Solutions to Hatcher

Labix

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Abstract

Solutions to the book Algebraic Topology authored by Allen Hatcher

Contents

1 The Fundamental Group

1.1 Basic Constructions

Exercise 1.1.1

Show that the composition of paths satisfy the following cancellation property: If $f_0 \cdot g_0 \simeq f_1 \cdot g_1$ and $g_0 \simeq g_1$ then $f_0 \simeq f_1$.

Proof. From the relation $g_0 \simeq g_1$ we have that $g_1 \cdot \overline{g}_0 \simeq e$. It follows that

$$\begin{aligned} f_0 \cdot g_0 &\simeq f_1 \cdot g_1 \\ f_0 \cdot g_0 \cdot \overline{g}_0 &\simeq f_1 \cdot g_1 \cdot \overline{g}_0 \\ &= f_0 \simeq f_1 \end{aligned}$$

and so we conclude. □

Exercise 1.1.2

Show that the change of basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. Recall that the isomorphism is defined by $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ sending $[\alpha] \in \pi_1(X, x_1)$ to $[h \cdot \alpha \cdot \overline{h}]$. We have that $h \overset{\partial}{\simeq} h'$ implies $h \cdot \alpha \cdot \overline{h} \simeq h' \cdot \alpha \cdot \overline{h'}$ so that $\beta_h([\alpha]) = \beta_{h'}([\alpha])$. □

Exercise 1.1.3

For a path connected space X, show that $\pi_1(X)$ is abelian if and only if all base point change homomorphisms β_h depend only on the endpoints of the path h.

Proof. Suppose that $\pi_1(X)$ is abelian. We want to show that $\beta_h = \beta_{h'}$ if $h(1) = h'(1)$. We have that

$$\begin{aligned} \beta_h([\alpha]) &= [h \cdot \alpha \overline{h}] \\ \beta_{h'}([\alpha]) &= [h' \cdot \alpha \cdot \overline{h'}] \end{aligned}$$

Since $\pi_1(X)$ is abelian, we have that

$$\begin{aligned} \beta_h([\alpha]) \cdot \overline{\beta_{h'}([\alpha])} &= [h \cdot \alpha \cdot \overline{h} \cdot h' \cdot \alpha \cdot \overline{h'}] \\ &= [h \cdot (\overline{h} \cdot h') \cdot \alpha \cdot \overline{\alpha} \cdot \overline{h'}] && (\overline{h} \cdot h' \text{ is a loop on } x_1) \\ &= [h' \cdot \overline{h'}] \\ &= [e_{x_0}] \end{aligned}$$

This implies that $[h \cdot \alpha \cdot \overline{\alpha}] = [h' \cdot \alpha \cdot \overline{h'}]$ which is what is required.

Now suppose that $\pi_1(X)$ is not abelian. Then there exists $[a], [b] \in \pi_1(X)$ such that $[a] \cdot [b] \neq [b] \cdot [a]$. In other words, $[\overline{b}] \cdot [a] \cdot [b] \neq [a]$. But clearly for the constant loop e, we have that $[\overline{e}] \cdot [a] \cdot [e] = [a]$ which implies that

$$\begin{aligned} [\overline{b}] \cdot [a] \cdot [b] &\neq [\overline{e}] \cdot [a] \cdot [e] \\ \beta_b([a]) &\neq \beta_e([a]) \end{aligned}$$

even though b and e have the same end points. □

Exercise 1.1.4

Show that if a subspace $X \subset \mathbb{R}^n$ is locally star-shaped, then every path in X is homotopic in X to a piecewise linear path. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

Proof. Let γ be a path in X. Consider the open cover of $\gamma([0,1])$ by the star-shaped neighbourhood of each $x \in \gamma([0,1])$. Since $[0,1]$ is compact, $\gamma([0,1])$ is compact so the open cover has a finite subcover $U_1, \dots, U_m$ which are neighbourhoods of $\gamma(t_1) = x_1, \dots, \gamma(t_m) = x_m$ for $t_1 < \cdots < t_m$. For any $U_i \cap U_{i+1}$ (nonempty since open cover), choose $\gamma(s_i) = y_i$ and $t_1 < s_1 < t_2 < s_2 < \cdots < s_{m-1} < t_m$. Since each U_i is star-shaped at x_i, there are straight paths from x_i to y_{i-1} and y_i, say $a_{i-1} : I \to X$ and $b_i : I \to X$. Since U_i is star-shaped at x_i, any point between the paths a_{i_1} and $\gamma_{[s_{i-1}, t_i]}$ (likewise $\gamma_{[t_i, s_i]}$ and b_i) is reachable via a straight line, so that $\gamma_{[s_{i-1}, t_i]}$ is homotopic to the straight path a_{i-1} and likewise $\gamma_{[t_i, s_i]}$ is homotopic to the straight path b_i and so we are done.

If X is a union of finitely many closed convex sets, then notice that each convex set is star-shaped. Each $x \in X$ must be contained in one of the convex sets and so X is locally star-shaped. $\square$

Exercise 1.1.5

Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected if and only if all maps $S^1 \to X$ are homotopic (Without regards to basepoint).

Proof.

- $(a) \implies (b)$: Suppose that $f \simeq e_{x_0}$. This means that there exists a homotopy $H : S^1 \times I \to X$ from f to e_{x_0}. Now by the universal property of quotient spaces, we have a factorization

$$\begin{tikzcd} S^1 \times I \arrow[r, "p"] \arrow[dr, "H"'] & \frac{S^1 \times I}{S^1 \times \{1\}} \arrow[d, "\tilde{H}"] \\ & X \end{tikzcd}$$

where p is the quotient map. This is possible because $H(S^1 \times I) = \{x_0\}$. Since $\frac{S^1 \times I}{S^1 \times \{1\}} \cong D^2$, we obtain an extension.

- $(a) \implies (c)$: $S^1 \cong \frac{I}{0 \sim 1}$ so that every map $f : S^1 \to X$ is just a loop in X. Since all loops in X is homotopic to the constant map, we must have $\pi_1(X, x) = 0$.

- $(b) \implies (a)$: Suppose that $f : S^1 \to X$ is a map. By assumption, f can be extended to a map $\tilde{f} : D^2 \to X$. Since D^2 is contractible, we have $\mathrm{id}_{D^2} \simeq e$, which implies that

$$\tilde{f} \simeq f \circ e_{x_0} = e_{f(x_0)}$$

In particular, f is also homotopic to a constant map by the same homotopy.

- $(c) \implies (a)$: Suppose that $\pi_1(X, x_0) = 0$. Then any loop $f : I \to X$ is such that $f \simeq e_{x_0}$. In particular, a loop with domain I is just a map from S^1 because $S^1 \cong \frac{I}{0\sim 1}$.

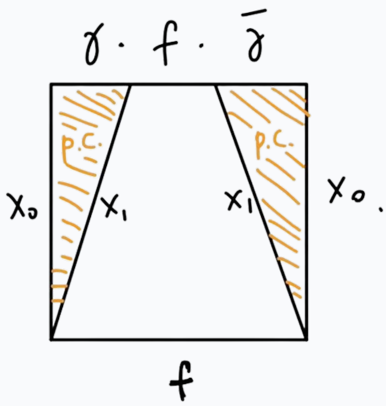
For the remainder of the question, suppose that X is simply connected. This means that $\pi_1(X, x_0) = 0$. This means that any loop $S^1 \to X$ is homotopic to a constant map. Since simply connectedness implies path connectedness, any constant paths are homotopic. This means that any $S^1 \to X$ are homotopic.

Now suppose that any $S^1 \to X$ are homotopic. Then in particular, they are all homotopic to the constant path. Thus $\pi_1(X, x_0) = 0$ for any $x_0 \in X$. □

Exercise 1.1.6

There is a natural map $\Psi : \pi_1(X, x_0) \to [S^1, X]$ obtained by ignoring basepoints. Show that Ψ is onto if X is path-connected, and that $\Psi([f]) = \Psi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

Proof. Suppose that X is path connected, and let $[f] \in [S^1, X]$. Let x_1 be the end point of the loop f. Since X is path connected, there exists $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. This means that $\gamma \cdot f \cdot \overline{\gamma}$ is a loop starting at x_0. But $[\gamma \cdot f \cdot \overline{\gamma}] = [f]$ via the homotopy



Thus $\Psi([\gamma \cdot f \cdot \overline{\gamma}) = [f]$.

Now suppose that $\Psi([f]) = \Psi([g])$. Then this implies that $f \simeq g$ are free homotopic where f and g have basepoint x_0. Let $H : I \times I \to X$ be the homotopy. Let $h : I \to X$ be defined as $h(t) = H(0, t)$. Then we have

$$h(0) = H(0, 0) = f(0) = x_0$$
$$h(1) = H(0, 1) = g(0) = x_0$$

so that h is a loop. By lemma 1.19, we have that $(H_0)_* = \beta_h \circ (H_1)_*$ if and only if $f_* = \beta_h \circ g_*$. Plugging in the generator ω_1 of $\pi_1(S^1, 1) \cong \mathbb{Z}$, we have that

$$f_*(\omega_1) = (\beta_h \circ g_*)(\omega_1)$$
$$f \circ \omega_1 = \beta_h(g \circ \omega_1)$$

But $f \simeq f \circ \omega_1$ and $\overline{h} \cdot g \cdot h \simeq \beta_h(g \circ \omega_1)$ and so we have that $[f] = [\overline{h}] \cdot [g] \cdot [h]$.

Suppose that $[f] = [\overline{\gamma} \cdot g \cdot \gamma]$ for some $\gamma : I \to X$ a loop so that f, g, γ are loops based at x_0. Applying Ψ, we have that $\Psi([f]) = \Psi([\overline{\gamma} \cdot g \cdot \gamma])$. Consider $\Psi([g]) \in [S^1, X]$. It is clear that $g \in \Psi([g])$. Moreover, we must have $g \simeq \overline{\gamma} \cdot g \cdot \gamma$ by the same homotopy given above (replace f with g). Thus we have that f is free homotopic to g. □

Exercise 1.1.7

Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$ so f restricts to the identity on the two boundary circles $S^1 \times I$. Show that f is homotopic to the identity by a homotopy f_t that is stationary on one of the boundary circles, but not by any homotopy f_t that is stationary on both boundary circles.

Proof. Define $H : (S^1 \times I) \times I \to S^1 \times I$ by

$$(\theta, s, t) \mapsto (\theta + 2\pi s(1-t), s)$$

Clearly H is continuous. Moreover,

$$\begin{aligned} H(\theta, s, 0) &= f(\theta, s) \\ H(\theta, s, 1) &= \mathrm{id}(\theta, s) \\ H(\theta, 0, t) &= (\theta, 0) \end{aligned}$$

Thus we have that

$$f \overset{S^1 \times \{0\}}{\simeq} \mathrm{id}$$

Now suppose that H is a homotopy from id to f that fixes $S^1 \times \{0\}$ and $S^1 \times \{1\}$. Let $\gamma : I \to S^1 \times I$ be a path defined as $\gamma(s) = \theta_0 + s$ for some fixed θ_0. Then the conditions on H implies that

$$\begin{aligned} H(\gamma(s), 0) &= \gamma(s) \\ H(\gamma(s), 1) &= (f \circ \gamma)(s) \end{aligned}$$

so that we have a homotopy $\gamma \simeq f \circ \gamma$.

Consider the projection $p : S^1 \times I \to S^1$. Then we have that

$$\begin{aligned} \gamma &\simeq f \circ \gamma \\ p \circ \gamma &\simeq p \circ f \circ \gamma \\ e &\simeq \omega_1 \end{aligned}$$

But ω_1 is a generator of $\pi_1(S^1)$ hence this is a contradiction. □

Exercise 1.1.8

Does the Borsak-Ulam theorem hold for the torus? In other words, for every map $f : S^1 \times S^1 \to \mathbb{R}^2$, must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?

Proof. The Borsak-Ulam theorem fails on the torus. Consider the map $f : S^1 \times S^1 \subset \mathbb{R}^3 \to \mathbb{R}^2$ that forgers the z coordinate of the torus. It is clear that for two points to have the same image under f, it must have the same y value in $S^1 \times S^1$ (Think of the first circle in $S^1 \times S^1$ having the y-axis passing through and the second circle having the z-axis passing through). Assume that the theorem holds. Then $f(x, y) = f(-x, -y)$ together with $y = -y$ implies that $y = 0$ in $\mathbb{R}^3$ coordinates. But not point in the torus has $\mathbb{R}^3$ coordinate $y = 0$, which is a contradiction. □

Exercise 1.1.9

Let A_1, A_2, A_3 be compact sets in $\mathbb{R}^3$. Use the Borsa-Ulam theorem to show that there is one plane $P \subset \mathbb{R}^3$ that simultaneously divides each A_i into two pieces of equal measure.

Proof. Consider $S^2 \subset \mathbb{R}^3$. Let v be a vector in S^2 and consider its span which I also denote by v. For any scalar p, there is a normal plane of v that passes through pv. In particular, there is a continuous collection of planes that slices through A_i for $i = 1, 2, 3$. Define a measure of volume in $\mathbb{R}^3$. Such a measure must be continuous so that the intermediate value theorem implies that there exists one such p_i for which the normal plane at $p_i v$ slices A_i in half by volume.

This is because as p increases in $\mathbb{R}$,

$$\text{Vol}\left(A \cap \text{lower of half of } \mathbb{R}^3 \text{ bounded by the normal plane}\right)$$

increases and eventually attains full volume (full volume is finite since A_i is compact) so that we can apply IVT.

Doing this for every vector v in $\mathbb{R}^3$, we obtain a function $f : S^2 \to \mathbb{R}^2$ defined by $f(v) = (p_1 - p_3, p_2 - p_3)$. By Borsak-Ulam theorem, there exists $v \in S^2$ such that $f(v) = f(-v)$ (Showing continuity of f is hard!). In other words, we have that $(p_1 - p_3, p_2 - p_3) = (p_3 - p_1, p_3 - p_2)$. This implies that $p_1 = p_2 = p_3$. But this means that the hyperplane at $p_1 v = p_2 v = p_3 v$ cuts through all A_1, A_2, A_3 and thus we conclude. $\square$