Group Cohomology

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Abstract

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1 Group Cohomology

1.1 G-Modules

Definition 1.1.1: G-Modules

Let G be a group. A G-module is an abelian group A together with a group action of G on A.

Definition 1.1.2: Morphisms of G-Modules

Let G be a group. Let M and N be G-modules. A function $f : M \to N$ is said to be a G-module homomorphism if it is an equivariant group homomorphism. This means that

$$f(g \cdot m) = g \cdot f(m)$$

for all $m \in M$ and $g \in G$.

1.2 The Group of Invariants

Definition 1.2.1: The Group of Invariants

Let G be a group and let M be a G-module. Define the group of invariants of G in M to be the subgroup

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$$

This is the largest subgroup of M for which G acts trivially.

Theorem 1.2.2

Let G be a group and let M be a G-module. Then there are canonical isomorphisms

$$M^G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

Definition 1.2.3: Functor of Invariants

Let G be a group. Define the functor of invariants by

$$(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$$

as follows.

- For each G-module M, M^G is the group of invariants
- For each morphism $f : M \to N$ of G-modules, $f^G : M^G \to N^G$ is the restriction of f to M^G.

Theorem 1.2.4

Let G be a group. The functor of invariants $(-)^G : {}_G\mathbf{Mod} \to \mathbf{Ab}$ is left exact.

1.3 The Different Forms of Cohomology of Groups

Definition 1.3.1: The nth Cohomology Group

Let G be a group. Define the nth cohomology group of G with coefficients in a G-module M to be

$$H^n(G;M) = (R^n(-)^G)(M)$$

the nth right derived functor of $(-)^G$.

Theorem 1.3.2

Let G be a group and let M be a G-module. Then there are natural isomorphisms

$$H^n(G;M) \cong \text{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$$

In the following theorem, we use the notation $(g_0, \dots, \hat{g}_i, \dots, g_n)$ as a shorthand for writing the element in G^n but with the ith term omitted.

Theorem 1.3.3

Let G be a group. Then the cochain complex

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{f_n} \mathbb{Z}[G^n] \xrightarrow{f_{n-1}} \mathbb{Z}[G^{n-1}] \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \qquad 0$$

where $f_n : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n]$ is defined by

$$(g_0, \dots, g_n) \mapsto \sum_{i=0}^n (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$$

is a projective resolution of $\mathbb{Z}$ lying in ${}_{\mathbb{Z}}[G]\textbf{Mod}$.

Corollary 1.3.4

Let G be a group and let M be a G-module. Then there is a natural isomorphism

$$H^n(G;M) \cong H^n(\text{Hom}_G(\mathbb{Z}[G^\bullet], M))$$

2 Group Homology

Definition 2.0.1: The Group of Coinvariants

Let G be a group and let M be a G-module. Define the group of coinvariants of G in M to be the quotient group

$$M_G = \frac{M}{\langle gm - m \mid g \in G, m \in M \rangle}$$

This is the largest quotient of M for which G acts trivially.