Geometric Group Theory

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Abstract

Potentially good books: Humphreys, Erdmann and Wildson

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1 Quasi-Isometries

1.1 Some Metric Properties

Definition 1.1.1: Proper Metric Spaces

Let X be a metric space. We say that X is proper if $B_r(x)$ is compact for all $r \in R$ and $x \in X$.

Definition 1.1.2

Let X be a metric space. We say that X is geodesic if for all $x, y \in X$, there exists a geodesic from x to y.

1.2 Quasi-Isometric Spaces

Definition 1.2.1: Quasi-Isometric Embeddings

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function of sets. We say that f is a quasi-isometric embedding if there exists $A \geq 1$ and $B \geq 0$ such that

$$\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x), f(y)) \leq A d_X(x, y) + B$$

for all $x_1, x_2 \in X$.

Definition 1.2.2: Quasi-Isometries

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a function of sets. We say that f is a quasi-isometry if the following are true.

- f is a quasi-isometric embedding.
- There exists $C \geq 0$ and $x_0 \in X$ such that

  $$d_Y(y, f(x_0)) \leq C$$

  for all $y \in Y$.

In this case we say that X and Y are quasi-isometric.

Proposition 1.2.3

Quasi-isometry is an equivalence relation on metric spaces.

Proposition 1.2.4

Let X and Y be metric spaces. If X and Y are Lipshitz equivalence, then X and Y are quasi-isometric.

1.3 Quasi-Geodesics

Definition 1.3.1: Quasi-Geodesics

Let X be a metric space. Let $\gamma : I \to X$ be as path in X. We say that γ is a quasi-geodesic if γ is a quasi-isometric embedding.

2 The Geometry of Presentations

2.1 The Cayley Graph of a Group

Definition 2.1.1: The Cayley Graph of a Group

Let G be a group. Let S be a generating set of G. Define the Cayley graph $\text{Cay}(G, S)$ of G with respect to S to consist of the following data.

- The vertices are given by $V(\text{Cay}(G, S)) = G$
- The edges are given by $E(\text{Cay}(G, S)) = \{(g, gs) \mid g \in G, s \in S\}$

Let (V, E) be a graph. Recall that a graph automorphism consists of a bijective map of vertices and a bjective map of edges such that

$$\{\phi(v), \phi(w)\} \in E$$

for all $\{v, w\} \in E$. They form a group by composition.

Lemma 2.1.2: The Action Lemma

Let G be a group. Let S be a generating set of G. Then G acts on the Cayley graph $\text{Cay}(G, S)$ of G with respect to S via tha map

$$\cdot : G \times \text{Cay}(G, S) \to \text{Cay}(G, S)$$

defined by $h \cdot g = hg$ and $h \cdot (g, gs) = (hg, hgs)$. Moreover, the action is faithful.

Proposition 2.1.3

Let G be a group. Let S be a generating set of G. Then the following are true regarding $\text{Cay}(G, S)$.

- $\text{Cay}(G, S)$ has no embedded cycles.
- $\text{Cay}(G, S)$ is connected.

Proposition 2.1.4

Let S be a set. Then $\text{Cay}(F_S, S)$ is a tree.

Proposition 2.1.5

Let G be a group. Let S be a generating set of G. Then $\text{Cay}(F_S, S)$ is a universal cover of $\text{Cay}(G, S)$.

2.2 The Word Metric on a Cayley Graph

Given a graph Γ, there are two ways to specify a path in Γ.

- We can define a path by a sequence $\gamma_V : [n] \to V(\Gamma)$ of adjacent vertices.
- We can also define a path by a sequence $\gamma_E : [n-1] \to E(\Gamma)$ of edges.

The above notation also indicates that any path is determined by either n vertices or $n - 1$ edges.

Definition 2.2.1: The Word Metric

Let G be a group. Let S be a generating set of G. Define the word metric on $\text{Cay}(G, S)$ to be the map

$$d_S : V(\text{Cay}(G, S)) \times V(\text{Cay}(G, S)) \to \mathbb{N}$$

given by

$$d_S(g,h) = \min\{n \in \mathbb{N} \mid \gamma_V : [n] \to V(\text{Cay}(G,S)) \text{ is a path from } g \text{ to } h\}$$

Lemma 2.2.2

Let G be a group. Let S be a generating set of G. Then d_S is a metric on Cay(G,S).

Proposition 2.2.3

Let G be a group. Let S be a generating set of G. Let $g \in G$ be fixed. Then the map

$$(h,k) \mapsto (gh,gk)$$

given by the action lemma is an isometry. In other words,

$$d_S(h,k) = d_S(gh,gk)$$

Let X be a metric space with two metrics d_1 and d_2. Recall that d_1 and d_2 are bilipschitz equivalent if there exists two constants $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y)$$

for all $x,y \in X$.

Lemma 2.2.4

Let G be a group. Let S,T be generating sets of G. Then d_S and d_T are bilipschitz equivalent.

Definition 2.2.5: The Word Norm

Let G be a group. Let S be a generating set of G. Let Cay(G,S) be the Cayley complex of G and S. Define the word norm of $g \in G$ to be

$$\|g\|_S = d_S(1_G, g)$$

Lemma 2.2.6

Let G be a group. Let S be a generating set of G. Then the following are true.

- $d_S(g,h) = \|g^{-1}h\|_S$ for all $g,h \in G$.
- $\|g^{-1}\|_S = \|g\|_S$ for all $g \in G$.
- $\|gh\|_S \leq \|g\|_S + \|h\|_S$ for all $g,h \in G$.

2.3 Realizing the Cayley Graph as a Connected Space

We have proved that Cayley graphs are connected as graphs, in the sense that any two vertices are connected by a path. But a priori the graph is not connected as a topological space, whose topology is generated by the metric.

Definition 2.3.1: Geometric Realization of Cayley Graphs

Let G be a group. Let S be a generating set of G. Define the geometric realization $|\text{Cay}(G,S)|$ of the Cayley graph to be the space

$$|\text{Cay}(G,S)| = \frac{E(\text{Cay}(G,S)) \times I}{\sim}$$

where $((g_1, g_1 s_1), t_1) \sim ((g_2, g_2 s_2), t_2)$ if one of the following are true.

- They describe the same vertex (with different representations of elements in G): $((g_1, g_1s_1), t_1) = ((g_2, g_2s_2), t_2)$.
- They describe the same vertex but they lie on different edges: Either one of the following
  - $g_1 = g_2$ and $t_1 = t_2 = 0$
  - $g_1 = g_2s_2$ and $t_1 = 0$, $t_2 = 1$
  - $g_1s_1 = g_2s_2$ and $t_1 = t_2 = 1$
  - $g_1s_1 = g_2$ and $t_1 = 1$, $t_2 = 0$
- They describe the same point on an edge but different orientations: $(g_1, g_1s_1) = (g_2, g_2s_2^{-1})$ and $t_1 = 1 - t_2$.

In particular, this gives a 1-dimensional CW complex.

We can also give a metric on the realization so that its restriction to the actual Cayley graph recovers the word metric.

Definition 2.3.2: Metric on realization

Let G be a group. Let S be a generating set of G. Define a metric $d : |\text{Cay}(G, S)| \times |\text{Cay}(G, S)| \to \mathbb{R}$ as follows.

$$d([((g_1, g_1s_1)t_1)], [((g_2, g_2s_2), t_2)]) = \begin{cases} |t_1 - t_2| & \text{if } (g_1, g_1s_1) = (g_2, g_2s_2) \\ |t_1 - (1 - t_2)| & \text{if } (g_1, g_1s_1) = (g_2s_2, g_2) \\ \min \left\{ \begin{smallmatrix} t_1 + d_S(g_1, g_2) + t_2 \\ t_1 + d_S(g_1, g_2s_2) + 1 - t_2 \\ 1 - t_1 + d_S(g_1s_1, g_2) + t_2 \\ 1 - t_1 + d_S(g_1s_1, g_2s_2) + 1 - t_2 \end{smallmatrix} \right\} & \text{otherwise} \end{cases}$$

We abuse notation sometimes and freely interchange the use of the Cayley graph and its geometric realization when the context is clear.

2.4 Realizing the Cayley Graph as a Presentation Complex

3 Metric Properties of Cayley Graphs

3.1 The Svarc-Milnor Lemma

Let G be a group acting on a space X. Recall that G is a properly discontinuous group action if for every compact set $K \subseteq X$, we have

$$(g \cdot K) \cap K \neq \emptyset$$

for finitely many $g \in G$.

Theorem 3.1.1: The Svarc-Milnor Lemma

Let X be a geodesic metric space such that $B_r(x)$ is compact for all $x \in X$ and $r \in \mathbb{R}_{\geq 0}$. Let G be a group acting on X such that the action is properly discontinuous and X/G is compact. Then the following are true.

- G is finitely generated.
- For any finite generating set S of G, $|\text{Cay}(G,S)|$ is quasi-isometric to (X,d)

Proposition 3.1.2

Let G be a group. Let S,T be a finite generating sets of G. Then $|\text{Cay}(G,S)|$ and $|\text{Cay}(G,T)|$ are quasi-isometric via the identity map on G.

3.2 Geodesics on Cayley Graphs

Proposition 3.2.1

Let G be a group. Let S be a finite generating set. Then the following are true.

- $|\text{Cay}(G,S)|$ is proper.
- $|\text{Cay}(G,S)|$ is complete.
- $|\text{Cay}(G,S)|$ is geodesic.

Definition 3.2.2: Geodesic Words

Let G be a group. Let S be a generating set. Let $\gamma_V : [n] \to V(\text{Cay}(G,S))$ be a path in $\text{Cay}(G,S)$. We say that γ_V is a geodesic word if

$$d_S(\gamma_V(0), \gamma_V(n)) = n$$

This is not the same definition as geodesics in metric spaces. (It doesn't make sense to talk about paths in $\text{Cay}(G,S)$ because it is a discrete topological space when we consider the topology generated by the metric).

Recalling that the metric on $\text{Cay}(G,S)$ is defined as the minimum length of a path between two elements, we see that geodesics are precisely paths that realizes such a distance minimizing path.

Proposition 3.2.3

Let G be a group. Let S be a generating set of G. Then $\gamma_V : [n] \to V(\text{Cay}(G,S))$ is a geodesic word if and only if $|\gamma_V| : [0,n] \to |\text{Cay}(G,S)|$ is a geodesic in the sense of metric spaces.

Lemma 3.2.4

Let G be a group. Let S be a generating set. If $\gamma_V : [n] \to \text{Cay}(G,S)$ is a geodesic, then $\gamma_V(0) * \cdots * \gamma_V(n)$ is a reduced word.

Note: The converse is not true. Consider $G = \langle a, b \rangle a^3 = b^2$. Both a^3 and b^2 are reduced words but they have different lengths.

3.3 Representing Geodesics in a Canonical Way

Note: geodesics are not the unique distance minimizing curve between two elements. Therefore we want to find a representative.

Definition 3.3.1: Short Lex Ordering

Let G be a group. Let S be a finite generating set of G. Let $u, v \in F(S)$. We say that

$$u <_{sl} v$$

if one of the following are true.

- $|u| < |v|$
- $|u| = |v|$ and there exists w such that $u = w * u'$, $v = w * v'$ and $u' <_{sl} v'$.

We call $<_{sl}$ the short lex ordering on $F(S)$.

Lemma 3.3.2

Let G be a group. Let S be a generating set. Then $<_{sl}$ is a total order on $F(S)$.

Definition 3.3.3: Short Lex Representative

Let G be a group. Let S be a generating set of G. Let $g \in G$. Define the short lex representative of g with respect to S to be

$$\min_{<_{sl}} \{s \in F(S) \mid s = g \text{ in G}\}$$

Lemma 3.3.4

Let G be a group. Let S be a generating set of G. Any subword of a short lex representative with respect to S is a short lex representative.

Corollary 3.3.5

Let G be a group. Let S be a generating set of G. Then the set of paths in Cay(G, S) consisting of short lex representatives form a spanning tree for Cay(G, S).

4 The Growth Type of Groups

4.1 Growth Function

Definition 4.1.1: Ball Around an Element

Let G be a group. Let S be a finite generating set of G. Let $R > 0$. Define the ball around $g \in G$ with radius n to be
$$B_n^{G,S}(g) = \{h \in G \mid d_S(g,h) \leq n\}$$

Proposition 4.1.2

Let G be a group. Let S be a finite generating set. Let $g, h \in G$. Then
$$\left|B_n^G(g)\right| = \left|B_m^G(h)\right|$$
for any $n \in \mathbb{N}$.

Definition 4.1.3: Growth Function

Let G be a group. Let S be a finite generating set of G. Let $R > 0$. Define the growth function $\Gamma_{G,S} : \mathbb{N} \to \mathbb{N}$ of G with respect to S to be
$$\Gamma_{G,S}(n) = \left|B_n^{G,S}(1_G)\right|$$
for $n \in \mathbb{N}$.

Proposition 4.1.4

Let G be a group Let S be a finite generating set of G. Then the following are true.
- $\Gamma_{G,S}(m+n) \leq \Gamma_{G,S}(m)\Gamma_{G,S}(n)$ for all $m, n \in \mathbb{N}$
- $\Gamma_{G,S}(n) \leq (2|S|+1)^n$ for all $n \in \mathbb{N}$.

Proof. For any pair (h,k) of elements of G such that $d_S(1,h) = m$ and $d_S(1,k) = n$, we have that
$$d_S(1_G, hk) \leq d_S(1_G, h) + d_S(h, hk) = d_S(1_G, h) + d_S(1_G, k) = m + n$$
This means that for any unique pair of elements (h,k) with $h \in B_m^{G,S}(1_G)$ and $k \in B_n^{G,S}(1_G)$, there exists a possibly non-unique element $hk \in B_{m+n}^{G,S}(1_G)$. Hence
$$\left|B_{m+n}^{G,S}(1_G)\right| \leq \left|B_m^{G,S}(1_G)\right| \cdot \left|B_n^{G,S}(1_G)\right|$$
and so $\Gamma_{G,S}(m+n) \leq \Gamma_{G,S}(m)\Gamma_{G,S}(n)$.

Notice that $\Gamma_{G,S}(1) = (2|S|+1)$ since the paths of the Cayley graph is given by S and their inverses. Together with the identity element which has zero norm gives the formula. We can then recursively apply the above inequality to get
$$\Gamma_{G,S}(n) \leq (\Gamma_{G,S}(1))^n = (2|S|+1)^n$$
$\square$

Lemma 4.1.5

Let G be a group Let S be a finite generating set of G. Then the following are true.
- $\Gamma_{G,S}(n) \leq \Gamma_{F(S),S}(n)$ for all $n \in \mathbb{N}$.
- $\Gamma_{G,S}(n) = \Gamma_{F(S),S}(n)$ for all $n \in \mathbb{N}$ if and only if $G \cong F(S)$.

Proof. The induced homomorphism $\phi : F(S) \to G$ sends $B_n^{F(S),S}(1_{F(S)})$ surjectively to $B_n^{F(S),S}(1_{F(S)})$. Indeed if $\gamma_V : [n] \to F(S)$ is a geodesic, then $\phi \circ \gamma_V$ may not be a geodesic so that $d_S(1_G, \phi \circ \gamma_V(n)) \leq n$. This means that $\phi \circ \gamma_V(n) \in B_n^{G,S}(1_G)$. Conversely, if $g \in B_n^{G,S}(1_G)$ then $g = w_1 \cdots w_n$ is a reduced word in G for $w_1, \dots, w_n \in S$. Then $w_1 \cdots w_n$ is also a reduced word in $F(S)$ and hence lie in $B_n^{F(S),S}(1_{F(S)})$. Moreover, $\phi(w_1 \cdots w_n) = g$. Hence ϕ is surjective on the two balls. Then we have

$$\Gamma_{G,S}(n) = |B_n^{G,S}(1_G)| = \left|\phi\left(B_n^{F(S),S}(1_{F(S)})\right)\right| \leq \left|B_n^{F(S),S}(1_{F(S)})\right| = \Gamma_{F(S),S}(n)$$

□

Lemma 4.1.6

Let S be a finite set. Then

$$\Gamma_{F(S),S}(n) = \frac{1 - |S|(2|S|-1)^n}{1-|S|}$$

Proof. I claim that the number of reduced words of length n is $2|S|(2|S|-1)^{n-1}$ when $n \geq 1$. We induct on n. When $n = 1$, then any reduced word is just the choice of a letter. Hence there are $2|S|$ number of reduced words of length 1. Now suppose that the number of reduced words of length k is given by $2|S|(2|S|-1)^{k-1}$. Any reduced word of length $k+1$ is given by the concatenation of a reduced word of length k and a choice of letter that is not the inverse of the last element of the given word. Thus there are $2|S|(2|S|-1)^{k-1} \cdot (2|S| - 1 = 2|S|(2|S|-1)^k)$ number of reduced words of length $k+1$. This completely the induction step.

Then we have

$$\begin{aligned}
\Gamma_{F(S),S}(n) &= 1 + \sum_{i=1}^{n} 2|S|(2|S|-1)^{n-1} \\
&= 1 + 2|S|\sum_{i=0}^{n-1}(2|S|-1)^i \\
&= 1 + 2|S|\frac{1-(2|S|-1)^n}{1-2|S|+1} \\
&= 1 + |S|\frac{1-(2|S|-1)^n}{1-|S|} \\
&= \frac{1-|S| + |S|\left(1-(2|S|-1)^n\right)}{1-|S|} \\
&= \frac{1-|S|(2|S|-1)^n}{1-|S|}
\end{aligned}$$

□

Proposition 4.1.7

Let G be a group Let S be a finite generating set of G. Then the following are equivalent.

- G is a finite group.
- $\Gamma_{G,S}$ is bounded.
- $\Gamma_{G,S}(n) = \Gamma_{G,S}(n+1)$ for some $n \in \mathbb{N}$.

Lemma 4.1.8

Let G be a group. Let S, T be finite generating sets of G. Then there exists $C, D > 0$ such that

$$\Gamma_{G,S}(n) \leq C\Gamma_{G,T}(n) \quad \text{and} \quad \Gamma_{G,T}(n) \leq D\Gamma_{G,S}(n)$$

for all $n \in \mathbb{N}$.

Theorem 4.1.9

There exists a finitely generated group G with finite generators S such that $\Gamma_{G,S}$ has super-polynomial growth but subexponential growth.

Theorem 4.1.10: [Hirsch 1958]

Let G be a finitely generated nilpotent group. Let $H \leq G$ be a subgroup of G. Then $[G : H]$ is finite and H is torsion-free.

Theorem 4.1.11: [Jennings 1955]

Let H be a finitely generated torsion-free and nilpotent group. Then H is isomorphic to a subgroup of $H_d(\mathbb{Z})$ for some $d \geq 1$.

Note: $H_d(\mathbb{Z})$ is the upper triangular matrices of $SL_d(\mathbb{Z})$.

Theorem 4.1.12: [Gromov 1981]

Let G be a finitely generated group such that $\Gamma_{G,S}$ has at most polynomial growth. Then there exists some subgroup $H \leq G$ such that $[G : H]$ is finite and H is nilpotent.

Theorem 4.1.13: [Bass 1972, Guivarch 1973]

Let G be a finitely generated nilpotent group. Then there exists $C, D, d \in \mathbb{N}$ such that

$$Cn^d \leq \Gamma_{G,S}(n) \leq Dp^d$$

($\Gamma_{G,S}$ has polynomial growth rate).

4.2 Distortion

Definition 4.2.1: Undistorted Subgroups

Let G be a group. Let S, T be generating sets of G. Let $H \leq G$ be a subgroup. We say that H is undistorted in G if there exists $C > 0$ such that

$$d_T(g, h) \leq Cd_S(g, h)$$

for all $g, h \in H$.

Intuitively, this means that when we restrict the metric to the subgroup, the shortest path when we had in H for two elements is still the shortest when we consider the two elements in G.

4.3 The Dehn Function

Definition 4.3.1: Area of an Element

Let G be a group. Let $G = \langle S \mid R \rangle$ be a finite presentation of G. Let $p : F_S \to G$ be the induced map by the universal property. Let $w \in F_S$ be such that $p(w) = 1_G$. Define the area of w to be

$$A(w) = \min\left\{ n \in \mathbb{N} \;\middle|\; w = \prod_{i=1}^{n} a_i r_i a_i^{-1} \text{ for } a_i \in F(S) \text{ and } r_i \in R \right\}$$

Definition 4.3.2: Dehn Functions

Let G be a group. Let $G = \langle S \mid R \rangle$ be a finite presentation of G. Let $p : F_S \to G$ be the induced map by the universal property. Define the Dehn function of G with respect to the presentation to be $\text{Dehn}_{\langle S \mid R \rangle} : \mathbb{N} \to \mathbb{N}$ given by

$$\text{Dehn}_{\langle S \mid R \rangle}(n) = \max\{A(s_1 \cdots s_k) \mid 0 \leq k \leq n, s_1, \dots, s_k \in F_S, p(s_1 \cdots s_k) = 1_G\}$$

5 The Word Problem for Groups

5.1 The Word Problem is Undecidable

Definition 5.1.1: Solvable Word Problem

Let G be a group. Let $G = \langle S \mid R\rangle$ be a finite presentation for G. Let W be the set of all words of S. We say that G has a solvable word problem if $\{w \in W \mid w = 1_G \text{ in } G\}$ is decidable.

Theorem 5.1.2

There exists a finitely presented group with an unsolvable word problem.

5.2 Dehn Presentations and The Word Problem

Recall that a group is finitely presented if $G \cong \langle S \mid R\rangle$ if S and R are finite sets.

Definition 5.2.1: Dehn Presentations

Let G be a group. Let $G = \langle S \mid R\rangle$ be a finite presentation of G. Let $p : F_S \to G$ be the induced map by the universal property. We say that $\langle S \mid R\rangle$ is a Dehn presentation for G if for all $w \in F(S)$ such that $p(w) = 1_G$, there exists $r_1r_2^{-1} \in F(S)$ such that the following are true.

- $\|r_1\| > \|r_2\|$.
- r_1 is a subword of w.
- Some cyclic permutation of the letters in $r_1r_2^{-1}$ or $r_2^{-1}r_1$ lies in R.

Proposition 5.2.2

The presentation

$$\pi_1(\Sigma_2) \cong \langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1}\rangle$$

for the fundamental group of the compact orientable surface of genus 2 is a Dehn presentation.

Proof. Recall that Σ_2 can be represented as an octagon whose edges are given by $abcda^{-1}b^{-1}c^{-1}d^{-1}$. Let p be the starting point of a. Then the starting and ending point of a, b, c, d are all p via the identification. In particular, the point p has 4 edges in and 4 edges out of it. By definition of the universal covering space (it is a local homeomorphism), every point in the inverse image of p of covering space intersects eight octagons at an vertex. This justifies why every vertex is the intersection of eight octagons at a vertex. This is the Cayley graph of $F(a, b, c, d)$ with generators a, b, c, d. In particular, it is the tiling of $\mathbb{H}^2$ using the octagon representing Σ_2.

Pick a lift of Σ_2 into the Cayley graph of $F(a, b, c, d)$. Call it R_0. Assuming R_n is defined, we define R_{n+1} to be the union of all octagons that intersect R_n but does not intersect R_{n-1} (by convention take $R_{-1} = \emptyset$.

Let $w \in F(a, b, c, d)$ such that w is the identity in $\pi_1(\Sigma_2)$. By translation, without loss of generality we can take w to start at the identity element of the free group. Moreover, without loss of generality take w to be a reduced word. If w is not the empty word, then since w has finite length, it reaches the outer boundary of R_n for some maximal $n \in \mathbb{N}$ and descends to R_{n-1}. It must run through the outermost boundary non-trivially (otherwise w is not reduced), and hence runs through at least 5 sides of some octagonal tile. Suppose it runs through $5 \leq n \leq 8$ sides. In this case, choose u to be the n (consecutive) sides and v to be the remaining sides such that $u * v$ is a cyclic permutation of $abcda^{-1}b^{-1}c^{-1}d^{-1}$ (this is possible since u is a consecutive subword of some cyclic permutation of $abcda^{-1}b^{-1}c^{-1}d^{-1}$). Then the pair u and v satisfies the requirements for $\langle a,b,c,d \mid abcda^{-1}b^{-1}c^{-1}d^{-1}\rangle$ to be a Dehn presentation for all reduced words w. Hence we are done. □

Theorem 5.2.3: Dehn's Algorithm

Let G be a group. If G admits a Dehn presentation, then the word problem for G is solvable.

6 The Geometry of Boundaries

6.1 Ends of a Group via Geodesic Rays

Definition 6.1.1: Geodesic Ray

Let G be a group. Let S be a finite generating set of G. A geodesic ray in $\text{Cay}(G,S)$ is a continuous map

$$\phi : [0,\infty) \to \text{Cay}(G,S)$$

such that if $B \subseteq \text{Cay}(G,S)$ is bounded then $\phi^{-1}(B)$ is bounded.

This is called proper rays in Loh.

Definition 6.1.2: Ends of a Group

Let G be a group. Let S be a finite generating set of G. Define

$$\text{Ends}(G,S) = \{\phi : [0,\infty) \to \text{Cay}(G,S) \mid \phi \text{ is a geodesic ray }\}/\sim$$

where $\phi_1 \sim \phi_2$ if for all $n \in \mathbb{N}$, there exists $t \in \mathbb{R}$ such that $\text{im}(\phi_1) \setminus B_n^{G,S}(1)$ and $\text{im}(\phi_2) \setminus B_n^{G,S}(1)$ lie in the same path component of $\text{Cay}(G,S) \setminus B_n^{G,S}(1)$.

6.2 Infinite Connected Components at Infinity

Definition 6.2.1: Infinite Connected Components

Let G be a group. Let S be a finite generating set of G. Define the set of infinite connected components of G with respect to S by

$$E_S(n) = \{[X] \in \pi_0(\text{Cay}(G,S) \setminus B_n^{G,S}(1)) \mid |X| = \infty\}$$

Definition 6.2.2

Let G be a group. Let S be a finite generating set of G. Define

$$C_S(n) = \text{Cay}(G,S) \setminus \bigcup_{A \in E_S(n)} A$$

6.3 Basic Properties of Ends

(ends are a quasi-isometry invariant)

Theorem 6.3.1: (Stallings)

Let G be a finitely generated group. Then

$$|\text{End}(G)| = 0, 1, 2 \text{ or } \infty$$

7 Hyperbolic Groups

7.1 Hyperbolic Metric Spaces

Definition 7.1.1: Geodesic Triangle

Let (X, d) be a geodesic metric space. Let $x, y, z \in X$. Define the geodesic triangle to be a triple

$$T(x, y, z) = \{\alpha, \beta, \gamma\}$$

such that $\alpha, \beta, \gamma : I \to X$ are geodesics with starting points x, y, z and ending points y, z, x respectively.

Let (M, d) be a metric space. Recall that the neighbourhood of a set $U \subseteq M$ of size r is the set

$$N_r(U) = \{x \in M \mid d(U, x) \leq r\}$$

Definition 7.1.2: Gromov Hyperbolic Metric Space

Let (X, d) be a geodesic metric space. Let $\delta > 0$. We say that X is a Gromov δ-hyperbolic metric space if for all geodesic triangles $T(x, y, z)$ for $x, y, z \in X$, we have

$$\alpha \subseteq N_\delta(\mathrm{im}(\beta)) \cup N_\delta(\mathrm{im}(\gamma))$$

7.2 Gromov Hyperbolic Groups

Definition 7.2.1: Gromov Hyperbolic Groups

Let G be a group. Let $\delta > 0$. We say that G is δ-hyperbolic if there exists a finite generating set S of G such that $|\mathrm{Cay}(G, S)|$ is a Gromov δ-hyperbolic metric space.

Examples: finite groups, free groups.

Theorem 7.2.2

Let G be a group. Then the following are equivalent.

- G is a Gromov hyperbolic group.
- G admits a Dehn presentation.
- There exists a finite presentation of G for which the Dehn function is linear.

Corollary 7.2.3

Let G be a group. If G is a Gromov hyperbolic group, then the following are true.

- G is finitely presented.
- The word problem for G is decidable.

7.3 The Conjugacy Problem

Definition 7.3.1: The Conjugacy Problem

Let G be a group. Let S be a finite generating set for G. The conjugacy problem asks for an algorithm to input $u, v \in F(S)$ and out put $w \in F(S)$ such that $uw = wv$ in G.

Theorem 7.3.2

Let G be a group. Let S be a finite generating set for G. If G is δ-hyperbolic, then the conjugacy problem admits a solution.