CS137

Labix

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Abstract

Contents

1 Asymptotic Relations

Definition 1.0.1. $g = O(f)$ means that g has at most run time f by a constant. $g = \Omega(f)$ means g has at least run time f by a constant. $g = \theta(f)$ means g has run time equal to f by a constant

Theorem 1.0.2 (Divide and Conquer)**.** The runtime of a divide and conquer algorithm. a is the number of subparts. $O(n)$ is the merging time.

$$T(n) = aT\left(\lfloor \frac{a}{2} \rfloor\right) + O(n)$$

Theorem 1.0.3 (Master Theorem)**.** If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$ for some constants $a > 0$, $b > 1$, $d \geq 0$. Then

$$T(n) = \begin{cases} O(n) & \text{if } d > \ln_b(a) \\ O(n^d \ln(n)) & \text{if } d = \ln_b(a) \\ O(n^{\ln_b(a)}) & \text{if } d < \ln_b(a) \end{cases}$$

Theorem 1.0.4 (Recurrence Relations)**.** Let

$$f_n = af_{n-1} + b^n$$

and $f_0 = 1$ f_n can be solved explicitly following the steps below.

$$\begin{aligned} \sum_{k=0}^{\infty} f_n x^n - 1 &= a \sum_{k=1}^{\infty} f_{n-1} x^n + \sum_{k=1}^{\infty} (bx)^n \\ &= ax \sum_{k=0}^{\infty} f_n x^n + \frac{1}{1-bx} - 1 \\ \sum_{k=0}^{\infty} f_n x^n &= \frac{1}{(1-ax)(1-bx)} \\ &= \frac{a}{(1-ax)(a-b)} - \frac{b}{(1-bx)(a-b)} \\ &= \frac{a}{a-b} \sum_{k=0}^{\infty} (ax)^n - \frac{b}{a-b} \sum_{k=0}^{\infty} (bx)^n \\ f_n &= \frac{a^{n+1}}{a-b} - \frac{b^{n+1}}{a-b} \end{aligned}$$

2 Graph Theory

2.1 Classification of Graphs

Definition 2.1.1. A graph $G = (V, E)$ is a set of vertices V and edges E.

Definition 2.1.2 (Multiple Edges)**.** A multiple edge are two edges such that both connectes a to b.

Definition 2.1.3 (Loops)**.** A loop is an edge that connects to itself.

Definition 2.1.4 (Simple Graph)**.** A graph with no loops and no multiple edges is a simple graph.

Definition 2.1.5 (Complete Graphs)**.** A graph is complete if every pair of vertices has an edge.

Definition 2.1.6 (Bipartite Graph)**.** A grpah is bipartite if its vertices can be partitioned into two sets such that every edge has one end point in each set.

2.2 Matchings

Definition 2.2.1 (Matching)**.** A matching M in G is a set of edges such that no two edge share common vertices.

Theorem 2.2.2 (Hall's Theorem)**.** Consider any bipartite graph $G = (L \cup R, E)$ with $|L| = n$. It contains a matching $M \subseteq E$ of size

$$|M| = |L|$$

if and only if

$$|N(A)| \geq |A|$$

for all $A \subseteq L$.

Definition 2.2.3 (Alternating Path)**.** An alternating path is a path that begins with an unmatched vertex, and whose edge belongs alternativelt to the matching and not to the matching.

Definition 2.2.4. A maximal matching is a matching M of a group that is not a subset of any other matching.

Definition 2.2.5 (Maximum Matching)**.** A maximum matching is a matching that contains the largest possible number of edges.

Theorem 2.2.6. If M is a matching in a bipartite graph that no alternating chain can exists, then M is a maximum matching.

Definition 2.2.7 (Vertex Cover)**.** Let G be a graph. A set of vertices V is said to be a vertex cover of G if every edge in G has at least one end point in V.

Proposition 2.2.8. Let M be a matching in a bipartite graph G and S a vertex cover, then $|M| \leq |S|$

Definition 2.2.9. Denote $\mu(G)$ the size of a maximum matching of G and $\tau(G)$ the size of a minimum vertex cover of G.

Theorem 2.2.10. For every bipartite graph G, $\mu(G) = \tau(G)$

2.3 Walks

Definition 2.3.1 (Degree Sequence)**.** The degree sequence of a graph is a list of its degrees.

Theorem 2.3.2. In any graph, let $d_1, \dots, d_n$ be a degree sequence. Then

$$\sum_{k=1}^{n} d_k = 2|E|$$

Definition 2.3.3. A sequence is graphical if it can be drawn into a graph

Theorem 2.3.4. A degree sequence $d_1 \geq \cdots \geq d_n$ is graphical if and only if $\sum_{k=1}^{n} d_k$ is even and for all $m \in \{1, \dots, n\}$,

$$\sum_{k=1}^{m} d_k \leq m(m-1) + \sum_{k=m+1}^{n} \min\{d_k, m\}$$

Definition 2.3.5 (Walk)**.** A walk on a graph is a sequence of alternating vertices and edges that start and end with vertices. (You are walking on a graph)

Definition 2.3.6 (Closed Walk)**.** A closed walk is a walk that starts and ends in the same vertex.

Definition 2.3.7 (Euler Walk)**.** An Euler walk is a walk that uses every vertex only once.

Definition 2.3.8 (Euler Circuit)**.** An Euler circuit is a closed walk that uses every vertex only once.

Theorem 2.3.9. Let G be connected. There exists a Euler circuit on G if and only if every vertex has even degree.

Theorem 2.3.10. Let G be connected. There exists a Euler walk on G if and only if exactly two vertices have odd degree.

2.4 Cliques and Independent Sets

Definition 2.4.1 (Cliques)**.** An l-clique is a complete graph on l vertices.

Definition 2.4.2 (Independent Sets)**.** Let $G = (V, E)$ be a graph and $X \subseteq V$. X is said to be independent if no two vertices in X are connected by an edge.

Proposition 2.4.3. G is a clique if and only if the complement of G is independent.

Proposition 2.4.4. Let $G = (V, E)$ be a graph and $S \subseteq V$. Then S is a vertex cover of G if and only if $G[V \setminus S]$ is independent.

Definition 2.4.5. Let $k, l \in \mathbb{N}$. $R(k, l)$ denotes the smallest positive integer such that every graph on $R(k, l)$ vertices contains a k-clique or an independent set on l vertices.

Proposition 2.4.6. For all $k \in \mathbb{N}$, $R(1, k) = R(k, 1) = 1$

Proposition 2.4.7. For all $k, l \in \mathbb{N}$, $R(k, l) = R(l, k)$

Theorem 2.4.8. Denote $\binom{V}{k}$ the set of all k-sized subsets of V. Let $n, k \in \mathbb{N}$ such that $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$. Then $R(k, k) > n$.

2.5 Pigeonhole Principle

Theorem 2.5.1 (Piegeonhole principle)**.** For every pair of non-empty finite sets A and B and for every function $f : A \to B$, if $|A| > (c-1)|B|$, then there exists an element $b \in B$ such that $\left|f^{-1}(b)\right| \geq c$.

Theorem 2.5.2 (Piegeonhole Principle 2)**.** For every $q_1, \dots, q_n \in \mathbb{N}$, for every pair of non-empty finite sets $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_n\}$ such that $r = \sum_{k=1}^{n} q_k - n + 1$, and for every function $f : A \to B$, there exists $i \in \{1, \dots, n\}$ such that $\left|f^{-1}(b_i)\right| \geq q_i$

Definition 2.5.3. A directed graph is an ordered pair (V, A) where the V is the set of vertices and $A \subseteq V \times V$ is the set of arcs.

Definition 2.5.4 (Tournament)**.** A directed graph $D = (V, A)$ is called a tournament if for every $i, v \in V$, only either one of (u, v) or (v, u) is in A.

Definition 2.5.5. Let $D = (V, A)$ be a tournament and let $S \subset V$. S is k-strong if $|S| = k$ and moreover, for every $v \in V \setminus S$, there is a $u \in S$ such that $(u, v) \in A$.

2.6 Probability

Theorem 2.6.1 (Markov's Inequality)**.** Let $X : \Omega \to \mathbb{R}^+$ be a random variable. Let $a > 0$ Then

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Proof.

$$\begin{aligned} E[X] &= \sum_{k=0}^{a-1} kP(X = k) + \sum_{k=a}^{\infty} kP(X = k) \\ &\geq \sum_{k=a}^{\infty} kP(X = k) \\ &\geq \sum_{k=a}^{\infty} aP(X = k) \\ &= aP(X \geq a) \end{aligned}$$

□

Theorem 2.6.2 (Law of Total Expectation)**.** Let $X : \Omega \to \mathbb{R}$ be a random variable. Let $A_1, \dots, A_n$ be a parition of Ω in the sample space Ω such that $P(A_i) \neq 0$ for all $i \in \{1, \dots, n\}$. Then

$$E[X] = \sum_{k=1}^{n} E[X|A_k]P(A_k)$$

Theorem 2.6.3. The expected number of cycles of length k in a permutation of S_n is given by $\frac{1}{k}$.

Proof. First choose k elements from n elements to form the cycle of length k. The number of possible cycles of length k is given by $\frac{k!}{k}$ because $(a_1, a_2 \dots, a_{k-1}, a_k) = (a_2, \dots, a_{k-1}, a_k, a_1)$ and vice versa. Then multiply it by the number of permutations that fixes the k elements we chose. And finally divide by the total number of possible permutations in n elements. We have that

$$\binom{n}{k}(k-1)!(n-k)!\frac{1}{n!} = \frac{1}{k}$$

$\square$