Algebraic Geometry 3

Labix

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Abstract

References:

- Fourier-Mukai Transforms in Algebraic Geometry (Derived Categories for Algebraic Geometry)

Contents

1 Sheaf Theory on Varieties

1.1 The Sheaf of Regular Functions

Recall that given an affine algebraic variety V and an open set U of V, we can equip a ring of rational functions over U. We can sort these rings into a sheaf.

Definition 1.1.1 (Structure Sheaf) Let X be a quasi-projective variety over $\mathbb{C}$. Define the structure sheaf of X of $\mathbb{C}$-algebras

$$\mathcal{O}_X : \mathbf{Open}(X) \to \mathbf{Rings}$$

as follows.

- On objects,
$$\mathcal{O}_X(U) = \{f \in k(X) \mid f \text{ is regular over } U\}$$
for each $U \subseteq X$ open.
- If $V \subseteq U$, then there is a unique morphism $\iota : V \to U$ given by the inclusion. Define a ring homomorphism
$$\mathcal{O}_X(\iota) : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$
in **Rings** that sends $f \in \mathcal{O}_X(U)$ to its restriction $f|_V \in \mathcal{O}_X(V)$.

Proposition 1.1.2 Let X be a quasi-projective variety over $\mathbb{C}$. Then the structure sheaf

$$\mathcal{O}_X(U) = \{f \in k(X) \mid f \text{ is regular over } U\}$$

defined above is a sheaf on X.

Proof We have seen from sheaf theory that this formula precisely gives the stalks of a sheaf as a colimit. ■

Lemma 1.1.3 Let X be a quasi-projective variety over $\mathbb{C}$. Then for each $p \in X$, the ring of germs of regular functions is given by

$$\mathcal{O}_{X,p} = \{(U, f) \mid U \subseteq X \text{ is open}, p \in U, f \text{ is regular on } U\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if $f = g$ on $U \cap V$.

Proof It is clear by definition that $\mathcal{O}_X(-)$ is a functor from $\mathbf{Open}(X)$ to **Rings**. Hence $\mathcal{O}_X(-)$ is indeed a presheaf. We check the identity and gluing axiom.

- Let $\{U_i \mid i \in I\}$ be an open cover of an open set $V \subseteq X$. Let $f_1, f_2 \in \mathcal{O}_X(V)$ such that $f_1|_{U_i} = f_2|_{U_i}$ for all $i \in I$. Let $v \in V$. Then $v \in U_i$ for some $i \in I$. Since $f_1|_{U_i} = f_2|_{U_i}$, we have that $f_1(v) = f_2(v)$. $v \in V$ is chosen arbitrary hence we conclude that $f_1 = f_2$.
- Let $\{U_i \mid i \in I\}$ be an open cover of an open set $V \subseteq X$. Suppose that $f_i \in \mathcal{O}_X(U_i)$ such that the restriction of f_i and f_j agree on $U_i \cap U_j$ for any $i, j \in I$. Define a function $f : V \to k$ as follows. For $v \in V$, there exists $i \in I$ such that $v \in U_i$. Then define $f(v) = f_i(v)$. It is clear that $f_i = f|_{U_i}$ for any $i \in I$ by definition. It is also well defined since if $v \in U_i \cap U_j$ then $f_i(v) = f_j(v)$. Finally, it is also regular at any point $v \in V$. This is because there exists $v \in U_i$ and locally on U_i, there exists $g, h \in \mathbb{C}[X]$ such that $h(v) \neq 0$ and
$$f(x) = f_i(x) = \frac{g(x)}{h(x)}$$
by considering a neighbourhood of v lying in U_i.

We conclude that $\mathcal{O}_X$ is indeed a sheaf on X. ■

Unfortunately one big problem in classical algebraic geometry is that the ringed space $(V, \mathcal{O}_V)$ for a variety V is not necessarily a locally ringed space. Ideally, we would want rational functions on a point p to be exactly the local ring $\mathbb{C}[V]_{m_p}$ where m_p is the maximal ideal corresponding to the point p by

Hilbert's nullstellensatz. We remedy this by using the spectrum of a ring as a topological space instead of a variety.

1.2 Morphisms of the Sheaf of Regular Functions

Definition 1.2.1 (Morphism of Varieties) Let X, Y be quasi-projective varieties over $\mathbb{C}$. A morphism from X to Y is a morphism of ringed spaces

$$(F : X \to Y, F^{\#} : \mathcal{O}_Y \to \mathcal{O}_X)$$

Explicitly, this consists of specifying a morphism $F : X \to Y$ of varieties. $F^{\#}$ is determined by the $\mathbb{C}$-algebra homomorphism

$$F^{\#}(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(F^{-1}(U))$$

given by the map $f \mapsto f \circ F$ for each open set $U \subseteq Y$.

2 The Construction of Schemes

2.1 Spectra and its Topology

Recall that for A a commutative ring, we defined

$$\text{Spec}(A) = \{P \subseteq A \mid P \text{ is a prime ideal of } A\}$$

We can similarly define the notion of vanishing loci of Spec(A).

Definition 2.1.1 (Zero Locus) Let A be a commutative ring. Let $T \subseteq A$. Define the vanishing locus of T to be

$$\mathbb{V}^{\mathrm{S}}(T) = \{p \in \text{Spec}(A) \mid T \subseteq p\}$$

Proposition 2.1.2 Let A be a commutative ring. Let $T \subseteq A$ be a subset. Let $I = (T)$ be the ideal generated by T. Then

$$\mathbb{V}^{\mathrm{S}}(T) = \mathbb{V}^{\mathrm{S}}(I)$$

The proposition shows that we only need to concern ourselves with the zero set of ideals of A.

Lemma 2.1.3 Let A be a commutative ring. The following are true.

- $\mathbb{V}^{\mathrm{S}}(1) = \emptyset$
- $\mathbb{V}^{\mathrm{S}}(0) = \text{Spec}(A)$
- Let $\{a_i | i \in I\}$ be a countable set of ideals of A, then

$$\mathbb{V}^{\mathrm{S}}\left(\sum_{i \in I} a_i\right) = \bigcap_{i \in I} \mathbb{V}^{\mathrm{S}}(a_i)$$

- Let $\{a_1, \dots, a_n\}$ be a finite set of ideals of A, then

$$\mathbb{V}^{\mathrm{S}}\left(\bigcap_{k=1}^{n} a_k\right) = \bigcup_{k=1}^{n} \mathbb{V}^{\mathrm{S}}(a_k)$$

Definition 2.1.4 (Zariski Topology on Spec) Let A be a commutative ring. Define the Zariski topology on Spec(A) to be the topology where the closed sets are exactly sets of the form $\mathbb{V}^{\mathrm{S}}(I)$ for $I \subseteq A$ an ideal of A.

Example 2.1.5 The Zariski topology on Spec$(\mathbb{Z})$ is given as follows.

- The points are given by Spec$(\mathbb{Z}) = \{(p) \mid p \text{ is a prime }\} \cup \{(0)\}$
- The closed subsets are of the form

$$\mathbb{V}^s((n)) = \{(p) \in \text{Spec}(\mathbb{Z}) \mid p \text{ divides } n\}$$

Example 2.1.6 The Zariski topology on Spec$(\mathbb{C}[x])$ is given as follows.

- The points are given by Spec$(\mathbb{C}[x]) = \{(x - a) \mid a \in \mathbb{C}\} \cup \{(0)\}$
- The closed subsets are precisely of the form

$$\mathbb{V}^s((p)) = \{(x - a) \mid a \text{ is a root of } p\}$$

  for $p \in \mathbb{C}[x]$ since $\mathbb{C}[x]$ is a PID.

Lemma 2.1.7 Let k be an algebraically closed field. Let V be an affine variety. Then the bijection given by the Hilbert's nullstellensatz

$$V \cong \text{maxSpec}(k[V])$$

is a homeomorphism where maxSpec$(k[V])$ inherits the subspace topology of Spec$(k[V])$.

Let k be an algebraically closed field. For a closed subset $V \subseteq \mathbb{A}_k^n$, we have the following.

- There is a bijection
$$V \overset{1:1}{\longleftrightarrow} \text{maxSpec}(V) \subseteq \text{Spec}(V)$$
The geometry of V is enhanced because we have more points in general, given by the irreducible subvarieties.
- In the Zariski topology of V, closed subsets are affine subvarieties of V. In the Zariski topology of Spec(V), closed subsets are collections of irreducible subvarieties.
- For a closed subset V of Spec$(k[V])$, we have $C = \mathbb{V}^s(I)$ for some $I \subseteq k[V]$ an ideal. There is a bijection of sets
$$C = \mathbb{V}^s(I) \overset{1:1}{\longleftrightarrow} \{W \subseteq \mathbb{V}(I) \mid W \text{ is irreducible }\}$$
- In particular, for an irreducible polynomial $f \in k[x_1, \dots, x_n]$, $(f) \in \text{Spec}(k[V])$ is a point. The closure of (f) is given by
$$\overline{(f)} = \{m \in \text{maxSpec}(k[V]) \mid (f) \subseteq m\} \cup \{(f)\}$$

Proposition 2.1.8 Let A be a commutative ring. Let $I \subseteq A$ be an ideal of A. Then
$$\text{Spec}(A/I) = \mathbb{V}^S(I)$$
where equal refers to the two spaces are equal.

We can also explicitly write out the open sets and a basis for the Zariski topology.

Definition 2.1.9 (Distinguished Open Sets) Let A be a commutative ring. Let $S \subseteq A$. Define the distinguished open set of S to be
$$D(S) = \{p \in \text{Spec}(A) | S \not\subseteq p\}$$
Let $f \in A$. Then the collection
$$D(f) = \{p \in \text{Spec}(A) | f \notin p\}$$
for f varying in A are called basic open sets.

Notice that from the definition we can directly see that $\mathbb{V}(S)$ and $D(S)$ partitions Spec(A) for every $S \subset A$. Moreover, if S generates the ideal a, $\mathbb{V}(a) = \mathbb{V}(M)$ hence we will only feed in ideals of A into $\mathbb{V}(-)$ from now on.

Theorem 2.1.10 Let A be a commutative ring. The open sets of the Zariski topology of Spec(A) are the sets $D(S)$ for $S \subseteq A$. Moreover, the collection $D(f)$ for $f \in A$ is a basis for the topology.

2.2 Induced Map on Spectrum

Definition 2.2.1 (Induced Map on Spectrum) Let R, S be commutative rings. Let $\phi : R \to S$ be a ring homomorphism. Define the function
$$\phi^\flat : \text{Spec}(S) \to \text{Spec}(R)$$
by the formula $P \mapsto \phi^{-1}(P)$

Proposition 2.2.2 Let A, B, C be commutative rings. Let $\phi : A \to B$ and $\psi : B \to C$ be ring homomorphisms. Then the following are true.
- $(\psi \circ \phi)^\flat = \phi^\flat \circ \psi^\flat$
- $(\text{id}_A)^\flat = \text{id}_{\text{Spec}(A)}$

Lemma 2.2.3 Let R, S be commutative rings. Let $\phi : R \to S$ be a ring homomorphism. Then the induced map
$$\phi^\flat : \text{Spec}(S) \to \text{Spec}(R)$$
is continuous.

Proposition 2.2.4 Let V, W be affine varieties over $\mathbb{C}$. Let $F : V \to W$ be a morphism of affine varieties. Let m_x be the maximal ideal of $\mathbb{C}[V]$ corresponding to the point x. Then
$$(F^*)^\flat(m_x) = m_{F(x)}$$

We thus now have the following nice diagram:
$$\begin{tikzcd}
\text{maxSpec}(\mathbb{C}[V]) \arrow[r, "\phi^\flat|_{\text{maxSpec}(\mathbb{C}[V])}"] \arrow[d, leftrightarrow, "1:1"'] & \text{maxSpec}(\mathbb{C}[W]) \arrow[d, leftrightarrow, "1:1"] \\
V \arrow[r, "\phi"'] & W
\end{tikzcd}$$

Proposition 2.2.5 Let V be an affine variety over $\mathbb{C}$. Let $X \subseteq V$ be an affine irreducible subvariety. Let P_X be the prime ideal of $\mathbb{C}[V]$ corresponding to X. Let $Y = \overline{F(X)}$. Then
$$(F^*)^\flat(P_X) = P_Y$$

Definition 2.2.6 (The Spec Functor) Define the spec functor
$$\text{Spec} : \mathbf{CRing} \to \mathbf{Top}$$
to consist of the following.

- For a commutative ring A, Spec(A) is the set of all prime ideals of A together with the Zariski topology.
- For a ring homomorphism $\phi : A \to B$,
$$\text{Spec}(\phi) = \phi^\flat : \text{Spec}(B) \to \text{Spec}(A)$$
is the continuous map defined by $P \mapsto \phi^{-1}(P)$.

Proposition 2.2.7 The spec functor Spec : $\mathbf{CRing} \to \mathbf{Top}$ is fully faithful. This means that for $A, B \in \mathbf{CRing}$, there is a natural bijection
$$\text{Hom}_{\mathbf{CRing}}(A, B) \cong \text{Hom}_{\mathbf{Top}}(\text{Spec}(B), \text{Spec}(A))$$

Example 2.2.8 Let $\sigma : \mathbb{C}[x] \to \mathbb{C}[[t]]$ be a map. Let $\iota : \mathbb{C}[[t]] \to \mathbb{C}((t))$ be the inclusion map. Consider the following diagram:
$$\begin{tikzcd}
\mathbb{C}[x] \arrow[rr, "\sigma"] & & \mathbb{C}((t)) \\
& \mathbb{C}[[t]] \arrow[ur, hook, "\iota"'] &
\end{tikzcd}$$
$$\begin{tikzcd}
\substack{\text{Spec}(\mathbb{C}[x])= \\ \{(0)\}\cup\{(x-a)\ \mid\ a\in\mathbb{C}\}} & & \text{Spec}(\mathbb{C}((t))) = \{(0)\} \arrow[ll, "\sigma^\flat"'] \arrow[dl, "\iota^\flat"] \\
& \text{Spec}(\mathbb{C}[[t]]) = \{(0), (t)\} \arrow[ul, dashed, "\rho"] &
\end{tikzcd}$$

- Let σ be the map $x \mapsto t$. Then ρ exists.
- Let σ be the map $x \mapsto t + 1$. Then ρ exists.

- Let σ be the map $x \mapsto 1/t$. Then ρ does not exist.
- Let σ be the map $x \mapsto t^2$. Then ρ exists.
- Let σ be the map $x \mapsto e^t$. Then ρ exists.

Proof We know that the Spec functor is fully faithful. This means that ρ exists if and only if there exists a map $\mathbb{C}[x] \to \mathbb{C}[[t]]$ making the top diagram commute.

For the first two cases, the images of the map lie inside $\mathbb{C}[[t]]$. Hence the map $\mathbb{C}[x] \to \mathbb{C}[[t]]$ exists, and hence ρ exists. In the first case, we have $\rho((t)) = (x)$ and in the second case we have $\rho((t)) = (x-1)$.

For the third case, no map makes the top diagram commute hence ρ does not exist.

For the fourth and fifth cases, the images of the map lie inside $\mathbb{C}[[t]]$. Hence ρ exists. In the fourth case we have $\rho((t)) = (x)$ and in the fifth case we have $\rho((t)) = (x-1)$. Indeed, $f \in \rho^{-1}(t)$ if and only if t divides $f(e^t)$. This means that $g(t) = f(e^t)$ has no constant term, or that $g(0) = 0$. This means that $f(1) = 0$, hence $x - 1$ divides f. ■

2.3 Hilbert's Nullstellensatz for Spec

Definition 2.3.1 (Ideals from a Zero Locus) Let A be a commutative ring. Let $V \subseteq \text{Spec}(R)$. Define
$$\mathbb{I}(V) = \{f \in A \mid f \in p \text{ for all } p \in V\}$$

Theorem 2.3.2 (Scheme-theoretic Nullstellensatz) Let A be a commutative ring. Let J be an ideal of A. Then $\mathbb{I}(\mathbb{V}(J)) = \sqrt{J}$.

Proposition 2.3.3 Let A be a commutative ring. Then $\mathbb{V}^{\text{S}}(-)$ and $\mathbb{I}^{\text{S}}(-)$ induce an inclusion reversing bijection
$$\begin{Bmatrix} \text{Radical ideals} \\ \text{of } A \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \text{Closed Subsets of} \\ \text{Spec}(A) \end{Bmatrix}$$

Proposition 2.3.4 Let A be a commutative ring. Then $\mathbb{V}^{\text{S}}(-)$ and $\mathbb{I}^{\text{S}}(-)$ induces a bijection
$$\begin{Bmatrix} \text{Irreducible Components} \\ \text{of Spec}(A) \end{Bmatrix} \overset{1:1}{\longleftrightarrow} \begin{Bmatrix} \text{Minimal prime} \\ \text{ideals of } A \end{Bmatrix}$$

2.4 Dimension Theory of Spectra

Proposition 2.4.1 Let R be a commutative ring. Then we have
$$\dim(R) = \dim(\text{Spec}(R))$$

Recall that if R is a commutative ring and M is an R-module, we defined
$$\text{Supp}(M) = \{P \in \text{Spec}(R) \mid M_P \neq \{0\}\}$$

We have seen that it is a closed subset because
$$\text{Supp}(M) = \mathbb{V}^S(\text{Ann}_R(M))$$

Proposition 2.4.2 Let R be a commutative ring. Let M be a finitely generated R-module. Then we have
$$\dim(M) = \dim(\text{Supp}(M))$$

2.5 The Structure Sheaf of a Ring

We now define the structure sheaf on a spectrum so that they form a ringed space.

Definition 2.5.1 (Structure Sheaf) Let A be a commutative ring and Spec(A) the spectrum of A as a topological space. Define the structure sheaf on Spec(A) to be the functor $\mathcal{O}_{\text{Spec}(A)} : \textbf{Open}(\text{Spec}(A)) \to \textbf{Rings}$ defined as follows.

- For each $U \subseteq X$ open, define

$$\mathcal{O}_{\text{Spec}(A)}(U) = \left\{ s : U \to \coprod_{p \in U} A_p \;\middle|\; \substack{\forall p \in U,\ s(p) \in A_p \text{ and} \\ \exists U_p \subset U \text{ s.t. } q \in U_p \text{ implies } s(q) \in A_p} \right\}$$

- For $V \subseteq U$ an inclusion, the unique morphism $\mathcal{O}_{\text{Spec}(A)}(U) \to \mathcal{O}_{\text{Spec}(A)}(V)$ sends $s \in \mathcal{O}_{\text{Spec}(A)}(U)$ to the restriction

$$s|_V : V \to \coprod_{p \in V} A_p$$

Note that each s as a function from U simply means that s is indexed by $U \subseteq \text{Spec}(A)$. Alternatively we can write each element of $\mathcal{O}(U)$ as $s = (s_p)_{p \in U}$ such that $s_p \in A_p$.

Theorem 2.5.2 Let A be a commutative ring. Then the structure sheaf

$$\mathcal{O}_{\text{Spec}(A)} : \textbf{Open}(\text{Spec}(A)) \to \textbf{Set}$$

defined above is indeed a sheaf on Spec(A).

The structure sheaf allows Spec(A) to be a ringed space. The ringed space on any spectrum is in fact a locally ringed space. But this is not true for the ringed space on varieties in the classical sense.

Proposition 2.5.3 Let A be a commutative ring. Then the following are true regarding the ringed space $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$.

- For any $p \in \text{Spec}(A)$, there is an isomorphism $\mathcal{O}_{\text{Spec}(A),p} \cong A_p$ on the level of stalks.
- $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ is a locally ringed space.
- For any element $f \in A$, there is an isomorphism $\mathcal{O}_{\text{Spec}(A)}(D(f)) \cong A_f$
- There is an isomorphism $\mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A)) \cong A$ on the global level.

Proof

- Define a homomorphism $\phi_p : \mathcal{O}_{\text{Spec}(A),p} \to A_p$ as follows. For $s \in \mathcal{O}_{\text{Spec}(A),p}$ a local section in a neighbourhood of p to $s(p) \in A_p$. This is well defined: If $(U, s) \sim (V, t)$, then there exists a neighbourhood $W \subseteq U \cap V$ of p such that $s|_W = t|_W$. Hence $s(p) = t(p)$. It is clear that ϕ_p is a ring homomorphism by definition of the sheaf. It remains to show that ϕ_p is a bijection.

  Assume that $a/f \in A_p$. Then $D(f)$ is an open neighbourhood of p and a/f becomes a section in $\mathcal{O}_{\text{Spec}(A)}(D(f))$. Hence ϕ_p is surjective. Now suppose that s and t be two local sections in $\mathcal{O}_{\text{Spec}(A),p}$ such that $s(p) = t(p)$. Assume that s is local on U and t is local on V, then $s(p) = a/f$ and $t(p) = b/g$ in $W \subseteq U \cap V$ for some $a, b \in A$ and $f, g \in A \setminus p$. Since $s(p) = t(p)$, we conclude that there exists $h \in A \setminus p$ such that $h(ag - bf) = 0$. For any $q \in D(f) \cap D(g) \cap D(h)$, $h(ag - bf) = 0$ still holds in A_q hence $a/f = b/g$ in $D(f) \cap D(g) \cap D(h)$, which is a neighbourhood of p. Hence $s = t$ in a neighbourhood of p. Thus s and t have the same stalk. Thus ϕ_p is injective.
- From the above we immediately conclude that every stalk of the ringed space is a local ring. Hence $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$ is a locally ringed space.

- Define a map $\phi : A_f \to \mathcal{O}_{\text{Spec}(A)}(D(f))$ by sending $a/f^n \in A_f$ to $\left(s : D(f) \to \coprod_{p \in D(f)} A_p\right) \in \mathcal{O}_{\text{Spec}(A)}(D(f))$ that assigns each $p \in D(f)$ to $a/f^n \in A_p$. This makes sense since $p \in D(f)$ implies $f \notin p$ so that $a/f^n \in A_p$. It is clear that this is a ring homomorphism. It remains to show that ϕ_p is a bijection.

  Suppose that $\phi(a/f^n) = \phi(b/f^m)$. For each $p \in D(f)$, $\phi(a/f^n)(p) = \phi(b/f^m)(p)$ implies that $a/f^n = b/f^m$ hence there exists some $h \in A$ such that $h(f^m a - f^n b)$. Notice that the annihilator $\text{Ann}_A(f^m a - f^n b)$ is such that h lies in it. Since $h \notin p$, we have that $\text{Ann}_A(f^m a - f^n b)$ is not a subset of p. This is true for any $p \in D(f)$ hence $V(\text{Ann}_A(f^m a - f^n b)) \cap D(f) = \emptyset$. We conclude that $f \in \sqrt{\text{Ann}_A(f^m a - f^n b)}$ so $f^l(f^m a - f^n b) = 0$ for some l. Since f is invertible in A_f, we can multiply the inverse on both sides to obtain $a/f^n = b/f^m$ and so ϕ is injective.

  Let $s \in \mathcal{O}_{\text{Spec}(A)}(D(f))$.
- Using the above applied to $f = 1_A$, we conclude that $\mathcal{O}_{\text{Spec}(A)}(\text{Spec}(A)) \cong A$

■

Definition 2.5.4 (The Upgraded Spec Functor) Define the new spec functor

$$\text{Spec} : \mathbf{CRing} \to \mathbf{LocallyRingedSpaces}$$

to consist of the following data.
- A commutative ring A is sent to the locally ringed space $(\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$.
- For a ring homomorphism $\phi : A \to B$,

  $$\text{Spec}(\phi) = (\phi^\flat, \phi^\#) : (\text{Spec}(B), \mathcal{O}_{\text{Spec}(B)}) \to (\text{Spec}(A), \mathcal{O}_{\text{Spec}(A)})$$

  is the map of locally ringed spaces defined by $\phi^\flat(P) = \phi^{-1}(P)$ and the morphism of sheaves

  $$\phi^\#(U) : \mathcal{O}_{\text{Spec}(A)}(U) \to \phi_* \left(\mathcal{O}_{\text{Spec}(B)}\right)(U) = \mathcal{O}_{\text{Spec}(B)}((\phi^\flat)^{-1}(U))$$

  for $U \subseteq \text{Spec}(A)$ open, defined by

  ?????

Proposition 2.5.5 The new spec functor $\text{Spec} : \mathbf{CRing} \to \mathbf{LocallyRingedSpaces}$ is fully faithful. This means that for $R, S \in \mathbf{CRing}$ there is a natural bijection

$$\text{Hom}_{\mathbf{CRing}}(R, S) \cong \text{Hom}_{\mathbf{LocallyRingedSpaces}}((\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)}), (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}))$$

3 Schemes, Subschemes and Morphism of Schemes

3.1 Affine Schemes

Definition 3.1.1 (Affine Schemes) Let R be a commutative ring. An affine scheme is a locally ringed space isomorphic to $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ where $\text{Spec}(R)$ is the topological space equipped with the Zariski Topology.

Definition 3.1.2 (The Category of Affine Schemes) The category of affine schemes **AffSch** consist of the following data.

- The objects are affine schemes isomorphic to $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ for some commutative ring R.
- Given two affine schemes $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$ and $(\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)})$, morphisms are given by morphisms of locally ringed spaces

$$(\text{Spec}(S), \mathcal{O}_{\text{Spec}(S)}) \to (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$$

By virtue of 2.5.5, giving a morphism of affine schemes is the same as giving a ring homomorphism of their underlying rings.

Proposition 3.1.3 There is an equivalence of categories between **AffSch** and **CRing** given by the functors

$$\text{Spec}(-) : \textbf{CRing} \overset{\cong}{\longleftrightarrow} \textbf{AffSch} : \Gamma$$

3.2 General Schemes

Definition 3.2.1 (Schemes) A scheme is a locally ringed space X such that every point $x \in X$ has an open neighbourhood U which is isomorphic to an affine scheme as a locally ringed space.

Definition 3.2.2 (Morphisms of Schemes) Let X and Y be schemes. We say that a map $f : X \to Y$ is a morphism of schemes if it is a morphism of locally ringed spaces.

Definition 3.2.3 (Affine Open Subsets) Let $(X, \mathcal{O}_X)$ be a scheme. Let $U \subset X$ be open. We say that U is an affine open subset of X if $(U, \mathcal{O}_X|_U)$ is an affine scheme.

Theorem 3.2.4 (Gluing Schemes) Let $(X_i, \mathcal{O}_{X_i})$ for $i \in I$ be a family of schemes. Denote U_{ij} an open subset of X_i for $i, j \in I$. Suppose that there is a system of morphisms

$$\theta_{ij} : (U_{ij}, \mathcal{O}_{X_i}|_{U_{ij}}) \to (U_{ji}, \mathcal{O}_{X_j}|_{U_{ji}})$$

such that $\theta_{ii} = \text{id}$, $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$. Then there exists a scheme $(X, \mathcal{O}_X)$ and an open cover $X = \bigcup_{i \in I} X_i'$ and a family of isomorphisms $\varphi_i : (X_i', \mathcal{O}_X|_{X_i'}) \to (X_i, \mathcal{O}_{X_i})$ such that

$$(\varphi_j|_{X_i \cap X_j})^{-1} \circ \theta_{ij}\varphi_i|_{X_i \cap X_j} = \text{id}$$

for all $i, j \in I$.

3.3 Projective Schemes

Definition 3.3.1 (Proj(S)) Let S be a graded ring. Denote $S_+ = \bigoplus_{d>0} S_d$ the irrelevant ideal. Define the set $\text{Proj}(S)$ to be the set of all homogeneous prime ideals p which do not contain all of S_+.

Definition 3.3.2 (Vanishing Set and Open Sets of a Homogeneous Ideal) Let a be a homogenous ideal of S, a graded ring. Define the vanishing set of a to be

$$V(a) = \{p \in \text{Proj}(S) \mid a \subseteq p\}$$

Define the open set of a to be

$$D(a) = \{p \in \text{Proj}(S) \mid a \not\subseteq p\}$$

Definition 3.3.3 (Basic Open Sets) Let S be a graded ring. Define

$$D_+(f) = \{p \in \text{Proj}(S) | f \notin p\}$$

for $f \in S_+$ to be a basic open set.

Proposition 3.3.4 Let R be a graded ring. The following are true.

- If $\{a_1, \dots, a_n\}$ are homogenous ideals of R, then

$$V(a_1, \dots, a_n) = \bigcup_{k=1}^n V(a_k)$$

- If $\{a_i | i \in I\}$ is a family of homogenous ideals of R, then

$$V\left(\sum_{i\in I} a_i\right) = \bigcap_{i \in I} V(a_i)$$

Similar to that of Spec(A) we can endow a topology on Proj(S).

Theorem 3.3.5 A topology can be defined on Proj(S) which is exactly the Zariski Topology. In particular,

- The closed sets of Proj(S) is exactly sets of the form $V(a)$ for $a \subseteq S$ for a a homogenous ideal
- The open sets of Proj(S) is exactly sets of the form$D(a)$ for $a \subseteq S$ for a a homogenous ideal
- The basic open sets of the form $D_+(f)$ for $f \in S_+$ form a basis for the topology.

Theorem 3.3.6 Let S be a graded ring. Let $p \in \text{Proj}(S)$. Consider the ring

$$S_{(p)} = T^{-1}S$$

where T is the multiplicative system consisting of all homogenous elements of S which are not in p. Construct $\mathcal{O}_{\text{Proj}(S)} : \textbf{Open}(\text{Proj}(S)) \to \textbf{Rings}$ as follows.

- For $U \subseteq \text{Proj}(S)$ an open set, define

$$\mathcal{O}_{\text{Proj}(S)}(U) = \left\{ s : U \to \coprod_{p \in U} S_{(p)} \;\middle|\; \substack{\forall p \in U,\ s(p) \in S_{(p)} \text{ and } \exists\ U_p \subseteq U \\ \text{s.t. } q \in U_p \text{ implies } s(q) = a/f \in S_{(q)} \text{ for } a \text{ and } f \text{ homogenous}} \right\}$$

- For $V \subseteq U$ the inclusion, define the unique map $\mathcal{O}_{\text{Proj}(S)}(U) \to \mathcal{O}_{\text{Proj}(S)}(V)$ by the restriction of elements.

Then $\mathcal{O}_{\text{Proj}(S)}$ is a sheaf on S.

Definition 3.3.7 (Projective Scheme) Let S be a graded ring. Define the projective scheme of S to be the locally ringed space $(\text{Proj}(S), \mathcal{O}_{\text{Proj}(S)})$

Proposition 3.3.8 Let S be a graded ring. Then the following are true.

- For any $p \in \text{Proj}(S)$, $\mathcal{O}_p \cong S_{(p)}$
- For $f \in S_+$, $\mathcal{O}|_{D_+(f)} \cong \text{Spec}(S_{(f)})$

- $\text{Proj}(S)$ is indeed a scheme.

Definition 3.3.9 (Projective Space over a Ring) Let A be a ring. Define the projective n-space over A to be the scheme
$$\mathbb{P}_A^n = \text{Proj}(A[x_0, \dots, x_n])$$

3.4 Subschemes

Definition 3.4.1 (Open Subschemes) Let $(X, \mathcal{O}_X)$ be a scheme. Let $U \subseteq X$ be an open subset. We say that $(U, \mathcal{O}_X|_U)$ an open subscheme of X.

Proposition 3.4.2 Let $(X, \mathcal{O}_X)$ be a scheme. Let $U \subseteq X$ be an open subset. Then the open subscheme $(U, \mathcal{O}_X|_U)$ is a scheme by it own right.

Definition 3.4.3 (Affine Morphism) A morphism $\pi : X \to Y$ is affine if for every affine open set U of Y, $\pi^{-1}(U)$ is an affine scheme.

Definition 3.4.4 (Closed Embedding) A morphism $\pi : X \to Y$ is a closed embedding or closed immersion if it is an affine morphism, and that for every affine open subset $\text{Spec}(B) \subset Y$, with $\pi^{-1}(\text{Spec}(B)) \cong \text{Spec}(A)$, the map $B \to A$ of rings is surjective.

Definition 3.4.5 (Closed Subscheme) A closed subscheme of a scheme X is an equivalence class of closed immersion, where $f : Y \to X$ and $f : Y' \to X$ are equivalent if there is an isomorphism $i : Y' \to Y$ such that $f' = f \circ i$.

Definition 3.4.6 (Locally Closed Embedding)

4 The Categorical Viewpoint of Schemes

4.1 The Category of Schemes

Definition 4.1.1 (The Category of Schemes) Define the category of schemes **Sch** to consist of the following data.

- The objects are schemes.
- Given two schemes, a morphism of schemes is a morphism of locally ringed spaces.
- Composition is given by the composition of functions.

Proposition 4.1.2 There is an adjunction

$$\Gamma : \mathbf{Sch} \rightleftarrows \mathbf{CRing} : \text{Spec}$$

Explicitly, there is a isomorphism

$$\text{Hom}_{\mathbf{CRing}}(\Gamma(X, \mathcal{O}_X), R) \cong \text{Hom}_{\mathbf{Sch}}((X, \mathcal{O}_X), (\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)}))$$

that is natural in $(X, \mathcal{O}_X)$ and R.

Definition 4.1.3 (Category of S-Schemes) Let S be a fixed scheme. Define the category of S-schemes to be the over category

$$\mathbf{Sch}_S = \mathbf{Sch}/S$$

Lemma 4.1.4 For any scheme X, the morphism $X \to \text{Spec}(\mathbb{Z})$ is a final object in Sch. Also, the identity morphism id : $S \to S$ is a final object in Sch_S

The problem is now is that the category of schemes is not good enough to work with in Algebraic Geometry. There are much more restriction to coordinate ring associated to a variety. If we use any arbitrary ring for the structure of schemes, it will be too broad to work with. For instance, in classical algebraic geometry we only work will coordinate rings, which are finitely generated.

4.2 Categorical Constructs of Schemes

Theorem 4.2.1 Let X and Y be two schemes over S. Then the fibered product $X \times_S Y$ exists and is unique up to unique isomorphism.

Proof We first prove the theorem for the case of affine schemes. Let $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and $S = \text{Spec}(C)$. I claim that $\text{Spec}(A \otimes_C B)$ is the fibered product of X and Y over S. Using the equivalence of categories, we have that

$$\begin{aligned}
\text{Hom}_{\mathbf{AffSch}}(Z, \text{Spec}(A \otimes_C B)) &\cong \text{Hom}_{\mathbf{Rings}}(A \otimes_C B, \Gamma(Z)) \\
&\cong \text{Hom}_{\mathbf{Rings}}(A, \Gamma(Z)) \times_{\text{Hom}_{\mathbf{Rings}}(C, \Gamma(Z))} \text{Hom}_{\mathbf{Rings}}(B, \Gamma(Z)) \\
&\cong \text{Hom}_{\mathbf{AffSch}}(Z, \text{Spec}(A)) \times_{\text{Hom}_{\mathbf{AffSch}}(Z, \text{Spec}(C))} \text{Hom}_{\mathbf{AffSch}}(Z, \text{Spec}(B))
\end{aligned}$$

Thus we have proved that $\text{Spec}(A \otimes_C B)$ is the fiber product of X and Y over S. ■

Recall that residue field of a point x in a scheme X is the field

$$k(p) = \frac{\mathcal{O}_{X,p}}{m}$$

where m is the maximal ideal of the local ring $\mathcal{O}_{X,p}$.

Definition 4.2.2 (Fiber of a Morphism) Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$ be a point. Let $k(y)$ be the residue field of y. Consider the natural morphism $\text{Spec}(k(y)) \to Y$. Then we define the fibre of the morphism f over the point y to be the scheme

$$X_y = X \times_Y \text{Spec}(k(y))$$

Notice that the underlying topological space of X_y is homeomorphic to the subspace $f^{-1}(y)$.

4.3 The Functor of Points

Definition 4.3.1 (The Functor of Points) Let X be a scheme. Define the functor of points to be the functor

$$\text{Hom}_{\mathbf{Sch}}(-, X) : \mathbf{AffSch}^{\text{op}} \to \mathbf{Set}$$

defined by sending each affine scheme Y to the set of maps $Y \to X$.

One classical usage of the functor of points is the following.

Lemma 4.3.2 Let X be a scheme. Let k be an algebraically closed field. Then the functor of points

$$\text{Hom}_{\mathbf{Sch}}(-, X) : \mathbf{AffSch}^{\text{op}} \to \mathbf{Set}$$

is faithful.

Definition 4.3.3 (k-Points of a Scheme) Let X be a scheme. Let k be a field. Define the k-points of X to be the set

$$X(k) = \text{Hom}_{\mathbf{Sch}}(\text{Spec}(k), X)$$

Intuitively, the k-points of a scheme are precisely the set of points on X that act like points in k. If $k = \mathbb{C}$, then $\mathbb{C}$-points of X (complex points) behave very similar as if they were points in $\mathbb{C}^n$.

5 Absolute Properties of Schemes

5.1 Integral Schemes

Definition 5.1.1 (Reduced Schemes) A scheme X is reduced if for every open set U, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

Proposition 5.1.2 A scheme X is reduced if and only if $\mathcal{O}_{X,p}$ is reduced at every point $p \in X$.

This definition indeed solves the problem. Suppose that f and g are functions on a reduced scheme that agrees on all points. ??? FOAG 5.2A
Thus $f = g$ and so the values of the function now define the function itself.

Definition 5.1.3 (Integral Schemes) A scheme X is integral if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain.

Proposition 5.1.4 A scheme X is integral if and only if it is irreducible and reduced.

Proof Suppose that $X = (\text{Spec}(A), \mathcal{O})$ is integral. We already know that the ring cannot have nilpotent elements from groups and rings. Suppose that $X = X_1 \cup X_2$ for some X_1, X_2 closed. We show that either $X_1 = X$ or $X_2 = X$. Suppose that $X_1 \neq X$. Then X_2 is closed means that $X_1 = V(S_1)$ and $X_2 = V(S_2)$ for some $S_1, S_2 \subset A$. ∎

Proposition 5.1.5 An affine scheme $\text{Spec}(A)$ is integral if and only if A is an integral domain.

5.2 The Generic Point and The Function Field

Definition 5.2.1 (Generic Points) Let X be a scheme with an affine cover $\bigcup_{i \in I} U_i$ with each $U_i = \text{Spec}(R_i)$. We say that $p \in X$ is a generic point if p corresponds to the (0) ideal in some $U_i = \text{Spec}(R_i)$. (Equivalently, p is a generic point if $\overline{p} = U_i$ for some i).

Note that this notion extends to general schemes since general schemes are covered by affine schemes. In particular, this means that a general scheme can have multiple generic points. A priori is the generic points of a subscheme does not necessarily coincide with any of the generic points of a scheme.

Proposition 5.2.2 Let X be a scheme. If X is irreducible, then X has a unique generic point. More generally, if X has irreducible components S_i for $i \in I$, then each S_i has a unique generic point.

Proposition 5.2.3 Let X be an integral scheme. Let ν be its unique generic point. Then $\mathcal{O}_{X,\nu}$ is a field.

Definition 5.2.4 (Function Field of an Integral Scheme) Let X be an integral scheme and let $\nu \in X$ be its unique generic point. Define the function field to be

$$K(X) = \mathcal{O}_{X,\nu}$$

which is a field.

5.3 Noetherian Schemes

Definition 5.3.1 (Quasi-compact) We say that a scheme X is quasicompact if every open cover of X has a finite subcover.

Definition 5.3.2 (Locally Noetherian Schemes) A scheme X is locally noetherian if it can be covered by an open affine subsets of $\text{Spec}(A_i)$, where each A_i is a noetherian ring. X is noetherian if it is locally noetherian and quasi-comapct.

Lemma 5.3.3 Let X be a scheme. If X is Noetherian, then the underlying space of X is Noetherian.

Proposition 5.3.4 Let X be a scheme. Then X is Noetherian if and only if for every affine subset $U \cong \text{Spec}(A)$ of X, A is Noetherian.

6 Relative Properties of Schemes

6.1 Morphisms of Finite Type and Finite Morphisms

Definition 6.1.1 (Quasi-compact Morphisms) We say that a scheme $f : X \to Y$ is a quasi-compact morphism if for every quasi-compact open subset V of Y, $f^{-1}(V)$ is quasi-compact.

Definition 6.1.2 (Locally of Finite Type) Let $f : X \to Y$ be a morphism of schemes. We say that f is locally of finite type if for every $y \in Y$, there exists an affine open subset $U = \text{Spec}(A)$ containing y such that $f^{-1}(U)$ is covered by open affine subsets $V_i = \text{Spec}(B_i)$ of X such that A_i is a finitely generated B-algebra.

f is said to be of finite type if in addition each $f^{-1}(U)$ can be covered by finitely many V_i.

6.2 Separated Morphisms

Separatedness is essentially the analog of the Hausdorff condition for schemes. Recall that a topological space X is Hausdorff if and only if the digonal morphism to $X \times X$ is closed.

Definition 6.2.1 (Diagonal Morphisms) Let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of X.

Definition 6.2.2 (Separated Morphisms and Schemes) Let $f : X \to Y$ be a morphism of schemes. We say that f is separated (or X is separated over Y) if the diagonal morphism δ is a closed immersion. A scheme X is separated if it is separated over Spec$(\mathbb{Z})$.

Proposition 6.2.3 If $f : X \to Y$ is a morphism of affine schemes, then f is separated.

Proposition 6.2.4 Let $f : X \to Y$ be a morphism of schemes. Then f is separated if and only if the image of the diagonal morphism is a closed subset of $X \times_Y X$.

Theorem 6.2.5 (Valuative Criterion of Separatedness) Let $f : X \to Y$ be a morphism of schemes. Let X be Noetherian. Then f is separated if and only if the following criterion is satisfied.

for any field K and any valuation ring R with quotient field K, let $i : \text{Spec}(K) \to \text{Spec}(R)$ be the morphism induced by the inclusion $K \subseteq R$. Given morphisms $\text{Spec}(R) \to Y$ and $\text{Spec}(K) \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
\text{Spec}(K) \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\text{Spec}(R) \arrow[r] \arrow[ur, dashed, "!"] & Y
\end{tikzcd}$$

there exists at most one morphism $\text{Spec}(R) \to X$ such that the above diagram commutes.

Proposition 6.2.6 Let X and Y be Noetherian schemes. Then any open or closed immersions $f : X \to Y$ are separated.

6.3 Proper Morphisms

A map of spaces $X \to Y$ is said to be proper if it preserves compact sets in point set topology. A proper morphism will mean a similar thing in algebraic geometry. However, perhaps the more surprising thing is that properness turns out to be the indicative criterion for a variety to be a group. This will be explored when we redefine varieties in the next chapter.

Definition 6.3.1 (Universally Closed Morphisms) Let $f : X \to Y$ be a morphism of schemes. We say that f is universally closed if for any morphism $g : Z \to Y$ of schemes, the morphism

$$g \otimes_Y f : Z \otimes_Y X \to Y$$

is closed.

Definition 6.3.2 (Proper Morphisms) Let $f : X \to Y$ be a morphism of schemes. We say that f is proper if it is separated, of finite type and is universally closed. A scheme X over a field k is said to be proper if the structure morphism $X \to \text{Spec}(k)$ is proper.

Theorem 6.3.3 Let X be a scheme of finite type over $\mathbb{C}$. Then X is proper if and only if $X(\mathbb{C})$ is compact and Hausdorff.

7 Varieties Redefined

7.1 Varieties Revisited

Proposition 7.1.1
Let k be an algebraically closed field. Then there is a fully faithful functor

$$\text{IrrVar}_k \to \text{Sch}_k$$

given by (Hartshorne 2.2.6). Moreover,, schemes in the image are Noetherian, integral separated schemes of finite type over k.

7.2 New Definition of Variety

Definition 7.2.1 (Varieties) Let k be a field. A variety over k is a scheme X such that X is an integral separated scheme of finite type over k.

Let us think again why we need all these extra properties of a scheme for a variety to make sense.

- We would like our scheme to be reduced because we would like functions on the variety to be determined by their points.
- Separatedness is analogous to the Hausdorff property.
- A scheme of finite type over k means that it has a finite cover, and each cover $U_i = \text{Spec}(A_i)$ is such that A_i is a finitely generated algebra (algebra of finite type) over k. Have a finite cover prevents any dimensional argument to blow to infinity. While finitely generated algebras has always been the main concern of Algebraic Geometry: Indeed coordinate rings and polynomial rings are finitely generated algebras. Moreover, the equivalence of categories given in section 1.1 concerns only finitely generated algebras as well.

Definition 7.2.2 (Irreducible Varieties) Let k be a field. Let X be a variety. We say that X is irreducible if its underlying topological space is irreducible.

Definition 7.2.3 (Complete Varieties) Let k be a field. We say that a variety X over k is complete if it is proper over k.

Definition 7.2.4 (The Category of Varieties) Define the category of varieties Var_k over a field k as follows.

- The objects are varieties X over k
- The morphisms are morphisms of schemes $X \to Y$ over k.
- Composition is given by the composition of morphisms.

Proposition 7.2.5 Let X, Y be varieties. Then any morphism $\phi : X \to Y$ of scheme are separated and of finite type.

7.3 Subvarieties and their Properties

Definition 7.3.1 (Subvarities) Let X be a variety. A subvariety of X is a closed subscheme of X such that X is also a variety.

Proposition 7.3.2 Let X and Y be varieties over an algebraically closed field k. Then

$$X \times_{\text{Spec}(k)} Y$$

is a variety.

7.4 The Category of Affine Varieties

Definition 7.4.1 (Affine Varieties)
Let k be a field. Let X be a variety over k. We say that X is affine if X is an affine scheme.

In other words, a scheme X is said to be an irreducible affine variety if the following are true.

- X is integral (reduced + irreducible)
- X is separated
- X is a scheme of finite type over k
- X is an affine scheme

Theorem 7.4.2 There is an equivalence of categories

$$\mathbf{AffAlgVar}_k \cong (\text{Integral Affine Varieties})$$

induced by the functor of points.

8 The Module Structure on Sheaves

Recall that in sheaf theory we also defined the analogue of modules for sheaves. The enrichment of a module structure provides deep insights on schemes.

8.1 Sheaves of Modules

We restate the definition here of a sheaf of modules here. Let $\mathcal{A}$ be a sheaf of rings over X. Let U be an open set of X. A sheaf of $\mathcal{A}$-modules over X is a sheaf $\mathcal{F}$ such that each $\mathcal{F}(U)$ is an $\mathcal{A}(U)$-modules. Moreover, for each inclusion of open sets $V \subseteq U$, the restriction homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] \arrow[d, "\text{res}_{U,V} \times \text{res}_{U,V}"'] & \mathcal{F}(U) \arrow[d, "\text{res}_{U,V}"] \\ \mathcal{A}(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] & \mathcal{F}(V) \end{tikzcd}$$

For a scheme $(X, \mathcal{O}_X)$ that is also a locally ringed space, we denote the category of sheaves of $\mathcal{O}_X$-modules as $\text{Mod}_{\mathcal{O}_X}$.

It is the analogue of a module over a ring for the following reason.

Definition 8.1.1 (Associated Sheaf) Let M be an A-module. Define a sheaf $\tilde{M}$ on Spec(A) as follows.

- For each open set $U \subseteq \text{Spec}(A)$, define

$$\tilde{M}(U) = \left\{ s : U \to \coprod_{p \in U} M_p \;\middle|\; \substack{\forall p \in U,\ s(p) \in M_p \text{ and } \exists\ U_p \subseteq U \text{ s.t.} \\ q \in V \text{ implies } s(q) = \frac{m}{f} \in M_q \text{ for } f \in A, m \in M} \right\}$$

- For $V \subseteq U$ an inclusion, define the unique morphism $\tilde{M}(U) \to \tilde{M}(V)$ by the restriction.

Lemma 8.1.2 Let M be an A-module. Then the associated sheaf is a sheaf of $\mathcal{O}_{\text{Spec}(A)}$-modules.

Lemma 8.1.3 If X is connected then the rank of a locally free sheaf on X is constant.

Theorem 8.1.4 Let M be an A-module. Then the following are true regarding the associated sheaf $\tilde{M}$.

- For each $p \in \text{Spec}(A)$, there is an isomorphism $\tilde{M}_p \cong M_p$
- For any $f \in A$, there is an isomorphism $\tilde{M}(D(f)) \cong M_f$ of A_f-modules
- $\Gamma(X, \tilde{M}) = M$

Theorem 8.1.5 The tilde construction is functorial in the following sense. Let R be a ring. The construction

$$\widetilde{(\,\cdot\,)} : \textbf{Mod}_R \to \textbf{Mod}_{\mathcal{O}_{\text{Spec}(R)}}$$

defined by $M \mapsto \widetilde{M}$ and $M \to N$ mapping to induced morphism of sheaves??? $\widetilde{M} \to \widetilde{N}$ is a functor.

Theorem 8.1.6 The tilde construction is left adjoint

$$\widetilde{(\,\cdot\,)} : \textbf{Mod}_R \rightleftarrows \textbf{Mod}_{\mathcal{O}_X} : \Gamma$$

to the global section functor Γ.

Definition 8.1.7 (The Ideal Sheaf) Let X be a scheme and let Y be a closed subscheme of X. Let $i : Y \to X$ be the inclusion. Define the ideal sheaf of Y to be

$$\mathcal{I}_Y = \ker(i^\# : \mathcal{O}_X \to i_*(\mathcal{O}_Y))$$

8.2 Quasicoherent Sheaves

Recall that sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasicoherent if for all $p \in X$, there exists an open neighbourhood $U \subseteq X$ such that there is an exact sequence:

$$\mathcal{O}_X^{\otimes I}|_U \longrightarrow \mathcal{O}_X^{\otimes J}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

for some countable indexing sets I and J. We will now give an explicit description of quasi-coherent sheaves for when $(X, \mathcal{O}_X)$ is a scheme.

Theorem 8.2.1 Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. Then the following are equivalent.

- $\mathcal{F}$ is quasi-coherent.
- For all affine open subschemes U, $\mathcal{F}|_U \cong \widetilde{M}$ for some $\mathcal{O}_X(U)$-module M.
- X can be covered by open affine subsets $U_i = \text{Spec}(A_i)$ such that for each i, there is an A_i-module M_i with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$.

Also recall from sheaf theory that we denoted the category of quasi-coherent sheaves over $\mathcal{O}_X$ by

$$\mathbf{QCoh}_{\mathcal{O}_X}$$

Moreover, this category is abelian. In some sense, the category of quasicoherent sheaves is the smallest abelian category for which it encompasses the category of locally free sheaves.

Now let us recall what it means to be a coherent sheaf. There are two definitions to recall. Let $(X, \mathcal{O}_X)$ be a ringed space. We say that a sheaf of $m\mathcal{O}_X$-module $\mathcal{F}$ is of finite type if for all $p \in X$, there exists an open neighbourhood $U \subseteq X$ such that there is a surjective morphism

$$\mathcal{O}_X^{\otimes n}|_U \to \mathcal{F}|_U$$

for some $n \in \mathbb{N}$. We say that $\mathcal{F}$ is a coherent sheaf if the following are true.

- $\mathcal{F}$ is a sheaf of finite type.
- For any $U \subseteq X$ and any morphism
$$\varphi : \mathcal{O}_X^{\otimes n}|_U \to \mathcal{F}|_U$$
of $\mathcal{O}_X$-modules, then kernel of φ is a sheaf of finite type.

Theorem 8.2.2 Let $(X, \mathcal{O}_X)$ be a scheme. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is a coherent sheaf if and only if $\mathcal{F}$ is quasi-coherent and each M_i in 6.2.1 is a finitely generated A_i-module.

In the case that A is locally Noetherian, the category of finite rank locally free sheaves sit inside the category of coherent sheaves, which is also an abelian category.

Theorem 8.2.3 Let R be a ring and let $X = \text{Spec}(R)$. The adjunction of the tilde functor

$$\widetilde{(\cdot)} : \mathbf{Mod}_R \rightleftarrows \mathbf{Mod}_{\mathcal{O}_X} : \Gamma$$

with the global section functor restricts to an equivalence of categories

$$\mathbf{Mod}_R \cong \mathbf{QCoh}_{\mathcal{O}_X}$$

If A is noetherian, the same functor gives an equivalence of categories

$$\mathbf{FGMod}_R \cong \mathbf{Coh}_{\mathcal{O}_X}$$

between the category of finitely generated R-modules and the category of coherent $\mathcal{O}_X$-modules.

Proposition 8.2.4 Let X be a space and let $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ be sheaves on X such that there is an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}'' \longrightarrow 0$$

of $\mathcal{O}_X$-modules. Assume that $\mathcal{F}$ is quasi-coherent. Then there is an exact sequence of the form

$$0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

In other words, quasi-coherent sheaves are acyclic for affine schemes and the global section functor.

8.3 Sheaves of Modules on Graded Rings

Definition 8.3.1 (Sheaves of Modules on Graded Rings) Let S be a graded ring. Let M be a graded S-module. Consider the module

$$M_{(p)} = T^{-1}M$$

where T is the multiplicative system of homogenous elements of S not in p. Define the sheaf associated to M on Proj(S),

$$\tilde{M} : \textbf{Open}(\text{Proj}(S)) \to \textbf{Rings}$$

as follows.

- For each $U \subseteq \text{Proj}(S)$ open, define

$$\mathcal{O}_{\text{Proj}(S)}(U) = \left\{ s : U \to \coprod_{p\in U} M_{(p)} \;\middle|\; \substack{\forall p\in U,\ s(p)\in M_p \text{ and } \exists\ U_p\subseteq U \text{ s.t. } q\in V \text{ implies} \\ s(q)=\frac{m}{f}\in M_q \text{ for } f\in S \text{ and } m\in M \text{ homogenous}} \right\}$$

- For $V \subseteq U$ an inclusion, define the unique morphism $\tilde{M}(U) \to \tilde{M}(V)$ by restriction.

Proposition 8.3.2 Let S be a graded ring and let M be a graded module over S. Then the following are true regarding the sheaf of modules $\tilde{M}$.

- For any $p \in \text{Proj}(S)$, there is an isomorphism $\tilde{M}_p \cong M_{(p)}$
- For any homogenous $f \in S_+$, there is an isomorphism

$$\tilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$$

  via the isomorphism of $D_+(f)$ with Spec$(S_{(f)})$

Lemma 8.3.3 Let S be a graded ring and let M be a graded module over S. Then $\tilde{M}$ is a $\mathcal{O}_{\text{Proj}(S)}$-module. Moreover, if S is Noetherian, then $\tilde{M}$ is a coherent $\mathcal{O}_{\text{Proj}(S)}$-module.

For a graded ring S, recall that we can shift the grading of S up and down, and it will still be a graded ring. This is the shifted $S(n)$ where n denotes shifting up n times. (Note that this is not true for algebras because S is an algebra over S_0, if the grading is shifted then $S(n)$ is an algebra over $S(n)_n$.

Definition 8.3.4 (The Twisting Sheaf of Serre) Let S be a graded ring. Let $X = \text{Proj}(S)$. For any $n \in \mathbb{Z}$, define the sheaf

$$\mathcal{O}_X(n) = \widetilde{S(n)}$$

We call $\mathcal{O}_X(1)$ the twisting sheaf of Serre. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, denote

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

The twisting sheaf of Serre is important because it is the prototypical example of an invertible sheaf on Proj(S).

Proposition 8.3.5 Let S be a graded ring and let $X = \text{Proj}(S)$. Suppose that S is generated by S_1 as an S_0-algebra. Then the following are true.

- The sheaf $\mathcal{O}_X(n)$ is invertible.
- If M is a graded S-module, then $\widetilde{M}(n) \cong \widetilde{M(n)}$
- There is an isomorphism $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$

Definition 8.3.6 (Graded Module Associated to a Sheaf of Modules) Let S be a graded ring and let $X = \text{Proj}(S)$. Let $\mathcal{F}$ be a sheaf of mO_X-module. Define the graded S-module associated to $\mathcal{F}$ to be the group

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

together with the structure of graded S-module as follows. If $s \in S_d$, then s determines a global section $s \in \Gamma(X, \mathcal{O}_X(d))$ naturally. For any $t \in \Gamma(X, \mathcal{F}(n))$, define $s \cdot t \in \Gamma(X, \mathcal{F}(n+d))$ by sending $s \otimes t \in \mathcal{F}(n) \otimes \mathcal{O}_X(d)$ to $\mathcal{F}(n+d)$ by the isomorphism

$$\mathcal{F}(n) \otimes \mathcal{O}_X(d) \cong \mathcal{F}(n+d)$$

Proposition 8.3.7 Let A be a ring and let $S = A[x_0, \dots, x_n]$ for $r \geq 1$. Then there is an isomorphism

$$\Gamma_*(\mathcal{O}_{\text{Proj}(S)}) \cong S$$

Note that this is not true if S is not a polynomial ring.

9 The Study of Smoothness

9.1 Codimension

Definition 9.1.1 (Codimension) Let X be a topological space and Y an irreducible subset of X. Define the codimension of Y in X to be

$$\operatorname{codim}_X(Y) = \sup_{\substack{Z_1,\dots,Z_n\subseteq X \\ \text{Closed and irreducible}}} \{n \in \mathbb{N} \mid \overline{Y} \subset Z_1 \subset \cdots \subset Z_n\}$$

Theorem 9.1.2 (Krull's Principal Ideal Theorem (Geometric)) Let X be a locally Noetherian scheme, and f is a function. The irreducible components of $V(f)$ are codimension 0 or 1.

9.2 Regular Schemes

Definition 9.2.1 (Zariski Cotangent Space)
Let k be a field. Let X be a scheme over k. Let $p \in X$. Let m be the unique maximal ideal of the local ring $\mathcal{O}_{X,p}$. Define the Zariski cotangent space at p to be the $k \cong \frac{\mathcal{O}_{X,p}}{m}$ vector space $\frac{m}{m^2}$.

Definition 9.2.2 (Regular Schemes)
Let X be a scheme. Let $p \in X$. We say that X is regular at p if $\mathcal{O}_{X,p}$ is a regular local ring. We say that X is regular if X is regular at all points $p \in X$.

9.3 The Sheaf of Differential Forms

Definition 9.3.1 (Sheaf of Relative Differentials) Let $f : X \to Y$ be a morphism of schemes. Define the sheaf of relative differentials to be the pullback sheaf

$$\Omega^1_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

where $\Delta : X \to X \times_Y X$ is the diagonal morphism and $\mathcal{I} = \ker(\Delta^\#)$ is the sheaf of ideals of $\Delta(X)$.

Proposition 9.3.2 Let $f : X \to Y$ be a morphism of schemes. Then $\Omega^1_{X/Y}$ is quasi-coherent. Moreover, there are isomorphisms

$$\Omega^1_{X/Y}|_U \cong \left(\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(U)}\right)^\sim$$

on the level of local sections and

$$\left(\Omega^1_{X/Y}\right)_{X,p} \cong \Omega^1_{\mathcal{O}_{X,p}/\mathcal{O}_{Y,f(p)}}$$

on the level of stalks.

Proposition 9.3.3 Let $f : X \to Y$ be a morphism of schemes. Then $\Omega^1_{X/Y}$ is a quasicoherent $\mathcal{O}_X$-module.

Proposition 9.3.4 Let $f : X \to Y$ and $g : Y \to Z$ be morphism of schemes. Then there is an exact sequence

$$f^*\Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Proposition 9.3.5 Let $f : X \to Y$ be a morphism of schemes. Let Z be a closed subscheme of X with ideal sheaf $\mathcal{I}$. Then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{X/Y} \otimes \mathcal{O}_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0$$

Lemma 9.3.6 Let $X = \mathbb{A}^n_Y$. Then $\Omega^1_{X/Y}$ is a free $\mathcal{O}_X$-module of rank n.

Theorem 9.3.7 Let X be an irreducible separated scheme of finite type over an algebraically closed field k. Then $\Omega^1_{X/k}$ is a locally free sheaf of rank $\dim(X)$ if and only if X is a nonsingular variety over k.

Definition 9.3.8 (The Canonical Sheaf) Let X be a non-singular variety over a field k of dimension $\dim(X) = n$. Define the canonical sheaf of X to be

$$\omega_X = \bigwedge_{i=1}^n \Omega^1_{X/k}$$

9.4 Smooth Schemes

Definition 9.4.1 (Smooth Schemes) A scheme X over a field k is said to be smooth of dimension d if there exists an open cover $\{U_i \mid i \in I\}$ such that each U_i is of the form Spec $\left(\frac{k[x_1,\dots,x_n]}{(f_1,\dots,f_r)}\right)$ and the Jacobian matrix

$$\text{rank}\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_r}{\partial x_1}(p) & \cdots & \frac{\partial f_r}{\partial x_n}(p) \end{pmatrix} \geq n - d$$

has rank greater than $n - d$ for all $p \in X$.

Proposition 9.4.2 Let X be a smooth scheme over k. Then the following are true.
- X is locally of finite type over k.
- X is regular.
- X is reduced.

Theorem 9.4.3 If k is a perfect field, then every regular schemes of finite type over k are smooth over k.

9.5 Smooth Morphisms

10 Cohomology of Schemes

10.1 Cohomology of a Noetherian Affine Scheme

Proposition 10.1.1 Let I be an injective module over a Noetherian ring A. Then the sheaf $\tilde{I}$ on Spec(A) is flasque.

Theorem 10.1.2 Let A be a Noetherian ring. Then for all quasi-coherent sheaves $\mathcal{F}$ on $X =$ Spec(A),

$$H^i(X,\mathcal{F}) = 0$$

for all $i > 0$.

Note that result is also true if we drop the requirement that A is Noetherian. But the proof is more difficult.

Theorem 10.1.3 Let X be a Noetherian scheme. Then the following are equivalent.

- X is an affine scheme
- $H^i(X,\mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > 0$
- $H^1(X,\mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$

10.2 Cohomology of Projective Space

Theorem 10.2.1 Let A be a Noetherian ring and let $S = A[x_0,\dots,x_r]$. Let $X =$ Proj(S) be the projective space over A. Let $\mathcal{O}_X(1)$ be the twisting sheaf of Serre. Then the following are true.

- The natural map
$$S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n\in\mathbb{Z}} H^0(X,\mathcal{O}_X(n))$$
is an isomorphism
- For all $n \in \mathbb{Z}$, $H^i(X,\mathcal{O}_X(n)) = 0$ for $0 < i < r$
- There is an isomorphism $H^r(X,\mathcal{O}_X(-r-1)) \cong A$
- For each $n \in \mathbb{Z}$, the natural map
$$H^0(X,\mathcal{O}_X(n)) \times H^r(X,\mathcal{O}_X(-n-r-1)) \to H^r(X,\mathcal{O}_X(-r-1)) \cong A$$
is a perfect pairing of finitely generated free A-modules

11 The Theory of Divisors

11.1 Weil Divisors

Definition 11.1.1 (Regular in Codimension 1) Let X be a scheme. We say that X is regular in codimension 1 if every local ring $O_{X,p}$ of dimension 1 is regular.

A primer on Hartshorne's Weil divisors: In order to develop the theory of Weil divisors on a scheme X, for every codimension 1-subscheme Y, we want to define a homomorphism $K(Y) \to \mathbb{Z}$. We want this to be analogous to the homomorphism ord : $k(x_1, \dots, x_n) \to \mathbb{Z}$ sending a rational function to its order. Such a homomorphism is precisely a discrete valuation. Therefore, we would like each $K(Y)$ to be a discrete valuation ring. This is possible precisely when every $K(Y) = \mathcal{O}_{Y,\eta}$ is a discrete valuation ring for η the generic point of Y. Therefore we would like X to be regular in codimension 1.

I have not yet found a reason for Hartshorne's requirement that X be separated.

Definition 11.1.2 (Prime Divisors) Let X be a noetherian integral separated scheme which is regular in codimension 1. A prime divisor on X is a closed integral subscheme Y of codimension 1.

Definition 11.1.3 (Weil Divisors) Let X be a noetherian integral separated scheme which is regular in codimension 1. A Weil divisor on X is an element of the free abelian group $Div(X)$, generated by the prime divisors. In other words, a Weil divisor is an element of the form

$$D = \sum_{i \in I} n_i Y_i$$

where Y_i is a prime divisor and n_i an integer and only finitely many n_i are nonzero.

Lemma 11.1.4 Let X be an integral scheme. Let Y be an integral subscheme of X with generic point η. Then

$$K(X) = \text{Frac}(\mathcal{O}_{X,\eta})$$

Proof https://math.stackexchange.com/questions/218767/relation-of-function-field-of-a-scheme-to-the-local-ring-of-its-prime-divisor?rq=1 ■

Definition 11.1.5 (Divisors of Functions) Let X be a noetherian integral separated scheme which is regular in codimension 1. Let P be a prime divisor of X and let ν be the unique generic point of P in X. Denote

$$v_P : \mathcal{O}_{X,\nu} \to \frac{\mathcal{O}_{X,\nu}}{m_\nu}$$

the valuation of the discrete valuation ring $\mathcal{O}_{X,\nu}$. Let f be a non-zero element of $\mathcal{O}_{X,\nu}$. Define the divisor of f to be

$$(f) = \sum_{\substack{P \in X \\ P \text{ a prime divisor of } X}} v_P(f) \cdot P$$

Lemma 11.1.6 Let X be a noetherian integral separated scheme which is regular in codimension 1. Let f be a non-zero element of $\mathcal{O}_{X,\nu}$ where ν is the generic point of X. Then $v_P(f) = 0$ for all but except finitely many prime divisors P of X.

The lemma shows that divisors of functions are well defined.

Definition 11.1.7 (Principal Divisors) Let X be a Noetherian integral separated scheme which is regular in codimension 1. We say that a divisor D on X is principal if $D = (f)$ for some function f.

Definition 11.1.8 (The Divisor Class Group) Let X be a Noetherian integral separated scheme which is regular in codimension 1. Let $\text{Prin}(X)$ be the subgroup of all principal divisors of X. Define the divisor class group of X as

$$\text{Cl}(X) = \frac{\text{Div}(X)}{\text{Prin}(X)}$$

Two elements of the same coset are said to be linearly equivalent.

Definition 11.1.9 (Degree Homomorphism) Let X be a Noetherian integral separated scheme which is regular in codimension 1. Define the degree homomorphism

$$\deg : \text{Div}(X) \to \mathbb{Z}$$

by $D = \sum_P n_P \cdot P \mapsto \sum_P n_P$. For each divisor D, define the degree of D to be

$$\deg(D) = \sum_P n_P$$

Lemma 11.1.10 Let $X = \mathbb{P}_k^n$ be the projective space over a field k. The degree homomorphism gives an isomorphism

$$\overline{\deg} : \text{Cl}(X) \xrightarrow{\cong} \mathbb{Z}$$

11.2 Cartier Divisors

Definition 11.2.1 (The Sheaf of Total Quotient Rings) Let $(X, \mathcal{O}_X)$ be a scheme. Define a presheaf $K : \textbf{Open}(X) \to \textbf{Ring}$ as follows.

- For each $U \subseteq X$ open, define $K(U) = \text{Frac}(\mathcal{O}_X(U))$
- For each inclusion $U \subseteq V$, define $K(V) \to K(U)$ by just the restriction map.

Define the sheaf of total quotient rings to be the associated sheaf of the presheaf K.

Definition 11.2.2 (Invertible Elements of a Sheaf) Let $(X, \mathcal{F})$ be a ringed space. Define the sheaf of invertible elements

$$\mathcal{F}^* : \textbf{Open} \to \textbf{Grp}$$

of $\mathcal{F}$ as follows.

- For $U \subseteq X$ an open set, define $\mathcal{F}^*(U) =$ The Group of Units of $\mathcal{F}(U)$
- For $U \subseteq V$, define $\mathcal{F}^*(V) \to \mathcal{F}^*(U)$ to just be the restriction map.

Definition 11.2.3 (Cartier Divisors) Let $(X, \mathcal{O}_X)$ be a scheme. A Cartier divisor on X is a global section of the sheaf

$$\frac{\mathcal{K}^*}{\mathcal{O}_X^*} : \textbf{Open} \to \textbf{Grp}$$

In other words, a Cartier divisor on X can be described as follows. If $\{U_i \mid i \in I\}$ is an open cover of X, a Cartier divisor is an element $f_i \in \mathcal{K}^*(U_i)$ for each $i \in I$ such that for each $i, j \in I$, $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$.

Lemma 11.2.4 Let X be a scheme. Then the set of all Cartier divisors of X form a group.

Definition 11.2.5 (Group of Cartier Divisor) Let X be a scheme. Define the group of Cartier divisors by

$$\text{CaCl}(X) = \{D \mid D \text{ is a Cartier divisor}\}$$

Definition 11.2.6 (Principal Cartier Divisors) Let $(X, \mathcal{O}_X)$ be a scheme. A Cartier divisor f of X is said to be principal if it is the image of the natural map

$$\mathcal{K}^*(X) \to \frac{\mathcal{K}^*}{\mathcal{O}_X^*}(X)$$

Two Cartier divisors are said to be linearly equivalent $D_1 \sim D_2$ if $D_1 - D_2$ is a principal divisor.

Proposition 11.2.7 Let X be an integral, separated Noetherian scheme such that all local rings are UFD. Then the group of Weil divisors is isomorphic

$$\text{Div}(X) \cong \frac{\mathcal{K}^*}{\mathcal{O}_X^*}(X)$$

to the group of Cartier divisors. Moreover, the principal Weil divisors correspond to the principal Cartier divisors.

11.3 Cartier Divisors and Invertible Sheaves

Definition 11.3.1 (Sheaf Associated to a Divisor) Let $(X, \mathcal{O}_X)$ be a scheme. Let D be a Cartier divisor on X, represented by $\{(U_i, f_i) \mid i \in I\}$. The sheaf associated to a divisor to be the subsheaf

$$\mathcal{L}(D) \subseteq \mathcal{K}$$

is the $\mathcal{O}_X$-module defined as follows. For each U_i, $\mathcal{L}(D)(U_i)$ is the $\mathcal{O}_X(U_i)$-module generated by f_i^{-1}.

Proposition 11.3.2 Let X be a scheme. Let D be a Cartier divisor of X. Then $\mathcal{L}(D)$ is an invertible sheaf.

The proposition motivates us to investigate the relations between Cartier divisors and invertible sheaves on a scheme X. This leads to a very fruitful and satisfying result.

Theorem 11.3.3 Let X be a scheme. For any Cartier divisor D of X, the association $D \mapsto \mathcal{L}(D)$ gives a bijection

$$\{\text{Cartier Divisors on } X\} \quad \overset{1:1}{\longleftrightarrow} \quad \{\text{Invertible Subsheaves of } \mathcal{K}\}$$

Proposition 11.3.4 Let X be a scheme. Let D_1 and D_2 be Cartier divisors of X. Then there is an isomorphism

$$\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$$

Moreover, D_1 and D_2 are linearly equivalent if and only if

$$\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$$

(disregarding them as subsheaves of $\mathcal{K}$).

Proposition 11.3.5 Let X be a scheme. For any Cartier divisor D, the association $D \mapsto \mathcal{L}(D)$ gives an injective group homomorphism

$$\frac{\text{CaCl}(X)}{\sim} \to \text{Pic}(X)$$

where $\sim$ is linear equivalence of Cartier divisors.

When X is integral, Cartier divisors and the Picard group is entirely the same invariant for X.

Theorem 11.3.6 Let X be a scheme. If X is integral, then the above homomorphism is an isomorphism

$$\frac{\text{CaCl}(X)}{\sim} \cong \text{Pic}(X)$$

Corollary 11.3.7 Let X be a scheme. If X is Noetherian, integral, separated and that all local rings are UFDs, then the above isomorphism

$$\frac{\text{CaCl}(X)}{\sim} \cong \text{Pic}(X)$$

is natural in X.

Corollary 11.3.8 Let $X = \mathbb{P}_k^n$ be the projective space over some field k. Then every invertible sheaf on X is isomorphic to $\mathcal{O}_X(m)$ for some $m \in \mathbb{Z}$.