Geometric Group Theory

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Abstract

Potentially good books: Humphreys, Erdmann and Wildson

Contents

1 The Geometry of Presentations

1.1 Cayley Graphs

Definition 1.1.1: The Cayley Graph of a Group

Let G be a group. Let S be a generating set of G. Define the Cayley graph $\Gamma = \Gamma(G, S)$ of G with respect to S to consist of the following data.

- The vertices are given by $V(\Gamma) = G$
- The edges are given by $E(\Gamma) = \{(g, gs) \mid g \in G, s \in S\}$

Let (V, E) be a graph. Recall that a graph automorphism consists of a bijective map of vertices and a bjective map of edges such that

$$\{\phi(v), \phi(w)\} \in E$$

for all $\{v, w\} \in E$. They form a group by composition.

Lemma 1.1.2: The Action Lemma

Let G be a group. Let S be a generating set of G. Then G acts on the Cayley graph Γ of G with respect to S via tha map

$$\cdot : G \times \Gamma \to \Gamma$$

defined by $h \cdot g = hg$ and $h \cdot (g, gs) = (hg, hgs)$.

Lemma 1.1.3

Let G be a group. Let S be a generating set of G. Then G acts on the Cayley graph faithfully.

Proposition 1.1.4

Let G be a group. Let S be a generating set of G. Then the following are true regarding the Cayley graph $\Gamma(G, S)$ of G with respect to S.

- $\Gamma(G, S)$ has no embedded cycles.
- $\Gamma(G, S)$ is connected.

Proposition 1.1.5

Let S be a set. Then $\Gamma(F_S, S)$ is a tree.

Proposition 1.1.6

Let G be a group. Let S be a generating set of G. Consider $\Gamma(G, S)$ as a CW complex. Then $\Gamma(F_S, S)$ is a universal cover of $\Gamma(G, S)$.

1.2 Giving the Cayley Graph a Metric

Definition 1.2.1: The Word Metric

Let G be a group. Let S be a generating set of G. Let Γ be the Cayley complex of G and S. Define the word metric to be the map $d_S : V(\Gamma) \times V(\Gamma) \to \mathbb{N}$ by

$$d_S(g, h) = \min\{n \in \mathbb{N} \mid \gamma : [0, n] \to \Gamma \text{ is a path from } g \text{ to } h\}$$

Lemma 1.2.2

Let G be a group. Let S be a generating set of G. Let Γ be the Cayley complex of G and S. Then d_Γ is a metric on $V(\Gamma)$.

Proposition 1.2.3

Let G be a group. Let S be a generating set of G. Let Γ be the Cayley complex of G and S. Then for each $k \in G$, the induced map $\Gamma \to \Gamma$ from the action of G on Γ is an isometry. In other words,

$$d_S(k \cdot g, k \cdot h) = d_S(g, h)$$

for all $g, h, k \in G$.

Definition 1.2.4: The Word Norm

Let G be a group. Let S be a generating set of G. Let Γ be the Cayley complex of G and S. Define the word norm of $g \in G$ to be

$$\|g\|_S = d_S(1_G, g)$$

Lemma 1.2.5

Let G be a group. Let S be a generating set of G. Let Γ be the Cayley complex of G and S. Then the following are true.

- $d_S(g, h) = \|g^{-1}h\|_S$ for all $g, h \in G$.
- $\|g^{-1}\|_S = \|g\|_S$ for all $g \in G$.
- $\|gh\|_S \leq \|g\|_S + \|h\|_S$ for all $g, h \in G$.