A Hybrid Polytope Exploration-Based Approach to Solve Mixed-Integer Bilevel Linear Programming Problems

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***Abstract*—To do**

Keywords: Bilevel Programming, n-Dimensional Polytope, Stackelberg Games.

I. Introduction

Very similar to all the other papers we've worked on and still not complete

A. What is MIBLP

Stackelberg Games are Game Theoretical scenarios where we find a leader (the principal) that makes decisions pursuing a certain objective and then, sequentially, a follower (the agent) counterpart makes decisions reacting to those of the leader [6]. Bilevel optimization allows to model such situations by incorporating constraints in the follower problem that depend on the leader's decision variables, as well as to model the influence of the follower decisions in the leader's objective function. Bilevel optimization is then a suitable framework for modeling hierarchical two-decision-maker scenarios since it allows to have both 'players' responding to a shared set of rules, but recognizing that each can act in pursuit of their own objectives.

B. Difficulties within MIBLPs

Due to the intrinsic nature of Bilevel Linear Programs (BLPs) where convexity does not hold for the problem's feasible region, these types of problems are classified as NP-Complete [5]. There has been a lot of work devoted to solve BLPs to optimality using single level reformulations that exploit the KKT optimality conditions of the follower sub-problem when all variables are continuous in their domain [9]. However, when integer constraints are enforced to some (if not all) variables in the BLPs, Mixed-Integer Linear Bilevel Programming problems (MIBLPs) arise, significantly increasing the difficulty to optimally solve them.

MIBLPs have drawn the attention of many researchers and academics for their usefulness to model game theoretical combinatorial decision-making processes that consider multiple decision-makers with (possibly) conflicting or diverging objective functions. A wide variety of solution strategies have been proposed to solve MIBLPs to optimality or to retrieve a good quality solution. Some of these approaches include single level reductions applying Branch & Bound methods [1]; descent methods, [4]; Branch & Cut techniques incorporating the ideas of the Corner Polyhedron and Intersection Cuts (ICs) [[3], [2]], among others.

C. Why our approach contributes to the OR community

There is not a standardized method to solve bilevel problems [8]

D. Paper's proposal

In this research project we propose a general approach that can address all the possible cases of a mixed integer linear bilevel program: only the leader variables take integer values (DCLB), both the leader and the follower variables are forced to be integers (DLB), only the follower variables can take integer values (CDLB), and both the leader and the follower variables are purely contonuous (CCLB) [10]. Our methodology retrieves the optimal solution for a set of instances from the literature, provides a list of feasible and possibly optimal solutions for a set of self-generated instances. Additionally, we apply our methodology to a Public Private Partnership case of study addressed in **CITEMYSELF**, where the interactions of a public contractor and a private party are modeled as a MIBLP in the context of an infrastructure operation project. Performance of the learning processes of our implementations are presented as well as the hyper parameters values of choice. Moreover, our methodology is able to provide local bilevel optimallity gaps in order to provide a measure for the quality of the incumbent for a local domain.

E. Structure

This document is organized as follows. Section **II** presents the literature review related to the problem's context; We review Bilevel Linear Programming problems (BLPs), MIBLPs, and the use of heuristic and hybrid approaches intended to solve them. Section **III** presents the polytope exploration based approach, which we named learn and cut (placeholder name). Section **IV** presents the results for a variety of testbed instances from the literature that we considered in our computational study as well as the set of self-generated instances. Section **V** reviews the PPP problem from the literature and presents its results under the proposed methodology. Section **VI** synthesizes the main contents of our research project; discusses the challenges of our solution approach, while it presents a brief analysis for the overall results of our instances testbed. Moreover, limitations, assumptions, and future work regarding our implementation is encouraged. Finally, the appendix presents the mathematical formulation for every testbed instances.

II. Literature Review

Same as all the other papers we've worked on

A. Mixed-Integer Bilevel Linear Programming

B. MIBLPs review

MIBLPs are nested optimization problems where the outer problem is often called the *upper level* or leader problem while the inner problem is called the *lower level* or follower problem. An example of a MIBLP structure, as shown in [3], is presented as follows:

$$\min_{x,y} c_x^T x + c_y^T y \tag{1}$$

$$G_x x + G_y y \leqslant q \tag{2}$$

$$x_j \text{ integer}, \ \forall\ j \in J_x \tag{3}$$

$$x^- \leqslant x \leqslant x^+ \tag{4}$$

$$y^- \leqslant y \leqslant y^+ \tag{5}$$

$$\begin{aligned} y \in \arg\min_{y \in \mathbb{R}^{n_2}} \{ & d^T y : Ax + By \leqslant b,\ l \leqslant y \leqslant u, \\ & y_j \text{ integer}, \ \forall\ j \in J_y \} \end{aligned} \tag{6}$$

$$J_x \subseteq N_x := \{1, \ldots, n_1\},\ J_y \subseteq N_y := \{1, \ldots, n_2\} \tag{7}$$

Where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, while c_x, c_y, G_x, G_y, q, d, A, B, b, l, u, x^-, x^+, y^- and y^+ are given matrices/vectors of appropriate size. Also, sets J_x and J_y represent the indexes for the integer-constrained x (leader) and y (follower) variables, respectively [3]. Expressions (1)-(5) correspond to the leader problem while expression (6) corresponds to the follower problem.

C. Importance of MIBLPs and disclaimer for optimistic position

MIBLPs have been granted special attention in recent years due to their challenging nature, depending on which level of the MIBLP (leader or follower) presents the discrete conditions for the corresponding decision variables. Our research assumes an optimistic position for MIBLPs, meaning that given multiple follower optimal solutions, the leader expects the follower to choose the solution that leads to the best objective function value for the leader. The reader can refer to [8] for a detailed review on BLPs history, solution approaches and the pessimistic position.

III. Methodology

Draft and has almost the same contents as previous papers

A. Methodology proposal

We describe an alternative way to formulate the MIBLP and some necessary concepts.

First the MIBLP has to be restated to its *value function formulation* ([7], [3]) as follows:

$$\min_{x,y} c_x^T x + c_y^T y \tag{8}$$

$$G_x x + G_y y \leqslant q \tag{9}$$

$$A_x x + B_y y \leqslant b \tag{10}$$

$$x_j \text{ integer}, \ \forall\ j \in J_x \tag{11}$$

$$y_j \text{ integer}, \ \forall\ j \in J_y \tag{12}$$

$$d^T y \leqslant \Phi(x) \tag{13}$$

Where $\Phi(x)$ in expression (13) denotes the *follower value function* for a given $x^* \in \mathbb{R}^{n_1}$ by computing the follower MILP:

$$\begin{aligned} \Phi(x^*) := \min_{y \in \mathbb{R}^{n_2}} \{ & d^T y : By \leqslant\ b - Ax^*,\ l \leqslant y \leqslant u, \\ & y_j \text{ integer}, \ \forall\ j \in J_y \} \end{aligned} \tag{14}$$

By dropping expression (13) from the model (8)-(13), the above formulation results in a MILP called the *high-point-relaxation* (HPR), whose Linear Programming (LP) relaxation is denoted by $\overline{\text{HPR}}$.

Bilevel feasibility is violated for a point (x, y) if it violates expression (13), whereas it is satisfied if it met conditions (8)-(13).

Two assumptions need to be considered to successfully retrieve bilevel (possibly optimal) solutions under our Q-learning approach:

1) The $\overline{\text{HPR}}$ feasible set is bounded; the follower MILP (14) has a finite optimal solution $\hat{y}$ for every feasible HPR $(x^*, \cdot)$ [3].
2) The $\overline{\text{HPR}}$ is feasible for at least one HPR $(x^*, \cdot)$ and its corresponding best-follower response $\hat{y}$.

B. Methodology description

The proposed approach consists of finding vertices of the n-dimensional polytope of the HPR's feasible region, that is, the convex hull of the HPR and not the one of $\overline{\text{HPR}}$). We find this vertices by generating random objective functions for the HPR, solving them and saving the resulting solution vertex. Additionally, we add a cut to the problem that guarantees that any found vertex must lie within an updated version of the feasible space of the polytope, which consists of satisfying that the original objective function has to be less than or equal to the incumbent objective function value. Since these polytopes are convex spaces in n-dimentions, we can derive points that belong to them as a convex combination of their vertices. Under the case of a DCLB or DDLB, With a certain probability, we round the leader variable values to the nearest integer for the generated points in the polytopes. The latter step retrieves a valid local dual bound for the underlying MIBLP since the follower variables values for such points are going to be the result of solving the follower problem for the respective leader variable values.

C. Implementation details

The implementation details of our polytope exploration-based approach **Algorithm 1** are presented as follows:

D. Hardware and software

We used **Gurobi Optimizer 9.0.1 in Sublime-Python IDE** to compute the initial positions, check HPR and bilevel feasibility, and retrieve the corresponding objectives for both the leader and the follower. Experiments were run in an **HP DESKTOP-OJKJ4ND** with processor **AMD Ryzen 3 2300U with Radeon Vega Mobile Gfx 2.00 GHz** and **12.00 GB RAM**.

Additionally, a possible extra step in 1 can be performed in the presence of DCLB or DDLB. Right before solving the HPR for a randomized objective function to find a vertex of the polytope, if LP exist and at least one of the leader variable values has a fractional solution, we randomly create a temporary bound constraint for one of these integer constraint fractional leader variables so we can temporary narrow the polytope's feasible space in order to possibly update the incumbent. However, if this procedure is performed, for the current exploration step, the retrieved vertex is not stored as a new vertex.

Algorithm 1 Learn & Cut

Input: t: A parameter that indicates how long will the exploration continue after updating the incumbent, ε: Probability of performing certain tasks in the exploration.
Output: A feasible bilevel solution.

Initialize a set of vertices $V = \varnothing$.
Initialize the incumbent objective value for $\overline{\text{HPR}}$, $z_{LP} = \infty$.
Initialize the incumbent objective value found for HPR, $z_{MILP} = \infty$.
while not *done* **do**
  add vertex = *True*.
  add constraint $c_x^T x + c_y^T y \leqslant z_{MILP}$
  if a random number $\leqslant \varepsilon$ **then**
    Solve the HPR for a randomly generated objective function.
  else
    Solve the $\overline{\text{HPR}}$ for a randomly generated objective function.
    add vertex $\leftarrow$ *False*
  end if
  let $\vec{\xi}$ be the solution found.
  if $\vec{\xi}$ not in V and *add vertex* is *True* **then**
    add $\vec{\xi}$ to V.
    Solve the follower problem to compute $\Phi(\vec{\xi}')$ and the best follower's response $\hat{y}$.
    update the incumbent, z_{LP}, and/or z_{MILP} if needed.
    if incumbent update **then**
      set $V \leftarrow \varnothing$.
    end if
  end if
  if $V \neq \varnothing$ **then**
    Randomly select a subset V', where $V' \subseteq V$.
  end if
  let $\vec{\xi}'$ be a convex combination of all the elements in V'.
  if a random number $\leqslant \varepsilon$ **then**
    round to the nearest integer every leader variable element in $\vec{\xi}'$ and update it.
  end if
  Solve the follower problem to compute $\Phi(\vec{\xi}')$ and the best follower's response $\hat{y}$.
  update the incumbent, z_{LP}, and/or z_{MILP} if needed.
  if incumbent update **then**
    set $V \leftarrow \varnothing$.
  end if
  if incumbent exists **and** t exceeded **then**
    done $\leftarrow$ *True*
  end if
end while

IV. TESTBED & RESULTS

Not ready

V. PPP INSTANCE

Not ready

VI. CONCLUSIONS

Not ready

VII. Bilevel vs. multi-bjective

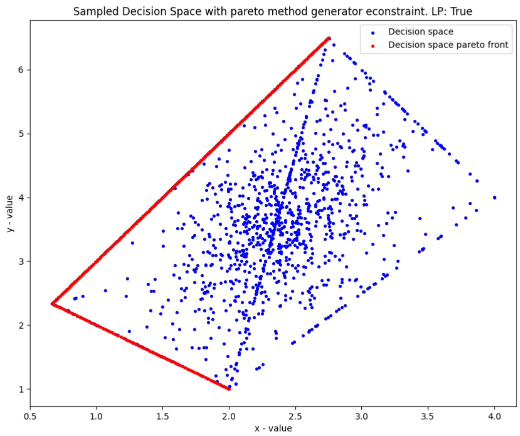

Fig. 1. Multi-objective pareto frontier instance 1

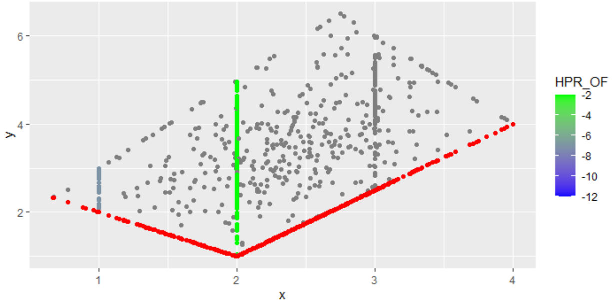

Fig. 2. Bilevel follower inducible region instance 1

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Instance problems:

1)

$$\min x - 4y$$

s.t.,

$$\min y$$

s.t.,

$$-x - y \leqslant -3$$

$$-2x + y \leqslant 1$$

$$2x + y \leqslant 12$$

$$3x - 2y \leqslant 4$$

$$10x + y \geqslant 20$$

$$x,\ y \in \mathbb{Z}^+ \cup \{0\}$$

2)

$$\min x + 3y$$

s.t.,

$$x \geqslant 0$$

$$x \leqslant 6$$

$$\min -y$$

s.t.,

$$x + y \leqslant 8$$
$$x + 4y \leqslant 8$$
$$x + 2y \geqslant 13$$
$$x,\ y\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

3)

$$\min -x - 10y$$

s.t.,

$$\min y$$

s.t.,

$$-25x + 20y \leqslant 30$$
$$x + 2y \leqslant 10$$
$$2x - y \leqslant 15$$
$$2x + 10y \geqslant 15$$
$$x,\ y\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

4)

$$\min x + 2y$$

s.t.,

$$\min -y$$

s.t.,

$$-x + 2.5y \leqslant 3.75$$
$$x + 2.5y \geqslant 3.75$$
$$2.5x + y \leqslant 8.75$$
$$x,\ y\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

5)

$$\min -2x_1 + x_2 + 0.5y_1$$

s.t.,

$$x_1 + x_2 \leqslant 2$$
$$\min -4y_1 + y_2$$

s.t.,

$$2x_1 - y_1 + y_2 \geqslant 2.5$$
$$-x_1 + 3x_2 - y_2 \geqslant -2$$
$$x_1,\ x_2,\ y_1,\ y_2\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

6)

$$\min -x - 2y_1 - 3y_2$$

s.t.,

$$0 \leqslant x \leqslant 8$$
$$\min -y_1 + y_2$$

s.t.,

$$x + y_1 + y_2 \leqslant 10$$
$$0 \leqslant y_1 \leqslant 9$$
$$0 \leqslant y_2 \leqslant 7$$
$$x,\ y_1,\ y_2\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

7)

$$\min -20x_1 - 60x_2 - 30x_3 - 50x_4 - 15y_1 - 10y_2 - 7y_3$$

s.t.,

$$\min -20y_1 - 60y_2 - 8y_3$$

s.t.,

$$5x_1 + 10x_2 + 30x_3 + 5x_4 + 8y_1 + 2y_2 + 3y_3 \leqslant 230$$
$$20x_1 + 5x_2 + 10x_3 + 10x_4 + 4y_1 + 3y_2 \leqslant 240$$
$$5x_1 + 5x_2 + 10x_3 + 5x_4 + 2y_1 + y_3 \leqslant 90$$
$$x_1,\ x_2,\ x_3,\ x_4\ \in\ \{0, 1\}$$
$$y_1,\ y_2,\ y_3\ \geqslant\ 0$$

8)

$$\min 2x - 11y$$

s.t.,

$$\min x + 3y$$

s.t.,

$$x - 2y \leqslant 4$$
$$2x - y \leqslant 24$$
$$3x + 4y \leqslant 96$$
$$x + 7y \leqslant 126$$
$$-4x + 5y \leqslant 65$$
$$-x - 4y \leqslant -8$$
$$x,\ y\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

9)

$$\min x - 4y$$

s.t.,

$$\min y$$

s.t.,

$$-2x + y \leqslant 0$$
$$2x + 5y \leqslant 108$$
$$2x - 3y \leqslant -4$$
$$x,\ y\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

10)

$$\min -4x_1 + 8x_2 + x_3 - x_4 + 9y_1 - 9y_2$$

s.t.,

$$-9x_1 + 3x_2 - 8x_3 + 3x_4 + 3y_1 \leqslant 1$$
$$4x_1 - 10x_2 + 3x_3 + 5x_4 + 8y_1 + 8y_2 \leqslant 25$$
$$4x_1 - 2x_2 - 2x_3 + 10x_4 - 5y_1 + 8y_2 \leqslant 21$$
$$9x_1 - 9x_2 + 4x_3 - 3x_4 - y_1 - 9y_2 \leqslant -1$$
$$-2x_1 - 2x_2 + 8x_3 - 5x_4 + 5y_1 + 8y_2 \leqslant 20$$
$$7x_1 + 2x_2 - 5x_3 + 4x_4 - 5y_1 \leqslant 11$$
$$\min -9y_1 + 9y_2$$

s.t.,

$$-6x_1 + x_2 + x_3 - 3x_4 - 9y_1 - 7y_2 \leqslant -15$$
$$4x_2 + 5x_3 + 10x_4 \leqslant 26$$
$$-9x_1 + 9x_2 - 9x_3 + 5x_4 - 5y_1 - 4y_2 \leqslant -5$$
$$5x_1 + 3x_2 + x_3 + 9x_4 + y_1 + 5y_2 \leqslant 32$$
$$x_1,\ x_2,\ x_3,\ x_4,\ y_1\ y_2\ \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

11)

$$\min \sum_{i=0}^{6} x_i$$

s.t.,

$$y_6 + x_6 \leqslant 1$$
$$3y_0 + 3y_1 + 3y_2 + 4y_3 + 4y_4 + 5y_5 + 6y_6 \leqslant 13$$
$$\min -3y_0 - 3y_1 - 3y_2 - 4y_3 - 4y_4 - 5y_5 - 6y_6$$

s.t.,

$$y_0 + 2y_1 + 2y_2 + 3y_3 + 3y_4 + 4y_5 + 5y_6 \leqslant 10$$
$$y_i + x_i \leqslant 1\ \forall\ i \in \{0, \ldots, 5\}$$
$$x_i,\ y_i,\ \in\ \mathbb{Z}^+\ \cup\ \{0\}\ \forall\ i \in \{0, \ldots, 6\}$$

12)

$$\min 8x_1 + 7x_2 + 10x_3 + 4x_4$$

s.t.,

$$\sum_{i=1}^{4} w_i \leqslant 2$$
$$\min -8x_1 - 7x_2 - 10x_3 - 4x_4$$

s.t.,

$$3x_1 + 4x_2 + 6x_3 + 3x_4 \leqslant 9$$
$$x_i \leqslant 1 - w_i\ \forall\ i \in \{1, 2, 3, 4\}$$
$$x_i,\ w_i\ \in\ \{0, 1\}\ \forall\ i \in \{1, 2, 3, 4\}$$

13)

$$\min -2x_1 + x_2 + x_3 - 2x_4 - x_5 + 3.5x_6 + y_1 + 1.5y_2 - 3y_3$$

s.t.,

$$\min -2x_2 + x_5 - 3y_1 + y_2 + 4y_3$$

s.t.,

$$-x_1 + 0.2x_2 + x_5 + 2x_6 - 4y_1 + 2y_2 + y_3 \leqslant 12$$
$$x1 + x_3 - 2x_4 - 4y_2 + y_3 \leqslant 10$$
$$5x_1 + x_4 + 3.2x_6 + 2y_1 + 2y_2 \leqslant 15$$
$$-3 * x_2 - x_4 + x_5 - 2y_1 \leqslant 12$$
$$-2x_1 - x_2 - y_2 + y_3 \leqslant -2$$
$$-y_1 - 2y_2 - y_3 \leqslant -2$$
$$-2x_2 - 3x_3 - x_5 \leqslant -3$$
$$x_i\ \in\ \mathbb{Z}^+\ \cup\ \{0\}\ \forall\ i \in \{1, \ldots, 6\}$$
$$y_i\ \in\ \mathbb{Z}^+\ \cup\ \{0\}\ \forall\ i \in \{1, \ldots, 3\}$$

14)

$$\min x - 4y$$

s.t.,

$$\min y$$

s.t.,

$$-\frac{12}{17}x + \frac{5}{6}y \leqslant 26$$
$$-\frac{5}{18}x + y \leqslant 37$$
$$-\frac{4}{19}x + \frac{4}{3}y \leqslant 55$$
$$-\frac{2}{6}x - y \leqslant -17.5$$
$$-x \leqslant -3.5$$
$$-\frac{8}{9}x - \frac{7}{30}y \leqslant -9.8$$
$$-\frac{12}{17}x - y \leqslant -22$$
$$-2x - 10y \leqslant -147$$
$$-y \leqslant -8.8$$
$$2x - y \leqslant 110$$
$$x + 5y \leqslant 303$$
$$\frac{17}{31}x + \frac{1}{10}y \leqslant 41.7$$
$$2x + 2y \leqslant 225$$
$$x - 4y \leqslant 6$$
$$x \leqslant 69$$
$$x - 2y \leqslant 30$$
$$y \leqslant 48.8$$
$$-\frac{7}{13}x + y \leqslant 33.2$$
$$-x - \frac{1}{2}y \leqslant -16$$
$$\frac{1}{4}x - \frac{1}{4}y \leqslant 11.2$$
$$x, y \in\ \mathbb{Z}^+\ \cup\ \{0\}$$

15)
$$\min 5x_1 + 7x_2 - 10y_1 + 3y_3 - 1y_3$$

s.t.,
$$\min 10y_1 - 3y_3 + 1y_3$$

s.t.,
$$-x_1 + 3x_2 - 5y_1 - y_2 + 4y_3 \leqslant -2$$
$$-2x_1 + 6x_2 - 3y_1 - 2y_2 + 2y_3 \leqslant -5$$
$$x2 - 2y_1 + y_2 - y_3 \leqslant -2$$
$$x_i, \in \{0,1\} \ \forall \ i \in \{1,2\}$$
$$y_i, \in \{0,1\} \ \forall \ i \in \{1,2,3\}$$

16)
$$\min -3x_1 - 2x_2 + 5x_3 + 2y_1 - 3y_2$$

s.t.,
$$\min -2y_1 + 3y_2$$

s.t.,
$$x_1 + x_2 + x_3 + 2y_1 + y_2 \leqslant 4$$
$$7x_1 + 3x_3 - 4y_1 + 3y_2 \leqslant 7$$
$$-11x_1 + 6x_2 + y_1 - 3y_2 \leqslant -10$$
$$x_i, \in \{0,1\} \ \forall \ i \in \{1,2,3\}$$
$$y_i, \geqslant 0 \ \forall \ i \in \{1,2\}$$

17)
$$\min 25x_1 - 1.5x_2 - 10x_3 - 0.5x_4 - 3x_5 - 2.5x_6 + 87y_1 + 101y_2 + 93y_3 + 55y_4$$

s.t.,
$$\min -3y_1 - 5y_2 - 7y_3 - 11y_4$$

s.t.,
$$2x_1+3x_2+7x_3+6x_4+x_5+3x_6+5y_1+5y_2-9y_3+4y_4 \leqslant 193$$
$$x_1 + 4x_2 + 7x_3 + 5x_4 + 7x_6 - 4y_1 + y_3 - 2y_4 \leqslant 173$$
$$-x_1-2x_2-x_3-x_4-3x_5-5x_6-7y_1-3y_2-4y_3-7y_4 \leqslant -31$$
$$4x_1+3x_2+7x_3+7x_5+7x_6+5y_1-3y_2+2y_3-3y_4 \leqslant 310$$
$$-4x_1-x_2-3x_3-5x_4-3x_5-7x_6-3y_1-2y_2-5y_3-6y_4 \leqslant -214$$
$$2x_1+7x_2+3x_3+4x_4+8x_5+8x_6-6y_1+y_2-4y_3+y_4 \leqslant 157$$
$$y_1 \leqslant 23$$
$$y_2 \leqslant 4$$
$$y_3 \leqslant 10$$
$$y_4 \leqslant 7$$
$$x_i \in \mathbb{Z}^+ \cup \{0\} \ \forall \ i \in \{1,\ldots,6\}$$
$$y_i \in \mathbb{Z}^+ \cup \{0\} \ \forall \ i \in \{1,\ldots,4\}$$