Exercise Sheet for Week 8

Question 1

Let (a_n) be a sequence. Prove that if (a_n) is bounded, then $\frac{a_n}{n}$ converges to 0. (Prove directly from the definition of convergence)

Proof. Fix $\epsilon > 0$. Since a_n is bounded, there exists $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. By definition of convergence of the sequence $\left(\frac{M}{n}\right)_{n\in\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that $\left|\frac{M}{n}\right| < \epsilon$ for all $n > N$. Now we have that

$$\left|\frac{a_n}{n}\right| \leq \frac{M}{n} < \epsilon$$

for all $n > N$. Hence $\frac{a_n}{n}$ converges to 0. □

Question 2

Let (a_n) and (b_n) be two sequences. Prove that if (a_n) converges to a and $(a_n - b_n)$ is a sequence converging to 0, then (b_n) converges to 0.

Question 3

Let (a_n) and (b_n) be Cauchy sequences. Prove that $(a_n + b_n)$ is a Cauchy sequence. (Prove directly from the definition of Cauchy)

Question 4

Let (a_n) and (b_n) be two sequences. Define

$$c_n = \begin{cases} a_{(n+1)/2} & \text{if } n \text{ is odd} \\ b_{n/2} & \text{if } n \text{ is even} \end{cases}$$

Prove that (c_n) converges to L if and only if (a_n) and (b_n) both converges to L.

Question 5

Hard: Let (a_n) be a bounded sequence. Show that if every convergent subsequence of (a_n) has the same limit a, then (a_n) converges to a. (Hint: Assume for a contradiction that (a_n) does not converge to a. Then...)