Algebraic Number Theory

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October 5, 2024

Abstract

Contents

1 Basic Field Theory Over $\mathbb{Q}$

1.1 Algebraic Number Fields

Definition 1.1.1: Algebraic Number Field

An algebraic number field is a finite field extension $\mathbb{Q} < K$.

Let K, L be two fields. Recall that any field homomorphism $K \to L$ is injective, and hence it makes sense to say that they are embeddings.

Definition 1.1.2: Embeddings of Number Fields

Let K be a number field. Let $\sigma : K \to \mathbb{C}$ be a field embedding.

- We say that σ is a real embedding if $\sigma(K) \subseteq \mathbb{R}$.
- Otherwise we say that σ is a complex embedding.

Lemma 1.1.3

Let K be a number field. Let $\sigma : K \to \mathbb{C}$ be an embedding. Then $\sigma(a) = a$ for all $a \in \mathbb{Q}$.

Definition 1.1.4: Signature of a Number Field

Let K be a number field. Define the signature of K to be $(r, s) \in \mathbb{N} \times \mathbb{N}$ where r is the total number of real embeddings, and s is one half of the total number of complex embeddings.

Definition 1.1.5: Quadratic Fields

Let K be a number field. We say that K is a quadratic field if

$$[K : \mathbb{Q}] = 2$$

Recall that a number of square free if d is not divisible by m^2 for any $m \in \mathbb{N}$.

Lemma 1.1.6

Let K be a quadratic field. Then $K = \mathbb{Q}(\sqrt{d})$ for some unique square free number $d \neq 1$.

1.2 The Trace and Norm Map of Number Fields

The trace and norm map simplifies (is it true?) when we consider number fields.

Recall in Fields and Galois Theory that $\text{Emb}_{\mathbb{Q}}(K, \mathbb{C})$ is defined to the the set of all field homomorphisms from K to $\mathbb{C}$ fixing $\mathbb{Q}$.

Proposition 1.2.1

Let K be a number field. Then for any $\alpha \in K$, we have that

$$\text{Tr}_{K/\mathbb{Q}}(\alpha) = \sum_{\sigma \in \text{Emb}_{\mathbb{Q}}(K,\mathbb{C})} \sigma(\alpha)$$

Similarly we have that

$$N_{K/\mathbb{Q}}(\alpha) = \prod_{\sigma \in \text{Emb}_{\mathbb{Q}}(K,\mathbb{C})} \sigma(\alpha)$$

2 Algebraic Integers

Definition 2.0.1: Ring of Integers

Let K be an algebraic number field. Define the ring of integers of K to be

$$\mathcal{O}_K = \{\alpha \in K \mid \alpha \text{ is integral over } \mathbb{Z}\}$$

Lemma 2.0.2

Let K be a number field. Then

$$K = \text{Frac}(\mathcal{O}_K)$$

Definition 2.0.3: Algebraic Integers

An algebraic integer is an element of $\mathcal{O}_{\mathbb{C}}$.

Proposition 2.0.4

Let $\alpha \in \mathbb{C}$ be an algebraic number. Then α is an algebraic integer if and only if $\min(\mathbb{Q}, \alpha) = \min(\mathbb{Z}, \alpha)$.