Riemannian Manifolds

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Abstract

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1 Riemannian Metrics

1.1 The Riemannian Metric

Definition 1.1.1: Riemannian Metric

Let M be a smooth manifold. A Riemannian metric on M is a function $g : TM \times TM \to \mathbb{R}$ such that for each $p \in M$, the restriction of g to

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

is an inner product.

Definition 1.1.2: Riemannian Manifold

A Riemannian manifold (M, g) is a manifold M together with a Riemannian metric g on M.

Theorem 1.1.3

Every smooth manifold admits a Riemannian metric and hence is a Riemannian manifold.

Definition 1.1.4: Isometries

Let (M, g) and (N, h) be two Riemannian manifolds. We say that (M, g) and (N, h) are isometric if there exists a diffeomorphisms $f : M \to N$ such that

$$h \circ f = g$$

In this case f is said to be an isometry.

Definition 1.1.5: Local Isometries

Let (M, g) and (N, h) be two Riemannian manifolds. We say that they are locally isometric if for all $p \in M$, there exists an open neighbourhood $U \subseteq M$ of p and $V \subseteq N$ open and an isometry $f : U \to V$.

Definition 1.1.6: Flat Manifolds

Let (M, g) be a Riemannian manifold. We say that (M, g) is flat if it is locally isometric to $\mathbb{R}^n$ with the standard metric.

In general, not every Riemannian manifold is flat. This can be shown once we discuss curvatures and torsions. However, this is true when $n = 1$.

Lemma 1.1.7

Every 1 dimensional Riemannian manifold is flat.

1.2 Lengths and Angles

Definition 1.2.1: Length of a Tangent Vector

Let (M, g) be a Riemannian manifold. Let $v \in T_p(M)$ be a tangent vector for $p \in M$. Define the length of v to be

$$|v|_g = \sqrt{g_p(v, v)}$$

Definition 1.2.2: Angle between two Tangent Vectors

Let (M, g) be a Riemannian manifold. Let $p \in M$. For $v, w \in T_pM$ two tangent vectors, define the angle between v and w to be the unique $\theta \in [0, \pi]$ such that

$$\cos(\theta) = \frac{g_p(v, w)}{|v|_g |w|_g}$$

Definition 1.2.3: Orthogonal Tangent Vectors

Let (M, g) be a Riemannian manifold. Let $p \in M$. We say that two tangent vectors $v, w \in T_pM$ are orthogonal if

$$g_p(v, w) = 0$$

Definition 1.2.4: Length of a Curve

Let (M, g) be a Riemannian manifold. Let $\gamma : (a, b) \to M$ be a curve. Define the length of the curve by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(s)}(\gamma'(s), \gamma'(s))}\, ds$$

Definition 1.2.5: Angle between two Curves

Let (M, g) be a Riemannian manifold. Let $\gamma_1 : (a, b) \to M$ and $\gamma_2 : (c, d) \to M$ be two curves that intersecting at $p = \gamma_1(t_1) = \gamma_2(t_2) \in M$ and that $\gamma_1'(t_1) \neq 0$ and $\gamma_2'(t) \neq 0$. Define the angle between γ_1 and γ_2 at p to be the unique $\theta \in [0, \pi]$ such that

$$\cos(\theta) = \frac{g_p(X_{\gamma_1,p}, X_{\gamma_2,p})}{|X_{\gamma_1,p}|_g |X_{\gamma_2,p}|_g}$$

1.3 Musical Isomorphism

Definition 1.3.1: The Flat Map

Let (M, g) be a Riemannian manifold. Let $p \in M$. For each $X \in T_pM$, define the flat map

$$\flat : T_pM \to T_p^*M$$

by sending $X \in T_pM$ to the map $X^\flat : T_pM \to \mathbb{R}$ by $X^\flat(Y) = g_p(X, Y)$.

Theorem 1.3.2: The Musical Isomorphism

Let (M, g) be a Riemannian manifold. Let $p \in M$. Then the flat map

$$\flat : T_pM \to T_p^*M$$

is an isomorphism.

Definition 1.3.3: The Sharp Map

Let (M, g) be a Riemannian manifold. Let $p \in M$. Define the sharp map

$$\# : T_p^*M \to T_pM$$

to be the inverse of the flat map.

1.4 Bundle Metric

Definition 1.4.1: Bundle Metric

Let M be a topological manifold and $p : E \to M$ a vector bundle on M. Then a bundle metric on E is a section of $E^* \otimes E^*$ such that it is nondegenerate and symmetric.

In other words, a bundle metric is an assignment to each fibre, an inner product. Bilinearity is seen from $E^* \otimes E^*$, which is exactly the set of all bilinear forms $E \times E \to \mathbb{R}$.

Proposition 1.4.2

Let M be a smooth manifold. Then a Riemannian metric give rise to a bundle metric on TM. A bundle metric on TM gives rise to a Riemannian metric.

2 Connections and Parallel Transports

2.1 Affine Connections

Recall that for a smooth vector bundle $p : E \to M$, we denote the space of smooth sections on E by $\Gamma(E)$. Moreover, $\mathfrak{X}(M)$ is the space of smooth sections on the tangent bundle TM.

Definition 2.1.1: Connections

Let M be a smooth manifold. Let $p : E \to M$ be a smooth vector bundle. A connection on p is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

where we denote $\nabla(V, T)$ by $\nabla_V(T)$, such that the following are true.

- $C^\infty(M)$-linearity in first variable: For each $T \in \Gamma(E)$, the map $V \mapsto \nabla_V(T)$ is $C^\infty(M)$-linear. This means that

$$\nabla_{fV+hW}(T) = f\nabla_V(T) + g\nabla_W(T)$$

  for $V, W \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$.
- $\mathbb{R}$-linearity in second variable: For each $V \in \mathfrak{X}(M)$, the map $T \mapsto \nabla_V(T)$ is $\mathbb{R}$-linear. This means that

$$\nabla_V(\lambda T + \mu S) = \lambda\nabla_V(T) + \mu\nabla_V(S)$$

- Product rule: The map ∇ satisfies the following product rule:

$$\nabla_V(fT) = V(f) \cdot T + f\nabla_V(T)$$

Definition 2.1.2: Affine Connections

Let M be a smooth manifold. An affine connection of M is a connection

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

on the tangent bundle TM.

The directional derivative is the canonical affine connection on $\mathbb{R}^n$. In fact, every smooth manifold has such a canonical connection that generalizes the directional derivative.

Theorem 2.1.3

Every smooth manifold admits an affine connection.

2.2 Metric Connections

Definition 2.2.1: Metric Connections

Let M be a smooth manifold. Let ∇ be an affine connection. We say that ∇ is a metric connection if the

$$\nabla(X, g(Y, Z)) = g(\nabla(X, Y), Z) + g(Y, \nabla(X, Z))$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

2.3 The Levi-Civita Connection

Definition 2.3.1: The Lie Bracket on Smooth Vector Fields

Let M be a smooth manifold. Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields. Define the Lie bracket of X and Y to be the vector field $[X, Y]$ given by the formula

$$[X, Y]_p(f) = X(Y_p(f)) = Y(X_p(f))$$

for $p \in M$ and $f \in \mathcal{C}^\infty_{M,p}$.

Proposition 2.3.2

Let M be a smooth manifold. Let $X, Y \in \mathfrak{X}(M)$ be smooth vector fields. Let (U, ϕ) be a chart on M. Let X and Y be given locally on the chart by the formula

$$X = \sum_{k=1}^n a_k \frac{\partial}{\partial x^k} \quad \text{and} \quad Y = \sum_{k=1}^n b_k \frac{\partial}{\partial x^k}$$

for $a_k, b_k : U \to \mathbb{R}$ smooth functions. Then the Lie bracket of X and Y is given locally on the chart by the formula

$$[X, Y] = \sum_{k=1}^n (X(b_k) - Y(a_k)) \frac{\partial}{\partial x^k} = \sum_{i=1}^n \left(\sum_{j=1}^n \left(a_j \frac{\partial b_i}{\partial x^i} - b_i \frac{\partial a_j}{\partial x^j} \right) \right) \frac{\partial}{\partial x^i}$$

Definition 2.3.3: Levi-Civita Connections

Let (M, g) be a Riemannian manifold. Let $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an affine connection on M. We say that ∇ is a Levi-Civita connection if the following are true.

- ∇ is a metric connection.
- For any $X, Y \in \mathfrak{X}(M)$, we have

$$\nabla(X, Y) - \nabla(Y, X) = [X, Y]$$

Theorem 2.3.4: Existence and Uniqueness of Levi-Civita Connections

Let (M, g) be a Riemannian manifold. Then M has a unique Levi-Civita connection.

3 Taking Derivatives of the Vector Fields

3.1 Covariant Derivatives

Let M be a smooth manifold. Let $\gamma : I \to M$ be a smooth curve. Then $\text{im}(\gamma)$ is an embedded submanifold of M. Therefore it makes sense to talk about smooth vector fields on $\text{im}(\gamma)$. We overload the notation and denote the $C^\infty(\text{im}(\gamma))$-algebra of smooth vector spaces by

$$\mathfrak{X}(\gamma) = \{X : \text{im}(\gamma) \to TM \mid X \text{ is a smooth vector field on } \text{im}(\gamma)\}$$

Definition 3.1.1: Covariant Derivatives Along a Curve

Let M be a smooth manifold. Let $\gamma : (a, b) \to M$ be a curve on M. Let $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ be an affine connection on M. The covariant derivative of γ is a map

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

such that

- $\mathbb{R}$-linearity: $D_t(aV + bW) = aD_tV + bD_tW$ for $a, b \in \mathbb{R}$.
- Product rule: $D_t(fV) = f'V + fD_tV$ for $f \in C^\infty(a, b)$.
- Extendable: If $V \in \mathfrak{X}(\gamma)$ and there exists $\tilde{V} \in \mathfrak{X}(M)$ such that $\tilde{V}|_{\gamma(t)} = V(t)$ for all $t \in (a, b)$, then $D_tV = \nabla_{\gamma'(t)}\tilde{V}$.

Theorem 3.1.2: Existence and Uniqueness of Covariant Derivatives

Let M be a smooth manifold. Let ∇ be an affine connection. For each smooth curve $\gamma : (a, b) \to M$, the connection ∇ determines a unique covariant derivative

$$D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$$

For each $t \in I$, choose a chart $(U, \phi = (x^1, \dots, x^n))$ for $\gamma(t) \in M$. Write V locally on the chart by

$$V_{\gamma(t)} = \sum_{i=1}^n a_i(\gamma(t)) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}$$

Then the unique associated covariant derivatives is given locally on the chart by the formula

$$(D_tV)_{\gamma(t_0)} = \sum_{i=1}^n \left(\frac{\partial a_i}{\partial x^i}\bigg|_{\gamma(t_0)} \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)} + a_i(\gamma(t_0))\nabla\left(\gamma'(t_0), \frac{\partial}{\partial x^i}\bigg|_{\gamma(t_0)}\right)\right)$$

for $t_0 \in I$.

3.2 Parallel Transports

Definition 3.2.1: Parallel Vector Fields along a Curve

Let M be a smooth manifold. Let $\gamma : I \to M$ be a curve. Let D_t be its associated covariant derivative. Let $X : M \to TM$ be a vector field. We say that X is parallel along to γ if $D_tX = 0$.

Theorem 3.2.2

Let M be a smooth manifold. Let $\gamma : I \to M$ be a curve. Let ∇ be an affine connection. Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field $V(t)$ along γ such that $V(t_0) = v_0$.

Definition 3.2.3: Parallel Transports

Let M be a smooth manifold. Let $\gamma : (a, b) \to M$ be a curve on M. Let $t_0, t \in (a, b)$. The map

$$P_{t_0,t} : T_{\gamma(t_0)}(M) \to T_{\gamma(t)}(M)$$

defined by $v \mapsto X(t)$ where $X(t)$ is the unique parallel vector field along γ with $X(t_0) = v$.

4 Geometry on Manifolds

4.1 Geodesics

Let M be a smooth manifold. Let $\gamma : I \to M$ be a smooth curve on M. Then we can compute its differential to obtain a tangent vector

$$\gamma'(t) \in T_{\gamma(t)}M$$

for each $t \in I$. This defines a vector field in $\mathfrak{X}(\gamma)$.

Definition 4.1.1: Geodesics

Let M be a smooth manifold. Let ∇ be an affine connection on M. A geodesic on M is a curve $\gamma : I \to M$ such that

$$D_t(\gamma'(t)) = 0$$

where D_t is the associated covariant derivative of ∇ and γ.

We have seen geodesics for metric spaces, but these are slightly different. We require geodesics on manifolds to only minimize the distance locally.

5 Measuring the Curvature

5.1 Gauss-Bonnet Theorem

Theorem 5.1.1: The Gauss-Bonnet Formula

Let (M, g) be an oriented smooth 2-manifold. Let γ be a positively oriented curved polygon in M and let Ω be its interior. Then

$$\int_\Omega K\,dA + \int_\gamma \kappa_N\,ds + \sum_{i=1}^k \varepsilon_i = 2\pi$$

where

- K is the Gaussian curvature of g
- dA is the Riemannian volume form
- ε_i are the exterior angles of γ
- The second integral is taken with respect to arc length

Theorem 5.1.2: Gauss-Bonnet Theorem

Let (M, g) be an smooth compact 2-dimensional Riemannian manifold. Let K be the Gaussian curvature of M and let k_g be the geodesic curvature of ∂M. Then

$$\int_M K\,dA + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$

Corollary 5.1.3

Let (M, g) be an smooth compact 2-dimensional Riemannian manifold without boundary. Let K be the Gaussian curvature of M. Then

$$\int_M K\,dA = 2\pi\chi(M)$$