Selected Topics

Labix

February 13, 2025

Abstract

Contents

1 Excisive Functors between Spaces

1.1 Homotopy Pushouts and Homotopy Pullbacks

Work in category CGWH all spaces pointed Σ= reduced suspension

Let me begin by slightly rephrasing what we learned in classical algebraic topology. Recall that if X is a space and $X = A \cup B$ are two open sets, then

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ B \arrow[r, hook] & X \end{tikzcd}$$

is a pushout in **Spaces**.

- Recall that amalgamated products of groups is precisely the pushout in **Grps** (when the two maps are embeddings). The Seifert-Van Kampen theorem then says that if $A \cap B$ is non-empty, then $\pi_1 : \textbf{Spaces} \to \textbf{Grp}$ sends pushouts to pushouts.
- If we upgrade the fundamental group π_1 into the fundamental groupoid Π_1 developed by Brown, again $\Pi_1 : \textbf{Spaces} \to \textbf{Spaces} \to \textbf{Grpoids}$ sends pushouts to pushouts, without the assumption of connectedness of $A \cap B$, and removing the dependence of the base point. (6.7.2 R.Brown)
- Applying the singular homology functor does not give a pushout, nor a pullback, but gives a long exact sequence in homology which is useful for computation.
- Similarly, applying the singular cohomology functor gives a useful computational skill.

When we say that X is a pushout of A and B along $A \cap B$, we mean that X is built by the piece A and B. The Mayer-Vietoris theorem is the prime example of why we care about this. It says that computing the homology of parts of the space can recover homological information of the overall space. This is similar for the Seifert-van Kampen theorem. But in general, homotopy groups do not enjoy any theorems reminiscent to these properties, unless we are given (co)fibrations in which case there are indeed long exact sequences.

It is natural to look for conditions to relax so that we obtain similar results. The approach that algebraic topologists takes considers the following viewpoint: Most of the invariants in algebraic topology detect differences up to homotopy equivalence, but pushouts and pullbacks are more rigid than homotopy equivalence:

- Their universal properties guarantee that they are unique up to homeomorphism.
- Pushouts and pullbacks are determined by three spaces and two maps. But if we supply homotopy equivalent spaces then the pushout / pullback is not homotopy equivalence.

This means that pushouts and pullbacks (and in general arbitrary limts and colimits!) are quite incompatible with homotopies. Therefore we concern ourselves with a homotopy invariant version of this concept. The ordinary pushouts and pullbacks are unique up to homeomorphism. We can explicitly define a set and its topology and show that it satisfies a universal property. We take a similar approach here and first introduce a model for homotopy pushouts.

Definition 1.1.1: The Standard Homotopy Pushout

Let $X, Y, Z \in \textbf{CGWH}$ be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. Define the standard homotopy pushout of f and g to be the quotient space

$$\text{hocolim}(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y) = \frac{X \amalg (Z \times I) \amalg Y}{\sim}$$

where $\sim$ is the equivalence relation generated by $f(z) \sim (z, 0)$ and $g(z) \sim (z, 1)$ for $z \in Z$.

Remember that we want our new pushout to be a homotopy invariant, not a homeomorphic invariant. Any homeomorphic spaces has one unique way of writing it down set theoretically, but homotopy equivalent spaces are not determined by its set theoretic representation. Given an arbitrary square in **Spaces**, we also want to know how to compare whether the square exhibits a homotopy pushout. In particular, we use the standard model as an anchor for comparison.

Definition 1.1.2: The Canonical Map of Homotopy Pushouts

Let $X, Y, Z \in$ **CGWH** be spaces. Let $f : Z \to X$ and $g : Z \to Y$ be maps. Define the canonical map of the homotopy pushout of the diagram to be the map

$$s : \text{hocolim}(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y) \to \text{colim}(X \overset{f}{\leftarrow} Z \overset{g}{\rightarrow} Y)$$

given by the formula

$$u \mapsto \begin{cases} u & \text{if } u \in X \\ f(z) = g(z) & \text{if } u = (z, t) \in Z \times I \\ u & \text{if } u \in Y \end{cases}$$

Definition 1.1.3: The Standard Homotopy Pullback

Let $X, Y, Z \in$ **CGWH** be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Define the standard homotopy pullback of f and g to be the subspace

$$\text{holim}(X \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} Y) = \{(x, \alpha, y) \in X \times \text{Map}(I, Z) \times Y \mid \alpha(0) = f(x), \alpha(1) = g(y)\}$$

Definition 1.1.4: The Canonical Map of Homotopy Pullbacks

Let $X, Y, Z \in$ **CGWH** be spaces. Let $f : X \to Y$ and $g : Y \to Z$ be maps. Define the canonical map from the pullback to the homotopy pullback

$$c : \lim(X \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} Y) \to \text{holim}(X \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} Y)$$

to be given by $(x, y) \mapsto (x, e_{f(x)=g(y)}, y)$ where e refers to the constant loop at $f(x) = g(y)$.

Proposition 1.1.5

Let $X, Y, Z \in$ **CGWH** be spaces. Let $f : X \to Z$ and $g : Y \to Z$ be maps. Then the there is a homeomorphism

$$\lim(X \overset{f}{\rightarrow} Z \leftarrow P_g) \cong \text{holim}(X \overset{f}{\rightarrow} Z \overset{g}{\leftarrow} Y)$$

given by the map $(x, (y, \gamma)) \mapsto (x, \gamma, y)$.

Definition 1.1.6: Homotopy Pushout Squares

Let $W, X, Y, Z \in$ **CGWH** be spaces such that there is a (not necessarily commutative) diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

- We say that the square is a homotopy pushout square if the map

$$\beta : \text{hocolim}(X \overset{f}{\leftarrow} W \overset{g}{\rightarrow} Y) \overset{s}{\longrightarrow} \text{colim}(X \overset{f}{\leftarrow} W \overset{g}{\rightarrow} Y) \to Z$$

  is a weak equivalence.
- We say that the diagram is k-cocartesian if β is k-connected.

Definition 1.1.7: Homotopy Pullback Squares

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that there is a (not necessarily commutative) diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

- We say that the diagram is a homotopy pullback if the map
$$\alpha : W \to \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \xrightarrow{c} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$
is a weak equivalence.
- We say that the diagram is k-cartesian if α is k-connected.

It may appear weird that we require a weak equivalence instead of a homotopy equivalence. But it turns out this is the correct notion because of the following.

Theorem 1.1.8: The Matching Lemma

Suppose that we have a commutative diagram of spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Z \arrow[d, "e_Z"'] & Y \arrow[l, "g"'] \arrow[d, "e_Y"] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$

in $\mathbf{CGWH}$. Define the map

$$\phi_{X,Z,Y}^{X',Z',Y'} : \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X' \xrightarrow{f'} Z' \xleftarrow{g'} Y')$$

by the formula $(x, \gamma, y) \mapsto (e_X(x), e_Z \circ \gamma, e_Y(y))$. Then the following are true.
- If each e_X, e_Y, e_Z are homotopy equivalences, then ϕ is a homotopy equivalence.
- If each e_X, e_Y, e_Z are weak equivalences, then ϕ is a weak equivalence.

Proof. We first prove the case for homotopy equivalence. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\text{id}_X"'] & Z \arrow[d, "e_Z"] & Y \arrow[l, "g"'] \arrow[d, "\text{id}_Y"] \\ X \arrow[r, "e_Z \circ f"] \arrow[d, "e_X"'] & Z' \arrow[d, "\text{id}_{Z'}"] & Y \arrow[l, "e_Z \circ g"'] \arrow[d, "e_Y"] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$

We prove that the homotopy pullback of the first row is homotopy equivalent to that of the second, and we prove that the homotopy pullback of the second row is homotopy equivalent to that of the third.

Since e_Z is a homotopy equivalence, we can find a homotopy inverse k for e_Z and a homotopy $H : Z \times I \to Z$ such that $H(-, 0) = \text{id}_Z$ and $H(-, 1) = k \circ e_Z$. Define a map

$$\rho : \text{holim}(X \xrightarrow{f} Z' \xleftarrow{g} Y) \to \text{holim}(X \xrightarrow{e_Z \circ f} Z \xleftarrow{e_Z \circ g} Y)$$

by the formula

$$(x, \gamma', y) \mapsto (x, H(f(x), -) * k(\gamma'(-)) * \overline{H(g(y), -)} : I \to Z, y)$$

where $*$ denotes concatenation of paths. The path concatenation is well defined because we have that $H(f(x),1) = (k \circ e_Z \circ f)(x) = (k \circ \gamma')(0)$ and $k(\gamma'(1)) = k(e_Z(g(y))) = H(g(y),1)$. This is well defined on the homotopy pullback because we have that

- $H(f(x),-) * k(\gamma'(-)) * \overline{H(g(y),-)}(0) = H(f(x),0) = \mathrm{id}_Z(f(x)) = f(x)$
- $H(f(x),-) * k(\gamma'(-)) * \overline{H(g(y),-)}(1) = H(g(y),0) = \mathrm{id}_Z(g(y)) = g(y)$

I claim that this map is the homotopy inverse to the map $\phi = \phi_{X,Y,Z}^{X,Y,Z'}$. We have that

$$\begin{aligned}\rho(\phi(x,\gamma,y)) &= \rho(x, e_Z \circ \gamma, y)\\ &= (x, H(f(x),-) * k(e_Z(\gamma(-)) * \overline{H(g(y),-)}, y)\end{aligned}$$

Now I claim that the middle path is homotopic to γ. For the first component of the concatenation, the path $H(f(x),t) : I \to Z$ can be contracted to $H(f(x),0) = f(x) = \gamma(0)$ so you can homotope the traversal along $H(f(x),-)$ to the single point $f(x) = \gamma(0)$. For the third component of the concatenation, this is similar so we can homotope the traversal of $\overline{H(g(y),-)}$ to the single point $g(y) = \gamma(1)$. The middle part of the path is homotopic to γ because $k \circ e_Z$ is homotopic to id_Z. Thus we conclude. $\square$

Note: there is a similar result for pushouts.

We can compute some examples of homotopy pushouts and pullbacks.

Example 1.1.9

Homotopy fiber as homotopy pushout and homotopy pullback

Example 1.1.10

Let X be a pointed space. There is a unique map $X \to *$ to the terminal object in **Spaces**. The homotopy pushout of $* \leftarrow X \to *$ is given by

$$\text{hocolim}(* \leftarrow X \to *) = \frac{* \amalg (X \times I) \amalg *}{\sim}$$

where $\sim$ is generated by $* \sim (x,0)$ and $* \sim (x,1)$. This is precisely the definition of suspension.

Similarly, there is a unique map $* \to X$ sending $*$ to the base point of X. The homotopy pullback of $* \to X \leftarrow *$ is given by

$$\begin{aligned}\text{holim}(* \to X \leftarrow *) &= \{(*,\gamma,*) \in * \times \text{Map}(I,X) \times * \mid \gamma(0) = * = \gamma(1)\}\\ &\cong \{\gamma \in \text{Map}(I,X) \mid \gamma \text{ is a loop at the base point }\}\end{aligned}$$

This is precisely the definition of loopspace.

Proposition 1.1.11

Consider the following (not necessarily commutative) square

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \end{tikzcd}$$

in **CGWH**. Let the right square be a homotopy pullback square. Then the left square is a homotopy pullback if and only if the rectangle is a homotopy pullback square. (6.4.4 Arkhowitz)

Proposition 1.1.12

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that there is a (not necessarily commutative) diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Then the following are true.

- If the square is a homotopy pullback, and $Y \to Z$ is a weak equivalence, then $W \to X$ is a weak equivalence.
- If $Y \to Z$ and $W \to X$ are weak equivalence, then the square is a homotopy pullback.

Proposition 1.1.13

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that following is a (not necessarily commutative) square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Then the following are true.

- The square is a homotopy pullback if and only if for all $x \in X$, the map
$$\text{hofiber}_x(W \to X) \to \text{hofiber}_{f(x)}(Y \to Z)$$
is a weak equivalence.
- The square is k-cartesian if and only if for all $x \in X$, the map
$$\text{hofiber}_x(W \to X) \to \text{hofiber}_{f(x)}(Y \to Z)$$
is k-connected.

Proof. Begin with the homotopy pullback square

$$\begin{tikzcd} \text{holim}(X \to Z \leftarrow Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

We know that there is a homotopy pullback square given by

$$\begin{tikzcd} \text{holim}_y(g) \arrow[rr] \arrow[d] & & Y \arrow[d, "g"] \\ * \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

We can view this as a homotopy pullback square where we consider the composition $\text{holim}_y(g) \to * \to X$ as one single map. The gives a comparison of a homotopy pullback square with the standard homotopy pullback. Hence we obtain

$$\begin{tikzcd} \text{holim}_y(g) \arrow[r] \arrow[d] & \text{holim}(X \to Z \leftarrow Y) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ * \arrow[r] & X \arrow[r] & Z \end{tikzcd}$$

By the above prp, we conclude that the square on the left is a homotopy pullback. By the same method, we can glue two more squares to obtain the diagram:

$$\begin{tikzcd}
\operatorname{holim}_y(\alpha) \arrow[r] \arrow[d] & \operatorname{holim}_y(f) \arrow[r] \arrow[d, "u"] & W \arrow[d, "\alpha"] & \\
* \arrow[r] & \operatorname{holim}_y(g) \arrow[r] \arrow[d] & \operatorname{holim}(X \to Z \leftarrow Y) \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& * \arrow[r] & X \arrow[r] & Z
\end{tikzcd}$$

If α is a weak equivalence, then since the top wide rectangle is a homotopy pullback we have $\operatorname{holim}_y(\alpha)$ is weakly equivalent to $*$. But the top left square is a homotopy pullback hence $\operatorname{holim}_y(f)$ is weakly equivalent to $\operatorname{holim}_y(g)$. Conversely, suppose that u is a weak equivalence. Since the top left square is a homotopy pullback, this implies that $\operatorname{holim}_y(\alpha)$ is weakly contractible for all y. In articular it is n-connected for all n. Then this implies that α is $(n+1)$-connected for all n. Hence α is a weak equivalence. $\square$

Need: 3.7.29 (for Blakers-Massey)

Definition 1.1.14: Excisive Functors

Let $F : \mathbf{Spaces} \to \mathbf{Spaces}$ be a functor. We say that F is excisive if the following are true.

- F is a homotopy functor. This means that if f is a weak equivalence, then $F(f)$ is a weak equivalence.
- F is finitary. This means that if I is a filtered category and $X : I \to \mathbf{Spaces}$ is a diagram, then
$$\operatorname*{hocolim}_{i\in I} F(X_i) \to F\left(\operatorname*{hocolim}_{i\in I} X_i\right)$$
is a weak equivalence.
- F sends homotopy pushouts to homotopy pullbacks.

The finitary requirement ensures that the functor F is determined by its value on a finite complexes. Then by taking homotopy colimits the value of F on spaces generated by colimits of finite complexes are determined. We will say something about this condition again in section 2.

A reasonable question might be to ask why excisive functors send homotopy pushouts to homotopy pullbacks instead to homotopy pushouts. Intuitively, pushouts are how we assemble spaces using smaller pieces. But interestingly enough, homotopy groups work better with homotopy pullbacks rather than homotopy pushouts.

Theorem 1.1.15

LES of homotopy groups https://math.stackexchange.com/questions/1262049/long-exact-sequence-of-homotopy-groups-pi-n-for-a-pointed-homotopy-pullback-s 5.6.9 Martin Arkowitz

As long as the identity functor is excisive, then this would pose no problem. But reality is often disappointing.

1.2 The Failure of the Identity Functor to be Excisive

It is natural to ask what kinds of functors are excisive. Instead of giving an example of an excisive functor, allow me to give an example of a functor that is not excisive. There are two reasons for this. The main theorem - Blakers-Massey theorem will assist us in giving a plethora of excisive functors. Moreover, it demonstrates the possibly simplest functor in existence - the identity functor is not excisive.

Blakers-Massey theorem itself is a powerful theorem that implies the connectivity bounds in the Freudenthal suspension theorem. Before the main theorem, we must present preparatory definitions and lemmas, and then prove a special case of the theorem, and finally generalize it to the arbitrary case. Let me first give the statement of the main theorem.

Theorem 1.2.1: Blakers-Massey Theorem

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

is a homotopy pushout. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cartesian. Explicitly, this means that

$$\alpha : X_0 \to \text{holim}(X_1 \to X_{12} \leftarrow X_2)$$

is $(k_1 + k_2 - 1)$-connected.

Firstly, recall that a square is a homotopy pullback if the map α above is n-connected for all $n \in \mathbb{N}$. The theorem implies that the identity functor is not excisive because after giving connectedness assumptions on maps in the square, we only get an upper bound of the connectedness of α, which means that the identity functor is excisive up to a certain dimension.

Definition 1.2.2: (Degenerative) Cubes

Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Let $\delta > 0$. Let $L \subseteq \{1, \dots, n\}$. A cube in $\mathbb{R}^n$ is a set of the form

$$W = W(a, \delta, L) = \{x \in \mathbb{R}^n \mid a_i \leq x \leq a_i + \delta \text{ for } i \in L \text{ and } x_i = a_i \text{ for } i \notin L\}$$

The notation is making the object more complicated than what it should look like. $a \in \mathbb{R}^n$ refers to the bottom left coordinate of the cube. δ is the length of the cube and L refers to the number of non-degenerate faces of the cube. In particular, any cube in $\mathbb{R}^n$ is homeomorphic to the standard cube I^k for some $k \leq n$.

- When $n = 3$, $W(0, 1, \{1, 2\})$ is the unit square on the xy-plane.
- When $n = 3$, $W(0, 1, \{1, 2, 3\})$ is the unit cube.
- When $n = 4$, $W(0, 1, \{1, 2, 3\})$ is the unit cube with nonzero first three coordinates and zero otherwise.

Definition 1.2.3: Special Sub-cube of a Cube

Let $W = W(a, \delta, L)$ be a cube in $\mathbb{R}^n$. Let $j = 1$ or 2. Suppose that $1 \leq p \leq |L|$. Define

$$K_p^j(W) = \left\{(x_1, \dots, x_n) \in W \;\middle|\; \frac{\delta(j-1)}{2} + a_i < x_i < \frac{\delta j}{2} + a_i \text{ for at least } p \text{ values of } i \in L\right\}$$

Again the notation is making the object more complicated. Taking $W = W(0, 1, \{1, 2, 3\}) = I^3$ in $\mathbb{R}^3$, we have

- $K_3^1(W) = W(0, 1/2, \{1, 2, 3\})$ is one eighth of the cube I^3 with bottom left corner at the origin.
- $K_3^2(W) = W((1/2, 1/2, 1/2), 1/2, \{1, 2, 3\})$ is one eighth of the cube I^3 with bottom left corner at $(1/2, 1/2, 1/2)$
- $K_2^1(W)$ allows for one coordinate to go beyond the bottom left one eighth of the cube, and is the union four of the $1/8$-sub-cubes that are adjacent to the xy-plane, the yz-plane and the xz-plane.
- $K_1^1(W)$ allows for two coordinate to go beyond the bottom left one eighth of the cube, and is equal to $W \setminus K_3^2(W)$.

- $K_0^1(W)$ allows for all coordinate to go beyond the bottom left one eighth of the cube, so the condition becomes vacuous and is equal to W.

Summarizing, we think of $K_p^j(W)$ as follows. Subdivide the $|L|$-dimensional cube into $2^{|L|}$ sub-cubes of equal volume. K_p^1 is the union of a number of sub-cubes closest to the bottom left sub-cube. K_p^2 is the union of a number of sub-cubes closed to the upper right sub-cube.

Lemma 1.2.4

Let Y be a space. Let $B \subseteq Y$ be a subspace of Y. Let $W = W(a, \partial, L)$ be a cube in $\mathbb{R}^n$. Let $f : W \to Y$ be a map. Let $j = 1$ or 2. Suppose that there exists some $p \leq |L|$ such that

$$f^{-1}(B) \cap C \subset K_p^j(C)$$

for all cubes $C \subset \partial W$. Then there exists a map $g : W \to Y$ such that $g \overset{\partial W}{\simeq} f$ and

$$g^{-1}(B) \subset K_p^j(W)$$

Proof. (Proof by Munson in Cubical Homotopy Theory)Firstly, notice that any cube W is homeomorphic to I^n for some n, so we can just prove the statement for when $W = I^n$. In this case, our parameters of the cube is given by $I^n = W(a = 0, \delta = 1, L = \{1, \dots, n\})$ and our $K_p^j(W)$ is given by

$$K_p^j(W) = \left\{(x_1, \dots, x_n) \in C \;\middle|\; \frac{j-1}{2} < x_i < \frac{j}{2} \text{ for at least } p \text{ values of } i \in \{1, \dots, n\}\right\}$$

Let p_j be the center of the sub-cube $\left[\frac{j-1}{2}, \frac{ji}{2}\right]^n$ inside I^n for $j = 1, 2$. Let R be a ray with starting point p_j. Let $P(R, p_j)$ be the intersection of R and $\partial\left[\frac{j-1}{2}, \frac{ji}{2}\right]^n$. Let $Q(R, p_j)$ be the intersection of R and ∂I^n. By construction, the points p_j, $P(R, p_j)$ and $Q(R, p_j)$ are collinear with $P(R, p_j)$ always being the mid point and $P(R, p_j)$ is possibly equal to $Q(R, p_j)$. Being a line, we can define a linear homotopy from the line $[p_j, P(R, p_j)]$ to the line $[p_j, Q(R, p_j)]$ that fixes the point p_j and sends $P(R, p_j)$ to $Q(R, p_j)$. Denote the homotopy by $h(y, t)$ for $y \in [p_j, P(R, p_j)]$ and t the time variable.

Now we can define a homotopy $H_j : I^n \times I \to I^n$ as follows: For each $y \in I^n$, there exists a unique ray R starting at p_j and passing through y. Then we obtain a homotopy h from $[p_j, P(R, p_j)]$ to $[p_j, Q(R, p_j)]$ as above. Define $H(y, t) = h(y, t)$. It is clear that $H_j(q, 1) = q$ for all $q \in \partial I^n$ so that H_j is a homotopy from the identity, relative to the boundary ∂I^n.

Let $g = f \circ H_j(-, 1)$. From the properties of the homotopy H_j, we notice that $f \circ H_j : I^n \times I \to Y$ is a homotopy from $f \circ H(-, 0) = f \circ \text{id} = f$ to $f \circ H(-, 1) = g$, relative to the boundary ∂I^n. Thus we now have a homotopy from f to a map g relative to the boundary. It remains to show that $g^{-1}(B) \subset K_p^j(C)$.

Let $z = (z_1, \dots, z_n) \in g^{-1}(B)$. If $z \in \left[\frac{j-1}{2}, \frac{j}{2}\right]^n$ then clearly $z \in K_p^j(C)$ is true. So suppose instead that $z = (z_1, \dots, z_n) \in g^{-1}(B)$ satisfies the fact that either $z_a \geq \frac{j}{2}$ or $z_b \leq \frac{j-1}{2}$ for $1 \leq a, b \leq n$. Let R be the ray from p_j passing through z. Then the condition on z means that $z \in [P(R, p_j), Q(R, p_j)]$. Hence under the homotopy H, z is mapped to ∂I^n. But ∂I^n is a union $n - 1$ dimensional faces of I^n which are cubes. So $H(z, 1)$ lies in some cube $C \subseteq \partial I^n$. By construction of g, $g(z) = f(H(z, 1))$ and $g(z) \in B$ implies that $H(z, 1) \in f^{-1}(B)$. Then $H(z, 1) \in f^{-1}(B)$ and $H(z, 1) \in C$ implies that

$$H(z, 1) \in f^{-1}(B) \cap C \subseteq K_p^j(C)$$

by the assumption on f. Write $H(z, 1) = (w_1, \dots, w_n) \in \partial I^n$. This means that $\frac{j-1}{2} < w_i < \frac{j}{2}$ for at least p of the coordinates of $H(z, 1)$.

Now the ray starting at p_j and passing through z is parametrized by the line $p_j + t(z - p_j)$ for $t \geq 0$. Since $H(z,1)$ lies behind the two points z and p_j, we can write $H(z,1) = p_j - t_0(z - p_j)$ for some $t_0 \geq 1$. By definition, p_j is the point given in coordinates by $\left(\frac{2j-1}{4}, \dots, \frac{2j-1}{4}\right)$. Hence the ith coordinate of $H(z,1)$ can be written as

$$w_i = \frac{2j-1}{4} + t_0\left(z_i - \frac{2j-1}{4}\right)$$

Recall that in the previous paragraph we found that $\frac{j-1}{2} < w_i < \frac{j}{2}$ for at least p of the coordinates of $H(z,1)$. Substituting w_i into the inequality and simplifying gives

$$-\frac{1}{4t_0} + \frac{2j-1}{4} < z_i < \frac{1}{4t_0} + \frac{2j-1}{4}$$

Since $t_0 \geq 1$, we get

$$-\frac{1}{4} + \frac{2j-1}{4} < -\frac{1}{4t_0} + \frac{2j-1}{4} < z_i < \frac{1}{4t_0} + \frac{2j-1}{4} < \frac{1}{4} + \frac{2j-1}{4}$$

The leftmost and rightmost terms bound z_i between $\frac{j-1}{2}$ and $\frac{j}{2}$ for at least p amount of coordinates z_i of z. Hence $z \in K_p^j(C)$. This completes the proof. $\square$

Lemma 1.2.5

Let X be a space. Let $X_0, X_1, X_2 \subseteq X$ be subspaces of X such that

$$X = X_1 \cup X_2$$

and $X_0 = X_1 \cap X_2$ is non-empty. Assume that for each $i = 1, 2$, (X_i, X_0) is k_i-connected with $k_i \geq 0$. Let $f : I^n \to X$ be a map. Let

$$I^n = \bigcup_k W_k$$

be the decomposition of I^n into cubes W_k such that $f(W_k) \subseteq X_i$ for one of $i = 0, 1, 2$ by the Lebesgue covering lemma. Then there exists a homotopy

$$H : I^n \times I \to X$$

such that the following are true.

- $f(-) = H(-, 0)$
- If $f(W) \subset X_i$, then $H(W, t) \subset X_i$ for all $t \in I$.
- If $f(W) \subset X_0$, then $H(W, t) = f(W)$ for all $t \in I$.
- If $f(W) \subset X_i$, then $\left((H(-,1))^{-1}(X_i \setminus X_0)\right) \cap W \subset K^i_{k_i+1}(W)$.

Proof. Let C^d be the union of all cubes of dimension $\leq d$. We induct on d, the existence of such a homotopy $H : C^d \times I \to X$ that holds the required conditions true for all cubes W with dimension $\leq d$.

We first construct the homotopy for all cubes of dimension 0. When $\dim(W) = 0$, there are two cases:

- If $f(W) \subset X_0$, define $H|_{W \times I}$ by $H(w,t) = f(w)$
- If $f(W) \subset X_j$ and $f(W) \not\subset X_i$ for $1 \leq i \neq j \leq 2$, (X_j, X_0) is $(k_j \geq 0)$-connected implies that there exists a path $\gamma : I \to X$ from $f(W)$ to a point in X_0. Define $H|_{W \times I}$ by $H(w,t) = \gamma(t)$ (again $W = \{w\}$ is a one point set).

Thus we now have a well defined map $H : C^0 \times I \to X$. We need to show that this map satisfies the required conditions.

- For each $z \in C^0$, either $H(z,0) = f(z)$ from the first case or $H(z,0) = \gamma(0) = f(z)$.

- If $f(W) \subset X_i$, then by construction $H(W,t) \subset X_i$ from the second case.
- If $f(W) \subset X_0$, then $H(W,t) = f(W)$ by the first case.
- $K^i_{k_i+1}(W) = \{w\}$ is a one point set and $(H(-,1))^{-1}(X_i \setminus X_0) \cap W \subseteq W$ means that this condition is satisfied.

Now H is built on three pieces: the union of cubes landing in X_i for $i = 0, 1, 2$. The second and third conditions guarantee that each of the three pieces define a homotopy on each piece respectively. Since $\partial W \hookrightarrow W$ is a cofibration, we can extend these pieces of homotopy from 0-dimensional cubes to 1-dimensional. Recursively we are able to define a homotopy for all cubes of all dimensions inside I^n that satisfy the first three conditions.

Therefore now we can invoke the inductive hypothesis, so that there exists a homotopy from f so that the new function satisfy all our required conditions for all cubes of dimension $< d$. With abuse of notation, call the restriction of our newly acquired function to C^{d-1} also by the name f. Let W be a cube of dimension d.

- If $f(W) \subseteq X_0$, define $H|_{W\times I}$ by $H(w,t) = f(w)$
- If $f(W) \subset X_1$ and $f(W) \not\subset X_2$ and $\dim(W) = d \leq k_1$, (X_j, X_0) is $(k_j \geq 0)$-connected implies there exists a homotopy $K : W \times I \to X$ from f relative to ∂W such that $K(W, 1) \subseteq X_0$. Define $H|_{W\times I}$ by $H = K$.
- If $f(W) \subset X_1$ and $f(W) \not\subset X_2$ and $\dim(W) = d > k_1$, then by induction we have
$$f^{-1}(X_1 \setminus X_0) \cap W' \subset K^1_d(W') \subset K^1_{k_1+1}(W')$$
for all $W' \subset \partial W$ (induction is applicable since $\dim(W') < \dim(W)$). By the above lemma, there exists a map $g : W \to X$ such that g and f are homotopic relative to ∂W such that $g^{-1}(X_1 \setminus X_0) \subset K^1_{k_1+1}(W)$. Call this homotopy from f to g by $R : W \times I \to X$. Then we define $H|_{W\times I}$ by $H = R$.

(WLOG the cases where X_1 is swapped with X_2 and k_1 is swapped with k_2 and $K^1_p(W)$ is swapped with $K^2_p(W)$ in the last two sub-cases has a symmetrical argument). Finally we show that our required conditions are satisfied.

- In all cases, $H(-,0) = f$ as one can immediately see.
- The second condition holds for all cubes of dimension $< d$ by inductive hypothesis. It also holds for our first and second case since $X_0 \subset X_1, X_2$. For the third case, H is a homotopy relative to ∂W. Since by induction hypothesis $f(\partial W) \subseteq X_1$, we also have $g(\partial W) \subseteq X_1$. Since g is continuous then $H(W,1) = g(W) \subseteq X_1$.
- The third condition holds for all cubes of dimension $< d$ by inductive hypothesis, and holds true for all cubes of dimension d by the first case.
- The fourth condition holds true by our argument in the third case, and is vacuously true in the second case since $H(W,1) \subseteq X_0$ implies that $(H(-,1))^{-1}(X_1 \setminus X_0) = \emptyset$.

Thus the proof is complete. □

We can now prove a weaker version of Blakers-Massey theorem.

Proposition 1.2.6

Let X be a space. Let e^{d_i} be a cell of dimension d_i for $i = 1, 2$. Then the following diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \cup e^{d_1} \arrow[d] \\ X \cup e^{d_2} \arrow[r] & X \cup e^{d_1} \cup e^{d_2} \end{tikzcd}$$

given by inclusion maps is $(d_1 + d_2 - 3)$-cartesian.

Proof. (Proof is by Munson in Cubical Homotopy Theory) Let $p_1 \in e^{d_1}$ and $p_2 \in e^{d_2}$ be interior points. Since $X \cup e^{d_2}$ is weakly equivalent to $X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1\}$ by inclusion (and similarly for $X \cup e^{d_1}$), the above square admits a weak equivalence to the following square:

$$\begin{tikzcd}
X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1, p_2\} \arrow[r, hook] \arrow[d, hook] & X \cup e^{d_1} \cup e^{d_2} \setminus \{p_2\} \arrow[d, hook] \\
X \cup e^{d_1} \cup e^{d_2} \setminus \{p_1\} \arrow[r, hook] & X \cup e^{d_1} \cup e^{d_2}
\end{tikzcd}$$

Let $Y = X \cup e^{d_1} \cup e^{d_2}$. By thm???? to show that the square is $(d_1 + d_2 - 3)$-cartesian is the same as showing the map

$$\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \to \text{hofiber}_y(Y \setminus \{p_2\} \to Y)$$

is $(d_1 + d_2 - 3)$-cartesian. Let

$$C = \text{hofiber}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\}) \simeq *$$

Now the intersections below can be calculated to give

$$C \cap \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) = \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1, p_2\})$$

and

$$C \cap \text{hofiber}_y(Y \setminus \{p_2\} \to Y) = \text{hofiber}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\}) \simeq *$$

(because C and the homotopy fibers are all subspaces of $Y \times \text{Map}(I, Y)$). By thm??? we conclude that the square

$$\begin{tikzcd}
C \cap \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \arrow[r, hook] \arrow[d, hook] & \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \arrow[d, hook] \\
C \arrow[r, hook] & C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\})
\end{tikzcd}$$

is a homotopy pullback. Since the left two terms are contractible, it is a weak equivalence. By thm??? we conclude that the map on the right is a weak equivalence. Similarly, we can consider the diagram

$$\begin{tikzcd}
C \cap \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \arrow[r, hook] \arrow[d, hook] & \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \arrow[d, hook] \\
C \arrow[r, hook] & C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y)
\end{tikzcd}$$

which is a homotopy pullback. And the fact that the terms on the left are contractible imply that the map on the right (which is in fact equal)

$$\text{hofiber}_y(Y \setminus \{p_2\} \to Y) \overset{=}{\hookrightarrow} \text{hofiber}_y(Y \setminus \{p_2\} \to Y)$$

is a weak equivalence by the same theorem. We now have a chain of maps

$$\begin{tikzcd}
\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \arrow[r, hook, "\text{weak eq.}"] & C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}) \arrow[dl, hook] \\
C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y) \arrow[r, "\text{weak eq.}"] & \text{hofiber}_y(Y \setminus \{p_2\} \to Y)
\end{tikzcd}$$

where the middle map is inclusion. It remains to show that the middle map is $(d_1 + d_2 - 3)$-connected. This is the same as showing that the pair

$$(C \cup \text{hofiber}_y(Y \setminus \{p_2\} \to Y), C \cup \text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\}))$$

is $(d_1 + d_2 - 3)$-connected.

To simplify notations let us write the pair as (A, B). Let $\phi : (I^n, \partial I^n) \to (A, B)$ be a map. Recall that $A = \{(x, \phi) \in Y \times \text{Map}(I, Y) \mid \phi(0) = y, \phi(1) = x\}$. The first variable is determined by the end point of ϕ so giving a map $I^n \to A$ is the same as giving a map $I^n \to \text{Map}(I, Y)$ for which all paths in the image has starting point y and ending point in $Y \setminus \{p_2\}$. By the hom-product adjunction, this is equivalent to giving a map $\psi : I^n \times I \to Y$ such that $\psi(z, 0) = y$ is the base point and $\psi(z, 1) \in Y \setminus \{p_2\}$. Similarly, we can consider the map $\phi : \partial I^n \to B$ and deduce that $\phi(z)$ is a path lying entirely in the codomain of the map of the homotopy fiber $C = \text{hofib}_y(Y \setminus \{p_2\} \to Y \setminus \{p_2\})$ or it is a path lying entirely in the codomain of the map of the homotopy fiber $\text{hofiber}_y(Y \setminus \{p_1, p_2\} \to Y \setminus \{p_1\})$). By the same adjunction we conclude that our ψ above must also satisfy for any fixed $z \in \partial I^n$, $\psi(z, t)$ lies entirely in either $Y \setminus \{p_1\}$ or $Y \setminus \{p_2\}$ (and conversely these information give a map $(I^n, \partial I^n) \to (A, B)$) by the adjunction).

To summarize: we have a map
$$\psi : I^n \times I \to Y$$
such that

- $\psi(z, 0) = y$ is the base point for all $z \in I^n$.
- $\psi(z, 1) \in Y \setminus \{p_2\}$ for all $z \in I^n$.
- For any fixed $z \in \partial I^n$, $\psi(z, t)$ lies entirely in $Y \setminus \{p_1\}$ or $Y \setminus \{p_2\}$ for varying t.

The goal is to make a homotopy from ψ to a map whose third condition holds for any $z \in I^n$ when $n \leq d_1 + d_2 - 3$. Then passing through the adjunction again we see that our original map $(I^n, \partial I^n) \to (A, B)$ is homotopic to the constant map as required. Apply 5.1.4 to obtain a homotopy $H : I^n \times I \times I \to Y$ from ψ to a new map $\eta : I^n \times I \to Y$, such that we have a decomposition of $I^n \times I$ into cubes W and the following are true.

1. $\psi(W) \subset Y \setminus \{p_2\}$ implies $H(W, r) \subset Y \setminus \{p_2\}$ for all $r \in I$.
2. $\psi(W) \subset Y \setminus \{p_1\}$ implies $H(W, r) \subset Y \setminus \{p_1\}$ for all $r \in I$.
3. $\psi(W) \subset Y \setminus \{p_1, p_2\}$ implies $H(W, r) = \psi(W)$ for all $r \in I$.
4. $\psi(W) \subset Y \setminus \{p_2\}$ then $(H(-, 1)^{-1}(\{p_1\})) \cap W \subset K^1_{d_1}(W)$
5. $\psi(W) \subset Y \setminus \{p_1\}$ then $(H(-, 1)^{-1}(\{p_2\})) \cap W \subset K^2_{d_2}(W)$

We claim that $H(z, t, r)$ satisfies the three bullet points for all fixed r.
Firstly, we already know that $\psi(z, 0) = y$ is the base point for all $z \in I^n$. So for all cubes $W \subseteq I^n \times \{0\}$, we have $\psi(W) = \{y\} \subset Y \setminus \{p_1, p_2\}$. By 3., we conclude that $H(W, r) = \psi(W) = \{y\}$. Secondly, we know that $\psi(z, 1) \in Y \setminus \{p_2\}$ for all $z \in I^n$. So for all cubes $W \subseteq I^n \times \{1\}$, we have $\psi(W) \subset Y \setminus \{p_2\}$. By 1., we conclude that $H(W, r) \subset Y \setminus \{p_2\}$ for all $r \in I$. Finally, according to the third bullet point, $\psi(z, I)$ lies entirely in $Y \setminus \{p_1\}$ or $Y \setminus \{p_2\}$ for $z \in \partial I^n$ WLOG lets say it lies entirely in $Y \setminus \{p_i\}$. Choose cubes $W_1, \dots, W_k$ in the decomposition of $I^n \times I$ so that it forms a minimal cover for $\{z\} \times I \subset W_1 \cup \cdots \cup W_k$. By definition of the decomposition, these cubes firstly contain at least one point in $\{z\} \times I$, and $\psi(\{z\} timesI) \subset Y \setminus \{p_j\}$ implies that $\psi(W_1), \dots, \psi(W_k) \subset Y \setminus \{p_j\}$. By 1., we conclude that $H(W_1, r), \dots, H(W_k, r) \subset Y \setminus \{p_j\}$ so that $H(W_1 \cup \cdots \cup W_k, r) \subset Y \setminus \{p_j\}$.

It remains to show that $\eta(-, -) = H(-, -, 1)$ satisfies the stronger condition of the third bullet point as desired. Let $n \leq d_1 + d_2 - 3$. We want to show that $\eta(z, I) \subset Y \setminus \{p_j\}$ for some j. I claim that this is equivalent to saying
$$\text{proj}\left(\eta^{-1}(\{p_1\})\right) \cap \text{proj}\left(\eta^{-1}(\{p_2\})\right) = \emptyset$$
where proj is the projection to the first coordinate. Indeed if $\eta(z, I)$ always lie inside one of $Y \setminus \{p_j\}$, $j = 1, 2$, then $\text{proj}\left(\eta^{-1}(\{p_1\})\right) = \{z \in I^n \mid \eta(z, I) \subset Y \setminus \{p_1\}\}$ and similarly for the other projection so that their intersection is empty. Conversely if one of $\eta(z, I)$ does not entirely in $Y \setminus \{p_2\}$ then the intersection is non-empty.

So it suffices to prove that the intersection given above is empty. So suppose it is non-empty with an element z_0. Then there exists $t_1, t_2 \in I$ such that $\eta(z_0, t_1) \in Y \setminus \{p_2\}$ and $\eta(z_0, t_2) \in Y \setminus \{p_1\}$. Choose cubes $W_1 = W(a_1, \delta_1, L_1), W_2 = W(a_2, \delta_2, L_2)$ in the given decomposition of $I^n \times I$ so that $(z_0, t_1) \in W_1$ and $(z_0, t_2) \in W_2$. By 4. and 5. we have $(z_0, t_1) \in \eta^{-1}(\{p_1\}) \cap W_1 \subset K^1_{d_1}(W_1)$ and similarly for (z_0, t_2). Then (z_0, t_j) has at least d_j

coordinates satisfying the inequalities to lie in $K^1_{d_j}$. Hence $z_0 = \text{proj}(z_0, t)$ has at least $d_j - 1$ coordinates satisfying those inequalities for each $j = 1, 2$. For each j, $\text{proj}(W_j)$ is a cube containing z_0. Subdivide the cubes W_1 and W_2 further so that $\text{proj}(W_1) = \text{proj}(W_2)$. Since $\text{proj}(z_0, t) = z_0$, this means that z_0 has at least $d_j - 1$ coordinates satisfying the inequalities of $K^1_{d_1}(W_1)$ and $K^2_{d_2}(W_2)$. But notice that the inequalities of $K^1_{d_1}(W_1)$ and $K^2_{d_2}(W_2)$ are disjoint (one concerns whether the points are at the front of the cube, the other at the back). So these conditions are disjoint and z_0 must have at least $d_1 + d_2 - 2$ conditions on its coordinates. This is impossible if z_0 has less than $d_1 + d_2 - 3$ coordinates. Hence we are done.

Notice that the proof required $d_1, d_2 \geq 1$. Assume WLOG that $d_2 = 0$, then right from the beginning we are considering the map of homotopy fibers

$$\text{hofiber}_y(X \to X \cup e^{d_2}) \to \text{hofiber}_y(X \cup ed_1 \to X \cup e^{d_1} \cup e^{d_2})$$

where $X \cup e^{d_1}$ is now the disjoint union of X with a base point. Then $\text{hofiber}_y(X \to X \amalg *)$ consists of pairs $(x, \phi) \in X \times \text{Map}(I, X \amalg *)$ such that $\phi(0) \in X$ and $\phi(1) = *$. But $*$ is disjoint from X means that no pairs satisfy this conditions and the homotopy fiber is the empty set. Similarly for the target homotopy fiber. Hence the map of homotopy fibers is the identity and it is trivially true. The proof is similar for $d_1 = 0$. ☐

Theorem 1.2.7: Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \textbf{CGWH}$ be spaces such that the square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} \end{tikzcd}$$

is a homotopy pushout. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cartesian. Explicitly, this means that

$$\alpha : X_0 \to \text{holim}(X_1 \to X_{12} \leftarrow X_2)$$

is $(k_1 + k_2 - 1)$-connected.

Proof. By 3.7.29, we only need to consider squares of the form

$$\begin{tikzcd} X_0 \arrow[rr] \arrow[d] & & X_1 \arrow[d] \\ X_2 \arrow[r] & X_{12} = X_1 \amalg_{X_0} X_2 & \end{tikzcd}$$

By Hatcher 4.16, if (X, A) is k-connected CW complex, then there exists a CW pair (Z, A) such that $Z \setminus A$ only has cells of dimension $\geq k$ that is weakly equivalent to (X, A). The set up of the theorem now simplifies to the following: X_1 is the obtained from X_0 by gluing cells of dimension $\geq k_1$, and likewise for X_2. Now showing that the above diagram is $(k_1 + k_2 - 1)$-cartesian is the same as showing that the map of homotopy fibers of the vertical maps is $(k_1 + k_2 - 1)$-connected. This is the same as saying $(\text{holim}(X_1 \to X_{12}), \text{holim}(X_0 \to X_1)$ is $(k_1 + k_2)$-connected, which is the same as saying any map from $(I^n, \partial I^n)$ to the pair of space is homotopic to a map mapping I^n into $\text{holim}(X_0 \to X_1)$ relative boundary. By a similar method in the above lemma, such a map, by the hom product adjunction, is the same as giving a map $I^n \times I \to X_{12}$ for which $I^n \times \{1\}$ lies in X_1. But X_1 is a CW complex and $I^n \times \{1\}$ is compact, which means that image of the map is contained in finitely many cells, and so WLOG we can take X_1, X_2 to be the union of finitely many cells of appropriate dimension.

Now by 1.1.10 and the above lemma we are done. □

This theorem directly generalizes the homotopy excision theorem in the following way. For X a CW complex and A, B two subcomplexes with non-empty intersection and $X = A \cup B$, consider the following square of inclusions:

$$\begin{array}{ccc} A \cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & X \end{array}$$

We have seen that such a square diagram is a homotopy pushout diagram. Now any inclusion map $W \hookrightarrow Z$ is k-connected if and only if (Z, W) is k-connected. So $(A, A \cap B)$ is k_1-connected and (X, B) is k_2-connected. Blaker's-Massey theorem implies that

$$\text{hofiber}(A \cap B \to A) \to \text{hofiber}(B, X)$$

is $(k_1 + k_2 - 1)$-connected. But by definition we have an isomorphism $\pi_k(\text{hofiber}(U \to V) \cong \pi_{k+1}(V, U)$. So we are really just saying that $\pi_k(A, A \cap B) \to \pi_k(X, B)$ given by the inclusion is $(k_1 + k_2)$-connected.

1.3 Excisive Functors Coming From Spectra

While the identity functor is not excisive, there is still a wide class of functors that are exicisive, and these functors arise from spectra.

Definition 1.3.1: Spectra

A spectra consists of a collection of spaces X_n and bonding maps $\sigma_n : \Sigma X_n \to X_{n+1}$ for $n \in \mathbb{N}$ such that σ_n is a weak equivalence.

By the suspension loop adjunction, we will interchangably refer to the bonding map either as $\Sigma X_n \to X_{n+1}$ or as $X_n \to \Omega X_{n+1}$.

Given a space X and a spectrum $\{Y_n, \sigma_n\}$, we can smash X with the spectrum at each level and smashing with the identity will give bonding maps

$$\Sigma(Y_n \wedge X) = S^1 \wedge Y_n \wedge X \stackrel{\sigma_n \wedge \text{id}_X}{\to} Y_{n+1} \wedge X$$

(we work in **CGWH**) which are usually not weak equivalences. We call them pre-spectra. (2.4.1 foundations of stable homotopy theory)

Theorem 1.3.2

Let $\{K_n, \sigma_n\}$ be a spectrum. Then the functor

$$X \mapsto \text{hocolim}(K_0 \wedge X \to \Omega(K_1 \wedge X) \to \Omega^2(K_2 \wedge X) \to \cdots)$$

is an excisive functor.

2 Spectra as Reduced and Excisive Functors

2.1 Stable Infinity Categories

Definition 2.1.1: Infinity Pushouts

Let $\mathcal{C}$ be an infinity category. Let $F : \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a morphism of simplicial sets. Let $X \in \mathcal{C}$ be an object. We say that X is a pushout in $\mathcal{C}$ if there exists a natural transformation $u : \Delta X \Rightarrow F$ such that there is a homotopy equivalence of Kan complexes:

Definition 2.1.2: Infinity Pullbacks

Why are these the correct analogue?

Definition 2.1.3: Stable Infinity Categories

Example in mind: spectra in ordinary categories: pushout=pullback.

Definition 2.1.4: Excisive Functors

Let $F : \textbf{Spaces} \to \textbf{Spaces}$ be a functor. We say that F is excisive if the following are true.

- F is a homotopy functor. This means that if f is a weak equivalence, then $F(f)$ is a weak equivalence.
- F is finitary. This means that

2.2 Suspension and Loop Functors

Own notes: Higher algebra 1.4

trivial kan fibration -> section (Kerodon 1.5.5.5)

2.3 Stable Infinity Categories

Recall that $\mathcal{S} = N_\bullet^{\text{hc}}(\textbf{Top}_*)$ is the infinity category of spaces.

Proposition 2.3.1

Let $\mathcal{C}$ be a pointed infinity category that admits all finite colimits. Then $\text{Exc}_*(\mathcal{C}, \mathcal{S})$ is stable.

Proof. Let $F : \mathcal{C} \to \mathcal{S}$ be excisive and reduced. Then $\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F) = F \circ \Sigma_{\mathcal{C}}$. By definition of the suspension functor,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma_{\mathcal{C}}(X) \end{tikzcd}$$

is a pushout in $\mathcal{C}$. Since F is excisive,

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & (F \circ \Sigma_{\mathcal{C}})(X) \end{tikzcd}$$

is a pullback in $\mathcal{S}$. On the other hand, $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F) = \Omega_{\mathcal{S}} \circ F$. By definition of the loop functor,

$$\begin{tikzcd}
(\Omega_{\mathcal{S}} \circ F \circ \Sigma_{\mathcal{C}})(X) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & (F \circ \Sigma_{\mathcal{C}})(X)
\end{tikzcd}$$

is a pullback in $\mathcal{S}$ for any $X \in \mathcal{C}$. Therefore $F(X)$ and $(\Omega_{\mathcal{S}} \circ F \circ \Sigma_{\mathcal{C}})(X)$ are equivalent. Hence F and $\Omega_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(\Sigma_{\text{Exc}_*(\mathcal{C},\mathcal{S})}(F))$ are equivalent. $\square$

Theorem 2.3.2

There is an equivalence of infinity categories

$$\text{Sp}(\mathcal{S}) \simeq \lim(\cdots \to \mathcal{S} \overset{\Omega}{\to} \mathcal{S} \overset{\Omega}{\to} \mathcal{S}) =: \overline{\mathcal{S}}$$

induced by the evaluation map $\text{ev}_{S^0} : \overline{\mathcal{S}} \to \mathcal{S}$.

Proof.
Since $\mathcal{S}$ is presentable and the infinity category of presentable infinity categories admit all small limits, $\overline{\mathcal{S}}$ is also presentable. Every presentable infinity category admits all small limits and colimits. Since $\mathcal{S}$ is pointed, $\overline{\mathcal{S}}$ is also pointed. Since all limits are computed term-wise, we have that in particular $\Omega_{\overline{\mathcal{S}}}$ is computed term wise. given $\{X_n \mid n \in \mathbb{N}\}$ an object of $\overline{\mathcal{S}}$, $\{\Omega X_n \mid n \in \mathbb{N}\}$ is equivalent to $\{X_n \mid n \in \mathbb{N}\}$ because we have that ΩX_{n+1} is equivalent to X_n for all n. By a prp we conclude that $\overline{\mathcal{S}}$ is stable.

Consider the canonical functor $G : \overline{\mathcal{S}} \to \mathcal{S}$ defined by recovering the first factor: $(X_0, X_1, \dots) \mapsto X_0$. It is clear that it commutes with finite limits since limits are computed term-wise.

Let $\mathcal{C}$ be an arbitrary stable infinity category. Any functor $\mathcal{C} \to \mathcal{S}$ is left exact if and only if it is exact so that $\text{Exc}_*(\mathcal{C}, \mathcal{S}) = \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$. 1.4.2.16 implies that $\text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$ is a stable infinity category. Thus $\Omega_{\mathcal{S}} \circ -$ is an equivalence.

On the other hand, since Ω are computed term-wise (like all limits) and since $\text{Func}(\mathcal{C}, \overline{\mathcal{S}})$ is right adjoint to products we know that Func commutes with finite limits . Thus we have that

$$\text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) = \lim(\cdots \to \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S}) \overset{\Omega\circ -}{\to} \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S}) \overset{\Omega\circ -}{\to} \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S}))$$

Since each $\Omega_{\overline{\mathcal{S}}} \circ -$ is an equivalence of infinity categories, we conclude that $\text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) \simeq \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$. Thus evaluation on the first factor $G \circ - : \text{Exc}_*^{\text{L}}(\mathcal{C}, \overline{\mathcal{S}}) \to \text{Exc}_*^{\text{L}}(\mathcal{C}, \mathcal{S})$ is an equivalence of infinity categories.

By a previous corollary, there is an equivalence of infinity categories given by

$$\Omega^\infty \circ - : \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}}, \text{Sp}(\mathcal{S})) \to \text{Exc}_*^{\text{L}}(\overline{\mathcal{S}}, \mathcal{S})$$

The fact that G is left exact means that there is a factorization

$$\begin{tikzcd}
\overline{\mathcal{S}} \arrow[rr, "G"] \arrow[dr, "G' \circ -"'] & & \mathcal{S} \\
& \text{Sp}(\mathcal{S}) \arrow[ur, "\Omega^\infty"'] &
\end{tikzcd}$$

By functoriality we obtain a similar factorization:

$$
\begin{tikzcd}
\mathrm{Exc}_*^{\mathrm{L}}(\mathcal{C}, \overline{\mathcal{S}}) \arrow[rr, "G \circ -"] \arrow[dr, "G' \circ -"'] & & \mathrm{Exc}_*^{\mathrm{L}}(\mathcal{C}, \mathcal{S}) \\
& \mathrm{Exc}_*^{\mathrm{L}}(\mathcal{C}, \mathrm{Sp}(\mathcal{S})) \arrow[ur, "\Omega^\infty \circ -"'] &
\end{tikzcd}
$$

Since $G \circ -$ and $\Omega^\infty \circ -$ are both equivalence of infinity categories, we conclude that $G' \circ -$ is an equivalence of infinity categories.

Since this is true for all stable infinity categories, the fact that

$$\mathrm{Exc}_*(\mathcal{C}, \overline{\mathcal{S}}) = \mathrm{Exc}_*^L(\mathcal{C}, \overline{\mathcal{S}}) \simeq \mathrm{Exc}_*^L(\mathcal{C}, \mathrm{Sp}(\mathcal{S})) = \mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{S}))$$

is an equivalence for all stable $\mathcal{C}$ together with the Yoneda embedding implies that $\overline{\mathcal{S}}$ and $\mathrm{Sp}(\mathcal{S})$ is an equivalence of infinity categories. $\square$

Beware that in the proof we also showed that $G \circ -$ is an equivalence of infinity categories for any stable infinity category $\mathcal{C}$. But this does not imply that $\overline{\mathcal{S}}$ and $\mathcal{S}$ are equivalent because we are applying the Yoneda embedding on the category of stable infinity categories, and a priori $\mathcal{S}$ is not stable.

3 From Functors to Excisive Functors

3.1 Goodwillie Calculus

Definition 3.1.1: T_1 of a Functor

Let F be a homotopy functor. Let X be a space. Define

$$T_1F(X) = \text{holim}(F(CX) \to F(\Sigma X) \leftarrow F(CX))$$

It is clear that given a map $X \to Y$, this induces maps $CX \to CY$ and $\Sigma X \to \Sigma Y$. And the definition of the homotopy pullback tells us that T_1F is a functor. Any natural transformation $F \Rightarrow G$ gives maps $F(CX) \to G(CX)$ and $F(\Sigma X) \to G(\Sigma X)$ so that T_1 itself is a functor that sends functors to functors. Finally, there is a natural transformation $t_1(F) : F \Rightarrow T_1F$ since for each X there is a natural map $F(X) \to \text{holim}(F(CX) \to F(\Sigma X) \leftarrow F(CX))$.

Definition 3.1.2: P_1 of a Functor

Let F be a homotopy functor. Let X be a space. Define

$$P_1F(X) = \text{hocolim}(F(X) \overset{t_1(F)(X)}{\to} T_1F(X) \overset{t_1(T_1F)(X)}{\to} T_1(T_1F)(X) \to \cdots)$$

Since t_1 is natural transformation, we again obtain appropriate commutative diagrams so that P_1F becomes a functor.

Example 3.1.3

If F is reduced, then there is an easy way to describe the two functors. Namely, CX is contractible so $T_1F(X) = \Omega F(\Sigma X)$. Then

$$T_1(T_1(F))(X) = \Omega(T_1F)(\Sigma X) = \Omega(\Omega F(\Sigma(\Sigma X))) = \Omega^2 F(\Sigma^2 X)$$

Generalizing, we obtain

$$P_1F(X) = \underset{n\in\mathbb{N}}{\text{hocolim}}\ \Omega^n F(\Sigma^n X)$$

Theorem 3.1.4

P1 is excisive.

Example 3.1.5

Id -> Infinite loop suspension

3.2 Spectra and (Co)Homology Theories

Definition 3.2.1: Reduced Homology Theory

A reduced homology theory is a collection of functors and natural trasnformations

$$H_n : \mathbf{CGWH} \to \mathbf{Ab} \quad \text{and } s_n : H_n \Rightarrow H_{n+1} \circ \Sigma$$

such that the following axioms are satisfied.

- If $f \simeq g : X \to Y$ are homotopic, then $H_n(f) = H_n(g)$
- If $f : X \to Y$ is a map then there is an exact sequence

  $$H_n(X) \overset{f_*}{\to} H_n(Y) \overset{j_*}{\to} H_n(C_f)$$

  where $j : X \to C_f$ is the inclusion.

- The natural transformation gives an isomorphism
$$s_n(X) : H_n(X) \to H_{n+1}(\Sigma X)$$
for all n.
- For any wedge product $X = \bigvee_k X_k$, the inclusion maps induces an isomorphism
$$\bigoplus_k H_n(X_k) \cong H_n(X)$$
- If $f : X \to Y$ is a weak homotopy equivalence, then $H_n(f)$ is an isomorphism.

(Davies / Switzer)

Definition 3.2.2: Homology Theory Associated to Spectra

Let $\{K_n \mid n \in \mathbb{N}\}$ be a spectrum. Define a functor E_n : **Spaces** $\to$ **Ab** by
$$E_n(X) = \lim_{k\to\infty} \pi_{n+k}(X \wedge K_k)$$
and natural transformations $s_n : E_n \Rightarrow E_{n+1} \circ \Sigma$ given by the structure maps of the spectrum.

Here, the maps defining the direct limit $\pi_{n+k}(X \wedge K_k) \to \pi_{n+1+k}(X \wedge K_{k+1})$ are given by
$$\pi_{n+k}(X \wedge K_k) \stackrel{\Sigma}{\to} \pi_{n+1+k}(S^1 \wedge X \wedge K_k) \stackrel{\mathrm{id}_X \wedge \sigma_n}{\to} \pi_{n+1+k}(X \wedge K_{k+1})$$
(we are working in **CGWH**) Moreover, it is a functor since any map $f : X \to Y$ wedges with the identity to give a map $X \wedge K_k \to Y \wedge K_k$. The homotopy group functor gives a map of homotopy groups. And the universal property of direct limits give the induced map. (Davies p.229)

Theorem 3.2.3: Brown's Representability Theorem

Let H_n :

https://mathoverflow.net/questions/63974/is-every-homology-theory-given-by-a-spectrum

switzer 14.35

Example 3.2.4: Singular Cohomology

Example 3.2.5: K theory

Example 3.2.6: Landweber-exact Spectra

Theorem 3.2.7: Landweber exact functor theorem

3.3 A Map From Functors to (Co)Homology Theories

Example 3.3.1

Identity Functor -> stable homotopy theory (it is a homology theory)

Example 3.3.2

Excisive functor F -> F(Sn) -> corresponding cohomolog theory

4 Appendix

4.1 Homotopical Connectivity

2.6 cubical homotopy theory

Definition 4.1.1: n-Connected Spaces

Let X be a space. Let $n \in \mathbb{N}$. We say that X is n-connected if for all $-1 \leq k \leq n$, any map $S^k \to X$ is homotopic to the constant map to the base point.

Every non-empty space is (-1)-connected.

Definition 4.1.2: n-Connected Spaces

Let (X, A) be a pointed pair of spaces. Let $n \in \mathbb{N}$. We say that (X, A) is n-connected if for all $-1 \leq k \leq n$, any map $(D^k, \partial D^k) \to (X, A)$ is homotopic to a map $(D^k, \partial D^k) \to (A, A)$ relative to boundary.

Definition 4.1.3: n-Connected Maps

Let X, Y be spaces. Let $f : X \to Y$ be a map. We say that f is n-connected if (M_f, X) is n-connected.

Proposition 4.1.4

Let X be a space. Let $n \in \mathbb{N}$. Then the following are equivalent.
- X is n-connected.
- $\pi_k(X) = 0$ for all $0 \leq k \leq n$.
- For all $-1 \leq k \leq n$, every map $S^k \to X$ extends to a map $D^{k+1} \to X$.
- (CX, X) is $(n+1)$-connected.

Proposition 4.1.5

Let (X, A) be a space. Let $n \in \mathbb{N}$. Then the following are equivalent.
- (X, A) is n-connected
- For all $0 < k \leq n$, $\pi_k(X, A) = 0$ and $\pi_0(A) \to \pi_0(X)$ is surjective.
- $\iota : A \hookrightarrow X$ is n-connected.

Proposition 4.1.6

Let X, Y be spaces. Let $f : X \to Y$ be a map. Let $n \in \mathbb{N}$. If X is not empty, then the following are equivalent.
- f is n-connected
- $\text{hofiber}_y(f)$ is n-connected for all $y \in Y$.
- For all $0 < k \leq n$, $\pi_k(f)$ is an isomorphism and $\pi_n(f)$ is surjective.

4.2 Some Important Facts

Proposition 4.2.1

The infinity categories $\mathcal{S}$ and $\mathbf{Cat}_\infty$ is complete and cocomplete.

Fabian I.35

Limits of functors can be computed pointwise:

Proposition 4.2.2

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be infinity categories. Let $F : \mathcal{D} \to \mathcal{E}$ be a functor. Then the induced functor

$$- \circ F : \text{Func}(\mathcal{E}, \mathcal{C}) \to \text{Func}(\mathcal{D}, \mathcal{C})$$

preserves limits and colimits.

In particular, by choosing the inclusion functor $\{d\} \hookrightarrow \mathcal{D}$, we obtain the (co)limit preserving functor

$$\text{ev}_d : \text{Func}(\mathcal{D}, \mathcal{C}) \to \text{Func}(\{d\}, \mathcal{C}) \simeq \mathcal{C}$$

Now given any diagram $X : K \to \text{Func}(\mathcal{D}, \mathcal{C})$ that admits a limit, $\lim_K X$ is an object $\text{Func}(\mathcal{D}, \mathcal{C})$. To compute the value of this functor at $d \in \mathcal{D}$, we use the evaluation map to get

$$\left(\lim_K X \right)(d) = \text{ev}_d \left(\lim_K X \right) = \lim_K (\text{ev}_d \circ X)$$

where the limit on the right is now an object in $\mathcal{C}$, and the diagram of the limit is given on objects by $F(d)$ for all F in the image of X. (I.39 Fabian)