Algebraic Geometry 2

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Abstract

Algebraic Geometry is such a messy subject in a sense that a different books and lecture notes introduce different materials in a different orders, as well as having different prerequisites. After understanding a bit more in the subject, I believe that there is the need to give a clear distinction between traditional algebraic geometry and contemporary algebraic geometry. Although there are undoubtedly many overlapping between the two, I attempt to separate them to make clear their motivations as well as their results.

This book will mainly cover traditional algebraic geometry in the sense that the construction of affine and projective varieties will be covered, as well as the Hilbert Nullstellensatz theorems, morphisms, tangent maps and smoothness as well as classical constructions of some varieties. Affine schemes and sheaf theory are left for another time where they attempt to reinvent the fundamentals of algebraic geometry.

Knowledge on commutative algebra is required as a prerequisite. These set of notes make use of

- Algebraic Geometry I by I. R. Shafarevich and V. I. Danilov
- Algebraic Geometry by R. Hartshorne
- An Invitation to Algebraic Geometry by Karen. S, Pekka. K, Lauri .K, William .T

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1 The Tangent Space and Smooth Points

1.1 The Tangent Space

Definition 1.1.1 (Tangent Space) Let k be a field. Let V be a variety over k. Let $p \in V$. Denote the unique maximal ideal of $\mathcal{O}_{V,p}$ by m_p. Define the tangent space of V at p to be

$$T_pV = \left(\frac{m_p}{m_p^2}\right)^*$$

Example 1.1.2
Let p be a point in curve $\mathbb{V}(y^2 - x^2(x+1)) \subseteq \mathbb{A}^2_{\mathbb{C}}$. Then the following are true.

- If $p \neq (0,0)$, then $\dim(T_pV) = 1$.
- If $p = (0,0)$, then $\dim(T_pV) = 2$.

Proof
For each $p \in V$, the maximal ideal m_p of $\mathcal{O}_{V,p}$ is the ideal $(x - p_1, y - p_2)$ generated over $\mathcal{O}_{V,p}$. Hence we have that

$$\frac{m_p}{m_p^2} = k\langle [x-p_1], [y-p_2]\rangle$$

There are now two cases.

Suppose that $p \neq (0,0)$. I claim that the generators $[x - p_1]$ and $[y - p_2]$ are linearly dependent. Indeed, consider the scalars $[-((x^2 + p_1x + p_1^2) + (x + p_1)]$ and $[y + p_2]$ in $\mathcal{O}_{V,p}/m_p \cong k$. We have that

$$\begin{aligned}
[-((x^2 + p_1x + p_1^2) + (x + p_1))][x - p_1] + [y + p_2][y - p_2] &= [-(x^3 - p_1^3) - (x^2 - p_1^2)] + [y^2 - p_2^2] \\
&= [y^2 - x^3 - x^2 - (p_2^2 - p_1^3 - p_1^2)] \\
&= [y^2 - x^3 - x^2] - [p_2^2 - p_1^3 - p_1^2] \\
&= 0
\end{aligned}$$

Where the element $[y^2 - x^3 - x^2]$ is 0 in m_p/m_p^2 because $y^2 - x^3 - x^2 \in m_p$ and m_p is an ideal of $\mathcal{O}_{V,p} \cong \frac{\mathbb{C}[x,y]_{m_p}}{(y^2-x^3-x^2)_{m_p}}$ so that $y^2 - x^3 - x^2 = 0$, and the element $[p_2^2 - p_1^3 - p_1^2] = 0$ in m_p/m_p^2 because (p_1, p_2) is a point in the curve.

It remains to show that the scalars are exhibited for the linear dependence are actually not both zero. We have that $[y + p_2] = [0]$ if and only if $y + p_2 \in m_p$. This happens if and only if p_2 is a root of $y + p_2$, which happens if and only if $p_2 = 0$. Since (p_1, p_2) is a point on the curve, we conclude that $p_1 = 0$ or 1. By case assumption $(p_1, p_2) \neq 0$ and so we just have to deal with the case $p_1 = 1$. But if $p_1 = 1$ then the other scalar becomes $[-x^2] \in \mathcal{O}_{V,p}/m_p$ and this is clearly non-zero since $[-x^2] = 0$ if and only if $x^2 \in m_p$ if and only if $p_1 = 0$.

We are left with the case $p = (0,0)$. We want to show that $[x]$ and $[y]$ in m_p/m_p^2 are linearly independent. Assume that they are not. Then there exists $f, g \in \mathcal{O}_{V,p}$ such that $[f], [g] \in \mathcal{O}_{V,p}/m_p \cong k$ are scalars such that $[f][x] + [g][y] = 0$ in m_p/m_p^2 and that not both of $[f]$ and $[g]$ are zero. This means that $xf + yg \in m_p^2 = (x^2, xy, y^2)$. Hence $xf + yg$ must be a polynomial with no terms of degree 0 and 1. Since the only relation with can possibly apply to $xf + yg$ is $y^2 - x^3 - x^2 = 0$ and it has no degree 0 and degree 1 terms, we conclude that f and g cannot have constant terms. Then $f, g \in m_p$ and so $[f]$ and $[g]$ are both zero as scalars, a contradiction. ■

Proposition 1.1.3
Let k be a field. Let V be a variety over k. Let $p \in V$. Denote the unique maximal ideal of $\mathcal{O}_{V,p}$ by

m_p. Then there is an isomorphism

$$\frac{m_p}{m_p^2} \cong \Omega^1_{\mathcal{O}_{V,p}/k} \otimes_{\mathcal{O}_{V,p}} k$$

given by $[f] \mapsto df \otimes 1$ for d the universal derivation.

Our definition of tangent spaces does not depend on the choice of embedding of the variety. When we have a closed subvariety of $\mathbb{A}^n_k$, the tangent space can be computed using the defining equations of the affine variety. In fact, the definition of tangent spaces makes for a more intuitive definition using the differentials of the defining equations.

Proposition 1.1.4
Let k be a field. Let $V \subseteq \mathbb{A}^n_k$ be an affine variety. Let $p = (p_1, \dots, p_n) \in V$. Suppose that $f_1, \dots, f_r, g_1, \dots, g_s \in k[x_1, \dots, x_n]$ such that $V = \mathbb{V}(f_1, \dots, f_r) = \mathbb{V}(g_1, \dots, g_s)$. Then we have

$$\mathbb{V}\left(\sum_{k=1}^n \frac{\partial f_1}{\partial x_k}\bigg|_p (x_k - p_k), \dots, \sum_{k=1}^n \frac{\partial f_r}{\partial x_k}\bigg|_p (x_k - p_k)\right) = \mathbb{V}\left(\sum_{k=1}^n \frac{\partial g_1}{\partial x_k}\bigg|_p (x_k - p_k), \dots, \sum_{k=1}^n \frac{\partial g_s}{\partial x_k}\bigg|_p (x_k - p_k)\right)$$

Proof
Without loss of generality, it suffices to prove that the former set is contained in the latter one. Since g_i vanishes on V, we must have that $g_i \in (f_1, \dots, f_r)$. Hence there exists $h_1, \dots, h_r \in k[x_1, \dots, x_n]$ such that $g_i = \sum_{i=1}^r h_i f_i$. Then by the chain rule, we have

$$\frac{\partial g}{\partial x_k}\bigg|_p = \sum_{i=1}^r h_j(p) \frac{\partial f}{\partial x_k}\bigg|_p$$

since $f_j(p) = 0$. Then we have

$$\sum_{k=1}^n \frac{\partial g}{\partial x_k}\bigg|_p (x_k - p_k) = \sum_{k=1}^n \sum_{i=1}^r h_j(p) \frac{\partial f}{\partial x_k}\bigg|_p (x_k - p_k) = \sum_{j=1}^r h_j(p) \sum_{k=1}^n \frac{\partial f}{\partial x_k}\bigg|_p (x_k - p_k)$$

If $q = (q_1, \dots, q_n)$ vanishes in the former set, then the above calculation shows that $\sum_{k=1}^n \frac{\partial g_i}{x_k}\Big|_p (q_k - p_k) = 0$. Hence q lies in the latter set. ∎

Definition 1.1.5 (The Jacobian Matrix) Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_m) \subseteq \mathbb{A}^n_k$ be an affine variety. Let $p \in V$. Define the Jacobian matrix of V at p to be the $m \times n$ matrix

$$J_{V,p} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_1}{\partial x_n}\Big|_p \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}\Big|_p & \cdots & \frac{\partial f_m}{\partial x_n}\Big|_p \end{pmatrix}$$

Example 1.1.6 Let $V = \mathbb{V}(y - x^2, z - x^3) \subseteq \mathbb{A}^3_{\mathbb{C}}$. The Jacobian matrix of V at $(1, 1, 1)$ is given by

$$J_{V,(1,1,1)} = \begin{pmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

Lemma 1.1.7

Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_r) \subseteq \mathbb{A}_k^n$ be an affine variety. Let $p \in V$. Then we have

$$\mathbb{V}\left(\sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}\bigg|_p (x_k - p_k), \dots, \sum_{k=1}^{n} \frac{\partial f_r}{\partial x_k}\bigg|_p (x_k - p_k)\right) = \left\{ q \in \mathbb{A}_k^n \;\middle|\; J_{V,p}(q - p) = 0 \right\}$$

Proposition 1.1.8
Let k be a field. Let $V = \mathbb{V}(f_1, \dots, f_r) \subseteq \mathbb{A}_k^n$ be an affine variety that passes through 0. Then there is a k-vector space isomorphism

$$T_0V \cong \ker(J_{V,0})$$

1.2 Smooth and Singular Points

Definition 1.2.1 (Smooth and Singular Points)
Let k be a field. Let V be a variety over k. Let $p \in V$. We say that p is a smooth point of V if $\mathcal{O}_{V,p}$ is a regular local ring. We say that p is a singular point of V otherwise.

Theorem 1.2.2
Let k be a field. Let V be a variety over k. Then the set of singular points of X is a proper closed subset of V.

Lemma 1.2.3
Let k be a field. Let V be a projective variety over k. Let $p \in V$. If p lies in the intersection of at least two irreducible components of V, then p is a singular point.

Proof It suffices to show that $\mathcal{O}_{V,p}$ is not a integral domain because every regular local ring is a UFD and hence an integral domain. ■

An arbitrary point on a projective variety is either contained in a unique irreducible component or lies in the intersection of at least two irreducible components. The first lemma says that points in the latter case cannot be smooth, we are left to deal with the former case, hence without loss of generality we assume for the rest of the section that we deal with irreducible projective varieties.

Lemma 1.2.4
Let k be an algebraically closed field. Let V be an irreducible projective variety over k. Let $p \in V$. Then p is a smooth point if and only if $\dim(T_pV) = \dim(V)$

Proof
If p is a smooth point, then we have $\dim(V) = \dim(\mathcal{O}_{V,p}) = \dim(m_p/m_p^2) = \dim(T_pV)$. Conversely, if $\dim(V) = \dim(T_pV)$, then $\mathcal{O}_{V,p}$ is a regular local ring because $\dim(\mathcal{O}_{V,p}) = \dim(V) = \dim(T_pV) = \dim(m_p/m_p^2)$. ■

Proposition 1.2.5
Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}_k^n$ be an irreducible affine variety. Let $p \in V$. Then the following are equivalent.

- p is a smooth point of V.
- $\dim(T_pV) = \dim(V)$.
- For any choice of defining equations $V = \mathbb{V}(f_1, \dots, f_r)$, $\operatorname{rank}(J_{V,p}) = n - \dim(V)$.

Proof
We have seen (1) $\iff$ (2) above. It suffices to prove (1) $\iff$ (3). For a point p, let $m_p \in k[V]$ be the corresponding maximal ideal given by the Hilbert nullstellensatz. Consider the map $\phi : m_pk[V]_{m_p} \to k^n$ by

$$\phi(f) = \left(\frac{\partial f}{\partial x_1}\bigg|_p, \dots, \frac{\partial f}{\partial x_n}\bigg|_p\right)$$

We first check that the map is well defined. Suppose that $f=\frac{g_1}{h_1}=\frac{g_2}{h_2}$ for $g_1,g_2\in m_pk[V]_{m_p}$ and $h_1,h_2\in k[V]_{m_p}\setminus m_pk[V]_{m_p}$. For $j=1$ or 2 we have that

$$\left.\frac{\partial g_j/h_j}{\partial x_i}\right|_p=\frac{\frac{\partial g_j}{\partial x_i}|_p}{h_j(p)}$$

Now since $k[V]$ is an integral domain, we have $\frac{g_1}{h_1}=\frac{g_2}{h_2}$ if and only if $h_2g_1-h_1g_2=0$. Then we have

$$\begin{aligned}\left.\frac{\partial h_2g_1-h_1g_2}{\partial x_i}\right|_p&=0\\ \left.\frac{\partial h_2}{\partial x_i}\right|_p g_1(p)+\left.\frac{\partial g_1}{\partial x_i}\right|_p h_2(p)-\left.\frac{\partial h_1}{\partial x_i}\right|_p g_2(p)-\left.\frac{\partial g_2}{\partial x_i}\right|_p h_1(p)&=0\\ \left.\frac{\partial g_1}{\partial x_i}\right|_p h_2(p)-\left.\frac{\partial g_2}{\partial x_i}\right|_p h_1(p)&=0\end{aligned}$$

Since $h_1(p)$ and $h_2(p)$ is non-zero, we are done. Clearly ϕ is a k-linear map since differentiation is k-linear. Now suppose that $g,h\in m_pk[V]_{m_p}$. I claim that $\phi(gh)=0$ so that ϕ descends to a well defined map on $\frac{m_pk[V]_{m_p}}{m_p^2k[V]_{m_p}}$. Indeed, we have

$$\left.\frac{\partial gh}{\partial x_i}\right|_p=\left.\frac{\partial g}{\partial x_i}\right|_p h(p)+\left.\frac{\partial h}{\partial x_i}\right|_p g(p)=0$$

because $g,h\in m_pk[V]_{m_p}$.

We now have a well defined k-linear map $\phi:\frac{m_pk[V]_{m_p}}{m_p^2k[V]_{m_p}}\to k^n$. ■

Example 1.2.6
A point $p=(p_1,p_2)$ in the affine variety $\mathbb{V}(y^2-x^3)\subseteq\mathbb{A}^2_{\mathbb{C}}$ is singular if and only if $p=(0,0)$.

Proof
The Jacobian at the point p is given by

$$J_{\mathbb{V}(y^2-x^3),(p_1,p_2)}=\begin{pmatrix}-3p_1^2 & 2p_2\end{pmatrix}$$

The Jacobian has rank strictly less than 1 if and only if it is the zero matrix. And this is true if and only if $p_1=p_2=0$. ■

Definition 1.2.7 (Smooth Varieties)
Let k be a field. Let V be a variety over k. We say that V is smooth if every point of V is smooth.

2 The Algebra of Rational Functions

2.1 Rational Functions and The Function Field

Definition 2.1.1 (Function Field) Let k be a field. Let V be a variety over k. Define the set of rational functions on V to be

$$K(V) = \{(U, f) \mid U \subseteq V \text{ open and } f : U \to k \text{ is regular }\}/\sim$$

where we say that $(U, f) \sim (V, g)$ if there exists $W \subseteq U \cap V$ open such that $f|_W = g|_W$. Elements of the function field are called rational functions.

Lemma 2.1.2 Let k be a field. Let V be an variety over k. Define the operations

$$(U, f) + (V, g) = (U \cap V, f + g) \quad \text{and} \quad (U, f) \cdot (V, g) = (U \cap V, fg)$$

Then they induce well defined operations on $K(V)$ so that it is a k-algebra. Moreover, it is a field.

Lemma 2.1.3 Let V be a variety over $\mathbb{C}$. Then the following are true.
- The map $\mathcal{O}_V(V) \to \mathcal{O}_{V,p}$ given by $f \mapsto (V, f)$ is injective for any $p \in V$.
- The map $\mathcal{O}_{V,p} \to k(V)$ given by $(V, f) \mapsto (V, f)$ is injective for any $p \in V$.

Proposition 2.1.4 Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}_k^n$ be an irreducible affine variety. Then there is an isomorphism

$$K(V) \cong \text{Frac}(k[V])$$

Moreover, $K(V)$ is a finitely generated field extension of k.

Proposition 2.1.5 Let k be an algebraically closed field. Let $V \subseteq \mathbb{A}_k^n$ be an irreducible affine variety. Then we have

$$\text{trdeg}_k(K(V)) = \dim(V)$$

Proposition 2.1.6 Let k be an algebraically closed field. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety over k. Then there is a k-algebra isomorphism

$$K(X) \cong (k[V]_{(0)})_0$$

where the zero refers to taking the degree zero part of the graded ring.

Proof ∎

Proposition 2.1.7
Let k be an algebraically closed field. Let X be an irreducible variety over k. Then the following are true.
- For any open subset $U \subseteq X$, we have $k(X) = k(U)$.
- $k(X)$ is a finitely generated field extension.

Proof If $[(V, f)] \in k(X)$, then $[(U \cap V, f|_{U \cap V})]$ by definition. But the latter is also an element of $k(U)$ so $k(X) \subseteq k(U)$. Conversely, if $(V, g) \in k(U)$, then V is also an open subset of X, hence (V, g) is an element of $k(X)$ and so $k(U) \subseteq k(X)$.

Let U be an open affine subset of X. Then $k(X) = k(U)$. Then by prp2.1.4 we know that $k(U)$ is a finitely generated field extension over k. ∎

Example 2.1.8
Let k be an algebraically closed field. Let $n \in \mathbb{N}$. Then we have

$$k(\mathbb{P}^n) \cong k(x_1, \dots, x_n)$$

Proposition 2.1.9
Let k be a field. Let K/k be a finitely generated field extension. Then there exists an irreducible variety X over k such that $K(X) = K$.

2.2 Rational Maps between Varieties

Definition 2.2.1 (An Equivalence Class of Maps) Let X, Y be irreducible varieties. Let $U_1, U_2 \subseteq X$ be open. Let $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ be morphisms of varieties. We say that f_1 and f_2 are equivalent if there exists an open subset $W \subseteq U_1 \cap U_2$ such that

$$f_1|_W = f_2|_W : W \to Y$$

Definition 2.2.2 (Rational Maps) Let X, Y be irreducible varieties. A rational map $f : X \to Y$ is an equivalent class of morphisms of varieties $f : U \to Y$ for some open subset $U \subseteq X$.

Since open subsets of a variety are dense, rational maps are maps that are defined almost entirely on X. In particular, notice that rational functions on an irreducible variety V is the same as a rational map $V \to k$.

Moreover, if $f : X \to Y$ is a regular map (morphism of varieties), then $[(X, f)]$ is also a rational map.

Lemma 2.2.3
Let k be a field. Let V, W be irreducible varieties over k. Let $\phi : V \to W$ be a rational map. Then

$$\phi(x) = [\phi_0(x) : \cdots \phi_n(x)]$$

for some $\phi_0, \dots, \phi_n \in K(V)$.

Proof
This means that there exists an open subset $U \subseteq V$ such that $\phi|_U$ is a morphism of varieties. Morphism of varieties into projective space are given in coordinates by $[\phi_0 : \cdots : \phi_n]$ where ϕ_i is a regular map, or that $\phi_i \in \text{Hom}_{\textbf{Var}}(U, k)$. Moreover, $\phi_0, \dots, \phi_n$ must not simultaneously vanish on U (but it is allowed to simultaneously vanish on some point in V). In particular, ϕ_i is an element of $K(V)$ by definition. ■

Example 2.2.4
Let k be a field. The map of sets $\phi : \mathbb{P}^1_k \to \mathbb{P}^2_k$ defined by

$$\phi([s:t]) = \left[\frac{s+t}{s} : \frac{st+t^2}{(s-t)^2} : \frac{s^2-t^2}{t^2} \right]$$

is a rational map.

Proof
The map of sets is not well defined when any denominator is 0, or when the three coordinates simultaneously vanish. For the former, this happens when $s = 0$ or $t = 0$ or $s = t$. This gives three points $[0:1], [1:0], [1:1]$. The numerators simultaneously vanish when $s = -t$, $t(s+t) = 0$ and $(s+t)(s-t)$. Solving this gives the point $[1:-1]$. Hence this map is a rational map, but not a morphism of varieties. ■

Definition 2.2.5 (Dominant Maps) Let X, Y be irreducible varieties. Let $f : X \to Y$ be a rational map defined on $U \subseteq X$. We say that f is dominant if $f(U)$ contains an open subset.

It only makes sense to compose rational maps if the former one is dominant.

Proposition 2.2.6 Let X, Y, Z be irreducible varieties. Let $f : X \to Y$ and $g : Y \to Z$ be rational maps. If f is dominant, then $g \circ f$ is rational.

Definition 2.2.7 (Induced Map on Rational Functions) Let k be a field. Let X, Y be irreducible varieties over k. Let $\phi : X \to Y$ be a dominant rational map. Define the induced map on rational functions to be

$$\phi^* : K(Y) \to K(X)$$

given by $(U, f) \mapsto (\phi^{-1}(U), f \circ \phi)$.

Proposition 2.2.8 Let k be a field. Let X, Y be irreducible varieties over k. Let $\phi : X \to Y$ be a dominant rational map. Then the induced map $\phi^* : K(Y) \to K(X)$ is a k-algebra homomorphism.

Proposition 2.2.9 Let k be a field. Let X, Y be irreducible varieties over k. Then there is a one-to-one correspondence

$$\left\{\substack{\text{Dominant Rational Maps}\\ X \to Y}\right\} \overset{1:1}{\longleftrightarrow} \left\{\substack{k\text{-algebra homomorphisms}\\ K(Y) \to K(X)}\right\}$$

given by $\phi \mapsto \phi^*$.

2.3 Birational Equivalence

Definition 2.3.1 (Birational Maps) Let X, Y be irreducible varieties. Let $f : X \to Y$ be a dominant rational map defined on $U \subseteq X$. We say that f is a birational map if there exists a dominant rational map $g : Y \to X$ such that

$$g \circ f = \text{id}_U \quad \text{and} \quad f \circ g = \text{id}_V$$

for some open subsets $U \subseteq X$ and $V \subseteq Y$. In this case, we say that X and Y are birational.

Proposition 2.3.2 Let k be a field. Let X, Y be irreducible varieties over k. The the following conditions are equivalent.

- X and Y are birationally equivalent
- There exists open subsets $U \subseteq X$ and $V \subseteq Y$ with U isomorphic to V
- $K(X)$ and $K(Y)$ are isomorphic k-algebras

Example 2.3.3
The cuspidal cubic $\mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is birational to $\mathbb{A}^1_{\mathbb{C}}$.

Proof
Recall that $\mathbb{C}(\mathbb{A}^1_{\mathbb{C}}) \cong \mathbb{C}(t)$. Define a $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}(t) \to \left(\frac{\mathbb{C}[x,y]}{(y^2-x^3)}\right)_{(0)}$ by $t \mapsto y/x$. Since ϕ is a map of fields it is injective. It remains to show that ϕ is surjective. Notice that x and y generate the latter $\mathbb{C}$-algebra and so it suffices to find preimages of x and y. But notice that $t^2 \mapsto y^2/x^2 = x$ and $t^3 \mapsto y^3/x^3 = yx^3/x^3 = y$. Hence we are done. ∎

Example 2.3.4
The nodal cubic $\mathbb{V}(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is birational to $\mathbb{A}^1_{\mathbb{C}}$.

Proof Again recall that $\mathbb{C}(\mathbb{A}^1_{\mathbb{C}}) \cong \mathbb{C}(t)$. Define a $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}(t) \to \left(\frac{\mathbb{C}[x,y]}{(y^2-x^3-x^2)}\right)_{(0)}$ by $t \mapsto y/x$. Since ϕ is a map of fields it is injective. It remains to show that ϕ is surjective. Notice that x and y generate the latter $\mathbb{C}$-algebra and so it suffices to find preimages of x and y. But notice that $t^2 - 1 \mapsto y^2/x^2 - 1 = (x^3 + x^2)/x^2 - 1 = x$ and $t^3 - t \mapsto y^3/x^3 - y/x = y$. Hence we are done. ■

The idea of the $\mathbb{C}$-algebra isomorphism above comes from the parameterization $t \mapsto (t^2 - 1, t^3 - t)$ of the nodal cubic.

Lemma 2.3.5 Let $n \in \mathbb{N}$. Then $\mathbb{A}^n$ is birationally equivalent to $\mathbb{P}^n$.

Proof The function field of $\mathbb{A}^n$ is given by $K(\mathbb{A}^n) = K(x_1, \dots, x_n)$. On the other hand, we can compute the function field of $\mathbb{P}^n$ to get

$$\begin{aligned} K(\mathbb{P}^n) &= \left(k[x_0, \dots, x_n]_{(0)}\right)_0 \\ &= (k(x_0, \dots, x_n))_0 \end{aligned}$$

The zeroth graded component of $k(x_0, \dots, x_n)$ is given by

$$(k(x_0, \dots, x_n))_0 = \left\{ \frac{f}{g} \in k(x_0, \dots, x_n) \mid \deg(f) = \deg(g) \right\}$$

Define a map $\phi : (k(x_0, \dots, x_n))_0 \to k(x_1, \dots, x_n)$ by the map

$$f(x_0, \dots, x_n)/g(x_0, \dots, x_n) \mapsto f(1, x_1, \dots, x_n)/g(1, x_1, \dots, x_n)$$

Evaluation of the zeroth variable with 1 is a well defined $\mathbb{C}$-algebra homomorphism. I claim that this is a bijection. Define another map $\psi : k(x_1, \dots, x_n) \to (k(x_0, \dots, x_n))_0$ by

$$h/k \mapsto x_0^d h(x_1/x_0, \dots, x_n/x_0)/x_0^d k(x_1/x_0, \dots, x_n/x_0)$$

where $d = \max\{\deg(f), \deg(g)\}$. By construction we see that $x_0^d h(x_1/x_0, \dots, x_n/x_0)$ has the same degree as $x_0^d k(x_1/x_0, \dots, x_n/x_0)$ and that they are homogeneous polynomials (similar to the homogenization of a polynomial). Moreover it is clear that ψ and ϕ are inverses of each other. Hence we obtain an isomorphism of $\mathbb{C}$-algebras. ■

Definition 2.3.6 (Rational Varieties) Let k be a field. Let X be a variety over k. We say that X is rational if X is birationally equivalent to $\mathbb{P}^n_k$ for some $n \in \mathbb{N}$.

3 The Categorical Viewpoint

3.1 The Category of Varieties with Regular Maps

Recall that coordinate rings are finitely generated algebras over k the ground field.

Definition 3.1.1 (The Category of Affine Varieties) Define the category of affine varieties AffVar_k over a field k as follows.

- The objects are the affine varieties $\mathbb{V}(F)$ for some set of polynomials F over k.
- The morphisms are the morphisms of affine varieties. This means that $\phi : V \to W$ is such that $\phi(p) = (f_1(p), \dots, f_m(p))$ for some $f_1, \dots, f_m \in k[V]$.

Proposition 3.1.2 Let k be a field. Then there is an equivalence of categories

$$(\text{AffVar}_k)^{\text{op}} \cong (\text{Reduced Finitely Generated } k\text{-algebras})$$

given as follows.

- For every affine algebraic set V, there corresponds a reduced finitely generated k-algebra which is the coordinate ring
$$k[V] = \frac{k[x_1, \dots, x_n]}{I(V)}$$
- For every regular map $\phi : V \to W$, there is a corresponding homomorphism of k-algebras $\phi^* : k[W] \to k[V]$ defined by $f \mapsto f \circ \phi$.

Definition 3.1.3 (The Category of Affine Algebraic Varieties) Define the category of affine algebraic varieties AffAlgVar_k over a field k to be the full subcategory of AffVar_k consisting of irreducible affine varieties.

Proposition 3.1.4 Let k be a field the above equivalence of categories restricts to an equivalence of categories

$$(\text{AffAlgVar}_k)^{\text{op}} \cong (\text{Integral Finitely Generated } k\text{-algebras})$$

Definition 3.1.5 (Category of Varieties) Let k be a field. Define the category of varieties $\mathbf{Var}_k$ as follows.

- The objects are the quasi-projective varieties over k.
- The morphisms are the morphism ofquasi-projective varieties.

Lemma 3.1.6 Let k be a field. Let V, W be varieties over k. Then the following are true.

- If V and W are affine, then $V \times W$ is the categorical product in $\mathbf{Var}_k$.
- If V and W are projective, then $V \times W$ is the categorical product in $\mathbf{Var}_k$.

The goal is now to remove the non-nilpotent condition. For example, we would like to distinguish between the variety $V(x = 0)$ and the variety $V(x^2 = 0)$.

3.2 The Category of Varieties with Dominant Rational Maps

Definition 3.2.1 (Category of Varieties) Let k be a field. Define the category of irreducible varieties $\mathbf{BIrrVar}_k$ as follows.

- The objects are the irreducible quasi-projective varieties over k.
- The morphisms are the dominant rational maps.

Proposition 3.2.2
Let k be an algebraically closed field. Then there is an equivalence of categories

$$(\text{AffAlgVar}_k)^{\text{op}} \cong \left(\substack{\text{Finitely generated} \\ \text{field extensions over } k}\right)$$

given by sending a variety to its function field. The morphisms in the latter category are k-algebra homomorphisms.

4 Differential Forms on Varieties

4.1 Differential Forms as the Module of Kahler Differentials

Definition 4.1.1 (Regular Differential Forms on Varieties) Let k be a field. Let X be a variety over k. Define the module of Kahler differentials of X to be the module

$$\Omega^1_X = \Omega^1_{\mathcal{O}_X(X)/k}$$

Elements of Ω^1_X are called regular differential forms on X.

Definition 4.1.2 (Differential n-Forms on Varieties) Let k be a field. Let X be a variety over k. Define the module of differential n-forms of X to be the module

$$\Omega^n_X = \bigwedge_{i=1}^n \Omega^1_X$$

4.2 The Geometric Genus

Definition 4.2.1 (Geometric Genus) Let k be a field. Let X be a variety over k. Define the geometric genus of X to be

$$p_g(X) = \dim_k\left(\Omega_X^{\dim(X)}\right)$$

5 Different Types of Morphisms

5.1 Embeddings

Definition 5.1.1 (Embeddings) Let k be a field. Let X, Y be varieties over k. Let $\phi : X \to Y$ be a regular map. We say that ϕ is an embedding if X is isomorphic to $\phi(X)$ via ϕ.

Proposition 5.1.2 Let k be a field. Let X, Y be varieties over k. Let $\phi : X \to Y$ be a regular map. Then ϕ is an embedding if and only if the following are true.
- ϕ is injective.
- For all $p \in X$, the induced linear map
$$\phi^* : \frac{m_{\phi(p)}}{m_{\phi(p)}^2} \to \frac{m_p}{m_p^2}$$
is surjective.

5.2 Proper Morphisms

5.3 Finite Morphisms

Definition 5.3.1 (Finite Morphisms) Let k be a field. Let X, Y be varieties over k. Let $f : X \to Y$ be a morphism of varieties. We say that f is finite if there exists an affine cover $W_1, \dots, W_k$ of Y such that $f^{-1}(W_i)$ is affine and that via the map
$$f|_{f^{-1}(W_i)} : f^{-1}(W_i) \to W_i$$
we have that $k[f^{-1}(W)]$ is a finitely generated $k[W]$-module .

Example 5.3.2
Let k be an algebraically closed field. The map $\varphi : \mathbb{V}(y^2 - x(x^2 - 1)) \subseteq \mathbb{A}_k^2 \to \mathbb{A}_k^1$ defined by $(x, y) \mapsto x$ is a finite morphism.

Proof
The induced k-algebra homomorphism is the map $k[x] \to \frac{k[x,y]}{(y^2 - x(x^2-1))}$ given by $x \mapsto x$. Then $\frac{k[x,y]}{(y^2-x(x^2-1)}$ is a finitely generated $k[x]$-module because it is generated by 1 and y. ■

Proposition 5.3.3
Let k be a field. Let X, Y be varieties over k. Let $f : X \to Y$ be a morphism of varieties. Then f is finite if and only if f is proper and $\left|\phi^{-1}(q)\right| < \infty$ is finite for all $q \in Y$.

Proof
Suppose first that f is finite. Without loss of generality suppose that X and Y are affine, otherwise work with affine charts. By definition, $k[X]$ is a finitely generated $k[Y]$-module. Hence $\frac{k[X]}{\phi^*(m_q)}$ is also a finitely generated $\frac{k[Y]}{m_q} \cong k$-module. Notice that we have $\phi^*(m_q) \subseteq \mathbb{I}(\phi^{-1}(q))$ because $\mathbb{V}(\phi^*(m_q)) \subseteq \mathbb{V}(\mathbb{I}(\phi^{-1}(q)))$. Hence the quotient map $k[X] \to k[\phi^{-1}(q)]$ induces a surjective homomorphism $\frac{k[X]}{\phi^*(m_q)} \to k[\phi^{-1}(q)]$. Hence $k[\phi^{-1}(q)]$ is also a finitely dimensional k-algebra. But finite dimensional k-algebras must have finitely many maximal ideals (consider the surjection $R \to \prod \frac{R}{m}$ given by the Chinese remainder theorem). Thus $\phi^{-1}(q)$ is a finite set. ■

Example 5.3.4
Let k be an algebraically closed field. Define the map $f : \mathbb{V}(xy - 1) \subseteq \mathbb{A}_k^2 \to \mathbb{A}_k^1$ by $(x, y) \mapsto x$. Then the following are true.
- For all $q \in \mathbb{A}_k^1$, $f^{-1}(q)$ is finite.
- f is not a finite morphism.

Proof Let $q \in \mathbb{A}^1_k$. Then $f^{-1}(q) = \{(q, y) \mid qy = 1\} = \{(q, 1/q)\}$ so that f has finite fibers. f is not a finite morphism because the induced map of k-algebras is the inclusion map $k[x] \to k\left[x, \frac{1}{x}\right]$ and we have seen that the latter is not a finitely generated $k[x]$-module. ∎

Proposition 5.3.5
Let k be a field. Let X, Y be irreducible varieties over k. Let $\phi : X \to Y$ be a morphism of varieties. If f is finite and dominant, then the induced map

$$k(Y) \to k(X)$$

is a finite field extension.

Proof Let $W \subseteq Y$ be an affine open subset such that $\phi^{-1}(W) \subseteq X$ is an affine open subset and $k[\phi^{-1}(W)]$ is finitely generated over $k[W]$. Let $a_1, \dots, a_r \in k[\phi^{-1}(W)]$ be generators as a $k[W]$-module. Let $S = k[W] \setminus \{0\}$. By definition, we have that $a_1, \dots, a_r$ span $S^{-1}k[\phi^{-1}(W)]$ as a $k(W)$ vector space. Since X is irreducible and $\phi^{-1}(W)$ is an open subset of X, $\phi^{-1}(W)$ is also irreducible and $S^{-1}k[\phi^{-1}(W)]$ is an integral domain that is an algebra over the field $k(W)$. Hence $S^{-1}k[\phi^{-1}(W)]$ is a field. But the smallest field containing an integral domain must be its field of fractions. Hence $S^{-1}k[\phi^{-1}(W)] = k(\phi^{-1}(W))$. Finally, since $\phi^{-1}(W)$ is an open subset of X, $k(\phi^{-1}(W)) = k(X)$ and so $k(X)$ is a finitely generated field extension over $k(W) = k(Y)$. ∎

Definition 5.3.6 (Degree of Finite Morphism) Let k be a field. Let X, Y be varieties over k. Let $f : X \to Y$ be a finite and dominant morphism of varieties. Define the the degree of f to be

$$\deg(f) = \dim_{K(Y)} K(X)$$

6 Normal Varieties

6.1 Normal Varieties

Definition 6.1.1 (Normal Varieties) Let k be a field. Let X be a variety over k. We say that X is normal if $\mathcal{O}_{X,p}$ is a normal domain for all $p \in X$.

Lemma 6.1.2
Let k be a field. Let V be a variety over k. If V is smooth, then V is normal.

Proof
If V is smooth, then $\mathcal{O}_{X,p}$ is a regular local ring, and so it is a UFD and hence a normal domain. ■

If V is a curve then the converse also holds.

Proposition 6.1.3 Let k be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then X is normal if and only if $k[X]$ is a normal domain.

Example 6.1.4 Let k be an algebraically closed field. Then $V = \mathbb{V}(y^2 - x^3) \subseteq \mathbb{A}_k^2$ is not a normal variety.

Proof The coordinate ring of the variety is given by $k[V] = \frac{k[x,y]}{(y^2-x^3)}$. I claim that $\overline{k[V]} = k[[y]/[x]]$.

Firstly, $k[t]$ is a normal domain since any element in $k \in k[t]$ is integral, and t is integral in $k[t]$ by the monic polynomial $z - t \in k[t][z]$. Since sums and products of integral elements are integral, we have that $\overline{k[t]} \subseteq k[t]$. Hence $\overline{k[t]} = k[t]$. Now we have that $k[t^2, t^3] \subseteq k[t]$ which implies that

$$\overline{k[t^2, t^3]} \subseteq \overline{k[t]} = k[t]$$

On the other hand, notice that any $a \in k$ is integral over $k[t^2, t^3]$ via the monic polynomial $z - a \in k[t^2, t^3][z]$. Also, $t \in k[t]$ is integral over $k[t^2, t^3]$ via the monic polynomial $z^2 - t^2 \in k[t^2, t^3][z]$. Since sums and products of integral elements are integral, we thus have that $k[t] \subseteq \overline{k[t^2, t^3]}$. Hence $k[t] = \overline{k[t^2, t^3]}$. Finally we have that $\text{Frac}(k[t^2, t^3]) = k(t)$.

Consider the following diagram:

$$\begin{tikzcd}
k[V] \arrow[r, hook] \arrow[d] & \overline{k[V]} \arrow[r, hook] \arrow[d] & k(V) \arrow[d] \\
k[t^2, t^3] \arrow[r, hook] & \overline{k[t^2, t^3]} = k[t] \arrow[r, hook] & k(t)
\end{tikzcd}$$

It is commutative since all the vertical maps are given by $[x] \mapsto t^2$ and $[y] \mapsto t^3$. I claim that the first map is an isomorphism. To see this, consider the map $\varphi : k[x, y] \to k[t]$ defined by $x \mapsto t^2$ and $y \mapsto t^3$. Notice that $(y^2 - x^3) \subseteq \ker(\varphi)$. Now let $f \in \ker(\varphi)$. By the division algorithm, $f = (y^2 - x^3)g(x, y) + h(x, y)$ for some $g, h \in k[x, y]$ and the degree of y in h is less than or equal to 1. Then $f \in \ker(\varphi)$ implies that $h(x, y) = 0$. But $h(t^2, t^3) = 0$ if and only if $h(x, y) = 0$ by inspecting coefficients. Hence $(y^2 - x^3) = \ker(\varphi)$. It then follows that the third map is an isomorphism and hence the second map is an isomorphism.

Now the preimage of $k[t]$ in the isomorphism is $k[[y]/[x]]$. Hence we have computed the integral closure of $k[V]$. By the above prp clearly $k[V]$ is a not a normal domain and so V is not a normal variety. ■

6.2 Normalization

Definition 6.2.1 (Normalization of an Affine Variety) Let k be a field. Let X be an affine variety over k. Define the normalization of X to be the affine variety $\widetilde{X}$ whose coordinate is given by $k[\widetilde{X}] = \overline{k[X]}$.

Proposition 6.2.2 (Universal Property of Normalization) Let k be a field. Let X be an affine variety over k. Then the normalization $\widetilde{X}$ of X satisfies the following universal property.

- Universal Property: If Z is a normal variety, then for any dominant map $\varphi : Z \to X$, there exists a unique morphism $\widetilde{\varphi} : Z \to \widetilde{X}$ such that the following diagram commutes:
$$\begin{tikzcd} Z \arrow[r, dashed, "\exists!\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$
where π is the induced map from the inclusion $k[X] \hookrightarrow \overline{k[X]} = k[\widetilde{X}]$.
- $\widetilde{X}$ is the unique normal variety (up to unique isomorphism) that satisfies this property.

Proposition 6.2.3 Let k be an algebraically closed field. Let X be an affine variety over k. Then the following are true regarding the induced map

$$\pi : \widetilde{X} \to X$$

from the inclusion $k[X] \hookrightarrow k[\widetilde{X}] = \overline{k[X]}$.

- The map is birational.
- The map is surjective.
- $\pi^{-1}(p)$ is finite for any $p \in X$.

6.3 Projectively Normal

7 Resolution of Singularities

7.1 Blowing Ups

Definition 7.1.1 (Blowing Up at $\mathbb{A}^n$) Let $n \in \mathbb{N}$. Define the blowing up of $\mathbb{A}^n$ at the point $0 \in \mathbb{A}^n$ to be

$$\text{BL}_0(\mathbb{A}^n) = \mathbb{V}\{(x_i y_j - x_j y_i \mid 1 \leq i, j \leq n\}) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

together with the projection map

$$\varphi : \text{BL}_0(\mathbb{A}^n) \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1} \overset{\text{proj.}}{\to} \mathbb{A}^n$$

defined by $(x_1, \dots, x_n, [y_1 : \cdots : y_n]) \mapsto (x_1, \dots, x_n)$.

Proposition 7.1.2 The following are true regarding the blowing up $\text{BL}_0(\mathbb{A}^n)$ of $\mathbb{A}^n$ at 0 and the projection map $\varphi : \text{BL}_0(\mathbb{A}^n) \to \mathbb{A}^n$.

- $\varphi^{-1}(p)$ is a single point for $0 \neq p \in \mathbb{A}^n$.
- $\varphi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$.
- $\text{BL}_0(\mathbb{A}^n)$ is irreducible.

Definition 7.1.3 (The Blowing Up of a Variety) Let $X \subseteq \mathbb{A}^n$ be a closed variety such that $0 \in X$. Define the blowing up of X at 0 to be the set

$$\widetilde{X} = \overline{\varphi^{-1}(X \setminus \{0\})}$$

If X passes through $p \in X$, then the blowing up of X at p is obtained by translating p to the origin and blowing up.

Note: if X is a curve we call this the exceptional divisor.

8 Theory of Divisors

8.1 Divisors of a Variety

Definition 8.1.1 (Divisors of a Variety) Let X be an irreducible variety. Define the free group of divisors of X by

$$\text{Div}(X) = \mathbb{Z}\left\langle C \mid C_{\substack{\text{is an irreducible closed} \\ \text{subvariety of codimension } 1}} \right\rangle$$

We call an element of the free group a divisor of X. We call generators of the free group prime divisors.

Definition 8.1.2 (Effective Divisor) Let X be an irreducible variety. We say that a divisor

$$D = \sum_{i=1}^{r} k_i C_i$$

of X is effective if $k_i \geq 0$ for all i. In this case we write $D > 0$.

Let Y be an irreducible closed subvariety of a variety X of codimension 1. Recall that $\mathcal{O}_{X,Y}$ is a local ring.

Definition 8.1.3 (Regular in Codimension 1) Let k be a field. Let X be an irreducible variety over k. We say that X is regular in codimension 1 if for all irreducible closed subvariety Y of X of codimension 1, we have that $\mathcal{O}_{X,Y}$ is a regular local ring.

Definition 8.1.4 (Divisor of a Function) Let k be an algebraically closed field. Let X be an irreducible variety that is regular in codimension 1. Let $f \in K(X)$. Define the divisor of f to be

$$\text{div}(f) = \sum_{p \in X} v_p(f) \cdot p$$

where v_p is the discrete valuation of the regular local ring $\mathcal{O}_{X,p}$.

A lot of varieties have the property that $\mathcal{O}_{X,p}$ is a regular local ring. A large class of them comes from smooth varieties, but normal? varieties also has this property.

Example 8.1.5

Let k be an algebraically closed field. Let $f = \frac{2x_0 - 2x_1}{x_0} \in k(\mathbb{P}^1)$. Then we have

$$\text{div}(f) = [1 : 1] - [0 : 1]$$

Proof

We compute the contributions of each point on each chart. On $U_0 \cong \mathbb{A}^1_k$, our function becomes $2 - 2x_1$. Let $p \in U_0$. Then $v_p(2 - 2x_1) \neq 0$ if and only if $2 - 2x_1 \in m_p\mathcal{O}_{U_0,p}$. This is true if and only if $2 - 2p = 0$ and so $p = 1$ is the only point in this chart giving a non-zero contribution. Since $m_1\mathcal{O}_{U_0,1}$ is generated by $(x_1 - 1)$, we see that $v_1(2 - 2x_1) = v_1(-2(x_1 - 1)) = 1$. Thus $[1 : 1]$ has a coefficient of 1 in $\text{div}(f)$.

On $U_1 \cong \mathbb{A}^1_k$, our function becomes $\frac{2x_0 - 2}{x_0}$. Let $p \in U_1$. Notice that $\text{val}_p\left(\frac{2x_0-2}{x_0}\right) = \text{val}_p(2x_0 - 2) - \text{val}_p(x_0)$, we can treat the numerator and denominator separately. Similar to the above, we have that $\text{val}_p(2x_0 - 2) \neq 0$ if and only if $p = 1$. In this case, we have $\text{val}_1(2x_0 - 2) = \text{val}_1(2(x_0 - 1)) = 1$. But in this case our point in projective coordinates is given by $[1 : 1]$, and we have already computed its contribution on the chart U_0. Now $-\text{val}_p(x_0) \neq 0$ if and only if $p = 0$. Since $m_0\mathcal{O}_{U_1,0}$ is generated by x, we see that $-\text{val}_0(x_0) = -1$. Hence $[0 : 1]$ has a coefficient of -1 in $\text{div}(f)$. Hence $\text{div}(f) = [1 : 1] - [0 : 1]$. ∎

Lemma 8.1.6 Let k be an algebraically closed field. Let X be an irreducible variety over k that is regular in codimension 1. Let $f \in K(X)$. Then we have $\text{div}(f) \in \text{Div}(X)$ is a divisor of X.

Definition 8.1.7 (Principal Divisors) Let k be an algebraically closed field. Let X be an irreducible variety over k that is regular in codimension 1. Define the subgroup of principal divisors of X to be

$$\text{Prin}(X) = \{\text{div}(f) \mid f \in K(X)\} \leq \text{Div}(X)$$

Definition 8.1.8 (Divisor Class Group) Let k be an algebraically closed field. Let X be an irreducible variety over k that is regular in codimension 1. Define the divisor class group of X to be

$$\text{Pic}(X) = \frac{\text{Div}(X)}{\text{Prin}(X)}$$

We say that two divisors D and D' of X are linearly equivalent if $[D] = [D'] \in \text{Pic}(X)$. In this case we write $D \sim D'$.

Definition 8.1.9 (Degree of a Divisor) Let X be an irreducible variety. Define the degree homomorphism $\deg : \text{Div}(X) \to \mathbb{Z}$ to be

$$\deg\left(\sum_{i=1}^{r} k_i Y_i\right) = \sum_{i=1}^{r} k_i \deg(Y_i)$$

where $\deg(Y_i)$ is the degree of the hypersurface Y_i.

Proposition 8.1.10

Let k be an algebraically closed field. The following are true regarding the projective space $\mathbb{P}_k^n$.

- For any divisor $D \in \text{Div}(\mathbb{P}_k^n)$, we have $D \sim \deg(D)\mathbb{V}(x_0)$.
- A divisor $D \in \text{Div}(\mathbb{P}_k^n)$ is principal if and only if $\deg(D) = 0$.
- There is an isomorphism $\text{Pic}(\mathbb{P}_k^n) \cong \mathbb{Z}$ given by $\deg : \text{Pic}(\mathbb{P}_k^n) \to \mathbb{Z}$.

Proof

Special case $n = 1$: Suppose that $D = \sum_{i=1}^{n} a_i p_i - \sum_{j=1}^{m} b_j q_j$ where $a_i, b_j \in \mathbb{N}$ and $p_i, q_j \in \mathbb{P}_k^1$. Let

$$f(x) = \frac{(x - p_1)^{a_1} \cdots (x - p_n)^{a_n}}{(x - q_1)^{b_1} \cdots (x - q_m)^{b_m}}$$

This is a well defined rational function on $\mathbb{P}^1$ because f is regular on the dense open set $\mathbb{P}_k^1 \setminus \{q_1, \dots, q_m\}$. Moreover, it is clear that $\text{div}(f) = D$. ■

9 Intersection Theory