Analytic Number Theory

Labix

April 24, 2022

Abstract

Contents

1 Arithmetic Functions

1.1 Mobius Function and Euler Totient Function

Definition 1.1.1 (Arithmetical Functions)**.** A function $f : \mathbb{N} \to \mathbb{C}$ is an arithmetical function.

Definition 1.1.2 (Mobius Function)**.** Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$. Define the mobius function as

$$\mu(n) = (-1)^k$$

if $\alpha_1 = \cdots = \alpha_k = 1$. And 0 otherwise.

Theorem 1.1.3. For $n \in \mathbb{N}$, we have

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.1.4 (Euler's Totient Function)**.** Let $\phi(n)$ denote the numeber of positive integers less than n and relatively prime to n.

Theorem 1.1.5. Let $n \geq 1$.

$$n = \sum_{d|n} \phi(d)$$

Theorem 1.1.6. For $n \in \mathbb{N}$ we have

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

Theorem 1.1.7. For $n \in \mathbb{N}$ we have

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Proposition 1.1.8. The Euler's Totient Function has the following properties.

- $\phi(p^n) = p^{n-1}(p-1)$
- $\phi(mn) = \phi(m)\phi(n)\left(\frac{d}{\phi(d)}\right)$, where $d = \gcd(m, n)$
- $a|b \implies \phi(a)|\phi(b)$
- $\phi(n)$ is even for $n \geq 3$. Moreover, if n has r distinct odd prime factors, then $2^r|\phi(n)$

1.2 Dirichlet Functions

Definition 1.2.1 (Dirichlet Product)**.** If f and g are two arithmetical functions we define their dirichlet product to be the arithmetical function h defined by the equation

$$h(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

Theorem 1.2.2. Dirichlet multiplication is commutative and associative.

Definition 1.2.3. The arithmetical function I given by

$$I(n) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

is called the identity function.

Proposition 1.2.4. For all f we have $I * f = f * I = f$

Theorem 1.2.5. Let f be an arithmetical function with $f(1) \neq 0$. There is a unique arithemetical function f^{-1}, called the Dirichlet inverse of f such that

$$f * f^{-1} = f^{-1} * f = I$$

Moreover f^{-1} is given by the recursion formulas

$$f^{-1}(1) = \frac{1}{f(1)}$$

and

$$f^{-1}(n) = \frac{-1}{f(n)} \sum_{d|n \text{ and } d<n} f\left(\frac{n}{d}\right) f^{-1}(d)$$

for $n > 1$.

Definition 1.2.6 (Unit Function)**.** Define the unit function u to be the arithmetical function such that $u(n) = 1$ for all n.

Proposition 1.2.7. The Dirichlet Inverse of the mobius function is the unit function.

Theorem 1.2.8 (Mobius Inversion Formula)**.** Let f, g be arithmetical functions.

$$f(n) = \sum_{d|n} g(d)$$

if and only if

$$g(n) = \sum_{d|n} f(d)\mu\left(\frac{n}{d}\right)$$

1.3 Mangoldt Function

Definition 1.3.1 (Mangoldt's Function)**.** For $n \in \mathbb{N}$ define

$$\Lambda(n) = \begin{cases} \ln(p) & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1.3.2. For $n \in \mathbb{N}$,

$$\ln(n) = \sum_{d|n} \Lambda(d)$$

Theorem 1.3.3. For $n \in \mathbb{N}$ we have

$$\Lambda(n) = \sum_{d|n} \mu(d) \ln\left(\frac{n}{d}\right) = -\sum_{d|n} \mu(d) \ln(d)$$

1.4 Multiplicative Functions

Definition 1.4.1 (Multiplicative Functions)**.** An arithmetical function f is called multiplicative if f is not identically 0 and if

$$f(mn) = f(m)f(n)$$

when $\gcd(m, n) = 1$ It is completely multiplicative if it is multiplicative regardless of the condition.

Proposition 1.4.2. If f is multiplicative then $f(1) = 1$.

Proposition 1.4.3. Let $f(1) = 1$ be an arithmetical function. f is multiplicative if and only if

$$f\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k} f\left(p_i^{\alpha_i}\right)$$

Proposition 1.4.4. Let $f(1) = 1$ be an arithmetical function. f is completely multiplicative if and only if $f(p)^\alpha = f(p)^\alpha$ for all primes p and all integers $\alpha \geq 1$.

Proposition 1.4.5. If f and g are multiplicative, so is their Dirichlet product.

Proposition 1.4.6. If f and $f * g$ are multiplicative, then g is multiplicative.

Proposition 1.4.7. If f is multiplicative, so is f^{-1}.

1.5 Completely Multiplicative Functions

Theorem 1.5.1. Let f be multiplicative. Then f is completely multiplicative if and only if

$$f^{-1}(n) = \mu(n)f(n)$$

for all $n \geq 1$.

Theorem 1.5.2. If f is multiplicative we have

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n}(1 - f(p))$$

1.6 Liouville's Function

Definition 1.6.1 (Liouville's Function)**.** Define $\lambda(1) = 1$ and if

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}$$

define

$$\lambda(n) = (-1)^{\alpha_1 + \cdots + \alpha_k}$$

Proposition 1.6.2. $\lambda(n)$ is completely mutiplicative.

Theorem 1.6.3. For $n \in \mathbb{N}$ we have

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases}$$

Also $\lambda^{-1}(n) = |\mu(n)|$ for all n.

1.7 The Divisor Function

Definition 1.7.1. For real and complex α and any $n \in \mathbb{N}$ define

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha$$

Proposition 1.7.2. $\sigma_\alpha(n)$ is multiplicative.

Theorem 1.7.3. For $n \in \mathbb{N}$ we have

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d)\mu\left(\frac{n}{d}\right)$$

1.8 Bell Series

Definition 1.8.1 (Bell Series)**.** Let f be an arithmetical function and p a prime. Denote

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n$$

the bell series of f modulo p.

Theorem 1.8.2. Let f, g be multiplicative functions. Then $f = g$ if and only if $f_p(x) = g_p(x)$ for all primes p.

Theorem 1.8.3. Let f, g be arithmetical functions and let $h = f * g$. Then for every prime p we have

$$h_p(x) = f_p(x)g_p(x)$$

1.9 Derivatives

Definition 1.9.1 (Derivatives of Arithmetical Functions)**.** For any arithmetical function f define f' to be its derivative where

$$f'(n) = f(n)\ln(n)$$

for $n \geq 1$.

Theorem 1.9.2. Let f, g be arithmetical functions.

- $(f + g)' = f' + g'$
- $(f * g)' = f' * g + f * g'$
- $(f^{-1})' = -f' * (f * f)^{-1}$ whenever $f(1) \neq 0$

1.10 Selberg Identity

Theorem 1.10.1 (Selberg Identity)**.** For $n \in \mathbb{N}$ we have

$$\Lambda(n)\ln(n) + \sum_{d|n} \Lambda(d)\Lambda\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)\ln^2\left(\frac{n}{d}\right)$$