to \mathbf{CG}$ and the unit S^0.

Note that this is not true if we do not restrict the spaces to the category of compactly generated spaces.

Lemma 1.3.4

Let X be a pointed space. Then the reduced suspension and the smash product with the circle

$$\Sigma X \cong X \wedge S^1$$

are homeomorphic spaces.

Theorem 1.3.5

Let X, Y, Z be compactly generated with a chosen basepoint. Then the functors $- \wedge Y : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(Y, -) : \mathcal{K}_* \to \mathcal{K}_*$ are adjoint functors with the adjunction formula

$$\text{Hom}_{\mathcal{K}_*}(X \wedge Y, Z) \cong \text{Hom}_{\mathcal{K}_*}(X, \text{Map}_*(Y, Z))$$

Moreover, by giving the Hom set the compact open topology and applying k, we obtain an isomorphism

$$\text{Map}_*(X \wedge Y, Z) \cong \text{Map}_*(X, \text{Map}_*(Y, Z))$$

By choosing $Y = I$ in the adjunction, we recover the usual suspension-loopspace adjunction in $\mathbf{Top}_*$.

Corollary 1.3.6

Let X be a compactly generated space with a chosen basepoint. Then there is a natural homeomorphism

$$\text{Map}_*(\Sigma X, Y) \cong \text{Map}_*(X, \Omega Y)$$

given by adjunction of the functors $- \wedge S^1 : \mathcal{K}_* \to \mathcal{K}_*$ and $\text{Map}_*(S^1, -) : \mathcal{K}_* \to \mathcal{K}_*$.

1.4 The Mapping Cylinder and the Mapping Path Space

Equipped with the Cartesian closed structure in **CG** together with a canonical topology on the mapping space Y^X, we can now talk about the duality between the mapping cylinder and the mapping path space.

Definition 1.4.1: Mapping Cylinder

Let X, Y be spaces and let $f : X \to Y$ a map. Define the mapping cylinder of f to be

$$M_f = \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)} = (X \times I) \amalg_f Y$$

for $f : X \times \{1\} \cong X \to Y$ together with the quotient topology. It is the push forward of f and the inclusion map $i_0 : X \cong X \times \{0\} \hookrightarrow X \times I$.

Lemma 1.4.2

Let X, Y be spaces and let $f : X \to Y$ be a map. Then Y is a deformation retract of M_f.

Definition 1.4.3: The Mapping Path Space

Let X, Y be spaces and let $f : X \to Y$ be a map. Define the mapping path space of f to be

$$P_f = \{(x, \gamma) \in X \times \text{Map}(I, Y) \mid \gamma(0) = f(x)\}$$

It is the pull back of f and $\pi_0 : \text{Map}(I, Y) \to Y$ given by $\pi_0(\gamma) = \gamma(0)$ in **CG**.

2 Fibrations and Cofibrations

2.1 The Relative Point of View

Definition 2.1.1: Fibers of a Map

Let X, Y be spaces. Let $f : X \to Y$ be a map. Define the fiber of f at $y \in Y$ to be

$$\text{Fib}_y(f) = f^{-1}(y)$$

Definition 2.1.2: Cofibers of a Map

Let X, Y be spaces. Let $f : X \to Y$ be a map. Define the cofiber of f to be

$$\text{Cofib}(f) = \frac{Y}{f(X)}$$

Up until this point, in algebraic topology we have asked questions relating to two spaces and tried to answer them. For instance, we can ask whether two spaces are homeomorphic, homotopy equivalent or weakly equivalent. We can also ask these questions in a relative setting, this involves considering maps of spaces as objects themselves, instead of just the spaces.

Definition 2.1.3: Maps of Maps

Let X, Y, A, B be spaces. Let $f : X \to Y$ and $g : A \to B$ be maps. A map from f to g is a pair of maps $(\alpha : X \to A, \beta : Y \to B)$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

Definition 2.1.4: Homotopy from Maps to Maps

Let X, Y, A, B be spaces. Let $f : X \to Y$ and $g : A \to B$ be maps. Let (a, b) and (c, d) be two maps from f to g. We say that (a, b) and (c, d) are homotopic if there exists maps $H : X \times I \to Y$ and $K : A \times I \to B$ such that the following diagram

$$\begin{tikzcd}
X \times I \arrow[r, "f\times\text{id}_I"] \arrow[d, "H"'] & Y \times I \arrow[d, "K"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}$$

commutes and the following are true.

- H is a homotopy from a to c
- K is a homotopy from b to d

We now restrict to the point of view where maps are over a fixed space Z (In other words we are considering objects of $\textbf{Top}_Z$). A map of maps becomes the following data: Let $f : X \to Z$ and $g : Y \to Z$ be maps. A map from f to g is a map $h : X \to Y$ such that $g \circ h = f$. In other words the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

A homotopy then becomes the following data: Let $f : X \to Z$ and $g : Y \to Z$ be two maps. Let $h, k : X \to Y$ be two maps from f to g. We say that h and k are homotopic if there exists a homotopy $H : X \times I \to Y$ from h to k such that $H(-, t)$ for each t is a map over Z. This means that the following diagram commutes:

$$\begin{tikzcd} X \times I \arrow[rr, "H"] \arrow[dr, "f \circ p_X"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

We can now discuss homotopy equivalences on these maps.

Definition 2.1.5: Fiber Homotopy Equivalence

Let X, Y, Z be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. We say that f and g are homotopy equivalent if there exists two maps $h : X \to Y$ and $k : Y \to X$ such that $k \circ h$ and $h \circ k$ are both homotopic to the identity over Z.

The reason that it is called a fiber homotopy equivalence is because it gives homotopy equivalences on fibers.

Proposition 2.1.6

Let X, Y, Z be spaces. Let $f : X \to B$ and $g : Y \to B$ be maps. If f and g are homotopy equivalent, then for any $b \in B$, the fibers

$$\text{Fib}_b(f) \simeq \text{Fib}_b(g)$$

are homotopy equivalent.

Proof. Suppose that f and g are homotopy equivalent via two maps $h : X \to Y$ and $k : Y \to X$. This means that there exists a homotopy $H : X \times I \to X$ such that $H(-, 0) = h \circ k$ and $H(-, 1) = \text{id}_X$. Similarly, there exists a homotopy $K : Y \times I \to Y$ such that $K(-, 0) = k \circ h$ and $K(-, 1) = \text{id}_Y$. Consider the map $h|_{f^{-1}(b)} : f^{-1}(b) \to g^{-1}(b)$ and similarly for $k|_{g^{-1}(b)}$. Define two maps $\overline{H} : H|_{f^{-1}(b) \times I}$ and $\overline{K} = K|_{g^{-1}(b) \times I}$. To show that they are homotopies, we just need to show that $\overline{H} \subseteq f^{-1}(b)$ and similarly for $\overline{K}$. Now by definition, each $H(-, t) : X \to Y$ is such that $g(H(-, t)) = f$. Choose $x \in f^{-1}(b)$. Then

$$g(H(x, t)) = f(x) = b$$

Hence the entire homotopy $\overline{H}$ stays in the fiber $f^{-1}(b)$. Hence $\overline{H}$ is a well defined homotopy on $f^{-1}(b)$. Similarly for $\overline{K}$. Hence the two fibers are homotopy equivalent. $\square$

Unfortunately for most maps $f : X \to Y$, the fibers themselves are not homeomorphic, and not even homotopy equivalent.

Example 2.1.7

The fibers of the projection map $S^1 \to \mathbb{R}$ to the x-axis are not homotopy equivalent.

Proof. It is clear that the fiber of $S^1 \to \mathbb{R}$ is either empty, consist of one point, or of two points. Neither two of the three are homotopy equivalent. $\square$

There are two ways to proceed from here. We first try to find a set of maps in which all fibers are homotopy equivalent. This is the content of this section. Otherwise, we try and define a new notion of fiber so that we obtain homotopy equivalence. This is the content of the next section.

2.2 Fibrations and The Homotopy Lifting Property

Definition 2.2.1: The Homotopy Lifting Property

Let $p : E \to B$ be a map and let X be a space. We say that p has the homotopy lifting property with respect to X if for every homotopy $H : X \times I \to B$ and a lift $\widetilde{H(-,0)} : X \to E$ of $H(-,0)$, there exists a homotopy $\widetilde{H} : X \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook, "\iota"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists \widetilde{H}"] & B \end{tikzcd}$$

Definition 2.2.2: Fibrations

We say that a map $p : E \to B$ is a fibration if it has the homotopy lifting property with respect to all topological spaces X. We call B the base space and E the total space.

Definition 2.2.3: The Hopf Fibration

Define the Hopf fibration $h : S^3 \to S^2$ as follows. Consider S^2 as the one point compactification of $\mathbb{C}$. Also consider $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Define the map h by

$$(z_1, z_2) \to \frac{z_2}{z_1}$$

Example 2.2.4

The Hopf fibration $h : S^3 \to S^2$ is a fibration. Moreover, the fibers of the Hopf fibration are circles S^1.

Proof. We can rewrite the coordinates of S^3 by $r_j e^{i\theta_j}$. Then

$$h(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \frac{r_2}{r_1} e^{i(\theta_2 - \theta_1)}$$

Fix $re^{i\theta} \in S^2$. Then there exists a unique pair (r_1, r_2) that solves the simultaneous equation $rr_1 = r_2$ and $r_1^2 + r_2^2 = 1$. □

We will see that fibrations are a very well behaved class of maps in **Top**.

Lemma 2.2.5

Let X, Y, Z be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Let $h : X \to Y$ be a map over Z. If f and g are fibrations, then h is a homotopy equivalence if and only if it is a fiber homotopy equivalence.

Proposition 2.2.6

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f : X \to Y$ be a map. Then the map

$$q : P_f \to Y$$

given by $q(x, \gamma) = \gamma(1)$ is a fibration.

Proof. Suppose that we are given a homotopy lifting problem:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g"] \arrow[d, hook, "\iota"'] & P_f \arrow[d, "q"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & Y \end{tikzcd}$$

We write $g(a) = (g_1(a), g_2(a))$ for the components of g. Now recall that the definition of the mapping path space implies that $f(g_1(a)) = g_2(a)(0)$. By commutativity of the diagram and definition of q we also have $g_2(a)(1) = H(a, 0)$. Define $\tilde{H} : A \times I \to P_f$ by the fomula

$$\tilde{H}(a,t) = (g_1(a), h_2(a,t))$$

where

$$h_2(a,t)(s) = \begin{cases} g_2(a)(1+t)(s) & \text{if } 0 \leq s \leq \frac{1}{1+t} \\ H(a, (1+t)s - 1) & \text{if } \frac{1}{1+t} \leq s \leq 1 \end{cases}$$

The definition of h_2 makes sense because $g_2(a)(1) = H(a, 0)$. By the gluing lemma $h_2(a,t)$ is continuous. h_2 is also continuous in a and t because $g_2(a)$ is a path and g_2 is continuous and H is continuous in both variables and the composite of continuous functions are continuous. Hence $\tilde{H}$ is continuous. Now $\tilde{H}(-,0) = (g_1(-), g_2(-)) = g(-)$. Thus $\tilde{H}(-,0)$ is a lift of g. It remains to show that $\tilde{H}$ is a lift of H. We have that

$$\begin{aligned} q(\tilde{H}(a,t)) &= q(g_1(a), h_2(a,t)) \\ &= h_2(a,t)(1) \\ &= H(a,t) \end{aligned}$$

and so we conclude. $\square$

We can factorize any continuous map into a fibration and a homotopy equivalence through the mapping path space. Because we are working with the mapping path space here, we need to restrict our attention to compactly generated space.

Theorem 2.2.7

Let $X, Y \in$ **CGWH**. Let $f : X \to Y$ be a map. Then there exists a homotopy equivalence $h : X \to P_f$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, dashed, "\exists h"'] & & Y \\ & P_f \arrow[ur, "q"'] & \end{tikzcd}$$

Proof. Define the map $h : X \to P_f$ by $h(x) = (x, e_{f(x)})$. It is easy to see that $q \circ h = f$. I claim that the projection map $p_X : P_f \to X$ gives the homotopy inverse of h. Define a map $H : P_f \times I \to P_f$ by

$$H(x, \gamma, t) = (x, \gamma_t)$$

where γ_s is the path $s \mapsto \gamma(st)$. It is continuous since the composition of continuous functions are continuous and each component of H is continuous. Also, we have that $h(p_X(x,\gamma)) = h(x) = (x, e_{f(x)})$ and $H(x,\gamma,0) = (x,\gamma_0) = (x, e_{f(x)})$ so that $H(-,0) = h \circ p_X$. When $t = 1$ we also have

$$H(-,1) = (x,\gamma,1) = (x,\gamma_1) = (x,\gamma) = \mathrm{id}_{P_f}(x,\gamma)$$

so that H is a homotopy. $\square$

Proposition 2.2.8

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f : X \to Y$ be a map. Let $h : X \to P_f$ be the map that gives a factorization $q \circ h = f$. If f is a fibration, then h is a fiber homotopy equivalence.

2.3 Cofibrations and The Homotopy Extension Property

Definition 2.3.1: The Homotopy Extension Property

Let $i : A \to X$ be a map and let Y be a space. Denote i_0 the inclusion map $A \times \{0\} \hookrightarrow A \times I$. We say that i has the homotopy extension property with respect to Y if for every homotopy $H : A \times I \to Y$ and every map $f : X \to Y$ such that

$$H \circ i_0 = f \circ i$$

there exists a homotopy $\widetilde{H} : X \times I \to Y$ such that the following diagram commute:

$$\begin{tikzcd}
A \cong A \times \{0\} \arrow[r, "\iota_0"] \arrow[d, hook, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}_I"] \arrow[ddr, bend left, "H"] & \\
X \cong X \times \{0\} \arrow[r, "\iota_0"'] \arrow[drr, bend right, "f"'] & X \times I \arrow[dr, dashed, "\exists \tilde{H}"] & \\
& & Y
\end{tikzcd}$$

Definition 2.3.2: Cofibrations

Let A, X be spaces. Let $i : A \to X$ be a map. We say that i is a cofibration if it has the homotopy extension property for all spaces Y.

Proposition 2.3.3

Let A, X be spaces. Let $i : A \to X$ be a cofibration. Then $i : A \to i(A)$ is a homeomorphism.

There is actually an easier way to write out cofibrations when (X, A) is a pair of spaces.

Lemma 2.3.4

Let (X, A) be a pair of spaces with A closed in X. Let $\iota : A \to X$ be the inclusion. Then ι is a cofibration if and only if for all spaces Y and maps $f : X \to Y$ and $H : A \times I \to Y$, there exists a map $\tilde{H} : X \times I \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "f \cup H"] \arrow[d, "\iota"'] & Y \\
X \times I \arrow[ur, "\tilde{H}"'] &
\end{tikzcd}$$

Cofibrations and fibrations are dual in the following sense. Recall from section 1 that if X and Y are in $\mathbf{CGWH}$, then there is a bijection

$$\mathrm{Hom}_{\mathbf{CGWH}}(X \times I, Y) \cong \mathrm{Hom}_{\mathbf{CGWH}}(X, \mathrm{Map}(I, Y))$$

Now under this bijection, we can rewrite the diagram in the homotopy lifting property:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}_X"'] & E^I \arrow[d, "p_*"] \\ X \arrow[r, "H"'] \arrow[ur, dashed, "\exists \tilde{H}"] & B^I \end{tikzcd}$$

Proposition 2.3.5

Let $A, X \in \mathbf{CGWH}$ be spaces. Let $f : A \to X$ be a map. Then the map

$$q : A \to M_f$$

given by $q(a) = [a, 0]$ is a cofibration.

Proof. Suppose that we are given a homotopy lifting problem:

$$\begin{tikzcd} A \cong A \times \{0\} \arrow[r, "\iota_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}_I"] \arrow[ddr, bend left, "H"] & \\ X \cong X \times \{0\} \arrow[r, "\iota_0"'] \arrow[drr, bend right, "f"] & X \times I \arrow[dr, dashed, "\tilde{H}"] & \\ & & Y \end{tikzcd}$$

□

Dual to the factorization through the mapping path space, we can factorize a map into a homotopy equivalence and a cofibration through the mapping cylinder

$$M_f = \frac{(X \times I) \amalg Y}{(x, 0) \sim f(x)} = (X \times I) \amalg_f Y$$

Theorem 2.3.6

Let $f : A \to X$ be a map. Then the inclusion map $i : A \to M_f$ defined by $i(a) = [a, 0]$ is a cofibration. Moreover, there exists a homotopy equivalence $h : M_f \to X$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & X \\ & M_f \arrow[ur, dashed, "\exists h"'] & \end{tikzcd}$$

2.4 Basic Properties of Fibrations and Cofibrations

Proposition 2.4.1

Let $X_1, X_2, Y_{,1}, Y_2 \in \mathbf{CGWH}$. Let $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ be maps. Then the following are true.

- If p_1 and p_2 are fibrations then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a fibration.
- If p_1 and p_2 are cofibrations then $p_1 \coprod p_2 : X_1 \coprod X_2 \to Y_1 \coprod Y_2$ is a cofibration.

Proposition 2.4.2

Let $X, Y, Z \in \mathbf{CGWH}$. Let $f : X \to Y$ be a map.

- Let f be a fibration. Consider the following lifting problem:

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "g"] \arrow[d, hook, "i_0"'] & X \arrow[d, "f"] \\ Z \times I \arrow[r, "h"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

If h_0 and h_1 are both solutions to the lifting problem, then h_0 and h_1 are homotopic relative to $Z \times \{0\}$.

- Let f be a cofibration. Consider the following extension problem:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"'] & Z \times \{0\} \\ Y \arrow[r, "h"'] \arrow[ur, dashed] & Z \times I \arrow[u, hook', "\text{ev}_0"'] \end{tikzcd}$$

If h_0 and h_1 are both solutions to the extension problem, then h_0 and h_1 are homotopic relative to Z.

Proposition 2.4.3

Let $X, Y, Z \in \mathbf{CGWH}$. Let $f : X \to Y$ be a map. Then the following are true.

- If f is a fibration, then the induced map

$$f_* : \text{Map}(Z, X) \to \text{Map}(Z, Y)$$

is a fibration.

- If f is a cofibration, then the map

$$f \times \text{id}_Z : X \times Z \to Y \times Z$$

is a cofibration.

Proposition 2.4.4

Let $X, Y, Z \in \mathbf{CGWH}$. Let $p : X \to Y$ be a map.

- If p is a fibration and $f : Z \to Y$ is a map, then the pullback $X \times_Y Z \to Z$ of p and f is a fibration

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d, dashed, "\text{fibration}"'] & X \arrow[d, "p"] \\ Z \arrow[r, "f"'] & Y \end{tikzcd}$$

- If p is a cofibration and $g : X \to Z$ is a map, then the push forward $Z \to Z \coprod_X Y$ of p and g is a cofibration

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & Z \arrow[d, dashed, "\text{cofibration}"] \\ Y \arrow[r] & Z \coprod_X Y \end{tikzcd}$$

Proposition 2.4.5

Let $X, Y, Z \in \mathbf{CGWH}$. Let $p : X \to Y$ be a map.

- If p is a fibration and $f : Z \to Y$ is a (homotopy) weak equivalence, then the pullback $X \times_Y Z \to X$ of p and f is a (homotopy) weak equivalence

$$\begin{tikzcd} X \times_Y Z \arrow[r, dashed, "\simeq"] \arrow[d] & X \arrow[d, "p"] \\ Z \arrow[r, "f{,}\simeq"'] & Y \end{tikzcd}$$

- If p is a cofibration and $g : X \to Z$ is a (homotopy) weak equivalence, then the push forward $Y \to Z \coprod_X Y$ of p and g is a (homotopy) weak equivalence

$$\begin{tikzcd} X \arrow[r, "g{,}\simeq"] \arrow[d, "p"'] & Z \arrow[d] \\ Y \arrow[r, dashed, "\simeq"'] & Z \coprod_X Y \end{tikzcd}$$

Proposition 2.4.6

Let E_1, E_2, B_1, B_2 be spaces. Let $p : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be fibrations. Let $(f : E_1 \to E_2, g : B_1 \to B_2)$ be a map from p_1 to p_2. If f and g are homotopy equivalences, then for any $b_1 \in B_1$, the fibers

$$\text{Fib}_{b_1}(p_1) \simeq \text{Fib}_{g(b_1)}(p_2)$$

are homotopy equivalent. This is displayed in the following diagram:

$$\begin{tikzcd} \text{Fib}_{b_1}(p_1) \arrow[r, "\simeq"] \arrow[d, hook] & \text{Fib}_{g(b_1)}(p_2) \arrow[d, hook] \\ E_1 \arrow[r, "f{,}\simeq"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g{,}\simeq"'] & B_2 \end{tikzcd}$$

2.5 Long Exact Sequences from (Co)Fibrations

Theorem 2.5.1: Homotopy Long Exact Sequence in Fibration

Let $p : E \to B$ be a fibration over a path connected space B with fiber F. Let $\iota : F \hookrightarrow E$ be the inclusion of the fiber. Then there is a long exact sequence in homotopy groups:

$$\cdots \longrightarrow \pi_{n+1}(B, b_0) \xrightarrow{\partial} \pi_n(F, e_0) \xrightarrow{\iota_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial} \pi_{n-1}(F, e_0) \longrightarrow \cdots \longrightarrow \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0)$$

for $e_0 \in E$ and $b_0 = p(e_0)$. Moreover, p_* is an isomorphism.

Theorem 2.5.2: Homology Long Exact Sequence in Cofibration

Let $p : X \to Y$ be a cofibration with cofiber $C = \frac{Y}{p(X)}$. Let proj : $Y \to C$ be the projection map. Then there is a long exact sequence in homology groups:

$$\cdots \longrightarrow \widetilde{H}_{n+1}(C) \xrightarrow{\partial} \widetilde{H}_n(X) \xrightarrow{f_*} \widetilde{H}_n(Y) \xrightarrow{\text{proj}_*} \widetilde{H}_n(C) \xrightarrow{\partial} \widetilde{H}_{n-1}(X \longrightarrow \cdots \longrightarrow \widetilde{H}_0(Y) \xrightarrow{\text{proj}_*} \widetilde{H}_0(B, b_0)$$

2.6 (Co)Fibers of a (Co)Fibration are Homotopic

The following definition is a supporting notion for our proof that fibers of a fibration are homotopy equivalent.

Definition 2.6.1: Induced Map of Fibers

Let $p : E \to B$. Let $\gamma : I \to B$ be a path from b_1 to b_2. Define the induced map of fibers of γ as follows: The map $H : E_{b_1} \times I \to B$ defined by $H(x,t) = \gamma(t)$ is a homotopy. Using the HLP of p, we obtain a lift:

$$\begin{tikzcd} E_{b_1} \times \{0\} \arrow[r, "\widetilde{H(-,0)}"] \arrow[d, hook] & E \arrow[d, "p"] \\ E_{b_1} \times I \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B \end{tikzcd}$$

Since $p \circ \widetilde{H}(x,t) = \gamma(t)$, we have that $\widetilde{H}(x,1) \in E_{b_2}$. The induced map of fibers is then the map

$$L_\gamma : E_{b_1} \to E_{b_2}$$

defined by $L_\gamma = \widetilde{H(-,1)}$

Lemma 2.6.2

Let $p : E \to B$ be a fibration. Let $\gamma : I \to B$ be a path from b_1 to b_2. Then the following are true regarding L_γ.

- If $\gamma \simeq \gamma'$ relative to boundary, then $L_\gamma \simeq L_{\gamma'}$.
- If $\gamma : I \to B$ and $\gamma' : I \to B$ are two composable paths, there is a homotopy equivalence $L_{\gamma\cdot\gamma'} \simeq L_{\gamma'} \circ L_\gamma$

Proof.

- Let $F : I \times I \to B$ be a homotopy equivalence from γ to γ'. Now consider the map $G : E_{b_1} \times I \times I \to B$ defined by $G(x,s,t) = F(s,t)$. Notice that $G(x,s,0) = F(s,0) = \gamma(s)$ and $G(x,s,1) = F(s,1) = \gamma'(s)$. Thus, we proceed as above by lifting $G(x,s,0)$ and $G(x,s,1)$ to obtain respectively $\widetilde{G(x,s,0)}$ and $\widetilde{G(x,s,1)}$ for which $\widetilde{G(x,1,0)} = L_\gamma$ and $\widetilde{G(x,1,1)} = L_{\gamma'}$. Now define $K : E_{b_1} \times I \times \partial I \to E$ by

$$K(x,s,t) = \begin{cases} \widetilde{G(x,s,1)} & \text{if } t = 0 \\ G(x,s,1) & \text{if } t = 1 \end{cases}$$

  We now obtain a homotopy called $\widetilde{G} : E_{b_1} \times I \times I \to E$ by the homotopy lifting property:

$$\begin{tikzcd} X \times I \times \partial I \arrow[r, "K"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \times I \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

  Now $\tilde{G}(-,1,-) : E_b \times I \to E$ is then a homotopy equivalence from $\tilde{G}(x,1,0) = L_\gamma$ to $\tilde{G}(x,1,1) = L_{\gamma'}$.
- We can repeat the above construction for γ and γ' to obtain homotopies $G : E_{b_1} \times I \to E$ and $G' : E_{b_1} \times I \to E$ such that when $t = 1$ we recover $\tilde{\gamma}$, $\tilde{\gamma'}$ and $\gamma \tilde{\cdot} \gamma'$ respectively. Now the composition of G and G' by traversing along $t \in I$ with twice the speed gives precisely a lift of $\gamma \cdot \gamma'$ (one can check the boundary conditions). Thus $L_{\gamma\cdot\gamma'}$ obtained in this manner coincides up to homotopy equivalence to $L_{\gamma'} \circ L_\gamma$ by invoking part a).

$\square$

Theorem 2.6.3

Let $p : E \to B$ be a fibration. Let b_1 and b_2 lie in the same path component of B. Then there is a homotopy equivalence

$$E_{b_1} \simeq E_{b_2}$$

given by the lift of any path $\gamma : I \to B$ from b_1 to b_2.

Proof. Let $\gamma : I \to B$ be a path from b_1 to b_2. From the above, it follows that $L_{\overline{\gamma}} \circ L_\gamma \simeq \mathrm{id}_{E_b}$ for any loop $\gamma : I \to B$ with basepoint b. We conclude that L_γ is a homotopy equivalence and so the fibers of $p : E \to B$ are homotopy equivalent. □

2.7 Serre Fibrations

Definition 2.7.1: Serre Fibration

We say that a map $p : E \to B$ is a Serre fibration if it has the homotopy lifting property with respect to all CW-complexes.

Lemma 2.7.2

Every (Hurewicz) fibration is a Serre fibration.

Proof. This is true since Hurewicz fibrations satisfies the homotopy lifting property with respect to all topological spaces, including CW complexes. □

Proposition 2.7.3

Let $p : E \to B$ be a fibration where B is path connected. Let F be the fiber of p. Let $b \in B$. Then the map

$$\cdot : \pi_1(B) \times E_b \to E_b$$

defined by $[\gamma]\cdot x = L_\gamma(x)$ induces an action of $\pi_1(B)$ on the homology groups $H_*(F; G)$ given by $[\gamma] \cdot [z] = (L_\gamma)_*([z])$ for any $g \in G$.

Proof. Notice first that such a map is well defined by lemma 6.3.3. Associativity follows from the second point of lemma 6.3.3. Identity follows the unique lift of the identity loop e_b that gives L_{e_b} is also the identity. □

3 Homotopy Fibers and Homotopy Cofibers

3.1 Basic Definitions

Definition 3.1.1: Homotopy Fibers and Cofibers

Let $f : X \to Y$ be a map. Define the homotopy fiber of f at $y \in Y$ to be

$$\text{hofiber}_y(f) = \{(x, \phi) \in X \times \text{Map}(I, Y) \mid f(x) = \phi(0), \phi(1) = y\} = \text{Fib}_y(P_f \to Y)$$

Define the homotopy cofiber of f to be

$$\text{hocofiber} = \frac{(X \times I) \amalg Y}{(x, 1) \sim f(x), (x, 0) \sim (x', 0)} = \text{Cofib}(X \to M_f) = C_f$$

TBA: hofiber = pullback $P_f \to Y \leftarrow *$ (time $t = 1$ and $* \mapsto y$).

Since the map $P_f \to Y$ is a fibration, the fibers of $P_f \to Y$, and hence the homotopy fibers of f are all homotopy equivalent.

Proposition 3.1.2

Let $p : E \to B$ be a fibration. Then the there is a homotopy equivalence

$$\text{Fib}_b(f) \simeq \text{Hofib}_b(f)$$

for each $b \in B$, given by the inclusion map $x \mapsto (x, e_x)$.

Proof. Consider the following diagram

$$\begin{tikzcd}
\text{Fib}_b(f) \arrow[r, "\simeq"] \arrow[d, hook] & \text{Hofib}_b(f) \arrow[d, hook] \\
X \arrow[r, "{h,\simeq}"] \arrow[d, "f"'] & P_f \arrow[d, "q"] \\
Y \arrow[r, "\text{id}_Y"'] & Y
\end{tikzcd}$$

and apply 2.4.6 to conclude. □

Proposition 3.1.3

Let X, Y be spaces. Let $f, g : X \to Y$ be maps. If f, g are homotopic, then for all $y \in Y$, there is a homotopy equivalence

$$\text{Hofib}_y(f) \simeq \text{Hofib}_y(g)$$

induced by the homotopy from f to g.

Example 3.1.4

Let X, Y be spaces. Let $f : X \to Y$ be a map.

- The homotopy fiber of $\{y\} \hookrightarrow Y$ is given by $\text{Hofib}_y(\{y\} \hookrightarrow Y) \cong \Omega Y$
- If f is null-homotopic, then the homotopy fiber of f is given by $\text{Hofib}_y(f) \cong X \times \Omega Y$

3.2 The Fiber and Cofiber Sequences

Definition 3.2.1: Path Spaces

Let (X, x_0) be a pointed space. Define the path space of (X, x_0) to be

$$PX = \{\phi : (I, 0) \to (X, x_0) \mid \phi(0) = x_0\} = \text{Map}_*((I, 0), (X, x_0))$$

together with the topology of the mapping space.

Theorem 3.2.2

Let X be a space. Then the following are true.

- The map $\pi : PX \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber ΩX
- The map $\pi : X^I \to X$ defined by $\pi(\phi) = \phi(1)$ is a fibration with fiber homeomorphic to PX.

We now write a fibration as a sequence $F \to E \to B$ for F the fiber of the fibration $p : E \to B$. This compact notation allows the following theorem to be formulated nicely.

Theorem 3.2.3

Let $f : X \to Y$ be a fibration with homotopy fiber F_f. Let $\iota : \Omega Y \to F_f$ be the inclusion map and $\pi : F_f \to X$ the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$\cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega F_f \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega_Y \xrightarrow{\iota} F_f \xrightarrow{\pi} X \xrightarrow{f} Y$$

where any two consecutive maps form a fibration. Moreover, $-\Omega f : \Omega X \to \Omega Y$ is defined as

$$(-\Omega f)(\zeta)(t) = (f \circ \zeta)(1 - t)$$

for $\zeta \in \Omega X$.

There is then the dual notion of loop spaces and the corresponding sequence. Write a cofibration $f : A \to X$ with homotopy cofiber B as $B \to A \to X$.

Theorem 3.2.4

Let $f : X \to Y$ be a cofibration with homotopy cofiber C_f. Let $i : Y \to C_f$ be the inclusion map and $\pi : C_f \to C_f/Y \cong \Sigma X$ be the projection map. Then up to homotopy equivalence of spaces, there is a sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C_f \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma C_f \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

where any two consecutive maps form a cofibration. Moreover, $-\Sigma f : \Sigma X \to \Sigma Y$ is defined by

$$(-\Sigma f)(x \wedge t) = f(x) \wedge (1 - t)$$

3.3 n-Connected Maps

Definition 3.3.1: n-Connected Maps

Let X, Y be spaces. Let $f : X \to Y$ be a map. We say that f is n-connected if the induced map

$$\pi_k(f) : \pi_k(X) \to \pi_k(Y)$$

is an isomorphism for $0 \leq k < n$ and a surjection for $k = n$.

We can rephrase some of the corner stone theorems of homotopy theory using n-connected maps.

- The homotopy excision theorem can be rephrased into the following. For X a CW-complex and A, B sub complexes of X such that $X = A \cup B$ and $A \cap B \neq \emptyset$. If $(A, A \cap B)$ is m-connected and $(B, A \cap B)$ is n-connected for $m, n \geq 0$, then the inclusion

$$\iota : (A, A \cap B) \to (X, B)$$

  is $(m + n)$-connected.

- The Freudenthal suspension theorem says that if X is an n-connected CW complex, then the map

$$\Omega\Sigma : X \to \Omega(\Sigma(X))$$

  is a $(2n + 1)$-connected map.

Proposition 3.3.2

Let X, Y be spaces. Let $f : X \to Y$ be a map. Then f is k-connected if and only if $\text{Hofib}_y(f)$ is $(k - 1)$-connected for all $y \in Y$.