Enriched Category Theory

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Abstract

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1 Algebraic Objects in a Category

1.1 Group Objects

Definition 1.1.1: Group Objects

Let $\mathcal{C}$ be a category with finite products. We say that $G \in \mathcal{C}$ is a group object if there exists three morphisms

- Multiplication: $m : G \times G \to G$
- Identity: $e : * \to G$ where $*$ is the terminal object
- Inverse: $\text{inv} : G \to G$

such that the following diagrams commute.

- Associativity:

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times \text{id}_G"] \arrow[d, "\text{id}_G \times m"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- Identity:

$$\begin{tikzcd} G \arrow[r, "(e, \text{id}_G)"] \arrow[d, "(\text{id}_G, e)"'] \arrow[dr, "\text{id}_G"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- Inverse:

$$\begin{tikzcd} G \arrow[r, "(\text{inv}, \text{id}_G)"] \arrow[d, "(\text{id}_G, \text{inv})"'] \arrow[dr, "e"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

Proposition 1.1.2

A group object in the category **Set** of sets is a group in the usual sense.

Proposition 1.1.3

A group object in the category **Grp** of groups is an abelian group.

2 Monoidal Categories

2.1 Strict and Weak Monoidal Categories

Definition 2.1.1: Strict Monoidal Categories

A strict monoidal category is a category $\mathcal{A}$ consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

- Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- Identity: $I \otimes A = A$ and $A \otimes I = A$

Notice that we require strict equality in the associativity and identity laws. Since we usually only consider objects up to isomorphism in a category, strict monoidal categories may seem quite rare in practise.

Definition 2.1.2: Weak Monoidal Category

A weak monoidal category is a category $\mathcal{A}$ consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

- Associativity: There are isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

  that is natural in A, B and C

- Identity: There are isomorphisms

$$\lambda_A : I \otimes A \xrightarrow{\cong} A \quad \text{and} \quad \rho_A : A \otimes I \xrightarrow{\cong} A$$

  that are both natural in A

Such natural isomorphisms must also satisfy the following commutative laws:

- The pentagon identity:

$$\begin{tikzcd}
 & (A \otimes B) \otimes (C \otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A\otimes B,C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes 1_D"'] & & A \otimes (B \otimes (C \otimes D)) \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B\otimes C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[u, "1_A \otimes \alpha_{B,C,D}"']
\end{tikzcd}$$

- The triangle identity:

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
 & A \otimes B &
\end{tikzcd}$$

It is clear that every strict monoidal category is also a weak monoidal category.

Lemma 2.1.3

Every category $\mathcal{C}$ with finite products is a monoidal category with product $\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and identity $*$ the terminal object.

Proposition 2.1.4

For any commutative ring R, the category $\mathbf{Mod}_R$ of R-modules is a monoidal category with the tensor product $\otimes$ and the identity object R.

Definition 2.1.5: Symmetric Monoidal Category

Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is a symmetric monoidal category if $\mathcal{C}$ is a weak monoidal category together with isomorphisms

$$s_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

that are natural in A and B such that the following are satisfied:

- Unit coherence: If I is the distinguished object of $\mathcal{C}$ as a weak monoidal category, then the following diagram commutes:

$$\begin{tikzcd} A \otimes I \arrow[rr, "s_{A,I}"] \arrow[dr, "\lambda_A"'] & & I \otimes A \arrow[dl, "\rho_A"] \\ & A & \end{tikzcd}$$

- The associativity coherence: For any $A, B, C \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} (A \otimes B) \otimes C \arrow[r, "s_{A,B} \otimes \mathrm{id}_C"] \arrow[d, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[d, "\alpha_{B,A,C}"] \\ A \otimes (B \otimes C) \arrow[d, "s_{A,B\otimes C}"'] & B \otimes (A \otimes C) \arrow[d, "\mathrm{id}_B \otimes s_{A,C}"] \\ (B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"'] & B \otimes (C \otimes A) \end{tikzcd}$$

- The inverse law: For any $A, B \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} & B \otimes A \arrow[dr, "s_{B,A}"] & \\ A \otimes B \arrow[ur, "s_{A,B}"] \arrow[rr, "\mathrm{id}_{A\otimes B}"'] & & A \otimes B \end{tikzcd}$$

2.2 Closed Categories

3 Enriched Categories