Cohomology of Schemes

Labix

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Abstract

References:

Contents

1 Symmetric Polynomials

1.1 Symmetric Polynomials

The theory of symmetric functions are important in combinatorics, representation theory, Galois theory and the theory of λ-rings.

Requirements: Groups and Rings
Books: Donald Yau: Lambda Rings

Definition 1.1.1: Symmetric Group Action on Polynomial Rings

Let R be a ring. Define a group action of S_n on $R[x_1, \dots, x_n]$ by

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

It is easy to check that this defines a group action.

Definition 1.1.2: Symmetric Polynomials

Let R be a ring. We say that a polynomial $f \in R[x_1, \dots, x_n]$ is symmetric if

$$\sigma \cdot f = f$$

for all $\sigma \in S_n$.

Definition 1.1.3: The Ring of Symmetric Polynomials

Let R be a ring. Define the ring of symmetric polynomials in n variables over R to be the set

$$\Sigma = \{f \in R[x_1, \dots, x_n] \mid \sigma \text{ is a symmetric polynomial } \}$$

Definition 1.1.4: Elementary Symmetric Polynomials

Let R be a ring. Define the elementary symmetric polynomials to be the elements $s_1, \dots, s_n \in R[x_1, \dots, x_n]$ given by the formula

$$s_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

Theorem 1.1.5: The Fundamental Theorem of Symmetric Polynomials

Let R be a ring. Then $s_1, \dots, s_n$ are algebraically independent over R. Moreover,

$$\Sigma = R[s_1, \dots, s_n]$$

2 λ-Rings

2.1 λ-Rings

Complex representation of a group is a λ-ring. Topological K theory is a λ-ring.

Requirements: Category Theory, Groups and Rings, Symmetric Functions
Books: Donald Yau: Lambda Rings

We need the theory of symmetric polynomials before defining λ-structures.

Definition 2.1.1: λ-Structures

Let R be a commutative ring. A λ-structure on R consists of a sequence of maps $\lambda^n : R \to R$ for $n \geq 0$ such that the following are true.

- $\lambda^0(r) = 1$ for all $r \in R$
- $\lambda^1 = \text{id}_R$
- $\lambda^n(1) = 0$ for all $n \geq 2$
- $\lambda^n(r+s) = \sum_{k=0}^n \lambda^k(r)\lambda^{n-k}(s)$ for all $r, s \in R$
- $\lambda^n(rs) = P_n(\lambda^1(r), \dots, \lambda^n(r), \lambda^1(s), \dots, \lambda^n(s))$ for all $r, s \in R$
- $\lambda^m(\lambda^n(r)) = P_{m,n}(\lambda^1(r), \dots, \lambda^{mn}(r))$ for all $r \in R$

Here P_n and $P_{m,n}$ are defined as follows.

- The coefficient of t^n in the polynomial
$$h(t) = \prod_{i,j=1}^n (1 + x_i y_j t)$$
is a symmetric polynomial in x_i and y_j with coefficients in $\mathbb{Z}$. P_n is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of x_i and y_j respectively.
- The coefficient of t^n in the polynomial
$$g(t) = \prod_{1 \leq i_1 \leq \dots \leq i_m \leq nm} (1 + x_{i_1} \cdots x_{i_m} t)$$
is a symmetric polynomial in x_i with coefficients in $\mathbb{Z}$. $P_{m,n}$ is precisely this polynomial written in terms of the elementary polynomials $e_1, \dots, e_n$ of x_i.

In this case, we call R a λ-ring.

Note that we do not require that the λ^n are ring homomorphisms.

Definition 2.1.2: Associated Formal Power Series

Let R be a λ-ring. Define the associated formal power series to be the function $\lambda_t : R \to R[[t]]$ given by
$$\lambda_t(r) = \sum_{k=0}^{\infty} \lambda^k(r) t^k$$
for all $r \in R$

Proposition 2.1.3

Let R be a λ-ring. Then the following are true regarding $\lambda_t(r)$.

- $\lambda_t(1) = 1 + t$
- $\lambda_t(0) = 1$
- $\lambda_t(r+s) = \lambda_t(r)\lambda_t(s)$
- $\lambda_t(-r) = \lambda(r)^{-1}$

Proposition 2.1.4

The ring $\mathbb{Z}$ has a unique λ-structure given by

$$\lambda_t(n) = (1+t)^n$$

Proposition 2.1.5

Let R be a λ-ring. Then R has characteristic 0.

Definition 2.1.6: Dimension of an Element

Let R be a λ-ring and let $r \in R$. We say that r has dimension n if $\deg(\lambda_t(r)) = n$. In this case, we write $\dim(r) = n$.

Proposition 2.1.7

Let R be a λ-ring. Then the following are true regarding the dimension of n.

- $\dim(r+s) \leq \dim(r) + \dim(s)$ for all $r, s \in R$
- If r and s both has dimension 1, then so is rs.

2.2 λ-Ring Homomorphisms and Ideals

Definition 2.2.1: λ-Ring Homomorphisms

Let R and S be λ-rings. A λ-ring homomorphism from R to S is a ring homomorphism $f : R \to S$ such that

$$\lambda^n \circ f = f \circ \lambda^n$$

for all $n \in \mathbb{N}$.

Definition 2.2.2: λ-Ideals

Let R be a λ-ring. A λ-ideal of R is an ideal I of R such that

$$\lambda^n(i) \in I$$

for all $i \in I$ and $n \geq 1$.

TBA:λ-ideal and subring. Ker, Im, Quotient Product, Tensor, Inverse Limit are λ-rings

Proposition 2.2.3

Let R be a λ-ring. Let $I = \langle z_i \mid i \in I \rangle$ be an ideal in R. Then I is a λ-ideal if and only if $\lambda^n(z_i) \in I$ for all $n \geq 1$ and $i \in I$.

Proposition 2.2.4

Every λ-ring R contains a λ-subring isomorphic to $\mathbb{Z}$.

2.3 Augmented λ-Rings

Definition 2.3.1: Augmented λ-Rings

Let R be a λ-ring. We say that R is an augmented λ-ring if it comes with a λ-homomorphism
$$\varepsilon : R \to \mathbb{Z}$$
called the augmentation map.

TBA: tensor of augmented is augmented

Proposition 2.3.2

Let R a λ-ring. Then R is augmented if and only if there exists a λ-ideal I such that
$$R = \mathbb{Z} \oplus I$$
as abelian groups.

2.4 Extending λ-Structures

Proposition 2.4.1

Let R be a λ-ring. Then there exists a unique λ-structure on $R[x]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[x]$ and $\varepsilon(r) = 0$ or 1.

Proposition 2.4.2

Let R be a λ-ring. Then there exists a unique λ-structure on $R[[x]]$ such that $\lambda_t(r) = 1 + rt$. Moreover, if R is augmented, then so is $R[[x]]$ and $\varepsilon(r) = 0$ or 1.

2.5 Free λ-Rings

2.6 The Universal λ-Ring

2.7 Adams Operations

3 Witt Vectors

3.1 Fundamentals of the Ring of Big Witt Vectors

Prelim: Symm Functions, Lambda Rings, Category theory, Frobenius endomorphism (Galois), Rings and Modules, Kaehler differentials (commutative algebra 2)
Leads to: K theory
Books: Donald Yau: Lambda Rings

Definition 3.1.1: Truncation Sets

Let $S \subseteq \mathbb{N}$. We say that S is a truncation set if for all $n \in S$ and $d|n$, then $d \in S$. For $n \in \mathbb{N}$ and S a truncation set, define

$$S/n = \{d \in \mathbb{N} \mid nd \in S\}$$

For instance, $\mathbb{N} \setminus \{0\}$ is a truncation set. We will also use $\{1, \dots, n\}$.

Theorem 3.1.2: Dwork's Theorem

Let R be a ring and let S be a truncation set. Suppose that for all primes p, there exists a ring endomorphism $\sigma_p : R \to R$ such that $\sigma_p(r) \equiv r^p \pmod{pR}$ for some $s \in R$. Then the following are equivalent.

- Every element $(b_i)_{i \in S} \in \prod_{i \in S} R$ has the form
$$(b_i)_{i \in S} = (w_i(a))_{i \in S}$$
for some $a \in R$
- For all primes p and all $n \in S$ such that $p|n$, we have
$$b_n \equiv \sigma_p(b_{n/p}) \pmod{p^n R}$$

In this case, a is unique, and a_n depends solely on all the b_k for $1 \le k \le n$ and $k \in S$.

We wish to equip $\prod_{i \in S} R$ with a non-standard addition and multiplication to make it into a ring.

Proposition 3.1.3

Consider the ring $R = \mathbb{Z}[x_i, y_i \mid i \in S]$. There exists unique polynomials

$$\xi_n(x_1, \dots, x_n, y_1, \dots, y_n), \pi_n(x_1, \dots, x_n, y_1, \dots, y_n), \iota_n(x_1, \dots, x_n)$$

for $n \in S$ such that
- $w_n(\xi_1, \dots, \xi_n) = w_n((x_i)_{i \in S}) + w_n((y_i)_{i \in S})$
- $w_n(\pi_1, \dots, \pi_n) = w_n((x_i)_{i \in S}) \cdot w_n((y_i)_{i \in S})$
- $w_n(\iota_1, \dots, \iota_n) = -w_n((x_i)_{i \in S})$

for all $n \in S$.

Note that the polynomials ξ_n, π_n have variables x_k and y_k for $k \le n$ and $k \in S$. This is similar for the variables of ι. From now on, this will be the convention: For S a truncation set, the sequence $a_1, \dots, a_n$ actually refers to the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \le d_1 \le \dots \le d_k \le n$ and $d_1, \dots, d_k$ are all divisors of n. The result of this is that sequences in $\mathbb{N}$ are now restricted to S.

Definition 3.1.4: The Ring of Truncated Witt Vector

Let R be a ring. Let S be a truncation set. Define the ring of big Witt vectors $W_S(R)$ of R to consist of the following.
- The underlying set $\prod_{i \in S} R$
- Addition defined by $(a_n)_{n \in S} + (b_n)_{n \in S} = (\xi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$
- Multiplication defined by $(a_n)_{n \in S} \times (b_n)_{n \in S} = (\pi_n(a_1, \dots, a_n, b_1, \dots, b_n))_{n \in \mathbb{N}}$

Theorem 3.1.5

Let R be a ring. Let S be a truncation set. Then the ring of big Witt vectors $W_S(R)$ of R is a ring with additive identity $(0, 0, \dots)$ and multiplicative identity $(1, 0, 0, \dots)$. Moreover, for $(a_n)_{n\in S} \in W(R)$, its additive inverse is given by $(\iota_n(a_1, \dots, a_n))_{n\in\mathbb{N}}$.

Proposition 3.1.6

Let $\phi : R \to R'$ be a ring homomorphism. Then the induced map $W_S(\phi) : W_S(R) \to W_S(R')$ defined by

$$W(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

is a ring homomorphism.

Definition 3.1.7: The Witt Functor

Define the Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ to consist of the following data.

- For each ring R, $W_S(R)$ is the ring of big Witt vectors
- For a ring homomorphism $\phi : R \to R'$, $W_S(\phi) : W_S(R) \to W_S(R')$ is the induced ring homomorphism defined by

$$W_S(\phi)((a_n)_{n\in S}) = (\phi(a_n))_{n\in S}$$

Proposition 3.1.8

Let S be a truncation set. The Witt functor is indeed a functor.

Definition 3.1.9: The Ghost Map

Let R be a ring. Let S be a truncation set. Define the ghost map to be the map

$$w : W_S(R) \to \prod_{k\in S} R$$

by the formula

$$w((a_n)_{n\in S}) = (w_n(a_1, \dots, a_n))_{n\in S}$$

Remember, by the sequence $a_1, \dots, a_n$ we mean the sequence $a_1, a_{d_1}, \dots, a_{d_k}, a_n$ where $1 \le d_1 \le \dots \le d_k \le n$ and $d_1, \dots, d_k$ the complete collection of divisors of n.

Proposition 3.1.10

Let S be a truncation set. Then the following are true.

- For each $n \in S$, the collection of maps $w_n : W_S(R) \to R$ for a ring R defines a natural transformation $w_n : W_S \to \text{id}$.
- The collection of ghost maps $w_R : W_S(R) \to \prod_{k\in S} R$ for R a ring defines a natural transformation $w : W_S \to (-)^S$.

Proposition 3.1.11

Let S be a truncation set. The truncated Witt functor $W_S : \textbf{Ring} \to \textbf{Ring}$ is uniquely characterized by the following conditions.

- The underlying set of $W_S(R)$ is given by $\prod_{k\in S} R$
- For a ring homomorphism $\phi : R \to S$, $W(\phi) : W(R) \to W(S)$ is the induced ring homomorphism defined by

$$W(\phi)((a_n)_{n\in\mathbb{N}}) = (\phi(a_n))_{n\in\mathbb{N}}$$

- For each $n \in S$, $w_n : W_S(R) \to R$ defines a natural transformation $w_n : W \to \text{id}$

This means that if there is another functor V satisfying the above, then W and V are naturally isomorphic.

Note that the above theorem implies that the ring structure on $\prod_{k \in S} R$ is unique under the above conditions.

3.2 Important Maps of Witt Vectors

Definition 3.2.1: The Forgetful Map

Let R be a ring. Let $T \subseteq S$ be truncation sets. Define the forgetful map $R_T^S : W_S(R) \to W_T(R)$ to be the ring homomorphism given by forgetting all elements $s \in S$ but $s \notin T$.

Definition 3.2.2: The nth Verschiebung Map

Let R be a ring. Let S be a truncation set. For $n \in \mathbb{N}$, define the nth Verschiebung map $V_n : W_{S/n}(R) \to W_S(R)$ by

$$V_n((a_d)_{d \in S/n})_m = \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise} \end{cases}$$

Note that this is not a ring homomorphism. However, it is additive.

Lemma 3.2.3

Let R be a ring. Let S be a truncation set. Then for all $a, b \in W_{S/n}(R)$, we have that

$$V_n(a + b) = V_n(a) + V_n(b)$$

Definition 3.2.4: Frobenius Map

Let S be a truncation set. Let R be a ring. Define the Frobenius map to be a natural ring homomorphism $F_n : W_S(R) \to W_{S/n}(R)$ such that the following diagram commutes:

$$\begin{tikzcd} W_S(R) \arrow[r, "w"] \arrow[d, "F_n"'] & \prod_{k \in S} R \arrow[d, "F_n^w"] \\ W_{S/n}(R) \arrow[r, "w"'] & \prod_{k \in S/n} R \end{tikzcd}$$

if it exists.

Lemma 3.2.5

Let S be a truncation set. Let R be a ring. Then the Frobenius map exists and is unique.

The following lemma relates this notion of Frobenius map to that in ring theory.

Lemma 3.2.6

Let A be an F_p algebra. Let S be a truncation set. Let $\varphi_p : A \to A$ denote the Frobenius homomorphism given by $a \mapsto a^p$. Then

$$F_p = R_{S/p}^S \circ W_S(\varphi) : W_S(A) \to W_{S/p}(A)$$

Definition 3.2.7: The Teichmuller Representative

Let R be a ring. Let S be a truncation set. Define the Teichmuller representative to be the map $[-]_S : R \to W_S(R)$ defined by

$$([a]_S)_n = \begin{cases} a & \text{if } n = 1 \\ 0 & b \text{ otherwise} \end{cases}$$

The Teichmuller representative is in general not a ring homomorphism, but it is still multiplicative.

Lemma 3.2.8

Let R be a ring. Let S be a truncation set. The for all $a, b \in R$, we have that

$$[ab]_S = [a]_S \cdot [b]_S$$

The three maps introduced are related as follows.

Proposition 3.2.9

Let R be a ring. Let S be a truncated set. Then the following are true.

- $r = \sum_{n \in S} V_n([r_n]_{S/n})$ for all $r \in W_S(R)$
- $F_n(V_n(a)) = na$ for all $a \in W_{S/n}(R)$
- $r \cdot V_n(a) = V_n(F_n(r) \cdot a)$ for all $r \in W_S(R)$ and all $a \in W_{S/n}(R)$
- $F_m \circ V_n = V_n \circ F_m$ if $\gcd(m, n) = 1$

The remaining section is dedicated to the example of $R = \mathbb{Z}$.

Proposition 3.2.10

Let S be a truncation set. Then the ring of big Witt vectors of $\mathbb{Z}$ is given by

$$W_S(\mathbb{Z}) = \prod_{n \in S} \mathbb{Z} \cdot V_n([1]_{S/n})$$

with multiplication given by

$$V_m([1]_{S/m}) \cdot V_n([1]_{S/n}) = \gcd(m, n) \cdot V_d([1]_{S/d})$$

and $d = \operatorname{lcm}(m, n)$.

3.3 The Ring of p-Typical Witt Vectors

For the ring of p-typical Witt vectors, we consider the truncation set $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$ for a prime p.

Definition 3.3.1: The Ring of p-Typical Witt Vectors

Let R be a ring. Let p be a prime. Let $P = \{1, p, p^2, \dots\} \subseteq \mathbb{N}$. Define the ring of p-typical Witt vectors to be

$$W_p(R) = W_P(R)$$

Define the ring of p-typical Witt vectors of length n to be

$$W_n(R) = W_{\{1, p, \dots, p^{n-1}\}}(R)$$

when the prime p is understood.

Theorem 3.3.2

Let R be a ring. Let p be a prime number. Let S be a truncation set. Write $I(S) = \{k \in S \mid k \text{ does not divide } p\}$. Suppose that all $k \in I(S)$ are invertible in R. Then there is a decomposition

$$W_S(R) = \prod_{k \in I(S)} W_S(R) \cdot e_k$$

where

$$e_k = \prod_{t \in I(S) \setminus \{1\}} \left(\frac{1}{k} V_k([1]_{S/k}) - \frac{1}{kt} V - kt([1]_{S/kt}) \right)$$

Moreover, the composite map given by

$$W_S(R) \cdot e_k \hookrightarrow W_S(R) \xrightarrow{F_k} W_{S/k} R \xrightarrow{R^{S/k}_{S/k \cap P}} W_{S/k \cap P}(R)$$

is an isomorphism.

3.4 The λ-structure on $\mathbf{W(R)}$

Lemma 3.4.1

Let R be a ring. Then every $f \in \Lambda(R)$ can be written uniquely as

$$f = \prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n)$$

Theorem 3.4.2: The Artin-Hasse Exponential

There is a natural isomorphism $E : \Lambda \to W$ given as follows. For a ring R, $E_R : \Lambda(R) \to W(R)$ is defined by

$$E_R\left(\prod_{k=1}^{\infty} (1 - (-1)^n a_n t^n) \right) = (a_n)_{n \in \mathbb{N}}$$

Corollary 3.4.3

Let R be a ring. Then $W(R)$ has a canonical λ-structure inherited from $\Lambda(R)$.

TBA: The forgetful functor $U : \Lambda\mathbf{Ring} \to \mathbf{CRing}$ has a left adjoint Symm and has a right adjoint W.

4 Formal Group Laws

Definition 4.0.1: Formal Group Laws

Let R be a ring. A formal group law over R is a power series

$$f(x,y) \in R[[x,y]]$$

such that the following are true.

- $f(x,0) = f(0,x) = x$
- $f(x,y) = f(y,x)$
- $f(x, f(y,z)) = f(f(x,y),z)$

Definition 4.0.2: The Formal Group Law Functor

Define the formal group law functor

$$FGL : \mathbf{Ring} \to \mathbf{Set}$$

by the following data.

- For each ring R, $FGL(R)$ is the set of all formal group laws over R
- For each ring homomorphism $f : R \to S$, $FGL(f)$ sends each formal group law $\sum_{i,j=0}^{\infty} c_{i,j} x^i y^j$ over R to the formal group law $\sum_{i,j=0}^{\infty} f(c_{i,j}) x^i y^j$ over S.

Definition 4.0.3: The Lazard Ring of a Formal Group Law

Define the lazard ring by

$$L = \frac{\mathbb{Z}[c_{i,j}]}{Q}$$

where Q is the ideal generated as follows. Write $f = \sum_{i,j=0}^{\infty} c_{i,j} x^i y^j$. Then Q is generated by the constraints on $c_{i,j}$ for which f becomes a formal group law.

Lemma 4.0.4

The Lazard ring $L = \mathbb{Z}[c_{i,j}]/Q$ has the structure of a graded ring where $c_{i,j}$ has degree $2(i + j - 1)$.

Theorem 4.0.5

The formal group law functor $FGL : \mathbf{Ring} \to \mathbf{Set}$ is representable

$$FGL(R) \cong \mathrm{Hom}_{\mathbf{Ring}}(L, R)$$

There exists a universal element $f \in L$ such that the map $\mathrm{Hom}_{\mathbf{Ring}}(L, R) \to FGL(R)$ given by evaluation on f is a bijection for any ring R.

Theorem 4.0.6

There is an isomorphism of the Lazard ring

$$L \cong \mathbb{Z}[t_1, t_2, \dots]$$

where each t_k has degree $2k$.

5 Homotopy Pullbacks and Pushouts

Homotopy pullbacks and pushouts are a special case of homotopy limits and colimits. It would be fruitful for us to first consider this case also because of how it is related to maps of spaces and (co)fibrations.

Definition 5.0.1: Homotopy Pullbacks

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in **CGWH**. Define the homotopy pullback $\text{holim}(X \overset{f}{\to} Z \overset{g}{\leftarrow} Y)$ of the diagram to be the subspace of $X \times \text{Map}(I, Z) \times Y$ consisting of

$$\{(x, \alpha, y) \in X \times \text{Map}(I, Z) \times Y \mid \alpha(0) = f(x), \alpha(1) = g(y)\}$$

The idea is that normally in pullbacks, we require that under f and g the elements of the pullback must arrive at the same point in Z. But here we relax the requirement by simply allowing elements of the homotopy pullback to arrive at the same path component of Z (so up to the existence of an homotopy of the two points in Z).

Definition 5.0.2: The Canonical Map of Homotopy Pullbacks

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in **CGWH**. Define the canonical map of the homotopy pullback of the diagram to be the map

$$c : \lim(X \overset{f}{\to} Z \overset{g}{\leftarrow} Y) \to \text{holim}(X \overset{f}{\to} Z \overset{g}{\leftarrow} Y)$$

defined by $(x, y) \mapsto (x, c_{f(x)=g(y)}, y)$.

Theorem 5.0.3: The Matching Lemma

Suppose that we have a commutative diagram of spaces

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e_X"'] & Z \arrow[d, "e_Z"'] & Y \arrow[l, "g"'] \arrow[d, "e_Y"] \\ X' \arrow[r, "f'"'] & Z' & Y' \arrow[l, "g'"] \end{tikzcd}$$

in **CGWH**. If each e_X, e_Y, e_Z are (homotopy) weak equivalences, then the induced map

$$\text{holim}(X \overset{f}{\to} Z \overset{g}{\leftarrow} Y) \to \text{holim}(X' \overset{f'}{\to} Z' \overset{g'}{\leftarrow} Y')$$

defined by $(x, \gamma, y) \mapsto (e_X(x), e_Z \circ \gamma, e_Y(y))$ is a (homotopy) weak equivalence.

When one of the maps f or g is a fibration, then the notion of a pullback coincides with that of homotopy pullback.

Definition 5.0.4: Homotopy Pullback Squares

Let $W, X, Y, Z \in \mathbf{CGWH}$ be spaces such that there is a (not necessarily commutative) diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

We say that the diagram is a homotopy pullback if $\alpha : W \to \lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$ is a weak equivalence.

The definition immediately suggests that the square

$$\begin{tikzcd} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is a homotopy pullback square. However, note that it is not a commutative square. However the following are indeed actual commutative squares. Moreover, they are all pullbacks:

$$\begin{tikzcd} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \arrow[r, "p_Y"] \arrow[d] & Y \arrow[d, "g"] \\ P_f \arrow[r, "p"'] & Z \end{tikzcd} \qquad \begin{tikzcd} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \arrow[r] \arrow[d] & P_g \arrow[d] \\ P_f \arrow[r] & Z \end{tikzcd} \qquad \begin{tikzcd} \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y) \arrow[r] \arrow[d, "p_X"'] & P_g \arrow[d, "p"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

This is the content of the next theorem.

Theorem 5.0.5

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in $\mathbf{CGWH}$. Then the following spaces are all homeomorphic.

- $\text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$
- $\lim(P_f \to Z \xleftarrow{g} Y)$
- $\lim(X \xrightarrow{f} Z \leftarrow P_g)$
- $\lim(P_f \to Z \leftarrow P_g)$

Proposition 5.0.6

Let $X, Y, Z \in \mathbf{CGWH}$ be spaces such that

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

is a diagram in $\mathbf{CGWH}$. If f or g is a fibration, then the canonical map

$$\lim(X \xrightarrow{f} Z \xleftarrow{g} Y) \to \text{holim}(X \xrightarrow{f} Z \xleftarrow{g} Y)$$

is a homotopy equivalence. In terms of diagrams, this means the following commutative square

$$\begin{tikzcd} \lim(X \to Z \leftarrow Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is a homotopy pullback if $X \to Z$ or $Y \to Z$ is a fibration.

Recall that the mapping path space P_f of a map $f : X \to Y$ is defined to be

$$P_f = f^*(\text{Map}(I, Y)) = \{(x, \phi) \subseteq X \times \text{Map}(I, Y) \mid f(x) = \pi_0(\phi) = \phi(0)\}$$

we can now prove that P_f is a homotopy invariance.

Corollary 5.0.7

Let $X, Y \in \mathbf{CGWH}$ be spaces. Let $f, g : X \to Y$ be maps. Then there is a homotopy equivalence

$$P_f \simeq P_g$$

Moreover, there is a homotopy equivalence

$$\text{hofiber}_y(f) = \text{hofiber}_y(g)$$

for any $y \in Y$.

Recall that the fiber of a map $f : X \to Y$ behaves poorly because the fibers are not homeomorphic and not even homotopy equivalent. However, we can now prove that the homotopy fibers are the correct notion of a fiber to study because they are homotopy equivalent.

Corollary 5.0.8

Let $X, Y \in \mathbf{CGWH}$ be space. Let $f : X \to Y$ be a map. If y_1 and y_2 lie in the same path component of Y then there is a homotopy equivalence

$$\text{hofiber}_{y_1}(f) = \text{hofiber}_{y_2}(f)$$

We now want to measure how far away is a square diagram from being a homotopy pullback.

5.1 Homotopy Squares

5.2 Connectedness of Homotopy Squares

6

Theorem 6.0.1: Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

is a homotopy pushout. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cartesian (this means that $\alpha : \lim(X_1 \to X_{12} \leftarrow X_2) \to \text{holim}(X_1 \to X_{12} \leftarrow X_2)$ is $(k_1 + k_2 - 1)$-connected).

Theorem 6.0.2: Dual Blakers-Massey Theorem for Squares

Let $X_0, X_1, X_2, X_{12} \in \mathbf{CGWH}$ be spaces such that the square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

is a homotopy pullback. Suppose the map $X_0 \to X_i$ is k_i-connected for $i = 1, 2$. Then the diagram is $(k_1 + k_2 - 1)$-cocartesian (this means that $\alpha : \lim(X_1 \to X_{12} \leftarrow X_2) \to \text{holim}(X_1 \to X_{12} \leftarrow X_2)$ is $(k_1 + k_2 - 1)$-connected).

7 n-Cubes

In algebraic topology, we have learnt about spaces, maps of spaces and maps of maps of spaces. We can say this in a more compact way. Namely, if we think of maps of maps of space as a square (2-cube), we can think of spaces as 0-cubes and maps of spaces as 1-cube. We have studied 2-cubes extensively under the guise of homotopy pullbacks and pushouts. We can now take this further and consider general n-cubes.

Definition 7.0.1: n-Cubes of Spaces

Let $n \in \mathbb{N}$. Let $P(n)$ denote the category of posets of the set $\{1, \dots, n\}$. An n-cube of spaces is a functor

$$X : P(n) \to \textbf{CGWH}$$

An n-cube of based spaces is a functor $X : P(n) \to \textbf{CGWH}_*$.

Explicitly, an n-cube of spaces $X : P(n) \to \textbf{CGWH}$ consists of the following data.

- For each $S \subseteq \{1, \dots, n\}$ a space X_S
- For each $S \subseteq T$, a map $f_{S\subseteq T} : X_S \to X_T$ such that $f_{S\subseteq S} = 1_{X_S}$ and for all $R \subseteq S \subseteq T$, we have a commutative diagram

$$\begin{tikzcd} X_R \arrow[r, "f_{R\subseteq S}"] \arrow[dr, "f_{R\subseteq T}"'] & X_S \arrow[d, "f_{S\subseteq T}"] \\ & X_T \end{tikzcd}$$

Omit drawing composite arrows and omit drawing identities.
Also: punctured cubes def

Definition 7.0.2: Cube of Cubes

An n-cube of m-cubes is a functor

$$X : P(n) \times P(m) \to \textbf{CGWH}$$

Lemma 7.0.3

An n-cube of m-cubes X is precisely an $(n + m)$-cube.

Definition 7.0.4: Map of n-Cubes

Let $X, Y : P(n) \to \textbf{CGWH}$ be n-cubes. A map of n-cubes is a natural transformation $F : X \to Y$ such that the assignment $Z : P(n + 1) \to \textbf{CGWH}$ given by

$$Z(S) = \begin{cases} X(S) & \text{if } S \subseteq \{1, \dots, n\} \\ Y(S \setminus \{1, \dots, n+1\}) & \text{if } \{1, \dots, n+1\} \subseteq S \end{cases}$$

defines an $(n + 1)$-cube.

objectwise (co)fibration, homotopy (weak) equivalence. homeomorphism

Definition 7.0.5: Strongly Homotopy Cartesian

Let X be an n-cube of spaces. We say that X is strongly homotopy cartesian if each of its faces of dimension $n \geq 2$ is homotopy cartesian.

8 Calculus of Functors

Definition 8.0.1: Homotopy Functors

Let $\mathcal{C}, \mathcal{D}$ be categories with a notion of weak equivalence. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopy functor if F preserves weak equivalences.