Homological Algebra

Labix

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Abstract

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1 Abelian Categories and its Properties

1.1 Additive Categories

Definition 1.1.1 (Pre-Additive Categories) Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is pre-additive if the following is true.

- For any $X, Y \in \mathcal{C}$, the Hom set
$$\text{Hom}_{\mathcal{C}}(X, Y) \in \mathbf{Ab}$$
has the structure of an abelian group.
- For any $X, Y, Z \in \mathcal{C}$, the composition of morphisms
$$\circ : \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) \to \text{Hom}_{\mathcal{C}}(X, Z)$$
is bilinear. This means that if $f, g \in \text{Hom}_{\mathcal{C}}(X, Y)$ and $h, k \in \text{Hom}_{\mathcal{C}}(Y, Z)$ are morphisms, then
$$h \circ (f + g) = h \circ f + h \circ g \quad \text{and} \quad (h + k) \circ f = h \circ f + k \circ f$$

We also say that $\mathcal{C}$ is enriched over $\mathbf{Ab}$. This relates to enriched category theory.

Definition 1.1.2 (Additive Categories) A category $\mathcal{A}$ is additive if in addition to being pre-additive, it also satisfies the following:

- $\mathcal{A}$ has a zero object, denoted 0
- $\mathcal{A}$ has finite products

Lemma 1.1.3 Let $\mathcal{A}$ be an additive category. Then the coproducts and products of $\mathcal{A}$ coincide. In other words, there is an isomorphism
$$X \times Y \cong X \amalg Y$$
for any $X, Y \in Obj(\mathbb{A})$.

1.2 Abelian Categories

Definition 1.2.1 (Abelian Categories) An additive category $\mathcal{A}$ is said to be abelian if it satisfies the following:

- Every morphism in $\mathcal{A}$ has a kernel and a cokernel
- Every monic morphism is the kernel of its cokernel
- Every epic morphism is the cokernel of its kernel

Theorem 1.2.2 Let R be a ring. Then the category ${}_R\mathbf{Mod}$ of R-modules is an abelian category.

Theorem 1.2.3 Let $\mathcal{A}$ be an abelian category whose objects form a set. Then there exists a ring R and an exact functor
$$F : \mathcal{A} \to {}_R\mathbf{Mod}$$
which is an embedding on objects and an isomorphism on Hom sets.

Definition 1.2.4 (Injectivity and Surjectivity) Let $f : X \to Y$ be a morphism in an abelian category.

- We say that f is injective if $\ker(f) = 0$
- We say that f is surjective if $\text{coker}(f) = 0$

In particular, these notions coincide that of epics and monics in an abelian category.

Proposition 1.2.5 Let $f : X \to Y$ be a morphism in an abelian category. Then the following are true.

- f is injective if and only if f is a monomorphism
- f is surjective if and only if f is epimorphism

1.3 Short Exact Sequences

Definition 1.3.1 (Short Exact Sequence) Let $\mathcal{A}$ be an abelian category. A short exact sequence in $\mathcal{A}$ consists of three objects $A, B, C \in \mathcal{A}$ and two morphisms $f : A \to B$ and $g : B \to C$ in $\mathcal{A}$ written in diagram form as

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

such that the following are true:

- f is a monomorphism
- g is an epimorphism
- $\mathrm{im}(f) = \ker(g)$

Definition 1.3.2 (Split Exact Sequence) Let $\mathcal{A}$ be an abelianc category. Let $A, B, C \in \mathcal{A}$ such that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence. We say that it is split exact if $B \cong A \oplus C$.

The following is an important equivalent characterization of split exact sequence.

Theorem 1.3.3 (The Splitting Lemma) Let $\mathcal{A}$ be an abelianc category. Let $A, B, C \in \mathcal{A}$. Then the following are equivalent for a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

- The short exact sequence is split exact sequence
- There exists a morphism $p : B \to A$ such that $p \circ f = \mathrm{id}_A$
- There exists a morphism $s : C \to B$ such that $g \circ s = \mathrm{id}_C$

Lemma 1.3.4 (Five Lemma) Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "l"] & B \arrow[r, "g"] \arrow[d, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "j"] \arrow[d, "p"] & E \arrow[d, "q"] \\
A' \arrow[r, "r"] & B' \arrow[r, "s"] & C' \arrow[r, "t"] & D' \arrow[r, "u"] & E'
\end{tikzcd}$$

where all the objects lie in an abelian group $\mathcal{A}$. If the two rows are exact, $m : B \to B', p : D \to D'$ are isomorphisms, $l : A \to A'$ is an epimorphism and $q : E \to E'$ is an monomorphism, then n is an isomorphism.

Lemma 1.3.5 (Snake Lemma) Consider the commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

where all the objects lie in an abelian group $\mathcal{A}$. If the two rows are exact, then there is an exact sequence relating the kernels and cokernels of a, b, c

$$\ker(a) \longrightarrow \ker(b) \longrightarrow \ker(c) \xrightarrow{d} \operatorname{coker}(a) \longrightarrow \operatorname{coker}(b) \longrightarrow \operatorname{coker}(c)$$

where d is called the connecting homomorphism.

1.4 Exact Functors

Definition 1.4.1 (Additive Functors) Let $\mathcal{A}, \mathcal{B}$ be abelian categories. We say that a functor $F : \mathcal{A} \to \mathcal{B}$ is additive if for every $X, Y \in \mathcal{A}$, the map

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \to \operatorname{Hom}_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Proposition 1.4.2 Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F preserves split exact sequences.

Definition 1.4.3 (Exact Functors) Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$. We say that F is exact if the sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact.

Lemma 1.4.4 Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. If F is an equivalence of categories, then F is an exact functor.

Definition 1.4.5 (Left and Right Exact Functors) Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}$.

- We say that F is right exact if the sequence

$$F(A) \to F(B) \to F(C) \to 0$$

is exact.
- We say that F is left exact if the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact.

Lemma 1.4.6 (L) t $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F is exact if and only if F is left exact and right exact.

Proposition 1.4.7 Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Then the following are true.

- F is additive and left exact if and only if F preserves all finite limits.
- F is additive and right exact if and only if F preserves all finite colimits.

Proposition 1.4.8 Let $\mathcal{A}$ be an abelian category. Let $M \in \mathcal{A}$. Then the following are true.

- The covariant functor $\operatorname{Hom}(M, -) : \mathcal{A} \to \mathbf{Ab}$ is left exact.
- The contravariant functor $\operatorname{Hom}(-, M) : \mathcal{A} \to \mathbf{Ab}$ is right exact.

Let $\mathcal{C}$ be a category and let $\mathcal{J}$ be a diagram. Recall that if $\mathcal{C}$ is complete, then the limit can be thought of as a functor $\lim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. Dually, if $\mathcal{C}$ is cocomplete, the colimit can be thought of as a functor $\operatorname{colim}_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$. If $\mathcal{C}$ is an abelian category, it makes sense to ask whether these two functors preserve exact sequences.

Proposition 1.4.9 Let $\mathcal{A}$ be an abelian category. Let $\mathcal{J}$ be a small category. Then the following are true.

- If $\mathcal{A}$ is complete, then the functor $\lim_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ is left exact
- If $\mathcal{A}$ is cocomplete, then the functor $\operatorname{colim}_{\mathcal{J}} : \mathcal{C}^{\mathcal{J}} \to \mathcal{C}$ is right exact

1.5 Freyd-Mitchell Embedding Theorem

Theorem 1.5.1 (Freyd-Mitchell Embedding Theorem) Let $\mathcal{A}$ be a small abelian category. Then there exists a ring R and an exact, fully faithful functor $F : \mathcal{A} \to {}_R\mathbf{Mod}$. This means that there is an isomorphism of sets

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(F(M), F(N))$$

Such a functor allows us think about diagrams in $\mathcal{A}$ as if they were diagrams in ${}_R\mathbf{Mod}$

Lemma 1.5.2 The Freyd-Mitchell embedding preserves kernels and cokernels. Moreover, it maps the zero object to the zero object.

2 Chain Complexes in an Abelian Category

2.1 The Category of Chain Complexes

Definition 2.1.1 (Chain Complexes) Let $\mathcal{A}$ be an abelian category. A chain complex $(C_\bullet, \partial_\bullet)$ in $\mathcal{A}$ is a family of objects $C_n \in \mathcal{A}$ for $n \in \mathbb{Z}$ and morphisms $\partial_n : C_n \to C_{n-1}$ in $\mathcal{A}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all n.

In other words, we have the diagram:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

for which we require that

$$\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$$

for each n.

Definition 2.1.2 (Chain Map) Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes in an abelian category $\mathcal{A}$. A chain map $f_\bullet : C_\bullet \to C'_\bullet$ is a family of maps

$$f_n : C_n \to C'_n$$

in $\mathcal{A}$ such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for all n.

In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 2.1.3 Let $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to E_\bullet$ be two chain maps. Then $g_\bullet \circ f_\bullet$ is also a chain map.

Definition 2.1.4 (Category of Chain Complexes) Let $\mathcal{A}$ be an abelian category. Define

$$\mathbf{Ch}(\mathcal{A})$$

to be the category of chain complexes where

- The objects are chain complexes of objects in $\mathcal{A}$.
- The morphisms are chain maps.
- Composition is given by composition of functions.

Definition 2.1.5 (Variants of the Category of Chain Complexes) Let $\mathcal{A}$ be an abelian category.

- Define the category
$$\mathbf{Ch}_{\geq 0}(\mathcal{A})$$
of non-negative chain complexes to be the full subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of chain complexes that is 0 in negative degrees.
- Define the category
$$\mathbf{Ch}_{\leq 0}(\mathcal{A})$$
of non-positive chain complexes to be the full subcategory of $\mathbf{Ch}(\mathcal{A})$ consisting of chain complexes that is 0 in positive degrees.

Theorem 2.1.6 Let $\mathcal{A}$ be an abelian category. Then $\mathbf{Ch}(\mathcal{A})$ is also an abelian category.

Definition 2.1.7 (The Homology Groups of a Chain Complex) Let $(C_\bullet,\partial_\bullet)$ be a chain complex in an abelian category $\mathcal{A}$. Define $Z_n(C_\bullet)=\ker(\partial_n)$ and $B_n(C_\bullet)=\operatorname{im}(\partial_{n+1})$. Define the nth homology of $(C_\bullet,\partial_\bullet)$ to be

$$H_n(C_\bullet)=\frac{Z_n(C_\bullet)}{B_n(C_\bullet)}=\frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$$

Elements of $Z_n(C_\bullet)=\ker(\partial_n)$ are called n-cycles and elements of $B_n(C_\bullet)=\operatorname{im}(\partial_{n+1})$ are called n-boundaries.

Definition 2.1.8 (The Homology Functor) Let $\mathcal{A}$ be an abelian category. Let $n\in\mathbb{N}$. Define the homology functors

$$H_n(-):\mathbf{Ch}(\mathcal{A})\to\mathcal{A}$$

for each n as follows.

- For each chain complex $C_\bullet$ over $\mathcal{A}$, $H_n(C_\bullet)$ is the nth homology group of $C_\bullet$.
- For each chain map $f:C_\bullet\to C'_\bullet$,

$$H_n(f):H_n(C_\bullet)\to H_n(C'_\bullet)$$

  is the induced map on homology.

Lemma 2.1.9 A chain map $f_\bullet:C_\bullet\to C'_\bullet$ induces group homomorphisms

$$f_*:H_n(C_\bullet)\to H_n(C'_\bullet)$$

between homology groups defined by $f_*([z])=[f(z)]$.

Proof For every map $f_n:C_n\to C'_n$, we can restrict the domain to cycles so that we obtain a map $f_n:Z_n(C_\bullet)\to C'_n$. Using the relation given between the boundary operator and the family of maps, we check that this map descends to a map in homology.

Firstly, $f_n(Z_n(C_\bullet))\subseteq Z_n(C'_\bullet)$. Indeed let $x\in Z_n(C_\bullet)$. Then we have that

$$\partial'_n(f_n(x))=f_{n-1}(\partial_n(x))=f_{n-1}(0)=0$$

which means that $f_n(x)$ lies in the kernel of ∂'_n. Now we have a map $f_n:Z_n(C_\bullet)\to Z_n(C'_\bullet)$. At the same time, f_n also restricts to a map $f_n:B_n(C_\bullet)\to B_n(C'_\bullet)$. Indeed if $b\in B_n(C_\bullet)$, then there exists some $c\in C_{n+1}$ such that $\partial_{n+1}(c)=b$. Applying f_n on both sides give

$$\begin{aligned}f_n(\partial_{n+1}(c))&=f_n(b)\\ \partial'_{n+1}(f_{n+1}(c))&=f_n(b)\end{aligned}$$

This means that $f_n(b)$ is the boundary of the element $f_{n+1}(c)\in C_{n+1}$, and so f_n restricts to a map of boundaries. Now $f_n:H_n(C_\bullet)\to H_n(C'_\bullet)$ is well defined. Indeed if $b_1,b_2\in B_n(C_\bullet)$ lie in the same coset, then $b_1B_n(C_\bullet)=b_2(C_\bullet)$ so that $b_1-b_2\in B_(C_\bullet)$. Then $f_n(b_1-b_2)\in B_n(C'_\bullet)$ so that $f_n(b_1)$ and $f_n(b_2)$ lie in the same coset. Thus f_n is well defined. ■

It is customary to drop the $n\in\mathbb{N}$ in the notation as it is usually implicit. So for example the condition for chain map becomes $\partial'\circ f=f\circ\partial$.

We then have functoriality of the induced map.

Proposition 2.1.10 Let $f_\bullet:C_\bullet\to D_\bullet$ and $g_\bullet:D_\bullet\to E_\bullet$ be two chain maps. Then $g_\bullet\circ f_\bullet$ is also a chain map. Moreover, the induced map on the homology groups satisfy the following:

- $g_*\circ f_*=(g\circ f)_*$
- $\mathrm{id}_*=\mathrm{id}_{H_n}$

Proof Firstly, we have that

$$\partial \circ g_n \circ f_n = g_{n-1} \circ \partial \circ f_n = g_{n-1} \circ f_{n-1} \circ \partial$$

so that $g \circ f$ is indeed a chain map.

We have that $g_*(f_*([z])) = g_*([f(z)]) = [g(f(z))] = (g_* \circ f_*)([z])$. Also, we have that

$$\text{id}_*([z]) = [\text{id}(z)] = [z] = \text{id}_{H^n}([z])$$

and so we conclude. ■

Definition 2.1.11 (Quasi-Isomorphisms) Let $C_\bullet, D_\bullet$ be chain complexes. Let $f : C_\bullet \to D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if the induced map in homology $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphism for all $n \in \mathbb{N}$.

2.2 The Category of Cochain Complexes

Definition 2.2.1 (Cochain Complexes) Let $\mathcal{A}$ be an abelian category. A cochain complex $(C^\bullet, \partial^\bullet)$ is a family objects $C^n \in \mathcal{A}$ for $n \in \mathbb{Z}$ and morphisms $\delta^n : C^{n-1} \to C^n$ such that $\delta^{n+1} \circ \delta^n = 0$ for all n. In other words, we have the diagram:

$$\cdots \longleftarrow C^{n+1} \xleftarrow{\delta^{n+1}} C^n \xleftarrow{\delta^n} C^{n-1} \longleftarrow \cdots$$

Notice that algebraically, there is no difference between a chain complex and a cochain complex, other than the fact that the boundary maps run in the other direction. For $(C_\bullet, \partial_\bullet)$ a chain complex, we can form a cochain complex by setting $C^n = C_{-n}$ and then using the same boundary maps.

Definition 2.2.2 (Category of Cochain Complexes) Let $\mathcal{A}$ be an abelian category. Define the category of cochain complexes

$$\mathbf{CCh}(\mathcal{A})$$

to consist of the following data.

- The objects are the cochain complexes over $\mathcal{A}$
- The morphisms are the chain maps
- Composition is given by the composition of maps on all levels of the chain map.

Because cochain complexes is the same data as a chain complex, we can also define homology groups for cochain complexes. We call them the cohomology groups of the cochain complex.

Definition 2.2.3 (Cohomology Groups) Let $\mathcal{A}$ be an abelian category. Let $(C^\bullet, \partial^\bullet)$ be a cochain complex. Define the nth cohomology group of the cochain complex to be

$$H^n(C^\bullet, \partial^\bullet) = \frac{\ker(\partial^{n+1} : C^n \to C^{n+1})}{\text{im}(\partial^n : C^{n-1} \to C^n)} = H_n(C^\bullet, \partial^\bullet)$$

for $n \in \mathbb{N}$.

The construction of cohomology groups are functorial because the homology groups of a chain complex also are.

Definition 2.2.4 (The Cohomology Functor) Let $\mathcal{A}$ be an abelian category. Let $n \in \mathbb{N}$. Define the nth cohomology functor to be the functor

$$H^n : \mathbf{CCh}(\mathcal{A}) \to \mathcal{A}$$

to consist of the following data.

- For each cochain complex $(C^\bullet, \delta^\bullet)$ over $\mathcal{A}$, define $H^n(C^\bullet, \delta^\bullet)$ to be the nth cohomology group of the cochain complex.
- For each chain map $f^\bullet : (C^\bullet, \delta^\bullet) \to (D^\bullet, \delta^\bullet)$, the induced map

  $$f^* : H^n(C^\bullet, \delta^\bullet) \to H^n(C^\bullet, \delta^\bullet)$$

  is defined by $[z] \mapsto [f^n(z)]$.

2.3 Chain Homotopy

Definition 2.3.1 (Chain Homotopy) Let $\mathcal{A}$ be an abelian category. Let $a_\bullet, b_\bullet : C_\bullet \to C'_\bullet$ be two chain maps in $\mathbf{Ch}(\mathcal{A})$. Then a chain homotopy from a to b is a collection of morphisms

$$\eta_n : C_n \to C'_{n+1}$$

in $\mathcal{A}$ such that

$$b_n - a_n = \partial'_{n+1}\eta_n + \eta_{n-1}\partial_n$$

for all $n \in \mathbb{Z}$. In this case, a and b are said to be chain homotopic.

In other words, we have the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "b_{n+1}-a_{n+1}"] \arrow[dl, "\eta_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "b_n-a_n"] \arrow[dl, "\eta_n"'] & C_{n-1} \arrow[r] \arrow[d, "b_{n-1}-a_{n-1}"] \arrow[dl, "\eta_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

In this case we write $f \simeq g$.

Lemma 2.3.2 Let a, b be chain homotopic. Then their induced maps in homology are equal. Meaning

$$a_n = b_n : H_n(X) \to H_n(Y)$$

Proof Let $c \in \ker(\partial_n)$ be an n-cycle. Using the equation for chain homotopy, we have that

$$\begin{aligned} b(c) - a(c) &= \partial'_{n+1}(\eta_n(c)) + \eta_{n-1}(\partial(c)) \\ &= \partial'_{n+1}(\eta(c)) \end{aligned}$$

is a boundary in $\operatorname{im}(\partial'_{n+1}) \subseteq C'_n$. Thus $b_n(c)$ and $a_n(c)$ are of the same coset in $H_n(X)$. ■

Proposition 2.3.3 Let $\mathcal{A}$ be an abelian group. Let $f_1, g_1 : C_\bullet \to D_\bullet$ and $f_2, g_2 : D_\bullet \to E_\bullet$ be chain maps in $\mathbf{Ch}(\mathcal{A})$. If f_1 and g_1 are chain homotopic and f_2 and g_2 are chain homotopic, then $f_2 \circ f_1$ is chain homotopic to $g_2 \circ g_1$.

Proof The chain homotopies between f_1 and g_1 imposes the identity

$$\partial\eta + \eta\partial = g_1 - f_1$$

for $\eta : C_\bullet \to D_\bullet$ the given chain homotopy. Similarly, for $\nu : D_\bullet \to E_\bullet$ we have the identity

$$\partial\nu + \nu\partial = g_2 - f_2$$

Then we have that

$$\begin{aligned} g_2 \circ g_1 - f_2 \circ f - 1 &= g_2 \circ g_1 - g_2 \circ f_1 + g_2 \circ f_1 - f_2 \circ f_1 \\ &= g_2(g_1 - f_1) + (g_2 - f_2) \circ f_1 \\ &= g_2(\partial\eta + \eta\partial) + (\partial\nu + \nu\partial) \circ f_1 \\ &= \partial g_2\eta + g_2\eta\partial + \partial\nu f_1 + \nu f_1\partial \\ &= \partial(g_2\eta + \nu f_1) + (g_2\eta + \nu f_1)\partial \end{aligned}$$

Thus $g_2\eta + \nu f_1 : C_n \to E_{n+1}$ would be a valid chain homotopy from $g_2 \circ g_1$ to $f_2 \circ f_1$. ■

Lemma 2.3.4 Let $\mathcal{A}$ be an abelian category. Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathbf{Ch}(\mathcal{A})$. Then the relation $\simeq$ on the chain maps from $C_\bullet$ to $D_\bullet$ is an equivalence relation.

Definition 2.3.5 (Chain Homotopy Equivalence) Let $\mathcal{A}$ be an abelian category. Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathbf{Ch}(\mathcal{A})$. We say that they are chain homotopy equivalence if there exists chain maps $a_\bullet : C_\bullet \to D_\bullet$ and $b_\bullet : C_\bullet \to D_\bullet$ such that there are chain homotopies

$$b_\bullet \circ a_\bullet \simeq \mathrm{id}_{C_\bullet} \quad \text{and} \quad a_\bullet \circ b_\bullet \simeq \mathrm{id}_{D_\bullet}$$

Lemma 2.3.6 Let $\mathcal{A}$ be an abelian category. Let $C_\bullet$ and $D_\bullet$ be chain homotopy equivalent in $\mathbf{Ch}(\mathcal{A})$. Then the chain maps induces an isomorphism

$$H_n(C_\bullet) \cong H_n(D_\bullet)$$

in all degrees $n \in \mathbb{N}$.

Proof We know that $b_\bullet \circ a_\bullet \simeq \mathrm{id}_{C_\bullet}$ which means that they induce the same map:

$$b_* \circ a_* = \mathrm{id}_{H_n(C_\bullet)}$$

Similarly the chain homotopies $a_\bullet \circ b_\bullet \simeq \mathrm{id}_{D_\bullet}$ induce the same map

$$a_* \circ b_* : \mathrm{id}_{H_n(D_\bullet)}$$

as the identity. Then these two identities mean that a_* is both injective and surjective. ■

Proposition 2.3.7 Let $\mathcal{A}$ be an abelian category. Then chain homotopy equivalence defines an equivalence relation on all chain complexes in $\mathbf{Ch}(\mathcal{A})$.

Proof Clearly any chain complex is chain homotopy equivalent to itself by the identity map. If $C_\bullet$ and $D_\bullet$ are chain homotopy equivalent by the chain maps $a_\bullet : C_\bullet \to D_\bullet$ and $b_\bullet : D_\bullet \to C_\bullet$, then we have the identities $b_\bullet \circ a_\bullet = \mathrm{id}_{C_\bullet}$ and $a_\bullet \circ b_\bullet = \mathrm{id}_{D_\bullet}$. We can then read them in reverse so that $D_\bullet$ and $C_\bullet$ are chain homotopy equivalence by the maps $b_\bullet$ and $a_\bullet$.

Suppose further that $D_\bullet$ and $E_\bullet$ are chain homotopy equivalent via the maps $u_\bullet : D_\bullet \to E_\bullet$ and $v_\bullet : E_\bullet \to D_\bullet$. Then the maps $u_\bullet \circ a_\bullet$ and $b_\bullet \circ v_\bullet$ give a chain homotopy equivalence between $C_\bullet$ and $E_\bullet$. Indeed, upon composition, we have that they are chain homtopic to the identity maps. ■

2.4 Mapping Cones and Mapping Cylinders

2.5 Sequences of Chain Complexes

One can even define short exact sequences of chain complexes themselves.

Definition 2.5.1 (Short Exact Sequence of Chain Complexes) Let $A_\bullet, B_\bullet, C_\bullet$ be chain complexes in an abelian category $\mathcal{A}$. Let $i : A_\bullet \to B_\bullet$ and $j : B_\bullet \to C_\bullet$ be chain maps in $\mathbf{Ch}(\mathcal{A})$. A short exact sequence of chain complexes is a diagram of the form

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_A"] \arrow[d, "i"] & A_n \arrow[r, "d_A"] \arrow[d, "i"] & A_{n-1} \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "d_B"] \arrow[d, "j"] & B_n \arrow[r, "d_B"] \arrow[d, "j"] & B_{n-1} \arrow[r] \arrow[d, "j"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_C"] \arrow[d] & C_n \arrow[r, "d_C"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

such that for each n (vertically in the diagram), the sequence

$$0 \longrightarrow A_n \xrightarrow{i} B_n \xrightarrow{j} C_n \longrightarrow 0$$

is a short exact sequence. We write this as

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$$

Theorem 2.5.2 Let $\mathcal{A}$ be an abelian category. Let $A_\bullet, B_\bullet, C_\bullet$ be a chain complexes in $\mathbf{Ch}(\mathcal{A})$ such that

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$$

is a short exact sequence of chain complexes. Then there exists a connecting homomorphism $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that the following sequence of homology

$$\cdots \longrightarrow H_{n+1}(C_\bullet) \xrightarrow{\partial} H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \longrightarrow \cdots$$

is an exact sequence.

Theorem 2.5.3 Let $A_\bullet, B_\bullet, C_\bullet, A'_\bullet, B'_\bullet, C'_\bullet$ be six chain complexes in an abelian category $\mathcal{A}$ and let the following

$$0 \longrightarrow A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \longrightarrow 0$$

$$0 \longrightarrow A'_\bullet \xrightarrow[i']{} B'_\bullet \xrightarrow[j']{} C'_\bullet \longrightarrow 0$$

be two short exact sequence of chain complexes. Let the following diagram be a morphism of the two short exact sequence of chain complexes in $\mathbf{Ch}(\mathcal{A})$.

$$
\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & \\
 & & 0 \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] & & & \\
 & \cdots \arrow[rr] & & A'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_n \arrow[rr, "\partial"] \arrow[dd, "i'"] & & A'_{n-1} \arrow[rr] \arrow[dd, "i'"] & & \cdots \\
\cdots \arrow[rr] & & A_{n+1} \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_n \arrow[ur, "\alpha"] \arrow[rr, "\partial"] \arrow[dd, "i"] & & A_{n-1} \arrow[ur, "\alpha"] \arrow[rr] \arrow[dd, "i"] & & \cdots & \\
 & \cdots \arrow[rr] & & B'_{n+1} \arrow[rr, "\partial"] \arrow[dd, "j'"] & & B'_n \arrow[rr, "\partial"] \arrow[dd, "j'"] & & B'_{n-1} \arrow[rr] \arrow[dd, "j'"] & & \cdots \\
\cdots \arrow[rr] & & B_{n+1} \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_n \arrow[ur, "\beta"] \arrow[rr, "\partial"] \arrow[dd, "j"] & & B_{n-1} \arrow[ur, "\beta"] \arrow[rr] \arrow[dd, "j"] & & \cdots & \\
 & \cdots \arrow[rr] & & C'_{n+1} \arrow[rr, "\partial"] \arrow[dd] & & C'_n \arrow[rr, "\partial"] \arrow[dd] & & C'_{n-1} \arrow[rr] \arrow[dd] & & \cdots \\
\cdots \arrow[rr] & & C_{n+1} \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_n \arrow[ur, "\gamma"] \arrow[rr, "\partial"] \arrow[dd] & & C_{n-1} \arrow[ur, "\gamma"] \arrow[rr] \arrow[dd] & & \cdots & \\
 & & & 0 & & 0 & & 0 & & \\
 & & 0 & & 0 & & 0 & & &
\end{tikzcd}
$$

Then the induced diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(B) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & H_n(C) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
\cdots \arrow[r] & H_n(A') \arrow[r, "i'_*"'] & H_n(B') \arrow[r, "j'_*"'] & H_n(C') \arrow[r, "\partial"'] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}
$$

is a commutative diagram.

2.6 Double Complexes

Definition 2.6.1 (Double Complexes) Let $\mathcal{A}$ be an abelian category. A double complex $(C_{\bullet,\bullet}, d^h, d^v)$ in $\mathcal{A}$ is a sequence of objects $C_{p,q} \in \mathcal{A}$ that is bigraded $p, q \in \mathbb{Z}$, together with the horizontal differentials $d^h : C_{p,q} \to C_{p+1,q}$ and vertical differentials $d^v : C_{p,q} \to C_{p,q+1}$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & C_{p-1,q+1} \arrow[u] \arrow[r, "d^h"] & C_{p,q+1} \arrow[u] \arrow[r, "d^h"] & C_{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & C_{p-1,q} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{p,q} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{p+1,q} \arrow[u, "d^v"] \arrow[r] & \cdots \\
\cdots \arrow[r] & C_{p-1,q-1} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{p,q-1} \arrow[u, "d^v"] \arrow[r, "d^h"] & C_{p+1,q-1} \arrow[u, "d^v"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

Definition 2.6.2 (Tensor Product Double Complex) Let $\mathcal{A}$ be an abelian category. Let $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ be two chain complexes. Define the tensor product of $C_\bullet$ and $D_\bullet$ to be the double complex $C_\bullet \otimes D_\bullet$ with the horizontal differential and vertical differential is defined by

$$d^h = d^{C_p} \otimes \mathrm{id}_{D_q} \quad \text{and} \quad d^v = (-1)^p \mathrm{id}_{C_p} \otimes d^{D_q}$$

respectively.

2.7 Total Complexes

Definition 2.7.1 (The Total Complex of a Double Complex) Let $\mathcal{A}$ be an abelian category where $\oplus$ denotes the product. Let $(C_{\bullet,\bullet}, d^h, d^v)$ be a double complex. Define the total complex $\mathrm{Tot}^{\oplus}(C)_\bullet$ to be the chain complex in $\mathcal{A}$ constructed as follows.

- For each $n \in \mathbb{Z}$, define
$$\mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}$$
- For each $n \in \mathbb{Z}$, define $d_n : \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$ by
$$d_n = d_n^v + (-1)^n d_n^h$$

In other words, the chain complex is of the form:

$$\cdots \longrightarrow \bigoplus_{p+q=n+1} C_{p,q} \xrightarrow{d_{n+1}^v + (-1)^{n+1} d_{n+1}^h} \bigoplus_{p+q=n} C_{p,q} \xrightarrow{d_n^v + (-1)^n d_n^h} \bigoplus_{p+q=n-1} C_{p,q} \longrightarrow \cdots$$

Definition 2.7.2 (Total Homology of a Double Chain Complex) Let $\mathcal{A}$ be an abelian category. Let $(C_{\bullet,\bullet}, d^h, d^v)$ be a double complex in $\mathcal{A}$. Define the total homology of C to be the homology

$$H_\bullet^{\mathrm{Tot}}(C_{\bullet,\bullet}, d^h, d^v) = H_\bullet(\mathrm{Tot}^{\oplus}(C)_\bullet, d)$$

of the total complex $(\mathrm{Tot}^{\oplus}(C)_\bullet, d)$

Definition 2.7.3 (Tensor Product Chain Complex) Let $\mathcal{A}$ be an abelian category. Let $(C_\bullet, d^C)$ and $(D_\bullet, d^D)$ be two chain complexes. Define the tensor product chain complex of $C_\bullet$ and $D_\bullet$ to be the chain complex

$$(C \otimes D)_\bullet = \mathrm{Tot}^{\oplus}(C_\bullet \otimes D_\bullet)$$

Note that this is different from the tensor product double complex of two chain complexes, although they are closely related. Although the tensor product double complex makes more intuitive sense, the canonical product for $\mathbf{Ch}(\mathcal{A})$ is in fact the tensor product chain complex. Moreover, this product is the symmetric monoidal product of $\mathbf{Ch}(\mathcal{A})$.

3 Derived Functors

3.1 Injective and Projective Objects

Injectivity and Projectivity objects are created just for the sake of allowing the Hom functor to be exact. Therefore its definition is also direct.

Definition 3.1.1 (Projective and Injective Objects) Let $\mathcal{A}$ be an abelian category. Let $Y \in \mathcal{A}$ be an object.
- We say that Y is injective if the functor $\text{Hom}(-, Y) : \mathcal{A} \to \mathbf{Ab}$ is exact.
- We say that Y is projective if the functor $\text{Hom}(Y, -) : \mathcal{A} \to \mathbf{Ab}$ is exact.

Definition 3.1.2 (Enough Injectives and Enough Projectives) Let $\mathcal{A}$ be an abelian category. $\mathcal{A}$ is said to have enough injectives if every object is the subobject of an injective object. $\mathcal{A}$ is said to have enough projectives if every object is the quotient of an projective object.

There are however equivalent definitions from the categorical point of view.

3.2 Resolutions of Objects

There are in general, four types of resolutions. Namely injective resolutions, projective resolutions, free resolutions and acyclic resolutions. We will study all four of them and their relations in this section.

Definition 3.2.1 (Injective Resolution) Let $\mathcal{A}$ be an abelian category. An injective resolution of an object $A \in \mathcal{A}$ is an exact sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

where each I^k is injective.

Proposition 3.2.2 Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ has enough injectives if and only if every object of $\mathcal{A}$ has an injective resolution.

Proposition 3.2.3 Let $\phi : A \to A'$ be a morphism in an abelian category $\mathcal{A}$. Suppose that there are injective resolutions

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\phi"'] & I^0 \arrow[r] & I^1 \arrow[r] & \cdots \\ 0 \arrow[r] & A' \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \end{tikzcd}$$

for A and A' respectively. Then there exists a chain map extending ϕ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\phi"'] & I^0 \arrow[r] \arrow[d, "\phi^0"'] & I^1 \arrow[r] \arrow[d, "\phi^1"'] & \cdots \\ 0 \arrow[r] & A' \arrow[r] & J^0 \arrow[r] & J^1 \arrow[r] & \cdots \end{tikzcd}$$

Moreover, any two such chain maps are homotopic.

Lemma 3.2.4 Let $\mathcal{A}$ be an abelian category. Then any two injective resolutions of an object A are homotopically equivalent.

Definition 3.2.5 (Projective Resolution) Let $\mathcal{A}$ be an abelian category. An projective resolution of an object A is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{d} A \longrightarrow 0$$

where each P_k is projective.

Proposition 3.2.6 Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ has enough projectives if and only if every object of $\mathcal{A}$ has a projective resolution.

Proposition 3.2.7 Let $\phi : A \to A'$ be a morphism in an abelian category $\mathcal{A}$. Suppose that there are projective resolutions

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & A \arrow[r] \arrow[d, "\phi"'] & 0 \\ \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

for A and A' respectively. Then there exists a chain map extending ϕ such that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "\phi_2"'] & P_1 \arrow[r] \arrow[d, "\phi_1"'] & A \arrow[r] \arrow[d, "\phi"'] & 0 \\ \cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & A' \arrow[r] & 0 \end{tikzcd}$$

Moreover, any two such chain maps are homotopic.

Lemma 3.2.8 Let $\mathcal{A}$ be an abelian category. Then any two projective resolutions of an object A are homotopically equivalent.

3.3 Derived Functors

Definition 3.3.1 (Right Derived Functors) Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose that $\mathcal{A}$ has enough injectives. Define the right derived functors $R^iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ as follows.

- On objects, $R^iF(A) = H^i(F(I^\bullet))$ where $d : A \to I^\bullet$ is an injective resolution of A
- On Morphisms, $R^iF(\phi : A \to B) = H^i(F(\phi^\bullet : I^\bullet \to (I')^\bullet))$ where $\phi^\bullet : I^\bullet \to (I')^\bullet$ is an extension of ϕ to resolutions.

Proposition 3.3.2 Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. The nth right derived functor R^nF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

Lemma 3.3.3 Let A be an injective object, then $R^nF(A) = 0$ for $n \neq 0$.

Corollary 3.3.4 If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor, then $R^0F = F$.

Theorem 3.3.5 Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough injectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. For any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is a canonical long exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow R^1(A) \longrightarrow R^1(B) \longrightarrow R^1(C) \longrightarrow R^2(A) \longrightarrow \cdots$$

Definition 3.3.6 (Left Derived Functors) Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Suppose that $\mathcal{A}$ has enough projectives. Define the left derived functors $L_iF : \mathcal{A} \to \mathcal{B}$ for $i \geq 0$ as follows.

- On objects, $L_iF(A) = H_i(F(P^\bullet))$ where $d : P_\bullet \to A$ is an projective resolution of A
- On Morphisms, $L_iF(\phi : A \to B) = L_i(F(\phi_\bullet : P_\bullet \to (P')_\bullet))$ where $\phi_\bullet : P_\bullet \to (P')_\bullet$ is an extension of ϕ to resolutions.

Proposition 3.3.7 Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. The nth left derived functor L_nF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

Lemma 3.3.8 Let A be a projective object, then $L_nF(A) = 0$ for $n \neq 0$.

Corollary 3.3.9 If $F : \mathcal{A} \to \mathcal{B}$ is a right exact functor, then $L_0F = F$.

Theorem 3.3.10 Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough projectives. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. For any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there is a canonical long exact sequence

$$\cdots \longrightarrow L_2(C) \longrightarrow L_1(A) \longrightarrow L_1(B) \longrightarrow L_1(C) \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

3.4 δ-Functors

Definition 3.4.1 (δ-Functors) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A homological δ-functor is a collection $\{T_n : \mathcal{A} \to \mathcal{B} \mid n \in \mathbb{N}\}$ of additive functors such that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there are morphisms $\delta_n : T_n(C) \to T_n(A)$ for $n \in \mathbb{N}$ such that the following are true.

- There is a long exact sequence

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta_n} T_{n-1}(A) \longrightarrow \cdots$$

- If there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

  the following diagram commutes:

$$\begin{tikzcd} T_n(C) \arrow[r, "\delta_n"] \arrow[d] & T_{n-1}(A) \arrow[d] \\ T_n(C') \arrow[r, "\delta'_n"] & T_{n-1}(A') \end{tikzcd}$$

4 A Second Course on Modules

4.1 The Hom and Tensor Functor

Theorem 4.1.1 (Tensor-Hom Adjunction) Let R be a ring and let M be an R-module. Then there is an adjunction $- \otimes_R M : {}_R\mathbf{Mod} \rightleftarrows {}_R\mathbf{Mod} : \mathrm{Hom}(M, -)$. Explicitly, there is a natural isomorphism

$$\mathrm{Hom}_{{}_R\mathbf{Mod}}(- \otimes_R M, -) \cong \mathrm{Hom}_{{}_R\mathbf{Mod}}(-, \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, -))$$

More generally, the tensor-hom adjunction is a phenomena that exhibits ${}_R\mathbf{Mod}$ as a closed monoidal category.

Proposition 4.1.2 Let R be a ring. Then $({}_R\mathbf{Mod}, \otimes_R, R)$ is a symmetric monoidal category. Moreover, it is a closed monoidal category with internal hom given by the usual hom functor $\mathrm{HOM}_{{}_R\mathbf{Mod}}(-,-) = \mathrm{Hom}_R(-,-)$.

We have seen that the left and right Hom functors

$$\mathrm{Hom}(M, -) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod} \quad \text{and} \quad \mathrm{Hom}(-, M) : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$$

are left exact functors. Using the tensor-hom adjunction, we can deduce the following.

Proposition 4.1.3 Let R be a ring and let M be an R-module. Then both

$$- \otimes_R M : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod} \quad \text{and} \quad M \otimes_R - : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$$

are right exact functors.

We find it important to notice the relation between all four functor here. Notice that because there is a canonical isomorphism $M \otimes_R N \cong N \otimes_R M$, we can replace $- \otimes_R M$ in the natural isomorphism

$$\mathrm{Hom}_{{}_R\mathbf{Mod}}(- \otimes_R M, -) \cong \mathrm{Hom}_{{}_R\mathbf{Mod}}(-, \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, -))$$

with $M \otimes_R -$. This leaves us with the question: What is the functor $\mathrm{Hom}(-, M)$ adjoint to? The answer is itself!

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, C))$$

Therefore it is more often to discuss $\mathrm{Hom}(M, -)$ in conjunction with $- \otimes_R M$, while $\mathrm{Hom}(-, M)$ is often left behind.

4.2 Projective and Injective Modules

Let R be a ring. Recall that an R-module P is projective if $\mathrm{Hom}(P, -)$ is an exact functor.

Lemma 4.2.1 Every free module is projective.

Proof Let R be a ring and let F be a free R-module. Suppose that F has basis B. Let M, N be R-modules. Suppose that $f : N \to M$ is surjective and there exists an R-module homomorphism $g : R^n \to M$. Since f is surjective, for each $b \in B$, we can choose a pre-image for $g(b)$ in N for all $b \in B$. Call it n_b. Now define $h : F \to N$ by $b \mapsto n_b$ and then extend it R-linearly. Now if

$\sum_{b\in B} k_b b \in F$, we have that

$$\begin{aligned}(f\circ h)\left(\sum_{b\in B} k_b b\right) &= f\left(\sum_{b\in B} k_b h(b)\right)\\ &= f\left(\sum_{b\in B} k_b n_b\right)\\ &= f\left(\sum_{b\in B} k_b n_b\right)\\ &= \sum_{b\in B} k_b f(n_b)\\ &= \sum_{b\in B} k_b g(b)\\ &= g\left(\sum_{b\in B} k_b b\right)\end{aligned}$$

so that $f\circ h = g$. Thus F is projective. ■

Proposition 4.2.2 Let R be a ring. Let P be an R-module. Then the following are equivalent.

- P is projective
- For every surjective homomorphism $f : N \twoheadrightarrow M$ and every R-module homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that the following diagram commutes:

$$\begin{tikzcd} & N \arrow[d, two heads, "f"] \\ P \arrow[ur, dashed, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

- $P \oplus Q$ is a free R-module for some R-module Q.

Proof

- $(3) \implies (1)$: Suppose that there exists a module Q such that $P\oplus Q$ is free. Let $f : N \to M$ be a surjective R-module homomorphism and let $g : P \to M$ be an R-module homomorphism.

■

Proposition 4.2.3 Let R be a ring. Let P_i be an R-module for each $i \in I$. Then the direct sum $\bigoplus_{i\in I} P_i$ is projective if and only if each P_i is projective.

Now recall that an R-module I is injective if $\text{Hom}(-, I)$ is an exact functor.

Proposition 4.2.4 Let R be a ring. Let I be an R-module. Then I is injective if and only if for every injective homomorphism $f : N \rightarrowtail M$ and every module homomorphism $g : N \to I$, there exists a module homomorphism $h : M \to I$ such that the following diagram commutes:

$$\begin{tikzcd} N \arrow[d, tail, "f"'] \arrow[dr, "g"] & \\ M \arrow[r, dashed, "\exists h"'] & I \end{tikzcd}$$

Example 4.2.5 Let k be a field. Then every k-module is injective.

Example 4.2.6 Let R be a DVR. Let t be a uniformizing parameter of R. Then $R[1/t]/R$ is an injective R-module.

Proposition 4.2.7 (Baer's Criterion) Let R be a ring. Let M be a left R-module. Then M is injective if and only if for every left ideal I of R and R-module homomorphism $\phi : I \to M$, there exists a map $\overline{\phi}$ such that the following diagram commutes:

$$\begin{tikzcd} I \arrow[r, "\phi"] \arrow[d, hook] & M \\ R \arrow[ur, dashed, "\exists\overline{\phi}"'] & \end{tikzcd}$$

4.3 Divisible Groups

(can be removed to Infinite Abelian Groups)

Definition 4.3.1 (Divisible Groups) Let A be an abelian group. Let $n \in \mathbb{N} \setminus \{0\}$. We say that A is n-divisible if for all $g \in A$, there exists $h \in A$ such that $nh = g$.

Proposition 4.3.2 Let A be an abelian group. Let $n \in \mathbb{N} \setminus \{0\}$. Then the following are equivalent.

- A is n-divisible.
- $nA = A$.
- A is an injective $\mathbb{Z}$-module.

Definition 4.3.3 (The Prüfer p-Group) Let p be a prime. Define the Prüfer p-Group to be the quotient group

$$\mathbb{Z}(p^\infty) = \frac{\mathbb{Z}[1/p]}{\mathbb{Z}}$$

Example 4.3.4 The following are divisible groups.

- $\mathbb{Q}$.
- $\mathbb{Z}[1/p]/\mathbb{Z}$.

Theorem 4.3.5 (Structure Theorem for Divisible Groups) Let A be an abelian group. If A is divisible, then there is an isomorphism

$$A \cong \left(\bigoplus_{p \text{ is prime}} \frac{\mathbb{Z}[1/p]}{\mathbb{Z}}^{k_p} \right) \oplus \mathbb{Q}^I$$

for some choice of indexing sets I and some $k_p \in \mathbb{N}$.

4.4 Flat Modules

While projective modules allows the exactness of $\text{Hom}(M, -)$, flat modules allows the exactness of $- \otimes_R M$.

Definition 4.4.1 (Flat Modules) Let R be a ring. An R-module M is said to be flat if for every injective linear map $\phi : K \to L$ of R-modules, the map

$$\phi \otimes \text{id}_M : K \otimes_R M \to L \otimes_R M$$

is also injective.

Proposition 4.4.2 Let R be a ring and M an R-module. Then M is a flat module if and only if for every short exact sequence $0 \to K \to L \to J \to 0$, the sequence

$$0 \to K \otimes_R M \to L \otimes_R M \to J \otimes_R M \to 0$$

is also exact.

Proposition 4.4.3 Let R be a ring. Then the following are true.

- Product: If A and B are flat over R then $A \otimes_R B$ is flat over R
- Base Change: Let S be an R-algebra ($R \to S$ a ring hom). Then $M \otimes_R S$ is flat over S for any flat R-module M
- Transitivity: Let S be an R-algebra such that S is flat over R. If C is flat over S then C is flat over R.

We have the following inclusion of modules

$$\text{Free Modules} \subset \text{Projective Modules} \subset \text{Flat Modules} \subset \text{Torsion Free Modules}$$

It is important to note that the "duality" between projective and injective modules is given from the bi-functor $\text{Hom}(-,-)$ while the "duality" given between projective and flat modules is given by the tensor-hom adjunction.

4.5 Finitely Presented Modules

Definition 4.5.1 (Finitely Presented Modules) Let R be a ring. Let M be a left R-module. We say that M is finitely presented if there exists an exact sequence of the form

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

Lemma 4.5.2 Let R be a ring. Let M be a left R-module. If M is finitely presented, then M is finitely generated.

Proposition 4.5.3 Let R be a ring. Let the following be an exact sequence of left R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then the following are true.

- If M_1 and M_3 are finitely presented, then M_2 is finitely presented.
- If M_2 is finitely presented and M_1 is finitely generated, then M_3 is finitely presented.
- If M_2 is finitely generated and M_3 is finitely presented, then M_1 is finitely generated.

Proposition 4.5.4 Let R be a commutative ring. Let S be an R-algebra. Let M, N be left R-modules. If S is flat over R and M is finitely presented, then there is an isomorphism

$$S \otimes_R \text{Hom}_R(M, N) \cong \text{Hom}_S(S \otimes_R M, S \otimes_R N)$$

4.6 Derived Functors in the Category of R-Modules

Definition 4.6.1 (The Ext Functor) Let R be a ring and let A be an R-module. Define the right derived functor of the left exact functor $\text{Hom}(A, -) : {}_R\mathbf{Mod} \to \mathbf{Ab}$ to be

$$\text{Ext}^i_R(A, -) = R^i(\text{Hom}(A, -)) : {}_R\mathbf{Mod} \to \mathbf{Ab}$$

Explicitly, for

$$0 \to A \to I^0 \to I^1 \to \cdots$$

an injective resolution, form the cochain complex

$$0 \to \mathrm{Hom}_R(A, I^0) \to \mathrm{Hom}_R(A, I^1) \to \cdots$$

and define Ext to be the cohomology group

$$\mathrm{Ext}^i_R(A, B) = \frac{\ker(\mathrm{Hom}_R(A, I^i) \to \mathrm{Hom}_R(A, I^{i+1}))}{\mathrm{im}(\mathrm{Hom}_R(A, I^{i-1}) \to \mathrm{Hom}_R(A, I^i))}$$

Notice how although A is viewed as a fixed R-module, it is written as if it was a variable for the functor Ext. Indeed, we can define the Ext functor in another manner.

Theorem 4.6.2 Let R be a ring and let A, B be R-modules. Then there is an isomorphism

$$\mathrm{Ext}^i_R(A, B) = R^i(\mathrm{Hom}(A, -))(B) \cong R^i(\mathrm{Hom}(-, B))(A)$$

that is natural in A and B.

Theorem 4.6.3 Let A, B be R-modules. Then the following are true regarding the Ext group.

- $\mathrm{Ext}^0_R(A, B) \cong \mathrm{Hom}_R(A, B)$
- $\mathrm{Ext}^i_R(A, B) = 0$ for all $i > 0$ if A is projective or B is injective
- $\mathrm{Ext}^i_R(A, B) = 0$ for all $i \geq 2$ if A, B are $\mathbb{Z}$-modules.

Definition 4.6.4 (The Tor Functor) Let R be a ring and let B be an R-module. Define the left derived functor of the right exact functor $- \otimes_R B : {}_R\mathbf{Mod} \to {}_R\mathbf{Mod}$ to be

$$\mathrm{Tor}^R_i(-, B) = L_i(- \otimes_R B) : {}_R\mathbf{Mod} \to \mathbf{Ab}$$

Explicitly, for

$$\cdots \to P_1 \to P_0 \to B \to 0$$

an injective resolution, form the chain complex

$$\cdots \to P_1 \otimes_R B \to P_0 \otimes_R B \to 0$$

and define Tor to be the homology group

$$\mathrm{Tor}^R_i(A, B) = \frac{\ker(P_i \otimes_R B \to P_{i-1} \otimes_R B)}{\mathrm{im}(P_{i+1} \otimes_R B \to P_i \otimes_R B)}$$

Theorem 4.6.5 Let R be a ring and let A, B be R-modules. Then there is an isomorphism

$$\mathrm{Tor}^R_i(A, B) = L_i(- \otimes_R B)(A) \cong L_i(A \otimes_R -)(B)$$

that is natural in A and B.

4.7 Extensions and Torsions

Definition 4.7.1 (Extensions) Let R be a ring. Let M, N be R-modules. An extension of M by N is a short exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

of R-modules.

Definition 4.7.2 (Equivalent Extensions) Let R be a ring. Let M, N be R-modules. We say that two extensions

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

are equivalent if there exists an R-module homomorphism $\phi : E \to F$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & E \arrow[r] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\ 0 \arrow[r] & N \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Proposition 4.7.3 Let R be a ring. Let M, N be R-modules. Suppose that the following two extensions are equivalent:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & E \arrow[r] \arrow[d, "\phi"] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\ 0 \arrow[r] & N \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Then ϕ is an isomorphism. Moreover, equivalent extensions is an equivalence relation.

Definition 4.7.4 (Split Extensions) Let R be a ring. Let M, N be R-modules. We say that an extension splits if it is equivalent to the following extension

$$0 \longrightarrow N \longrightarrow N \oplus M \longrightarrow M \longrightarrow 0$$

of R-modules. The above extension is called a trivial extension.

Theorem 4.7.5 Let R be a ring. Let M, N be R-modules. There is a bijection

$$\frac{\{\text{Extensions of } M \text{ by } N\}}{\cong} \quad \overset{1:1}{\longleftrightarrow} \quad \mathrm{Ext}^1_R(M, N)$$

where $\cong$ means equivalence of extensions. Moreover, the trivial extension corresponds to the zero element of $\mathrm{Ext}^1_R(M, N)$.

5 (Co)Homology Theories for Rings

5.1 Koszul Complexes

The following definitions requires the use of central elements. Recall that when R is commutative, this condition is null and so we can choose any element in R.

Definition 5.1.1 (Koszul Complexes) Let R be a ring. Let $x \in Z(R)$ be a central element. Define the Koszul complex $K(x)$ of x in R to be the chain complex

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$$

Definition 5.1.2 (Generalized Koszul Complexes) Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Define the generalized Koszul complex $K(x_1, \dots, x_n)$ of $x_1, \dots, x_n$ in R to be the chain complex given by

$$K(x_1, \dots, x_n) = \text{Tot}^{\oplus}\left(K(x_1) \otimes_R \cdots \otimes_R K(x_n)\right)$$

If M is an R-module, define the generalized Koszul complex of M to be

$$K(x_1, \dots, x_n; M) = K(x_1, \dots, x_n) \otimes_R M$$

Proposition 5.1.3 Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Then there is an isomorphism

$$\bigwedge_{i=1}^{k} R^n \cong K(x_1, \dots, x_n)_k$$

given on the basis elements by

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto 1 \otimes \cdots \otimes 1 \otimes e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes 1 \otimes \cdots \otimes 1$$

Theorem 5.1.4 Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Then the Koszul complex $K(x_1, \dots, x_n)$ is given explicitly as

$$0 \longrightarrow \bigwedge_{i=1}^{n} R^n \xrightarrow{d_n} \bigwedge_{i=1}^{n-1} R^n \longrightarrow \cdots \longrightarrow R^n \xrightarrow{d_1} R \longrightarrow 0$$

where the differential $d_k : \bigwedge_{i=1}^{k} R^n \to \bigwedge_{i=1}^{k-1} R^n$ is given on basis elements by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} x_{i_j} e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_{i_k}$$

where $e_1, \dots, e_n \in R^n$ are the standard generators of R^n.

For an example, let R be a commutative ring. Let $x, y \in R$. Then the Koszul complex $K(x, y)$ is given by

$$0 \longrightarrow R^2 \wedge R^2 \longrightarrow R^2 \longrightarrow R \longrightarrow 0$$

The differentials are given as follows.

- The first differential $R^2 \to R$ is given by $(r, s) \mapsto rx + sy$. It can also be given as a 1×2 matrix as $\begin{pmatrix} x & y \end{pmatrix}$. Also alternatively, we can write an R-basis for R^2 with $(1, 0)$ and $(0, 1)$. Then define the map $R^2 \to R$ by $(1, 0) \mapsto x$ and $(0, 1) \mapsto y$.
- Now recall that $R^2 \wedge R^2$ has R-basis given by $(1, 0) \wedge (0, 1)$ and so there is an isomorphism $R^2 \wedge R^2 \cong R$. Then the second differential $R \to R^2$ is given by $1 \mapsto (x, -y)$.

Definition 5.1.5 (Koszul (Co)Homology) Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Let M be an R-module. Define the Koszul homology of M with respect to $x_1, \dots, x_n$ by

$$H_k^{\text{Kos}}(x_1, \dots, x_n; M) = H_k(K(x_1, \dots, x_n; M))$$

Define the Koszul cohomology of M with respect to the central elements by

$$H^k_{\text{Kos}}(x_1, \dots, x_n; M) = H^k(\text{Hom}_R(K(x_1, \dots, x_n)), M)$$

Lemma 5.1.6 Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Let M be an R-module. Then the following are true.

- $H_0^{\text{Kos}}(x_1, \dots, x_n; M) = \frac{M}{(x_1, \dots, x_n)M}$
- $H^0_{\text{Kos}}(x_1, \dots, x_n; M) = \{m \in M \mid x_i m = 0 \text{ for all } i\}$

Theorem 5.1.7 (Kunneth Theorem) Let R be a ring. Let $x_1, \dots, x_n \in R$ be central elements. Let $C_\bullet$ be a chain complex of R-modules. Then there is an exact sequence given by

$$0 \longrightarrow H_0(x_1, \dots, x_n; H_q(C_\bullet)) \longrightarrow H_q^{\text{Tot}}(K(x_1, \dots, x_n) \otimes_R C_\bullet) \longrightarrow H_1(x_1, \dots, x_n; H_{q-1}(C_\bullet)) \longrightarrow 0$$

Theorem 5.1.8 (Self Duality of the Koszul Complex) Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Let M be a left R-module. Then there is a natural isomorphism of chain complexes

$$\text{Hom}(K(x_1, \dots, x_n); M) \cong K(x_1, \dots, x_n; M)$$

The self duality of the Koszul complex means that we could have equally defined the differential to be going up the chain instead of down. In this case, the complex can be defined as

$$0 \longrightarrow \bigwedge_{i=1}^n R^n \xrightarrow{d_n} \bigwedge_{i=1}^{n-1} R^n \longrightarrow \cdots \longrightarrow R^n \xrightarrow{d_1} R \longrightarrow 0$$

where the differential $d_k : \bigwedge_{i=1}^{k-1} R^n \to \bigwedge_{i=1}^k R^n$ is now defined by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}) = \sum_{j=1}^n x_j e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_{k-1}}$$

Corollary 5.1.9 Let R be a ring. Let $x_1, \dots, x_n \in Z(R)$ be central elements. Let M be a left R-module. Then there is an isomorphism

$$H_p^{\text{Kos}}(x_1, \dots, x_n; M) \cong H^{n-p}_{\text{Kos}}(x_1, \dots, x_n; M)$$

5.2 Local Cohomology

6 Spectral Sequences

6.1 General Spectral Sequences

Definition 6.1.1 (Homological Spectral Sequences) Let $\mathcal{A}$ be an abelian category. A homological spectral sequence consists of the following data.

- A collection of objects $E^r_{\bullet,\bullet} = \{E^r_{p,q} \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$ called pages for each $r \in \mathbb{N}$. So that there is a sequence
$$E^1_{\bullet,\bullet}, E^2_{\bullet,\bullet}, E^3_{\bullet,\bullet}, \dots$$
of family of objects
- A degree (p, q) map
$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$
for each $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$ such that $d^r \circ d^r = 0$
- Isomorphisms of the form $E^{r+1}_{\bullet,\bullet} = H_\bullet(E^r_{\bullet,\bullet}, d^r)$. This means that
$$E^{r+1}_{p,q} = \frac{\ker(d^r : E^r_{p,q} \to E^r_{p-r,q+r-1})}{\operatorname{im}(d^r : E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

We say that the total degree of $E^r_{p,q}$ is $n = p + q$.

Definition 6.1.2 (Cohomological Spectral Sequences) Let $\mathcal{A}$ be an abelian category. A cohomological spectral sequence consists of the following data.

- A collection of objects $E_r^{\bullet,\bullet} = \{E_r^{p,q} \in \mathcal{A} \mid p, q \in \mathbb{Z}\}$ called pages for each $r \in \mathbb{N}$. So that there is a sequence
$$E_1^{\bullet,\bullet}, E_2^{\bullet,\bullet}, E_3^{\bullet,\bullet}, \dots$$
of family of objects
- A degree (p, q) map
$$d_r^{p,q} : E_r^{p,q} \to E_r^{p-r,q+r-1}$$
for each $p, q \in \mathbb{Z}$ and $r \in \mathbb{N}$ such that $d_r \circ d_r = 0$
- Isomorphisms of the form $E_{r+1}^{\bullet,\bullet} = H^\bullet(E_r^{\bullet,\bullet}, d_r)$. This means that
$$E^{p,q}_{r+1} = \frac{\ker(d_r : E_r^{p,q} \to E_r^{p-r,q+r-1})}{\operatorname{im}(d_r : E_r^{p+r,q-r+1} \to E_r^{p,q})}$$

Notice that cohomological spectral sequences are really the same thing as homological spectral sequences, just reindex the objects by $E_r^{p,q} = E^r_{-p,-q}$.

Definition 6.1.3 (Bounded Spectral Sequences) Let $(E^r_{\bullet,\bullet}, d^r)$ be a homological spectral sequence. We say that it is bounded if for each $n \in \mathbb{N}$, there are only finitely many non-zero terms of total degree n in $E^r_{\bullet,\bullet}$ for each $r \in \mathbb{N}$.

We say that it is bounded below if there exists $s_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$ such that terms $E^r_{\bullet,\bullet}$ of total degree n are 0 for all $p < s$.

Lemma 6.1.4 Let $(E^r_{\bullet,\bullet}, d^r)$ be a bounded homological spectral sequence. Then for each $(p, q) \in \mathbb{Z}^2$, there exists $r_0 \in \mathbb{N}$ such that $E^{r+1}_{p,q} \cong E^r_{p,q}$ for all $r \geq r_0$.

Definition 6.1.5 (Stable Values) Let $(E^r_{\bullet,\bullet}, d^r)$ be a bounded homological spectral sequence. Let $(p, q) \in \mathbb{Z}^2$ and $r_0 \in \mathbb{N}$ such that $E^{r+1}_{p,q} = E^r_{p,q}$ for all $r \geq r_0$. Define the stable values of the sequence to be
$$E^\infty_{p,q} = E^{r_0}_{p,q}$$

6.2 The Spectral Sequence of Exact Couples

Definition 6.2.1 (Exact Couple) An exact couple consists of bigraded abelian groups $E_{\bullet,\bullet}$ and $A_{\bullet,\bullet}$ and maps $i : A_{\bullet,\bullet} \to A_{\bullet,\bullet}$ of degree (a, a'), $j : A_{\bullet,\bullet} \to E_{\bullet,\bullet}$ of degree (b, b') and $k : E_{\bullet,\bullet} \to A_{\bullet,\bullet}$ of degree (c, c') such that the triangle

$$\begin{tikzcd} A_{\bullet,\bullet} \arrow[rr, "i"] & & A_{\bullet,\bullet} \arrow[dl, "j"] \\ & E_{\bullet,\bullet} \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each vertex ($\text{im}(i) = \ker(j)$ and so on). We write the exact couple as (A, E, i, j, k).

Notice that this actually just the data of a long exact sequence. If we look at what happens nearby $E_{p,q}$, we see that we can expand out the triangle ad infinum:

$$\cdots \longrightarrow A_{p-a,q-a'} \longrightarrow E_{p,q} \longrightarrow A_{p+a,q+a'} \longrightarrow A_{p+a+b,q+a'+b'} \longrightarrow \cdots$$

Definition 6.2.2 (Derived Couple) Suppose that there is an exact couple of the form

$$\begin{tikzcd} A_{\bullet,\bullet} \arrow[rr, "i"] & & A_{\bullet,\bullet} \arrow[dl, "j"] \\ & E_{\bullet,\bullet} \arrow[ul, "k"] & \end{tikzcd}$$

for some gradation of i, j, k. Define the derived couple with the following data. Write $d = j \circ k$.

- For each $p, q \in \mathbb{Z}$, define
$$E'_{p,q} = \frac{\ker(d : E_{p,q} \to E_{p+c+b,q+c'+b'})}{\text{im}(d : E_{p-c-b,q-c'-b'} \to E_{p,q})}$$
so that $E'_{\bullet,\bullet}$ is bigraded.
- For each $p, q \in \mathbb{Z}$, define
$$A'_{p,q} = \text{im}(i : A_{p,q} \to A_{p+a,q+a'})$$
so that $A'_{\bullet,\bullet}$ is bigraded.
- For each $p, q \in \mathbb{Z}$, define
$$i' : A'_{p,q} \subseteq A_{p+a,q+a'} \to A'_{p+a,q+a} \subseteq A_{p+2a,q+2a'}$$
by $i' = i|_{A'_{p,q}}$. We simplify and write it as $i' : A'_{\bullet,\bullet} \to A'_{\bullet,\bullet}$
- For each $p, q \in \mathbb{Z}$, define
$$j' : A'_{p-b,q-b'} \to E'_{p,q}$$
as follows. For all $i(t) \in A'_{p-b,q-b'}$ where $t \in A_{p-b,q-b'}$, $j'(i(t)) = [j(t)] \in E'_{p,q}$. We simplify and write it as $j' : A'_{\bullet,\bullet} \to E'_{\bullet,\bullet}$
- For each $p, q \in \mathbb{Z}$, define
$$k' : E'_{p,q} \to A'_{p+c,q+c'}$$
as follows. For all $[e] \in E'_{p,q}$, $k'([e]) = k(e)$. We simplify and write it as $k' : E'_{\bullet,\bullet} \to A'_{\bullet,\bullet}$

We write the derived couple as $(A^1, E^1, i^1, j^1, k^1)$

Theorem 6.2.3 The derived couple of any exact couple is also an exact couple with the same grading of maps.

Theorem 6.2.4 Let (A, E, i, j, k) be an exact couple. Then $E, E^1, E^2, \dots$ together with maps $d^r = j^r \circ k^r$ for $r \in \mathbb{N}$ defines a homological spectral sequence.

6.3 The Spectral Sequence of Filtrations

Definition 6.3.1 (Filtered Chain Complexes) Let $\mathcal{A}$ be an abelian category. Let $C \in \mathbf{Ch}(\mathcal{A})$ be a chain complex. A filtered chain complex is a sequence of subchain complexes of C with inclusions

$$\cdots \subseteq F_pC \subseteq F_{p+1}C \subseteq \cdots$$

such that the boundary map $d : C_n \to C_{n-1}$ of C has the property that

$$d(F_pC_n) \subseteq F_pC_{n-1}$$

Definition 6.3.2 (Exhaustive Filtered Chain Complexes) Let $\mathcal{A}$ be an abelian category. Let $C \in \mathbf{Ch}(\mathcal{A})$ be a chain complex. A filtered chain complex F_pC is said to be exhaustive if $\bigcup_{p\in\mathbb{Z}} F_pC = C$.

Definition 6.3.3 (Spectral Sequence Arising from Filtered Chain Complexes) Let $\mathcal{A}$ be an abelian category. Let $(C_\bullet, d_\bullet) \in \mathbf{Ch}(\mathcal{A})$ be a chain complex. Let F_pC be a filtered chain complex that is exhaustive. Define the following objects and subobjects in $\mathcal{A}$.

- Define an object $E^0_{p,q} = \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}$ and a chain complex $E^0_p = \frac{F_pC}{F_{p-1}C}$.
- For each $p, r \in \mathbb{N}$, define
$$A^r_p = \{c \in F_pC \mid d(c) \in F_{p-r}C\}$$
called approximately cycles
- Write $\eta_p : F_pC \to \frac{F_pC}{F_{p-1}C} = E^0_p$ for the sujection, which is a chain map.
- For each $p, r \in \mathbb{N}$, define
$$Z^r_p = \eta_p(A^r_p) \subseteq E^0_p$$
- For each $p, r \in \mathbb{N}$, define
$$B^{r+1}_{p-r} = \eta_{p-r}(d(A^r_p)) \subseteq E^0_{p-r}$$
- For each $p \in \mathbb{N}$, define
$$Z^\infty_p = \bigcap_{r=1}^{\infty} Z^r_p \quad \text{and} \quad B^\infty_p = \bigcup_{r=1}^{\infty} B^r_p$$

Evidently, there is a tower of subobjects of E^0_p given by

$$0 = B^0_p \subseteq B^1_p \subseteq \cdots B^\infty_p \subseteq Z^\infty_p \subseteq \cdots Z^1_p \subseteq Z^0_p = E^0_p$$

Thus we finally define

$$E^r_p = \frac{Z^r_p}{B^r_p} \quad \text{for all } r \in R \quad \text{and} \quad E^\infty_p = \frac{Z^\infty_p}{B^\infty_p}$$

together with $d : E^r_p \to E^r_{p-r}$ to be the differential induced by the differential of C.

Lemma 6.3.4 Let $\mathcal{A}$ be an abelian category. Let $(C_\bullet, d_\bullet) \in \mathbf{Ch}(\mathcal{A})$ be a chain complex. Let F_pC be a filtered chain complex that is exhaustive. Then the following are true.

- For any $p, r \in \mathbb{N}$, $A^r_p + F_{p-1}C = A^{r-1}_{p-1}$
- For any $p, r \in \mathbb{N}$, $Z^r_p \cong A^r_p / A^{r-1}_{p-1}$
- For any $p, r \in \mathbb{N}$, there are isomorphisms
$$E^r_p = \frac{Z^r_p}{B^r_p} \cong \frac{A^r_p + F_{p-1}C}{d(A^{r-1}_{p+r-1}) + F_{p-1}C} \cong \frac{A^r_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}}$$

Theorem 6.3.5 Let $\mathcal{A}$ be an abelian category. Let $(C_\bullet, d_\bullet) \in \mathbf{Ch}(\mathcal{A})$ be a chain complex. Let F_pC be a filtered chain complex that is exhaustive. Then the map $d : E_p^r \to E_{p-r}^r$ determines an isomorphism

$$\frac{Z_p^r}{Z_p^{r+1}} \cong \frac{B_{p-r}^{r+1}}{B_{p-r}^r}$$

Moreover, this concludes that $(E_{p,q}^r, d)$ is a spectral sequence.

6.4 Convergence

Definition 6.4.1 (Weakly Convergent) Let $(E_{\bullet,\bullet}^r, d^r)$ be a spectral sequence. We say that it is weakly convergent if there is a graded object $H_\bullet$ together with an exhaustive filtration $F_\bullet H_n$ for every $n \in \mathbb{N}$, together with isomorphisms

$$\beta_{p,q} : E_{p,q}^\infty \xrightarrow{\cong} \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}}$$

Definition 6.4.2 (Convergent Spectral Sequences) Let $(E_{\bullet,\bullet}^r, d^r)$ be a spectral sequence. We say that it is convergent if the following are true.

- It is weakly convergent with filtrations $F_\bullet H_n$ for each $n \in \mathbb{N}$
- $\bigcap_{k=0}^\infty F_kH_\bullet = 0$

6.5 The Spectral Sequence of Double Complexes

6.6 Hyperhomology and Hypercohomology

Definition 6.6.1 (Cartan-Eilenberg Resolution) Let $\mathcal{A}$ be an abelian category with enough projectives. Let $A_\bullet \in \mathbf{Ch}(\mathcal{A})$ be a chain complex as follows:

$$\cdots \longrightarrow A_{p-1} \longrightarrow A_p \longrightarrow A_{p+1} \longrightarrow \cdots$$

A Cartan-Eilenberg resolution of $A_\bullet$ is an upper half double complex $P_{\bullet,\bullet}$ together with augmentation maps $\varepsilon : P_{P,\bullet} \to A_p$ as follows:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & P_{p-1,1} \arrow[l] \arrow[d] & P_{p,1} \arrow[l] \arrow[d] & P_{p+1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & P_{p-1,0} \arrow[l] \arrow[d, "\varepsilon"] & P_{p,0} \arrow[l] \arrow[d, "\varepsilon"] & P_{p+1,0} \arrow[l] \arrow[d, "\varepsilon"] & \cdots \arrow[l] \\
\cdots & A_{p-1} \arrow[l] & A_p \arrow[l] & A_{p+1} \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

such that the following are true.

- If $A_p = 0$ then $P_{p,\bullet} = 0$
- For each $p \in \mathbb{Z}$, the chain complex $(B_\bullet, d^v)$ of boundaries where $B_q = d^h(P_{p+1,q})$ form a projective resolution of A_p. This means that the vertical chain complexes

$$\begin{array}{c} \vdots \\ \downarrow \\ d^h(P_{p,0}) \subseteq P_{p,1} \\ \downarrow \\ d^h(P_{p+1,0}) \subseteq P_{p,0} \\ \downarrow{\scriptstyle \varepsilon} \\ A_p \end{array}$$

are projective resolutions.

- For each $p \in \mathbb{Z}$, the chain complex $(H_\bullet, d^v)$ of the homology groups where

$$H_q = H_p(P_{\bullet,q}, d^h)$$

form a projective resolution of $H_p(A_\bullet)$. This means that the vertical chain complexes

$$\begin{array}{c} \vdots \\ \downarrow \\ H_p(P_{\bullet,1}, d^h) \\ \downarrow{\scriptstyle d^v} \\ H_p(P_{\bullet,0}, d^h) \\ \downarrow{\scriptstyle \varepsilon} \\ H_p(A_\bullet) \end{array}$$

are projective resolutions.

Theorem 6.6.2 Let $\mathcal{A}$ be an abelian category. Then every chain complex $A_\bullet \in \mathbf{Ch}(\mathcal{A})$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \to A_\bullet$.

Definition 6.6.3 (Left Hyperderived Functors) Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor. Define the left hyperderived functors

$$\mathbb{L}_i F : \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$$

by the following.

- For $A_\bullet$ a chain complex in $\mathcal{A}$, choose a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \to A_\bullet$ of $A_\bullet$. Then

$$\mathbb{L}_i F(A_\bullet) = H_i(\text{Tot}^{\oplus}(F(P_{\bullet,\bullet})))$$

- For $f : A_\bullet \to B_\bullet$ a chain map, choose Cartan-Eilenberg resolutions for both complexes and consider the induced map $\overline{f} : P_{\bullet,\bullet} \to Q_{\bullet,\bullet}$. Define

$$\mathbb{L}_i F(f) = H_i(\text{Tot}^{\oplus}(\overline{f})) : \mathbb{L}_i F(A_\bullet) \to \mathbb{L}_i F(B_\bullet)$$

Definition 6.6.4 (Right Hyperderived Functors) Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Define the right hyperderived functors

$$\mathbb{R}^i F : \mathbf{CCh}(\mathcal{A}) \to \mathcal{B}$$

by the following.

- For $A_\bullet$ a cochain complex in $\mathcal{A}$, choose a Cartan-Eilenberg resolution $A_\bullet \to I_{\bullet,\bullet}$ of $A_\bullet$. Then

$$\mathbb{R}^i F(A_\bullet) = H^i(\text{Tot}^{\oplus}(F(I_{\bullet,\bullet})))$$

- For $f:A_\bullet\to B_\bullet$ a chain map, choose Cartan-Eilenberg resolutions for both complexes and consider the induced map $\overline{f}:I_{\bullet,\bullet}\to J_{\bullet,\bullet}$. Define

$$\mathbb{R}^iF(f)=H^i(\text{Tot}^\oplus(\overline{f})):\mathbb{R}^iF(A_\bullet)\to\mathbb{R}^iF(B_\bullet)$$

Lemma 6.6.5 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $X\in\mathcal{A}$ be an object. Then the following are true.

- Let $F:\mathcal{A}\to\mathcal{B}$ be right exact. By considering X as a chain complex with only non-zero term in degree 0, the hyperderived left functor $\mathbb{L}_iF(X)=L_iF(X)$ is the same as the usual left derived functor.
- Let $F:\mathcal{A}\to\mathcal{B}$ be left exact. By considering X as a cochain complex with only non-zero term in degree 0, the hyperderived right functor $\mathbb{R}^iF(X)=R^iF(X)$ is the same as the usual right derived functor.

Lemma 6.6.6 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Then the following are true.

- Let $F:\mathcal{A}\to\mathcal{B}$ be right exact. Then the restriction of $\mathbb{L}_iF:\mathbf{Ch}(\mathcal{A})\to\mathcal{B}$ to $\mathbf{Ch}^+(\mathcal{A})$ are precisely the left derived functor $L_i(H_0F):\mathbf{Ch}^+(\mathcal{A})\to\mathcal{B}$ of the functor $H_0F:\mathbf{Ch}(\mathcal{A})^+\to\mathcal{B}$.
- Let $F:\mathcal{A}\to\mathcal{B}$ be left exact. Then the restriction of $\mathbb{R}^iF:\mathbf{CCh}(\mathcal{A})\to\mathcal{B}$ to $\mathbf{CCh}^+(\mathcal{A})$ are precisely the left derived functor $R^i(H^0F):\mathbf{CCh}^+(\mathcal{A})\to\mathcal{B}$ of the functor $H^0F:\mathbf{CCh}(\mathcal{A})^+\to\mathcal{B}$.

Lemma 6.6.7 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let the following

$$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$$

be a short exact sequence of chain complexes in $\mathbf{Ch}^+(\mathcal{A})$. Let $F:\mathcal{A}\to\mathcal{B}$ be right exact. Then there exists a long exact sequence

$$\cdots\longrightarrow\mathbb{L}_{i+1}F(C)\xrightarrow{\delta}\mathbb{L}_iF(A)\longrightarrow\mathbb{L}_iF(B)\longrightarrow\mathbb{L}_iF(C)\xrightarrow{\delta}\mathbb{L}_{i-1}F(A)\longrightarrow\cdots$$

Proposition 6.6.8 There is always a convergent spectral sequence

$$E^2_{pq}=(L_pF)(H_q(A))\Rightarrow\mathbb{L}_{p+q}F(A)$$

Corollary 6.6.9 Let $\mathcal{A},\mathcal{B}$ be abelian categories. Let $F:\mathcal{A}\to\mathcal{B}$ be a right exact functor. Then the following are true.

- If A is exact, then $\mathbb{L}_iF(A)=0$ for all i.
- If $f:A\to B$ is a quasi-isomorphism, then it induces isomorphisms $\mathbb{L}_iF(A)\cong\mathbb{L}_iF(B)$

Next: dual versions of the above propositions.

7 Triangulated Categories

7.1 Axioms of a Triangulated Category

Definition 7.1.1 (Triangles) Let $\mathcal{C}$ be a category and $T : \mathcal{C} \to \mathcal{C}$ an automorphism functor. Let $A, B, C \in \mathcal{C}$. A triangle on (A, B, C) is a triple (u, v, w) of morphisms in $\mathcal{C}$ where $u : A \to B$, $v : B \to C$, $w : C \to T(A)$.

Define similarly the homotopy categories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ for $\mathbf{CCh}^+(\mathcal{A})$, $\mathbf{CCh}^-(\mathcal{A})$ and $\mathbf{CCh}^b(\mathcal{A})$ respectively.

Definition 7.1.2 (Morphisms of Triangles) Let $\mathcal{C}$ be a category and $T : \mathcal{C} \to \mathcal{C}$ an automorphism functor. Let (u, v, w) and (u', v', w') be triangles in $\mathcal{C}$. A morphism of triangles is a triple (f, g, h) such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

Definition 7.1.3 (Triangulated Categories) Let $\mathcal{C}$ be an additive category. We say that $\mathcal{C}$ is a triangulated category if there is a functor $T : \mathcal{C} \to \mathcal{C}$ and a family $\{(u, v, w) \mid u, v, w \in \text{Mor}(\mathcal{C})\}$ of triangles called exact triangles such that the following hold.

- For any morphism $u : A \to B$, there exists an exact triangle (u, v, w):

  $$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, dashed, "\exists w"] & T(A) \end{tikzcd}$$

  If (u, v, w) is a triangle on (A, B, C) isomorphic to an exact triangle (u', v', w') on (A', B', C'), then (u, v, w) is also exact:

  $$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\cong"'] & B \arrow[r, "v"] \arrow[d, "\cong"'] & C \arrow[r, "w"] \arrow[d, "\cong"'] & T(A) \arrow[d, "\cong"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

  Finally, $(\text{id}_A, 0, 0)$ is exact:

  $$\begin{tikzcd} A \arrow[r, "\text{id}_A"] & A \arrow[r] & 0 \arrow[r] & T(A) \end{tikzcd}$$

- Rotations: If (u, v, w) is an exact triangle on (A, B, C), then both rotations $(v, w, -T(u))$ and $(-T^{-1}(w), u, v)$ are exact triangles on $(B, C, T(A))$ and $(T^{-1}(C), A, B)$ respectively.
- Morphisms: Let the following be exact triangles:

  $$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A) \end{tikzcd}$$

  $$\begin{tikzcd} A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A') \end{tikzcd}$$

  Suppose that there exists morphisms $f : A \to A'$ and $g : B \to B'$ such that $g \circ u = u' \circ f$. Then there exists $h : C \to C'$ such that (f, g, h) is a morphism of triangles:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"'] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

- The Octahedral Axiom: Let the following be exact triangles:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} T(A)$$

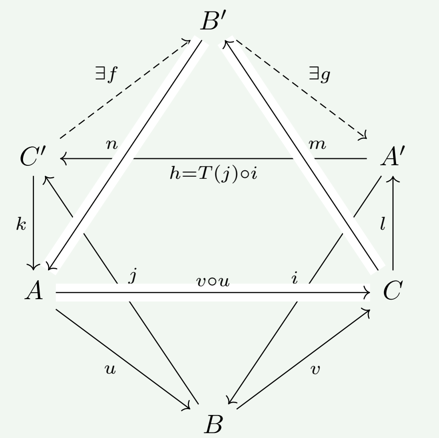
$$B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} T(B)$$

$$A \xrightarrow{v \circ u} C \xrightarrow{m} B' \xrightarrow{n} T(A)$$

  Then there exists an exact triangle:

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} T(C')$$

  such that $l = g \circ m$, $k = n \circ f$, $h = T(j) \circ i$, $i \circ g = T(u) \circ n$ and $f \circ j = m \circ v$. In other words, the following diagram commutes:



  Where we abused notation by drawing $k : C' \to T(A)$ as a morphism $C' \to A$ etc so that the drawing becomes compact.

Lemma 7.1.4 Let $(\mathcal{C}, T)$ be a triangulated category. Let (u, v, w) be an exact triangle. Then $v \circ u$, $w \circ v$ and $T(u) \circ w$ are 0 in $\mathcal{C}$.

Lemma 7.1.5 Let $(\mathcal{C}, T)$ be a triangulated category. Let (f, g, h) be a morphism of exact triangles. If both f and g are isomorphisms, then h is an isomorphism.

7.2 Morphisms of Triangulated Categories

Definition 7.2.1 (Morphisms of Triangulated Categories) Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. A morphism from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that the following are true.

- F is an additive functor
- F commutes with the translation functor. If T is the automorphism of $\mathcal{C}$ and S is the automorphism of $\mathcal{D}$, then
$$F \circ T = S \circ F$$
- F sends exact triangles to exact triangles

8 Derived Categories

8.1 The Homotopy Category of Cochain Complexes

Definition 8.1.1 (Homotopy Category of Cochain Complexes) Let $\mathcal{A}$ be an abelian category. Define the homotopy category of chain complexes

$$K(\mathcal{A})$$

to be the category defined as follows.
- The objects are the objects of $\mathbf{CCh}(\mathcal{A})$
- The morphisms are homotopy classes of chain maps
- Composition is given by composition of chain maps

Lemma 8.1.2 Let $\mathcal{A}$ be an abelian category. Then the cohomology functors $H^\bullet : \mathbf{CCh}(\mathcal{A}) \to \mathcal{A}$ induces a well defined functor from $K(\mathcal{A})$ to $\mathcal{A}$.

Proposition 8.1.3 Let $\mathcal{A}$ be an abelian category. The homotopy category of cochain complexes satisfy the following universal property.

If $F : \mathbf{CCh}(\mathcal{A}) \to \mathcal{D}$ is a functor that sends chain homotopy equivalences to isomorphisms, then F factors uniquely through $K(\mathcal{A})$:

$$\begin{tikzcd} \mathbf{CCh}(\mathcal{A}) \arrow[r] \arrow[dr, "F"'] & K(\mathcal{A}) \arrow[d, dashed, "\exists!"] \\ & \mathcal{D} \end{tikzcd}$$

Definition 8.1.4 (Distinguished Triangles in K($\mathcal{A}$)) Let $\mathcal{A}$ be an abelian category. We say that a triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

is distinguished in $K(\mathcal{A})$ if it is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow T(X)$$

Lemma 8.1.5 Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$, $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ are all triangulated categories with distinguished triangles given by the above definition.

8.2 Localization of Categories

Definition 8.2.1 (Localization of a Category) Let $\mathcal{C}$ be a category and let S be a collection of morphisms in $\mathcal{C}$. A localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ together with a functor $q : \mathcal{C} \to S^{-1}\mathcal{C}$ such that the following are true.
- For all $s \in S$, $q(s)$ is an isomorphism in $S^{-1}\mathcal{C}$
- If $F : \mathcal{C} \to \mathcal{D}$ is a functor such that $F(s)$ is an isomorphism for all $s \in S^{-1}\mathcal{C}$, then there exists a unique functor $G : S^{-1}\mathcal{C} \to \mathcal{D}$ such that the following diagram commute:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[dr, "F"'] & S^{-1}\mathcal{C} \arrow[d, dashed, "\exists! G"] \\ & \mathcal{D} \end{tikzcd}$$

Lemma 8.2.2 Let $\mathcal{A}$ be an abelian category. Then $K(\mathcal{A})$ is a localization of $\mathcal{A}$ with respect to all homotopy equivalences.

Not all localizations are well defined by set-theoretic issues. Morphisms that one wants to invert may not form a set or even a collection. We will give a way of explicitly constructing the localization of some specific categories below.

Definition 8.2.3 (Multiplicative System)

Definition 8.2.4 (Locally Small Multiplicative System)

Theorem 8.2.5 (Gabriel-Zisman Theorem)

Corollary 8.2.6 Let $\mathcal{C}$ be a category containing the zero object 0 and let $q : \mathcal{C} \to S^{-1}\mathcal{C}$ be a localization of $\mathcal{C}$. Then $q(X) \cong 0$ if and only if the S contains the 0 map $0 : X \to X$.

Corollary 8.2.7 Let $\mathcal{C}$ be a category and let $q : \mathcal{C} \to S^{-1}\mathcal{C}$ be a localization of $\mathcal{C}$. If $\mathcal{C}$ is additive, then $S^{-1}\mathcal{C}$ and q are both additive.

8.3 Derived Categories

Definition 8.3.1 (Derived Categories) Let $\mathcal{A}$ be an abelian category and let $\mathbf{CCh}(\mathcal{A})$ be any category of chain complexes of $\mathcal{A}$. Define the derived category

$$D(\mathcal{A}) = \mathbf{CCh}(\mathcal{A})[\mathcal{W}^{-1}]$$

of $\mathcal{A}$ where $\mathcal{W}$ is all the quasi-isomorphisms in $\mathbf{CCh}(\mathcal{A})$.

Respectively, for $\mathbf{CCh}^+(\mathcal{A})$ and $\mathbf{CCh}^b(\mathcal{A})$ define their derived categories to be the localization of the categories with respect to quasi-isomorphisms, denoted by $D^+(\mathcal{A})$ and $D^b(\mathcal{A})$ respectively.

Theorem 8.3.2 Let $\mathcal{A}$ be an abelian category. Let $K(\mathcal{A})$ be the homotopy category of chain complexes of $\mathcal{A}$. Let $\mathcal{W}$ be all the quasi-isomorphisms in $\mathbf{CCh}(\mathcal{A})$. Then there is an equivalence of categories

$$D(\mathcal{A}) = K(\mathcal{A})[\mathcal{W}^{-1}]$$

Definition 8.3.3 (Distinguished Triangles in D($\mathcal{A}$)) Let $\mathcal{A}$ be an abelian category. We say that a triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

is distinguished in $D(\mathcal{A})$ if it is isomorphic to a triangle of the form

$$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow T(X)$$

Theorem 8.3.4 Let $\mathcal{A}$ be an abelian category. Then $D(\mathcal{A})$ is a triangulated triangles with distinguished triangles given by the above definition.

Definition 8.3.5 (Full Subcategory of Complexes) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories such that $\mathcal{A}$ is a subcategory of $\mathcal{B}$. Define

$$D_{\mathcal{A}}^{(}\mathcal{B})$$

to be the full triangulated subcategory of complexes with cohomology in $\mathcal{A}$.

8.4 Derived Functors of Derived Categories

Definition 8.4.1 (Total Right Derived Functors) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. A total right derived functor of F is a morphism
$$RF : D(\mathcal{A}) \to D(\mathcal{B})$$
together with a natural transformation $\xi : (K(\mathcal{A}) \to K(\mathcal{B}) \to D(\mathcal{B})) \Rightarrow (K(\mathcal{A}) \to D(\mathcal{A}) \to D(\mathcal{B}))$ using the following diagram:

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\exists!"'] & K(\mathcal{B}) \arrow[d, "\exists!"] \\ D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B}) \end{tikzcd}$$

from the top-right path to the lower-left path which is universal in the following sense. If $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another morphism equipped with the same natural transformation χ, then there exists a unique natural transformation
$$\eta : RF \Rightarrow G$$
such that $\chi = \eta \circ \xi$.

Definition 8.4.2 (Total Left Derived Functors) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. A total right derived functor of F is a morphism
$$LF : D(\mathcal{A}) \to D(\mathcal{B})$$
together with a natural transformation $\xi : (K(\mathcal{A}) \to D(\mathcal{A}) \to D(\mathcal{B})) \Rightarrow (K(\mathcal{A}) \to K(\mathcal{B}) \to D(\mathcal{B}))$ using the following diagram:

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\exists!"'] & K(\mathcal{B}) \arrow[d, "\exists!"] \\ D(\mathcal{A}) \arrow[r, "RF"'] & D(\mathcal{B}) \end{tikzcd}$$

from the lower-left path to the top-right path which is universal in the following sense. If $G : D(\mathcal{A}) \to D(\mathcal{B})$ is another morphism equipped with the same natural transformation χ, then there exists a unique natural transformation
$$\eta : G \Rightarrow LF$$
such that $\chi = \eta \circ \xi$.

Lemma 8.4.3 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. Let $F : K(\mathcal{A}) \to K(\mathcal{B})$ be a morphism of triangulated categories. If F is exact, then F is its own left and right total derived functor.

Theorem 8.4.4 Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Then the following are true.

- If F is left exact and $\mathcal{A}$ has enough injectives, then
$$\mathbb{R}^i F = H^i(RF) : \mathbf{CCh}(\mathcal{A}) \to \mathcal{B}$$
- If F is right exact and $\mathcal{A}$ has enough projectives, then
$$\mathbb{L}_i F = H_i(LF) : \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$$

Notice that in particular, if $X \in \mathcal{A}$ is an object considered as a chain complex in degree 0, we have seen that $\mathbb{L}_i F = L_i F : \mathcal{A} \to \mathcal{B}$ hence $L_i F = H_i(LF) : \mathcal{A} \to \mathcal{B}$. This is similar for the dual version.