

Commutative Algebra 1

Labix

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Abstract

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1 Basic Notions of Commutative Rings

1.1 Local Rings

Definition 1.1.1: Local Rings

Let R be a commutative ring. We say that R is a local ring if it has a unique maximal ideal m . In this case, we say that R/m is the residue field of R .

Example 1.1.2

Consider the following commutative rings.

- $\mathbb{Z}/6\mathbb{Z}$ is not a local ring.
- $\mathbb{Z}/8\mathbb{Z}$ is a local ring.
- $\mathbb{Z}/24\mathbb{Z}$ is not a local ring.
- $\mathbb{R}[x]$ is not a local ring.

Proof.

- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$. They do not contain each other and so they are both maximal.
- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2 + 8\mathbb{Z})$ and $(4 + 8\mathbb{Z})$. But $(2 + 8\mathbb{Z}) \supseteq (4 + 8\mathbb{Z})$. Hence $\mathbb{Z}/8\mathbb{Z}$ has a unique maximal ideal.
- A similar proof as above ensues.
- Any irreducible polynomial $f \in \mathbb{R}[x]$ is such that (f) is a maximal ideal. Indeed the evaluation homomorphism gives an isomorphism $\frac{\mathbb{R}[x]}{(f)} \cong \mathbb{R}$.

□

Proposition 1.1.3

Let R be a ring and I an ideal of R . Then I is the unique maximal ideal of R if and only if I is the set containing all non-units of R .

Proof. Let I be the unique maximal ideal of R . Clearly I does not contain any unit else $I = R$. Now suppose that r is a non-unit. Suppose that $r \notin I$. Define $J = \{sr | s \in R\}$. Clearly J is an ideal. It must be contained in some maximal ideal. Since I is the unique maximal ideal, $J \subseteq I$. But this means that $r \in I$, a contradiction. Thus every non-unit is in I .

Suppose that I contains all non-units of R . Let $r \notin I$. Then there exists $s \notin I$ such that $rs = 1$. Then $(r + I)(s + I) = 1 + I$ in R/I . This means that every element of R/I has a multiplicative inverse which means that R/I is a field and thus I is a maximal ideal. Now let $J \neq I$ be another maximal ideal. Then J contains some unit r . This implies that $J = R$ and thus I is the unique maximal ideal.

□

Example 1.1.4

Let k be a field. Then the ring of power series $k[[x]]$ is a local ring.

Proof. Let M be the set of all non-units of $k[[x]]$. I first show that $f \in M$ if and only if the constant term of f is non-zero. Let g be a power series. Then the n th coefficient of $f \cdot g$ is given by

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

If the constant term of f is 0, then $c_0 = 0$ and so $f \cdot g \neq 1$. Now if the constant term of f is

$a_0 \neq 0$, then set $b_0 = \frac{1}{a_0}$. Now we can use the formula $0 = c_n$ to deduce

$$b_n = -\frac{\sum_{k=1}^n a_k b_{n-k}}{a_0}$$

. This is such that $a_n \cdot b_n = 0$. Define $g = \sum_{k=0}^{\infty} b_k x^k$. Then $f \cdot g = 1$. Thus f is a unit.

By the above proposition, we conclude that M is the unique maximal ideal of $k[[x]]$. \square

We will discuss more of local rings in the topic of localizations.

1.2 Noetherian Commutative Rings

We recall some facts about Noetherian rings. In the following, let R be a commutative ring, although they are also true if R is non-commutative if we take all modules defined below to be left (right) R -modules.

- If we have a short exact sequence of R -modules:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.

- If M and N are R -modules, then $M \oplus N$ is Noetherian if and only if M and N are Noetherian.
- If M is an R -module and N is an R -submodule of M , then M is Noetherian if and only if N and M/N are Noetherian.
- If R is Noetherian and I is an ideal of R , then R/I is Noetherian.
- Later when once has seen localization, we can also prove that: If R is Noetherian then $S^{-1}R$ is Noetherian for any multiplicative subset S of R .

Theorem 1.2.1: Hilbert's Basis Theorem

Let R be a commutative ring. If R is Noetherian, then

$$R[x_1, \dots, x_n]$$

is a Noetherian ring.

Proposition 1.2.2

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring. Then R is Noetherian if and only if R_0 is Noetherian and R is finitely generated as an R_0 -module.

1.3 Artinian Commutative Rings

We recall some facts about Artinian rings. In the following, let R be a commutative ring, although they are also true if R is non-commutative if we take all modules defined below to be left (right) R -modules.

- If we have a short exact sequence of R -modules:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then M_2 is Artinian if and only if M_1 and M_3 are Artinian.

- If M and N are R -modules, then $M \oplus N$ is Artinian if and only if M and N are Artinian.

- If M is an R -module and N is an R -submodule of M , then M is Artinian if and only if N and M/N are Artinian.
- If R is Artinian and I is an ideal of R , then R/I is Artinian.

Proposition 1.3.1

Let R be an integral domain. Then R is Artinian if and only if R is a field.

Proposition 1.3.2

Let R be a commutative ring. If R is Artinian, then the following are true.

- Every prime ideal of R is maximal
- There is an equality $N(R) = J(R)$
- R has only a finite number of maximal ideals

Theorem 1.3.3: (Hopkins)

Let R be a commutative ring. If R is Artinian then R is Noetherian.

1.4 Spectra of a Ring**Definition 1.4.1: Max Spectrum of a Ring**

Let A be a commutative ring. Define the max spectrum of A to be

$$\max\text{Spec}(A) = \{m \subseteq A \mid m \text{ is a maximal ideal of } A\}$$

Definition 1.4.2: Spectrum of a Ring

Let A be a commutative ring. Define the spectrum of A to be

$$\text{Spec}(A) = \{p \subseteq A \mid p \text{ is a prime ideal of } A\}$$

Example 1.4.3

Consider the following commutative rings.

- $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2 + 6\mathbb{Z}), (3 + 6\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/8\mathbb{Z}) = \{(2 + 8\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/24\mathbb{Z}) = \{(2 + 24\mathbb{Z}), (3 + 24\mathbb{Z})\}$
- $\text{Spec}(\mathbb{R}[x]) = \{(f) \mid f \text{ is irreducible}\}$

Proof.

- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$. We need to find which ones are prime ideals. Now $\mathbb{Z}/6\mathbb{Z} \setminus (2 + 6\mathbb{Z})$ consists of $1 + 6\mathbb{Z}$, $3 + 6\mathbb{Z}$ and $5 + 6\mathbb{Z}$. No multiplication of these elements give an element of $(2 + 6\mathbb{Z})$. So any two elements in $\mathbb{Z}/6\mathbb{Z}$ which multiply to an element of $(2 + 6\mathbb{Z})$ must contain one element that lie in $(2 + 6\mathbb{Z})$. Hence $(2 + 6\mathbb{Z})$ is prime. This is similar for $(3 + 6\mathbb{Z})$. Hence $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2 + 6\mathbb{Z}), (3 + 6\mathbb{Z})\}$.
- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2 + 8\mathbb{Z})$ and $(4 + 8\mathbb{Z})$. A similar argument as above shows that $(2 + 8\mathbb{Z})$ is a prime ideal. However, $6 + 8\mathbb{Z} \notin (4 + 8\mathbb{Z})$ while $(6 + 8\mathbb{Z})^2 = 4 + 8\mathbb{Z} \in (4 + 8\mathbb{Z})$ which shows that $(4 + 8\mathbb{Z})$ is not a prime ideal.
- A similar proof as above ensues.
- Recall that $\mathbb{R}[x]$ is a principal ideal domain. Let $I = (f)$ be a prime ideal of $\mathbb{R}[x]$. Then f is irreducible. Thus every prime ideal of $\mathbb{R}[x]$ is of the form (f) for f an irreducible

polynomial. □

Lemma 1.4.4

Let R, S be commutative rings. Let $f_1 : R \times S \rightarrow R$ and $f_2 : R \times S \rightarrow S$ denote the projection maps. Then the map

$$f_1^* \amalg f_2^* : \operatorname{Spec}(R) \amalg \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R \times S)$$

is a bijection.

Proof. The core of the proof is the fact that P is a prime ideal of $R \times S$ if and only if $P = R \times Q$ or $P = V \times S$ for either a prime ideal Q of R or a prime ideal V of S . It is clear that if Q is a prime ideal of S and V is a prime ideal of R , then $R \times Q$ and $V \times S$ are both prime ideals of $R \times S$.

So suppose that P is a prime ideal in $R \times S$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Since $P \neq R \times S$, at least one of e_1 or e_2 is not in P . Without loss of generality assume that $e_1 \notin P$. But $e_1 e_2 = 0 \in P$ and P being prime implies that $e_2 \in P$. Since e_2 is the identity of $\{0\} \times S \cong S$, we conclude that $\{0\} \times S \subseteq P$. By the correspondence theorem, the projection map $f_1 : R \times S \rightarrow R$ gives a bijection between prime ideals of $R \times S$ that contain $\{0\} \times S$ and prime ideals of R . So $f_1(P)$ is a prime ideal of R . Thus $P = f_1(P) \times S$ which is exactly what we wanted.

Now the bijection is clear. $f_1^* \amalg f_2^*$ sends a prime ideal P of R to $P \times S$ and it sends a prime ideal Q of S to $R \times Q$. This map is surjective by the above argument. It is injective by inspection. □

2 Ideals Of a Commutative Ring

2.1 Operations on Ideals

Proposition 2.1.1

Let R be a commutative ring. Let $S, T \subseteq R$ be subsets of R . Then

$$\langle S \cup T \rangle = \langle S \rangle + \langle T \rangle$$

Proposition 2.1.2

Let R be a commutative ring. Let I, J be ideals of R . Suppose that $I \subseteq J$. Let \bar{J} denote the ideal of R/I corresponding to J under the correspondence theorem. Then there is an isomorphism

$$\frac{R/I}{\bar{J}} \cong \frac{R}{I+J}$$

given by the formula $(r+I) + \bar{J} \mapsto r + (I+J)$.

Example 2.1.3

There is an isomorphism given by

$$\frac{\mathbb{Z}[x]}{(x+1, x^2+2)} \cong \mathbb{Z}/3\mathbb{Z}$$

Proof. Using the above propositions, we have that

$$\begin{aligned} \frac{\mathbb{Z}[x]}{(x+1, x^2+2)} &= \frac{\mathbb{Z}[x]}{(x+1) + (x^2+2)} \\ &\cong \frac{\mathbb{Z}[x]/(x+1)}{(3)} \end{aligned}$$

Indeed, the ideal (x^2+2) corresponds to the ideal (3) in $\frac{\mathbb{Z}[x]}{(x+1)}$ because the remainder of x^2+2 divided by $(x+1)$ is (3) . Now $\mathbb{Z}[x]/(x+1) \cong \mathbb{Z}$ by the evaluation homomorphism. Thus quotienting by the ideal (3) gives the field $\mathbb{Z}/3\mathbb{Z}$. \square

Some more important results from Groups and Rings and Rings and Modules include:

- If I and J are coprime, then $IJ = I \cap J$
- Chinese Remainder Theorem: If I and J are coprime, then there is an isomorphism

$$\frac{R}{I \cap J} \cong \frac{R}{I} \times \frac{R}{J}$$

2.2 Radical Ideals

The radical of an ideal is a very different notion from the radical of module.

Definition 2.2.1: Radical of an Ideal

Let I be an ideal of a ring R . Define the radical of I to be

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$$

Proposition 2.2.2

Let R be a commutative ring. Let I be an ideal. Then the following are true.

- $I \subseteq \sqrt{I}$
- $\sqrt{\sqrt{I}} = \sqrt{I}$
- $\sqrt{I^m} = \sqrt{I}$ for all $m \geq 1$
- $\sqrt{I} = R$ if and only if $I = R$

Proof.

- Let $r \in I$. Then $r^1 \in I$. Thus by choosing $n = 1$ we show that $r^n \in I$. Thus $r \in \sqrt{I}$.
- By the above, we already know that $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. So let $r \in \sqrt{\sqrt{I}}$. Then there exists some $n \in \mathbb{N}$ such that $r^n \in \sqrt{I}$. But $r^n \in \sqrt{I}$ means that there exists some $m \in \mathbb{N}$ such that $(r^n)^m \in I$. But $nm \in \mathbb{N}$ is a natural number such that $r^{nm} \in I$. Hence $r \in \sqrt{I}$ and so we conclude. \square

Proposition 2.2.3

Let R be a commutative ring. Let I, J be ideals of R . Then the following are true.

- If $I \subseteq J$ then $\sqrt{I} \subseteq \sqrt{J}$
- $\sqrt{IJ} = \sqrt{I \cap J}$
- $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$

Proof.

- Let $x \in \sqrt{IJ}$. Then $x^n \in IJ$. This means that there exists $i \in I$ and $j \in J$ such that $x^n = ij$. Since I and J are two sided ideals, we can conclude that $x^n = ij \in I, J$. Hence $x^n = ij \in I \cap J$. We conclude that $x \in \sqrt{I \cap J}$. Now let $x \in \sqrt{I \cap J}$. Then there exists $n \in \mathbb{N}$ such that $x^n \in I \cap J$. Then $x^n \in I$ and $x^n \in J$ implies that $x^{2n} = x^n \cdot x^n \in IJ$. We conclude that $x \in \sqrt{IJ}$. \square

Proposition 2.2.4

Let R be a commutative ring. Let I be an ideal. Then

$$\sqrt{I} = \bigcap_{\substack{p \text{ a prime ideal} \\ I \subseteq p \subseteq R}} p$$

Definition 2.2.5: Radical Ideals

Let R be a commutative ring. Let I be an ideal of R . We say that I is radical if

$$\sqrt{I} = I$$

In particular, by the above lemma it follows that the radical of an ideal is a radical ideal.

Lemma 2.2.6

Let R be a ring. Let P be a prime ideal of R . Then P is radical.

We conclude that there is an inclusion of types of ideal in which each inclusion is strict:

$$\text{Maximal ideals} \subset \text{Prime ideals} \subset \text{Radical ideals}$$

Theorem 2.2.7

Let R be a commutative ring. Let I be an ideal of R . Denote φ to be the inclusion preserving one-to-one bijection

$$\{\text{Ideals of } R \text{ containing } I\} \xleftrightarrow{1:1} \{\text{Ideals of } R/I\}$$

from the correspondence theorem for rings. In other words, $\varphi(A) = A/I$. Let $J \subseteq R$ be an ideal containing I . Then the following are true.

- J is a radical ideal if and only if $\varphi(J) = J/I$ is a radical ideal.
- J is a prime ideal if and only if $\varphi(J) = J/I$ is a prime ideal.
- J is a maximal ideal if and only if $\varphi(J) = J/I$ is a maximal ideal.

Proof.

- Let J be a radical ideal. Suppose that $r + I \in \sqrt{J/I}$. This means that $(r + I)^n = r^n + I \in J/I$ for some $n \in \mathbb{N}$. But this means that $r^n \in J$. This implies that $r \in \sqrt{J} = J$. Thus $r + I \in J/I$ and we conclude that $\sqrt{J/I} \subseteq J/I$. Since we also have $J/I \subseteq \sqrt{J/I}$, we conclude.

Now suppose that J/I is a radical ideal. Let $r \in \sqrt{J}$. This means that $r^n \in J$ for some $n \in \mathbb{N}$. Now $r^n + I = (r + I)^n \in J/I$ implies that $r + I \in \sqrt{J/I} = J/I$. Hence $r \in J$ and so $\sqrt{J} \subseteq J$. Since we also have that $J \subseteq \sqrt{J}$, we conclude.

- Let J be a prime ideal. Then R/J is an integral domain. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also an integral domain. Hence J/I is a prime ideal. The converse is also true.
- Let J be a maximal ideal. Then R/J is a field. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also a field. Hence J/I is a maximal ideal. The converse is also true.

□

Another way to write the bijections is via spectra:

$$\text{Spec}(R/I) \xleftrightarrow{1:1} \{P \in \text{Spec}(R) \mid I \subseteq P\}$$

and

$$\text{maxSpec}(R/I) \xleftrightarrow{1:1} \{m \in \text{maxSpec}(R) \mid I \subseteq m\}$$

2.3 Nilradical and Jacobson Ideals

Let R be a ring. Recall that an element $r \in R$ is nilpotent if $r^n = 0_R$ for some $n \in \mathbb{N}$. When R is commutative, we can form an ideal out of nilpotent elements.

Definition 2.3.1: Nilradicals

Let R be a ring. Define the nilradical of R to be

$$N(R) = \{r \in R \mid r \text{ is nilpotent}\}$$

Note that this is different from nilpotent ideals, as nilpotency is a property of an ideal. However the Nilradical ideal is a nil ideal and every sub-ideal of the nilradical is a nil ideal.

Proposition 2.3.2

Let R be a ring and $N(R)$ its nilradical. Then the following are true.

- $N(R)$ is an ideal of R
- $N(R/N(R)) = 0$

Proof.

- Suppose that r, s are nilpotent, meaning that $r^n = 0$ and $s^m = 0$. Then $(r + s)^{n+m} = 0$. Moreover, if $t \in R$ then $t \cdot r$ is also nilpotent
- Let $r \notin N(R)$. Every element $r + N(R) \in R/N(R)$ has the property that $r^n \neq 0$. Consider $(r + N(R))^n = r^n + N(R)$. If $r^n \in N(R)$ then $r^n = u$ for some nilpotent u , which means that r^n is nilpotent and thus r is nilpotent, a contradiction. This means that $r + N(R) \notin N(R/N(R))$ for all $r \notin N(R)$ and thus $N(R/N(R)) = 0$

□

Proposition 2.3.3

Let R be a commutative ring. The nilradical of R is the intersection of all prime ideals of R .

Proof. We want to show that

$$N(R) = \bigcap_{P \in \text{Spec}(R)} P$$

Trivially $N(R)$ is a prime ideal. Now suppose that $r \in R$ is in the intersection of all prime ideals. Then r^n also lies in every prime ideal.

□

Example 2.3.4

Consider the ring

$$R = \frac{\mathbb{C}[x, y]}{(x^2 - y, xy)}$$

Then its nilradical is given by $N(R) = (x, y)$.

Proof. Notice that in the ring R , $x^3 = x(x^2) = xy = 0$ and $y^3 = x^6 = (x^3)^2 = 0$ and hence x and y are both nilpotent elements of R . By definition of the nilradical, we conclude that $(x, y) \subseteq N(R)$. Now (x, y) is a maximal ideal of $\mathbb{C}[x, y]$ because $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$. Also notice that $(x, y) \supseteq (x^2 - y, xy)$ because for any element $f(x)(x^2 - y) + g(x)(xy) \in (x^2 - y, xy)$, we have that

$$\begin{aligned} f(x)(x^2 - y) + g(x)(xy) &\in (x^2 - y, xy) = (xf(x))x - f(x)y + (g(x)x)y \\ &= (xf(x))x + (xg(x) - f(x))y \in (x, y) \end{aligned}$$

By the correspondence theorem, $(x, y)/(x^2 - y)$ is a maximal ideal of R . In particular, (x, y) is also a prime ideal. But the $N(R)$ is the intersection of all prime ideals and hence $N(R) \subseteq (x, y)$. We conclude that $N(R) = (x, y)$.

□

Definition 2.3.5: Reduced Rings

Let R be a commutative ring. We say that R is reduced if $N(R) = 0$.

Proposition 2.3.6

Let R be a commutative ring. Let I be an ideal of R . Then R/I is reduced if and only if I is a radical ideal.

So radical, prime and maximal ideals all have characterizations using the quotient ring:

- I is maximal if and only if R/I is a field.
- I is prime if and only if R/I is an integral domain.
- I is radical if and only if R/I is reduced.

Recall the notion of the Jacobson radical from Rings and Modules. Let R be a ring. The Jacobson radical of R is the radical

$$J(R) = \text{rad}(R) = \bigcap_{\substack{S \trianglelefteq R \\ R \text{ is cosimple}}} S$$

of R considered as a left R -module. But when R is a commutative ring, this description can be simplified.

Proposition 2.3.7

Let R be a commutative ring. Then

$$J(R) = \bigcap_{m \in \max\text{Spec}(R)} m$$

Proof. Submodules of R are precisely ideals of R and cosimple ideals are ideals I of R for which R/I is simple. But if R/I is simple, then R/I contains no ideals which means that R/I is a field. So I is a maximal ideal. \square

Recall some properties of the Jacobson radical from Rings and Modules. For a (not necessarily commutative ring R),

- $J(R/J(R)) = 0$

Proposition 2.3.8

Let R be a commutative ring. Then $x \in J(R)$ if and only if $1 - xy \in R^\times$ for all $y \in R$.

Proof. \square

2.4 Extensions and Contractions of Ideals

Definition 2.4.1: Extension of Ideals

Let R, S be commutative rings. Let $f : R \rightarrow S$ be a ring homomorphism. Let I be an ideal of R . Define the extension I^e of I to S to be the ideal

$$I^e = \langle f(i) \mid i \in I \rangle$$

Proposition 2.4.2

Let R, S be commutative rings. Let $f : R \rightarrow S$ be a ring homomorphism. Let I, I_1, I_2 be an ideal of R . Then the following are true regarding the extension of ideals.

- Closed under sum: $(I_1 + I_2)^e = I_1^e + I_2^e$
- $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$
- Closed under products: $(I_1 I_2)^e = I_1^e I_2^e$
- $(I_1/I_2)^e \subseteq I_1^e/I_2^e$
- $\text{rad}(I)^e \subseteq \text{rad}(I^e)$

Definition 2.4.3: Contraction of Ideals

Let R, S be commutative rings. Let $f : R \rightarrow S$ be a ring homomorphism. Let J be an ideal of S . Define the contraction J^c of J to R to be the ideal

$$J^c = f^{-1}(J)$$

Proposition 2.4.4

Let R, S be commutative rings. Let $f : R \rightarrow S$ be a ring homomorphism. Let J, J_1, J_2 be an ideal of S . Then the following are true regarding the extension of ideals.

- $(J_1 + J_2)^e \supseteq J_1^e + J_2^e$
- Closed under intersections: $(J_1 \cap J_2)^e = J_1^e \cap J_2^e$
- $(J_1 J_2)^e \supseteq J_1^e J_2^e$
- $(J_1/J_2)^e \subseteq J_1^e/J_2^e$
- Closed under taking radicals: $\text{rad}(J)^e = \text{rad}(J^e)$

Proposition 2.4.5

Let R, S be commutative rings. Let $f : R \rightarrow S$ be a ring homomorphism. Let I be an ideal of R and let J be an ideal of S . Then the following are true.

- $I \subseteq I^{ec}$
- $J^{ce} \subseteq J$
- $I^e = I^{ece}$
- $J^c = J^{cec}$

2.5 Revisiting the Polynomial Ring**Proposition 2.5.1**

Let R be a commutative ring. Then we have

$$N(R[x]) = N(R)[x]$$

Proof. Let $f = \sum_{k=0}^n a_k x^k \in N(R)[x]$. Then each a_k is nilpotent in R , and there exists $n_k \in \mathbb{N}$ such that $a_k^{n_k} = 0$. This also proves that $a_k x^k$ is nilpotent. Since the sum of nilpotents is a nilpotent, we conclude that f is nilpotent.

Now suppose that $f \in N(R[x])$. We induct on the degree of f . Let $\deg(f) = 0$. Then f is nilpotent and f lies in R . Thus $f \in N(R)[x]$. Now suppose that the claim is true for $\deg(f) \leq n-1$. Let $\deg(g) = n$ with leading coefficient b_n . Since g is nilpotent in $R[x]$, there exists $m \in \mathbb{N}$ such that $g^m = 0$. Then in particular, $b_n^m = 0$ so that b_n is nilpotent. Then $b_n x^n$ is also nilpotent. Now since $N(R[x])$ is an ideal of $R[x]$, we have that $g - b_n x^n \in N(R[x])$. By inductive hypothesis, $g - b_n x^n \in N(R)[x]$. Since $N(R)$ is an ideal of R , we have that $N(R)[x]$ is an ideal of $R[x]$. So $g = (g - b_n x^n) + b_n x^n \in N(R)[x]$. Thus we are done. \square

Some more important results from Groups and Rings and Rings and Modules include:

- If R is an integral domain, then $R[x]$ is an integral domain.
- R is a UFD if and only if $R[x]$ is a UFD
- If F is a field, then $F[x]$ is an Euclidean domain, a PID and a UFD
- If F is a field, then the ideal generated by p is maximal if and only if p is irreducible.

Regarding ideals of the polynomial ring, the following maybe useful:

- $I[x]$ is an ideal of R
- There is an isomorphism $\frac{R[x]}{I[x]} \cong \frac{R}{I}[x]$ given by the map

$$\left(f = \sum_{k=0}^n a_k x^k + I[x] \right) \mapsto \left(\sum_{k=0}^n (a_k + I) x^k \right)$$

- If I is a prime ideal of R , then $I[x]$ is a prime ideal of $R[x]$.

3 Modules over a Commutative Ring

Recall from Rings and Modules that a module consists of an abelian group M and a ring R such that there is a binary operation $\cdot : R \times M \rightarrow M$ that mimic the notion of a group action:

- For $r, s \in R$, $s \cdot (r \cdot m) = (sr) \cdot m$ for all $m \in M$.
- For $1_R \in R$ the multiplicative identity, $1_R \cdot m = m$ for all $m \in M$.

When R is a commutative ring, the first axiom is relaxed so that the resulting element of M makes no difference whether you apply r first or s first. This makes module act even more similarly than fields (although one still need the notion of a basis, which appears in free modules). Therefore the first section concerns transferring techniques in linear algebra such as the Cayley Hamilton theorem to module over a ring that mimic the notion of vector spaces.

3.1 Cayley-Hamilton Theorem

Definition 3.1.1: Characteristic Polynomial

Let R be a commutative ring. Let $A \in M_{n \times n}(R)$ be a matrix. Define the characteristic polynomial of A to be the polynomial

$$c_A(x) = \det(A - xI)$$

Theorem 3.1.2: Cayley-Hamilton Theorem

Let R be a commutative ring. Let $A \in M_{n \times n}(R)$ be a matrix. Then $c_A(A) = 0$.

Corollary 3.1.3

Let R be a commutative ring. Let M be a finitely generated R -module. Let I be an ideal of R . Let $\varphi \in \text{End}_R(M)$. If $\varphi(M) \subseteq IM$, then there exists $a_1, \dots, a_n \in I$ such that

$$\varphi^n + a_1\varphi^{n-1} + \dots + a_{n-1}\varphi + \text{id}_M = 0 : M \rightarrow M$$

Proof. Suppose that M is generated by x_1, \dots, x_n . There exists a surjective map $\rho : R^n \rightarrow M$ given by $(r_1, \dots, r_n) \mapsto \sum_{k=1}^n r_k x_k$. Since $\varphi(M) \subseteq IM$, we have that

$$\varphi(x_k) = \sum_{i=1}^n r_{ki} x_i$$

for some $r_{ki} \in I$. Write A to be the matrix $A = (a_{ki})$. We now have a commutative diagram:

In other words, we have the diagram:

$$\begin{array}{ccc} R^n & \xrightarrow{\rho} & M \\ A \downarrow & & \downarrow \varphi \\ R^n & \xrightarrow{\rho} & M \end{array}$$

By Cayley-Hamilton theorem, we have that $c_A(A) = 0$ is the zero function. For all $x \in R^n$, we have that

$$\begin{aligned} c_A(A)(x) &= 0 \\ c_A(Ax) &= 0 \\ \rho(c_A(Ax)) &= \rho(0) \\ c_A(\rho(Ax)) &= 0 && (\rho \text{ is } R\text{-linear}) \\ c_A(\varphi(\rho(x))) &= 0 && (\text{Diagram is commutative}) \end{aligned}$$

Since ρ is surjective, we conclude that for any $m \in M$, the above calculation gives $c_A(\varphi(m)) = 0$ so that $c_A(\varphi)$ is the zero map. \square

3.2 Nakayama's Lemma

Lemma 3.2.1: Nakayama's Lemma I

Let R be a commutative ring. Let M be a finitely generated R -module. Let I be an ideal of R . If $IM = M$, then there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I$.

Proof. Choose $\varphi = \text{id}_M$. Then φ is surjective so that $M = \varphi(M) \subseteq IM$. By cor 4.1.3, there exists $r_1, \dots, r_n \in I$ such that $(1 + r_1 + \dots + r_n)M = 0$. By choosing $r = 1 + r_1 + \dots + r_n$, we see that $rM = 0$ and $r - 1 \in I$ so that we conclude. \square

Lemma 3.2.2: Nakayama's Lemma II

Let R be a commutative ring. Let M be a finitely generated R -module. Let I be an ideal of R such that $I \subseteq J(R)$ and $IM = M$. Then $M = 0$.

Proof. By Nakayama's lemma I, there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I \subseteq J(R)$. By 2.3.8, we have that $1 - (r - 1)(-1) = r \in R^\times$. This means that r is invertible. Hence $rM = 0$ implies $M = r^{-1}rM = 0$. \square

Corollary 3.2.3

Let R be a commutative ring. Let M be a finitely generated R -module. Let I be an ideal of R such that $I \subseteq J(R)$. Let N be an R -submodule of M . If

$$M = IM + N$$

then $M = N$.

Proof. Since quotients of finitely generated modules are finitely generated, we know that M/N is finitely generated. Define the map

$$\phi : IM + N \rightarrow I \frac{M}{N}$$

by $\phi(im + n) = i(m + N)$. This map is clearly surjective. Now I claim that $\ker(\phi) = N$. For any $im + n \in \ker(\phi)$, we see that $i(m + N) = N$ means that $im \in N$. Hence $im + n \in N$. On the other hand, if $im + n \in N$ then $im \in N$. But this means that $im + N = N$. Hence $im + n \in \ker(\phi)$. By the first isomorphism theorem for modules, we conclude that

$$\frac{M}{N} = \frac{IM + N}{N} \cong I \frac{M}{N}$$

We can now apply Nakayama's lemma II to conclude that $M/N = 0$ so that $M = N$. \square

Corollary 3.2.4

Let (R, m) be a local ring. Let M be a finitely generated R -module. Then the following are true.

- M/mM is a finite dimensional vector space over R/m .
- $a_1, \dots, a_n \in M$ generates M as an R -module if and only if $a_1 + mM, \dots, a_n + mM$

generates M/mM as a R/m vector space.

Proof. For the first part, we already know that M/mM is an R -module. We notice that for any $k \in m$ and $t + mM \in M/mM$ we have that $k(t + mM) = kt + kmM$. But $kt \in m$ means that $kt + kmM = mM$. Hence M/mM is well defined as an R/m -module. Now suppose that M is finitely generated by the elements a_1, \dots, a_n . Let $x + mM \in M/mM$. Then there exists $r_k \in R$ such that $x = r_1 a_1 + \dots + r_n a_n$. But this means that

$$x + mM = r_1(a_1 + mM) + \dots + r_n(a_n + mM)$$

This means that M/mM is generated by $a_1 + mM, \dots, a_n + mM$. We conclude that M/mM is finite dimensional.

Suppose that $a_1, \dots, a_n \in M$ generates M as an R -module. By the same argument as above, we can see that $a_1 + mM, \dots, a_n + mM$ is a set of generators for M/mM . For the other direction, suppose that $a_1 + mM, \dots, a_n + mM$ generates M/mM as an R/m -vector space. Define $N = Ra_1 + \dots + Ra_n \leq M$. Set $I = J(R) = m$. We want to show that $M = IM + N$. It is clear that $IM + N \leq M$. If $x \in M$, then there exists $r_k \in R$ such that $x + mM = r_1(a_1 + mM) + \dots + r_n(a_n + mM)$. In particular, this means that

$$x - \sum_{k=1}^n r_k a_k \in mM$$

Hence $x \in IM + N$. We can now apply the above corollary to deduce that $M = N = Ra_1 + \dots + Ra_n$ so that M is generated by a_1, \dots, a_n . And so we are done. \square

3.3 Change of Rings

Definition 3.3.1: Extension of Scalars

Let R, S be commutative rings. Let $\varphi : R \rightarrow S$ be a ring homomorphism. Let M be an R -module. Define the extension of M to the ring S to be the S -module

$$S \otimes_R M$$

Definition 3.3.2: Restriction of Scalars

Let R, S be commutative rings. Let $\varphi : R \rightarrow S$ be a ring homomorphism. Let M be an S -module. Define the restriction of M to the ring R to be the R -module M equipped with the action

$$r \cdot_R m = \varphi(r) \cdot_S m$$

for all $r \in R$.

Theorem 3.3.3

Let R, S be commutative rings. Let $\varphi : R \rightarrow S$ be a ring homomorphism. Then there is an isomorphism

$$\text{Hom}_S(S \otimes_R M, N) \cong \text{Hom}_R(M, N)$$

for any R -module M and S -module N given as follows.

- For $f \in \text{Hom}_S(S \otimes_R M, N)$, define the map $f^+ \in \text{Hom}_R(M, N)$ by

$$f^+(m) = f(1 \otimes m)$$

- For $g \in \text{Hom}_R(M, N)$, define the map $g^- \in \text{Hom}_S(S \otimes_R M, N)$ by

$$g^-(s \otimes m) = s \cdot g(m)$$

3.4 Properties of the Hom Set

Let R be a ring. Let M, N be R -modules. Recall that in Rings and Modules that $\text{Hom}_R(M, N)$ is a $Z(R)$ -modules. When R is commutative, $Z(R) = R$ so that the Hom set becomes an R -module.

Proposition 3.4.1

Let R be a commutative ring. Let M, N be R -modules. Then

$$\text{Hom}_R(M, N)$$

is an R -module with the following binary operations.

- For $\phi, \varphi : M \rightarrow N$ two R -module homomorphisms, define $\phi + \varphi : M \rightarrow N$ by $(\phi + \varphi)(m) = \phi(m) + \varphi(m)$ for all $m \in M$
- For $\phi : M \rightarrow N$ an R -module homomorphism and $r \in R$, define $r\phi : M \rightarrow N$ by $(r\phi)(m) = r \cdot \phi(m)$ for all $m \in M$.

In particular, it is an abelian group.

Proof. We first show that the addition operation gives the structure of a group.

- Since M is associative as an additive group, associativity follows
- Clearly the zero map $0 \in \text{Hom}_R(M, N)$ acts as the additive inverse since for any $\phi \in \text{Hom}_R(M, N)$, we have that $\phi(m) + 0 = 0 + \phi(m) = \phi(m)$ since 0 is the additive identity for M
- For every $\phi \in \text{Hom}_R(M, N)$, the map taking m to $-\phi(m)$ also lies in $\text{Hom}_R(M, N)$. Since $-\phi(m)$ is the inverse of $\phi(m)$ in M for each $m \in M$, we have that $-\phi$ is the inverse of ϕ

We now show that

- Let $r, s \in R$, we have that $((sr)\phi)(m) = (sr) \cdot \phi(m) = s \cdot (r \cdot \phi(m)) = s(r(\phi))(m)$ and hence we showed associativity.
- It is clear that $1_R \in R$ acts as the identity of the operation.

Thus we are done. □

Proposition 3.4.2

Let R be a ring. Let I be an indexing set. Let M_i, N be R -modules for $i \in I$. Then the following are true.

- There is an isomorphism

$$\text{Hom} \left(\bigoplus_{i \in I} M_i, N \right) \cong \bigoplus_{i \in I} \text{Hom}(M_i, N)$$

- There is an isomorphism

$$\text{Hom} \left(\prod_{i \in I} M_i, N \right) \cong \prod_{i \in I} \text{Hom}(M_i, N)$$

Definition 3.4.3: Induced Map of Hom

Let R be a commutative ring. Let M_1, M_2, N be R -modules. Let $f : M_1 \rightarrow M_2$ be an R -module homomorphism. Define the induced map

$$f^* : \text{Hom}_R(M_2, N) \rightarrow \text{Hom}_R(M_1, N)$$

by the formula $\varphi \mapsto \varphi \circ f$

Lemma 3.4.4

Let R be a commutative ring. Let M_1, M_2, N be R -modules. Let $f : M_1 \rightarrow M_2$ be an R -module homomorphism. Then the induced map

$$f^* : \text{Hom}(M_2, N) \rightarrow \text{Hom}(M_1, N)$$

is an R -module homomorphism.

3.5 Failure of Exactness of Hom and Tensoring**Proposition 3.5.1**

Let R be a commutative ring. Let the following be an exact sequence of R -modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Let N be an R -module. Then the following two sequences

$$0 \longrightarrow \text{Hom}_R(M_3, N) \longrightarrow \text{Hom}_R(M_2, N) \longrightarrow \text{Hom}_R(M_1, N)$$

$$\text{Hom}_R(N, M_1) \longrightarrow \text{Hom}_R(N, M_2) \longrightarrow \text{Hom}_R(N, M_3) \longrightarrow 0$$

are exact.

Proof.

- We first show that g^* is injective. Let $\phi, \rho \in \text{Hom}(C, G)$ such that $g^*(\phi) = g^*(\rho)$. This means that $\phi \circ g = \rho \circ g$. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Then

$$\phi(c) = \phi(g(b)) = \rho(g(b)) = \rho(c)$$

Hence $\phi = \rho$.

Now we show that $\text{im}(g^*) \subseteq \ker(f^*)$. Let $g^*(\phi) \in \text{Hom}(B, G)$ for $\phi \in \text{Hom}(C, G)$. We want to show that $f^*(g^*(\phi)) = 0$. But we have that

$$(\phi \circ g \circ f)(a) = \phi(g(f(a))) = \phi(0) = 0$$

since $\text{im}(f) = \ker(g)$. Thus we conclude.

Finally we show that $\ker(f^*) \subseteq \text{im}(g^*)$. Let $f^*(\phi) = 0$ for $\phi \in \text{Hom}(B, G)$. This means that $\phi \circ f = 0$ or in other words, $\text{im}(f) \subseteq \ker(\phi)$. Since $\phi(k) = 0$ for all $k \in \text{im}(f)$, ϕ descends to a map $\bar{\phi} : \frac{B}{\text{im}(f)} \rightarrow G$. But $\text{im}(f) = \ker(g)$ hence this is equivalent to a map $\bar{\phi} : \frac{B}{\ker(g)} \rightarrow G$. But by the first isomorphism theorem and the fact that g is surjective, we conclude that $\bar{g} : \frac{B}{\ker(g)} \xrightarrow{g} C$, where $b + \ker(g) \mapsto g(b)$. Thus we have constructed a map $\bar{\phi} \circ \bar{g}^{-1} : C \rightarrow G$ given by $g(b) \mapsto b + \ker(g) \mapsto \phi(b)$. But now $g^*(\bar{\phi} \circ \bar{g}^{-1})$ is the map defined by

$$b \mapsto g(b) \mapsto b + \ker(g) \mapsto \phi(b)$$

and so this map is exactly ϕ . Thus $\phi \in \text{im}(g^*)$. □

Proposition 3.5.2

Let R be a commutative ring. Let the following be an exact sequence of R -modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Let N be an R -module. Then the following sequence

$$M_1 \otimes N \xrightarrow{f \otimes \text{id}_N} M_2 \otimes N \xrightarrow{g \otimes \text{id}_N} M_3 \otimes N \longrightarrow 0$$

is exact.

However, one can observe that we did not imply that $M_1 \otimes N \rightarrow M_2 \otimes N$ is injective. Indeed, this is because tensoring does not preserve injections.

4 Algebra Over a Commutative Ring

4.1 Commutative Algebras

Definition 4.1.1: Commutative Algebras

Let R be a commutative ring. A commutative R -algebra is an R -algebra A that is commutative.

Proposition 4.1.2

Let R be a commutative ring. Then the following are equivalent characterizations of a commutative R -algebra.

- A is a commutative R -algebra
- A is a commutative ring together with a ring homomorphism $f : R \rightarrow A$

Proof. Suppose that A is an R -algebra. Then define a map $f : R \rightarrow A$ by $f(r) = r \cdot 1$ where $r \cdot 1$ is the module operation on A . Then clearly this is a ring homomorphism.

Suppose that A is a commutative ring together with a ring homomorphism $f : R \rightarrow A$. Define an action $\cdot : R \times A \rightarrow A$ by $r \cdot a = f(r)a$. Then this action clearly allows A to be an R -module. \square

Under the correspondence of associative algebra, the above proposition gives a another correspondence between the first one.

$$\left\{ (A, R) \mid \begin{array}{l} A \text{ is a commutative} \\ R\text{-algebra} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \phi : R \rightarrow A \mid \begin{array}{l} \phi \text{ is a ring homomorphism} \\ \text{such that } f(R) \subseteq Z(A) = A \end{array} \right\}$$

In particular, the construction above are inverses of each other so that it gives the one-to-one correspondence.

4.2 Finitely Generated Algebra

Definition 4.2.1: Finitely Generated Algebras

Let R be a commutative ring. Let A be an R -algebra. We say that A is finitely generated if there exists $a_1, \dots, a_n \in A$ such that every element $a \in A$ can be written as a polynomial in a_1, \dots, a_n . This means that

$$a = \sum_{i_1, \dots, i_n} r_{i_1, \dots, i_n} a_1^{i_1} \cdots a_n^{i_n}$$

Finitely generated algebras are also called algebra of finite type.

Theorem 4.2.2

Let A be a commutative algebra over a ring R . Then the following are equivalent.

- A is a finitely generated algebra over R
- There exists elements $a_1, \dots, a_n \in A$ such that the evaluation homomorphism

$$\phi : R[x_1, \dots, x_n] \rightarrow A$$

given by $\phi(f) = f(a_1, \dots, a_n)$ is a surjection

- There is an isomorphism

$$A \cong \frac{R[x_1, \dots, x_n]}{I}$$

for some ideal I

Definition 4.2.3: Finitely Presented Algebra

Let R be a ring. Let $A = R[x_1, \dots, x_n]/I$ be a finitely generated algebra over R for some ideal I . We say that A is finitely presented if I is finitely generated.

Lemma 4.2.4

Let R be a ring, considered as an algebra over \mathbb{Z} . If R is finitely generated over \mathbb{Z} , then R is finitely presented.

Proof. Trivial since \mathbb{Z} is a principal ideal domain. □

4.3 Finite Algebras**Definition 4.3.1: Finite Algebras**

Let R be a commutative ring. Let A be an R -algebra. We say that A is finite if A is finitely generated as an R -module.

Example 4.3.2

Let R be a commutative ring. Then $R[x]$ is a finitely generated algebra over R but is not a finite R -algebra.

4.4 Zariski's Lemma**Lemma 4.4.1**

Let F be a field. Let $f \in F[x]$. Then the localization $F[x]_f$ is not a field.

Theorem 4.4.2: Zariski's Lemma

Let F be a field. Let K be a field that is also a finitely generated algebra over F . Then K is a finite algebra. In particular, K is a finitely generated vector space over F .

Corollary 4.4.3

Let F be an algebraically closed field. Let K be a field that is also a finitely generated algebra over F . Then the inclusion homomorphism $F \hookrightarrow K$ is an F -algebra isomorphism.

Corollary 4.4.4

Let F be an algebraically closed field. Then every maximal ideal of $F[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in F$.

5 Localization

5.1 Localization of a Ring

Definition 5.1.1: Multiplicative Set

Let R be a commutative ring. $S \subseteq R$ is a multiplicative set if $1 \in S$ and S is closed under multiplication: $x, y \in S$ implies $xy \in S$

Definition 5.1.2: Localization of a Ring

Let R be a commutative ring and $S \subseteq R$ be a multiplicative set. Define the ring of fractions of R with respect to S by

$$S^{-1}R = \left\{ \frac{r}{s} \mid r \in R, s \in S \right\} / \sim$$

where \sim is defined by

$$\frac{r}{s} \sim \frac{r'}{s'} \text{ if and only if } \exists v \in S \text{ such that } v(ru' - r'u) = 0$$

If $S = \{1, f, f^2, \dots\}$ then we write

$$S^{-1}R = R_f = R[1/f]$$

Proposition 5.1.3

Let $S^{-1}R$ be a ring of fractions.

- \sim as defined in the ring of fractions is an equivalence relation
- $(S^{-1}R, +, \times)$ is a ring
- The map $k : R \rightarrow S^{-1}R$ defined by $r \mapsto r/1$ is a ring homomorphism, called the localization map.

Proof.

- Trivial
- Define addition by $\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'}$ and multiplication by $\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$. Clearly addition is abelian, and has identity $\frac{0}{1}$ and inverse $\frac{-r}{s}$ for any $\frac{r}{s} \in S^{-1}R$. Multiplication also has identity $\frac{1}{1}$.

□

Proposition 5.1.4: Universal Property

Let R be a commutative ring. Let S be a multiplicative subset of R . Let $k : R \rightarrow S^{-1}R$ be localization map. Then the following is true.

For all commutative rings B and ring homomorphisms $\phi : R \rightarrow B$ such that $\phi(s) \in B^\times$ for all $s \in S$, there exists a unique ring homomorphism $\psi : S^{-1}R \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{k} & S^{-1}R \\ & \searrow \phi & \downarrow \exists! \psi \\ & & B \end{array}$$

Lemma 5.1.5

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset of R . If R is Noetherian, then $S^{-1}R$ is Noetherian.

5.2 Localization at a Prime Ideal**Lemma 5.2.1**

Let R be a ring and P a prime ideal of R . Then $R \setminus P$ is a multiplicative set.

Proof. By definition, $xy \in P$ implies $x \in P$ or $y \in P$, since $R \setminus P$ removes all these elements, we have that $x \notin P$ and $y \notin P$ implies that $xy \notin P$. \square

Definition 5.2.2: Localization on Prime Ideals

Let R be a commutative ring. Let P be a prime ideal. Denote

$$R_p = (R \setminus P)^{-1}R$$

the localization of R at P .

Lemma 5.2.3

Let R be an integral domain. Then we have

$$\text{Frac}(R) = R_{(0)}$$

Proposition 5.2.4

Let R be a commutative ring and let p be a prime ideal of R . Then R_p is a local ring with unique maximal ideal given by

$$pR_p = \left\{ \frac{r}{s} \mid r \in p, s \notin p \right\}$$

Proof. Let I be the set of all non-units of R_p . It is sufficient to show that I is an ideal by the above lemma. Clearly if $i \in I$ then $r \cdot i$ is also not invertible. Explicitly, we have

$$I = \left\{ \frac{r}{s} \in R_p \mid r \in p \right\}$$

Let $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in I$, then $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$ is in I since $r_1, r_2 \in p$ and p being an ideal implies $r_1 s_2 + r_2 s_1 \in p$. \square

Be wary that in general localizations does not result in a local ring. This happens only when we are localizing with respect to a prime ideal. The importance of prime ideals is not explicit in the above because only using prime ideals P can $R \setminus P$ be a multiplicative set which ultimately allows localization to make sense.

Proposition 5.2.5

Let R be an integral domain. Then there is an equality

$$R = \bigcap_{m \text{ a maximal ideal}} R_m$$

when considered as subrings of $\text{Frac}(R)$.

5.3 Localization of a Module

Definition 5.3.1: Localization of a Module

Let R be a commutative ring and $S \subseteq R$ be a multiplicative set. Let M be a R -module. Define the ring of fractions of M with respect to S by

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \sim$$

where \sim is defined by

$$\frac{m}{s} \sim \frac{m'}{s'} \text{ if and only if } \exists v \in S \text{ such that } v(mu' - m'u) = 0$$

If $S = \{1, f, f^2, \dots\}$ then we write

$$S^{-1}M = M_f = M[1/f]$$

Lemma 5.3.2

Let R be a commutative ring. Let M be an R -module. Let $S \subseteq R$ be a multiplicative subset. Then $S^{-1}M$ is an $S^{-1}R$ -module with operation given by

$$\left(\frac{r}{s_1}, \frac{m}{s_2} \right) \mapsto \frac{r \cdot m}{s_1 s_2}$$

Definition 5.3.3: Induced Map of Localization

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Define the induced map

$$S^{-1}\phi : S^{-1}M \rightarrow S^{-1}N$$

by the formula $\frac{m}{s} \mapsto \frac{\phi(m)}{s}$.

Lemma 5.3.4

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Then the induced map

$$S^{-1}\phi : S^{-1}M \rightarrow S^{-1}N$$

is a well defined ring homomorphism.

Proposition 5.3.5

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let the following be an exact sequence of R -modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then the following is an exact sequence of $S^{-1}R$ -modules.

$$0 \longrightarrow S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3 \longrightarrow 0$$

Corollary 5.3.6

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M be an R -module. Then the following are true.

- If N_1, N_2 are R -submodules of M , then

$$S^{-1}(N_1 + N_2) = S^{-1}N_1 + S^{-1}N_2$$

as $S^{-1}R$ -submodules of $S^{-1}M$.

- If N_1, N_2 are R -submodules of M , then

$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2$$

as $S^{-1}R$ -submodules of $S^{-1}M$.

- If N is an R -submodule of M , then

$$S^{-1} \frac{M}{N} \cong \frac{S^{-1}M}{S^{-1}N}$$

as $S^{-1}R$ -modules.

- If N is an R -module, then

$$S^{-1}(M \oplus N) \cong S^{-1}M \oplus S^{-1}N$$

as $S^{-1}R$ -modules.

Proposition 5.3.7

Let R be a commutative ring. Let M be an R -module. Then there is an isomorphism

$$S^{-1}M \cong S^{-1}R \otimes_R M$$

of $S^{-1}R$ -modules given by $\frac{m}{s} \mapsto \frac{1}{s} \otimes m$.

Lemma 5.3.8

Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Then the following are true.

- Localization commutes with kernels:

$$S^{-1} \ker(\phi) \cong \ker(S^{-1}\phi)$$

- Localization commutes with images:

$$S^{-1}(\operatorname{im} \phi) \cong \operatorname{im}(S^{-1}\phi)$$

- Localization commutes with cokernels:

$$S^{-1} \frac{N}{\operatorname{im}(\phi)} \cong \frac{S^{-1}N}{\operatorname{im}(S^{-1}\phi)}$$

5.4 Ideals of a Localization

Definition 5.4.1: Ideals Closed Under Division

Let R be a commutative ring. Let I be an ideal of R . Let $S \subseteq R$ be a multiplicative subset. We say that I is closed under division by s if for all $s \in S$ and $a \in R$ such that $sa \in I$, we have $a \in I$.

Lemma 5.4.2: L

Let R be a commutative ring. Let I be an ideal of R . Let $S \subseteq R$ be a multiplicative subset. Let $\iota : R \hookrightarrow S^{-1}R$ be the inclusion homomorphism. Then we have

$$I^e = S^{-1}I$$

Theorem 5.4.3

Let R be a commutative ring. Let I be an ideal of R . Let $S \subseteq R$ be a multiplicative subset. Let $\iota : R \hookrightarrow S^{-1}R$ be the inclusion homomorphism. Then there is a one-to-one bijection

$$\{J \mid J \text{ is an ideal of } S^{-1}R\} \xrightarrow{1:1} \left\{ I \mid \begin{array}{l} I \text{ is an ideal of } R \text{ and} \\ I \text{ is closed under division by } S \end{array} \right\}$$

given by $J \mapsto J^c$.

Proposition 5.4.4

Let R be a commutative ring. Let I be an ideal of R . Let $S \subseteq R$ be a multiplicative subset. Then the above bijection restricts to the following bijection

$$\{J \mid J \text{ is a prime ideal of } S^{-1}R\} \xrightarrow{1:1} \left\{ I \mid \begin{array}{l} I \text{ is a prime ideal of } R \text{ and} \\ I \cap S = \emptyset \end{array} \right\}$$

5.5 Local Properties

Definition 5.5.1: Local Properties of Modules

Let R be a commutative ring. A property of R -modules is local if for any R -modules M , the following are equivalent.

- M has the property
- M_P has the property for all prime ideals P
- M_m has the property for all maximal ideals m

Proposition 5.5.2: Injectivity and Surjectivity are Local Properties

Let R be a commutative ring. Let M, N be R -modules. Let $\phi : M \rightarrow N$ be an R -module homomorphism. Let S be a multiplicative subset of R . Then the following are equivalent.

- ϕ is injective (surjective)
- For each prime ideal P of R , the induced map $\phi_P : S^{-1}M \rightarrow S^{-1}N$ is injective (surjective)
- For each maximal ideal m of R , the induced map $\phi_m : S^{-1}M \rightarrow S^{-1}N$ is injective (surjective)

More local properties: zero, nilpotent

Non-local properties: freeness, domain

Proposition 5.5.3: Exactness is Local

Let R be a commutative ring. Let M_1, M_2, M_3 be R -modules. Let $f : M_1 \rightarrow M_2$ and $g : M_2 \rightarrow M_3$ be R -module homomorphisms. Then the following conditions are equivalent.

- The following sequence is exact:

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

- The following sequence is exact:

$$0 \longrightarrow (M_1)_P \xrightarrow{f_P} (M_2)_P \xrightarrow{g_P} (M_3)_P \longrightarrow 0$$

for all prime ideals P of R .

- The following sequence is exact:

$$0 \longrightarrow (M_1)_m \xrightarrow{f_m} (M_2)_m \xrightarrow{g_m} (M_3)_m \longrightarrow 0$$

for all maximal ideals m of R .

Definition 5.5.4: Local Properties of Elements

A property of an element of M is local if the following is true. $m \in M$ has the property if and only if $m \in M_P$ has the property.

6 Primary Decomposition

6.1 Support of a Module

Definition 6.1.1: Support of a Module

Let A be a commutative ring. Let M be an A -module. The support of M is the subset

$$\text{Supp}(M) = \{P \text{ a prime ideal of } A \mid M_P \neq 0\}$$

Let R be a commutative ring. Let M be an R -module. Recall that the annihilator of an element $m \in M$ is the ideal

$$\text{Ann}_R(m) = \{r \in R \mid r \cdot m = 0\}$$

Moreover, we define

$$\text{Ann}_R(M) = \{r \in R \mid r \cdot m = 0 \text{ for all } m \in M\} = \bigcap_{m \in M} \text{Ann}_R(m)$$

Proposition 6.1.2

Let R be a commutative ring. Let M be an R -module. Let P be a prime ideal of R . Then $P \in \text{Supp}(M)$ if and only if $\text{Ann}_R(M) \subseteq P$.

6.2 Associated Prime

Definition 6.2.1: Associated Prime

Let M be an A -module. An associated prime of M is a prime ideal P of A such that

$$P = \text{Ann}_A(m)$$

for some $m \in M$. Also define

$$\text{Ass}(M) = \{P \text{ a prime ideal of } A \mid P \text{ is an associated prime of } M\}$$

Proposition 6.2.2

Let R be a commutative ring. Let M be an R -module. Then

$$\text{Ass}(M) \subseteq \text{Supp}(M)$$

Proposition 6.2.3

Let R be a commutative ring. Let M be an R -module. Then the following are true.

- $\text{Ass}(M)$ is a finite set.
- For $P \in \text{Ass}(M)$, $\text{Ann}_R(M) \subseteq P$.
- We have

$$\text{Ass}(M) = \{P \in \text{Spec}(R) \mid \text{For any prime ideal } Q \subseteq P, Q \text{ does not contain } \text{Ann}_R(M)\}$$

Proof.

-
- We have seen that every $P \in \text{Supp}(M)$ is such that $\text{Ann}_R(M) \subseteq P$. Since $\text{Ass}(M) \subseteq \text{Supp}(M)$, we are done.

□

Proposition 6.2.4

Let R be a commutative ring. Let M be an R -module. Then

$$\bigcup_{P \in \text{Ass}(M)} P = \{m \in M \mid m \text{ is a zero divisor of } M\} \cup \{0\}$$

Theorem 6.2.5: Disassembly of an R -Module

Let R be a Noetherian commutative ring. Let M be a finitely generated R -module. Then there exists a chain of R -submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that

$$\frac{M_{i+1}}{M_i} \cong \frac{R}{P_i}$$

for some prime ideal P_i of R .

6.3 Primary Ideals**Definition 6.3.1: Primary Ideals**

Let R be a commutative ring. Let Q be a proper ideal of R . We say that Q is a primary ideal of R if $fg \in Q$ implies $f \in Q$ or $g^m \in Q$ for some $m > 0$.

Lemma 6.3.2

Let A be a commutative ring. Let Q be a primary ideal of A . Then \sqrt{Q} is the smallest prime ideal containing Q .

Lemma 6.3.3

Let R be a Noetherian ring and I be a proper ideal that is not primary. Then

$$I = J_1 \cap J_2$$

for some ideals $J_1, J_2 \neq I$.

Definition 6.3.4: P -Primary Ideals

Let A be a commutative ring. Let P be a prime ideal. Let Q be an ideal. We say that Q is a P -primary ideal of A if

$$Q = \sqrt{P}$$

Theorem 6.3.5

Let A be a Noetherian ring and Q an ideal of A . Then Q is P -primary if and only if $\text{Ann}(A/Q) = \{P\}$.

6.4 Primary Decomposition

We want to express ideal I in R as $I = P_1^{e_1} \cdots P_n^{e_n}$ similar to a factorization of natural numbers, for some prime ideals P_1, \dots, P_n . However this notion fails and thus we have the following new type of ideal.

Definition 6.4.1: Primary Decompositions

Let A be a commutative ring. Let I be an ideal of A . A primary decomposition I consists of primary ideals Q_1, \dots, Q_r of A such that

$$I = Q_1 \cap \dots \cap Q_r$$

Definition 6.4.2: Minimal Primary Decompositions

Let A be a commutative ring. Let I be an ideal of A . Let

$$I = Q_1 \cap \dots \cap Q_r$$

be a primary decomposition of I . We say that the decomposition is minimal if the following are true.

- Each $\sqrt{Q_i}$ are distinct for $1 \leq i \leq r$
- Removing a primary ideal changes the intersection. This means that for any i ,

$$I \neq \bigcap_{j \neq i} Q_j$$

Theorem 6.4.3

Every proper ideal in a Noetherian ring has a primary decomposition.

Lemma 6.4.4

Let $\phi : R \rightarrow S$ be a ring homomorphism and Q be a primary ideal in S . Then $\phi^{-1}(Q)$ is primary in R .

7 Integral Dependence

7.1 Integral Elements

Definition 7.1.1: Integral Elements

Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b \in B$. We say that b is integral over A if there exists a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$ such that $p(b) = 0$.

When A and B are field, this is a familiar notion in Field and Galois theory.

Lemma 7.1.2

Let K be a field. Let $F \subseteq K$ be a subfield. Let $k \in K$. Then k is integral over F if and only if k is algebraic over F .

Proposition 7.1.3

Let B be a commutative ring and let $A \subseteq B$. Let $b \in B$. Then the following are equivalent.

- b is integral over A
- $A[b] \subseteq B$ is finitely generated A -submodule.
- There exists an A sub-algebra $A' \subseteq B$ such that $A[b] \subseteq A'$ and A' is finitely generated as an A -module.

Proposition 7.1.4

Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b_1, b_2 \in B$ be integral over A . Then $b_1 + b_2$ and $b_1 b_2$ are both integral over A .

7.2 Integral Closure

Definition 7.2.1: Integral Closure

Let B be a commutative ring. Let $A \subseteq B$ be a subring. Define the subring

$$\overline{A} = \{b \in B \mid b \text{ is integral over } A\}$$

to be the integral closure of A in B .

Definition 7.2.2: Integral Extensions

Let B be a commutative ring and let $A \subseteq B$ be a subring. We say that B is integral over A if $\overline{A} = B$. We also say that B is the integral extension of A .

Lemma 7.2.3

Let B be a commutative ring and let $A \subseteq B$ be a subring. Then \overline{A} is an integral extension of A .

Lemma 7.2.4

Let $A \subseteq B \subseteq C$ be commutative rings. If C is integral over B and B is integral over A , then C is integral over A .

Proposition 7.2.5

Let A, B be commutative rings such that $A \subset B$ is an integral extension. Let J be an ideal of B . Then $\frac{B}{J}$ is integral over $\frac{A}{J \cap A}$.

Proposition 7.2.6

Let A, B be commutative rings such that $A \subset B$ is an integral extension. Let S be a multiplicative subset of B . Then $S^{-1}B$ is integral over $S^{-1}A$.

Lemma 7.2.7

Let A, B be integral domains such that $A \subset B$ is an integral extension. Then A is a field if and only if B is a field.

Proposition 7.2.8

Let B be a commutative ring. Let $A \subseteq B$ be a subring. Let S be a multiplicatively closed subset of A . Then

$$\overline{S^{-1}A} = S^{-1}\overline{A}$$

Definition 7.2.9: Integrally Closed

Let B be a commutative ring. Let $A \subseteq B$ be a subring. We say that A is integrally closed in B if $\overline{A} = A$.

7.3 The Going-Up and Going-Down Theorems

We want to compare prime ideals between integral extensions.

Proposition 7.3.1

Let A, B be rings such that $A \subset B$ is an integral extension. Let Q be a prime ideal of B . Then $Q \cap A$ is a maximal ideal of A if and only if Q is a maximal ideal of B .

Proposition 7.3.2

Let A, B be rings such that $A \subset B$ is an integral extension. Let P be a prime ideal of A . Then the following are true.

- There exists a prime ideal Q of B such that $P = Q \cap A$
- If Q_1, Q_2 are prime ideals of B such that $Q_1 \cap A = P = Q_2 \cap A$ and $Q_1 \subseteq Q_2$, then $Q_1 = Q_2$.

Theorem 7.3.3: The Going-Up Theorem

Let A, B be rings such that $A \subset B$ is an integral extension. Let $0 \leq m < n$. Consider the following situation

$$\begin{array}{ccc} B & Q_1 \subseteq \cdots \subseteq Q_m & \text{(Prime ideals of } B) \\ \uparrow & & \\ A & P_1 \subseteq \cdots \subseteq P_m \subseteq P_{m+1} \subseteq \cdots \subseteq P_n & \text{(Prime ideals of } A) \end{array}$$

where $Q_i \cap A = P_i$ for $1 \leq i \leq m$. Then there exists prime ideals Q_{m+1}, \dots, Q_n of B such that the following are true.

- $Q_{m+1} \subseteq \cdots \subseteq Q_n$

- $Q_i \cap A = P_i$ for $m+1 \leq i \leq n$

7.4 Normal Domains

We now concern ourselves with integral domains. Let R be an integral domain. A special fact about R is that the canonical homomorphism $R \rightarrow R_{(0)} = \text{Frac}(R)$ is an injection. This means that we can think of R as living inside of $\text{Frac}(R)$ while preserving all the structure of R .

Definition 7.4.1: Normal Domains

Let R be an integral domain. We say that R is normal if R is integrally closed in $\text{Frac}(R)$.

Proposition 7.4.2

Let R be a normal domain. Let S be a multiplicative subset of R . Then $S^{-1}R$ is a normal domain.

Proof. We want to show that $S^{-1}R$ is integrally closed in $\text{Frac}(R) = \text{Frac}(S^{-1}R)$. This means that we want to show $\overline{S^{-1}R} = S^{-1}R$. It is clear that $S^{-1}R \subseteq \overline{S^{-1}R}$. So let $g \in \overline{S^{-1}R}$. Suppose that $p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in (S^{-1}R)[x]$ such that $p(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for $0 \leq i \leq n-1$. Then notice that $sg \in S^{-1}R$ satisfies the monic polynomial

$$q(x) = x^n + \sum_{k=0}^{n-1} s^{n-k} a_k x^k$$

since $q(sg) = s^n g^n + \sum_{k=0}^{n-1} s^n a_k x^k = s^n p(g) = 0$. But q is a polynomial in R since $s^{n-k} a_k \in R$. Thus we have that $sg \in R = R$ since R is normal. This means that $g \in S^{-1}R$ and hence we conclude. \square

Proposition 7.4.3: Normal is a Local Property

Let R be an integral domain. Then the following are equivalent.

- R is normal
- R_P is normal for all prime ideals P
- R_m is normal for all maximal ideals m .

Proof. Notice that an integral domain R is normal if and only if the canonical inclusion map $R \hookrightarrow \overline{R}$ is surjective. Since surjectivity is a local property, this map is surjective if and only if for all prime ideals P of R , $R_P \hookrightarrow \overline{R_P}$ is surjective. But $\overline{R_P} = \overline{R_P}$ by the above. Hence $R \hookrightarrow \overline{R}$ is surjective if and only if $R_P \hookrightarrow \overline{R_P}$ is surjective. Hence R is normal if and only if R_P is normal for all prime ideals P of R . The similar holds for all maximal ideals. \square

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8 Introduction to Dimension Theory for Rings

8.1 Krull Dimension

Definition 8.1.1: Krull Dimension

Let R be a commutative ring. Define the Krull dimension of R to be

$$\dim(R) = \max\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t \text{ for } p_0, \dots, p_t \text{ prime ideals}\}$$

Lemma 8.1.2

Let R, S be commutative rings such that $R \subseteq S$ is an integral extension. Then $\dim(R) = \dim(S)$.

Proposition 8.1.3

Let F be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Then the following are true.

- $\dim(F[x_1, \dots, x_n]) = n$.
- Every maximal chain prime ideals in $F[x_1, \dots, x_n]$ is of length n .

8.2 Height of Prime Ideals

Definition 8.2.1: Height of a Prime Ideal

Let R be a commutative ring. Let p be a prime ideal of R . Define the height of p to be

$$\text{ht}(p) = \sup\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t = p \text{ for } p_0, \dots, p_t \text{ prime ideals}\}$$

Lemma 8.2.2

Let p be a prime ideal in a ring R . Then

$$\text{ht}(p) = \dim(R_p)$$

Lemma 8.2.3

Let R be a commutative ring. Let P be a prime ideal of R . Then

$$\dim(R) \geq \dim(R/P) + \text{ht}_R(P)$$

Proposition 8.2.4

Let k be a field. Let A be an integral domain that is a finitely generated k -algebra. Then the following are true.

- $\dim(A) = \text{trdeg}_k(\text{Frac}(A))$
- For any prime ideal P of A , we have

$$\dim(A) = \dim(A/P) + \text{ht}_A(P)$$

Proposition 8.2.5: Dimension is a Local Concept

Let R be a commutative ring. Then the following numbers are equal.

- The Krull dimension $\dim(R)$
- The supremum $\sup\{\dim(R_m) \mid m \text{ is a maximal ideal of } R\}$
- The supremum $\sup\{\text{ht}_R(m) \mid m \text{ is a maximal ideal of } R\}$

Theorem 8.2.6: Krull's Principal Ideal Theorem

Let R be a Noetherian ring. Let I be a proper and principal ideal of R . Let p be the smallest prime ideal containing I . Then

$$\text{ht}_R(p) \leq 1$$

8.3 Length of a Module**Definition 8.3.1: Length of a Module**

Let R be a ring and let M be an R -module. Define the length of M to be

$$l_R(M) = \sup\{n \in \mathbb{N} \mid 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M\}$$

Lemma 8.3.2

Let R be a ring. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be a short exact sequence of R -modules. Then

$$l_R(M) = l_R(M') + l_R(M'')$$

Lemma 8.3.3

Let (A, m) be a local ring and let M be an A -module. If $mM = 0$, then

$$l_A(M) = \dim_{A/m}(M)$$

Proposition 8.3.4

Let R be a ring and let M be an R -module. Then the following are equivalent.

- M is simple
- $l_R(M) = 1$
- $M \cong A/m$ for some maximal ideal m of A

8.4 The Hilbert Polynomial**Definition 8.4.1: The Hilbert Polynomial**

Let $R = \bigoplus_{k=0}^{\infty} R_k$ be a Noetherian graded ring. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a graded R -module. Define the Hilbert function $H_M : \mathbb{N} \rightarrow \mathbb{N}$ of R to be the function defined by

$$H_M(n) = l_{R_0}(M_n)$$

Definition 8.4.2: The Hilbert Series

Let $R = \bigoplus_{k=0}^{\infty} R_k$ be a Noetherian graded ring. Let $M = \bigoplus_{k=0}^{\infty} M_k$ be a graded R -module. Define the Hilbert series $HS_M \in \mathbb{Z}[[t]]$ of M to be the formal series

$$HS_M(t) = \sum_{k=0}^{\infty} H_M(k)t^k = \sum_{k=0}^{\infty} l_{R_0}(M_k)t^k$$

Theorem 8.4.3

Let $R = \bigoplus_{k=0}^{\infty} R_k$ be a Noetherian graded ring such that R_0 is Artinian. Let $M = \bigoplus_{k=0}^{\infty} M_k$

be a graded R -module. Let $\lambda : \{M_i \mid i \in I\} \rightarrow \mathbb{Z}$ be an additive function. Then the function

$$g(t) = \sum_{k=0}^{\infty} \lambda(M_k) t^k$$

is a rational function and can be written in the form

$$g(t) = \frac{f(t)}{\prod_{i=1}^r (1 - t^{d_i})}$$

for some $f(t) \in \mathbb{Z}[t]$ and $d_i \in \mathbb{N}$.

Theorem 8.4.4: The Fundamental Theorem of Dimension Theory

Let (R, m) be a local Noetherian ring. Let I be an m -primary ideal. Then the following numbers are equal.

- Let $J = \bigoplus_{k=0}^{\infty} \frac{I^k}{I^{k+1}}$. The order of the pole at 1 of the rational function HS_J .
- The minimum number of elements of R that can generate an m -primary ideal of R
- The dimension $\dim_{R/m}(R)$

The following is a generalization of Krull's principal ideal theorem. Both of the theorems can actually be deduced directly from the fundamental theorem.

Theorem 8.4.5: Krull's Height Theorem

Let R be a Noetherian ring. Let I be a proper ideal generated by n elements. Let p be the smallest prime ideal containing I . Then

$$\text{ht}_R(p) \leq n$$

Theorem 8.4.6

Let (R, m) be a Noetherian local ring and let $k = R/m$ be the residue field. Then

$$\dim(R) \leq \dim_k(m/m^2)$$

8.5 Structure Theorem for Artinian Rings

Lemma 8.5.1

Let R be a commutative ring. Then the following are true.

- If R is a field, then $\dim(R) = 0$
- If R is Artinian, then $\dim(R) = 0$

Proposition 8.5.2

Let $R \neq 0$ be a commutative ring. Then R is Artinian if and only if R is Noetherian and $\dim(R) = 0$.

Theorem 8.5.3: Structure Theorem for Commutative Artinian Rings

Let R be an Artinian commutative ring. Then there exists Artinian local rings A_1, \dots, A_k such that

$$R \cong \bigoplus_{i=1}^k A_i$$

Moreover, the decomposition is unique up to reordering of the direct product.

9 Valuation and Valuation Rings

9.1 Valuation Rings

Definition 9.1.1: Valuation Rings

Let R be an integral domain. We say that R is a valuation ring if for all $x \in \text{Frac}(R)$ and $x \neq 0$, then either x or x^{-1} is in R .

Lemma 9.1.2

Let R be a valuation ring. Then the following are true.

- R is a local ring
- R is integrally closed

9.2 Valuations on a Field

Definition 9.2.1: Totally Ordered Group

Let G be an abelian group. We say that G is a totally ordered group if there is a total order " \leq " on G such that $a \leq b$ implies $ca \leq cb$ for all $a, b, c \in G$.

Definition 9.2.2: Valuation on a Field

Let K be a field. Let G be a totally ordered abelian group. A valuation on K with values in G is a group homomorphism $v : K^\times \rightarrow G$ such that for all $x, y \in K^*$, we have

- $v(xy) = v(x) + v(y)$
- $v(x + y) \geq \min\{v(x), v(y)\}$

We use the convention that $v(0) = \infty$.

Definition 9.2.3: Associated Valuation Ring

Let K be a field and $v : K \rightarrow \mathbb{Z}$ a discrete valuation. Define the associated valuation ring of K to be the subring

$$R_v = \{x \in K \mid v(x) \geq 0\}$$

Lemma 9.2.4

Let K be a field. Let v be a discrete valuation on K . Then R_v is a valuation ring.

9.3 Discrete Valuations and Normalizations

Definition 9.3.1: Discrete Valuations

Let K be a field. A discrete valuation on K is a valuation $v : K^\times \rightarrow \mathbb{Z}$.

Definition 9.3.2: Normalized Discrete Valuations

Let (K, v) be a discrete valuation ring. We say that it is normalized if v is surjective.

Lemma 9.3.3

Let K be a field with a discrete valuation v . Then $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$.

Lemma 9.3.4: Normalization of a Discrete Valuation

Let K be a field with a discrete valuation v such that $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Define the normalization of v to be the valuation $v_N : K^\times \rightarrow \mathbb{Z}$ defined by

$$v_N(k) = \frac{1}{n}v(k)$$

for all $k \in K^\times$.

Therefore we always work on normalized discrete valuation rings.

9.4 Discrete Valuation Rings**Definition 9.4.1: Discrete Valuation Rings**

Let R be a commutative ring. We say that R is a discrete valuation ring if there exists a field K and a discrete valuation v on K such that

$$R = R_v$$

is the associated valuation ring of K .

Proposition 9.4.2

Let R be a discrete valuation ring with valuation v . Let $t \in R$ be such that $v(t) = 1$. Then the following are true.

- A nonzero element $u \in R$ is a unit if and only if $v(u) = 0$
- $\dim(R) = 1$

Proof.

- Let R be a discrete valuation ring. Suppose that $x \in R$ is a unit. Then $v(x^{-1}) = -v(x)$. Then $-v(x), v(x) \geq 0$ implies $v(x) = 0$. Now if $v(y) > 0$, suppose for contradiction that $u \in R$ is an inverse of y , then

$$0 = v(1) = v(uy) = v(u) + v(y)$$

But $v(y) > 0$ implies that $v(u) < 0$ which implies that $u \notin R$, a contradiction. □

Definition 9.4.3: Uniformizing Parameter

Let R be a discrete valuation ring with valuation v . A uniformizing parameter for R is an element $t \in R$ such that $v(t) = 1$.

Proposition 9.4.4

Let R be a discrete valuation ring with valuation v . Let $t \in R$ be a uniformizing parameter of R . Then the following are true.

- Every non-zero ideal of R is a principal ideal of the form (t^n) for some $n \geq 0$
- Every $r \in R \setminus \{0\}$ can be written in the form $r = ut^n$ for some unit u and $n \geq 0$.

Proof.

- Let $t \in R$ such that $v(t) = 1$. Let $x \in m$ where $v(x) = n > 0$. Then $v(x) = nv(t) = v(t^n)$ means that every $x \in m$ is of the form t^n . Thus $m = (t)$. Since every ideal I is a subset of this maximal ideal, any ideal is of the form $I = (t^n)$ for some $n > 0$.

- Follows from the fact that (t^n) is the unique maximal ideal.

□

The rest of the section devotes efforts to recognizing discrete valuation rings.

Proposition 9.4.5

Let R be a valuation ring. Then the following are equivalent.

- R is a discrete valuation ring.
- R is a principal ideal domain.
- R is Noetherian.

Proposition 9.4.6

Let R be an integral domain. Then the following are equivalent.

- R is a discrete valuation ring
- R is a UFD with a unique irreducible element up to multiplication of a unit
- R is Noetherian, local with principal maximal ideal
- R is Noetherian, local and $\dim(R) = 1 = \dim_{R/m}(m/m^2) = 1$ for m the unique maximal ideal of R
- R is Noetherian, local and $I = m^k$ for all non-zero ideals I of R and m the unique maximal ideal of R
- R is Noetherian, local and there exists $t \in R$ and $k > 0$ such that $I = (t^k)$ for all non-zero ideal I of R

Proof.

- (1) \implies (3): We have seen that the set of non-units is precisely the set $m = \{x \in R \mid v(x) > 0\}$. We show that this is an ideal. Clearly $x, y \in m$ implies $v(x + y) = \min\{v(x), v(y)\} > 0$. Let $u \in R$. Then $v(ux) = v(u) + v(x) > 0$ since $v(x) > 0$ and $v(u) \geq 0$.

We have seen that every ideal is of the form (t^n) for some $n > 0$. Thus every ascending chains of ideal must be of the form

$$(t^{n_1}) \subset (t^{n_2}) \subset \dots$$

for $n_1 > n_2 > \dots$. Since n_1, n_2, \dots is strictly decreasing, the chain must eventually stabilizes. This proves that R is Noetherian and has principal maximal ideal.

- (1) \implies (3):

□

In summary, if R is a discrete valuation ring, then R has the following properties.

- R is integrally closed and in particular is normal.
- R is a PID and in particular is a UFD and an integral domain.
- R is Noetherian and local
- R has Krull dimension 1.
- $\dim_{R/m}(m/m^2) = 1$ (these are called regular local rings as we will see in Commutative Algebra 2)
- Every ideal I of R is equal to the power m^k of the maximal ideal m . In particular if m is generated by the uniformizing parameter t , then $I = (t^k)$ in this case.
- Such a t is an irreducible element (that is unique up to multiplication by a unit), and every element of R can be written as ut^n for u a unit and $n \in \mathbb{N}$.

There is a simple diagram of relationships between DVRs and some other standard types of commutative rings.

DVRs \subset PIDs \subset UFDs \subset Normal Domains \subset Integral Domains

10 Dedekind Domains

10.1 Fractional Ideals

Definition 10.1.1: Fractional Ideal

Let R be an integral domain. Let I be a R -submodule of $\text{Frac}(R)$. We say that I is a fractional ideal of R if there exists $r \in R \setminus \{0\}$ such that $rI \subseteq R$.

While I is not exactly an ideal of R , we can think of it as if it were an ideal because it is isomorphic to an actual ideal of R .

Lemma 10.1.2

Let R be an integral domain. Let I be a fractional ideal of R where $rI \subseteq R$ for some $r \in R \setminus \{0\}$. Then there is an R -module isomorphism

$$I \cong rI \subseteq R$$

given by $i \mapsto ri$.

Proof. I claim that there is an R -module isomorphism $I \cong rI$ for $rI \subseteq R$ given by $i \mapsto ri$. The kernel of this R -module homomorphism is given by $\{i \in I \mid ri = 0\}$. But $ri = 0$ if and only if $r = 0$ or $i = 0$. Since $r \neq 0$ we must have $i = 0$ so that the kernel is trivial. Moreover, this R -module homomorphism is surjective since for any $k \in rI$ it can be written as $k = ri$ for some i . Then $i \in I$ maps to ri under the morphism. Hence $I \cong rI$ as R -modules. \square

Lemma 10.1.3

Let R be an integral domain. Let I be a fractional ideal of R . If R is Noetherian, then I is finitely generated.

Proof. Let R be Noetherian. Since I is isomorphic to rI for some non-zero $r \in R$, and rI is an ideal of R , R being Noetherian implies that rI is finitely generated and hence I is finitely generated. \square

10.2 Invertible Ideals

Definition 10.2.1: Invertible Ideals

Let R be an integral domain. Let I be an R -submodule of $\text{Frac}(R)$. We say that I is invertible if there exists an ideal J of R such that $JI = R$.

Lemma 10.2.2

Let R be an integral domain. Let I be an R -submodule of $\text{Frac}(R)$. Then I is invertible if and only if $I^{-1}I = R$ where we define

$$I^{-1} = \{s \in \text{Frac}(R) \mid sI \subseteq R\}$$

Proposition 10.2.3

Let R be an integral domain. Let I be an R -submodule of $\text{Frac}(R)$. Then the following are true.

- If I is a non-zero principal ideal of R , then I is invertible.
- If I is invertible, then I is fractional.

Proposition 10.2.4

Let R be an integral domain. Let I be a fractional ideal. Then I is invertible if and only if I is finitely generated, and for any maximal ideal m of R , IR_m is a principal ideal of R_m .

Proposition 10.2.5

Let R be an integral domain. Let P be a non-zero prime ideal of R . If R is Noetherian and P is invertible, then R_P is a discrete valuation ring.

Proof. Let R be a Noetherian integral domain and P a non-zero invertible prime ideal. We know that PR_P is the unique maximal ideal of the local ring R_P . By the above prp, PR_P is a principal ideal. Thus R_P is now a Noetherian local ring with principal maximal ideal. By prp10.4.6 in Commutative Algebra 1, we conclude that R_P is a discrete valuation ring. \square

10.3 Dedekind Domains**Definition 10.3.1: Dedekind Domains**

Let R be an integral domain. We say that R is a dedekind domain if every non-zero ideal can be expressed uniquely as a direct product of finitely many prime ideals of R .

Dedekind sought for an integral domain whose ideals can be factorized uniquely as a product of primes.

Proposition 10.3.2

Let R be an integral domain that is not a field. Then the following are equivalent.

- R is a Dedekind domain.
- Every non-zero fractional ideal I of R is invertible ($I^{-1}I = R$).
- R is Noetherian, $\dim(R) = 1$ and normal
- R is Noetherian, $\dim(R) = 1$ and for any non-zero maximal ideal m of R , R_m is a discrete valuation ring.
- R is Noetherian, $\dim(R) = 1$ and every primary ideal in R is a prime power.

Proof.

- (2) \implies (3): Let I be an ideal of R . Since I is invertible, by 1.1.5 we conclude that I is finitely generated. Hence R is Noetherian. Let P be a prime ideal of R . By assumption, P is invertible. prp1.2.5 implies that R_P is a DVR. In particular, it is integrally closed and $\dim(R_P) = 1$. This means that $\text{ht}_R(P) = 1$. Thus R is either a field or $\dim(R) = 1$. By assumption R is not a field. Hence $\dim(R) = 1$. We know that $R = \bigcap_{m \text{ a maximal ideal}} R_m$. Since prime ideals are maximal ideals in one dimensional rings, we can rewrite the intersection as

$$R = \bigcap_{P \text{ a prime ideal}} R_P$$

But each R_P is a DVR. Hence R is a DVR and we conclude that R is normal.

- (3) \implies (2): m be a maximal ideal of R . We have seen from Commutative Algebra 1 that R_m is a Noetherian local ring. By 7.4.2 in Commutative Algebra 1 we also conclude that R_m is normal. By 9.3.2 of Commutative Algebra 1 we know that $\dim(R_m) = \text{ht}_R(m) = 1$. By 10.4.6 of Commutative Algebra 1, R_m is a DVR and in particular m is a principal ideal.

Let I be a fractional ideal of R . We know by 1.1.3 that I is finitely generated. Since R_m is a normal Noetherian local ring of dimension 1, the ideal I_m of R_m must be principal. By 1.1.5 we conclude that I is invertible.

- (3) \implies (4):
- (4) \implies (3): Let m be a maximal ideal of R . We know that R_m is a DVR. In particular, it is a normal domain.

□

By virtue of the fourth item, we can think of Dedekind domains as a patching up of local discrete valuation rings.

Proposition 10.3.3

Let R be a Dedekind domain. Let I and J be ideals of R whose prime factorization is given by

$$I = P_1^{a_1} \times \cdots \times P_n^{a_n} \quad \text{and} \quad J = P_1^{b_1} \times \cdots \times P_n^{b_n}$$

for P_1, \dots, P_n distinct prime ideals of R . Then the following are true.

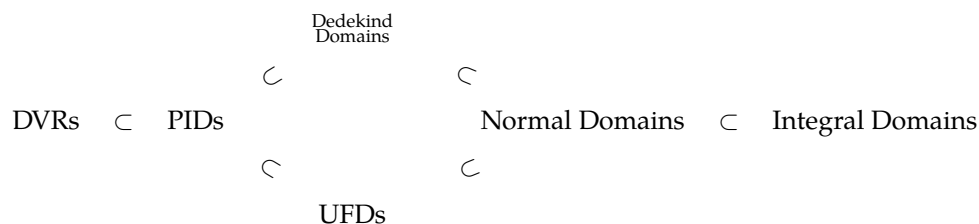
- $I + J = P_1^{\min\{a_1, b_1\}} \times \cdots \times P_n^{\min\{a_n, b_n\}}$
- $I \cap J = P_1^{\max\{a_1, b_1\}} \times \cdots \times P_n^{\max\{a_n, b_n\}}$
- $IJ = P_1^{a_1+b_1} \times \cdots \times P_n^{a_n+b_n}$

Proposition 10.3.4

Let R be a Dedekind domain. Let I be an ideal of R . Then the following are true.

- For any $a \in I$, there exists $b \in R$ such that $I = (a, b)$.
- I is can be finitely generated by two elements.

We summarize the relation between Dedekind domains and other types of domains in the following diagram:



In particular, DVRs, PIDs and Dedekind domains are 1-dimensional.