Algebraic Curves

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Abstract

Contents

1 Algebraic Curves in Classical Algebraic Geometry

1.1 Basic Properties of Curves

Definition 1.1.1: Curves

Let k be a field. Let X be a variety over k. We say that X is a curve if $\dim(X) = 1$.

Proposition 1.1.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Let $p \in C$ be a non-singular point. Then $\mathcal{O}_{C,p}$ is a DVR. Moreover, the valuation is given by the degree of the regular function.

Proof. Since p is non-singular, by definition $\mathcal{O}_{C,p}$ is a regular local ring. Moreover, we know that $1 = \dim(C) = \dim(\mathcal{O}_{C,p})$ so that $\mathcal{O}_{C,p}$ has Krull dimension 1. By the equivalent characterization of DVR we conclude. □

Example 1.1.3

Consider the projective curve $C = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{\mathbb{C}}$. Let $p = [p_0 : p_1 : p_2]$ be a point on the curve.

If $p_2 \neq 0$, then $p \in U_2$. Under the affine chart (U_2, φ_2), we find that $C_2 = \varphi_2(C \cap U_2) = \mathbb{V}(x^2 + y^2 - 1)$. The corresponding coordinate ring is given by $\frac{\mathbb{C}[x,y]}{(x^2+y^2-1)}$. The formula for the local ring in the affine case gives

$$\mathcal{O}_{C,p} \cong \left(\frac{\mathbb{C}[x, y]}{(x^2 + y^2 - 1)} \right)_{m_{(p_0/p_2, p_1/p_2)}}$$

Recall that the unique maximal ideal of the local ring is given as the $\mathcal{O}_{X,p}$-module $m_p = \{f \in \mathbb{C}[C_2] \mid f(p_0/p_2, p_1/p_2) = 0\}$, which under the nullstellensatz is the maximal ideal corresponding to the point $(p_0/p_2, p_1/p_2)$ and is given by $m_p = (x - r, y - s)$ where $r = p_0/p_1$ and $s = p_0/p_2$. By Nakayama's lemma, since $x - r, y - s$ generate m_p we know that $x-r+m_p^2, y-s+m_p^2$ span the vector space m_p/m_p^2 over $\mathcal{O}_{X,p}/m_p$. I claim that they are linearly dependent. This mean that I want to find $f + m_p^2$ and $g + m_p^2$ in $\mathcal{O}_{X,p}/m_p$ that are non-trivial, and that $(x - r)f + (y - s)g + m_p^2 = m_p^2$. This means that we want to find $f, g \in \mathcal{O}_{X,p} \setminus m_p$ such that $(x - r)f + (y - s)g \in m_p^2$. Choose $f = x + r$ and $g = y + s$ to get

$$(x - r)(x + r) + (y - s)(y + s) = x^2 - r^2 + y^2 - s^2 = 1 - 1 = 0$$

since (r, s) lie on the curve. Moreover, $x + r, y + s\mathcal{O}_{X,p} \setminus m_p$ since evaluating at (r, s) at the functions are non-zero. This verifies that $\mathcal{O}_{X,p}$ is a regular local ring of dimension 1, hence is a DVR.

We can even find its uniformizer and valuation. Since $x - r$ and $y + s$ are linearly dependent and spans m_p/m_p^2, any one of the two is a basis for the vector space. WLOG take $x - r$ to be a basis. Nakayama's lemma implies that $x - r$ generates m_p. Being a DVR means that for all $f \in \mathcal{O}_{X,p}$, $f = u(x - r)^n$ where u is invertible. Then the valuation of f is n.

Proposition 1.1.4

Let C be an affine irreducible curve over $\mathbb{C}$. Then C is smooth if and only if C is a normal variety.

Proposition 1.1.5

Let k be a field. Let C be a smooth curve over k. Then for any projective variety $X \subseteq \mathbb{P}^n$ and rational map $\phi : C \to X$, there exists a regular map

$$\overline{\phi} : C \to X$$

such that $\overline{\phi}|_U = \phi|_U$ for some dense subset $U \subseteq C$.

Proposition 1.1.6

Let k be an algebraically closed field. Let X, Y be smooth irreducible projective curves over k. Let $\phi : X \to Y$ be a non-constant regular map. Then ϕ is a finite morphism.

1.2 Blowing Up Curves and Normalization

Recall that by taking the integral closure of the coordinate ring $k[C]$ of an irreducible affine curve $C \subseteq \mathbb{A}^n$, we obtain a corresponding variety $\widetilde{C}$ called the normalization of C.

Proposition 1.2.1

Let k be an algebraically closed field. Let $C \subseteq \mathbb{A}_k^n$ be an irreducible affine curve over k. Then the normalization $\widetilde{C}$ is smooth.

Theorem 1.2.2

Let k be an algebraically closed field. Let C be an irreducible curve over k. Then C is birational to a unique non-singular projective irreducible curve.

1.3 Divisors on Curves

2 Algebraic Curves in the Context of Schemes

Definition 2.0.1: Algebraic Curves

Let k be an algebraically closed field. A curve over k is an integral separated scheme X of finite type over k that has dimension 1.

Proposition 2.0.2

Let X be an algebraic curve. Then the arithmetic and geometric genus coincide. In particular,

$$p_a(X) = p_g(X) = \dim_k H^1(X, \mathcal{O}_X)$$

We will simply call the genus of a curve g from now on since the arithmetic genus is the same as the geometric genus.

2.1 Riemann-Roch Theorem

Definition 2.1.1: Canonical Divisor

Let X be an algebraic curve. The canonical divisor K of X is a divisor in the linear equivalence class of

$$\Omega^1_{X/k} = \omega_X$$

Theorem 2.1.2: Riemann-Roch Theorem

Let X be an algebraic curve. Let D be a divisor on X and let K be the canonical divisor of X. Let $\mathcal{L}(D)$ be the associated sheaf of the divisor D. Then

$$\dim_k(H^0(X, \mathcal{L}(D))) + \dim_k(H^0(X, \mathcal{L}(K-D))) = \deg(D) + 1 - p_g(X)$$

2.2 Classification of Curves in $\mathbb{P}^3$