Metric Space

Labix

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Abstract

References

- A course in Point Set Topology by John B. Conway
- Lecture Notes of MAT327 at the University of Toronto by Ivan Khatchatourian

Contents

1 Metric Spaces

1.1 Basic Definitions

A lot of the times we would like to add a structure of a metric to space so that analysis such as continuity and integration can be performed on it.

Definition 1.1.1 (Metric) Let X be a set. Let $x, y, z \in X$. A metric is a function $d : X \times X \to \mathbb{R}$ satisfying the following.

- $d(x, y) \geq 0$ with equality if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

Definition 1.1.2 (Metric Space) A metric space is an oredered pair (X, d) where X is a set and d is a metric on X.

Definition 1.1.3 (Open Balls) Let X be a metric space. Let $a \in X$. Define the open ball of radius r around a to be

$$B_r(a) = \{x \in X | d(x, a) < r\}$$

Lemma 1.1.4 (Metric Subspace) Let (X, d) be a metric space. Let $A \subseteq X$, then $(A, d|_A)$ is also a metric space.

Proof $d|_A$ inherits the metric properties of X while being restricted to A. ■

1.2 Open and Closed Subsets

Definition 1.2.1 (Open Sets) Let M be a metric space. Let $U \subset M$. We say that U is open if for every $a \in U$, there exists r such that

$$B_r(a) \subseteq U$$

Definition 1.2.2 (Closed Sets) Let M be a metric space. Let $U \subset M$. We say that U is closed if $M \setminus U$ is open.

Lemma 1.2.3 Let M be a metric space. Let $a \in M$ and $r \in \mathbb{R}_{\geq 0}$. Then $B_r(a)$ is an open set.

Proof Let $B_r(a)$ be our open ball. Let $x \in B_r(a)$. Then

$$B_{(r-d(x,a))/2}(x) \subseteq B_r(a)$$

thus we are done. ■

Proposition 1.2.4 Let M be a metric space. Then the following are true.

- Let $\{U_i \mid i \in I\}$ be a family of open subsets of M. Then $\bigcup_{i \in I} U_i$ is an open subset of M.
- Let $U_1, \dots, U_n$ be open subsets of M. Then $\bigcap_{i=1}^n U_i$ is an open subset of M.

Proof Let $U_1, U_2, \dots$ be a sequence of open sets. Let $U = \bigcup_{n \in \mathbb{N}} U_n$. Let $x \in U$. Then there exists $k \in \mathbb{N}$ such that $x \in U_k$. Since U_k is open, there exists $r \in \mathbb{R}^+$ such that

$$B_r(x) \subseteq U_k \subseteq U$$

and we are done.

Observe that

$$X \setminus \bigcup_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} (X \setminus U_n)$$

By definition of closed sets, $X \setminus U$ is closed and we are done. ■

Proposition 1.2.5 Let M be a metric space. Then the following are true.

- Let $U_1, \dots, U_n$ be closed subsets of M. Then $\bigcup_{i=1}^n U_i$ is an open subset of M.
- Let $\{U_i \mid i \in I\}$ be a family of closed subsets of M. Then $\bigcap_{i \in I} U_i$ is an open subset of M.

Proof Let $U_1, \dots, U_n$ be opens sets. Then let $x \in \bigcap_{k=1}^n U_k$. Then $x \in U_k$ for all $k \in \{1, \dots, n\}$ and there exists $r_k > 0$ such that $B_{r_k}(x) \subseteq U_k$ for each k. Take $r = \min\{r_1, \dots, r_n\}$. Then

$$B_r(x) \subseteq B_{r_k}(x) \subseteq U_k$$

for each k and thus $B_r(x) \subseteq \bigcap_{k=1}^n U_k$ and we are done.

Observe that

$$X \setminus \bigcap_{k=1}^n U_k = \bigcup_{k=1}^n (X \setminus U_k)$$

and by definition of closed sets, $X \setminus \bigcap_{k=1}^n U_k$ is closed and we are done. ■

Definition 1.2.6 (Distance Between Subsets) Let (M, d) be a metric space. Let $U, T \subseteq M$. Let $x \in M$. Define the distance between U and T to be

$$d(U, T) = \inf\{d(u, t) \mid u \in U, t \in T\}$$

Definition 1.2.7 (Diameter of a Subset) Let (M, d) be a metric space. Let $U \subseteq M$. Define the diameter of U to be

$$\text{diam}(U) = \sup\{d(x, y) \mid x, y \in U\}$$

Definition 1.2.8 (Bounded Subsets)
Let X be a metric space. Let $U \subseteq X$ be a subset. We say that U is bounded if $\text{diam}(U) < \infty$.

Definition 1.2.9 (Neighbourhood of a Subset) Let (M, d) be a metric space. Let $U, T \subseteq M$. Let $r \in \mathbb{R}_{\geq 0}$. Define the neighbourhood of U with size r to be

$$N_r(U) = \{x \in M \mid d(U, \{x\}) \leq r\}$$

1.3 Points in a Subset

Definition 1.3.1 (Interior Points) Let M be a metric space. Let $x \in U \subset M$. We say that x is an interior point of U if there exists r such that

$$B_r(x) \subset U$$

Denote the set of all interior points by U°.

Definition 1.3.2 (Boundary) Let M be a metric space. Let $x \in U \subset M$. We say that x is a boundary point of U if for every r,

$$B_r(x) \cap U \neq \emptyset \text{ and } B_r(x) \cap M \setminus U \neq \emptyset$$

Denote the set of all boundary points by ∂U.

Proposition 1.3.3 Let M be a metric space. Let $U \subset M$. Then U is open if and only

$$U \cap \partial U = \emptyset$$

Proof Suppose that U is open. Let $x \in U \cap \partial U$. This means that $x \in \partial U$ and $B_r(x) \cap M \setminus U \neq \emptyset$ for all r. But this means that $B_r(x)$ cannot be a subset of U is it always contains point outside U, thus $x \notin U$ and thus $U \cap \partial U = \emptyset$.

Let $U \cap \partial U = \emptyset$. Let $x \in U$. Then $x \notin \partial U$. Thus by negation of the definition of boundary, there exists $r > 0$ such that $B_r(x) \cap M \setminus U = \emptyset$. Thus $B_r(x) \subseteq U$ and we are done. ∎

Definition 1.3.4 (Closure) Let M be a metric space. Let $U \subset M$. Define the closure of U to be

$$\overline{U} = U \cup \partial U$$

Proposition 1.3.5 Let M be a metric space. Let $U \subset M$. Then U is closed if and only

$$\overline{U} = U$$

1.4 Sequences, Limits and Continuity

Definition 1.4.1 (Sequences) Let X be a metric space. A sequence in X is an ordered set of numbers $x_0, x_1, x_2, \dots$ such that they all are in X. We denote this sequence by $(x_n)_{n\in\mathbb{N}}$.

Definition 1.4.2 (Convergence) Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ a metric space is said to converge to $x \in X$ if for every $\epsilon > 0$ there exists N such that $n > N$ implies

$$d(x_n, x) < \epsilon$$

Lemma 1.4.3 (Uniqueness of Limit) Let X be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ and $y \in X$, then $x = y$.

Proposition 1.4.4 Let X be a metric space. $U \subseteq X$ is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subseteq U$ that converges, it converges to some $x \in U$.

Proof Suppose that U is closed. Then $X \setminus U$ is open by definition. Let $\{x_n\} \subset U$ converge to $x \notin U$. Then $x \in X \setminus U$. By definition of convergence, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for $n > N$. But since $X \setminus U$ is open, there should be some ϵ such that $B_\epsilon(x) \subset X \setminus U$. In this case, we would have $x_n \in B_r(x) \subset X \setminus U$ which is a contradiction.
Suppose that the right hand side is true. Suppose for a contradiction that $X \setminus U$ is not open. Then for every $\epsilon > 0$, $B_\epsilon(x)$ is not a subset of $X \setminus U$ for some $x \in X \setminus U$. Let $y_k \in B_{1/k}(x)$ but $y_k \notin X \setminus U$. Then $y_k \in U$ and $y_k \to x \in X \setminus U$, a contradiction. ∎

Definition 1.4.5 (Continuity) Let (U, d_1), (V, d_2) be metric spaces. $f : U \to V$ is said to be continuous at $p \in U$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$x \in B_\delta(p) \implies f(x) \in B_\epsilon(f(p))$$

Or equivalently,

$$f(B_\delta(p)) \subset B_\epsilon(f(p))$$

Proposition 1.4.6 Let $f : X \to Y$ be a funciton between metric spaces. Then f is continuous at a if and only if for every sequence x_n such that $\lim_{n\to\infty} x_n \to a$, we have

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Proposition 1.4.7 Let U, V be metric spaces. Let $f : U \to V$ be a function. Then f is continuous if and only if for every open subsets $\Omega \subset V$, $f^{-1}(\Omega)$ is open.

Proof Suppose that f is continuous. Let $\Omega \subset V$ such that Ω is open. Then for every $p \in f^{-1}(V)$, there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset V$. By continuity, there exists $\delta > 0$ such that $f(B_{\delta(p)}) \subset B_\epsilon(f(p))$. This implies $B_\delta(p) \subset f^{-1}(B_\epsilon(f(p)))$. But also since $B_\epsilon(f(p)) \subset V$, we have

$$B_\delta(p) \subset f^{-1}(B_\epsilon(f(p))) \subset f^{-1}(V)$$

Since this is true for every p, $f^{-1}(V)$ must be open.
Now suppose that $\Omega \subset V$ is open imply $f^{-1}(\Omega)$ is open. Let $p \in \Omega$. Then there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset \Omega$. By assumption, we must have $f^{-1}(B_\epsilon(f(p)))$ is open. The fact that this is open means there exists $\delta > 0$ such that $B_\delta(p) \subset f^{-1}(B_\epsilon(f(p)))$. Then we have

$$f(B_\delta(p)) \subset B_\epsilon(f(p))$$

and we are done. ■

Proposition 1.4.8
Let X, Y, Z be metric spaces. Let $f : X \to Y$ and $g : Y \to Z$ be functions. If f and g are continuous, then $g \circ f$ is continuous.

1.5 Product Metrics

Definition 1.5.1 (The p Metric) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $p \in [1, \infty)$. Define the p metric to be the map $d_p : (X \times Y) \times (X \times Y) \to \mathbb{R}$ given by

$$d_p((x_1, y_1), (x_2, y_2)) = (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{1/p}$$

Proposition 1.5.2 Let (X, d_X) and (Y, d_Y) be metric spaces. Let $p \in [1, \infty)$. Then d_p is a metric on $X \times Y$.

Proposition 1.5.3 Let X, Y be metrics spaces. Let $((x_n, y_n))_{n\in\mathbb{N}}$ be a sequence in $X \times Y$. Let $(x, y) \in X \times Y$. Then $((x_n, y_n))_{n\in\mathbb{N}}$ converges to (x, y) if and only if $(x_n)_{n\in\mathbb{N}}$ converges to x and $(y_n)_{n\in\mathbb{N}}$ converges to y.

2 Geometry of Metric Spaces

2.1 Linear Isometries

Definition 2.1.1 (Distance Preserving Maps) Let (X, d_1) and (X, d_2) be metric spaces. Let $f : X \to Y$ be a function. We say that f is distance preserving if for any $p, q \in X$, we have

$$d_2(f(p), f(q)) = d_1(p, q)$$

Definition 2.1.2 (Isometries) Let (X, d_1) and (X, d_2) be metric spaces. Let $f : X \to Y$ be a distance preserving map. We say that f is an isometry if there exists a distance preserving map $g : Y \to X$ such that

$$g \circ f = \text{id}_X \quad \text{and} \quad f \circ g = \text{id}_Y$$

Lemma 2.1.3 Let (X, d_1) and (X, d_2) be metric spaces. Let $f : X \to Y$ be a distance preserving map. Then f is an isometry if and only if f is bijective.

2.2 Geodesics

Definition 2.2.1 (Geodesics) Let (X, d) be a metric space. Let $\gamma : I = [0, 1] \to X$ be a function. We say that γ is a geodesic if

$$d(\gamma(a), \gamma(b)) = |b - a|$$

for all $a, b \in I$.

Proposition 2.2.2 Let (X, d) be a metric space. Let $\gamma : I \to X$ be a geodesic. Then the following are true.

- γ is continuous and injective
- I is homeomorphic to $\text{im}(\gamma)$.

Definition 2.2.3 (Geodesic Metric Spaces) Let (X, d) be a metric space. We say that X is geodesic if for all $a, b \in X$, there exists a geodesic $\gamma : I \to X$ from a to b.

3 Connectedness

3.1 Definitions and Properties

Definition 3.1.1 (Connectedness) Let X be metric space. We say that X is disconnected if there exists open subsets $\emptyset \neq U, V \subseteq X$ such that $X = U \cup V$. We say that X is connected otherwise.

Notice that the definition of connectedness is implicit from the definition of disconnectedness. We give an explicit criteria to prove connectedness.

Proposition 3.1.2 Let X be a metric space. Then the following are equivalent.
- X is connected
- If $f : X \to \{0, 1\}$ is a continuous function then f is constant.
- The only subsets of X which are both open and closed are X and $\emptyset$.

Proof

- $(1) \iff (2)$: We prove the contrapositive. Namely X is disconnected if and only if there exists a continuous function $f; X \to \{0, 1\}$ that is non-constant. Suppose that X is disconnected. Then there exists $A, B \subset X$ that are open such that $A \cap B = \emptyset$ and $A \cup B = X$. Define f by
$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$
This function is continuous since every open set in $\{0, 1\}$ is mapped to an open set in X. It clearly is non-constant thus we are done.
Now suppose that $f : X \to \{0, 1\}$ is non-constant continuous function. Then by defining $A = f^{-1}(0)$ and $B = f^{-1}(1)$, we are done.
- $(1) \iff (3)$: Suppose that X is connected but there exists non-empty $A \subset X$ such that it is both open and closed. Then $X \setminus A$ is open and is disjoint with A and $A \cup X \setminus A = X$. This is a contradiction to X being connected.
Now suppose that the only subsets which are both open and closed are X and $\emptyset$. Suppose for a contradiction that X is disconnected. Then there exists open sets $A, B \subset X$ such that $A \cap B = \emptyset$ and $A \cup B = X$. Then clearly $B = X \setminus A$ is open, but $X \setminus A$ is the set difference of an open set thus it should be closed. Then B is both open and closed and we have a contradiction.

■

These two criteria will prove themselves to be particularly useful in proving theorems related to connectedness as well as begin able to identify concrete examples on connectedness.

Proposition 3.1.3 Let X be a metric space. Let $S \subset X$ be a metric subspace. Then S is connected if and only if the following is true. If U, V are open subsets of X and $U \cap V \cap S = \emptyset$ and $S \subseteq U \cup V$ implies $S \subseteq U$ or $S \subseteq V$.

Proof ■

Lemma 3.1.4 If $C \subset (X, d)$ is connected then so is any set S satisfying $C \subset S \subset \overline{C}$.

Lemma 3.1.5 Let X be a metric space. The countable union of connected subsets of X such that they have a nonempty intersection is connected.

Proof Suppose that $\{A_i | i \in I\}$ are all connected and has a nonempty intersection $x \in X$. Suppose that $f : X \to \{0, 1\}$ is a continuous function such that $f(x) = 0$. For every A_i, $f|_{A_i}$ is a constant function since f is continuous. This means that $f|_{A_i}(x) = 0$ for all $x \in A_i$. Then f when only restricted to the countable union of A_i, it will be identically zero. Thus we are done. ■

Proposition 3.1.6 Continuity preserves connectedness. That is, if $f : X \to Y$ is a continuous function between metric spaces and X is connected, then $f(X)$ is conneted.

Proof Suppose that $f(X)$ is disconnected. Then there exists a non-empty $A \subset f(X)$ that is both open and closed. By continuity, $f^{-1}(A)$ is also both open and closed, which is a contradiction since X is connected. ■

Proposition 3.1.7 The product of two connected spaces is connected.

Notice that none of the above propositions involve any notion of distance. This is baecause these are topological properties rather than metric properties, which will be discussed more on a topology course.

3.2 Path-Connectedness

Definition 3.2.1 (Path-Connected Metric Space) Let X be a metric space. Then we say that X is path-connected if the following are true. For any $a, b \in X$, there exists a continuous map $\gamma : [0, 1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. γ is called a path.

Lemma 3.2.2 Let X be a metric space. Define a relation $\sim$ on X as $a \sim b$ if and only if there exists a path $\gamma : [0, 1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Then $\sim$ is an equivalent relation.

Proposition 3.2.3 Every path-connected metric space is connected.

Proposition 3.2.4 A connected open subset of a normed space is path-connected.

3.3 Connectedness on $\mathbb{R}^n$

Theorem 3.3.1 A subset of $\mathbb{R}$ is connected if and only if it is an interval.

Below is a partial converse of path connectedness implying connectedness over $\mathbb{R}^n$.

Theorem 3.3.2 Connected open subsets of $\mathbb{R}^n$ are path connected.

Theorem 3.3.3 Open subsets of $\mathbb{R}^n$ have open connected components.

Theorem 3.3.4 A subset U of $\mathbb{R}$ is open if and only if it is the disjoint union of countably many open intervals.

4 Compactness

4.1 Compactness and Sequential Compactness

Definition 4.1.1 (Open Cover) An open cover of a metric space X is a collection $\mathcal{U}$ of open subsets of X such that E$X = \bigcup_{U \in \mathcal{U}} U$E

Definition 4.1.2 (Compact Metric Spaces) Let X be a metric space. Let $K \subseteq X$. K is said to be compact if every open cover of K contains a finite subcover.

Definition 4.1.3 (Lebesgue Number) Let $\mathcal{U}$ be an open cover of a metric space X. A number $\delta > 0$ is called a Lebesgue number for $\mathcal{U}$ if for any $x \in X$ there exists $U \in \mathcal{U}$ such that $B_\delta(x) \subset U$.

Lemma 4.1.4 Every open cover $\mathcal{U}$ of a compact metric space X has a Lebesgue number.

Definition 4.1.5 (Sequential Compactness) Let X be a metric space. Then X is said to be sequentially compact if any sequence of elements in X has a convergent subsequence.

Lemma 4.1.6 If X is sequentially compact that any open cover of X has a Lebesgue number.

Proposition 4.1.7 Let (X, d) be a metric space. Then the following are equivalent.

- X is compact
- X is sequentially compact
- X is closed and totally bounded

4.2 Properties of Compactness

Proposition 4.2.1 A compact subset of a metric space is closed.

Proof Let $K \subset X$ be compact. Let $a \in X \setminus K$. For every $x \in K$, $B_{d(a,x)/2}(a)$ and $B_{d(a,x)/2}(x)$ are disjoint open balls. Then $\{B_{d(a,x)/2}(x) | x \in K\}$ is an open cover of K. Since K is compact, it has a finite subcover $B_{d(a,x_1)/2}(x_1), \dots, B_{d(a,x_n)/2}(x_n)$. But

$$K \cap \bigcap_{k=1}^{n} B_{d(a,x_k)/2}(a) = \emptyset$$

since the two types of balls are disjoint. Thus K is closed. ■

Proposition 4.2.2 A compact subset of a metric space is bounded.

Proof Let $a \in X$. Let $x \in K$. Then $x \in B_r(a)$ for all $r > d(a, x)$. Thus K is covered by the collection of open balls $B_r(a)$. Thus it has a finite subcover $B_{r_1}(a), \dots, B_{r_n}(a)$. Thus

$$K \subset \bigcup_{k=1}^{n} B_{r_k}(a) = B_{\max\{r_1,\dots,r_n\}}(a)$$

and we are done. ■

Proposition 4.2.3 Let X be a compact metric space. Let $C \subseteq X$ be a closed subset. Then C is compact.

Proof Let U be a cover of C by open subsets of X. Then $U \cup X \setminus C$ is an open cover of X, thus has a finite subcover. This provides an open subcover of C since $X \setminus C$ is open and you can remove this element fromt the subcover. ∎

4.3 Compactness and Continuity

Theorem 4.3.1 Continuity preserves compactness.

Proof Let $f : X \to Y$ be a continuous function between metric spaces. Suppose that X is compact. ∎

Lemma 4.3.2 Let X, Y be metric spaces. A sequence $\{(x_n, y_n)\} \subset X \times Y$ converges if and only if $\{x_n\} \subset X$ converges in X and $\{y_n\} \subset Y$ converges in Y.

Proposition 4.3.3 The product of two compact metric spaces is compact.

Theorem 4.3.4 (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof Let K be a compact subset of $\mathbb{R}^n$. K is closed by proposition 3.2.1 and K is bounded by proposition 3.2.2.
Let K be a closed and bounded subset of $\mathbb{R}^n$. If K is bounded then $K \subset [-r, r]^n$ for some $r > 0$. I claim that $[-r, r]^n$ is compact. Once it is compact, applying 3.2.3 to the closed subset K of $[-r, r]^n$, we have that K is compact.
Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[-r, r]$ by bolzano weierstrass it has a convergent subsequence. Thus $[-r, r]$ is sequentially compact and thus compact. Using the productivity of compact metric spaces, we have that $[-r, r]^n$ is compact thus we are done. ∎

4.4 Uniform Continuity

Definition 4.4.1 (Uniformly Continuous) A map $f : X \to Y$ is uniformaly continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$$

for any $x, y \in X$.

Theorem 4.4.2 A continuous map from a compact metric into a metric space is uniformly continuous.

5 Completeness

5.1 Motivation and Definitions

Definition 5.1.1 (Cauchy Sequence) We say that $\{x_n\} \subset (X, d)$ is a Cauchy sequence if for every $\epsilon > 0$, there exists some N such that $d(x_n, x_m) < \epsilon$ for all $n, m > \epsilon$.

Proposition 5.1.2 Every convergent sequence is Cauchy.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in a metric space X. Let $\epsilon > 0$, then from convergence we have that for $d(x_n, x) < \frac{\epsilon}{2}$ for all $n > N$ for some $N \in \mathbb{N}$. Then choosing the same N, we have that

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

thus we are done. ■

We now give the definition of a complete space in terms of Cauchy sequences.

Definition 5.1.3 (Complete Spaces) A metric space (X, d) is complete if any Cauchy sequence in X converges.

Proposition 5.1.4 Every compact metric space is complete.

Proof Suppose that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in a compact metric space X. Then X being sequentially compact means that there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ such that it converges in X. But then clearly

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_j}, x)$$

implies that $x_n \to x$ since in the inequality, the first part of the sum corresponds to the sequence being Cauchy and thus tends to 0, while the latter part correponds to the subsequence being convergent and thus tends to 0. ■

5.2 Properties of Complete Spaces

Proposition 5.2.1 A subspace of a metric space is complete if and only if it is closed under a complete metric space.

Proof Suppose that X is a metric space and $U \subset X$ is a complete metric space. Let $(x_n)_{n\in\mathbb{N}} \subset U$ and that $x_n \to x \in X$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy thus it convergence to some $y \in U$. We will show that in fact $x = y$. This is true from the fact that

$$d|_U(x_n, y) = d(x_n, y)$$

Thus $(x_n)_{n\in\mathbb{N}}$ is in fact a sequence that converges in U. This shows that U is closed.
Now suppose that U is closed under a complete metric space X. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in U. Then trivially it is also a Cauchy sequence in X and thus is convergent. Since U is closed, the limit is necessarily in U and thus U is complete. ■

Theorem 5.2.2 (Cantor's Intersection Theorem) Let X be a complete metric space. Let $S_1 \supseteq S_2 \supseteq \dots$ form a nested sequence of non-empty closed sets in X with the property that $\text{diam}(S_n) \to 0$ as $n \to \infty$. Then

$$\bigcap_{n=1}^{\infty} S_n \neq \emptyset$$

Proof For each $N \in \mathbb{N}$, choose $x_N \in S_N$. Then for all $n > N$, $x_n \in S_N$. Thus for $n, m > N$, we have that $d(x_n, x_m) \leq \text{diam}(S_n)$. It follows that $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Thus $x_n \to x$ for some $x \in X$. Since each S_n is closed and $x_n \in S_N$ for all $n > N$, we must have that $x \in S_n$ for each n. Thus

$x \in \bigcap_{k=1}^{\infty} S_n$ is nonempty. ∎

Below are a few examples of complete spaces.

Proposition 5.2.3 $\mathbb{R}^n$ and $\mathbb{C}$ are both complete.

Proof Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^n$. Denote the ith component of x_k by $x_{k,i}$. Then for every $\epsilon > 0$, there exists N such that

$$\|x_k - x_m\| = \left(\sum_{i=1}^{n} |x_{k,i} - x_{m,i}|^2\right)^{\frac{1}{2}} < \epsilon$$

for $k, m > N$. In particular, we have that each individual

$$|x_{k,i} - x_{m,i}| < \epsilon$$

for $m, n > N$. Thus $(x_{k,i})_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. But we know that Cauchy sequences in $\mathbb{R}$ converges, thus $(x_{k,i})_{k\in\mathbb{N}}$ converges to $x_i \in \mathbb{R}$. Now define $x = (x_1, \dots, x_n)$, then

$$\|x_k - x\| = \left(|x_{k,i} - x_i|^2\right)^{\frac{1}{2}} < n\epsilon$$

by convergence of each individual component. Thus $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence.
The proof for $\mathbb{C}$ is the same in considering $\mathbb{R}^2$. ∎

Proposition 5.2.4 Every normed vector space is complete.

5.3 Completion

The goal of this section is to attempt to complete a metric space by adding in the missing limits of a metric space.

Definition 5.3.1 (Space of Bounded Real Functions) Denote $B(X)$ the space of all bounded real valued functions on a metric (topological) space X. This means that

$$B(X) = \{f : X \to \mathbb{R} \,|\, |f| \leq M \text{ for some } M \in \mathbb{R}\}$$

Proposition 5.3.2 The metric space with distance induced by the supremum norm

$$\|f\|_\infty = \sup_{x\in X} |f(X)|$$

for $f \in B(X)$ is complete.

Proof Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $B(X)$. Then for every $\epsilon > 0$, there exists N such that

$$\|f_n - f_m\|_\infty = \sup_{x\in X} |f_n(x) - f_m(x)| < \epsilon$$

for all $n, m > N$. In particular, for each $x \in X$, the property of supremum implies that $|f_n(x) - f_m(x)| < \epsilon$ for $n, m > N$. Thus $(f_n(x))_{n\in\mathbb{N}} \subset \mathbb{R}$ is Cauchy for each x. Since $\mathbb{R}$ is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges for each $x \in X$.
Now define the function $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Then fix $\epsilon > 0$, we have that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n > N$ by letting $m \to \infty$ from the fact that $|f_n(x) - f_m(x)| < \epsilon$. This N does not depend on x. Fix $\epsilon = 1$, then there exists $N_1 \in \mathbb{N}$ such that

$$\begin{aligned} |f(x) - f_n(x)| &\le |f(x) - f_{N_1}(x)| \\ &\le 1 + |f_{N_1}(x)| \end{aligned}$$

for all $x \in X$ and $n > N_1$ thus f is bounded. This means that $f \in B(X)$ and that $\|f_n - f\|_\infty < \epsilon$ for all $n > N$. ■

Proposition 5.3.3 Any metric space X can be isometrically embedded into the complete metric space $B(X)$.

5.4 Compactness, Completeness and Totally Bounded

Definition 5.4.1 (Totally Bounded) A metric space X is totally bounded if for any $\epsilon > 0$, there exists $B_\epsilon(p_k)$ for $k \in \{1, \dots, n\}$ such that

$$X \subseteq \bigcup_{k=1}^{n} B_\epsilon(p_k)$$

Theorem 5.4.2 A subspace Y of a metric space X that is complete is compact if and only if it is closed and totally bounded.

Theorem 5.4.3 A subspace Y of a complete metric space is totally bounded if and only if its closure is compact.

5.5 Contraction Mapping and Completion

Definition 5.5.1 (Lipschitz Continuous) Let (X, d_X) and (Y, d_Y) be metric spaces and suppose that $f : X \to Y$. We say that f is a Lipschitz map if there is a constant $K \ge 0$ such that

$$d_Y(f(x), f(y)) \le K d(x, y)$$

for all x, y in X.
If $Y = X$ and $K \in [0, 1)$ then f is a contraction mapping.

Lemma 5.5.2 If $f : X \to Y$ is Lipschitz continuous then it is continuous.

Theorem 5.5.3 (Contraction Mapping Theorem) Let X be a nonempty complete metric space and suppose that $f : X \to X$ is a contraction. Then f has a unique fixed point, meaning there is a unique $x \in X$ such that $f(x) = x$.

Proof Let $x_0 \in X$ and define a sequence by $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Then we have that

$$d(x_{n+1}, x_n) \le K d(x_n, x_{n-1}) \le \cdots \le K^n d(x_1, x_0)$$

Then for any $k > n$, we have that

$$d(x_k, x_n) \leq \sum_{i=n}^{k-1} d(x_{i+1}, x_i)$$
$$\leq \sum_{i=n}^{k-1} K^i d(x_1, x_0)$$
$$\leq \frac{K^i}{1-K} d(x_1, x_0)$$

This is Cauchy since we can choose $\epsilon > 0$ such that $\frac{K^i}{1-K} < \epsilon$. Since X is complete, we have that $x_n \to x$ for some $x \in X$. Since f is continuous we have that $f(x_n) \to f(x)$. Now taking limits on

$$x_{n+1} = f(x_n)$$

we have that $x = f(x)$.
To prove uniqueness, note that if $f(x) = x$ and $f(y) = y$, then

$$d(x, y) = d(f(x), f(y)) \leq Kd(x, y)$$

which implies that $(1 - K)d(x, y) = 0$. Thus $x = y$. ■

Another name for this theorem would be Banach's Fixed Point Theorem.

Theorem 5.5.4 (Picard-Lindelof Theorem) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous with

$$|f(x) - f(y)| \leq L|x - y|$$

where $x, y \in \mathbb{R}^n$. Then for any $x_0 \in \mathbb{R}^n$, the differential equation

$$\frac{dx}{dt} = f(x)$$

with initial condition $x(0) = x_0$ has a unique solution on $[-t, t]$ for any $Lt < 1$.

5.6 Cantor's Theorem

Theorem 5.6.1 If X is a complete metric space and $\{F_n | n \in \mathbb{N}\}$ is a collection of open dense subsets of X, then

$$F = \bigcap_{k=1}^{\infty} F_n t$$

is dense in X. Equivalently, if $\{G_n | n \in \mathbb{N}\}$ is a collection of nowhere dense subsets of a nonempty complete metric space X, then

$$\bigcup_{k=1}^{\infty} F_k \neq X$$

Lemma 5.6.2 The Cantor set is uncountable.