Commutative Algebra 1

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Abstract

Contents

1 Ideals Of a Commutative Ring

1.1 Basic Operations on Ideals

Recall that $(R,+,\cdot)$ is a ring if the following axioms hold.

- $(R,+)$ is an abelian group.
- Multiplicative Associativity: $x\cdot(y\cdot z)=(x\cdot y)\cdot z$.
- Multiplicative Identity: There exists $1_R\in R$ such that $x\cdot 1_R=x=1_R\cdot x$ for all $x\in R$.
- Left distributivity: $r\cdot(x+y)=r\cdot x+r\cdot y$ for all $r,x,y\in R$.
- Right distributivity: $(x+y)\cdot r=x\cdot r+y\cdot r$ for all $r,x,y\in R$.

A ring R is commutative if
$$x\cdot y=y\cdot x$$
for all $x,y\in R$.

Let R be a commutative ring. Recall that an ideal of R is a subset $I\subseteq R$ such that

- If $a,b\in I$, then $a+b\in I$.
- If $r\in R$ and $a\in I$, then $ra\in I$.

Lemma 1.1.1 Let R be a commutative ring. Let I,J be ideals of R. Let P be a prime ideal of R. Then the following are equivalent.
- $IJ\subseteq P$.
- $I\cap J\subseteq P$.
- $I\subseteq P$ or $J\subseteq P$.

Proof

- (1) $\implies$ (2): Let $f\in I\cap J$. Then $f\in I$ and $f\in J$ implies that $f^2\in IJ\subseteq P$. Since P is prime, we conclude that $f\in P$.
- (2) $\implies$ (3): Suppose that $f\in I$ and $f\notin P$. For any $g\in J$, we have $fg\in I\cap J\subseteq P$. Since P is prime and $f\in I$, we have $J\in P$.
- (3) $\implies$ (1): Without loss of generality suppose that $I\subseteq P$. Then $IJ\subseteq I\subseteq P$.

■

Proposition 1.1.2 (Prime Avoidance) Let R be a commutative ring. Let $I_1,\dots,I_n$ be ideals of R. Let $P_1,\dots,P_k$ be prime ideals of R.
- Let I be an ideal of R. If $I\subseteq\bigcup_{i=1}^k P_i$, then $I\subseteq P_i$ for some i.
- Let P be an ideal of R. If $\bigcap_{i=1}^n I_i\subseteq P$, then $I_i\subseteq P$ for some i.
- Let P be an ideal of R. If $P=\bigcap_{i=1}^n I_i$, then $I_i=P$ for some i.

Proof

- We prove the contrapositive by induction k. When $k=1$, the case is clear. Suppose that $I\not\subseteq P_i$ for $1\le i\le k-1$ implies $I\not\subseteq\bigcup_{i=1}^{k-1}P_i$. Now suppose that $I\not\subseteq P_i$ for $1\le i\le k$. By induction hypothesis, for each i, there exists $x_j\in I$ such that $x_j\notin\bigcup_{i\neq j}P_i$. So $x_j\notin P_i$ for $j\neq i$. There are two cases. If $x_j\notin P_j$ for some j, then $x_j\notin\bigcup_{j\neq i}P_i\cup P_j=\bigcup_{i=1}^k P_i$ so we are done. If $x_j\in P_j$ for all j, then consider the element $y=\sum_{i=1}^k\prod_{j\neq i}x_j\in I$. Notice that $x_j \in P_j$ for $j \neq i$ implies that $\prod_{j\neq i} x_j$ lie in P_k for any $k \neq i$. It is not an element of P_i because P_i is prime and $x_j \notin P_i$ for $j \neq i$. Then we conclude that y does not lie in P_i for any i. Hence $y \notin \bigcup_{i=1}^k P_i$ and we are done.

- We prove the contrapositive. Suppose that $I_i \not\subseteq P$ for all i. Then for each i, there exists $x_i \in I_i$ such that $x_i \notin P$. Then $\prod_{i=1}^n x_i \in \bigcap_{i=1}^n I_i$ is not an element of P since P is a prime ideal. Hence we are done.
- By the above, we have that $P = \bigcap_{i=1}^n I_i$ implies that $I_i \subseteq P$ for some i. Then $P = \bigcap_{i=1}^n I_i \subseteq I_i$ implies that $P = I_i$.

■

Example 1.1.3 There is an isomorphism given by

$$\frac{\mathbb{Z}[x]}{(x+1, x^2+2)} \cong \mathbb{Z}/3\mathbb{Z}$$

Proof Using the above propositions, we have that

$$\begin{aligned}\frac{\mathbb{Z}[x]}{(x+1, x^2+2)} &= \frac{\mathbb{Z}[x]}{(x+1)+(x^2+2)}\\ &\cong \frac{\mathbb{Z}[x]/(x+1)}{(3)}\end{aligned}$$

Indeed, the ideal (x^2+2) corresponds to the ideal (3) in $\frac{\mathbb{Z}[x]}{(x+1)}$ because the remainder of x^2+2 divided by $(x+1)$ is (3). Now $\mathbb{Z}[x]/(x+1) \cong \mathbb{Z}$ by the evaluation homomorphism. Thus quotienting by the ideal (3) gives the field $\mathbb{Z}/3\mathbb{Z}$. ■

Let R be a commutative ring. Recall that two ideals I, J are coprime if $I + J = R$. In particular, this implies that $IJ = I \cap J$. Then the Chinese Remainder theorem reads as

$$\frac{R}{\prod_{i=1}^k I_i} = \frac{R}{\bigcap_{i=1}^k I_i} \cong \prod_{i=1}^k \frac{R}{I_i}$$

1.2 The Nilradical of Commutative Rings

Let R be a ring. Recall that an element $r \in R$ is nilpotent if $r^n = 0_R$ for some $n \in \mathbb{N}$. When R is commutative, we can form an ideal out of nilpotent elements.

Definition 1.2.1 (Nilradicals) Let R be a commutative ring. Define the nilradical of R to be

$$N(R) = \{r \in R \mid r \text{ is nilpotent}\}$$

Note that this is different from nilpotent ideals, as nilpotency is a property of an ideal. However the Nilradical ideal is a nil ideal and every sub-ideal of the nilradical is a nil ideal.

Proposition 1.2.2 Let R be a ring and $N(R)$ its nilradical. Then the following are true.
- $N(R)$ is an ideal of R
- $N(R/N(R)) = 0$

Proof

- Suppose that r, s are nilpotent, meaning that $r^n = 0$ and $s^m = 0$. Then $(r+s)^{n+m} = 0$. Moreover, if $t \in R$ then $t \cdot r$ is also nilpotent
- Let $r \notin N(R)$. Every element $r + N(R) \in R/N(R)$ has the property that $r^n \neq 0$. Consider $(r + N(R))^n = r^n + N(R)$. If $r^n \in N(R)$ then $r^n = u$ for some nilpotent u, which means that r^n is nilpotent and thus r is nilpotent, a contradiction. This means that $r + N(R) \notin N(R/N(R))$ for all $r \notin N(R)$ and thus $N(R/N(R)) = 0$

■

Proposition 1.2.3 Let R be a commutative ring. Then we have

$$N(R) = \bigcap_{\substack{P \text{ is a prime} \\ \text{ideal of } R}} P$$

Proof Let $x \in N(R)$. Let P be an arbitrary prime ideal. Since x is nilpotent, $x^n = 0$ for some $n \in \mathbb{N}$. If $x \notin P$, then $x^2 \notin P$ since P is a prime ideal. Recursively we see that $x^k \notin P$ for all $k \in N \setminus \{0\}$. But $x^n = 0 \in P$ is a contradiction. Hence $N(R) \subseteq \bigcap_{P \in \text{Spec}(R)} P$.

Now suppose that $x \in R$ is not nilpotent. Consider the set

$$\Sigma = \{I \trianglelefteq R \mid x^k \notin I \text{ for all } k \geq 1\}$$

Notice that $(0) \in \Sigma$ and hence it is non-empty. Let $I_1 \subseteq I_2 \subseteq \cdots$ be a chain in Σ. Define $I = \bigcup_{k=1}^{\infty} I_k$. I claim that $I \in \Sigma$. First of all if $a, b \in I$ and $r \in R$, then $a \in I_m$ and $b \in I_n$ for some $m, n \geq 1$. Then $a, b \in I_{\max\{m,n\}}$ so that $a + b \in I_{\max\{m,n\}} \subseteq I$. Also $ra \in I_m \subseteq I$ since I_m is an ideal. Hence I itself is an ideal of R. Suppose for a contradiction that $x^n \in I$ for some n. Then $x^n \in I_k$ for some k. This is a contradiction since $I_k \in \Sigma$. Thus we know that $I \in \Sigma$. In particular, I is an upper bound of $I_1 \subseteq I_2 \subseteq \cdots$. By Zorn's lemma, we conclude that Σ has a maximal element, say P.

Suppose for a contradiction that P is not a prime ideal. Let $ab \in P$ and $a, b \notin P$. Then $P \subset P + (a), P + (b)$. Since P is maximal in Σ, $P + (a)$ and $P + (b)$ cannot be in Σ, and there exists $x^m \in P + (a)$ and $x^n \in P + (b)$ for some m, n. Then

$$x^{m+n} = x^m \cdot x^n \in (P + (a))(P + (b)) = P + (ab)$$

Hence $P + (ab) \notin \Sigma$. But $ab \in P$ implies that $P + (ab) = P$. We have reached a contradiction. Thus P is a prime ideal that does not contain x. We show that $x \notin N(R)$ implies $x \notin P$ for some prime ideal P. The contrapositive of this statement is $x \in P$ for all prime ideals P implies $x \in N(R)$. Hence we are done. ■

Example 1.2.4 Consider the ring

$$R = \frac{\mathbb{C}[x, y]}{(x^2 - y, xy)}$$

Then its nilradical is given by $N(R) = (x, y)$.

Proof Notice that in the ring R, $x^3 = x(x^2) = xy = 0$ and $y^3 = x^6 = (x^3)^2 = 0$ and hence x and y are both nilpotent elements of R. By definition of the nilradical, we conclude that $(x, y) \subseteq N(R)$. Now (x, y) is a maximal ideal of $\mathbb{C}[x, y]$ because $\mathbb{C}[x, y]/(x, y) \cong \mathbb{C}$. Also notice that $(x, y) \supseteq$

$(x^2 - y, xy)$ because for any element $f(x)(x^2 - y) + g(x)(xy) \in (x^2 - y, xy)$, we have that

$$\begin{aligned} f(x)(x^2 - y) + g(x)(xy) \in (x^2 - y, xy) &= (xf(x))x - f(x)y + (g(x)x)y \\ &= (xf(x))x + (xg(x) - f(x))y \in (x, y) \end{aligned}$$

By the correspondence theorem, $(x, y)/(x^2 - y)$ is an maximal ideal of R. In particular, (x, y) is also a prime ideal. But the $N(R)$ is the intersection of all prime ideals and hence $N(R) \subseteq (x, y)$. We conclude that $N(R) = (x, y)$. ∎

Definition 1.2.5 (Reduced Rings) Let R be a commutative ring. We say that R is reduced if $N(R) = 0$.

1.3 The Jacobson Radical of Commutative Rings

Let R be a commutative ring. Recall that the Jacobson radical of a ring is defined to be

$$J(R) = \bigcap_{m \text{ a maximal ideal}} m$$

since left and right maximal ideals coincide in R. Properties of the Jacobson radical include:

- $J(R/J(R)) = 0$.

Lemma 1.3.1 Let R be a commutative ring. Then $x \in J(R)$ if and only if $1 - xy \in R^\times$ for all $y \in R$.

Proof Suppose that $x \notin J(R)$. Then $x \notin m$ for some maximal ideal m. Then $R = m + (x)$ since m is maximal. Then there exists $p \in m$ and $y \in R$ such that $1 = p + xy$. Then $1 - xy = p \in m \notin R^\times$.

Suppose that $1 - xy \notin R^\times$ for some $y \in R$. Then $(1 - xy)$ is a proper ideal of R. Then there exists a maximal ideal m such that $(1 - xy) \subseteq m$. If $x \in m$ then $yx \in m$ which implies that $1 = xy + 1 - xy \in m$. This is a contradiction and so $x \notin m$. Hence $x \notin J(R)$. ∎

Lemma 1.3.2 Let R be a commutative ring. Then $x \in R$ is a unit if and only if $[x] \in R/J(R)$ is a unit.

Proof Suppose that $x \in R$ is a unit. Then there exists $y \in R$ such that $xy = 1$. Then $[x][y] = [1]$ so we are done. Now suppose that $[x][y] = [1]$ for some $y \in R$. Then there exists $m \in J(R)$ such that $xy = 1 + m$. By the above lemma, $1 + m$ is a unit hence x is a unit. ∎

1.4 The Radical of an Ideal

The radical of an ideal is a very different notion from the radical of module.

Definition 1.4.1 (Radical of an Ideal) Let I be an ideal of a ring R. Define the radical of I to be

$$\sqrt{I} = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N}\}$$

Proposition 1.4.2 Let R be a commutative ring. Let I be an ideal. Then the following are true.

- $I \subseteq \sqrt{I}$
- $\sqrt{\sqrt{I}} = \sqrt{I}$
- $\sqrt{I^m} = \sqrt{I}$ for all $m \geq 1$
- $\sqrt{I} = R$ if and only if $I = R$

Proof

- Let $r \in I$. Then $r^1 \in I$ Thus by choosing $n = 1$ we shows that $r^n \in I$. Thus $r \in \sqrt{I}$.
- By the above, we already know that $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. So let $r \in \sqrt{\sqrt{I}}$. Then there exists some $n \in \mathbb{N}$ such that $r^n \in \sqrt{I}$. But $r^n \in \sqrt{I}$ means that there exists some $m \in \mathbb{N}$ such that $(r^n)^m \in I$. But $nm \in \mathbb{N}$ is a natural number such that $r^{nm} \in I$. Hence $r \in \sqrt{I}$ and so we conclude.
- Since $I^m \subseteq I$, we know that $\sqrt{I^m} \subseteq \sqrt{I}$. Let $x \in \sqrt{I}$. Then $x^n \in I$ for some $n \in \mathbb{N}$. Then we have $(x^n)^m = x^{n+m} \in I^m$ so that $x \in \sqrt{I^m}$.
- Clearly if $I = R$ then $I \subseteq \sqrt{I}$ implies that $\sqrt{I} = R$. Conversely, $\sqrt{I} = R$ implies that $1 \in \sqrt{I}$ and hence $1 \in I$. Hence $I = R$.

■

Proposition 1.4.3 Let R be a commutative ring. Let I, J be ideals of R. Then the following are true.

- If $I \subseteq J$ then $\sqrt{I} \subseteq \sqrt{J}$
- $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$
- $\sqrt{I + J} = \sqrt{\sqrt{I} + \sqrt{J}}$

Proof

- Let $x \in \sqrt{I}$. Then $x^n \in I$ for some $n \in \mathbb{N}$. Then $x^n \in J$ so $x \in \sqrt{J}$.
- Since $IJ \subseteq I \cap J \subseteq I, J$, we already have $\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}$. Let $x \in \sqrt{I} \cap \sqrt{J}$. Then there exists $n, m \in \mathbb{N}$ such that $x^n \in I$ and $x^m \in J$. Then $x^n \cdot x^m = x^{n+m} \in IJ$ implies that $x \in \sqrt{IJ}$.
- Since $I, J \subseteq I + J$, we have $\sqrt{I} + \sqrt{J} \subseteq \sqrt{I + J}$ so that $\sqrt{\sqrt{I} + \sqrt{J}} \subseteq \sqrt{I + J}$. On the other hand, $I \subseteq \sqrt{I}$ and $J \subseteq \sqrt{J}$ implies that $I + J \subseteq \sqrt{I} + \sqrt{J}$. Then $\sqrt{I + J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}}$ and so we are done.

■

Lemma 1.4.4 Let R be a commutative ring. Then we have

$$N(R) = \sqrt{(0)}$$

Proof True from definitions. ■

Lemma 1.4.5 Let R be a commutative ring. Let I be an ideal of R. Let $\pi : R \to R/I$ be the quotient homomorphism. Then we have

$$\sqrt{I} = \pi^{-1}\left(N\left(\frac{R}{I}\right)\right)$$

Proof Let $x \in R$. Then we have that $x^n \in I$ if and only if $\pi(x^n) = x^n + I = I$ if and only if $x + I \in N(R/I)$. ■

Proposition 1.4.6 Let R be a commutative ring. Let I be an ideal. Then

$$\sqrt{I} = \bigcap_{\substack{p \text{ a prime ideal} \\ I \subseteq p \subseteq R}} p$$

Proof Write $\pi : R \to R/I$ the quotient homomorphism. Using prp1.2.3 and the correspondence theorem, we have that

$$\sqrt{I} = \pi^{-1}\left(\bigcap_{\substack{P \text{ is a prime} \\ \text{ideal of } R}} P\right) = \bigcap_{\substack{P \text{ is a prime} \\ \text{ideal of } R}} \pi^{-1}(P) = \bigcap_{\substack{p \text{ a prime ideal} \\ I \subseteq p \subseteq R}} p$$

Definition 1.4.7 (Radical Ideals) Let R be a commutative ring. Let I be an ideal of R. We say that I is radical if

$$\sqrt{I} = I$$

In particular, by the above lemma it follows that the radical of an ideal is a radical ideal.

Lemma 1.4.8 Let R be a ring. Let P be a prime ideal of R. Then P is radical.

Proof We already know that $P \subseteq \sqrt{P}$. Let $x \in \sqrt{P}$. Then $x^n \in P$ for some $n \in \mathbb{N}$. Since P is prime, by inducting downwards we deduce that $x \in P$. Thus P is radical.

We conclude that there is an inclusion of types of ideal in which each inclusion is strict:

$$\substack{\text{Maximal} \\ \text{ideals}} \subset \substack{\text{Prime} \\ \text{ideals}} \subset \substack{\text{Radical} \\ \text{ideals}}$$

Proposition 1.4.9 Let R be a commutative ring. Let I be an ideal of R. Then R/I is reduced if and only if I is a radical ideal.

So radical, prime and maximal ideals all have characterizations using the quotient ring:

- I is maximal if and only if R/I is a field.
- I is prime if and only if R/I is an integral domain.
- I is radical if and only if R/I is reduced.

1.5 The Correspondence between Ideals and the Quotient

Definition 1.5.1 (Max Spectrum of a Ring) Let A be a commutative ring. Define the max spectrum of A to be

$$\text{maxSpec}(A) = \{m \subseteq A \mid m \text{ is a maximal ideal of } A\}$$

Definition 1.5.2 (Spectrum of a Ring) Let A be a commutative ring. Define the spectrum of A to be

$$\text{Spec}(A) = \{p \subseteq A \mid p \text{ is a prime ideal of } A\}$$

Example 1.5.3 Consider the following commutative rings.

- $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2 + 6\mathbb{Z}), (3 + 6\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/8\mathbb{Z}) = \{(2 + 8\mathbb{Z})\}$
- $\text{Spec}(\mathbb{Z}/24\mathbb{Z}) = \{(2 + 24\mathbb{Z}), (3 + 24\mathbb{Z})\}$
- $\text{Spec}(\mathbb{R}[x]) = \{(f) \mid f \text{ is irreducible }\}$

Proof

- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2 + 6\mathbb{Z})$ and $(3 + 6\mathbb{Z})$. We need to find which ones are prime ideals. Now $\mathbb{Z}/6\mathbb{Z} \setminus (2+6\mathbb{Z})$ consists of $1+6\mathbb{Z}$, $3+6\mathbb{Z}$ and $5+6\mathbb{Z}$. No multiplication of these elements give an element of $(2 + 6\mathbb{Z})$. So any two elements in $\mathbb{Z}/6\mathbb{Z}$ which multiply to an element of $(2+6\mathbb{Z})$ must contain one element that lie in $(2+6\mathbb{Z})$. Hence $(2+6\mathbb{Z})$ is prime. This is similar for $(3+6\mathbb{Z})$. Hence $\text{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2+6\mathbb{Z}), (3+6\mathbb{Z})\}$.

- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2+8\mathbb{Z})$ and $(4+8\mathbb{Z})$. A similar argument as above shows that $(2+8\mathbb{Z})$ is a prime ideal. However, $6+8\mathbb{Z} \notin (4+8\mathbb{Z})$ while $(6+8\mathbb{Z})^2 = 4+8\mathbb{Z} \in (4+8\mathbb{Z})$ which shows that $(4+8\mathbb{Z})$ is not a prime ideal.
- A similar proof as above ensues.
- Recall that $\mathbb{R}[x]$ is a principal ideal domain. Let $I = (f)$ be a prime ideal of $\mathbb{R}[x]$. Then f is irreducible. Thus every prime ideal of $\mathbb{R}[x]$ is of the form (f) for f an irreducible polynomial.

■

Lemma 1.5.4 Let R, S be commutative rings. Let $f_1 : R \times S \to R$ and $f_2 : R \times S \to S$ denote the projection maps. Then the map

$$f_1^* \amalg f_2^* : \text{Spec}(R) \amalg \text{Spec}(S) \to \text{Spec}(R \times S)$$

is a bijection.

Proof The core of the proof is the fact that P is a prime ideal of $R \times S$ if and only if $P = R \times Q$ or $P = V \times S$ for either a prime ideal Q of P or a prime ideal V of S. It is clear that if Q is a prime ideal of S and V is a prime ideal of R, then $R \times Q$ and $V \times S$ are both prime ideals of $R \times S$.

So suppose that P is a prime ideal in $R \times S$. Let $e_1 = (1,0)$ and $e_2 = (0,1)$. Since $P \neq R$, at least one of e_1 or e_2 is not in P. Without loss of generality assume that $e_1 \notin P$. But $e_1 e_2 = 0 \in P$ and P being prime implies that $e_2 \in P$. Since e_2 is the identity of $\{0\} \times S \cong S$, we conclude that $\{0\} \times S \subseteq P$. By the correspondence theorem, the projection map $f_1 : R \times S \to R$ gives a bijection between prime ideals of $R \times S$ that contain $\{0\} \times S$ and prime ideals of R. So $f_1(P)$ is a prime ideal of R. Thus $P = f_1(P) \times S$ which is exactly what we wanted.

Now the bijection is clear. $f_1^* \amalg f_2^*$ sends a prime ideal P of R to $P \times S$ and it sends a prime ideal Q of S to $R \times Q$. This map is surjective by the above argument. It is injective by inspection. ■

Theorem 1.5.5 Let R be a commutative ring. Let I be an ideal of R. Denote φ to be the inclusion preserving one-to-one bijection

$$\left\{\substack{\text{Ideals of } R\\ \text{containing } I}\right\} \overset{1:1}{\longleftrightarrow} \{\text{Ideals of } R/I\}$$

from the correspondence theorem for rings. In other words, $\varphi(A) = A/I$. Let $J \subseteq R$ be an ideal containing I. Then the following are true.

- J is a radical ideal if and only if $\varphi(J) = J/I$ is a radical ideal.
- J is a prime ideal if and only if $\varphi(J) = J/I$ is a prime ideal.
- J is a maximal ideal if and only if $\varphi(J) = J/I$ is a maximal ideal.

Proof

- Let J be a radical ideal. Suppose that $r+I \in \sqrt{J/I}$. This means that $(r+I)^n = r^n + I \in J/I$ for some $n \in \mathbb{N}$. But this means that $r^n \in J$. This implies that $r \in \sqrt{J} = J$. Thus $r+I \in J/I$ and we conclude that $\sqrt{J/I} \subseteq J/I$. Since we also have $J/I \subseteq \sqrt{J/I}$, we conclude.

  Now suppose that J/I is a radical ideal. Let $r \in \sqrt{J}$. This means that $r^n \in J$ for some $n \in \mathbb{N}$. Now $r^n + I = (r+I)^n \in J/I$ implies that $r+I \in \sqrt{J/I} = J/I$. Hence $r \in J$ and so $\sqrt{J} \subseteq J$. Since we also have that $J \subseteq \sqrt{J}$, we conclude.

- Let J be a prime ideal. Then R/J is an integral domain. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also an integral domain. Hence J/I is a prime ideal. The converse is also true.
- Let J be a maximal ideal. Then R/J is a field. By the second isomorphism theorem, we have that $R/J \cong (R/I)/(J/I)$ and hence $(R/I)/(J/I)$ is also a field. Hence J/I is a maximal ideal. The converse is also true.

Another way to write the bijections is via spectra:

$$\text{Spec}(R/I) \overset{1:1}{\longleftrightarrow} \{P \in \text{Spec}(R) \mid I \subseteq P\}$$

and

$$\text{maxSpec}(R/I) \overset{1:1}{\longleftrightarrow} \{m \in \text{maxSpec}(R) \mid I \subseteq m\}$$

1.6 Extensions and Contractions of Ideals

Definition 1.6.1 (Extension of Ideals) Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I be an ideal of R. Define the extension I^e of I to S to be the ideal

$$I^e = \langle f(i) \mid i \in I \rangle$$

Proposition 1.6.2 Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I, I_1, I_2 be an ideal of R. Then the following are true regarding the extension of ideals.
- If $I_1 \subseteq I_2$, then $I_1^e \subseteq I_2^e$.
- Closed under sum: $(I_1 + I_2)^e = I_1^e + I_2^e$
- $(I_1 \cap I_2)^e \subseteq I_1^e \cap I_2^e$
- Closed under products: $(I_1 I_2)^e = I_1^e I_2^e$
- $(\sqrt{I})^e \subseteq \sqrt{I^e}$

Proof

- Let $x \in I_1^e$. Then $x = \sum s_k f(i_k)$ for some $i_k \in I_1$. Then $i_k \in I_2$ implies that $x \in I_2^e$.
- Since $I_1, I_2 \subseteq I_1 + I_2$, we have $I_1^e + I_2^e \subseteq (I_1 + I_2)^e$. Conversely, let $x, \in (I_1 + I_2)^e$. Then $x = \sum s_k f(i_k)$ for $i_k \in I_1 + I_2$. Then we have

$$x = \sum_{i_k \in I_1} s_k f(i_k) + \sum_{i_k \in I_2} s_k f(i_k) \in I_1^e + I_2^e$$

  so we conclude.
- Since $I_1 \cap I_2 \subseteq I_1, I_2$ we are done.
- It suffices to check the generators lie in each other. Let $x \in I_1 I_2$. Then $x = \sum i_k j_k$ for some $i_k \in I_1$ and $j_k \in I_2$. Then $f(x) = \sum f(i_k) f(j_k)$. Since $f(i_k) \in I_1^e$ and $f(j_k)^e$, then $f(x) \in I_1^e I_2^e$ so we conclude that $(I_1 I_2)^e \subseteq I_1^e I_2^e$. Conversely, suppose that $x \in I_1^e I_2^e$. Then $x = \sum f(i_k)(j_k)$ for $i_k \in I_1$ and $j_k \in I_2$. Since f is a ring homomorphism, we have that

$$x = \sum f(i_k) f(j_k) = f\left(\sum i_k j_k\right)$$

  Since $\sum i_k j_k \in I_1 I_2$, we conclude that $x \in I_1^e I_2^e$.
- We have that

$$(\sqrt{I})^e = \left(f(i) \;\middle|\; i \in \bigcap_{\substack{P \text{ prime} \\ I \subseteq P}} P \right) \subseteq f\left(\bigcap_{\substack{P \text{ prime} \\ I \subseteq P}} f(P) \right) \subseteq f\left(\bigcap_{\substack{Q \text{ prime} \\ I^e \subseteq Q}} f(f^{-1}(Q)) \right)$$

The last inclusion follows since for $I^e \subseteq Q$, we must have that $I \subseteq f^{-1}(Q)$. Then we have that

$$(\sqrt{I})^e = f\left(\bigcap_{\substack{Q \text{ prime} \\ I^e \subseteq Q}} Q\right) = \sqrt{I^e}$$

and so we are done.

■

Definition 1.6.3 (Contraction of Ideals) Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let J be an ideal of S. Define the contraction J^c of J to R to be the ideal

$$J^c = f^{-1}(J)$$

Proposition 1.6.4 Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let J, J_1, J_2 be an ideal of S. Then the following are true regarding the extension of ideals.

- If $J_1 \subseteq J_2$, then $J_1^c \subseteq J_2^c$.
- $(J_1 + J_2)^c \supseteq J_1^c + J_2^c$
- Closed under intersections: $(J_1 \cap J_2)^c = J_1^c \cap J_2^c$
- $(J_1 J_2)^c \supseteq J_1^c J_2^c$
- Closed under taking radicals: $\text{rad}(J)^c = \text{rad}(J^c)$

Proof

- Clear since $f^{-1}(J_1) \subseteq f^{-1}(J_2)$ for $J_1 \subseteq J_2$.
- Since $J_1, J_2 \subseteq J_1 + J_2$, we have that $J_1^c + J_2^c \subseteq (J_1 + J_2)^c$.
- Since $J_1 \cap J_2 \subseteq J_1, J_2$, we have that $(J_1 \cap J_2)^c \subseteq J_1^c \cap J_2^c$. Let $x \in J_1^c \cap J_2^c$. Then we have $f(x) \in J_1, J_2$ so that $f(x) \in J_1 \cap J_2$. Hence $x \in (J_1 \cap J_2)^c$.
- Suppose that $x \in J_1^c$ and $y \in J_2^c$. Then $f(xy) = f(x)f(y) \in J_1^c J_2^c$. Hence $xy \in J_1^c J_2^c$.
-

■

Proposition 1.6.5 Let R, S be commutative rings. Let $f : R \to S$ be a ring homomorphism. Let I be an ideal of R and let J be an ideal of S. Then the following are true.

- $I \subseteq I^{ec}$
- $J^{ce} \subseteq J$
- $I^e = I^{ece}$
- $J^c = J^{cec}$

Proof

- Let $x \in I$. Then $f(x) \in I^e$. Thus $x \in f^{-1}(I^e)$.
- Since J^{ce} is generated by $f(x)$ for all $x \in J^c$, it suffices to check that $f(x) \in J$ for all $x \in J^c$. But $x \in J^c$ implies that $f(x) \in J$ so we are done.
- Since $I \subseteq I^{ec}$, we know that $I^e \subseteq I^{ece}$. Also, from the second item we take $J = I^e$ to get $I^{ece} \subseteq I^e$.
- From the first item, take $I = J^c$ to get $J^c \subseteq J^{cec}$. Also, since $J^{ce} \subseteq J$, we have that $J^{cec} \subseteq J^c$.

■

Example 1.6.6 Let S be a commutative ring and let $R \subseteq S$ be a subring. Let $f : R \to S$ be the inclusion map. Let $I \subseteq R$ be an ideal of R and let $J \subseteq S$ be an ideal of S. Then the following are true.

- $I^e = S \cdot I$.
- $J^c = J \cap R$.

1.7 Minimal Prime Ideals

Definition 1.7.1 (Minimal Prime Ideals) Let R be a commutative ring. Let I be an ideal of R. Let P be a prime ideal of R. We say that P is a minimal prime ideal over I if for any other prime ideal $Q \supseteq I$ containing I, we have $P \subseteq Q$.

Proposition 1.7.2 Let R be a commutative ring. Let I be an ideal of R. Then a minimal prime ideal over I exists.

2 Basic Notions of Commutative Rings

2.1 Noetherian Commutative Rings

We recall some facts about Noetherian rings. In the following, let R be a commutative ring, although they are also true if R is non-commutative if we take all modules defined below to be left (right) R-modules.

- If we have a short exact sequence of R-modules:
$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$
Then M_2 is Noetherian if and only if M_1 and M_3 are Noetherian.
- If M and N are R-modules, then $M \oplus N$ is Noetherian if and only if M and N are Noetherian.
- If M is an R-module and N is an R-submodule of M, then M is Noetherian if and only if N and M/N are Noetherian.
- If R is Noetherian and I is an ideal of R, then R/I is Noetherian.
- Later when once has seen localization, we can also prove that: If R is Noetherian then $S^{-1}R$ is Noetherian for any multiplicative subset S of R.

Proposition 2.1.1 Let R be a Noetherian commutative ring. Let I be an ideal of R. Then there exists $n \in \mathbb{N}$ such that
$$\sqrt{I}^n \subseteq I \subseteq \sqrt{I}$$

Proof It is clear that $I \subseteq \sqrt{I}$. Since R is Noetherian, $\sqrt{I}$ is finitely generated by say $x_1, \dots, x_n$. Then $x_i^{n_i} \in I$ for some $n_i \in \mathbb{N}$. Let $m = 1 + \sum_{i=1}^n (n_i - 1)$. Then $\sqrt{I}^m$ is generated by $x_1^{r_1} \cdots x_n^{r_n}$ for $\sum_{i=1}^n r_i = m$. If $r_i < n_i$ for i then
$$m = \sum_{i=1}^n r_i \leq \sum_{i=1}^n (n_i - 1) < m$$
is a contradiction. Hence there exists some i for which $r_i \geq n_i$. Thus $x_1^{r_1} \cdots x_n^{r_n} \in I$. Thus $\sqrt{I}^m \subseteq I$. ■

Proposition 2.1.2 Let R be a Noetherian commutative ring. Then $N(R)$ is a nilpotent ideal.

Proof By the above, there exists $n \in \mathbb{N}$ such that $(N(R))^n = \sqrt{(0)}^n \subseteq (0) \subseteq \sqrt{(0)}$. Hence $(N(R))^n = (0)$ for some $n \in \mathbb{N}$. ■

Proposition 2.1.3
Let R be a Noetherian commutaive ring. Let I be an ideal of R. Then I has only finitely many minimal prime ideals.

Proof
Define the set
$$\mathcal{F} = \{I \subseteq R \mid I \text{ is an ideal of } R \text{ that has infinitely many minmal prime ideals } \}$$
Assume for a contradiction that the set is non-empty. Since R is Noetherian, every chain in $\mathcal{F}$ has a maximal element. Let I be such a maximal element of the set. I is not a prime ideal because prime ideals only have one minimal prime ideal by definition. This means that there exists $a, b \notin I$ such that $ab \in I$. Now $I + (a)$ and $I + (b)$ both contain I, and since $ab \in I$ we have that $(I + (a))(I + (b)) \subseteq I$. For any minimal prime ideal P of I, we have that $(I + (a))(I + (b)) \subseteq I \subseteq P$ implies that $(I + (a)) \subseteq P$ or $(I + (b)) \subseteq P$. This shows that the set of minimal prime ideals of I is a subset of the union of the set of minimal prime ideals of $I + (a)$ and $I + (b)$, and so without loss of generality, $I + (a)$ has infinitely many minimal prime ideals. But then $I + (a)$ strictly contains I and $I + (a) \in \mathcal{F}$. This is a contradiction since I is assumed to be maximal. ■

2.2 Artinian Commutative Rings

Let R be a commutative ring. Recall that R is Artinian if any descending chain of ideals

$$I_1 \supseteq I_2 \supseteq \cdots$$

terminates at finitely many steps, meaning $I_k = I_k + n$ for some $k \in \mathbb{N}$.

Generally, if R is Artinian then the following are true.

- $J(R)$ is a nilpotent ideal.
- R is Noetherian.
- R has finite length.

There are also properties of Artinian rings that only commutative rings can realize.

Proposition 2.2.1 Let R be an integral domain. Then R is Artinian if and only if R is a field.

Proof It is clear that every field is Artinian. Conversely, let R be Artinian. Consider the following descending chain of ideals in R:

$$R \supseteq (x) \supseteq (x^2) \supseteq$$

for any $0 \neq x \in R$. Since R is Artinian, the chain terminates and $(x^n) = (x^{n+1})$ for some $n \in \mathbb{N}$. Then there exists $y \in R$ such that $x^n = yx^{n+1}$. This means that $x^n(1 - yx) = 0$. Since R is an integral domain, R has no nilpotents. Hence x^n is non-zero and $1 = xy$. Thus x has an inverse so that R is a field. ■

Proposition 2.2.2 Let R be an Artinian commutative ring. Then the following are true.

- $\text{Spec}(R) = \text{maxSpec}(R)$.
- $N(R) = J(R)$

Proof Let P be a prime ideal. Since quotients of Artinian rings are Artinian, R/P is Artinian. Since R/P is also an integral domain, we conclude by the above that R/P is a field. Hence P is maximal.

Since every prime ideal in R is maximal, we have that

$$N(R) = \bigcap_{P \text{ a prime ideal}} P = \bigcap_{P \text{ a maximal ideal}} P = J(R)$$

and so we conclude. ■

Proposition 2.2.3 Let R be a commutative ring. If R is Artinian, then R has finitely many maximal ideals.

Proof Consider the collection

$$\{m_1 \cap \cdots \cap m_k \mid m_1, \dots, m_k \text{ are maximal ideals of } R\}$$

of R-submodules of R. Since R is Artinian, every collection of R-submodules of R has a minimal element. Hence this collection also has a minimal element, say $m_1 \cap \cdots \cap m_k$. Let m be another maximal ideal of R. Then

$$m \cap m_1 \cap \cdots \cap m_k \subseteq m_1 \cap \cdots \cap m_k$$

Since $m_1 \cap \cdots \cap m_k$ is minimal, they are equal. By prime avoidance, we conclude that $m \supseteq m_i$ for some i. Since they are maximal, we have $m = m_i$. Hence $m_1, \dots, m_k$ gives the full list of distinct maximal ideals of R. ■

2.3 Local Rings

Definition 2.3.1 (Local Rings) Let R be a commutative ring. We say that R is a local ring if it has a unique maximal ideal m. In this case, we say that R/m is the residue field of R.

Example 2.3.2 Consider the following commutative rings.
- $\mathbb{Z}/6\mathbb{Z}$ is not a local ring.
- $\mathbb{Z}/8\mathbb{Z}$ is a local ring.
- $\mathbb{Z}/24\mathbb{Z}$ is not a local ring.
- $\mathbb{R}[x]$ is not a local ring.

Proof

- The only ideals of $\mathbb{Z}/6\mathbb{Z}$ are $(2+6\mathbb{Z})$ and $(3+6\mathbb{Z})$. They do not contain each other and so they are both maximal.
- The only ideals of $\mathbb{Z}/8\mathbb{Z}$ are $(2+8\mathbb{Z})$ and $(4+8\mathbb{Z})$. But $(2+8\mathbb{Z}) \supseteq (4+8\mathbb{Z})$. Hence $\mathbb{Z}/8\mathbb{Z}$ has a unique maximal ideal.
- A similar proof as above ensues.
- Any irreducible polynomial $f \in \mathbb{R}[x]$ is such that (f) is a maximal ideal. Indeed the evaluation homomorphism gives an isomorphism $\frac{\mathbb{R}[x]}{(f)} \cong \mathbb{R}$.

■

Proposition 2.3.3 Let R be a ring and I an ideal of R. Then I is the unique maximal ideal of R if and only if I is the set containing all non-units of R.

Proof Let I be the unique maximal ideal of R. Clearly I does not contain any unit else $I = R$. Now suppose that r is a non-unit. Suppose that $r \notin I$. Define $J = \{sr | s \in R\}$ Clearly J is an ideal. It must be contained in some maximal ideal. Since I is the unique maximal ideal, $J \subseteq I$. But this means that $r \in I$, a contradiction. Thus every non-unit is in I.
Suppose that I contains all non-units of R. Let $r \notin I$. Then there exists $s \notin I$ such that $rs = 1$. Then $(r+I)(s+I) = 1 + I$ in R/I. This means that every element of R/I has a multiplicative inverse which means that R/I is a field and thus I is a maximal ideal. Now let $J \neq I$ be another maximal ideal. Then J contains some unit r. This implies that $J = R$ and thus I is the unique maximal ideal. ■

Example 2.3.4 Let k be a field. Then the ring of power series $k[[x]]$ is a local ring.

Proof Let M be the set of all non-units of $k[[x]]$. I first show that $f \in M$ if and only if the constant term of f is non-zero. Let g be a power series. Then the nth coefficient of $f \cdot g$ is given by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

If the constant term of f is 0, then $c_0 = 0$ and so $f \cdot g \neq 1$. Now if the constant term of f is $a_0 \neq 0$, then set $b_0 = \frac{1}{a_0}$. Now we can use the formula $0 = c_n$ to deduce

$$b_n = -\frac{\sum_{k=1}^{n} a_k b_{n-k}}{a_0}$$

This is such that $a_n \cdot b_n = 0$. Define $g = \sum_{k=0}^{\infty} b_k x^k$. Then $f \cdot g = 1$. Thus f is a unit.

By the above proposition, we conclude that M is the unique maximal ideal of $k[[x]]$. ■

Proposition 2.3.5 Let R be a commutative ring. Then the following are equivalent.

- R has exactly one prime ideal. (It is given by $N(R)$).
- Every element of R is either a unit or nilpotent.
- $N(R)$ is a maximal ideal.

Under these equivalent assumptions, $(R, N(R))$ is a local ring.

Proof

- $(1) \implies (2)$: We know that $N(R)$ is a prime ideal, hence it is the unique prime ideal and unique maximal ideal. Thus R is a local ring. By the above, elements of $R \setminus N(R)$ are units and element of $N(R)$ are nilpotent.
- $(2) \implies (3)$: It is clear that every nilpotent is a non-unit. By assumption, non-units of R are nilpotents. Hence $N(R)$ is the set of all non-units. Since $N(R)$ is an ideal, by the above we conclude that $(R, N(R))$ is a local ring. In particular, $N(R)$ is the unique maximal ideal of R.
- $(3) \implies (1)$: Suppose that $N(R)$ is a maximal ideal. Let $P \neq R$ be a prime ideal of R. Since $N(R)$ is the intersection of all prime ideals, we have $N(R) \subseteq P$. By the correspondence theorem, P corresponds to a prime ideal of $R/N(R)$. But $R/N(R)$ is a field, and since $P \neq R$ we must have that $P = N(R)$. Thus $N(R)$ is the unique prime ideal of R.

■

Proposition 2.3.6 Let R be a Noetherian commutative ring. Then the following are equivalent.

- R is an Artinian local ring.
- R has a nilpotent maximal ideal.
- R has a unique proper radical ideal.
- R has a unique prime ideal.
- $N(R)$ is a maximal ideal of R.

Proof

- $(1) \implies (2)$: Let R be Artinian and local. By 2.1.4 we have $N(R) = J(R) = m$ since $J(R)$ is the intersection of all maximal ideals. Since R is Noetherian, by 2.1.3 $N(R) = m$ is nilpotent.

■

Since every Artinian ring is Noetherian, the above proposition implies the following.

Corollary 2.3.7 Let R be an Artinian commutative ring. Then the following are true.

- R is local.
- $N(R)$ is the unique maximal ideal of R.
- $N(R)$ is the unique prime ideal of R.
- $N(R)$ is the unique radical ideal of R.
- $N(R)$ is a nilpotent ideal.

We will discuss more of local rings in the topic of localizations.

2.4 Revisiting the Polynomial Ring

Lemma 2.4.1 Let R be a commutative ring. Then $R[x]$ has infinitely many irreducible polynomials.

Proof If not, then there exists a finite list of irreducible polynomials $f_1, \dots, f_k$. Then $1+f_1, \dots, f_k$ is not divisible by $f_1, \dots, f_k$ and so must contain a monic irreducible factor not equal to $f_1, \dots, f_k$. This is a contradiction. ■

Proposition 2.4.2 Let R be a commutative ring. Then we have

$$N(R[x]) = N(R)[x]$$

Proof Let $f = \sum_{k=0}^n a_k x^k \in N(R)[x]$. Then each a_k is nilpotent in R, and there exists $n_k \in \mathbb{N}$ such that $a_k^{n_k} = 0$. This also proves that $a_k x^k$ is nilpotent. Since the sum of nilpotents is a nilpotent, we conclude that f is nilpotent.

Now suppose that $f \in N(R[x])$. We induct on the degree of f. Let $\deg(f) = 0$. Then f is nilpotent and f lies in R. Thus $f \in N(R)[x]$. Now suppose that the claim is true for $\deg(f) \leq n-1$. Let $\deg(g) = n$ with leading coefficient b_n. Since g is nilpotent in $R[x]$, there exists $m \in \mathbb{N}$ such that $g^m = 0$. Then in particular, $b_n^m = 0$ so that b_n is nilpotent. Then $b_n x^n$ is also nilpotent. Now since $N(R[x])$ is an ideal of $R[x]$, we have that $g - b_n x^n \in N(R[x])$. By inductive hypothesis, $g - b_n x^n \in N(R)[x]$. Since $N(R)$ is an ideal of R, we have that $N(R)[x]$ is an ideal of $R[x]$. So $g = (g - b_n x^n) + b_n x^n \in N(R)[x]$. Thus we are done. ■

Theorem 2.4.3 (Hilbert's Basis Theorem) Let R be a commutative ring. If R is Noetherian, then $R[x]$ is a Noetherian ring.

Proof It suffices to show that every ideal of $R[x]$ is finitely generated. Let I be an ideal of $R[x]$. Let $I^{\leq n}$ be the ideal generated by

$$I^{\leq n} = (f \in I \mid \deg(f) \leq n)$$

Notice that $I^{\leq n}$ is an R-submodule of $\bigoplus_{i=0}^n R \cdot x^i$. Since R is Noetherian, $I^{\leq n}$ is finitely generated as an R-module. In particular, $I^{\leq n}$ is finitely generated as an $R[x]$-module with the same finite generating set.

I claim that the chain of ideals

$$I^{\leq 0} \subseteq I^{\leq 1} \subseteq \cdots \subseteq I^{\leq k} \subseteq I = \bigcup_{i=0}^{\infty} I^{\leq i}$$

of $R[x]$ eventually stabilizes. Let $LC(f)$ be the leading coefficient of $f \in R[x]$. The define

$$LC(I) = \{LC(f) \mid f \in I\}$$

Notice that $LC(I)$ is an ideal of R. Since R is Noetherian, $LC(I)$ is finitely generated as an R-module by say $a_1, \dots, a_r$. This means that there exists $f_1, \dots, f_r \in R[x]$ such that $LC(f_i) = a_i$. Let $d = \max\{\deg(f_1), \dots, \deg(f_r)\}$. Without loss of assumption we can replace f_i with $x^{d-\deg(f_i)} f_i$ so that $f_1, \dots, f_r$ have the same degree d.

I claim that $I^{\leq n} = I^{\leq n+1}$ for $n \geq d$. $I^{\leq n} \subseteq I^{\leq n+1}$ is trivial. Suppose that $f \in I^{\leq n+1}$. If $\deg(f) \leq n$ then we are done. So suppose that $\deg(f) = n+1$. Then the leading coefficient of f is a linear combination of the leading coefficients of $f_1, \dots, f_r$. So there exists $b_1, \dots, b_r \in R$ such that $LC(f) = \sum_{i=1}^r b_i LC(f_i)$. Then $f - \left(\sum_{i=1}^r b_i f_i\right) x^{n+1-d} \in I^{\leq n}$. Since $\sum_{i=1}^r b_i f_i \in I^{\leq d} \subseteq I^{\leq n}$, we conclude that $f \in I^{\leq n}$. We conclude. ■

Some more important results from Groups and Rings and Rings and Modules include:

- If R is an integral domain, then $R[x]$ is an integral domain.
- R is a UFD if and only if $R[x]$ is a UFD
- If F is a field, then $F[x]$ is an Euclidean domain, a PID and a UFD
- If F is a field, then the ideal generated by p is maximal if and only if p is irreducible.

Regarding ideals of the polynomial ring, the following maybe useful:

- $I[x]$ is an ideal of R
- There is an isomorphism $\frac{R[x]}{I[x]} \cong \frac{R}{I}[x]$ given by the map

$$\left(f = \sum_{k=0}^{n} a_k x^k + I[x]\right) \mapsto \left(\sum_{k=0}^{n} (a_k + I) x^k\right)$$

- If I is a prime ideal of R, then $I[x]$ is a prime ideal of $R[x]$.

3 Modules over a Commutative Ring

Recall from Rings and Modules that a module consists of an abelian group M and a ring R such that there is a binary operation $\cdot : R \times M \to M$ that mimic the notion of a group action:

- For $r, s \in R$, $s \cdot (r \cdot m) = (sr) \cdot m$ for all $m \in M$.
- For $1_R \in R$ the multiplicative identity, $1_R \cdot m = m$ for all $m \in M$.

When R is a commutative ring, the first axiom is relaxed so that the resulting element of M makes no difference whether you apply r first or s first. This makes module act even more similarly than fields (although one still need the notion of a basis, which appears in free modules). Therefore the first section concerns transferring techniques in linear algebra such as the Cayley Hamilton theorem to module over a ring that mimic the notion of vector spaces.

3.1 Cayley-Hamilton Theorem

Definition 3.1.1 (Characteristic Polynomial) Let R be a commutative ring. Let $A \in M_{n\times n}(R)$ be a matrix. Define the characteristic polynomial of A to be the polynomial

$$c_A(x) = \det(A - xI)$$

Theorem 3.1.2 (Cayley-Hamilton Theorem for Rings) Let R be a commutative ring. Let $A \in M_{n\times n}(R)$ be a matrix. Then $c_A(A) = 0$.

Theorem 3.1.3 (Cayley-Hamiliton Theorem for Modules) Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R. Let $\varphi \in \text{End}_R(M)$. If $\varphi(M) \subseteq IM$, then there exists $a_1, \dots, a_{n-1} \in I$ such that

$$\varphi^n + a_1\varphi^{n-1} + \cdots + a_{n-1}\varphi + \text{id}_M = 0 : M \to M$$

Proof Suppose that M is generated by $x_1, \dots, x_n$. There exists a surjective map $\rho : R^n \to M$ given by $(r_1, \dots, r_n) \mapsto \sum_{k=1}^n r_k x_n$. Since $\varphi(M) \subseteq IM$, we havt that

$$\varphi(x_k) = \sum_{i=1}^n r_{ki}x_i$$

for some $r_{ki} \in I$. Write A to be the matrix $A = (a_{ki})$. We now have a commutative diagram: In other words, we have the diagram:

$$\begin{tikzcd} R^n \arrow[r, two heads, "\rho"] \arrow[d, "A"'] & M \arrow[d, "\varphi"] \\ R^n \arrow[r, two heads, "\rho"'] & M \end{tikzcd}$$

By Cayley-Hamilton theorem, we have that $c_A(A) = 0$ is the zero function. For all $x \in R^n$, we have that

$$\begin{aligned} c_A(A)(x) &= 0 \\ c_A(Ax) &= 0 \\ \rho(c_A(Ax)) &= \rho(0) \\ c_A(\rho(Ax)) &= 0 && (\rho \text{ is } R\text{-linear}) \\ c_A(\varphi(\rho(x))) &= 0 && (\text{Diagram is commutative}) \end{aligned}$$

Since ρ is surjective, we conclude that for any $m \in M$, the above calculation gives $c_A(\varphi(m)) = 0$ so that $c_A(\varphi)$ is the zero map. ∎

Proposition 3.1.4 Let R be a commutative ring. Let M be a finitely generated R-module. Let $\phi : M \to M$ be a surjective R-module homomorphism. Then ϕ is an isomorphism.

Proof Consider M as an $R[\phi]$-module via the action $\phi \cdot m = \phi(m)$. Notice that $(\phi)M = M$ since ϕ is surjective. By the Cayley-Hamilton theorem, there exists $\alpha_1, \dots, \alpha_{n-1} \in R$ such that

$$\mathrm{id}^n + \alpha_1\phi\mathrm{id}^{n-1} + \cdots + \alpha_{n-1}\phi\mathrm{id} + \mathrm{id} = 0 : M \to M$$

This simplifies to the equation

$$(\alpha_1 + \cdots + \alpha_{n-1})\phi(m) + m = 0$$

for all $m \in M$.

We want to show that ϕ is injective. Suppose that $\phi(m) = 0$ for some $m \in M$. From the above equation, we see that $m = 0$. Hence ϕ is an isomorphism. ■

3.2 Nakayama's Lemma

Lemma 3.2.1 (Nakayama's Lemma I) Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R. If $IM = M$, then there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I$.

Proof Choose $\varphi = \mathrm{id}_M$. Then φ is surjective so that $M = \varphi(M) \subseteq IM$. By crl 4.1.3, there exists $r_1, \dots, r_n \in I$ such that $(1 + r_1 + \cdots + r_n)M = 0$. By choosing $r = 1 + r_1 + \cdots + r_n$, we see that $rM = 0$ and $r - 1 \in I$ so that we conclude. ■

Lemma 3.2.2 (Nakayama's Lemma II) Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R such that $I \subseteq J(R)$ and $IM = M$. Then $M = 0$.

Proof By Nakayama's lemma I, there exists $r \in R$ such that $rM = 0$ and $r - 1 \in I \subseteq J(R)$. By 2.3.8, we have that $1 - (r-1)(-1) = r \in R^\times$. This means that r is invertible. Hence $rM = 0$ implies $M = r^{-1}rM = 0$. ■

Corollary 3.2.3 Let R be a commutative ring. Let M be a finitely generated R-module. Let I be an ideal of R such that $I \subseteq J(R)$. Let N be an R-submodule of M. If

$$M = IM + N$$

then $M = N$.

Proof Since quotients of finitely generated modules are finitely generated, we know that M/N is finitely generated. Define the map

$$\phi : IM + N \to I\frac{M}{N}$$

by $\phi(im + n) = i(m + N)$. This map is clearly surjective. Now I claim that $\ker(\phi) = N$. For any $im + n \in \ker(\phi)$, we see that $i(m + N) = N$ means that $im \in N$. Hence $im + n \in N$. On the other hand, if $im + n \in N$ then $im \in N$. But this means that $im + N = N$. Hence $im + n \in \ker(\phi)$. By the first isomorphism theorem for modules, we conclude that

$$\frac{M}{N} = \frac{IM + N}{N} \cong I\frac{M}{N}$$

We can now apply Nakayama's lemma II to conclude that $M/N = 0$ so that $M = N$. ■

Corollary 3.2.4 Let (R, m) be a local ring. Let m be a maximal ideal of R. Let M be a finitely generated R-module. Then the following are true.

- M/mM is a finite dimensional vector space over R/m.
- $a_1, \dots, a_n \in M$ generates M as an R-module if and only if $a_1 + mM, \dots, a_n + mM$ generates M/mM as a R/m vector space.
- $a_1, \dots, a_n \in M$ is a minimal set of generators of M as an R-module if and only if $a_1 + mM, \dots, a_n + mM$ is a basis for M/mM as a R/m vector space.

Proof Since the projection map $\pi : M \to M/mM$ is surjective, clearly any set of generators of M is a set of generators for M/mM. This also shows that if M is finitely generated then M/mM is a finite dimensional R/m-vector space.

For the other direction, suppose that $a_1 + mM, \dots, a_n + mM$ generates M/mM as an R/m-vector space. Define $N = Ra_1 + \cdots + Ra_n \leq M$. Set $I = J(R) = m$. We want to show that $M = IM + N$. It is clear that $IM + N \leq M$. If $x \in M$, then there exists $r_k \in R$ such that $x + mM = r_1(a_1 + mM) + \cdots + r_n(a_n + M)$. In particular, this means that

$$x - \sum_{k=1}^{n} r_k a_k \in mM$$

Hence $x \in IM + N$. We can now apply the above corollary to deduce that $M = N = Ra_1 + \cdots + Ra_n$ so that M is generated by $a_1, \dots, a_n$. And so we are done.

Suppose that $a_1, \dots, a_n$ generate M. The above shows that $a_1 + mM, \dots, a_n + mM$ spans M/mM. So suppose for a contradiction that $a_1, \dots, a_n$ is a minimal generating set but $a_1 + mM, \dots, a_n + mM$ is not a basis for m/m^2. This means that after relabelling, $a_1 + mM, \dots, a_{n-1} + mM$ spans M/mM. By the above, this means that $a_1, \dots, a_{n-1}$ generate M. This is a contradiction of the minimality of the generating set $a_1, \dots, a_n$. Hence $a_1 + mM, \dots, a_n + mM$ is a basis for m/m^2.

Now suppose that $a_1 + mM, \dots, a_n + mM$ is a basis for M/mM. We have seen above that $a_1, \dots, a_n$ generate M. If this is not minimal, then there is some smaller generating set $b_1, \dots, b_k$ that still generates M where $k < n$. By the above, $b_1 + mM, \dots, b_k + mM$ spans M/mM hence $n = \dim_{R/m}(M/mM) \leq k$. This is a contradiction since $k < n$. Hence we are done. ■

3.3 Change of Rings

Definition 3.3.1 (Extension of Scalars) Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Let M be an R-module. Define the extension of M to the ring S to be the S-module

$$S \otimes_R M$$

Definition 3.3.2 (Restriction of Scalars) Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Let M be an S-module. Define the restriction of M to the ring R to be the R-module M equipped with the action

$$r \cdot_R m = \varphi(r) \cdot_S m$$

for all $r \in R$.

Theorem 3.3.3 Let R, S be commutative rings. Let $\varphi : R \to S$ be a ring homomorphism. Then there is an isomorphism

$$\mathrm{Hom}_S(S \otimes_R M, N) \cong \mathrm{Hom}_R(M, N)$$

for any R-module M and S-module N given as follows.

- For $f \in \text{Hom}_S(S \otimes_R M, N)$, define the map $f^+ \in \text{Hom}_R(M, N)$ by

$$f^+(m) = f(1 \otimes m)$$

- For $g \in \text{Hom}_R(M, N)$, define the map $g^- \in \text{Hom}_S(S \otimes_R M, N)$ by

$$g^-(s \otimes m) = s \cdot g(m)$$

3.4 Properties of the Hom Set

Let R be a ring. Let M, N be R-modules. Recall that in Rings and Modules that $\text{Hom}_R(M, N)$ is a $Z(R)$-modules. When R is commutative, $Z(R) = R$ so that the Hom set becomes an R-module.

Proposition 3.4.1 Let R be a commutative ring. Let M, N be R-modules. Then

$$\text{Hom}_R(M, N)$$

is an R-module with the following binary operations.
- For $\phi, \varphi : M \to N$ two R-module homomorphisms, define $\phi + \varphi : M \to N$ by $(\phi + \varphi)(m) = \phi(m) + \varphi(m)$ for all $m \in M$
- For $\phi : M \to N$ an R-module homomorphism and rR, define $r\phi : M \to N$ by $(r\phi)(m) = r \cdot \phi(m)$ for all $m \in M$.

Proof We first show that the addition operation gives the structure of a group.

- Since M is associative as an additive group, associativity follows
- Clearly the zero map $0 \in \text{Hom}_R(M, N)$ acts as the additive inverse since for any $\phi \in \text{Hom}_R(M, N)$, we have that $\phi(m) + 0 = 0 + \phi(m) = \phi(m)$ since 0 is the additive identity for M
- For every $\phi \in \text{Hom}_R(M, N)$, the map taking m to $-\phi(m)$ also lies in $\text{Hom}_R(M, N)$. Since $-\phi(m)$ is the inverse of $\phi(m)$ in M for each $m \in M$, we have that $-\phi$ is the inverse of ϕ

We now show that

- Let $r, s \in R$, we have that $((sr)\phi)(m) = (sr) \cdot \phi(m) = s \cdot (r \cdot \phi(m)) = s(r(\phi))(m)$ and hence we showed associativity.
- It is clear that $1_R \in R$ acts as the identity of the operation.

Thus we are done. ■

Proposition 3.4.2 Let R be a ring. Let I be an indexing set. Let M_i, N be R-modules for $i \in I$. Then the following are true.
- There is an isomorphism

$$\text{Hom}\left(\bigoplus_{i \in I} M_i, N\right) \cong \bigoplus_{i \in I} \text{Hom}(M_i, N)$$

- There is an isomorphism

$$\text{Hom}\left(\prod_{i \in I} M_i, N\right) \cong \prod_{i \in I} \text{Hom}(M_i, N)$$

Definition 3.4.3 (Induced Map of Hom) Let R be a commutative ring. Let M_1, M_2, N be R-modules. Let $f : M_1 \to M_2$ be an R-module homomorphism. Define the induced map

$$f^* : \text{Hom}_R(M_2, N) \to \text{Hom}(M_1, N)$$

by the formula $\varphi \mapsto \varphi \circ f$

Lemma 3.4.4 Let R be a commutative ring. Let M_1, M_2, N be R-modules. Let $f : M_1 \to M_2$ be an R-module homomorphism. Then the induced map

$$f^* : \text{Hom}(M_2, N) \to \text{Hom}(M_1, N)$$

is an R-module homomorphism.

3.5 More on Exact Sequences

Proposition 3.5.1 Let R be a commutative ring. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Let N be an R-module. Then the following two sequences

$$0 \longrightarrow \text{Hom}_R(M_3, N) \longrightarrow \text{Hom}_R(M_2, N) \longrightarrow \text{Hom}_R(M_1, N)$$

$$\text{Hom}_R(N, M_1) \longrightarrow \text{Hom}_R(N, M_2) \longrightarrow \text{Hom}_R(N, M_3) \longrightarrow 0$$

are exact.

Proof

- We first show that g^* is injective. Let $\phi, \rho \in \text{Hom}(C, G)$ such that $g^*(\phi) = g^*(\rho)$. This means that $\phi \circ g = \rho \circ g$. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Then

  $$\phi(c) = \phi(g(b)) = \rho(g(b)) = \rho(c)$$

  Hence $\phi = \rho$.

  Now we show that $\text{im}(g^*) \subseteq \ker(f^*)$. Let $g^*(\phi) \in \text{Hom}(B, G)$ for $\phi \in \text{Hom}(C, G)$. We want to show that $f^*(g^*(\phi)) = 0$. But we have that

  $$(\phi \circ g \circ f)(a) = \phi(g(f(a)) = \phi(0) = 0$$

  since $\text{im}(f) = \ker(g)$. Thus we conclude.

  Finally we show that $\ker(f^*) \subseteq \text{im}(g^*)$. Let $f^*(\phi) = 0$ for $\phi \in \text{Hom}(B, G)$. This means that $\phi \circ f = 0$ or in other words, $\text{im}(f) \subseteq \ker(\phi)$. Since $\phi(k) = 0$ for all $k \in \text{im}(f)$, ϕ descends to a map $\overline{\phi} : \frac{B}{\text{im}(f)} \to G$. But $\text{im}(f) = \ker(g)$ hence this is equivalent to a map $\overline{\phi} : \frac{B}{\ker(g)} \to G$. But by the first isomorphism theorem and the fact that g is surjective, we conclude that $\overline{g} : \frac{B}{\ker(g)} \overset{g}{\cong} C$, where $b + \ker(g) \mapsto g(b)$. Thus we have constructed a map $\overline{\phi} \circ \overline{g}^{-1} : C \to G$ given by $g(b) \mapsto b + \ker(g) \mapsto \phi(b)$. But now $g^*(\overline{\phi} \circ \overline{g}^{-1})$ is the map defined by

  $$b \mapsto g(b) \mapsto b + \ker(g) \mapsto \phi(b)$$

  and so this map is exactly ϕ. Thus $\phi \in \text{im}(g^*)$.

■

Example 3.5.2 Applying $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p\mathbb{Z})$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0$$

does not give a sequence that is exact on the right.

Proof The new sequence is now

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{\mathrm{id}_{\mathbb{Z}/p\mathbb{Z}}} \mathbb{Z}/p\mathbb{Z} \xrightarrow{0} \mathbb{Z}/p\mathbb{Z}$$

Evidently the 0 map is not surjective. ■

Proposition 3.5.3 Let R be a commutative ring. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Let N be an R-module. Then the following sequence

$$M_1 \otimes N \xrightarrow{f \otimes \mathrm{id}_N} M_2 \otimes N \xrightarrow{g \otimes \mathrm{id}_N} M_3 \otimes N \longrightarrow 0$$

is exact.

However, one can observe that we did not imply that $M_1 \otimes N \to M_2 \otimes N$ is injective. Indeed, this is because tensoring does not preserve injections.

4 Algebra Over a Commutative Ring

4.1 Commutative Algebras

Definition 4.1.1 (Commutative Algebras) Let R be a commutative ring. A commutative R-algebra is an R-algebra A that is commutative.

Proposition 4.1.2 Let R be a commutative ring. Then the following are equivalent characterizations of a commutative R-algebra.

- A is a commutative R-algebra
- A is a commutative ring together with a ring homomorphism $f : R \to A$

Proof Suppose that A is an R-algebra. Then define a map $f : R \to A$ by $f(r) = r \cdot 1$ where $r \cdot 1$ is the module operation on A. Then clearly this is a ring homomorphism.
Suppose that A is a commutative ring together with a ring homomorphism $f : R \to A$. Define an action $\cdot : R \times A \to A$ by $r \cdot a = f(r)a$. Then this action clearly allows A to be an R-module. ∎

Under the correspondence of associative algebra, the above proposition gives a another correspondence between the first one.

$$\left\{(A, R) \;\middle|\; \begin{matrix}A \text{ is a commutative}\\ R\text{-algebra}\end{matrix}\right\} \overset{1:1}{\longleftrightarrow} \left\{\phi : R \to A \;\middle|\; \begin{matrix}\phi \text{ is a ring homomorphism}\\ \text{such that } f(R) \subseteq Z(A) = A\end{matrix}\right\}$$

In particular, the construction above are inverses of each other so that it gives the one-to-one correspondence.

4.2 Free Commutative Algebras

Let R be a commutative ring. Let X be a set. Recall that we defined $R\langle X\rangle$ to be the free (non-commutative) R-algebra over X. Explicitly, if $W = \{x_1 \cdots x_n \mid x_1, \dots, x_n \in X\}$ is the set of words on X, then

$$R\langle X\rangle = \bigoplus_{w \in W} R \cdot w$$

together with multiplication defined by $(x_1 \cdots x_n) \cdot (y_1 \cdots y_n) = x_1 \cdots x_n \cdot y_1 \cdots y_m$.

Definition 4.2.1 (Free Commutative Algebra over a Ring) Let R be a commutative ring. Let X be a set. Define the free commutative R-algebra over X to be the quotient

$$\text{Free}_R(X) = \frac{R\langle X\rangle}{\langle x_i x_j - x_j x_i \mid x_i, x_j \in X\rangle}$$

Proposition 4.2.2 (Universal Property of Free Commutative Algebras) Let R be a commutative ring. Let X be a set. The free commutative algebra $\text{Free}_R(X)$ satisfies the following universal property.

- Universal Property: If A is a commutative R-algebra, then for every $f : X \to A$ a map of sets, there exists a unique homomorphism of algebras $\varphi : \text{Free}_R(X) \to A$ such that $\varphi(x_i) = f(x_i)$ for each $x_i \in X$. In other words, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \text{Free}_R(X) \arrow[d, dashed, "\exists!\varphi"] \\ & A \end{tikzcd}$$

  where $\iota : X \to \text{Free}_R(X)$ is the inclusion.
- $\text{Free}_R(X)$ is the unique R-algebra (up to unique isomorphism) that satisfies this property.

Proposition 4.2.3 Let R be a commutative ring. Let X be a set. Then there is an R-algebra isomorphism

$$\text{Free}_R(X) \cong R[X]$$

with the polynomial ring over X.

4.3 Finiteness Properties of Algebras

Definition 4.3.1 (Finitely Generated Algebras) Let R be a commutative ring. Let A be a commutative R-algebra. We say that A is finitely generated if there exists $a_1, \dots, a_n \in A$ such that every element $a \in A$ can be written as a polynomial in $a_1, \dots, a_n$. This means that

$$a = \sum_{i_1, \dots, i_n} r_{i_1, \dots, i_n} a_1^{i_1} \cdots a_n^{i_n}$$

Finitely generated algebras are also called algebra of finite type.

Theorem 4.3.2 Let A be a commutative algebra over a ring R. Then the following are equivalent.

- A is a finitely generated algebra over R
- There exists elements $a_1, \dots, a_n \in A$ such that the evaluation homomorphism

$$\phi : R[x_1, \dots, x_n] \to A$$

  given by $\phi(f) = f(a_1, \dots, a_n)$ is a surjection
- There is an isomorphism

$$A \cong \frac{R[x_1, \dots, x_n]}{I}$$

  for some ideal I

Definition 4.3.3 (Finitely Presented Algebra) Let R be a ring. Let $A = R[x_1, \dots, x_n]/I$ be a finitely generated algebra over R for some ideal I. We say that A is finitely presented if I is finitely generated.

Lemma 4.3.4 Let R be a ring, considered as an algebra over $\mathbb{Z}$. If R is finitely generated over $\mathbb{Z}$, then R is finitely presented.

Proof Trivial since $\mathbb{Z}$ is a principal ideal domain. ■

Definition 4.3.5 (Finite Algebras) Let R be a commutative ring. Let A be an R-algebra. We say that A is finite if A is finitely generated as an R-module.

Example 4.3.6 Let R be a commutative ring. Then $R[x]$ is a finitely generated algebra over R but is not a finite R-algebra.

5 Localization

5.1 Localization of Modules

Definition 5.1.1 (Multiplicative Set) Let R be a commutative ring. $S \subseteq R$ is a multiplicative set if $1 \in S$ and S is closed under multiplication: $x, y \in S$ implies $xy \in S$

Definition 5.1.2 (Localization of a Module) Let R be a commutative ring and $S \subseteq R$ be a multiplicative set Let M be a R-module. Define the ring of fractions of M with respect to S by

$$S^{-1}M = \left\{ \frac{m}{s} \;\middle|\; m \in M, s \in S \right\} / \sim$$

where $\sim$ is defined by

$$\frac{m}{s} \sim \frac{m'}{s'} \text{ if and only if } \exists v \in S \text{ such that } v(mu' - m'u) = 0$$

Lemma 5.1.3 Let R be a commutative ring. Let M be an R-module. Let $S \subseteq R$ be a multiplicative subset. Then $S^{-1}M$ is a well defined $S^{-1}R$-module with operation given by

$$\left(\frac{r}{s_1}, \frac{m}{s_2} \right) \mapsto \frac{r \cdot m}{s_1 s_2}$$

Definition 5.1.4 (Induced Map of Localization) Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Define the induced map

$$S^{-1}\phi : S^{-1}M \to S^{-1}N$$

by the formula $\frac{m}{s} \mapsto \frac{\phi(m)}{s}$.

Lemma 5.1.5 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Then the induced map

$$S^{-1}\phi : S^{-1}M \to S^{-1}N$$

is a well defined ring homomorphism.

Lemma 5.1.6 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N, K be R-modules. Let $f : M \to N$ and $g : N \to K$ be R-module homomorphisms. Then the following are true.

- Composition: $S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f : S^{-1}M \to K$.
- Identity: $S^{-1}\text{id}_M = \text{id}_{S^{-1}M}$

Proposition 5.1.7 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let the following be an exact sequence of R-modules.

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

Then the following is an exact sequence of $S^{-1}R$-modules.

$$S^{-1}M_1 \xrightarrow{f} S^{-1}M_2 \xrightarrow{g} S^{-1}M_3$$

Proof Since $\text{im}(f) = \ker(g)$, we have that $g \circ f = 0$ which implies that $0 = S^{-1}0 = S^{-1}(g \circ f) = S^{-1}g \circ S^{-1}f$. Hence $\text{im}(S^{-1}f) \subseteq \ker(S^{-1}g)$. Conversely, let $m_2/s \in \ker(S^{-1}g)$. Then $g(m_2)/s = 0$ and so $g(tm_2) = tg(m_2) = 0$ for some $t \in S$. Since $\text{im}(f) = \ker(g)$, there exists $m_1 \in M_1$ such that $f(m_1) = tm_2$. Then we have

$$(S^{-1}f)(m_1/ts) = f(m_1)/ts = tm_2/ts = m_2/s$$

Hence $m_2/s \in \text{im}(S^{-1}(f))$. ■

Corollary 5.1.8 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M be an R-module. Then the following are true.

- Localization commutes with quotients: If N is an R-submodule of M, then
$$S^{-1}\frac{M}{N} \cong \frac{S^{-1}M}{S^{-1}N}$$
as $S^{-1}R$-modules.
- Localization commutes with products: If N is an R-module, then
$$S^{-1}(M \times N) \cong S^{-1}M \times S^{-1}N$$
as $S^{-1}R$-modules.
- Localization commutes with internal sums: If N_1, N_2 are R-submodules of M, then
$$S^{-1}(N_1 + N_2) \cong S^{-1}N_1 + S^{-1}N_2$$
as $S^{-1}R$-submodules of $S^{-1}M$.
- Localization commutes with intersections: If N_1, N_2 are R-submodules of M, then
$$S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2$$
as $S^{-1}R$-submodules of $S^{-1}M$.

Proof Consider the exact sequences:
$$0 \longrightarrow N \xrightarrow{\text{incl.}} M \xrightarrow{\text{proj.}} M/N \longrightarrow 0$$
$$0 \longrightarrow M \xrightarrow{\text{incl.}} M \times N \xrightarrow{\text{proj.}} N \longrightarrow 0$$
$$0 \longrightarrow N_1 \xrightarrow{\text{incl.}} N_1 + N_2 \xrightarrow{\text{proj.}} N_2 \longrightarrow 0$$
$$0 \longrightarrow N_1 \cap N_2 \xrightarrow{n \mapsto (n,n)} N_1 \times N_2 \xrightarrow{(n_1,n_2) \mapsto n_1 - n_2} N_1 + N_2 \longrightarrow 0$$
respectively and apply the above proposition. ■

Lemma 5.1.9 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Then the following are true.

- Localization commutes with kernels:
$$S^{-1}\ker(\phi) \cong \ker(S^{-1}\phi)$$
- Localization commutes with cokernels:
$$S^{-1}\frac{N}{\text{im}(\phi)} \cong \frac{S^{-1}N}{\text{im}(S^{-1}\phi)}$$
- Localization commutes with images:
$$S^{-1}(\text{im}\,\phi) \cong \text{im}(S^{-1}\phi)$$

Proof Consider the exact sequences:

$$0 \longrightarrow \ker(\phi) \hookrightarrow M \xrightarrow{\phi} N$$

$$M \xrightarrow{\phi} N \longrightarrow \frac{M}{\operatorname{im}(\phi)} \longrightarrow 0$$

$$0 \longrightarrow \ker(\phi) \longrightarrow M \longrightarrow \operatorname{im}(\phi) \longrightarrow 0$$

respectively and apply 5.3.6. ■

Proposition 5.1.10 Let R be a commutative ring. Let M be an R-module. Then there is an isomorphism

$$S^{-1}M \cong S^{-1}R \otimes_R M$$

of $S^{-1}R$-modules given by $\frac{m}{s} \mapsto \frac{1}{s} \otimes m$.

5.2 Localization at Single Elements and Away from Prime Ideals

Lemma 5.2.1 Let R be a commutative ring. Let $f \in R$ be non-zero. Then the set $\{f^n \mid n \in \mathbb{N}\}$ is a multiplicative set.

Definition 5.2.2 (Localization at an Element) Let R be a commutative ring. Let M be an R-module. Let $f \in R$ be non-zero. Define the localization of M at f to be the ring

$$M_f = \{f^n \mid n \in \mathbb{N}\}^{-1}R$$

Lemma 5.2.3 Let R be a commutative ring. Let $f \in R$ be non-zero. Then there is an R-algebra isomorphism

$$R_f \cong R\left[\frac{1}{f}\right]$$

given by $\frac{a}{f^k} \mapsto a \cdot \frac{1}{f^k}$.

Lemma 5.2.4 Let R be a commutative ring and P a prime ideal of R. Then $R \backslash P$ is a multiplicative set.

Proof By definition, $xy \in P$ implies $x \in P$ or $y \in P$, since $R \setminus P$ removes all these elements, we have that $x \notin P$ and $y \notin P$ implies that $xy \notin P$. ■

Definition 5.2.5 (Localization at Prime Ideals) Let R be a commutative ring. Let M be an R-module. Let P be a prime ideal. Denote

$$M_p = (R \setminus P)^{-1}M$$

the localization of M at P.

5.3 The Localization Map

Proposition 5.3.1 Let R be a commutative ring. Let S be a multiplicative subset of R. Then the following are true.

- $(S^{-1}R, +, \times)$ is a ring
- The map $k : R \to S^{-1}R$ defined by $r \mapsto r/1$ is a ring homomorphism, called the localization map.

Proof

- Define addition by $\frac{r}{s}+\frac{r'}{s'}=\frac{rs'+r's}{ss'}$ and multiplication by $\frac{r}{s}\cdot\frac{r'}{s'}=\frac{rr'}{ss'}$. Clearly addition is abelian, and has identity $\frac{0}{1}$ and inverse $\frac{-r}{s}$ for any $\frac{r}{s}\in S^{-1}R$. Multiplication also has identity $\frac{1}{1}$.

■

Lemma 5.3.2 Let R be a commutative ring. Let S be a multiplicative subset of R. Then localization map $R\to S^{-1}R$ is injective if and only if S does not contain zero divisors.

Proof Suppose that $R\to S^{-1}R$ is injective. Then $sr=0$ implies $r=0$ for all $s\in S$. Hence S does not contain zero divisors. Suppose that S does not contain zero divisors. Then $sr=0$ implies that $r=0$ since S has no zero divisors. Hence the localization map is injective. ■

Proposition 5.3.3 (Universal Property) Let R be a commutative ring. Let S be a multiplicative set. Then $S^{-1}R$ and the localization map $k:R\to S^{-1}R$ satisfies the following universal property.

- For any commutative ring B and ring homomorphism $\phi:R\to B$ such that $\phi(s)\in B^\times$ for all $s\in S$, there exists a unique ring homomorphism $\phi:S^{-1}R\to B$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "k"] \arrow[dr, "\phi"'] & S^{-1}R \arrow[d, dashed, "\exists!\psi"] \\ & B \end{tikzcd}$$

- $S^{-1}R$ is the unique commutative ring (up to unique isomorphism) that has such a property.

Lemma 5.3.4 Let R be a commutative ring. If R is an integral domain, then then following are true.

- If S is a mutliplicative subset of R such that $0\notin S$, then $S^{-1}R$ is an integral domain.
- $\text{Frac}(R)=(0)$.
- The localization map induces a ring isomorphism

$$R\cong\bigcap_{m\text{ a maximal ideal}}R_m$$

Proof

- Suppose that $0=\frac{a}{s}\cdot\frac{b}{t}$. By the equivalence relation this is the same as saying that $uab=0$ for some $u\in S$. Since R is an integral domain and $0\neq S$, we conclude that $u\notin S$ so that $ab=0$. Again since R is an integral domain this implies that $a=0$ or $b=0$. Hence either $a/s=0$ or $b/t=0$ in $S^{-1}R$. Hence $S^{-1}R$ is an integral domain.

- Trivial.

- Clearly the map is well defined. Moreover, since for each maximal ideal m, $0\notin R\setminus m$. Hence the localization map is injective. Suppose for a contradiction that the localization map is not surjective. Then there exists x in the intersection such that $x\neq r/1$ for all $r\in R$. Consider the ideal $I=\{r\in R\mid rx=s/1\text{ for some }s\in R\}$. Since $1\notin R$, I is a proper ideal. So there exists a maximal ideal m containing I. But x also cannot lie in R_m and hence the intersection. Indeed, if $x\in R_m$, then $x=a/b$ for some $a\in R$ and $b\notin m$. Then $bx=a\in R$ implies that $b\in I$. This is a contradiction to $b\notin m$. Thus no such x exists. Hence the localization map is surjective.

■

5.4 Ideals of a Localization

Definition 5.4.1 (Ideals Closed Under Division) Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. We say that I is closed under division by s if for all $s \in S$ and $a \in R$ such that $sa \in I$, we have $a \in I$.

Lemma 5.4.2 Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Then we have

$$I^e = S^{-1}I$$

Proof Let $f : R \to S^{-1}R$ be the localization map. Then $f(I) \subseteq S^{-1}I$ implies that $I^e \subseteq S^{-1}I$. Conversely, suppose that $i/s \in S^{-1}I$. Then $i/s = (1/s) \cdot f(i) \in I^e$. Hence $I^e = S^{-1}I$. ■

Proposition 5.4.3 Let R be a commutative ring. Let S be a multiplicative subset of R. Let P be a prime ideal of R. Then the following are true.

- $S^{-1}P$ is a prime ideal of $S^{-1}R$ if and only if $S \cap P = \emptyset$.
- $S^{-1}P = S^{-1}R$ if and only if $S \cap P \neq \emptyset$.

Proof Recall that R/P is an integral domain if P is prime. Since S^{-1} commutes with quotients, we have that

$$\frac{S^{-1}R}{S^{-1}P} \cong S^{-1}\frac{R}{P}$$

If $S \cap P = \emptyset$, then $0 \in P$ implies that $0 \notin S$. This means that $0 \notin \phi(S)$. By 5.3.1 we conclude that $S^{-1}(R/P)$ is an integral domain. Hence $S^{-1}P$ is a prime ideal. If $S \cap P \neq \emptyset$, suppose that $x \in S \cap P$. Then ????? ■

Theorem 5.4.4 Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Let $\phi : R \to S^{-1}R$ denote the localization map. Then there is a one-to-one bijection

$$\{J \mid J \text{ is an ideal of } S^{-1}R\} \overset{1:1}{\longleftrightarrow} \left\{I \;\middle|\; \substack{I \text{ is an ideal of } R \text{ and} \\ I \text{ is closed under division by } S}\right\}$$

whose map is given by $J \mapsto J^c = \phi^{-1}(J)$ and inverse is given by $I \mapsto I^e = S^{-1}I$.

Proof We first show that our map of sets are well defined. Let J be an ideal of $S^{-1}R$. We first show that $\phi^{-1}(J)$ is closed under division by S. Suppose that $s \in S$ and $r \in R$ such that $sr \in \phi^{-1}(J)$. Then $sr/1 \in J$. Now since J is an ideal of $S^{-1}R$, we know that $1/s \cdot sr/1 \in J$. But $1/s \cdot sr/1 = r/1 = \phi(r)$. This means that $\phi(r) \in J$ and hence $r \in \phi^{-1}(J)$. Thus $\phi^{-1}(J)$ is an ideal closed under division by S.

Now let I be an ideal of R closed under division. I claim that $S^{-1}I$ is an ideal of $S^{-1}R$. Let $a/s, b/t \in S^{-1}I$. Then $a/s + b/t = (at + bs)/st$. Since I is an ideal, we know that $at + bs \in I$. Also since S is a multiplicative subset, $st \in S$. Hence $(at + bs)/st \in I$. Now let $a/s \in S^{-1}I$ and $r/t \in S^{-1}R$. Then $(a/s) \cdot (r/t) = ar/st$. Since I is an ideal, $ar \in I$. Thus $ar/st \in S^{-1}I$ so that I is an ideal.

It remains to show that the two maps are inverses of each other. Let J be an ideal of $S^{-1}R$. We want to show that $J = S^{-1}(\phi^{-1}(J))$. Let $a/s \in J$. Since J is an ideal, we have $\phi(a) = a/1 = 1/s \cdot a/s \in J$. Hence $a \in \phi^{-1}J$ so that $a/s \in S^{-1}\phi^{-1}(J)$. Thus $J \subseteq S^{-1}(\phi^{-1}(J))$. Now by 1.5.5 the extension of the contraction of J is a subset of J. Hence we conclude.

On the other hand, we also want to show that $I = \phi^{-1}(S^{-1}I)$. Again by 1.5.5 we know that $I \subseteq \phi^{-1}(S^{-1}I)$. Conversely, let $x \in \phi^{-1}(S^{-1}I)$. Then $\phi(x) = x/1 \in S^{-1}I$. This means that $x/1 = b/t$ for some $b \in I$ and $t \in S$. Then there exists $u \in S$ such that $uxt = ub$. Since $b \in I$,

$ub \in I$ hence $uxt \in I$. Since $ut \in S$ and I is closed under division, we have $x \in I$.

This shows that $S^{-1}(-)$ and $\phi^{-1}(-)$ are mutual inverses of each others. Thus we conclude.

Using the theorem we conclude that every ideal of $S^{-1}R$ is of the form $S^{-1}I$ for some ideal I of R such that I is closed under division by S.

Proposition 5.4.5 Let R be a commutative ring. Let I be an ideal of R. Let $S \subseteq R$ be a multiplicative subset. Then the above bijection restricts to the following bijection

$$\text{Spec}(S^{-1}R) \overset{1:1}{\longleftrightarrow} \left\{ I \;\middle|\; \begin{matrix} I \text{ is a prime ideal of } R \\ \text{and } I \cap S = \emptyset \end{matrix} \right\}$$

Proof Let $\phi : R \to S^{-1}R$ be the localization map. From the above we know that $Q = S^{-1}\phi^{-1}(Q)$ for any prime ideal Q of $S^{-1}R$. This implies that $S^{-1}\phi^{-1}(Q)$ is prime. By 5.4.3 this implies that $\phi^{-1}(Q) \cap S = \emptyset$. Thus the map $J \mapsto \phi^{-1}(J)$ induces a well defined map on our given sets of prime ideals.

Conversely, by 5.4.3 we know that if P is a prime ideal of R such that $S \cap P = \emptyset$, then $S^{-1}P$ is a prime ideal of $S^{-1}R$. Hence the inverse map is also well defined on our domain and codomain. By the above theorem it is already a bijection, hence we are done.

Proposition 5.4.6 Let R be a commutative ring. Let P be a prime ideal of R. Then the above bijection gives

$$\text{Spec}(R_P) \overset{1:1}{\longleftrightarrow} \left\{ I \;\middle|\; \begin{matrix} I \text{ is a prime ideal of } R \\ \text{and } I \subseteq P \end{matrix} \right\}$$

Proof Notice that the condition that $I \cap S = \emptyset$ in the above proposition translates to $I \cap (R \setminus P) = \emptyset$, which is the same as saying $I \subseteq P$.

Proposition 5.4.7 Let R be a commutative ring and let P be a prime ideal of R. Then R_P is a local ring with unique maximal ideal given by

$$PR_P = \left\{ \frac{r}{s} \mid r \in P, s \notin P \right\}$$

Proof We show that PR_P is the only unique maximal ideal. Suppose that I is an ideal in R_P such that I is not a subset of PR_P. Then there exists $a/s \in I$ such that $a \notin P$ and $s \notin P$. It is clear that s/a is then an element of R_P. So a/s is invertible. Hence $I = R_P$.

Be wary that in general localizations does not result in a local ring. This happens only when we are localizing with respect to a prime ideal. The importance of prime ideals is not explicit in the above because only using prime ideals P can $R \setminus P$ be a multiplicative set which ultimately allows localization to make sense.

Proposition 5.4.8 (Localization of a Localization) Let R be a commutative ring. Let S be a multiplicative subset of R. Let P be a prime ideal of R such that $S^{-1}P$ is a prime ideal of $S^{-1}R$. Then

$$(S^{-1}R)_{S^{-1}P} \cong R_P$$

Proof Define a map $S^{-1}R \to R_P$ by the identity map. This is well defined because if $s \in S$, then we know $S^{-1}P$ is a prime ideal implies $S \cap P = \emptyset$, so $s \notin P$. Thus r/s is a well defined fraction in R_P. Since it is just the identity map, it is a well defined ring homomorphism. Now let $r/s \in S^{-1}R \setminus S^{-1}P$. Then $r \notin P$ implies that r is invertible in R_P. Hence $r/s \cdot s/r = 1$ in R_P.

Thus r/s is invertible in R_P. Thus we can invoke the universal property to obtain a unique map

$$(S^{-1}R)_{S^{-1}P} \to R_P$$

Conversely, define a map $R \to (S^{-1}R)_{S^{-1}P}$ by the identity map $r \mapsto (r/1)/(1/1)$. This is well defined because $1 \notin P$ implies $1/1 \in S^{-1}R \setminus S^{-1}P$. Clearly this is a well defined ring homomorphism. For $s \in S$, notice that $(s/1)/(1/1)$ is invertible in $(S^{-1}R)_{S^{-1}P}$ via the element $(1/s)/(1/1)$. Thus we can invoke the universal property of $S^{-1}R$ to obtain a unique map

$$S^{-1}R \to (S^{-1}R)_{S^{-1}P}$$

We now have two unique maps going both directions between $S^{-1}R$ and $(S^{-1}R)_{S^{-1}P}$. This implies that they are isomorphic. ■

Lemma 5.4.9 Let R be a commutative ring. Let $S \subseteq R$ be a multiplicative subset of R. If R is Noetherian, then $S^{-1}R$ is Noetherian.

Proof Follows from the correspondence of ideals in localizations. ■

5.5 Localization of Graded Rings

Proposition 5.5.1 Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Then R_P is a graded ring in which the grading structure is given as follows: $f/g \in R_P$ has degree $\deg(f) - \deg(g)$.

Definition 5.5.2 (Localization of a Graded Ring) Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Define the localization of R with respect to P to be

$$(R_P)_0 = \{f \in R_P \mid f \text{ lies in the 0th graded component of } R_P\}$$

Proposition 5.5.3 Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a commutative ring that is graded. Let P be a homogeneous prime ideal of R. Then $(R_P)_0$ is a local ring with unique maximal ideal given by

$$(PR_P) \cap (R_P)_0$$

5.6 Local Properties

Definition 5.6.1 (Local Properties of Elements) Let R be a commutative ring. Let M be an R-module. A property of an element of M is local if the following is true. $m \in M$ has the property if and only if $m \in M_P$ has the property.

Lemma 5.6.2 Let R be a commutative ring. Let M be an R-module. Then $x \in M$ being the zero element is a local property.

Proof Suppose that $x = 0$ in M. Then clearly $x = 0$ in both M_P and M_m for all prime ideals P and maximal ideals m. Now let $x = 0$ in M_m for all maximal ideals m. This means that there exists $a_m \in R \setminus m$ such that $a_m x = 0$. Let I be the ideal

$$I = \sum_{m \text{ a maximal ideal}} a_m R \subseteq R$$

Since $a_m \in I$ but $a_m \notin m$, we must have that I is not contained in any maximal ideals. Hence $I = R$. Then there exists $r_i \in R$ such that $1 = \sum_{i=1}^{n} r_i a_{m_i}$ for some $a_{m_i} \in R \setminus m_i$. Then we have

$$x = \sum_{i=1}^{n} (r_i a_{m_i} x) = 0 \in M$$

Definition 5.6.3 (Local Properties of Modules) Let R be a commutative ring. A property of R-modules is local if for any R-modules M, the following are equivalent.
- M has the property
- M_P has the property for all primes ideals P
- M_m has the property for all maximal ideals m

Lemma 5.6.4 Let R be a commutative ring. Let M be an R-module. Then the module being 0 is a local property.

Proof If $M = 0$, then clearly $M_P = 0$ and $M_m = 0$ for all prime ideals P and maximal ideals m. Then using 5.6.2 we conclude that if $M_m = 0$ for all maximal ideals m, then $M = 0$. ■

Proposition 5.6.5 (Injectivity and Surjectivity are Local Properties) Let R be a commutative ring. Let M, N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism. Let S be a multiplicative subset of R. Then the following are equivalent.
- ϕ is injective (surjective)
- For each prime ideal P of R, the induced map $\phi_P : S^{-1}M \to S^{-1}N$ is injective (surjective)
- For each maximal ideal m of R, the induced map $\phi_m : S^{-1}M \to S^{-1}N$ is injective (surjective)

More local properties: nilpotent
Non-local properties: freeness, domain

Proposition 5.6.6 (Exactness is Local) Let R be a commutative ring. Let M_1, M_2, M_3 be R-modules. Let $f : M_1 \to M_2$ and $g : M_2 \to M_3$ be R-module homomorphisms. Then the following conditions are equivalent.
- The following sequence is exact:
$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$
- The following sequence is exact:
$$(M_1)_P \xrightarrow{f_P} (M_2)_P \xrightarrow{g_P} (M_3)_P$$
for all prime ideals P of R.
- The following sequence is exact:
$$(M_1)_m \xrightarrow{f_m} (M_2)_m \xrightarrow{g_m} (M_3)_m$$
for all maximal ideals m of R.

Proof $(1) \implies (2), (3)$ is clear since localization preserves exact sequences. It remains to show that $(3) \implies (1)$. Let $x \in M$. Then we have that $g_m(f_m(x)) = 0$ for all maximal ideals m. Since being 0 is a local property, we conclude that $g(f(x)) = 0$. Hence $\operatorname{im}(f) \subseteq \ker(g)$. Since kernels and images and quotients commute with localizations, we have that
$$\left(\frac{\ker(g)}{\operatorname{im}(f)}\right)_m \cong \frac{\ker(g_m)}{\operatorname{im}(f_m)} = 0$$
Since being a zero module is a local property, we conclude that $\operatorname{im}(f) = \ker(g)$. ■

6 Primary Decomposition

6.1 The Annihilator and Associated Primes

Let R be a commutative ring. Let M be an R-module. Recall that we define the annihilator of a subset $S \subseteq M$ to be the ideal
$$\text{Ann}_R(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}$$
When R is a commutative ring, the annihilator is a two sided ideal and consequently has some nice properties.

Proposition 6.1.1 Let R be a commutative ring. Let M be an R-module. Let $\text{Ann}_R(x)$ for $x \in M$ be a maximal element in the set
$$\{\text{Ann}_R(x) \mid 0 \neq x \in M\}$$
Then $\text{Ann}_R(x)$ is a prime ideal.

Proof Suppose that $ab \in \text{Ann}_R(x)$ and $b \notin \text{Ann}_R(x)$. Notice that if $rx = 0$ then $r(bx) = brx = 0$ so that r annihilates bx. Hence $\text{Ann}_R(x) \subseteq \text{Ann}_R(bx)$. Since x is non-zero and $b \notin I$, bx is also non-zero hence $\text{Ann}_R(bx)$ lies in the given set of annihilators. Since $\text{Ann}_R(x)$ is maximal we conclude that
$$\text{Ann}_R(x) = \text{Ann}_R(bx)$$
But ab annihilates x by definition so that a annihilates bx. Hence $a \in \text{Ann}_R(bx) = \text{Ann}_R(x)$. Hence $\text{Ann}_R(x)$ is prime. ■

Recall that if $S \subseteq M$ is a subset and R is not a commutative ring, then in general we only have the relation
$$\text{Ann}_R(\langle S \rangle) \subseteq \text{Ann}_R(S)$$

Proposition 6.1.2 Let R be a commutative ring. Let M be an R-module. Let $S \subseteq M$ be a subset. Then
$$\text{Ann}_R(\langle S \rangle) = \text{Ann}_R(S)$$

Definition 6.1.3 (Associated Prime) Let R be a commutative ring. Let M be an R-module. Let P be a prime ideal of R. We say that P is an associated prime of M if
$$\text{Ann}_R(m) = P$$
for some $m \in M$.

Lemma 6.1.4 Let R be a Noetherain commutative ring. Let M be an R-module. If $M \neq 0$, then $\text{Ass}(M) \neq \emptyset$.

Proof By 6.1.1, there exists $x \in M$ such that $\text{Ann}_R(x)$ is a prime ideal. ■

Lemma 6.1.5 Let R be a commutative ring. Let M be an R-module. Let P be a prime ideal of R. Then P is an associated prime of M if and only if R/P is isomorphic to a submodule of M.

Proof If P is an associated prime, then $P = \text{Ann}_R(m)$ for some $0 \neq m \in M$. Then $\langle m \rangle \cong \frac{R}{\text{Ann}_R(m)}$ so that R/P is isomorphic to a submodule of M. Conversely, if $R/P \cong N \leq M$ for some submodule N, notice that R/P is cyclic and so N is generated by one element $n \in N$. Then $P = \text{Ann}_R(n)$. ■

Definition 6.1.6 (Set of Associated Prime) Let R be a commutative ring. Let M be an R-module. Define the set of associated primes of M to be
$$\text{Ass}(M) = \{P \in \text{Spec}(R) \mid P \text{ is an associated prime of } M\}$$

Another way to think about the set of associated primes of M is that

$$\text{Ass}(M) = \{\text{Ann}_R(m) \mid \text{Ann}_R(m) \in \text{Spec}(R)\}$$

Lemma 6.1.7 Let R be a Noetherian commutative ring. Let M be an R-module. Then we have

$$\bigcup_{P \in \text{Ass}(M)} P = \{r \in R \mid r \text{ is a zero divisor of } M\} \cup \{0\}$$

Proof If $r \in R$ is a non-zero zero divisor of M, then $rm = 0$ for some $0 \neq m \in M$. Then $r \in \text{Ann}_R(m)$. By 6.1.1, r is contained some prime ideal that is an annihilator. Hence r lies in the union in the left. Conversely, if r lies in some annihilator then clearly r is a zero divisor, or $r = 0$. ■

Proposition 6.1.8
Let R be a commutative ring. Let S be a multiplicative subset of R. Let M be an $S^{-1}R$-module. Then we have

$$\text{Ass}_{S^{-1}R}(S^{-1}M) = \text{Ass}_R(S^{-1}M)$$

Proof ■

Proposition 6.1.9
Let R be a Noetherian commutative ring. Let S be a multiplicative subset of R. Let M be an R-module. Then the following are true.

- Considering $\text{Spec}(S^{-1}R)$ as a subset of $\text{Spec}(R)$ by the correspondence of prime ideals of localization, we have

$$\text{Ass}_R(S^{-1}M) = \text{Ass}_R(M) \cap \text{Spec}(S^{-1}R)$$

- Let P be a prime ideal of R. Then $P \in \text{Ass}_R(M)$ if and only if $PR_P \in \text{Ass}_{R_P}(M_P)$.

Proof
Suppose that $P \in \text{Ass}_R(S^{-1}M)$. Let φ denote the localization map. Then $P = \text{Ann}_R(m/s)$ for some $m/s \in S^{-1}M$. But $p/t \in P$ annihilates m/s if and only if p annihilates m. Hence by lifting P back to R by the correspondence, $\varphi^{-1}(P) = \text{Ann}_R(m)$ and must also be a prime lying in $\text{Spec}(S^{-1}R)$.

Conversely, if $P \in \text{Spec}(S^{-1}R)$ annihilates $m \in M$, then it must also annihilate $m/1$. And $m/1$ is annihilated by $r/s \in S^{-1}R$ if r annihilates m in M. Thus we conclude. ■

Proposition 6.1.10 Let R be a commutative ring. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then we have

$$\text{Ass}(M_2) \subseteq \text{Ass}(M_1) \cup \text{Ass}(M_3)$$

Example 6.1.11 Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Then $\text{Ass}(\mathbb{Z}) \subset \text{Ass}(\mathbb{Z}) \cup \text{Ass}(\mathbb{Z}/2\mathbb{Z})$ is a strict subset.

Proof Clearly $(2) \subseteq \mathbb{Z}$ annihilates $\mathbb{Z}/2\mathbb{Z}$ but does not annihilate $\mathbb{Z}$. ■

Theorem 6.1.12 (Disassembly of an R-Module) Let R be a Noetherian commutative ring. Let M be a finitely generated R-module. Then there exists a chain of R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that

$$\frac{M_i}{M_{i-1}} \cong \frac{R}{P_i}$$

for some prime ideal P_i of R.

Proof If M is trivial then we are done. So suppose that $M \neq \{0\}$. We define the R-submodules inductively.

- When $n = 1$, $\text{Ass}(M) \neq \emptyset$, say $P_1 \in \text{Ass}(M)$. Since P_1 is an annihilator, $M_1 = R/P_1$ is an R-submodule of M.
- Assume that $M_1 \subset \cdots \subset M_i$ is constructed. If $M_i = M$ then we are done. If not, then $M/M_i \neq \{0\}$ and $P_{i+1} \in \text{Ass}(M/M_i) \neq \emptyset$. Then $N = R/P_{i+1}$ is an R-submodule of M/M_i. By the correspondence theorem for R-modules, N corresponds to an R-submodule M_{i+1} of M containing M_i.

The process eventually terminates since M is Noetherian. ■

Proposition 6.1.13 Let R be a Noetherian commutative ring. Let M be a finitely generated R-module. Let $P_1, \dots, P_n \in \text{Spec}(R)$ be the prime ideals appearing in the disassembly of M. Then

$$\text{Ass}(M) \subseteq \{P_1, \dots, P_n\}$$

Proof We induct on the length of the disassembly. When $n = 0$ the result is trivial. Suppose that the result holds true for all R-modules whose length of disassembly is $\leq k$. Let M be an R-module whose disassembly has length $k+1$. Let $\varphi : M/M_k \to R/P_k$ be the isomorphism given in the disassembly. Let $m \in M$ be such that $\text{Ann}_R(m)$ is a prime idea. If $m \in M_k$ then by inductive hypothesis we are done. So suppose that $m \notin M_k$. If r annihilates m, then r annihilates $\varphi(m)$ in R/P_k. Hence ■

Definition 6.1.14 (Embedded Associated Primes) Let R be a commutative ring. Let M be an R-module. Let $I \in \text{Ass}(M)$ be an associated prime. We say that I is embedded if I is not minimal in $\text{Ass}(M)$.

6.2 The Support of a Module

Definition 6.2.1 (Support of a Module) Let A be a commutative ring. Let M be an A-module. The support of M is the subset

$$\text{Supp}(M) = \{P \text{ a prime ideal of } A \mid M_P \neq 0\}$$

Lemma 6.2.2 Let R be a commutative ring. Let the following be an exact sequence of R-modules.

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

Then we have

$$\text{Supp}(M_2) = \text{Supp}(M_1) \cup \text{Supp}(M_2)$$

Proposition 6.2.3 Let R be a commutative ring. Let M be a finitely generated R-module. Then

$$\text{Supp}(M) = \{P \in \text{Spec}(R) \mid \text{Ann}_R(M) \subseteq P\}$$

Proof We first show the case when M is generated by one element $m\in M$. Let $P\in\text{Supp}(M)$. Then $M_P\neq 0$ and so $m/1\neq 0\in M_P$. This means that for all $s\in R\setminus P$, we have $sm\neq 0$. Then $R\setminus P\cap\text{Ann}_R(m)=\emptyset$. Then $P\supseteq\text{Ann}_R(m)=\text{Ann}_R(M)$. Conversely, suppose that $P\notin\text{Supp}(M)$. Then $M_P=0$ and so $m/1=0$. So there exists $s\in R\setminus P$ such that $sm=0$. Hence $R\setminus P\cap\text{Ann}_R(m)\neq\emptyset$ and so $\text{Ann}_R(M)=\text{Ann}_R(m)$ is not a subset of P.

Now suppose that M is finitely generated by $m_1,\dots,m_k$. Then we have

$$\begin{aligned}\text{Supp}(M)&=\bigcup_{i=1}^k\text{Supp}(R\cdot m_i)\\&=\bigcup_{i=1}^k\{P\in\text{Spec}(R)\mid\text{Ann}_R(m_i)\subseteq P\}\\&=\bigcup_{i=1}^k\{P\in\text{Spec}(R)\mid\text{Ann}_R(m_i)\subseteq P\}\\&=\left\{P\in\text{Spec}(R)\;\middle|\;\bigcap_{i=1}^k\text{Ann}_R(m_i)\subseteq P\right\}\qquad(\text{lmm1.1.2})\\&=\{P\in\text{Spec}(R)\mid\text{Ann}_R(M)\subseteq P\}\end{aligned}$$

■

Lemma 6.2.4 Let R be a commutative ring. Let M be a finitely generated R-module. Let $P_1,\dots,P_n\in\text{Spec}(R)$ be a complete list of distinct minimal prime ideals over $\text{Ann}_R(M)$. Then we have

$$\text{Supp}(M)=\bigcup_{k=1}^n\{P\in\text{Spec}(R)\mid P_k\subseteq P\}$$

Proof We induct on the length of the diassembly of M. If $n=1$, then M is simple, and $M\cong R/P$ with $P=\text{Ann}_R(M)$. Now suppose the result is true for $\leq n-1$. Let $0=M_0\subset\cdots M_n=M$ be the diassembly of M. Then we obtain an exact sequence of the form

$$0\longrightarrow M_{n-1}\xrightarrow{f}M\xrightarrow{g}\frac{M}{M_{n-1}}\longrightarrow 0$$

In particular, we have $\text{Supp}(M)=\text{Supp}(M/M_{n-1})\cup\text{Supp}(M_{n-1})$. By induction, we have $\text{Supp}(M_{n-1})=\bigcup_{i=1}^{n-1}\{P\in\text{Spec}(R)\mid P_i\subseteq P\}$, and similarly for the simple module M/M_{n-1} we have the result of the case $n=1$. Hence we are done. ■

Let R be a commutative ring. Let M be an R-module. Let $P_1,\dots,P_n\in\text{Spec}(R)$ be the prime ideals appearing in the disassembly of M. Then summarizing the above, we have

$$\text{Ass}(M)\subseteq\{P_1,\dots,P_n\}\subseteq\text{Supp}(M)=\{P\in\text{Spec}(R)\mid\text{Ann}_R(M)\subseteq P\}$$

It turns out that the minimal primes of the four sets coincide.

Proposition 6.2.5 Let R be a Noetherian commutative ring. Let M be a finitely generated R-module. Let $P_1,\dots,P_n\in\text{Spec}(R)$ be the prime ideals appearing in the disassembly of M. Then the following sets are equal.

- $\{P\in\text{Spec}(R)\mid P\text{ is minimal in }\text{Supp}(M)\}$.
- $\{P\in\text{Spec}(R)\mid P\text{ is minimal in }\text{Ass}(M)\}$.
- $\{P\in\text{Spec}(R)\mid P\text{ is a minmal prime ideal over }\text{Ann}_R(M)\}$.
- $\{P\in\text{Spec}(R)\mid P\text{ is minimal in }\{P_1,\dots,P_n\}\}$.

Proof

- $(1) = (4)$: By the above lemma, we have

$$\text{Supp}(M) = \bigcup_{i=1}^{n}\{P \in \text{Spec}(R) \mid P_i \subseteq P\} = \bigcup_{P \in \{P_1,\dots,P_n\} \text{ minimal}} \{Q \in \text{Spec}(R) \mid P_k \subseteq Q\}$$

If P is minimal in $\text{Supp}(M)$, then it is minimal in the union. Then $P \in \{Q \in \text{Spec}(R) \mid P_k \subseteq Q\}$ for some minimal P_k. If $P \neq P_k$ then evidently P is not minimal, hence $P = P_k$. The converse is similar.

- $(1) = (3)$: By 6.2.3, $\text{Supp}(M) = \{P \in \text{Spec}(R) \mid \text{Ann}_R(M) \subseteq P\}$ means that P is a minimal prime ideal over $\text{Ann}_R(M)$ if and only if P is minimal in $\text{Supp}(M)$.

■

6.3 Primary Ideals

Definition 6.3.1 (Primary Ideals) Let R be a commutative ring. Let Q be a proper ideal of R. We say that Q is a primary ideal of R if $fg \in Q$ implies $f \in Q$ or $g^m \in Q$ for some $m > 0$.

Lemma 6.3.2 Let R be a commutative ring. Let Q be an ideal of R. Then the following are true.
- If Q is a prime ideal, then Q is a primary ideal.
- If $\sqrt{Q}$ is a maximal ideal, then Q is a primary ideal.

Proof Let $fg \in Q$. Since Q is prime, $f \in Q$ or $g \in Q$ and so we are done.

Let $fg \in Q$ and $f \notin Q$. Let $I = \{g \in R \mid fg \in Q\}$. Clearly $Q \subseteq I$. Moreover $1 \notin I$. Hence I is a proper ideal. Then we have $m = \sqrt{Q} \subseteq \sqrt{I}$. Hence $I \subseteq \sqrt{I} = m$ since m is maximal. This shows that $g \in I$ implies $g \in m = \sqrt{Q}$. Hence we are done. ■

Lemma 6.3.3 Let $\phi : R \to S$ be a ring homomorphism and Q be a primary ideal in S. Then $\phi^{-1}(Q)$ is primary in R.

Proposition 6.3.4 Let R be a commutative ring. Let Q be a proper ideal of R. Then Q is primary if and only if every zero divisor in R/Q is nilpotent.

Lemma 6.3.5 Let R be a commutative ring. Let Q be a primary ideal of R. Then the following are true.
- $\sqrt{Q}$ is a prime ideal.
- $\sqrt{Q}$ is minimal among primes that contain Q.

Definition 6.3.6 (P-Primary Ideals) Let R be a commutative ring. Let P be a prime ideal. Let Q be an ideal. We say that Q is a P-primary ideal of R if the following are true.
- Q is a primary ideal.
- $\sqrt{Q} = P$.

Lemma 6.3.7 Let R be a commutative ring. Let P be a prime ideal. Let Q_1, Q_2 be P-primary ideals. Then $Q_1 \cap Q_2$ is a P-primary ideal.

Proposition 6.3.8 Let R be a Noetherian commutative ring. Let P be a prime ideal of R. Let Q be a proper ideal. Then Q is P-primary if and only if $\text{Ass}(R/Q) = \{P\}$.

Proof Let Q be a P-primary ideal. We know that $\text{Ass}(R/Q)$ is non-empty. So let I be a prime ideal such that $I \in \text{Ass}(R/Q)$. Clearly $Q \subseteq I$. There exists $[r] \in R/Q$ where $[r] \neq 0$ such that $\text{Ann}_R([r]) = I$. Let $x \in I \setminus \{0\}$. Then $[xr] = [x] \cdot [r] = 0 \in R/Q$ implies that $[x]$ is a zero divisor of R/Q. By 6.3.4, we conclude that $[x] \in N(R/Q)$. Then by lemma 1.4.5, we have $x \in \sqrt{Q} = P$. Hence we have $Q \subseteq I \subseteq \sqrt{Q} = P$. Taking radical gives $I = P$ since I is a prime ideal. Hence $\text{Ass}(R/Q) = \{P\}$.

Now suppose that $\text{Ass}(R/Q) = \{P\}$. Let $xy \in Q$. Suppose that $x \notin Q$. Then we have $[x] \cdot [y] = [xy] = 0 \in R/Q$. Hence $y \in \text{Ann}_R([x])$. But we also have

$$\sqrt{\text{Ann}_R([x])} = \bigcap_{\substack{I \text{ is a minimal prime} \\ \text{ideal over } \text{Ann}_R([x])}} I = \bigcap_{\substack{I \text{ is minimal} \\ \text{in } \text{Ass}([x])}} I = P$$

Similarly, we know that

$$\sqrt{\text{Ann}_R(R/Q)} = \bigcap_{\substack{I \text{ is a minimal prime} \\ \text{ideal over } \text{Ann}_R(R/Q)}} I = \bigcap_{\substack{I \text{ is minimal} \\ \text{in } \text{Ass}(R/Q)}} I = P$$

Then $y \in \text{Ann}_R([x])$ implies that $y \in \sqrt{\text{Ann}_R([x])} = P = \sqrt{\text{Ann}_R(R/Q)}$. This means that $y^n \in \text{Ann}_R(R/Q)$ for some $n \in \mathbb{N}$. Hence $y^n \in Q$. ∎

Lemma 6.3.9 Let R be a Noetherian commutative ring. Let P be a prime ideal. Let Q be P-primary. Then we have

$$P^n \subseteq Q \subseteq P$$

for some $n \in \mathbb{N}$.

Proof Since R is Noetherian, P is finitely generated. Suppose that $P = (f_1, \dots, f_k)$. Since $\sqrt{Q} = P$, we have $f_i^{n_i} \in Q$ for some $n_i \in \mathbb{N}$. Then for any monomial of degree $m > \sum_{i=1}^k (n_i - 1)$ is a multiple of $f_i^{n_i}$ for some $1 \leq i \leq k$. Hence $P^m \subseteq Q$. ∎

Example 6.3.10 Let k be a field. Let $I = (x^2, xy) \subseteq k[x, y]$. Then we have

$$(x^2) \subseteq I \subseteq (x)$$

but I is not primary. In particular, this shows that the condition in the above lemma is not a sufficient condition for ideals to be primary.

Proof I is not primary because $xy \in I$ but $x \notin I$ and $y^n \notin I$ for any $n \in \mathbb{N}$. ∎

Corollary 6.3.11 Let R be a Noetherian commutative ring. Let m be a maximal ideal of R. Let Q be a proper ideal. Then the following are equivalent.

- Q is m-primary.
- $\text{Ass}(R/Q) = \{m\}$
- There exists $n \in \mathbb{N}$ such that $m^n \subseteq Q \subseteq m$.

Proof By the above proposition we have (1) $\iff$ (2). The above lemma also shows that (1) $\implies$ (3). Finally, suppose that $m^n \subseteq Q \subseteq m$. Then taking radicals give $m = \sqrt{m^n} \subseteq \sqrt{Q} \subseteq \sqrt{m} = m$. By 6.3.3 we conclude that Q is m-primary. ∎

6.4 Primary Decomposition

We want to express ideal I in R as $I = P_1^{e_1} \cdots P_n^{e_n}$ similar to a factorization of natural numbers, for some prime ideals $P_1, \dots, P_n$. However this notion fails and thus we have the following new type of ideal.

Definition 6.4.1 (Primary Decompositions) Let A be a commutative ring. Let I be an ideal of A. A primary decomposition I consists of primary ideals $Q_1, \dots, Q_r$ of A such that

$$I = Q_1 \cap \cdots \cap Q_r$$

Example 6.4.2 Let k be a field. For any $\alpha \in k$, the ideal $(x^2, xy) \subseteq k[x, y]$ has a primary decomposition given by

$$(x^2, xy) = (x) \cap (x^2, y - \alpha x)$$

Proof Since (x) is a prime ideal, it is a (x)-primary ideal. ■

Definition 6.4.3 (Minimal Primary Decompositions) Let A be a commutative ring. Let I be an ideal of A. Let

$$I = Q_1 \cap \cdots \cap Q_r$$

be a primary decomposition of I. We say that the decomposition is minimal if the following are true.

- Each $\sqrt{Q_i}$ are distinct for $1 \leq i \leq r$
- Removing a primary ideal changes the intersection. This means that for any i, $I \neq \bigcap_{j \neq i} Q_j$

Theorem 6.4.4 Let R be a Noetherian commutative ring. Let I be a proper ideal of R. Then I admits a minimal primary decomposition.

Definition 6.4.5 (Prime Divisors of an Ideal) Let R be a commutative ring. Let I be an ideal of R. We say that a prime ideal P of R is a prime divisor of I if $P = \sqrt{Q}$ for some ideal Q that lies in a minimal primary decomposition of I.

6.5 The Two Uniqueness Theorems

Theorem 6.5.1 (First Uniqueness Theorem)
Let R be a Noetherian commutative ring. Let I be an ideal of R. Let

$$I = \bigcap_{i=1}^{k} Q_i$$

be a minimal primary decomposition of I, where each Q_i is P_i-primary. Then we have

$$\operatorname{Ass}(R/I) = \{P_1, \dots, P_k\}$$

Moreover, $\operatorname{Ass}(R/I)$ is unique.

Theorem 6.5.2 (Second Uniqueness Theorem)
Let R be a Noetherian commutative ring. Let I be an ideal of R. Let

$$I = \bigcap_{i=1}^{k} Q_i$$

be a minimal primary decomposition of I, where each Q_i is P_i-primary. Let P be a minimal element of $\operatorname{Ass}(R/I) = \{P_1, \dots, P_n\}$. Let $\varphi : R \to R_P$ be the localization map. Then we have

$$Q_i = \varphi^{-1}(I_P)$$

Moreover, the primary decomposition is uniquely determined by the minimal P in $\operatorname{Ass}(R/I)$.

6.6 Symbolic Powers

Definition 6.6.1 (Symbolic Powers)
Let R be a Noetherian commutative ring. Denote $\varphi : R \to R_P$ the localization map with respect to the prime ideal P. Let I be an ideal of R. Define the nth symbolic power of I to be

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(R/I)} \varphi^{-1}(I^n R_P)$$

7 Integral Dependence

7.1 Integral Elements

Definition 7.1.1 (Integral Elements) Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b \in B$. We say that b is integral over A if there exists a monic polynomial $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$ such that $p(b) = 0$.

When A and B are field, this is a familiar notion in Field and Galois theory.

Lemma 7.1.2 Let K be a field. Let $F \subseteq K$ be a subfield. Let $k \in K$. Then k is integral over F if and only if k is algebraic over F.

Proposition 7.1.3 Let B be a commutative ring and let $A \subseteq B$. Let $b \in B$. Then the following are equivalent.

- b is integral over A
- $A[b] \subseteq B$ is finitely generated A-submodule.
- There exists an A sub-algebra $A' \subseteq B$ such that $A[b] \subseteq A'$ and A' is finitely generated as an A-module.

Proof

- (1) $\implies$ (2): Since b is integral over A, $b^n = a_{n-1}b^{n-1} + \cdots + a_1 b + a_0$. Hence $A[b] = \bigoplus_{i=0}^{n-1} A \cdot b^i$ is a finitely generated A-module.
- (2) $\implies$ (3): Choose $A' = A[b]$.
- (3) $\implies$ (1). By assumption, A' is a finitely generated A-module. Let $\phi : A' \to A'$ be the ring homomorphism defined by $\phi(x) = bx$. By Cayley-Hamilton theorem, there exists $a_1, \dots, a_{n-1} \in A$ such that

$$\phi^n + a_{n-1}\phi^{n-1} + \cdots + a_1\phi + a_0 = 0$$

Since ϕ is the multiplication by b map, we have

$$(b^n + a_{n-1}b^{n-1} + \cdots + a_1 b_+ a_0)(y) = 0$$

for all $y \in A'$. Choosing $y = 1$, we see that b is integral over A.

■

Lemma 7.1.4 Let $A \subseteq B$ be commutative rings. Then B is a finitely generated A-module if and only if $B = A[x_1, \dots, x_n]$ for some $x_1, \dots, x_n \in B$ that is integral over A.

Proof Induct on n and use the fact that x_i is integral over A if and only if $A[x_i]$ is a finitely generated A-module, and the fact that x_i is integral over $A[x_1, \dots, x_{i-1}]$. ■

Proposition 7.1.5 Let B be a commutative ring and let $A \subseteq B$ be a subring. Let $b_1, b_2 \in B$ be integral over A. Then $b_1 + b_2$ and $b_1 b_2$ are both integral over A.

7.2 Integral Closure

Definition 7.2.1 (Integral Closure) Let B be a commutative ring. Let $A \subseteq B$ be a subring. Define the subring
$$\overline{A} = \{b \in B \mid b \text{ is integral over } A\}$$
to be the integral closure of A in B.

Example 7.2.2 The integral closure of $\mathbb{Z} \subseteq \mathbb{Q}$ is $\mathbb{Z}$.

Proposition 7.2.3 Let B be a commutative ring. Let $A \subseteq B$ be a subring. Let S be a multiplicatively closed subset of A. Then
$$\overline{S^{-1}A} = S^{-1}\overline{A}$$

Definition 7.2.4 (Integral Extensions) Let B be a commutative ring and let $A \subseteq B$ be a subring. We say that B is integral over A if $\overline{A} = B$. We also say that B is the integral extension of A.

Lemma 7.2.5 Let $A \subseteq B \subseteq C$ be commutative rings. Then C is integral over B and B is integral over A if and only if C is integral over A.

Proposition 7.2.6 Let A, B be commutative rings such that $A \subset B$ is an integral extension. Then the following are true.

- Let J be an ideal of B. Then $\frac{B}{J}$ is integral over $\frac{A}{J \cap A}$.
- Let S be a multiplicative subset of B. Then $S^{-1}B$ is integral over $S^{-1}A$.

Proof Suppose that J is an ideal of B. Let $b + J \in B/J$. Since $b \in B$ and B is integral over A, there exists $a_0, \dots, a_{n-1} \in A$ such that
$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$
Reduction to J gives
$$(b + J)^n + (a_{n-1} + J)(b + J)^{n-1} + \cdots + (a_1 + J)(b + J) + (a_0 + J) = J$$
This shows that $b + J$ is an integral element of $A/J \cap A$ because each $a_i + J$ is an element of $A/J \cap A$ by restriction to A.

Let $b/s \in S^{-1}B$. Since B is integral over A, there exists $a_0, \dots, a_{n-1} \in A$ such that
$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$
Dividing s^n on both sides give
$$\frac{b^n}{s^n} + \frac{a_{n-1}}{s}\frac{b^{n-1}}{s^{n-1}} + \cdots + \frac{a_1}{s^{n-1}}\frac{b}{s} + \frac{a_0}{s^n} = 0$$
This shows that b/s is an integral element of $S^{-1}A$. ■

Lemma 7.2.7 Let A, B be integral domains such that $A \subset B$ is an integral extension. Then A is a field if and only if B is a field.

Proof Suppose that A is a field. Let $0 \neq b \in B$. Then there exists $a_0, \dots, a_{n-1} \in A$ such that
$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0$$
for smallest of such $n \in \mathbb{N}$. Rearranging gives
$$b(b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1) = -a_0$$
Notice that $a_0 \neq 0$ because otherwise it contradicts the minimality of n. Since A is a field, we can

divide $-a_0 \neq 0$ on both sides to find an inverse of b. Hence B is a field.

Now assume that B is a field. Let $0 \neq a \in A$. Since B is a field, $a^{-1} \in B$ is such that there exists $a_0, \dots, a_{n-1} \in A$ such that

$$a^{-n} + a_{n-1}a^{-(n-1)} + \cdots + a_1 a^{-1} + a_0 = 0$$

Multiplying a^{n-1} on both sides and rearranging, we get

$$a^{-1} = -\left(a_{n-1} + \cdots + a_1 a^{n-2} + a_0 a^{n-1}\right)$$

This shows that $a^{-1} \in A$. Hence A is a field. ■

Definition 7.2.8 (Integrally Closed) Let B be a commutative ring. Let $A \subseteq B$ be a subring. We say that A is integrally closed in B if $\overline{A} = A$.

Theorem 7.2.9 (Gauss's Lemma) Let B be a commutative ring. Let $A \subseteq B$ be a subring. Suppose that A is integrally closed in B. Then the following are true.

- If $f, g \in B[x]$ are monic polynomials such that $fg \in A[x]$, then $f, g \in A[x]$.
- If $f \in A[x]$ is irreducible, then f is irreducible as a polynomial in $B[x]$.

Proof Clearly the first statement implies the second. We first prove that for any monic polynomial $f \in B[x]$, there exists a ring C such that $B \subseteq C$ and f factorizes as a product of linear terms in $C[x]$. To show this, we induct on n. If $n = 1$ then we are done. Suppose that the hypothesis is true for some $k \in \mathbb{N}$. Suppose that $\deg(f) = k + 1$. ■

7.3 The Going-Up and Going-Down Theorems

We want to compare prime ideals between integral extensions.

Lemma 7.3.1 Let A, B be rings such that $A \subset B$ is an integral extension. Let Q be a prime ideal of B. Then $Q \cap A$ is a maximal ideal of A if and only if Q is a maximal ideal of B.

Proof By 7.2.6, we know that B/Q is integral over $A/Q \cap A$. By 7.2.7, B/Q is a field if and only if $A/Q \cap A$ is a field. Hence Q is a maximal ideal of B if and only if $Q \cap A$ is a maximal ideal of A. ■

Proposition 7.3.2 Let A, B be rings such that $A \subset B$ is an integral extension. Let P be a prime ideal of A. Then the following are true.

- There exists a prime ideal Q of B such that $P = Q \cap A$
- If Q_1, Q_2 are prime ideals of B such that $Q_1 \cap A = P = Q_2 \cap B$ and $Q_1 \subseteq Q_2$, then $Q_1 = Q_2$.

Proof Let $\alpha : A \to A_P$ and $\beta : B \to B_P$ be the localization maps. Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A_P \arrow[r, hook] & B_P \end{tikzcd}$$

Since PB_P is the unique maximal ideal of B_P, we know that $PA_P = PB_P \cap A_P$ is the unique maximal ideal of A_P. On the other hand, we also know that $\beta^{-1}(PB_P)$ is a prime ideal of B. By commutativity of the diagram, we have that P is mapped to $\beta^{-1}(PB_P)$. Then by definition of extension we have that $\beta^{-1}(PB_P) \cap B = P$.

Let Q_1, Q_2 be as given. We have that

$$(Q_1 \cap A)A_P = PA_P = (Q_2 \cap A)A_P$$

is the same maximal ideal of A_P since they both contract to P in A. By the above lemma, $(Q_1 \cap A)B_P$ and $(Q_2 \cap A)B_P$ are both maximal ideals of B_P. By commutativity of the diagram, $(Q_1 \cap A)B_P = Q_1B_P$ and $(Q_2 \cap A)B_P = Q_2B_P$. Since $Q_1 \subseteq Q_2$, we have that $Q_1B_P \subseteq Q_2B_P$. Since Q_1B_P and Q_2B_P are both maximal ideals, they must be equal. Hence by contraction we deduce that $Q_1 = Q_2$. ■

Theorem 7.3.3 (The Going-Up Theorem) Let A, B be rings such that $A \subset B$ is an integral extension. Let $0 \leq m < n$. Consider the following situation

$$\begin{array}{llll} B & Q_1 \subseteq \cdots \subseteq Q_m & & \text{(Prime ideals of } B\text{)} \\ \uparrow & & & \\ A & P_1 \subseteq \cdots \subseteq P_m & \subseteq P_{m+1} \subseteq \cdots \subseteq P_n & \text{(Prime ideals of } A\text{)} \end{array}$$

where $Q_i \cap A = P_i$ for $1 \leq i \leq m$. Then there exists prime ideals $Q_{m+1}, \dots, Q_n$ of B such that the following are true.

- $Q_{m+1} \subseteq \cdots \subseteq Q_n$
- $Q_i \cap A = P_i$ for $m + 1 \leq i \leq n$

Proof By induction, it suffices to prove the case $m = 1$ and $n = 2$. This means that we want to find a prime ideal Q_2 such that $Q_1 \subseteq Q_2$ and $Q_2 \cap A = P_2$. By 7.2.6, B/Q_1 is integral over A/P_1. Since P_2/P_1 is a prime in A/P_1 by the correspondence theorem, by 7.3.2 there exists a prime ideal Q_2/Q_1 in B/Q_1 such that $Q_2/Q_1 \cap A/P_1 = P_2/P_1$. This implies that $Q_2 \cap A = P_2$. Hence we are done. ■

7.4 Zariski's Lemma

Lemma 7.4.1 Let F be a field. Let $f \in F[x]$ be a polynomial. Then the localization $F[x]_f$ is not a field.

Proof By 1.8.1, $F[x]$ has infinitely many irreducible polynomials. Then there exists a monic irreducible polynomial g that does not divide f. Assume for a contradiction that $F[x]_f$ is a field. Then $g/1$ is invertible. So there exists $h \in F[x]$ and $n \in \mathbb{N}$ such that $1 = g \cdot \frac{h}{f^n}$. This means that there exists $m \in \mathbb{N}$ such that $ghf^m = f^{n+m} \in F[x]$. If $n + m = 0$, then g is a unit, a contradiction. Otherwise, g divides f^{n+m}. Since g is irreducible, g divides f and is also a contradiction. Hence $F[x]_f$ is not a field. ■

Theorem 7.4.2 (Zariski's Lemma) Let F be a field. Let K/F be a field extension. Then K/F is a finite field extension if and only if K is finitely generated as an F-algebra.

Proof Since K is finitely generated as an F-algebra, there exists $x_1, \dots, x_n \in K$ such that every element in K can be written as a polynomial in $x_1, \dots, x_n$. This means that $K = F(x_1, \dots, x_n)$ as fields. Suppose for a contradiction that K/F is not an algebraic (integral) extension. Without loss of generality, suppose that $F(x_1, \dots, x_r)/F$ is transcendental (not integral) and $K/F(x_1, \dots, x_r)$ is algebraic (integral).

Let $L = F(x_1, \dots, x_{r-1})$. Consider the transcendental (not integral) extension $L(x_r)/L$. Now K is generated as an L-algebra by the elements $x_1, \dots, x_n$. Since $K/L(x_r)$ is integral, there exists monic polynomials $p_i \in L(x_r)[y]$ such that $p_i(x_i) = 0$. Since $L(x_r)$ is the field of fractions of the polynomial ring $L[x_r]$, each coefficient of p_i can be expressed as a fraction g/h for $g, h \in L(x_r)$ and $h \neq 0$. Let f be the product of all denominators of the coefficient of p_i for all i. Then $p_i \in L[x_r]_f[y]$. So every $x_1, \dots, x_n$ satisfies a monic polynomial with coefficients in $L[x_r]_f$. Hence the $L[x_r]_f$ subalgebra of K generated by $x_1, \dots, x_n$ is integral over $L[x_r]_f$. By 7.2.7, $L[x]_f$ is a field. This is a contradiction to the above lemma. Hence we are done. ■

There is a correspondence between the different terms used in Field and Galois Theory and Commutative Algebra

Field Extension K/F	B an A-algebra
$x \in K$ is algebraic	$b \in B$ is integral
K/F is an algebraic extension	$A \subseteq B$ is an integral extension
The algebraic closure $F < \overline{F} < K$	The integral closure $A \subseteq \overline{A} \subseteq B$
K/F is a finite extension	S is a finitely generated R-algebra

Corollary 7.4.3 Let F be an algebraically closed field. Let K be a field that is also a finitely generated algebra over F. Then $K = F$.

Proof By Zariski's lemma, K/F is a finite field extension. Let $x \in K$. Let f be the minimal polynomial of x. Since F is algebraically closed, f is linear. Hence $x \in F$. ■

Corollary 7.4.4 Let F be an algebraically closed field. Then we have

$$\text{maxSpec}(F[x_1, \dots, x_n]) = \{(x_1 - a_1, \dots, x_n - a_n) \mid (a_1, \dots, a_n) \in F^n\}$$

Proof Let m be a maximal ideal of $F[x_1, \dots, x_n]$. Then $F[x_1, \dots, x_n]/m$ is a finitely generated F-algebra that is a field. By the above, we have that $F[x_1, \dots, x_n]/m \cong F$. Then there exists $a_i \in F$ such that a_i corresponds to $x_i + m$ by the isomorphism. This means that $a_i + m = x_i + m$, or $(x_i - a_i) \in m$. Hence $(x_1 - a_1, \dots, x_n - a_n) \subseteq m$. Since $(x_1 - a_1, \dots, x_n - a_n)$ is maximal by the evaluation homomorphism, we conclude that $m = (x_1 - a_1, \dots, x_n - a_n)$. ■

7.5 Normal Domains

We now concern ourselves with integral domains. Let R be an integral domain. A special fact about R is that the canonical homomorphism $R \to R_{(0)} = \text{Frac}(R)$ is an injection. This means that we can we can think of R as living inside of Frac(R) while preserving all the structure of R.

Definition 7.5.1 (Normal Domains) Let R be an integral domain. We say that R is normal if R is integrally closed in Frac(R).

Proposition 7.5.2 Let R be a normal domain. Let S be a multiplicative subset of R. Then $S^{-1}R$ is a normal domain.

Proof We want to show that $S^{-1}R$ is integrally closed in Frac$(R) = \text{Frac}(S^{-1}R)$. This means that we want to show $\overline{S^{-1}R} = S^{-1}R$. It is clear that $S^{-1}R \subseteq \overline{S^{-1}R}$. So let $g \in \overline{S^{-1}R}$. Suppose that $p(x) = x^n + \sum_{k=0}^{n-1} a_k x^k \in (S^{-1}R)[x]$ such that $p(g) = 0$. Choose $s \in S$ such that $sa_i \in R$ for $0 \le i \le n-1$. Then notice that $sg \in S^{-1}R$ satisfies the monic polynomial

$$q(x) = x^n + \sum_{k=0}^{n-1} s^{n-k} a_k x^k$$

since $q(sg) = s^n g^n + \sum_{k=0}^{n-1} s^n a_k x^k = s^n p(g) = 0$. But q is a polynomial in R since $s^{n-k} a_k \in R$. Thus we have that $sg \in \overline{R} = R$ since R is normal. This means that $g \in S^{-1}R$ and hence we conclude. ■

Proposition 7.5.3 Let R be a commutative ring. If R is a UFD, then R is normal.

Proof Let $a/b \in \text{Frac}(R)$ that is integral. Assume that a, b do not have common factors. Then there exists $r_0, \dots, r_{n-1} \in R$ such that

$$\frac{a^n}{b^n} + r_{n-1}\frac{a^{n-1}}{b^{n-1}} + \dots + r_1 \frac{a}{b} + r_0 = 0$$

Rearranging, we get

$$a^n = -b\left(r_{n-1}a^{n-1} + \dots + r_1 a^1 b^{n-2} + r_0 b^{n-1}\right)$$

This shows that any irreducible element dividing b also divides a^n, and hence a. Since a and b do not have common factors, this means that no irreducible element divides b. Since R is a UFD, b must be a unit. Hence $a/b \in R$. ■

Example 7.5.4 The integral closure of $\mathbb{Z}$ in $\mathbb{Q}[i]$ is $\mathbb{Z}[i]$.

Proof If $a+bi \in \mathbb{Z}[i]$, then $p(x) = x^2 - 2ax + a^2 + b^2$ is a monic polynomial such that $p(a+bi) = 0$. Conversely, let $z \in \mathbb{Q}[i]$ lie in the integral closure of $\mathbb{Z}$. Then z is also an integral element of $\mathbb{Z}[i]$. Since $\mathbb{Z}[i]$ is a UFD, $\mathbb{Z}[i]$ is a normal domain and so is integrally closed in $\text{Frac}(\mathbb{Z}[i]) = \mathbb{Q}[i]$. So $z \in \overline{\mathbb{Z}[i]} = \mathbb{Z}[i]$ shows that $\overline{\mathbb{Z}} \subseteq \overline{\mathbb{Z}[i]}$. ■

Proposition 7.5.5 (Normal is a Local Property) Let R be an integral domain. Then the following are equivalent.

- R is normal
- R_P is normal for all prime ideals P
- R_m is normal for all maximal ideals m.

Proof Notice that an integral domain R is normal if and only if the canonical inclusion map $R \hookrightarrow \overline{R}$ is surjective. Since surjectivity is a local property, this map is surjective if and only if for all prime ideals P of R, $R_P \hookrightarrow \overline{R}_P$ is surjective. But $\overline{R}_P = \overline{R_P}$ by the above. Hence $R \hookrightarrow \overline{R}$ is surjective if and only if $R_P \to \overline{R_P}$ is surjective. Hence R is normal if and only if R_P is normal for all prime ideals P of R. The similar holds for all maximal ideals. ■

Atiyah-Macdonald

Proposition 7.5.6 Let R be a normal domain. Then $R[x]$ is a normal domain.

Proposition 7.5.7 Let R be a normal domain. Let $\text{Frac}(R) < K$ be an algebraic extension. Let $a \in K$. Then a is integral over R if and only if the minimal polynomial $\min(\text{Frac}(R), a) \in R[x]$.

Proof Suppose that $\min(\text{Frac}(R), a) \in R[x]$. Then $\min(\text{Frac}(R), a)(a) = 0$ and $\min(\text{Frac}(R), a)$ is monic by definition. Hence a is integral over R.

Now suppose that $a \in K$ is integral over R. Let $\overline{K}$ be the algebraic closure of K. Then $\min(\text{Frac}(R), a)$ splits into monic irreducible polynomials

$$\min(\text{Frac}(R), a)(x) = (x - a_1) \cdots (x - a_n) \in \overline{K}[x]$$

for $a_1, \dots, a_n \in \overline{K}$. Since a is integral over R, there exists a monic polynomial $g \in R[x]$ such that $g(a) = 0$. By definition of the minimal polynomial, we have $\min(\text{Frac}(R), a)$ divides g. Hence $g(a_i) = 0$ for each i and that $a_1, \dots, a_n$ are integral over R. Now the coefficients of $\min(\text{Frac}(R), a)$ are sums and products of $a_1, \dots, a_n$, and hence are also integral over R. But R is a normal domain so the coefficients of $\min(\text{Frac}(R), a)$ lie in R. ■

8 Introduction to Dimension Theory for Rings

8.1 Krull Dimension

Definition 8.1.1 (Krull Dimension) Let R be a commutative ring. Define the Krull dimension of R to be

$$\dim(R) = \max\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t \text{ for } p_0, \dots, p_t \text{ prime ideals}\}$$

In particular, notice that a commutative ring R has $\dim(R) = 0$ if and only if every prime ideal is maximal.

Lemma 8.1.2 Let R be a commutative ring. Let I be an ideal of R. Then we have $\dim(R) \geq \dim(R/I)$.

Lemma 8.1.3 Let R, S be commutative rings such that $R \subseteq S$ is an integral extension. Then $\dim(R) = \dim(S)$.

Proposition 8.1.4 Let F be a field. Let $n \in \mathbb{N} \setminus \{0\}$. Then the following are true.
- $\dim(F[x_1, \dots, x_n]) = n$.
- Every maximal chain prime ideals in $F[x_1, \dots, x_n]$ is of length n.

Lemma 8.1.5 Let R be a commutative ring. Then the following are true.
- If R is a field, then $\dim(R) = 0$
- If R is Artinian, then $\dim(R) = 0$

Proof Let R be a field. Then the only proper prime ideal of R is (0). In particular, (0) forms the only chain of prime ideals in R. Hence $\dim(R) = 0$.

Now let R be Artinian. Let P be a prime ideal of R. Then R/P is an integral domain. Moreover, every quotient of an Artinian ring is Artinian. Hence R/P is Artinian. By prp1.3.1, we conclude that R/P is a field. Hence P is a maximal ideal. Any chain of prime ideals of R must terminate at the first prime ideal since it is maximal. Hence $\dim(R) = 0$. ■

Definition 8.1.6 (Dimension of Modules) Let R be a commutative ring. Let M be an R-module. Define the dimension of M to be

$$\dim(M) = \dim\left(\frac{R}{\text{Ann}_R(M)}\right)$$

Proposition 8.1.7
Let R be a commutative ring. Let M be an R-module. Then we have

$$\dim(M) = \sup\{\dim(R/P) \mid P \in \text{Ass}(M)\}$$

Proof We know that the set of all minimal prime ideals of $\text{Ass}(M)$ is equal to the set of all minimal prime ideals over $\text{Ann}_R(M)$ by 6.2.5. Any maximal chain of prime ideals over $\text{Ann}_R(M)$ must also start at one of these minimal prime ideals. Hence by the correspondence theorem we have $\sup\{\dim(R/P) \mid P \in \text{Ass}(M)\} = \dim(R/\text{Ann}_R(M))$. ■

8.2 Height of Prime Ideals

Definition 8.2.1 (Height of a Prime Ideal) Let R be a commutative ring. Let p be a prime ideal of R. Define the height of p to be

$$\text{ht}(p) = \max\{t \in \mathbb{N} \mid p_0 \subset \cdots \subset p_t = p \text{ for } p_0, \dots, p_t \text{ prime ideals }\}$$

Lemma 8.2.2 Let R be a commutative ring. Then

$$\dim(R) = \max\{\text{ht}(P) \mid P \in \text{Spec}(R)\}$$

Lemma 8.2.3 Let R be a commutative ring. Let P be a prime ideal of R. Then

$$\text{ht}(P) = \dim(R_P)$$

Proof Let $\dim(R_P) = n$. Then there exists a strict chain of prime ideals of R_P of length n (and no chain of prime ideals of length $> n$). By prp5.4.6, prime ideals of R_P are in bijection with prime ideals of R that P contains. Hence the maximal chain of prime ideals of length n correspond to a chain of prime ideals in R that contain P, of length n. Hence $\dim(R_p) = n \leq \text{ht}(P)$. Conversely, let $m = \text{ht}(P)$. Then there exists a strict chain of prime ideals that are subsets of P, that are of length m. By the same correspondence, the chain of prime ideals correspond to a chain of prime ideals in R_P of length m. Hence $\text{ht}(P) = m \leq \dim(R_P)$.

The two inequalities combine to show that $\dim(R_P) = \text{ht}(P)$. ■

Lemma 8.2.4 Let R be a commutative ring. Let P be a prime ideal of R. Then

$$\dim(R) \geq \dim(R/P) + \text{ht}_R(P)$$

Proposition 8.2.5 Let k be a field. Let A be an integral domain that is a finitely generated k-algebra. Then the following are true.

- $\dim(A) = \text{trdeg}_k(\text{Frac}(A))$
- For any prime ideal P of A, we have

$$\dim(A) = \dim(A/P) + \text{ht}_A(P)$$

Proposition 8.2.6 (Dimension is a Local Concept) Let R be a commutative ring. Then the following numbers are equal.

- The Krull dimension $\dim(R)$
- The supremum $\sup\{\dim(R_m) \mid m \text{ is a maximal ideal of } R\}$
- The supremum $\sup\{\text{ht}_R(m) \mid m \text{ is a maximal ideal of } R\}$

Corollary 8.2.7 Let (R, m) be a local ring. Then

$$\dim(R) = \dim(R_m) = \text{ht}_R(m)$$

Theorem 8.2.8 (Krull's Principal Ideal Theorem) Let R be a Noetherian ring. Let I be a proper and principal ideal of R. Let p be the smallest prime ideal containing I. Then

$$\text{ht}_R(p) \leq 1$$

8.3 The Length of Modules over Commutative Rings

Let R be a ring. Recall that the length of an R-module M is defined to be the supremum

$$l_R(M) = \sup\{n \in \mathbb{N} \mid 0 = M_0 \subset M_1 \subset \cdots \subset M_n = M\}$$

Lemma 8.3.1 Let (R, m) be a local ring and let M be an R-module. If $mM = 0$, then

$$l_A(M) = \dim_{A/m}(M)$$

Example 8.3.2
Let (R, m) be a local ring. Then we have

$$l_R\left(\frac{m^n}{m^{n+1}}\right) = \dim_{R/m}\left(\frac{m^n}{m^{n+1}}\right)$$

Proof
Follows from above. ∎

Lemma 8.3.3
Let R be a commutative ring. Let M be a finitely generated R-module. Let m be a maximal ideal such that $m^{n-1}M \neq 0$ and $m^n M = 0$. Then $l_R(M) < \infty$.

Proof
For $1 \leq i \leq n$, consider the exact sequence

$$0 \longrightarrow m^i M \longrightarrow m^{i-1}M \longrightarrow \frac{m^{i-1}M}{m^i M} \longrightarrow 0$$

We have that $l_R(m^{i-1}M/m^iM) = l_R(m^{i-1}M) - l_R(m^iM)$. Then summing over i gives

$$\sum_{i=1}^n l_R\left(\frac{m^{i-1}M}{m^iM}\right) = l_R(M) - l_R(m^nM) = l_R(M)$$

Since m^iM is R-submodule of M it is finitely generated. Hence $m^iM/m^{i+1}M$ is a finite dimensional R/m vector space. Hence $l_R(M)$ is also finite. ∎

Lemma 8.3.4
Let R be a commutative ring. Let M be an R-module. Let S be a multiplicative subset of R. Then we have

$$l_R(M) \geq l_R(S^{-1}M)$$

8.4 Structure Theorem for Artinian Rings

Let R be a ring. Let M be an R-module. Recall that a composition series for M is a sequence of R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

such that $\frac{M_{i+1}}{M_i}$ is a simple R-module for $1 \leq i < k$.

Proposition 8.4.1 Let $R \neq 0$ be a commutative ring. Then R is Artinian if and only if R is Noetherian and $\dim(R) = 0$.

Proof Let R be Artinian. In Rings and Modules, the Akizuki-Hopkins-Levitzki theorem proves that R is Noetherian. Moreover, lmm8.1.4 shows that $\dim(R) = 0$.

Now let R be Notherian and $\dim(R) = 0$. This means that every prime ideal of R is maximal. Let S be the set of all ideals of R that admit a composition series. I claim that S is non-empty. Let $T = \{\text{Ann}_R(x) \mid 0 \neq x \in R\}$. By 6.1.1, the maximal element $\text{Ann}_R(x)$ in T is a prime ideal. Since $\dim(R) = 0$ we have $\text{Ann}(x)$ is a maximal ideal. $R/\text{Ann}(x)$ is a field (and hence a simple R-module). The multiplication map $r \mapsto rx$ has kernel $\text{Ann}(x)$. Hence the induced map $R/\text{Ann}(x) \to R$ is injective, and we can consider $R/\text{Ann}(x)$ as a subring of R. Together with the fact that it is a simple R-module makes it an R-submodule with composition series length of 1. Hence S is non-empty.

Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain in S. Since R is Noetherian, the chain terminates with some ideal $I \in S$. If $I = R$, then R has a composition series. If $I \neq R$, then R/I is non-zero. Choose a prime ideal P of R such that $I \subseteq P \neq R$ (this always exists since we can choose maximal ideals). Then we have $0 \neq R/P \subseteq R/I$. Let $p : R \to R/I$ be the projection map. Let $T = p^{-1}(R/P)$. Then we have that $N \subset T \subseteq M$ and $T/N \cong R/P$. Since $\dim(R) = 0$, P is maximal hence R/P is a field (and a simple R-module). This proves that $T \in S$. But this contradicts the maximality of N. Hence $N = R \in T$. Thus R has a composition series. From Rings and Modules we know that this implies R is Noetherian. Hence we conclude. ■

Example 8.4.2 Let k be a field. The $k[x]$-module $\frac{k[x,x^{-1}]}{k[x]}$ is Artinian but not Noetherian.

Proof It is not Noetherian because it is not finitely generated. Write $M = \frac{k[x,x^{-1}]}{k[x]}$. For the Artinian result, we first show that if $N \leq \frac{k[x,x^{-1}]}{k[x]}$ and for all $n \in \mathbb{N}$ there exists $f + k[x] \in \frac{k[x,x^{-1}]}{k[x]}$ such that f contains the term $1/x^n$, then $N = \frac{k[x,x^{-1}]}{k[x]}$.

By assumption, for any $n \in \mathbb{N}$, there exists $f \in k[x, x^{-1}]$ such that f contains the term $1/x^n$. If $\deg(f) < -n$, then we can multiply f with $x^{\deg(f)-n}$ to get a polynomial g such that $\deg(g) = -n$. So denote $f_n + N$ the element in N such that $\deg(f_n) = -n$. Then by multiplying with a suitable coefficient α, $f_n - \alpha f_{n-1}$ contains only $1/x^n$. Hence N contains $1/x^n$ for all $n \in \mathbb{N}$ as $k[x]$-module. Since these elements generate $\frac{k[x,x^{-1}]}{k[x]}$ as a k-module, they also generate as a $k[x]$-module. Hence $N = \frac{k[x,x^{-1}]}{k[x]}$.

This means that if N is a proper sub-module, there exists a minimal $n \in \mathbb{N}$ such that $1/x^n \in N$ and $1/x^{n+1} \notin N$. Hence every N is a finitely generated k-module, or in other words, N is a finite dimensional vector space. Thus any decreasing chain of $k[x]$-submodules must terminate by a dimension argument. ■

Theorem 8.4.3 (Structure Theorem for Commutative Artinian Rings) Let R be an Artinian commutative ring. Then R decomposes into a direct product of Artinian local rings

$$R \cong \bigoplus_{i=1}^{k} R_i$$

Moreover, the decomposition is unique up to reordering of the direct product.

Proof Let $m_1, \dots, m_k$ be the full list of distinct maximal ideals of R. Then

$$\prod_{i=1}^{k} m_i^n = 0$$

for some $n \in \mathbb{N} \setminus \{0\}$. The ideals m_i^n and m_j^n are pairwise coprime for $i \neq j$. Hence by the Chinese Remainder Theorem we obtain ring isomorphisms

$$\begin{aligned} R &\cong \frac{R}{0} \\ &\cong \frac{R}{\prod_{i=1}^{k} m_i^n} \\ &\cong \frac{R}{\bigcap_{i=1}^{k} m_i^n} && (m_i^n \text{ and } m_j^n \text{ pairwise coprime}) \\ &\cong \bigoplus_{i=1}^{k} \frac{R}{m_i^n} && \text{(CRT)} \end{aligned}$$

By the correspondence of maximal ideals, R/m_i^n has a unique maximal ideal m_i/m_i^n. Hence it is local. Also since R is Artinian, R/m_i^n is Artinian. Thus we are done. ■

9 Valuation and Valuation Rings

9.1 Valuation Rings

Definition 9.1.1 (Valuation Rings) Let R be an integral domain. We say that R is a valuation ring if for all $x \in \text{Frac}(R)$ and $x \neq 0$, then either x or x^{-1} is in R.

Lemma 9.1.2 Let R be an integral domain. Then R is a valuation ring if and only if the ideals of R are totally ordered by inclusion.

Proof Let R be a valuation ring. Let I, J be ideals of R. If I is not a subset of J, there exists $x \in I$ such that $x \notin J$. Then for any $0 \neq y \in J$, $x/y \in \text{Frac}(R) \setminus R$ since otherwise y is a unit in J so that $J = R$ and $I \subseteq R$. Then $y/x \in R$ so that $y = x(y/x) \in I$. Hence $J \subseteq I$.

Now suppose that the ideals of R are totally ordered by inclusion. ■

Lemma 9.1.3 Let R be a valuation ring. Then the following are true.

- R is a local ring.
- R is normal.

Proof Since all ideals of R are totally ordered, there is only one unique maximal ideal.

Let $x \in \text{Frac}(R)$ be integral over R. Then

$$x^n + r_{n-1}x^{n-1} + \cdots + r_1 x + r_0 = 0$$

for some $r_0, \dots, r_{n-1} \in R$. If $x \in R$ then we are done. If $x \notin R$ then since R is a valuation ring, $x^{-1} \in R$. Then

$$x = -(r_1 + r_2 x^{-1} + \cdots + r_n x^{1-n}) \in R$$

so that R is normal. ■

Definition 9.1.4 (Totally Ordered Group) Let G be an abelian group. We say that G is a totally ordered group if there is a total order "$\leq$" on G such that $a \leq b$ implies $ca \leq cb$ for all $a, b, c \in G$.

Definition 9.1.5 (Valuation on a Field) Let K be a field. Let G be a totally ordered abelian group. A valuation on K with values in G is a map $v : K^\times \to G$ such that for all $x, y \in K^*$, we have

- $v(xy) = v(x) + v(y)$ (v is a group homomorphism)
- $v(x + y) \geq \min\{v(x), v(y)\}$

We use the convention that $v(0) = \infty$.

Definition 9.1.6 (Associated Valuation Ring) Let K be a field and $v : K \to \mathbb{Z}$ a discrete valuation. Define the associated valuation ring of K to be the subring

$$R_v = \{x \in K \mid v(x) \geq 0\}$$

Lemma 9.1.7 Let K be a field. Let v be a discrete valuation on K. Then R_v is a valuation ring.

Definition 9.1.8 (Discrete Valuations) Let K be a field. A discrete valuation on K is a valuation $v : K^\times \to \mathbb{Z}$.

Definition 9.1.9 (Normalized Discrete Valuations) Let (K, v) be a discrete valuation ring. We say that it is normalized if v is surjective.

Lemma 9.1.10 Let K be a field with a discrete valuation v. Then $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$.

Lemma 9.1.11 (Normalization of a Discrete Valuation) Let K be a field with a discrete valuation v such that $v(K^\times) = n\mathbb{Z}$ for some $n \in \mathbb{N}$. Define the normalization of v to be the valuation $v_N : K^\times \to \mathbb{Z}$ defined by

$$v_N(k) = \frac{1}{n}v(k)$$

for all $k \in K^\times$.

Therefore we always work on normalized discrete valuations.

9.2 Discrete Valuation Rings

Definition 9.2.1 (Discrete Valuation Rings) Let R be a commutative ring. We say that R is a discrete valuation ring if there exists a field K and a discrete valuation v on K such that

$$R = R_v$$

is the associated valuation ring of K.

Lemma 9.2.2 Let R be a discrete valuation ring with valuation v. Then $0 \neq u \in R$ is a unit if and only if $v(u) = 0$. In particular, the maximal ideal of R is given by

$$\{r \in R \mid v(r) > 0\}$$

Proof Let R be a discrete valuation ring. Suppose that $x \in R$ is a unit. Then $v(x^{-1}) = -v(x)$. Then $-v(x), v(x) \geq 0$ implies $v(x) = 0$. Now if $v(y) > 0$, suppose for contradiction that $u \in R$ is an inverse of y, then

$$0 = v(1) = v(uy) = v(u) + v(y)$$

But $v(y) > 0$ implies that $v(u) < 0$ which implies that $u \notin R$, a contradiction. ∎

Example 9.2.3 Let $n \in \mathbb{N}$. Define $\text{ord}_n : \mathbb{Q} \to \mathbb{Z}$ as follows. For $p/q \in \mathbb{Q}$, let $p = p'n^i$ and $q = q'n^j$ such that $\gcd(p', n) = \gcd(q', n) = 1$. Then define

$$\text{ord}_n\left(\frac{p}{q}\right) = \text{ord}_n\left(n^{i-j}\frac{p'}{q'}\right) = i - j$$

Then ord_n is a discrete valuation if and only if n is prime. In this case, the valuation ring of ord_n is given by

$$R_{\text{ord}_n} = \mathbb{Z}_n$$

Proof Suppose that n is a prime. Let $n^s p_1/q_1 \in \mathbb{Q}$ and $n^t p_2/q_2$ be in lowest terms. Then $n^{s+t}(p_1p_2/q_2q_2)$ is in lowest terms since n is prime. Then we have

$$\text{ord}_n(n^{s+t}(p_1p_2/q_2q_2)) = s + t = v(n^s p_1/q_1) + v(n^t p_2/q_2)$$

Without loss of generality, suppose that $s \leq t$. Then $n^s p_1/q_1 + n^t p_2/q_2 = n^s(p_1/q_1 + n^{t-s}p_2/q_2)$ is in lowest terms since n is prime. Then we have

$$v(n^s p_1/q_1 + n^t p_2/q_2) = v(n^s(p_1/q_1 + n^{t-s}p_2/q_2)) = s = \min\{v(n^s p_1/q_1), v(n^t p_2/q_2)\}$$

Thus ord_n is a discrete valuation.

If n is composite, without loss of generality suppose that $n = pq$ for p and q primes.

The valuation ring of ord_n for n prime is given by

$$R_{\text{ord}_n} = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; n \text{ does not divide } q \right\}$$

Hence $R_{\text{ord}_n} = \mathbb{Z}_n$. ■

Definition 9.2.4 (Uniformizing Parameter) Let R be a discrete valuation ring with valuation v. A uniformizing parameter for R is an element $t \in R$ such that $v(t) = 1$.

Proposition 9.2.5 Let R be a discrete valuation ring with valuation v. Let $t \in R$ be a uniformizing parameter of R. Then the following are true.

- Every $r \in R \setminus \{0\}$ can be written in the form
$$r = ut^n$$
for some unit u and $n \geq 0$.
- The valuation of any element $r = ut^n \in R$ is given by
$$v(ut^n) = n$$
- The set of all ideals of R is given by
$$\{(t^n) \mid n \in \mathbb{N} \setminus \{0\}\}$$
In particular, the unique maximal ideal of R is (t).

Proof

- If $x \in R$ is a unit then we are done. If not, then consider the element $u = t^{-n}x$ for $n = v(x)$. Then we have
$$v(u) = v(t^{-n}x) = -n + v(x) = 0$$
Hence u is a unit. Multiplying t^n on both sides of $u = t^{-n}x$ proves that $x = ut^n$ for some unit u and $n \in \mathbb{N}$.
- It follows that the valuation of $r = ut^n$ is n.
- Let I be an ideal of R. Let $n = \min\{v(x) \mid x \in I\}$. or all $x \in I$, we can write x as $x = ut^k$ for some unit u and $k \geq n$. Hence $I \subseteq (t^n)$. Since n is a minimum, there exists $x \in I$ such that $x = ut^n$ for some unit u and $n \in \mathbb{N}$. Then $u^{-}x = t^n \in I$ since I is an ideal. Hence $I = (t^n)$. It follows that the unique maximal ideal of R is given by (t).

■

The rest of the section devotes efforts to recognizing discrete valuation rings.

Proposition 9.2.6 (Equivalent Characterizations of DVRs I) Let R be an integral domain. Then the following are equivalent.

- R is a discrete valuation ring.
- R is local, a PID and not a field.
- R is Noetherian, local, $\dim(R) = 1$ and normal.
- R is Noetherian, local, $\dim(R) > 0$ and the unique maximal ideal m is principal.
- R is a UFD with a unique irreducible element up to multiplication of a unit

Proof

- $(1) \implies (2)$: We have seen that valuation rings are local. It is a PID by 9.2.5. It is not a field since R is a local ring with non-trivial unique maximal ideal.

- $(2) \implies (3)$: Every PID is Noetherian and normal and every prime ideal is maximal. But local rings have a unique maximal ideal. The maximal ideal is non-trivial since R is not a field. Hence $\dim(R) = 1$.
- $(3) \implies (4)$: By Nakayama's lemma, $m \neq m^2$. I claim that any $x \in m \setminus m^2$ generates m. Since $\dim(R) = 1$, we have $\text{Spec}(R) = \{(0), m\}$. Assume for a contradiction that $m/(x) \neq \{0\}$. By lmm6.2.4, we have $\text{Ass}(m/(x)) \neq \{0\}$. By our assumption for contradiction, we can only have $\text{Ass}(m/(x)) = \{m\}$. By definition, this means that there exists $0 \neq [y] \in m/(x)$ such that $\text{Ann}_R([y]) = m$. In other words, $ym \subseteq (x)$. Considering everything inside $\text{Frac}(R)$, we have $y/x \in \text{Frac}(R)$ is such that $y/x \notin R$ and $y/x \cdot m \subseteq R$. There are now two cases.

  Case 1: $y/x \cdot m = R$.
  Then $1 = yt/x$ for some $t \in m$, which means that $x = yt$ and $x \in ym \subseteq m^2$. This is a contradiction.

  Case 2: $y/x \cdot m = m$. Then the multiplication map $z \mapsto y/x \cdot z$ satisfies the hypothesis of the Cayley-Hamilton theorem, and there exists $a_0, \dots, a_{n-1} \in R$ such that

  $$(y/x)^n + a_{n-1}(y/x)^{n-1} + \cdots + a_1(y/x) + a_0 = 0$$

  But then this proves that y/x is integral over R. Since R is normal, $y/x \in R$. This is also a contradiction.

  Thus m is a PID.
- $(4) \implies (1)$: Suppose that $m = (x)$ for some $x \in R$. If x is nilpotent, then $\dim(R) = 0$ and a contradiction. I claim that $\bigcap_{i=1}^{\infty}(x^i) = \{0\}$. Suppose that t lies in the intersection. Then $t = yx$ for some $y \in R$. If y is not in the intersection, then there exists $n \in \mathbb{N}$ such that y is non-zero in $(x^n)/(x^{n+1})$. By Nakayama's lemma, y generates (x^n) and so t generates (x^{n+1}). Then $t \notin (x^{n+2})$ is a contradiction. In particular, there for any $y \in R$, we have $y \in (x^n) \setminus (x^{n+1})$ for some $n \in \mathbb{N}$. This means that $y = ux^n$ for some $u \notin (x)$. In particular, u is a unit. Similarly, $z = vx^m$ for v a unit. Then $yz = uvx^{n+m}$ is non-zero. Hence R is an integral domain. Then the map $ux^n \mapsto n$ is a valuation.
- $(5) \implies (1)$: Let t be the unique irreducible element. Define a map $v : \text{Frac}(R) \to \mathbb{Z}$ as follows. Since R is a UFD, every element in R can be uniquely written as zt^n for z a unit and $n \in \mathbb{N}$. Also, every element in $\text{Frac}(R)$ can be uniquely written as zt^n for z a unit in $n \in \mathbb{Z}$. Then define $v(zt^n) = n$. It is clear that v is a valuation. Its associated valuation ring is then precisely R.

■

Proposition 9.2.7 (Equivalent Characterizations of DVRs II) Let R be an integral domain that is Noetherian and local with unique maximal ideal m. Then the following are equivalent.
- R is a discrete valuation ring.
- $\dim(R) = 1$ and R is normal.
- R is not a field and m is principal.
- $\dim(R) = 1$ and $\dim_{R/m}(m/m^2) = 1$ (R is a regular local ring)
- $I = m^k$ for all non-zero ideals I of R
- There exists $t \in R$ and $k > 0$ such that $I = (t^k)$ for all non-zero ideal I of R

Proof The proposition is an immediate consequence of the above. ■

Proposition 9.2.8 Let R be a Noetherian integral domain and $\dim(R) = 1$. Then R is normal if and only if R_m is a discrete valuation ring for all maximal ideals m.

In summary, if R is a discrete valuation ring, then R has the following properties.

- R is integrally closed and in particular is normal.
- R is a PID and in particular is a UFD and an integral domain.
- R is Noetherian and local
- R has Krull dimension 1.
- $\dim_{R/m}(m/m^2) = 1$ (these are called regular local rings as we will see in Commutative Algebra 2)
- Every ideal I of R is equal to the power m^k of the maximal ideal m. In particular if m is generated by the uniformizing parameter t, then $I = (t^k)$ in this case.
- Such a t is an irreducible element (that is unique up to multiplication by a unit), and every element of R can be written as ut^n for u a unit and $n \in \mathbb{N}$.

There is a simple diagram of relationships between DVRs and some other standard types of commutative rings.

$$\text{DVRs} \subset \text{PIDs} \subset \text{UFDs} \subset \text{Normal Domains} \subset \text{Integral Domains}$$

10 Dedekind Domains

10.1 Fractional Ideals

Definition 10.1.1 (Fractional Ideal) Let R be an integral domain. Let I be a R-submodule of $\text{Frac}(R)$. We say that I is a fractional ideal of R if there exists $r \in R \setminus \{0\}$ such that $rI \subseteq R$.

While I is not exactly an ideal of R, we can think of it as if it were an ideal because it is isomorphic to an actual ideal of R.

Lemma 10.1.2 Let R be an integral domain. Let I be a fractional ideal of R where $rI \subseteq R$ for some $r \in R \setminus \{0\}$. Then there is an R-module isomorphism

$$I \cong rI \subseteq R$$

given by $i \mapsto ri$.

Proof I claim that there is an R-module isomorphism $I \cong rI$ for $rI \subseteq R$ given by $i \mapsto ri$. The kernel of this R-module homomorphism is given by $\{i \in I \mid ri = 0\}$. But $ri = 0$ if and only if $r = 0$ or $i = 0$. Since $r \neq 0$ we must have $i = 0$ so that the kernel is trivial. Moreover, this R-module homomorphism is surjective since for any $k \in rI$ it can be written as $k = ri$ for some i. Then $i \in I$ maps to ri under the morphism. Hence $I \cong rI$ as R-modules. ■

Example 10.1.3 The $\mathbb{Z}$-submodule $\mathbb{Z} \cdot \frac{1}{2}$ of $\mathbb{Q}$ is a fractional ideal.

Proof Indeed, we have $2\left(\mathbb{Z} \cdot \frac{1}{2}\right) = \mathbb{Z}$, and we think of $\mathbb{Z} \cdot \frac{1}{2}$ as a $\mathbb{Z}$-module isomorphic to $\mathbb{Z}$. ■

Lemma 10.1.4 Let R be an integral domain. Let I be a fractional ideal of R. If R is Noetherian, then I is finitely generated.

Proof Let R be Noetherian. Since I is isomorphic to rI for some non-zero $r \in R$, and rI is an ideal of R, R being Noetherian implies that rI is finitely generated and hence I is finitely generated. ■

10.2 Invertible Ideals

Definition 10.2.1 (Inverse of an Ideal) Let R be an integral domain. Let I be an R-submodule of $\text{Frac}(R)$. Define

$$I^{-1} = \{s \in \text{Frac}(R) \mid sI \subseteq R\}$$

Lemma 10.2.2 Let R be an integral domain. Let I be an R-submodule of $\text{Frac}(R)$. Then there is an R-module isomorphism

$$I^{-1} \cong \text{Hom}_R(I, R)$$

given by $s \mapsto (r \mapsto sr)$.

Proof Denote $\varphi_s : I \to R$ the multiplication by s map for $s \in I^{-1}$. It is clear that the given map is an R-module homomorphism. The map is injective since R is an integral domain. It remains to show that the map is surjective. Let $\varphi \in \text{Hom}_R(I, R)$. For any $r \in R$ and $i \in I$, we have

$$\varphi(r \cdot i) = r \cdot \varphi(i)$$

■

Definition 10.2.3 (Invertible Ideals) Let R be an integral domain. Let I be an R-submodule of $\text{Frac}(R)$. We say that I is invertible if there exists an R-submodule of J of R such that $JI = R$.

Lemma 10.2.4 Let R be an integral domain. Let $I \subseteq R$ be a subset. Then I is an ideal if and only if I is a fractional ideal.

Proof Clearly if I is a fractional ideal, then I is an ideal. Conversely, if I is an ideal then $rI \subseteq R$ for all $r \in R$ implies that I is a fractional ideal. ∎

Proposition 10.2.5 Let R be an integral domain. Let I be an R-submodule of Frac(R). Then I is invertible if and only if $I^{-1}I = R$.

Proof Clearly if $I^{-1}I = R$ then I is invertible. Now suppose that $JI = R$ for some R-submodule J of Frac(R). Then we have

$$R = JI \subseteq I^{-1}I = R$$

by definition of I^{-1}. Hence $JI = I^{-1}I$. Multiplying J on both sides and using the fact that R is commutative, we have that $J = I^{-1}$. ∎

Lemma 10.2.6 Let R be an integral domain. Let I be an invertible ideal of R. Then for any prime ideal P of R, the ideal IR_P of R_P is a principal ideal.

Proof Since $I^{-1}I = r$, write $1 = \sum_{i=1}^k s_ia_i$ for $s_i \in I^{-1}$ and $a_i \in I$. Since $1 \notin P$, at least one of s_ia_i is not in P. Then s_ia_i is a unit in PR_P and so a_i generates IR_P. ∎

Proposition 10.2.7 Let R be an integral domain. Let I be an R-submodule of Frac(R). Then the following are true.

- If I is a non-zero principal ideal of R, then I is invertible.
- If I is invertible, then I is fractional.
- If I is invertible, then I is finitely generated.

Proof

- Suppose that $I = (a)$ for $a \in R$. Then clearly we have $(1/a)(a) = R$.
- Let I be invertible. Since $I^{-1}I = R$, we can write $1 = \sum_{i=1}^n s_ia_i$ for $s_i \in I^{-1}$ and $a_i \in I$. Then for any $r \in R$, we have $b = \sum_{i=1}^k s_i(a_ib)$ where $a_ib \in R$. Let s be the product of the denominators of s_i. Then $sb \in R$. Hence I is a fractional ideal.
- Let I be invertible. Since $I^{-1}I = R$, we can write $1 = \sum_{i=1}^n s_ia_i$ for $s_i \in I^{-1}$ and $a_i \in I$. Then for any $x \in R$, we have $x = \sum_{i=1}^n (s_ix)a_i$. Since $s_i \in I^{-1}$ and $x \in R$, we have $s_ix \in R$. Hence x can be written as a R-linear combination of $a_1, \dots, a_n$. Hence I is finitely generated.

∎

Proposition 10.2.8 Let R be an integral domain. Let I be an R-submodule of Frac(R). Then I is invertible if and only if the following are true.

- I is fractional.
- I is finitely generated.
- For any prime ideal P of R, IR_P is a principal ideal of R_P.

Proof We have seen the forward direction already. Now suppose that I satisfies the three listed conditions. I claim that $(I^{-1})_P = (I_P)^{-1}$. Let $r/s \in (I^{-1})_P$ and $a/b \in I_P$. Then clearly $r/s \cdot a/b \in R_P$ so that $r/s \in (I_P)^{-1}$. Conversely, suppose that $I = R(a_1, \dots, a_n)$. Let $x \in (I_P)^{-1}$. Then $xa_i \in R_P$. This means that there exists $c_i \in R \setminus P$ such that $xa_ic_i \in R$. Set $c = c_1 \cdots c_n$. Then clearly $cx \in I^{-1}$ so that $x \in (I^{-1})_P$.

Suppose that $I^{-1}I \neq R$. Since $I^{-1}I$ is a proper ideal of R, there exists a maximal ideal m of R containing $I^{-1}I$. By the correspondence of ideals for localization, we have $(I^{-1})_mI_m = (I_m)^{-1}I_m \subseteq$

mR_m. This is a contradiction since the above proposition together with the fact that IR_m is a principal ideal of R_m should imply that $(I_m)^{-1}I_m = R_m$. ■

Proposition 10.2.9 Let R be an integral domain. Let P be a non-zero prime ideal of R. If R is Noetherian and P is invertible, then R_P is a discrete valuation ring.

Proof Let R be a Noetherian integral domain and P a non-zero invertible prime ideal. We know that PR_P is the unique maximal ideal of the local ring R_P. By the above prp, PR_P is a principal ideal. Thus R_P is now a Noetherian local ring with principal maximal ideal. By prp10.4.6 in Commutative Algebra 1, we conclude that R_P is a discrete valuation ring. ■

10.3 Dedekind Domains

Definition 10.3.1 (Dedekind Domains) Let R be an integral domain. We say that R is a dedekind domain if every non-zero ideal I of R is invertible.

Dedekind sought for an integral domain whose ideals can be factorized uniquely as a product of primes.

Proposition 10.3.2 Let R be an integral domain that is not a field. Then the following are equivalent.

- Every non-zero ideal I of R is invertible ($I^{-1}I = R$).
- R is Noetherian, $\dim(R) = 1$ and normal
- R is Noetherian, $\dim(R) = 1$ and for any non-zero maximal ideal m of R, R_m is a discrete valuation ring.
- R is Noetherian, $\dim(R) = 1$ and every primary ideal in R is a prime power.

Proof

- (1) $\implies$ (2): For any ideal I of R, I is invertible. By 10.2.6, I is finitely generated. Then every R-submodule of R is finitely generated and so R is Noetherian. For any prime ideal P of R, 10.2.8 implies that R_P is a discrete valuation ring since P by assumption is invertible. Then R_P is a normal domain for any prime ideal P. Since normality is a local condition, we conclude that R is a normal domain.
- (2) $\implies$ (3): For any maximal ideal m of R, R_m is Noetherian since localization preserves Noetherianess. Also, R_m is local. Since normality is a local condition, R_m is also normal. Finally, we have $\dim(R_m) = \dim(R) = 1$. Hence by the equivalent characterizations of DVRs, we conclude that R_m is a DVR.
- (3) $\implies$ (1): Let $I \subseteq R$ be a fractional ideal of R. We know by 10.1.4 that I is finitely generated. Since R_m is a normal Noetherian local ring of dimension 1, the ideal I_m of R_m must be principal. By 10.2.7 we conclude that I is invertible.

■

By virtue of the fourth item, we can think of Dedekind domains as a patching up of local discrete valuation rings.

We summarize the relation between Dedekind domains and other types of domains in the following diagram:

$$\begin{tikzcd}
 & & \text{Dedekind Domains} \arrow[dr, phantom, "\subset" sloped] & & \\
\text{DVRs} \arrow[r, phantom, "\subset"] & \text{PIDs} \arrow[ur, phantom, "\subset" sloped] \arrow[dr, phantom, "\subset" sloped] & & \text{Normal Domains} \arrow[r, phantom, "\subset"] & \text{Integral Domains} \\
 & & \text{UFDs} \arrow[ur, phantom, "\subset" sloped] & &
\end{tikzcd}$$

In particular, DVRs, PIDs and Dedekind domains are 1-dimensional. Moreover, notice that the only difference between DVRs and Dedekind domains is that DVRs are local rings. They both share the fact that they are Noetherian, $\dim(R) = 1$ and normal.

10.4 Prime Factorization of Ideals

Definition 10.4.1 (Prime Factorization of Ideals) Let R be a commutative ring. Let I be an ideal of R. A prime factorization of I consists of maximal ideals $P_1, \dots, P_k$ such that the following are true.

- For some $n_1, \dots, n_k \in \mathbb{N} \setminus \{0\}$, we have

$$I = P_1^{n_1} \cdots P_k^{n_k}$$

- Each $P_1, \dots, P_n \in \text{Ass}(I)$ is an associated prime ideal of I.
- The factorization is unique up to permutation,.

Proposition 10.4.2 Let R be an integral domain. Then R is a Dedekind domain if and only if every ideal of R has a prime factorization.

Proposition 10.4.3 Let R be a Dedekind domain. For any prime ideal P of R, denote $v_i : \text{Frac}(R_P) \to \mathbb{Z}$ the discrete valuation of R_P. Then for any $a \in R \setminus \{0\}$, we have

$$(a) = P_1^{v_1(a)} \cdots P_n^{v_n(a)}$$

for $P_1, \dots, P_n \in \text{Ass}((a))$.

Proposition 10.4.4 Let R be a Dedekind domain. Let I and J be ideals of R whose prime factorization is given by

$$I = P_1^{a_1} \times \cdots \times P_n^{a_n} \quad \text{and} \quad J = P_1^{b_1} \times \cdots \times P_n^{b_n}$$

for $P_1, \dots, P_n$ distinct prime ideals of R. Then the following are true.

- $I + J = P_1^{\min\{a_1, b_1\}} \times \cdots \times P_n^{\min\{a_n, b_n\}}$
- $I \cap J = P_1^{\max\{a_1, b_1\}} \times \cdots \times P_n^{\max\{a_n, b_n\}}$
- $IJ = P_1^{a_1 + b_1} \times \cdots \times P_n^{a_n + b_n}$

Proposition 10.4.5 Let R be a Dedekind domain. Let I be an ideal of R. Then the following are true.

- For any $a \in I$, there exists $b \in R$ such that $I = (a, b)$.
- I is can be finitely generated by two elements.