Enriched Category Theory

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Abstract

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1 Algebraic Objects in a Category

1.1 Group Objects

Definition 1.1.1: Group Objects

Let $\mathcal C$ be a category with finite products. We say that $G\in\mathcal C$ is a group object if there exists three morphisms

• Multiplication: $m: G \times G \rightarrow G$

• Identity: $e:* \rightarrow G$ where * is the terminal object

• Inverse: inv : $G \rightarrow G$

such that the following diagrams commute.

• Associativity:

$$\begin{array}{ccc} G \times G \times G \xrightarrow{m \times \mathrm{id}_G} G \times G \\ \mathrm{id}_G \times m \Big\downarrow & & \Big\downarrow m \\ G \times G \xrightarrow{m} & G \end{array}$$

• Identity:

$$G \xrightarrow{(e, \mathrm{id}_G)} G \times G$$

$$\downarrow^{\mathrm{id}_G, e} \downarrow^{m}$$

$$G \times G \xrightarrow{m} G$$

• Inverse:

$$G \xrightarrow{(\mathrm{inv},\mathrm{id}_G)} G \times G$$

$$\downarrow^{(\mathrm{id}_G,\mathrm{inv})} G$$

$$G \times G \xrightarrow{m} G$$

Proposition 1.1.2

A group object in the category Set of sets is a group in the usual sense.

Proposition 1.1.3

A group object in the category **Grp** of groups is an abelian group.

2 Monoidal Categories

2.1 Strict and Weak Monoidal Categories

Definition 2.1.1: Strict Monoidal Categories

A strict monoidal category is a category \mathcal{A} consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

- Associativity: $(A \otimes B) \otimes C = A \otimes (B \otimes C)$
- Identity: $I \otimes A = A$ and $A \otimes I = A$

Notice that we require strict equality in the associativity and identity laws. Since we usually only consider objects up to isomorphism in a category, strict monoidal categories may seem quite rare in practise.

Definition 2.1.2: Weak Monoidal Category

A weak monoidal category is a category \mathcal{A} consisting of a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ together with an object $I \in \mathcal{A}$ such that the following are true.

• Associativity: There are isomorphisms

$$\alpha_{A,B,C}: (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$

that is natural in A, B and C

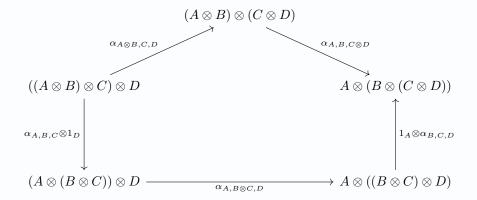
• Identity: There are isomorphisms

$$\lambda_A: I\otimes A\stackrel{\cong}{\longrightarrow} A \quad \text{ and } \quad \rho_A: A\otimes I\stackrel{\cong}{\longrightarrow} A$$

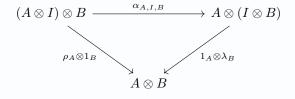
that are both natural in A

Such natural isomorphisms must also satisfy the following commutative laws:

• The pentagon identity:



• The triangle identity:



It is clear that every strict monoidal category is also a weak monoidal category.

Lemma 2.1.3

Every category \mathcal{C} with finite products is a monoidal category with product $\times: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and identity * the terminal object.

Proposition 2.1.4

For any commutative ring R, the category \mathbf{Mod}_R of R-modules is a monoidal category with the tensor product \otimes and the identity object R.

Definition 2.1.5: Symmetric Monoidal Category

Let $\mathcal C$ be a category. We say that $\mathcal C$ is a symmetric monoidal category if $\mathcal C$ is a weak monoidal category together with isomorphisms

$$s_{A,B}:A\otimes B\stackrel{\cong}{\longrightarrow} B\otimes A$$

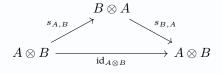
that are natural in A and B such that the following are satisfied:

• Unit coherence: If I is the distinguished object of C as a weak monoidal category, then the following diagram commutes:

• The associativity coherence: For any $A,B,C\in\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{s_{A,B} \otimes \operatorname{id}_{C}} & (B \otimes A) \otimes C \\ & & & & \downarrow^{\alpha_{B,A,C}} \\ A \otimes (B \otimes C) & & & B \otimes (A \otimes C) \\ & & & \downarrow^{\operatorname{id}_{B} \otimes s_{A,C}} \\ & & & \downarrow^{\operatorname{id}_{B} \otimes s_{A,C}} \\ & & & (B \otimes C) \otimes A & \xrightarrow{\alpha_{B,C,A}} & B \otimes (C \otimes A) \end{array}$$

• The inverse law: For any $A, B \in \mathcal{C}$, the following diagram commutes:



2.2 Closed Categories

3 Enriched Categories